# CEE 631
# CONSTRUCTION DECISIONS UNDER UNCERTAINTY

# Class Notes

## COMPETITIVE BIDDING MODELS



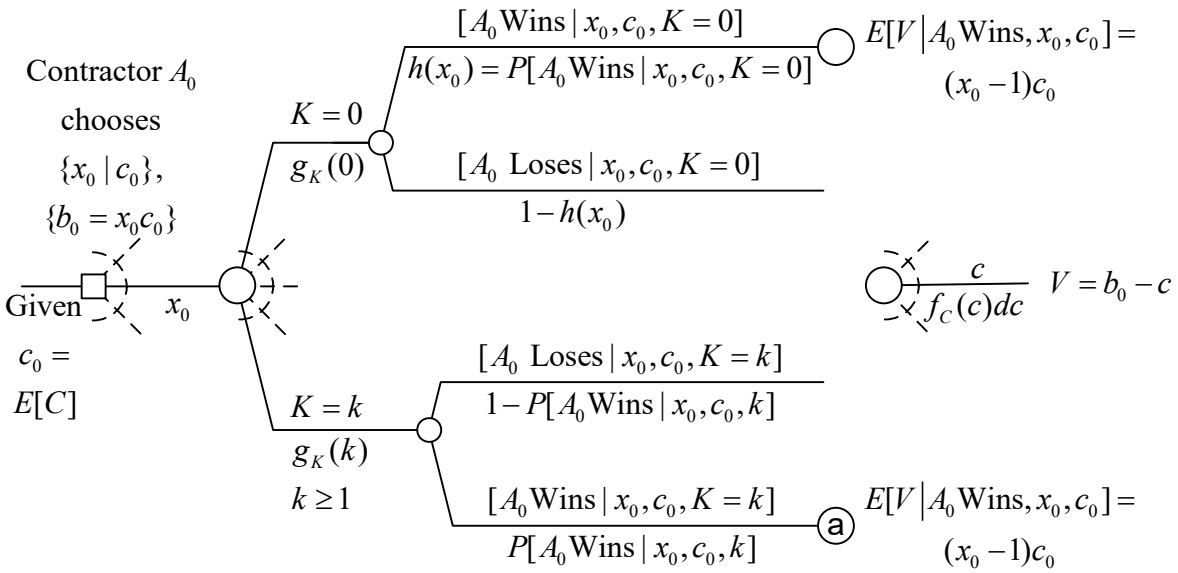

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## Contracting Firm's Bidding Decision

Contractor $A_0$ must decide its own *lump sum bid* $b_0$ when bidding to win a new project against an unknown number of opponents $K$. Alternatively, contractor $A_0$ must decide the bid-to-cost ratio $x_0$ (where, $x_0 = FBC$ = "*Firm's Bid-to-Cost ratio*") to use for this project and compute its *bid* $b_0$, by multiplying $FBC = x_0$ by its estimate of the project's expected cost, $E[C] = c_0$.

$$\text{Contractor } A_0\text{'s } Bid = b_0 = (FBC)\ E[C] = x_0\ c_0$$

The decision tree below shows that contractor $A_0$ must decide that value of $FBC = x_0$ (and thus calculate its bid $b_0 = x_0\ c_0$), that maximizes its expected profit for this project $V$ (given by the *EMV* expression @ (d) below) when bidding against an unknown number of opponents $K \geq 0$.




○ $E[V \mid A_0 \text{Wins}, x_0, c_0] = \int_c (b_0 - c) f_C(c) dc = b_0 - E[C] = b_0 - c_0 = (x_0 - 1)c_0$

$P[A_0 \text{Wins} \mid x_0, c_0] = g_K(0)h(x_0) + \sum_{k=1}^{\infty} g_K(k) P[A_0 \text{Wins} \mid x_0, c_0, k]$

○ $E[V \mid x_0, c_0] = E[V \mid A_0 \text{Wins}, x_0, c_0] P[A_0 \text{Wins} \mid x_0, c_0]$

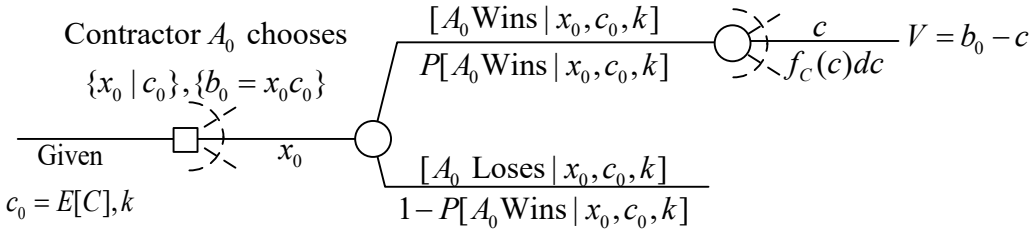
$= \{g_K(0)h(x_0) + \sum_{k=1}^{\infty} g_K(k) P[A_0 \text{Wins} \mid x_0, c_0, k]\}(x_0 - 1)c_0$

*FBC* = $x_0$= *Firm's Bid-to-Cost* ratio = *Bid* / $E[Cost]$ = *Bid* /$E[C]$ = $b_0 / c_0$.
*FBC* = $x_0$ is the main decision variable for contractor $A_0$.
(Note that in the above notation, "*Firm*" = "Contractor $A_0$ as a company".)

*Cost C* = Project's direct cost whose expected value is estimated to be $E[C] = c_0$.
Note: *C* includes only *direct* project costs (which include project OH).
*I.e.*, *C* includes those costs that do not exist if the project is not performed.
The actual value $c$ ($\neq c_0$) of the project cost *C* is unknown. The actual cost $c$ will only be known if $A_0$ wins, and then only after the project is constructed.

## Contracting Firm's Bidding Decision with Known number of Opponents *k*

Bidding models often assume that contractor $A_0$ knows (a) the identities of the $k$ opponents, or (b) just the number of opponents $k$, that he will bid against on an upcoming project. In such cases, the above decision tree is simplified as follows:




- $E[V \mid A_0 \text{Wins}, x_0, c_0] = \int_c (b_0 - c) f_C(c) dc = b_0 - E[C] = b_0 - c_0 = (x_0 - 1)c_0$
- $E[V \mid x_0, c_0, k] = E[V \mid A_0 \text{Wins}, x_0, c_0] P[A_0 \text{Wins} \mid x_0, c_0, k]$
  $= P[A_0 \text{Wins} \mid x_0, c_0, k](x_0 - 1)c_0$

## How is *FBC* related to the Firm's Markup?

Markup$ is the amount of money which is added to the project's estimated expected direct cost, $E[C] = c_0$, to arrive at a bid price. Markup% is a percentage which when multiplied by the project's estimated expected direct cost, $E[C] = c_0$, gives Markup$:

$$Bid = b_0 = E[C] + Markup\$$$

$$Markup\% = \frac{Markup\$}{E[C]} = \frac{Bid - E[C]}{E[C]} = \frac{b_0}{c_0} - 1 = FBC - 1 = x_0 - 1$$

Markup$ covers:

(a) "General & Administrative Overhead" (G&A OH) and
(b) Net profit = Gross profit ($V$) - (G&A OH allocated to the project).

## What is the Firm's Expected Gross Profit *V* on a Project?

After contractor $A_0$ wins the job, its expected profit is "*EMV* @ (a)":

$$E[V \mid A_0 \text{Wins}, x_0, c_0] = \int_c (b_0 - c) f_C(c) dc = b_0 - E[C] = b_0 - c_0 = (x_0 - 1)c_0$$

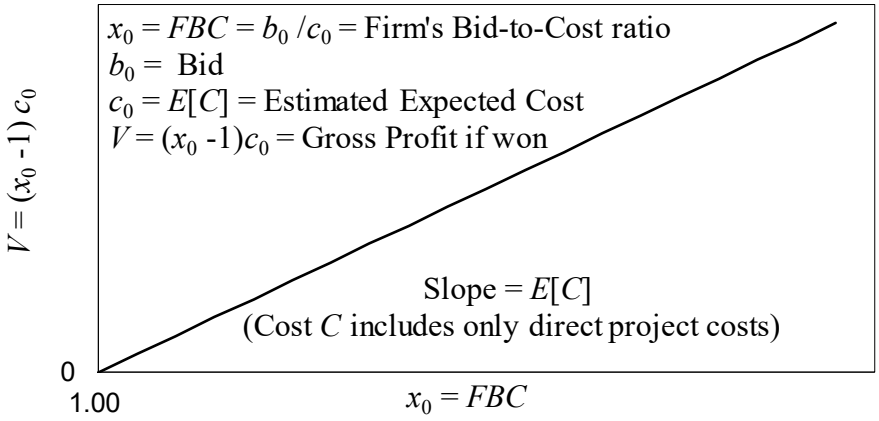
Before contractor $A_0$ bids (and possibly wins) the project, its expected profit is "*EMV* @ (d)":

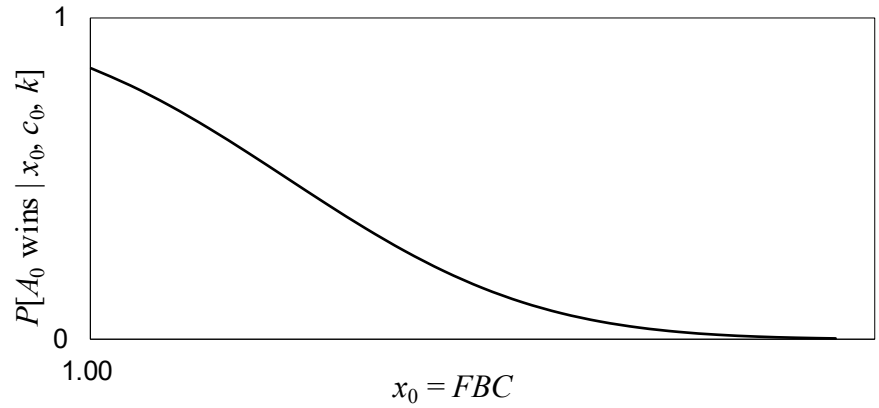
$$E[V \mid x_0, c_0, k] = P[A_0 \text{Wins} \mid x_0, c_0, k](x_0 - 1)c_0$$

The graph of the expected profit, "*EMV* @ (d)", as a function of $FBC = x_0$, is shown below.

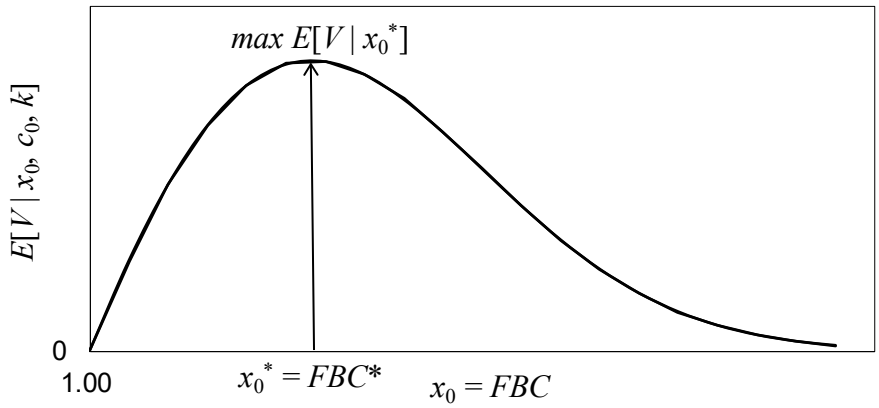
## Expected Gross Profit *before* winning the job, "*EMV* @ (d)"

$$E[V \mid x_0, c_0, k] = P[A_0 \text{Wins} \mid x_0, c_0, k] \cdot (x_0 - 1)c_0$$



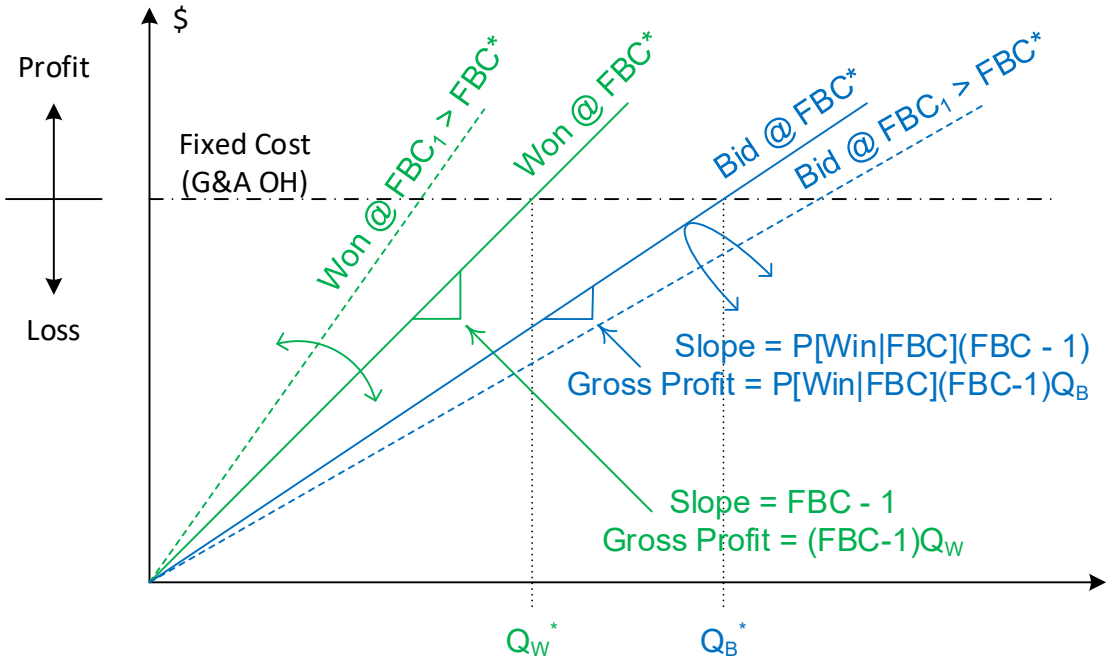

$$E[V \mid x_0, c_0, k] = P[A_0 \text{ wins} \mid x_0, c_0, k]\ (x_0 - 1)\ c_0$$




The optimum $x_0^* = FBC^*$ gives the maximum expected gross profit.

## Break-Even Analysis

The following simplified graph shows how much volume of work (measured in "direct cost $") a contractor must (a) bid and (b) win each year to cover its fixed costs (i.e., its general and administrative overhead for the year) and be profitable.




$Q_B^*$ = Minimum dollar volume bid at FBC* to be profitable
$Q_W^*$ = Minimum dollar volume won at FBC* to be profitable

Break Even Analysis

## Historical Bidding Data Available to Contractor $A_0$

| Pr. $j$ | Est.Cost $c_{0j}$ | $i^{th}$ Oppnt's Bid on Project $j$, $b_{ij}$ $i=1$ | $i=2$ | $i=3$ | … | $i=k_j$ | Bid-to-Cost Ratio, $BC_{ij}=b_{ij}/c_{0j}$ $i=1$ | $i=2$ | $i=3$ | … | $i=k_j$ | $LBC_j=\min(BC_{ij})$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $c_{01}$ | $b_{11}$ | $b_{21}$ | | | | $b_{11}/c_{01}$ | $b_{21}/c_{01}$ | | | | $LBC_1$ |
| 2 | $c_{02}$ | $b_{12}$ | $b_{22}$ | $b_{32}$ | … | $b_{k2}$ | $b_{12}/c_{02}$ | $b_{22}/c_{02}$ | $b_{32}/c_{02}$ | … | $b_{k2}/c_{02}$ | $LBC_2$ |
| 3 | $c_{03}$ | $b_{13}$ | $b_{23}$ | $b_{33}$ | | | $b_{13}/c_{03}$ | $b_{23}/c_{03}$ | $b_{33}/c_{03}$ | | | $LBC_3$ |
| … | … | … | … | … | … | … | … | … | … | … | … | … |
| $n$ | $c_{0n}$ | $b_{1n}$ | $b_{2n}$ | | | | $b_{1n}/c_{0n}$ | $b_{2n}/c_{0n}$ | | | | $LBC_n$ |
| Number of opponents for project $j=k_j$ | | | | | | | Avg($BC$) | | | | | Avg($LBC$) |
| Total number of opponents for all $n$ projects $=K$ | | | | | | | SD($BC$) | | | | | SD($LBC$) |

Contractor $A_0$ has bid $n$ projects in the past. For each past project $j$, contractor $A_0$ knows:

1. His estimated expected cost, $c_{0j}$
2. His cost to prepare the estimate (a *sunk cost*, not relevant)
3. His own bid price, $b_{0j}$
4. The number of opponents that he had to face $k_j$
5. His opponents' bid prices, $b_{ij}$, where $i$ is the opponent and $j$ is the project
6. His actual costs on projects that he won and constructed (not relevant)

We will not consider (2) and (6) above. Contractor $A_0$ has already estimated the project for which he is about to submit a bid and thus (2) is a *sunk cost*. The resulting estimate of the expected cost of the project he is about to bid is *unbiased*, $c_0 = E[C]$, so (6) can be ignored.

Since the bids and estimated costs vary considerably from project to project, Contractor $A_0$ needs a way to standardize the above data so that they may be comparable on the same scale. The obvious way to standardize bidding data is to divide each opponent's bid $b_{ij}$ by contractor $A_0$'s own cost estimate $c_{0j}$. This gives a bid-to-cost ratio $BC_{ij}$ for each opponent $i$ on project $j$:

| $BC_{ij} = b_{ij} / c_{0j}$ | $LBC_j = \min_i BC_{ij} = \min_i \{b_{ij} / c_{0j}\}$ |
|---|---|

Each bid-to-cost ratio $BC_{ij}$ represents the *apparent* $FBC_{ij}$ used by opponent $i$ on project $j$. This is an "*apparent FBC*" because it is computed using contractor $A_0$'s own cost estimate $c_{0j}$, instead of the $i^{th}$ opponent's cost estimate $c_{ij}$ (which is not known to contractor $A_0$).

Total number of bid-to-cost ratios $BC_{ij}$ available from $n$ past projects $= K = \sum_{j=1}^{n} k_j$ .

Contractor $A_0$'s own bid-to-cost ratios, $FBC_{0j} = b_{0j} / c_{0j}$, are not included in the set of $K$ bid-to-cost ratios because $FBC_{0j}$, $b_{0j}$ and $c_{0j}$, are *decision variables* for contractor $A_0$. They cannot be subject to statistical and probabilistic analysis.

## Method I: "Brute Force" Method — A Bidding Model Based Only on Statistical Data

Objective: Find the value $FBC^*$ that (in hindsight) would have maximized contractor $A_0$'s profit, if $A_0$ had used this particular $FBC^*$ to bid each and every one of the $n$ past projects.

Reasoning: $FBC^*$ would have been contractor $A_0$'s best choice, if he had to use the same $FBC$ value to bid all past $n$ projects. So, $A_0$ should use the same $FBC^*$ to bid the next project, too.

1. Arrange the available historical information from all $n$ past projects in a table (see above).
2. In each row compute $LBC_j$, the *Low Bid-to-Cost* ratio for project $j$ (see above).
3. Sort the projects (rows) in this table by descending $LBC$. Renumber the $n$ projects so that, $LBC_1 = \max LBC$, is in the first row, and $LBC_n = \min LBC$, is in the last row of the table.
4. Consider table row $j$: If $A_0$ had bid all $n$ projects using $FBC = LBC_j - \varepsilon$, ($\varepsilon$ = small amount)
   a. $A_0$ would have won all the projects $w$ in the first $j$ rows because their $LBC_w > FBC$.
   b. $A_0$ would have lost all the projects in the remaining $n - j$ rows because their $LBC < FBC$.
5. Compute contractor $A_0$'s profit if he had actually used the value $FBC = LBC_j - \varepsilon$ to bid all $n$ past projects and had thus won all $j$ projects in the first $j$ rows of the table (and lost the rest). There are two ways to compute this profit:
   a. The formula below gives profit in dollars. However, it is sensitive to the magnitude of the estimated costs (measured by $c_{0w}$) of the first $j$ projects ($j$ rows) that $A_0$ would have won.

$$E[V \mid FBC] = \sum_{w=1}^{j} [(FBC - 1)c_{0w}] = (FBC - 1)\sum_{w=1}^{j} c_{0w}$$

   b. The formula below considers only the relative proportion of projects won, $j/n$, which is an estimate for $P[Win|FBC]$. Thus, it is not sensitive to the size, i.e., $c_{0w}$, of the particular projects that are won at this $FBC$. (Units = % of estimated cost $c_0$, of project to bid next)

$$E[V \mid FBC] = (FBC - 1)\, j / n$$

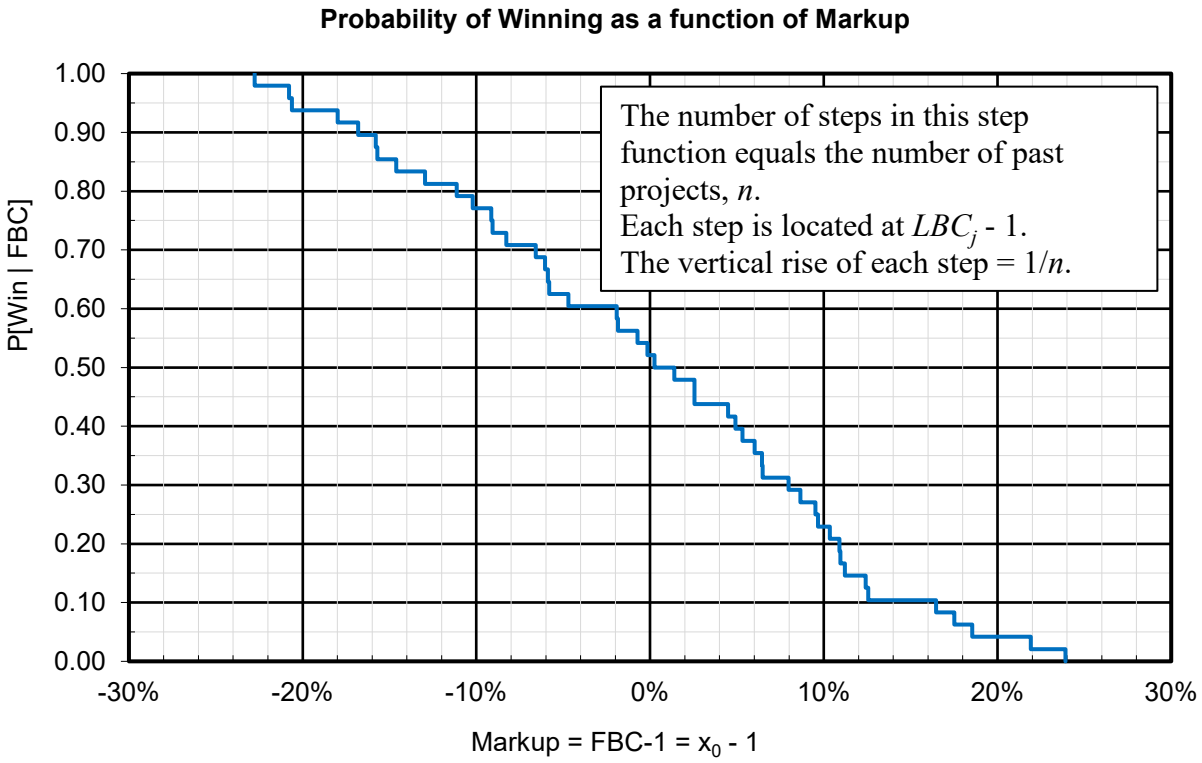
6. Repeat steps 4-5 by setting $FBC$ to its next trial value in row $j$+1, i.e., $FBC = LBC_{j+1} - \varepsilon$.
7. After $FBC$ has been set to all $n$ trial values(*), $FBC = LBC_j - \varepsilon$, the optimal $FBC^*$ is the one that produces the maximum profit value $E[V \mid FBC^*]$ (using either (a) or (b) above).

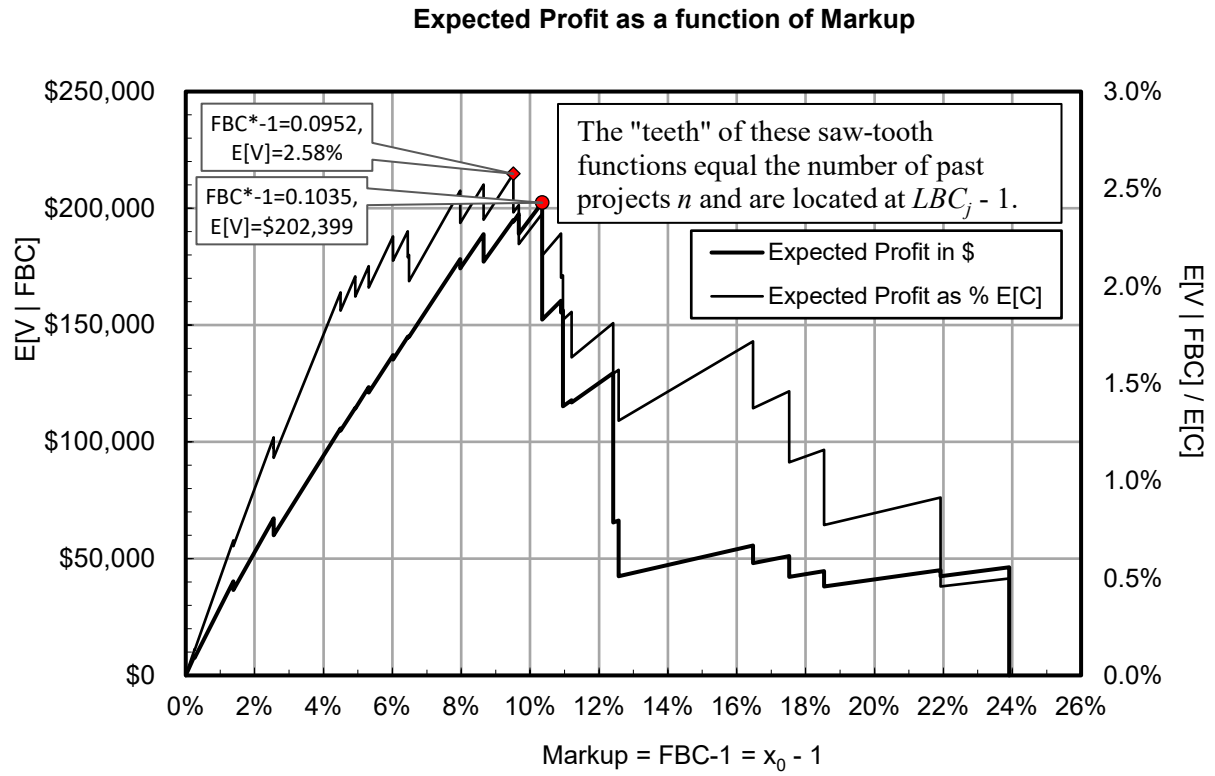
(*) When the table of past projects has been sorted in *descending* order of $LBC$ (step 3 above), any value of $FBC$ between the $LBC$ values of two successive rows (row $j$ and row $j$+1)

$$LBC_j - \varepsilon > FBC > LBC_{j+1}$$

would have won the same projects (i.e., those in the first $j$ rows of the table) as when using $FBC = LBC_j - \varepsilon$, but at less profit. Thus, the best value of $FBC$ in any interval ($LBC_j$, $LBC_{j+1}$) is $FBC = LBC_j - \varepsilon$. Clearly, the optimal $FBC^*$ *must* be one of the $n$ values $FBC = LBC_j - \varepsilon$.

Also notice that the expected profit $E[V \mid FBC]$ given by (5.a) or (5.b) above is a "saw-tooth" function. In the interval, $LBC_{j+1} < FBC < LBC_j$, the expected profit $E[V \mid FBC]$ increases *linearly* with $FBC$. But it has two values at $FBC = LBC_j$ (and at $FBC = LBC_{j+1}$). When $FBC = LBC_j - \varepsilon$, (5.a) or (5.b) give a high value because they include project $j$. When $FBC = LBC_j + \varepsilon$, (5.a) or (5.b) give a low value because they do not include project $j$. Thus, a correct graph of expected profit $E[V \mid FBC]$ has a "tooth" (two values) at each $FBC = LBC_j$.


Probability of Winning as a function of Markup
P[Win | FBC]
1.00
0.90
0.80
0.70
0.60
0.50
0.40
0.30
0.20
0.10
0.00
-30%
-20%
-10%
0%
10%
20%
30%
Markup = FBC-1 = x0 - 1
The number of steps in this step function equals the number of past projects, n.
Each step is located at LBCj - 1.
The vertical rise of each step = 1/n.


Expected Profit as a function of Markup
E[V | FBC]
$250,000
$200,000
$150,000
$100,000
$50,000
$0
E[V | FBC] / E[C]
3.0%
2.5%
2.0%
1.5%
1.0%
0.5%
0.0%
0%
2%
4%
6%
8%
10%
12%
14%
16%
18%
20%
22%
24%
26%
Markup = FBC-1 = x0 - 1
FBC*-1=0.0952, E[V]=2.58%
FBC*-1=0.1035, E[V]=$202,399
The "teeth" of these saw-tooth functions equal the number of past projects n and are located at LBCj - 1.
Expected Profit in $
Expected Profit as % E[C]

## METHOD II – Use the distribution of *LBC* to find $P[A_0 \text{ wins} \mid x_0]$

### Probabilistic Model

1. Let $X_{min}$ be a random variable that represents the lowest apparent bid-to-cost ratio to be used by the opponents on the project to bid.
   Suppose that $X_{\min} \sim N(m,\sigma^2)$ follows a Normal distribution.
   Let $x_0$ be the *FBC* to be used by contractor $A_0$ to compute its bid, $b_0=x_0c_0$, where $c_0$ is contractor $A_0$'s estimated expected cost of the project to bid.
2. The probability that contractor $A_0$ would win the project is then given by:
$$P[A_0\text{Wins} \mid x_0, c_0] = P[X_{\min} > x_0] = 1 - F_{X_{\min}}(x_0)$$
3. Contractor $A_0$ should compute its optimum bid, $b_0^*=x_0^*c_0$, using the optimum $x_0^*$ that maximizes its expected profit:
$$E[V \mid x_0^*, c_0] = \max_{x_0}\{[1 - F_{X_{\min}}(x_0)](x_0 - 1)c_0\}$$
4. To find the optimum $x_0^*$, set the derivative of the expected profit to zero:
$$x_0^* - 1 - \frac{1 - F_{X_{\min}}(x_0^*)}{f_{X_{\min}}(x_0^*)} = 0$$
   This equation can be entered in Excel using the function NORM.DIST (or the older function NORMDIST) that gives the CDF and PDF of the Normal distribution.
$$x_0^* - 1 - \frac{1\text{-NORM.DIST}(x_0^*,m,\sigma,\text{TRUE})}{\text{NORM.DIST}(x_0^*,m,\sigma,\text{FALSE})} = 0$$
   This nonlinear equation can be solved for $x_0^*$ in Excel easily by using either *goal-seek* or the *Solver* add-in.

### Statistical Estimation

5. Arrange the available historical information from all $n$ past projects in a table (see above).
6. In each row $j$ compute $LBC_j$, the *Low Bid-to-Cost* ratio for project $j$ (see above).
7. Use the $n$ values $LBC_j$ to compute estimates of the mean $m$ and variance $\sigma^2$ of $X_{min}$:

| Estimator | Excel Formula |
|---|---|
| $\hat{m} = \overline{LBC} = \frac{1}{n}\sum_{j=1}^{n} LBC_j$ | $\hat{m} = \overline{LBC} = AVERAGE(LBC_j)$ |
| $\hat{\sigma}^2 = s_{n-1}^2 = \frac{1}{n-1}\sum_{j=1}^{n}(LBC_j - \overline{LBC})^2$ | $\hat{\sigma} = s_{n-1} = STDEV(LBC_j)$ |

## METHOD III - Use the distribution of *bid-to-cost ratios* $X_i=BC_i$ to find $P[A_0 \text{ wins} \mid x_0]$

### Probabilistic Models

1. Let $X_i$ be a Normally distributed random variable $X_i \sim N(E[X_i], Var[X_i])$ for the apparent bid-to-cost ratio used by the $i^{th}$ opponent on the project that contractor $A_0$ is about to bid. Let $x_0$ be the *FBC* to be used by contractor $A_0$ to compute its bid, $b_0=x_0c_0$.
2. The probability of winning against opponent $i$ when using an *FBC* value $x_0$ is:

$$P[A_0 \text{Wins against opponent } i \mid x_0, c_0] = 1 - F_{X_i}(x_0)$$

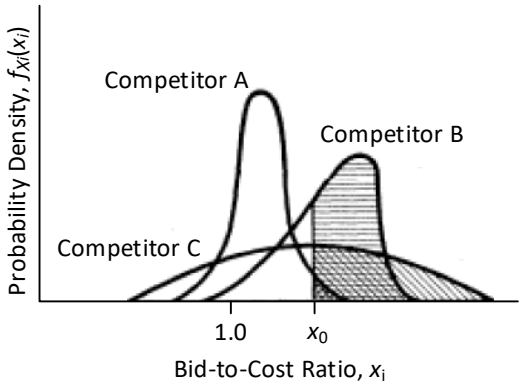
Four common models for the probability of winning when bidding against $k$ opponents are:

a. **Friedman's formulas**:

1. $P[A_0 \text{Wins} \mid x_0, c_0, k] = \prod_{i=1}^{k} [1 - F_{X_i}(x_0)]$
2. $P[A_0 \text{Wins} \mid x_0, c_0, k] = [1 - F_X(x_0)]^k$

b. **Gates' formulas**:

1. $P[A_0 \text{Wins} \mid x_0, c_0, k] = 1 / \{1 + \sum_{i=1}^{k} \frac{F_{X_i}(x_0)}{1 - F_{X_i}(x_0)}\}$
2. $P[A_0 \text{Wins} \mid x_0, c_0, k] = 1 / \{1 + k \frac{F_X(x_0)}{1 - F_X(x_0)}\}$




(a.1) and (b.1) require that $E[X_i]$ and $Var[X_i]$ be estimated separately for each opponent $i$.

(a.2) and (b.2) use the $m$ and $\sigma^2$ estimated for the *average* opponent (see below).

3. Contractor $A_0$'s bid, $b_0=x_0c_0$, should use the optimum $x_0^*$ that maximizes expected profit:

$$E[V \mid x_0^*, c_0, k] = \max_{x_0} \{P[A_0 \text{Wins} \mid x_0, c_0, k](x_0 - 1)c_0\}$$

### Statistical Estimation for formulas (a.2) and (b.2)

4. Arrange the available historical information from all $n$ past projects in a table. (See above).
5. For each project $j$ that contractor $A_0$ has bid in the past, compute the bid-to-cost ratios, $BC_{ij}$, used by the $k_j$ opponents ($i$ shows the opponent and $j$ the past project). (See above).
   The total number of available bid-to-cost ratios is (see the example table): $K = \sum_{j=1}^{n} k_j$
6. Use the $K$ values $BC_{ij}$ to compute estimates of the mean and standard deviation for $X$:

| Estimator | Excel Formula |
|---|---|
| $\hat{m} = \overline{BC} = \frac{1}{K} \sum_{j=1}^{n} \sum_{i=1}^{k_j} BC_{ij}$ | $\hat{m} = \overline{LBC} = AVERAGE(BC_{ij})$ |
| $\hat{\sigma}^2 = s_{n-1}^2 = \frac{1}{K-1} \sum_{j=1}^{n} \sum_{i=1}^{k_j} (BC_{ij} - \overline{BC})^2$ | $\hat{\sigma} = s_{n-1} = STDEV(BC_{ij})$ |

7. Use the estimates in (6) to define the parameters of the Normal distribution of $X \sim N(m, \sigma^2)$ for the *average* opponent that are needed to calculate formulas (a.2) and (b.2)

## GATES' MODEL – ROSENSHINE'S PROOF

(*Historical Note*: Marvin Gates never proved his formula nor specified the assumptions for it. For this reason, others stepped in to identify the required necessary and sufficient conditions.)

Let contractor $A_0$ compete against *k* opponents $A_1, \ldots, A_k$.

$$P[A_0 \text{ wins}] + P[A_1 \text{ wins}] + \ldots + P[A_k \text{ wins}] = 1$$

$$P[b_0 < B_i] = \frac{P[b_0 < B_i]P[(A_0 \text{ wins}) \cup (A_i \text{ wins})]}{P[(A_0 \text{ wins}) \cup (A_i \text{ wins})]}$$

**Assumption: events** $[b_0 < B_i]$ **and** $[(A_0 \text{ wins}) \cup (A_i \text{ wins})]$ **are independent**

$$P[b_0 < B_i] = \frac{P[(b_0 < B_i) \cap \{(A_0 \text{ wins}) \cup (A_i \text{ wins})\}]}{P[(A_0 \text{ wins}) \cup (A_i \text{ wins})]}$$

$$P[b_0 < B_i] = \frac{P[A_0 \text{ wins}]}{P[A_0 \text{ wins}] + P[A_i \text{ wins}]}$$

From here we get

$$P[A_i \text{ wins}] = \frac{P[A_0 \text{ wins}](1 - P[b_0 < B_i])}{P[b_0 < B_i]}$$

Substituting this into the first equation gives

$$P[A_0 \text{ wins}] + \frac{P[A_0 \text{ wins}](1 - P[b_0 < B_1])}{P[b_0 < B_1]} + \cdots + \frac{P[A_0 \text{ wins}](1 - P[b_0 < B_k])}{P[b_0 < B_k]} = 1$$

which can be solved to yield Gates' *general* formula

$$P[A_0 \text{ wins}] = \frac{1}{1 + \sum_{i=1}^{k} \frac{1 - P[b_0 < B_i]}{P[b_0 < B_i]}} = \frac{1}{1 + \sum_{i=1}^{k} \frac{1 - P[x_0 < X_i]}{P[x_0 < X_i]}} = \frac{1}{1 + \sum_{i=1}^{k} \frac{F_{X_i}(x_0)}{1 - F_{X_i}(x_0)}}$$

If $P[b_0 < B_i]$ is the same for all $i = 1, \ldots, k$, we get the *simplified* version of Gates' formula

$$P[A_0 \text{ wins}] = \frac{1}{1 + k\frac{1 - P[b_0 < B_i]}{P[b_0 < B_i]}} = \frac{1}{1 + k\frac{1 - P[x_0 < X]}{P[x_0 < X]}} = \frac{1}{1 + k\frac{F_X(x_0)}{1 - F_X(x_0)}}$$

Note: The above formulas used the following substitutions:

$$b_0 = x_0 c_0; \quad B_i = X_i c_0; \quad P[x_0 < X_i] = 1 - F_{X_i}(x_0); \quad P[x_0 < X] = 1 - F_X(x_0)$$

## GATES' MODEL – ANOTHER PROOF

Let contractor $A_0$ compete against *k* opponents $A_1, \ldots, A_k$.

**Assumption:**

$$\frac{\mathrm{P}[A_i \text{ wins}]}{P[A_0 \text{ wins}]} = \frac{P[B_i < b_0]}{P[b_0 < B_i]}$$

The above is unquestionably true if the number of opponents $k = 1$.

If the above assumption is accepted for $k > 1$, we have

$$\mathrm{P}[A_i \text{ wins}] = P[A_0 \text{ wins}] \frac{1 - P[b_0 < B_i]}{P[b_0 < B_i]}$$

Substituting this into the following equation

$$P[A_0 \text{ wins}] = 1 - \sum_{i=1}^{k} P[A_i \text{ wins}]$$

gives

$$P[A_0 \text{ wins}] = 1 - P[A_0 \text{ wins}] \sum_{i=1}^{k} \frac{1 - P[b_0 < B_i]}{P[b_0 < B_i]}$$

This equation can be solved to yield Gates' *general* formula

$$P[A_0 \text{ wins}] = \frac{1}{1 + \sum_{i=1}^{k} \frac{1 - P[b_0 < B_i]}{P[b_0 < B_i]}} = \frac{1}{1 + \sum_{i=1}^{k} \frac{1 - P[x_0 < X_i]}{P[x_0 < X_i]}} = \frac{1}{1 + \sum_{i=1}^{k} \frac{F_{X_i}(x_0)}{1 - F_{X_i}(x_0)}}$$

If $P[b_0 < B_i]$ is the same for all $i = 1, \ldots, k$, we get the *simplified* version of Gates' formula

$$P[A_0 \text{ wins}] = \frac{1}{1 + k \frac{1 - P[b_0 < B_i]}{P[b_0 < B_i]}} = \frac{1}{1 + k \frac{1 - P[x_0 < X]}{P[x_0 < X]}} = \frac{1}{1 + k \frac{F_X(x_0)}{1 - F_X(x_0)}}$$

Note: The above formulas used the following substitutions:

$$b_0 = x_0 c_0; \quad B_i = X_i c_0; \quad P[x_0 < X_i] = 1 - F_{X_i}(x_0); \quad P[x_0 < X] = 1 - F_X(x_0)$$

## Optimum Markup by Direct Solution

The following illustrates how to solve directly for the optimum *FBC,* $x_0^*$.

$$E[V\,|x_0,c_0,k] = E[V\,|A_0\text{Wins},x_0,c_0]P[A_0\text{Wins}\,|\,x_0,c_0,k]$$
$$= P[A_0\text{Wins}\,|\,x_0,c_0,k](x_0-1)c_0$$
$$\frac{dE[V\,|x_0,c_0,k]}{dx_0} = P[A_0\text{Wins}\,|\,x_0,c_0,k]c_0 + \frac{dP[A_0\text{Wins}\,|\,x_0,c_0,k]}{dx_0}(x_0-1)c_0 = 0$$

$$x_0^* - 1 + \frac{P[A_0\text{Wins}\,|\,x_0^*,c_0,k]}{\dfrac{dP[A_0\text{Wins}\,|\,x_0^*,c_0,k]}{dx_0}} = 0$$

The last formula is the basic equation for solving directly for the optimum *FBC,* $x_0^*$.

**Application to Friedman's simplified model:**

$$P[A_0\text{Wins}|\,x_0,c_0,k] = [1-F_X(x_0)]^k$$
$$\frac{dP[A_0\text{Wins}\,|\,x_0,c_0,k]}{dx_0} = -kf_X(x_0)[1-F_X(x_0)]^{k-1}$$
$$x_0^* - 1 + \frac{[1-F_X(x_0)]^k}{-kf_X(x_0)[1-F_X(x_0)]^{k-1}} = x_0^* - 1 - \frac{1-F_X(x_0)}{kf_X(x_0)} = 0$$
$$x_0^* - 1 - \frac{1\text{-NORM.DIST}(x_0^*,m,\sigma,\text{TRUE})}{k\text{*NORM.DIST}(x_0^*,m,\sigma,\text{FALSE})} = 0$$

For $k = 1$, the above formula is identical to the one that gives the optimum *FBC,* $x_0^*$, for "Method II" where *LBC* follows a Normal pdf. The difference lies in the statistics used to estimate the mean $m$ and variance $\sigma^2$ used in each case.

**Application to Gates' simplified model:**

$$P[A_0\text{Wins}|\,x_0,c_0,k] = \{1+k\frac{F_X(x_0)}{1-F_X(x_0)}\}^{-1} = \frac{1-F_X(x_0)}{1+(k-1)F_X(x_0)}$$
$$\frac{dP[A_0\text{Wins}\,|\,x_0,c_0,k]}{dx_0} = \frac{-kf_X(x_0)}{\{1-F_X(x_0)+kF_X(x_0)\}^2}$$
$$x_0^* - 1 - \frac{\{1-F_X(x_0)\}\{1+(k-1)F_X(x_0)\}}{kf_X(x_0)} = 0$$

$$x_0^* - 1 - \frac{\{1\text{-NORM.DIST}(x_0^*,m,\sigma,\text{TRUE})\}*\{1+(k\text{-}1)*\text{NORM.DIST}(x_0^*,m,\sigma,\text{TRUE})\}}{k\text{*NORM.DIST}(x_0^*,m,\sigma,\text{FALSE})} = 0$$

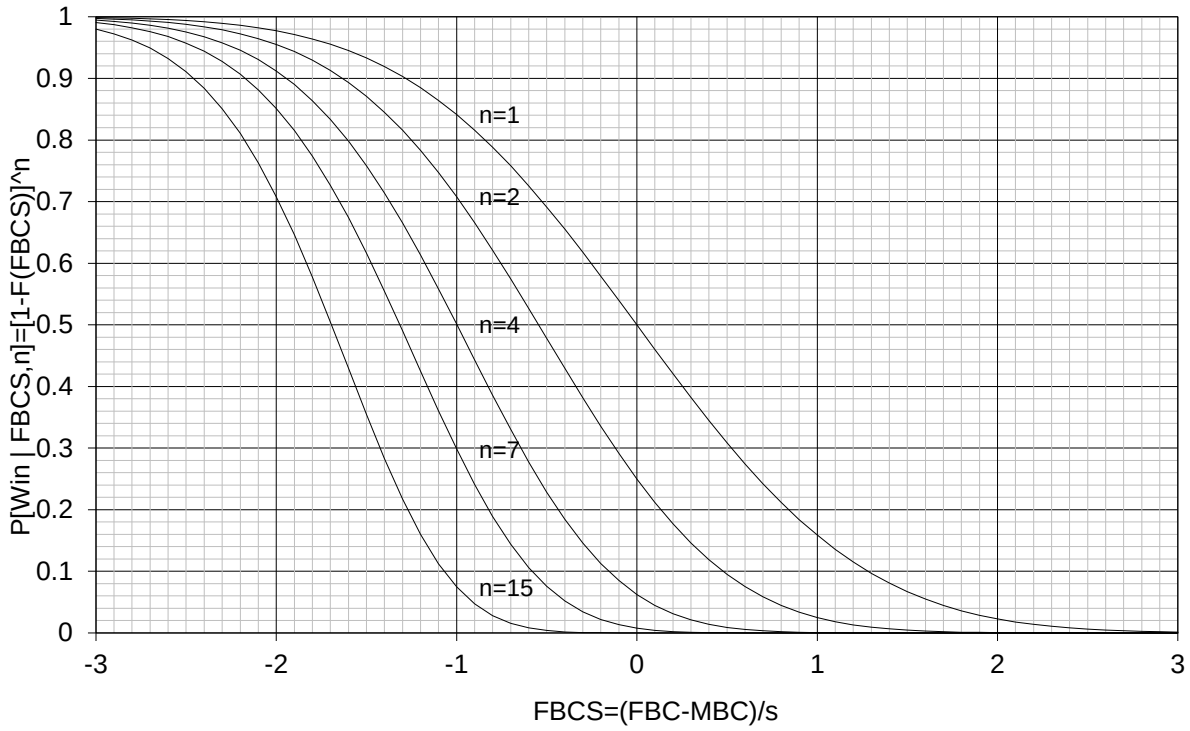
**Excel Solver**

Alternatively, the Excel add-in "*Solver*" (or the more powerful "*Analytic Solver Platform*") can be used to find the optimum value of *FBC,* $x_0^*$, directly, by maximizing the expected profit shown below, without first finding and setting its derivative to zero.

$$E[V\,|\,x_0^*,c_0,k] = \max_{x_0} P[A_0\text{Wins}|\,x_0,c_0,k](x_0-1)c_0$$

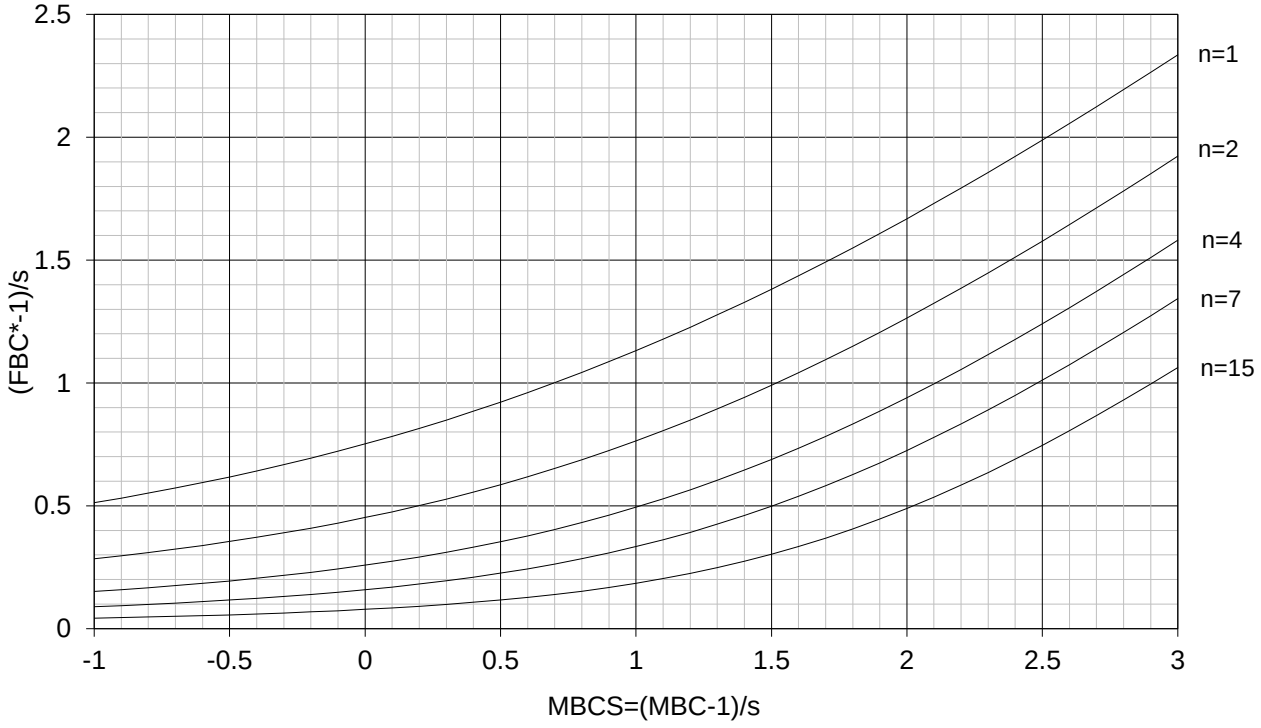
## Friedman's Bidding Model

Probability to Win given FBC and No. of Competitors, n



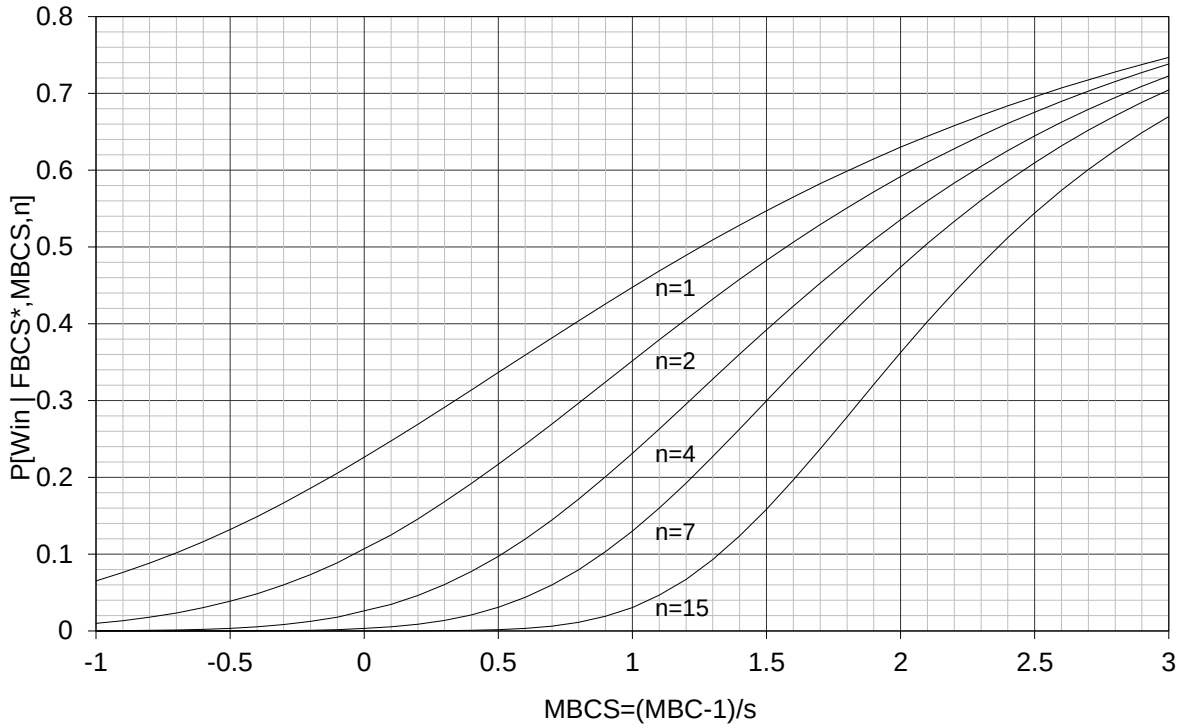

## Friedman's Bidding Model

Optimum FBC given No. of Competitors, n



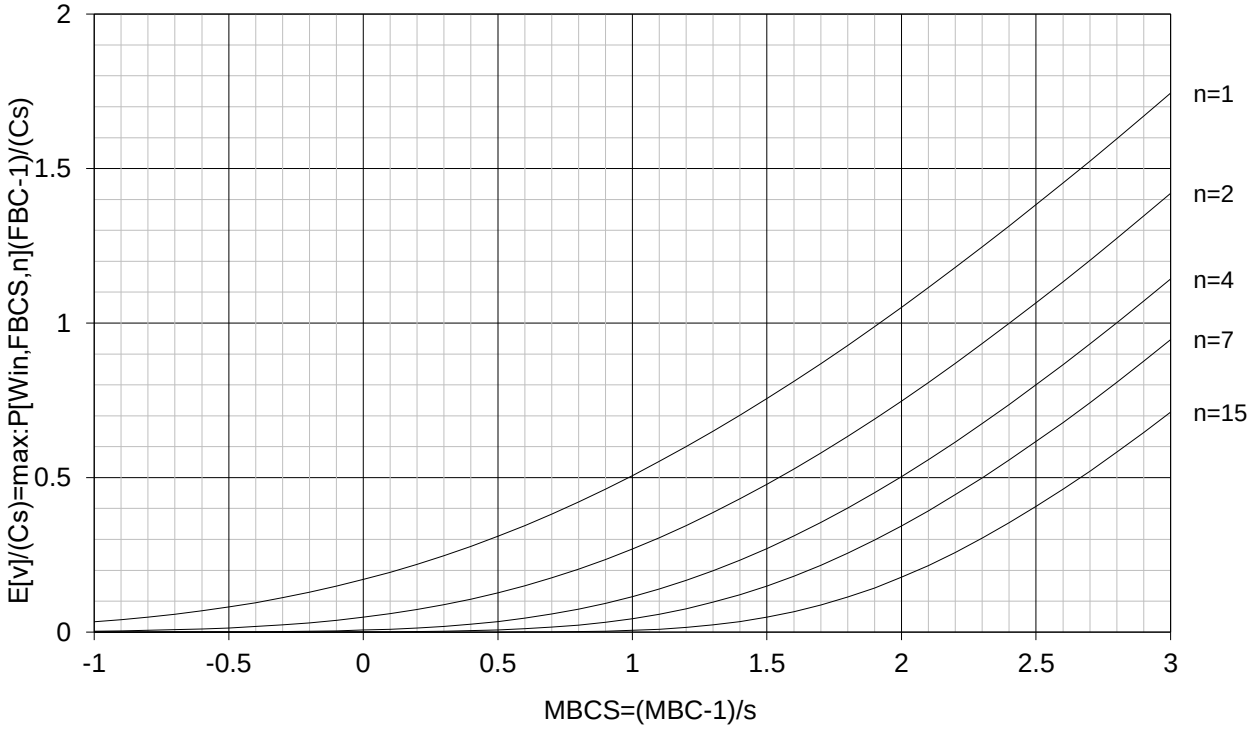
## Friedman's Bidding Model

Probability to Win at Optimum FBC given No. of Competitors, n




## Friedman's Bidding Model

Expected Profit at Optimum FBC given No. of Competitors, n



## LOW BID GIVES WAY TO MID-BID

A contract-letting procedure that awards projects to the bid nearest the average encourages Florida contractors to bid a true and reasonable cost. The Florida Department of Transportation (FDOT) hopes that the bid-averaging method (BAM) will yield higher quality projects and fewer monetary squabbles.

"Using standard low-bid methods, we're not sure we're getting the best price now anyway," said Paul Theberge, project administrator at FDOT's Fort Myers office. Overruns often bring total project cost above the contract price and, worse, low-bid encourages contractors to cut costs wherever possible to get jobs and maintain profit margins, he said. The result, though difficult to quantify, may be lower quality work that will cost more in maintenance and early replacement.

BAM bidding theoretically sets the contractor's bid target at an accurate price for the job, rather than the lowest. The statistically based method has its origins in Europe, where contracting considered nonstandard in the U.S. is the norm. The owner of a BAM contract throws out the high and low bids and averages the rest if there are more than five, averages all bids if there are three or four and readvertises the job if fewer than three bids are received.

"This is not the good old capitalist way of bidding, so there is probably some apprehension on the contractor's part," said Charles R. Glagola, a University of Florida civil engineering professor who is studying BAM and other alternative contract procedures. A primary question is whether low-bid methods give contractors enough money to do a quality job, Glagola said. Reduction in claims or overall construction costs would be obvious indicators of BAM's success, but comparing the "effective final costs" of BAM and traditional contracts by factoring in durability, administrative costs and other hidden expenses would be more conclusive, he said.

In 1996, at the urging of the secretary of transportation, the Florida legislature passed a statute allowing use of alternative techniques on $60 million out of the approximately $1.25 billion worth of annual FDOT contracts. "The jury is still out" on BAM bidding, said Ken Leuderalbert, FDOT manager of quality initiatives, but some project managers have noted that contractors are less likely to request more money for minor changes to BAM projects.

BAM is one of many alternative contracting practices getting FDOT test runs. Others include cost-plus-time, which applies a formula to adjust monetary bids using a contractor's estimate of days to complete a project; long-term warranty clauses; no-excuse bonuses; and incentive/disincentive contracts, which reward contractors for early completion but increase penalties for lateness.

BAM will be used primarily for straightforward, "cookbook" projects such as bridge rehabilitations and pavement resurfacing, Theberge said. About 10 BAM projects are in progress, with three scheduled for completion by this fall, he said.

# FRIEDMAN'S BIDDING MODEL—ERRORS AND CORRECTIONS

By Photios G. Ioannou[1], Ph.D., Dipl. Eng.

## ABSTRACT

Friedman's classic 1956 bidding model for the case where a contractor is bidding against an unknown number of opponents applies Friedman's multiplicative model for the probability to win, which applies only when bidding against at least one opponent, to the case where the number of opponents is zero, to which it does not apply. Thus, both the resulting general equations and the examples for the probability to win, as well as those for expected profit are incorrect. Moreover, the example graphs offered are not true representations of these incorrect equations. The causes and consequences of these errors are described in detail and new equations are offered as replacements. Exact graphs of the incorrect equations as well as of their replacements are presented. Four new bidding models are developed by which a contractor can find the optimum bid price that maximizes expected profit when the number of opponents is unknown and could be zero, including the case of an owner with a reserve price that leads to a distinct and novel bidding strategy.

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Note. This manuscript was submitted for review to the ASCE Journal of Construction Engineering and Management on April 24, 2018, and was accepted for publication on February 15, 2019.

## Friedman's bidding model

Friedman's classic bidding model (Friedman 1956) is arguably one of the best-known and most-cited competitive bidding models (Benjamin and Meador 1979, Carr 1982, Clough et al. 2015, Dixie 1974, Gates 1967, Grogan 1992, Ioannou et al. 1988 1993 2010, Rosenshine 1972, Rothkopf 2007, Takano et al. 2018). It examines a situation common in construction where several contractors compete against each other by submitting bids for a particular project. The winner of the project is the contractor submitting the lowest responsible bid. The objective of Friedman's model is to determine the optimal bid price that a risk-neutral contractor should submit to maximize the expected profit for this project. Overall, Friedman's model follows a classic decision analysis approach (Rothkopf 2007) and examines the competitive bidding problem in three different cases:

(a) When the contractor knows the *number* and *identities* of his competitors.

(b) When the contractor knows only the *number* of competitors but not their identities.

(c) When the number of competitors is uncertain.

The original description of Friedman's bidding model in (Friedman 1956) contains two obvious typographical errors. In Eq. (10) and (11) the factorial in the denominator should be $i!$ (and not 1!). More importantly, however, the formulation for case (c) where "*the number of competitors that a contractor is about to face is unknown*" contains modeling mistakes and its application contains several errors. These are described and corrected below.

The following discussion begins by formulating Friedman's bidding model for cases (a) and (b) which are used as building blocks for examining and correcting case (c). This leads to a new formulation for the general probability to win for case (c) that considers explicitly the probability that the owner may choose to award the project to the contractor even though no other bids have been submitted, and which can be used with other bidding models and not just Friedman's. Four alternative bidding model formulations are developed based on the new probability to win, where the first three do not require additional assumptions other than those already in (Friedman 1956). The fourth model is new and examines the case where the owner may award the contract to a single bidder whose price does not exceed the owner's reserve price. This bidding model leads to a new bidding strategy that could not have been modeled before and which can result in even higher expected profit, especially when the number of expected opponents is small.

Friedman's bidding model has been criticized before, especially about its assumptions and

the multiplicative expressions for the probability to win as applied in cases (a) and (b). Such criticism appears in several publications including (Benjamin and Meador 1979, Carr 1982, Dixie 1974, Gates 1967, Rosenshine 1972) and others. Some of this criticism is unfounded as pointed out in (Ioannou 1988). Because of this history, it should be made clear that none of the above references considered case (c) as in (Friedman 1956) and as a result their contributions were independent and not negatively affected in any way by the errors described below.

## Basic Definitions

The notation adopted in this paper is to indicate random variables and the identity of contractors using capitalized letters and to show decision variables and the values of random variables using lower-case letters. As is often the practice in contracts, a contractor is referred to as if though singular in number and masculine in gender to eliminate some of the ambiguity in the few instances where the use of pronouns is necessary.

Friedman's model examines the bidding process from the perspective of a risk-neutral contractor $A_0$ who is about to bid on a new project against $n$ competitors, $A_1, A_2, \ldots, A_n$.

It is assumed that contractor $A_0$ has already determined a cost estimate $c_0$ for the project. The actual cost of the project, if performed by contractor $A_0$, is uncertain and is represented by the random variable $C$, with probability density function $f_C(c)$.

Contractor $A_0$ is considered to be risk-neutral and needs to select a bid price $b_0$ that maximizes the expected profit $V$ for this project, which is given by Eq. (1) and (2) in (Friedman 1956), and shown as Eq. (1) below:

$$\begin{aligned} E[V\,|b_0, c_0] &= P[A_0 \text{ Wins}|\, b_0]\int_c (b_0 - c)\, f_C(c)dc \\ &= P[A_0 \text{ Wins}|\, b_0](b_0 - E[C]) \\ &= P[A_0 \text{ Wins}|\, b_0](b_0 - c_0) \end{aligned} \tag{1}$$

Based on Eq. (1), (Friedman 1956) concludes that the cost estimate $c_0$ that contractor $A_0$ should use to arrive at the optimum bid price $b_0$ should be the contractor's unbiased estimate of the project's expected cost, $E[C]$.

Contractor $A_0$ can arrive at the bid price $b_0$ by choosing a monetary markup $m_0$ to add to the unbiased cost estimate $c_0$. Alternatively, contractor $A_0$ can follow the approach in (Friedman 1956) and determine the bid price $b_0$ by multiplying the cost estimate $c_0$ by the selected *bid-to-cost ratio*, $x_0$, which is a simple function of $m_0$ and $c_0$, as shown below:

$$b_0 = c_0 + m_0 = (1 + m_0 / c_0)c_0 = x_0 c_0 \tag{2}$$

Thus, in (Friedman 1956) contractor $A_0$ determines the bid price, $b_0 = x_0 c_0$, by selecting a value for the bid-to-cost ratio $x_0$ that maximizes the project's expected profit given by Eq. (1).

For contractor $A_0$, the values $c_0$ and $x_0$ (and hence, the corresponding bid price $b_0 = c_0 x_0$) are decision variables. Similarly, the cost estimate $C_i$ and the bid-to-cost ratio $X_i$ (and thus the bid price $B_i = C_i X_i$) used by competitor $A_i$, are decision variables for that contractor. However, because these quantities are unknown to contractor $A_0$, they are treated as random variables from $A_0$'s perspective.

At this point, (Friedman 1956) introduces the assumption that contractor $A_0$ can use the bidding results from previous contracts to study the bidding patterns of potential opponents. Specifically, contractor $A_0$ can estimate the bid-to-cost ratio $x_{ij}$ that was used by competitor $A_i$ to bid past project $j$ by computing the ratio of the competitor's submitted bid $b_{ij}$ to contractor $A_0$'s own cost estimate $c_{0j}$ for the same project $j$:

$$x_{ij} = b_{ij} / c_{0j} \tag{3}$$

Clearly, each value $x_{ij}$ calculated using Eq. (3) is an "apparent bid-to-cost ratio" because it divides the competitor's bid $b_{ij}$ by contractor $A_0$' s own cost estimate $c_{0j}$ for past project $j$. This is necessary because the cost estimate $c_{ij}$ that was used by competitor $A_i$ on project $j$ is unknown. These apparent bid-to-cost ratios $x_{ij}$ from past projects $j$ are the basic data from which contractor $A_0$ can estimate the probability of winning on future projects.

The key random variables for determining the probability that contractor $A_0$ will win an upcoming project are the apparent bid-to-cost ratios $X_i$ that will be used by opponents $A_i$, which when multiplied by contractor $A_0$'s own cost estimate $c_0$, give the competing bids $B_i$ to be submitted:

$$B_i = X_i c_0 \quad i = 1, 2, \ldots, n \tag{4}$$

Contractor $A_0$'s expected profit for an upcoming project can be expressed in terms of his bid-to-cost ratio $x_0$ by substituting Eq. (2) into Eq. (1):

$$E[V \,|\, x_0, c_0] = P[A_0 \text{ Wins} \,|\, x_0, c_0](x_0 - 1)c_0 \tag{5}$$

The optimal bid-to-cost ratio $x_0$ that contractor $A_0$ should use to determine his own bid, $b_0 = x_0 c_0$, is found by maximizing the expected profit given by Eq. (5).

## Independence Assumptions

The computation of the probability that contractor $A_0$ wins under Friedman's bidding model requires two important assumptions:

- The apparent bid-to-cost ratios $X_i$, $i = 1, 2,\ldots, n$, are mutually independent.
- The probability distribution of each apparent bid-to-cost ratio $X_i$, $i = 1, 2,\ldots, n$, does not depend on the values $x_0$ and $c_0$ chosen by $A_0$:

$$P[X_i < x_i \mid x_0, c_0] = P[X_i < x_i] \quad i = 1, 2, \ldots, n \tag{6}$$

The modeling and statistical reasons for these assumptions are discussed in detail under several different bidding conditions in (Ioannou 1988) and are not presented here.

## Probability of Winning when the Number and Identities of Opponents are Known

Friedman's first formula for determining the probability that "Contractor $A_0$ wins" considers the case where the number and identities of opponents are known. In this case, it is assumed that contractor $A_0$ has estimated probability density functions $f_{X_i}(x_i)$ for the apparent bid-to-cost ratios $X_i$ for each competitor $A_i$, $i = 1, 2,\ldots, n$, based on historical apparent bid-to-cost ratios $b_{ij} / c_{0j}$ obtained from past projects $j$.

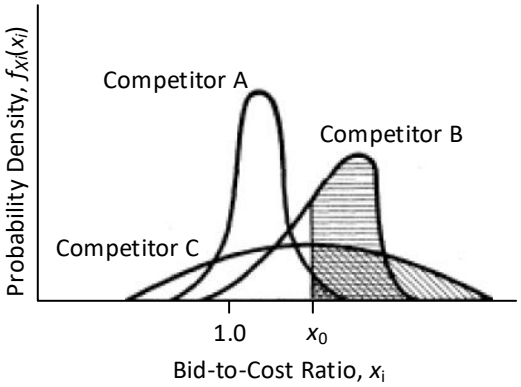



**Fig. 1.** Bidding patterns of competitors [adapted from Fig. 3 in (Friedman 1956)]

Fig. 1 shows three such probability density functions for the apparent bid-to-cost ratios $X_i$ of three opponent contractors *A*, *B*, and *C*. Fig. 1 is similar to Fig. 3 in (Friedman 1956) but uses the notation adopted in this article for clarity. The probability that "Contractor $A_0$ wins against a single competitor $A_i$ when using a bid-to ratio $x_0$" (i.e., when using a bid price $b_0 = x_0 c_0$) is given by the shaded area under the probability density function for the apparent bid-to-cost ratio $X_i$, for

the corresponding competitor, *A*, *B*, or *C*, in Fig. 1:

$$P[A_0 \text{Wins against } A_i \mid x_0, c_0] = \int_{x_0}^{\infty} f_{X_i}(x_i)dx_i = 1 - F_{X_i}(x_0) \tag{7}$$

When contractor $A_0$ knows exactly which competitors are going to submit bids for an upcoming project, Friedman's classic formula for the probability that contractor $A_0$ would win against all $n$ opponents when submitting a bid $b_0 = x_0 c_0$, is given by the product of the probabilities of defeating each of the opponents separately:

$$P[A_0 \text{Wins} \mid x_0, c_0] = \prod_{i=1}^{n} \int_{x_0}^{\infty} f_{X_i}(x_i)dx_i = \prod_{i=1}^{n} [1 - F_{X_i}(x_0)] \tag{8}$$

## Probability of Winning when Only the Number of Opponents is Known

When the identities of the individual opponents are not known, (Friedman 1956) introduces the concept of an "average" competitor for whom the apparent bid-to-cost ratio $X$ follows a probability density function $f_X(x)$ that represents the distribution of all historical apparent bid-to-cost ratios $b_{ij}/c_{0j}$, for all competitors that were faced on past projects $j$. In this case, the probability that "contractor $A_0$ wins against a single average competitor when using a bid-to ratio $x_0$" is given by:

$$P[A_0 \text{Wins against 1 average competitor} \mid x_0, c_0] = \int_{x_0}^{\infty} f_X(x)dx = 1 - F_X(x_0) \tag{9}$$

Thus, the probability that contractor $A_0$ would win against exactly $k$ opponents when submitting a bid $b_0 = x_0 c_0$ is given by the probability in Eq. (9) raised to the $k^{\text{th}}$ power:

$$P[A_0 \text{Wins against } k \text{ average competitors} \mid x_0, c_0] = [\int_{x_0}^{\infty} f_X(x)dx]^k = [1 - F_X(x_0)]^k \tag{10}$$

## Probability of Winning when the Identities & Number of Opponents are Unknown

At this point (Friedman 1956) examines the general case where the identities, as well as the number of opponents, are unknown. This case is indeed the focal point of the bidding model presented in (Friedman 1956). The previous models serve primarily as introductory building blocks leading to the realistic situation where the number of opponents $K$ is a random variable with its own probability mass function:

$$P[K = k] = g_K(k), \quad k \geq 0 \tag{11}$$

Eq. (5) in (Friedman 1956) states that the probability that contractor $A_0$ would win against an

unknown number of "average" opponents when submitting a bid, $b_0 = x_0 c_0$, is given by the Total Probability theorem and Eq. (10) and (11) as follows:

$$P[A_0 \text{Wins} \mid x_0, c_0] = \sum_{k=0}^{\infty} g_K(k)[\int_{x_0}^{\infty} f_X(x)dx]^k = \sum_{k=0}^{\infty} g_K(k)[1 - F_X(x_0)]^k \tag{12}$$

Eq. (12), above, is identical to Eq. (5) in (Friedman 1956) and is the underlying cause of the mistakes in (Friedman 1956) as explained below.

## Example Application for Unknown Number of Competitors

To proceed further, (Friedman 1956) introduces specific distributions for $f_X(x)$ and $g_K(k)$ that when substituted in Eq. (12) provide the probability that contractor $A_0$ would win, when submitting a bid $b_0 = x_0 c_0$, against an unknown number of average competitors $K$.

Specifically, (Friedman 1956) assumes that the number of opponents $K$ follows a Poisson distribution whose expected value, $E[K] = \lambda$, equals the "estimated number of opponents":

$$P[K = k] = g_K(k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0,1,\ldots \tag{13}$$

Furthermore, (Friedman 1956) assumes that the probability density function, $f_X(x)$, for the apparent bid-to-cost ratio $X$ of the "average" competitor, is a Gamma distribution with parameters alpha, $\alpha$, and beta, $\beta$, that are estimated from past bidding data:

$$f_X(x) = \frac{\beta^{-\alpha} x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)} = \frac{\beta^{-\alpha} x^{\alpha-1} e^{-x/\beta}}{(\alpha-1)!}, \quad x > 0 \tag{14}$$

Although not stated explicitly, (Friedman 1956) also assumes that the parameter alpha, $\alpha$, in Eq. (14) is a positive integer (as implied by the factorial in Eq. (14)), so that the cumulative of the Gamma distribution (which is a non-trivial integral of Eq. (14)) can be given in closed form by the cumulative of a Poisson distribution (the Gamma and the Poisson are conjugate distributions), as shown below:

$$1 - F_X(x_0) = \sum_{j=0}^{\alpha-1} \frac{(x_0/\beta)^j e^{-x_0/\beta}}{j!} = F_S(\alpha - 1) \tag{15}$$

Eq. (15) introduces the *Poisson* random variable $S$, whose specific distribution is defined by its mean, $E[S] = x_0/\beta$. The Poisson random variable $S$ was introduced in (Friedman 1956), because its cumulative $F_S(\alpha - 1)$ was available in probability tables and provided a convenient

way to calculate $1-F_X(x_0)$. It is important to note that the parameters $\alpha$ and $\beta$ are constants typically estimated from past bidding data. Hence, to obtain the values of $F_X(x_0)$ for different values of $x_0$, one must read $F_S(\alpha-1)$ from *different* cumulative Poisson tables that correspond to different means $E[S] = x_0/\beta$ depending on $x_0$, but that are all read at the same constant value $\alpha$-1.

The fact that (Friedman 1956) uses the Poisson and Gamma distributions simplifies the application of Eq. (12) considerably, because, as shown below, the summation in Eq. (12) becomes the Taylor series expansion for the exponential function $e^z$, where $z = \lambda F_S(\alpha\text{-}1)$.

Thus, for the specific Poisson and Gamma distributions chosen in (Friedman 1956), the probability that contractor $A_0$ will win against an unknown number of opponents, when using a bid-to-cost ratio $x_0$, can be determined by applying Eq. (12) as follows:

$$\begin{aligned} P[A_0\text{Wins}\,|\,x_0, c_0] &= \sum_{k=0}^{\infty} g_K(k)[1-F_X(x_0)]^k = \sum_{k=0}^{\infty} \lambda^k \frac{e^{-\lambda}}{k!}\left[F_S(\alpha-1)\right]^k \\ &= e^{-\lambda}\sum_{k=0}^{\infty}\frac{1}{k!}\left[\lambda F_S(\alpha-1)\right]^k = e^{-\lambda}\exp\left[\lambda F_S(\alpha-1)\right] \\ &= \exp\left[-\lambda\left(1-F_S(\alpha-1)\right)\right] \end{aligned} \tag{16}$$

Eq. (16), above, is identical to Eq. 10 in (Friedman 1956), and continues the modeling mistake introduced in Eq. (12). Moreover, the graphs of Eq. (16), which are shown in Fig. 5 of (Friedman 1956), are incorrect, as explained below.

### Incorrect Graphs for the Probability of Winning (Unknown Number of Opponents)

Fig. 5 in (Friedman 1956) claims to show graphs of the curves for the probability of winning given by Eq. (16) for the specific expected number of opponents: $\lambda$ = 1, 3, 5. For easy reference, Fig. 5 in (Friedman 1956) is reproduced as Fig. 2 in this article. As shown in Fig. 2, the three curves in Fig. 5 of (Friedman 1956) approach the value zero for large values of the bid-to-cost ratio $x_0$. As explained below, that is not the true asymptotic behavior of Eq. (16).

The exact curves of Eq. (16), when plotted numerically for $\lambda$ = 1, 3, 5, are shown in Fig. 3. For large values of the bid-to-cost ratio $x_0$, each of the true curves of Eq. (16) approaches its own horizontal asymptote whose ordinate equals the probability of having zero opponents:

$$\lim_{x_0\to\infty} P[A_0\text{Wins}\,|\,x_0, c_0] = P[K=0] = e^{-\lambda} \tag{17}$$

For small values of the expected number of opponents, $\lambda$ = 1, 2, 3, the probability of having zero opponents $P[K=0] = e^{-\lambda}$ is significantly greater than zero. Thus, the exact numerical graphs

of Eq. (16) in Fig. 3 make it clear that the curves sketched in (Friedman 1956) and shown in Fig. 2 are not true representations of Eq. (16).

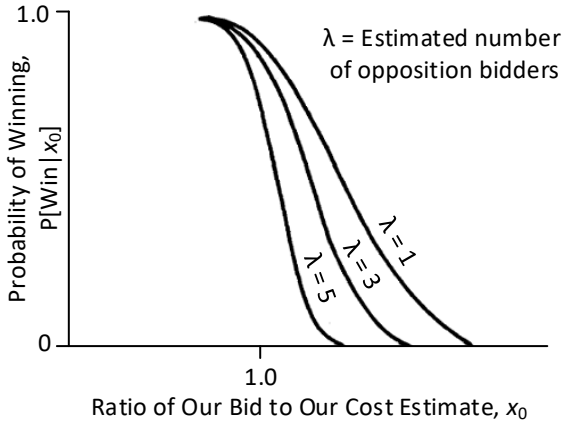



**Fig. 2.** Probability of winning vs. bid-to-cost ratio, for various estimated number of opponents [incorrect graph; adapted from Fig. 5 in (Friedman 1956)]

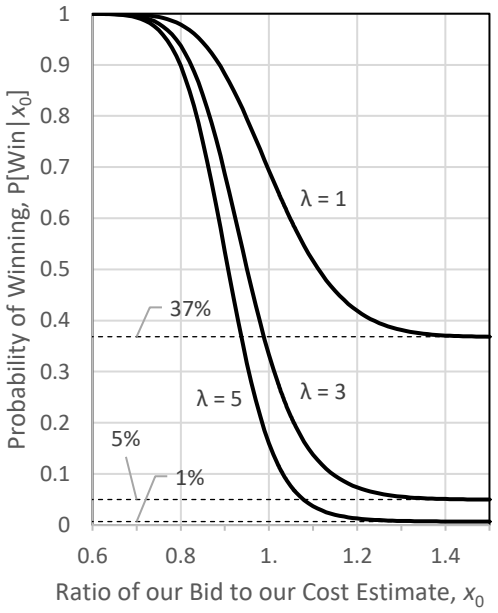



**Fig. 3.** True graphs of the (incorrect) probability of winning (proposed in Friedman 1956) vs. bid-to-cost ratio, $x_0$

For reference, the three exact graphs of Eq. (16) in Fig. 3 are based on a Gamma distribution (for the apparent bid-to-cost ratio, $X$, of the average competitor) with the following parameters:

$$\alpha = 63, \quad \beta = 1/60, \quad E[X] = \alpha\beta = 1.05, \quad SD[X] = \alpha^{0.5}\beta = 0.1323 \tag{18}$$

## Absence of the Required Global Maximum for Expected Profit

Contractor $A_0$'s expected project profit $E[V|\, x_0,c_0]$, over his estimated-cost $c_0$, when using a bid-to-cost ratio $x_0$, and facing an unknown number of opponents $K$, is obtained by substituting Eq. (16) into Eq. (5):

$$\begin{aligned} E[V\,|x_0,c_0]/\,c_0 &= (x_0-1)P[A_0 \text{ Wins}|\, x_0,c_0] \\ &= (x_0-1)\exp\left[-\lambda\left(1-F_S(\alpha-1)\right)\right] \end{aligned} \tag{19}$$

Eq. (19) is identical to Eq. (11) in (Friedman 1956) where it is simply asserted that it has a global maximum and that contractor $A_0$ should determine his bid $b_0 = x_0c_0$ by multiplying the expected cost $c_0$ by the bid-to-cost ratio $x_0$ that maximizes the expected profit function given by Eq. (19). No graphs or numerical examples of Eq. (19) are provided.

Unfortunately, the assertions that Eq. (19) exhibits a maximum and that there exists a corresponding optimum bid-to-cost ratio $x_0$ are not true. This is easily shown by plotting Eq. (19) numerically and observing its true behavior.

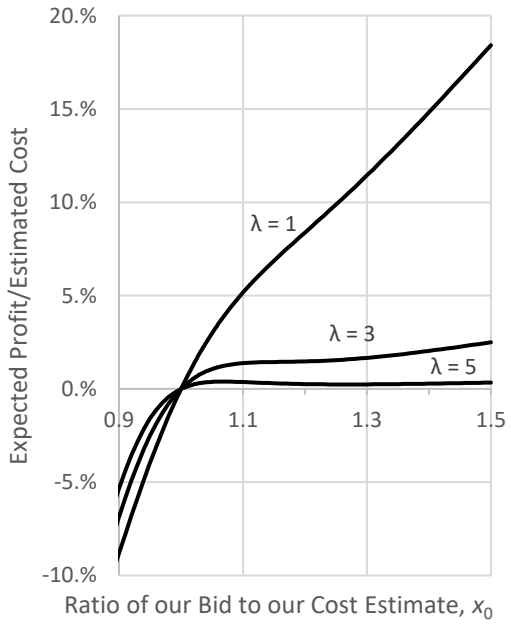



**Fig. 4.** True graphs of the (incorrect) expected profit over estimated cost (proposed in Friedman 1956) vs. bid-to-cost ratio, $x_0$

The curves in Fig. 4 show that the expected profit given by Eq. (19) for $\lambda = 1$ and $\lambda = 3$ increases monotonically as the bid-to-cost ratio $x_0$ increases, and as a result these curves have no

local or global maxima. The graph for $\lambda = 5$ has a local maximum at $x_0$=1.07 followed by a local minimum at $x_0$=1.26 and for larger values of $x_0$ continues to grow monotonically to infinity. Thus, for all three cases, $\lambda = 1, 3, 5$, the global maximum profit would occur when the bid-to-cost ratio $x_0$, and the bid $b_0$ that contractor $A_0$ should submit, become infinitely large.

The conclusion from the above discussion is that Eq. (16) for the probability of winning, and Eq. (19) for the expected profit, as given in (Friedman 1956), do not lead to a bidding model that can be used to determine the optimum bid-to-cost ratio $x_0$, and thus, the optimum bid price $b_0 = x_0 c_0$. This is because Eq. (16) and Eq. (19) cannot be used to determine a realistic optimal value $x_0$ (other than infinity) that maximizes $A_0$'s expected profit when the number of competitors $K$ is unknown.

## The First Modeling Mistake

The first modeling mistake in (Friedman 1956) is introduced in Eq. (12) and becomes clear when taking the limit of Eq. (12) as the bid-to-cost ratio $x_0$, and hence the bid price $b_0 = x_0 c_0$, become very large. For large values of the bid-to-cost ratio $x_0$, the probability of winning against a single opponent, $[1-F_X(x_0)]$, approaches zero for any distribution $F_X(x_0)$. Hence, all terms $[1-F_X(x_0)]^k$ in Eq. (12) that give the probability that contractor $A_0$ would win, given that there will be exactly $k$ opponents, $k = 1, 2,\ldots$, also approach zero. The notable exception (and the cause of the mistake) is the probability of winning against zero opponents. When $k = 0$, Eq. (12) asserts that the conditional probability of winning is $[1-F_X(x_0)]^0 = 1$.

Consequently, the probability given by Eq. (12) in (Friedman 1956) that contractor $A_0$ would win when using a very large bid-to-cost ratio $x_0$, and thus submitting a very high bid price $b_0 = x_0 c_0$, does not approach zero, but instead approaches the probability of having zero opponents, $g_K(0)$, which is positive (i.e., non-zero):

$$\lim_{x_0 \to \infty} P[A_0 \text{Wins} \mid b_0 = x_0 c_0] = g_K(0)[1-F_X(x_0)]^0 = g_K(0) \quad > 0 \tag{20}$$

This asymptotic behavior is illustrated clearly in Fig. 3 where each of the true graphs of Eq. (16) for $\lambda = 1, 3, 5$, approaches a different horizontal asymptote that depends on $\lambda$ and equals the probability of having no opponents $P[K=0] = g_K(0) = e^{-\lambda}$ as shown by Eq. (17).

The modeling mistake in (Friedman 1956) stems from the fact that the summation in Eq. (12) starts at $k = 0$, which represents the case where the number of competitors is zero. The probability of winning, $[1-F_X(x_0)]^k$, given by Eq. (10), is applicable when the number of

opponents $k$ is an integer greater than zero. Thus, the mistake in Eq. (12) is that it applies Eq. (10) for the probability of winning against exactly $k$ opponents ($k \geq 1$), to the inappropriate case where the number of opponents is zero, $k = 0$. Clearly, Eq. (10) was not developed to be used for $k = 0$, to give $[1-F_X(x_0)]^0 = 1$.

By applying Eq. (10) to the case where $K = 0$, Eq. (12) asserts with certainty that the project owner will choose to award the contract to contractor $A_0$, when $A_0$ is the only bidder, even when the bid price submitted by contractor $A_0$, $b_0 = x_0 c_0$, is unrealistically high.

While it is possible for an owner to decide and award the contract to contractor $A_0$, when $A_0$ is the only bidder, the modeling mistake in Eq. (12) is that it ensures that the owner is certain to do so with probability 1, irrespective of how high contractor $A_0$ has decided to bid.

## Missing: Probability Model that Owner will Award Contract to Single Bidder

The significant shortcoming of the bidding model in (Friedman 1956) is that does not recognize that the probability that the owner would award the contract to contractor $A_0$ when no other bids have been received, i.e., when $K = 0$, is a special case that needs to be modeled separately. This probability does not depend on the bids of other opponents but rests entirely on the provisions of the bidding documents and the decisions of the project owner. For example, bidding documents often state that the bidding process would be considered null and void if the number of received bids is below an acceptable minimum, and almost always state that the owner reserves the right to reject any and all bids (AIA-A701 1997). It is also possible that because there is an urgent need for the project to be built, the project owner might be willing to accept a single bid, even if it is somewhat high. For all these possibilities, contractor $A_0$ must use a separate model for the probability of winning when $K = 0$ and cannot use Friedman's multiplicative model that applies only when the number of opponents is at least one, $K \geq 1$.

## The Correct Form for the Probability to Win

To arrive at a correct form for the probability to win when the number of opponents is unknown, it is necessary to model explicitly the conditional probability $h(x_0)$ that contractor $A_0$ would win when submitting a bid $b_0 = x_0 c_0$ and when the number of opponents is zero, $K = 0$:

$$h(x_0) = P[A_0 \text{Wins} \mid x_0, c_0, K = 0] \tag{21}$$

Having arrived at an appropriate model for the conditional probability $h(x_0)$, the correct form for the general probability of winning against an unknown number of opponents would be:

$$P[A_0\text{Wins} \mid x_0, c_0] = g_K(0)h(x_0) + \sum_{k=1}^{\infty} g_K(k)P[A_0\text{Wins} \mid x_0, c_0, k] \tag{22}$$

Eq. (22) is general and applies to any bidding model that estimates the probability of winning against exactly $k$ opponents, $k \geq 1$, such as the bidding models in (Friedman 1956, Gates 1967, Carr 1982). For Friedman's model, the expanded and correct form for Eq. (12) is:

$$P[A_0\text{Wins} \mid x_0, c_0] = g_K(0)h(x_0) + \sum_{k=1}^{\infty} g_K(k)[1 - F_X(x_0)]^k \tag{23}$$

Thus, Eq. (23) should replace Eq. (12) as the correct form given by Friedman's model for the probability that contractor $A_0$ would win the project when submitting a bid $b_0 = x_0 c_0$, against an unknown number $K$ of "average" competitors.

## The Probability of Winning against Zero Opponents

There are many ways to model the new term in Eq. (23):

$$P[(A_0\text{Wins} \mid x_0, c_0) \cap (K = 0)] = P[A_0\text{Wins} \mid x_0, c_0, K = 0]P[K = 0] = h(x_0)g_K(0) \tag{24}$$

Cases 1-3, described below, set $g_K(0)$ or $h(x_0)$ equal to zero and lead to three possible bidding models that require only the assumptions and models already in (Friedman 1956). Case 4 extends Friedman's model by introducing a new explicit model for the probability $h(x_0)$, which is required to determine the complete Eq. (23).

The following four Cases are investigated in the sections below:

Case 1: The probability of having no opponents is zero, $P[K=0] = g_K(0) = 0$, and the number of opponents $K$ follows a *shifted* Poisson distribution with parameter $\mu$.

Case 2: The probability of having no opponents is zero, $P[K=0] = g_K(0) = 0$, and the number of opponents $K$ follows a *truncated* Poisson distribution with parameter $\rho$.

Case 3: The probability that contractor $A_0$ would be awarded the contract, given that there are no other competitors, is zero: $h(x_0) = 0$.

Case 4: The probability that contractor $A_0$ would be awarded the contract, given that there are no other opponents, is given by $h(x_0)$, a function that depends on the owner's reserve price and approaches zero as the bid-to-cost ratio $x_0$ becomes large.

## Probability of Having No Opponents is Zero, $P[K=0] = 0$

One way to correct Friedman's model *without* a new model for $h(x_0)$, is to select a

distribution $g_K(k)$ for which the probability that contractor $A_0$ will have no opponents is zero, $P[K=0] = g_K(0) = 0$.

Cases 1 and 2 below follow this approach and modify the same Poisson distribution adopted as $g_K(k)$ in (Friedman 1956), and given by Eq. (13) above. The shifted Poisson distribution in Case 1, and the truncated Poisson distribution in Case 2, are special versions of Eq. (13) that eliminate the term for $k$=0 in Eq. (12), and thus provide correct bidding models *within* the scope of (Friedman 1956). They also produce curves for the probability of winning that decline asymptotically to zero as intended by Friedman and shown by the sketches in Fig. 2.

### Case 1 — The Number of Opponents *K* follows a Shifted Poisson distribution

In Case 1, the unknown number of opponents $K$ follows a shifted Poisson distribution whose possible outcomes, $k$ = 1, 2, 3, …, do not include the value zero. The distribution of $K$ is obtained by defining the random variable $M = K$-1, that follows the same Poisson distribution used in (Friedman 1956) and given by Eq. (13), but with mean $\mu$.

The distribution of $K = M + 1$ is then given by:

$$P[K=k] = g_K(k) = p_M(k-1) = \frac{\mu^{k-1}e^{-\mu}}{(k-1)!}, \quad k = 1, 2, 3\ldots \tag{25}$$

The expected value of $K$ is that of a shifted Poisson random variable:

$$E[K] = E[M+1] = E[M]+1 = \mu+1 \tag{26}$$

The correct version of Eq. (16) in this case is obtained by substituting Eq. (25) into Eq. (23):

$$\begin{aligned} P[A_0\text{Wins}\,|\,x_0, c_0] &= g_K(0)h(x_0) + \sum_{k=1}^{\infty} g_K(k)[1-F_X(x_0)]^k \\ &= \sum_{k=1}^{\infty} \mu^{k-1}\frac{e^{-\mu}}{(k-1)!}\left[F_S(\alpha-1)\right]^k \\ &= F_S(\alpha-1)\exp[-\mu\left(1-F_S(\alpha-1)\right)] \end{aligned} \tag{27}$$

Thus, Eq. (27) multiplies the incorrect probability to win given in (Friedman 1956) and shown in Eq. (16), by $F_S(\alpha$-1), i.e., by [1-$F_X(x_0)$]. This ensures that the corrected probability that contractor $A_0$ wins approaches zero for large values of the bid-to-cost ratio $x_0$.

### Case 2 — The Number of Opponents *K* follows a Truncated Poisson distribution

In Case 2, the unknown number of opponents $K$ follows a zero-truncated Poisson distribution

whose outcomes, $k$ = 1, 2, 3, …, do not include the value zero. The distribution of $K$ is obtained from Eq. (13) as the *conditional* distribution of the standard Poisson random variable $R$, with mean $\rho$, given that $R > 0$:

$$P[K=k] = g_K(k) = P[R=k \mid R>0] = \frac{\rho^k}{k!(e^\rho - 1)}, \quad k = 1, 2, 3\ldots \tag{28}$$

The expected value of the truncated Poisson random variable, $E[K]$, is:

$$E[K] = \frac{\rho}{1-e^{-\rho}} \tag{29}$$

The correct version of Eq. (16) in this case is obtained by substituting Eq. (28) into Eq. (23):

$$\begin{aligned} P[A_0\text{Wins} \,|\, x_0, c_0] &= g_K(0)h(x_0) + \sum_{k=1}^{\infty} g_K(k)[1-F_X(x_0)]^k \\ &= \sum_{k=1}^{\infty} \frac{\rho^k}{k!(e^\rho - 1)} \left[F_S(\alpha - 1)\right]^k \\ &= \frac{\exp[\rho F_S(\alpha - 1)] - 1}{e^\rho - 1} \end{aligned} \tag{30}$$

Fig. 5 shows graphs of the probability to win for Cases 1-4. The graphs for Cases 1 and 2 illustrate that although Eq. (27) and Eq. (30) are different, they give almost identical probabilities when evaluated numerically, provided that the Poisson parameters $\mu$ and $\rho$ are chosen so that the expected number of opponents $E[K]$ are the same. The relationship between $\mu$ and $\rho$ is obtained by setting Eq. (26) and Eq. (29) for the expected number of opponents $E[K]$ to be equal:

$$E[K] = \mu + 1 = \frac{\rho}{1-e^{-\rho}} \tag{31}$$

To allow comparison, the graphs in Fig. 5 use the same parameters for the Gamma distribution of the apparent bid-to-cost ratios $X$, as Fig. 3, i.e., $\alpha$ = 63 and $\beta$ = 1/60.

### Case 3 — Contract Not Awarded when Only One Bid is Submitted

Case 3 considers the not uncommon situation where the contract will not be awarded if only one bid is submitted. In this case, the probability that $A_0$ would win the contract given that no other bids are submitted is zero:

$$P[A_0\text{Wins} \mid x_0, c_0, K=0] = h(x_0) = 0 \tag{32}$$

Like Cases 1 and 2, Case 3 eliminates the term for $k$=0 in Eq. (12) and provides a correct

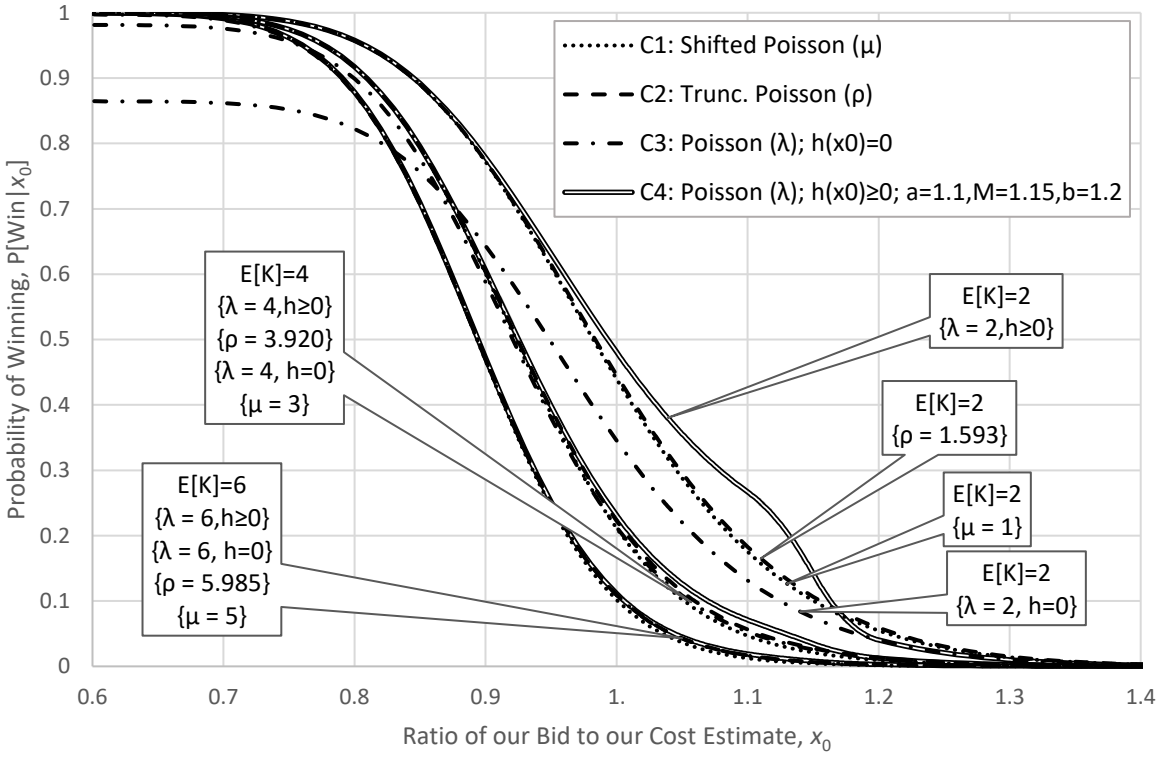



**Fig. 5.** Probability of winning vs. bid-to-cost ratio, $x_0$, for Cases 1-4

bidding model *within* the scope of (Friedman 1956), without the need for a new model for $h(x_0)$. Because $h(x_0) = 0$, Case 3 can model the unknown number of opponents $K$ using the same Poisson distribution with parameter $\lambda$ adopted in (Friedman 1956) and shown by Eq. (13), for which the probability of having zero opponents is non-zero, $P[K=0] = g_K(0) = e^{-\lambda}$.

Also, Case 3 is important because it provides *lower bounds* for both the *probability of winning* and for the *expected profit* given by Case 4 where $h(x_0) \geq 0$ (other things being equal).

For Case 3, the probability that contractor $A_0$ would win the project when submitting a bid $b_0 = x_0 c_0$ against an unknown number of competitors $K$, when the contract will not be awarded if only one bid is submitted, is obtained by substituting Eq. (32) and Eq. (13) into Eq. (23):

$$\begin{aligned} P[A_0\text{Wins}\,|\,x_0, c_0] &= g_K(0)h(x_0) + \sum_{k=1}^{\infty} g_K(k)[1-F_X(x_0)]^k \\ &= \sum_{k=1}^{\infty} \lambda^k \frac{e^{-\lambda}}{k!}\left[F_S(\alpha-1)\right]^k \\ &= \exp\left[-\lambda\left(1-F_S(\alpha-1)\right)\right] - e^{-\lambda} \end{aligned} \tag{33}$$

The probability given by Eq. (33) is the same as Eq. (16) in (Friedman 1956) but reduced by the probability of having no opponents, $P[K=0] = e^{-\lambda}$.

Graphs of the probability of winning for Case 3 given by Eq. (33) are shown in Fig. 5 for $E[K] = \lambda = 2, 4, 6$. Each curve for Eq. (33) in Fig. 5 is identical in shape to the corresponding incorrect curve of Eq. (16) in Fig 3 but appears lower by a vertical offset equal to the probability of having no opponents, $P[K=0] = e^{-\lambda}$. This makes the curves for Eq. (33) in Fig. 5 behave as expected and approach zero as the bid-to-cost ratio $x_0$ increases. The subtraction of the constant $e^{-\lambda}$ also implies that the probability of winning does not approach 1.0 as the bid-to-cost ratio $x_0$ decreases, but rather approaches the probability of having at least one opponent, $P[K \geq 1] = 1-e^{-\lambda}$, i.e., that the contract will be awarded, which is again as expected.

### Case 4: Contract May be Awarded When no other Bids are Submitted, $h(x_0) \geq 0$

In general, to apply Eq. (23) for the probability to win, contractor $A_0$ must use all pertinent historical data and all the information available at the time of bidding about the project and about the project owner to estimate the following:

(a) The distribution $g_K(k)$ for the unknown number of opponents.

(b) The distribution $F_X(x)$ for the apparent bid-to-cost ratio of the average opponent.

(c) The conditional probability $h(x_0)$ that the owner will award the contract to $A_0$, at a bid price $b_0 = x_0 c_0$, when no other bids are submitted.

Clearly, there are countless possibilities for $g_K(k)$, $F_X(x)$ and $h(x_0)$. The following discussion continues the example in (Friedman 1956) where $F_X(x)$ is modeled by a Gamma distribution and $g_K(k)$ is modeled by a Poisson distribution with parameter $\lambda$. For these distributions, the general probability that contractor $A_0$ will win against an unknown number of opponents $K$, for *any* probability $h(x_0) \geq 0$, is obtained by substituting Eq. (13) and (14) into Eq. (23):

$$\begin{aligned} P[A_0 \text{Wins} \,|\, x_0, c_0] &= g_K(0)h(x_0) + \sum_{k=1}^{\infty} g_K(k)[1-F_X(x_0)]^k \\ &= e^{-\lambda}h(x_0) + \sum_{k=1}^{\infty} \lambda^k \frac{e^{-\lambda}}{k!}\left[F_S(\alpha-1)\right]^k \\ &= \exp\left[-\lambda\left(1-F_S(\alpha-1)\right)\right] - e^{-\lambda}[1-h(x_0)] \end{aligned} \tag{34}$$

Eq.(34) is the difference of two functions, the first of which is Eq. (16), the incorrect probability of winning given in (Friedman 1956), while the second is the required correction, $e^{-\lambda}[1-h(x_0)]$, which equals the probability that contractor $A_0$ will have no opponents and will

not be awarded the contract. Thus, Eq.(34) is always less than Eq. (16).

Moreover, the probability to win given by Eq.(34) is always greater than the probability to win given by Eq. (33) for Case 3 by an amount equal to $e^{-\lambda}h(x_0)$, which is the probability that contractor $A_0$ will have no opponents and will be awarded the contract. For large values of the bid-to-cost ratio $x_0$, the probability $h(x_0)$ approaches zero, and consequently this probability approaches zero, too. Thus, the probability of winning also approaches zero, as in Case 3.

## Owner's Reserve Price

The owner's decision to award the contract to contractor $A_0$ when no other bids have been submitted depends on many factors that include the bid price $b_0 = x_0c_0$, the value of the project to the owner, the importance of the project's timing, the impact of having to advertise the project again for bids, the past and future relationship between the owner and contractor $A_0$, and the perceived ability of contractor $A_0$ to construct the project according to plans and specifications, on time, and without undue disputes, at a price $b_0 = x_0c_0$.

Recognizing the complexities involved, the following analysis will concentrate on the monetary question whether the bid price $b_0 = x_0c_0$ submitted by $A_0$ is less than the owner's reserve price $T$, which is the maximum that the owner would be willing to pay to have the project constructed by contractor $A_0$. Moreover, to make the analysis easier, the owner's reserve price $T$ will be modeled as the product of the unknown reserve-price-to-cost ratio $Q$, and the expected project cost $c_0$ estimated by contractor $A_0$:

$$T = Qc_0 \tag{35}$$

## Contract Awarded to Single Bid when less than Owner's Reserve Price

From contractor $A_0$'s perspective, the owner's reserve-price-to-cost ratio $Q$ is a random variable whose distribution $f_Q(q)$ is defined between a lower bound $a$ and an upper bound $b$. For values of the bid-to-cost ratio $x_0$ that are less than the lower bound $a$, the bid price $b_0 = x_0c_0$ would be so low that it is certain to be less than the owner's reserve price $T = Qc_0$, and the project will be awarded to $A_0$. Conversely, for values of the bid-to-cost ratio $x_0$ that are greater than the upper bound $b$, the bid price $b_0 = x_0c_0$ would be so high that it is certain to be greater than the owner's reserve price $T = Qc_0$, and the bid would be rejected. The probability $h(x_0)$ that the project would be awarded when $a \le x_0 \le b$, depends on the distribution $F_Q(q)$ of the reserve-price-to-cost ratio $Q$, as defined between the bounds $a$ and $b$. Thus, the relationship between the

probability $h(x_0)$ and the distribution $F_Q(q)$ is as follows:

$$h(x_0) = P[T > b_0] = P[Q > x_0] = \begin{cases} 1 - F_Q(x_0) = 1 & , \quad x_0 < a \\ 1 - F_Q(x_0) & , \quad a \le x_0 < b \\ 1 - F_Q(x_0) = 0 & , \quad b < x_0 \end{cases} \tag{36}$$

The probability function $h(x_0)$ in Eq. (36) consists of two horizontal lines joined by a descending S-shaped curve. For small values $x_0 < a$, the function $h(x_0)$ is the horizontal line $h(x_0) = 1$ and Eq. (34) is identical to the incorrect Eq. (16) suggested in (Friedman 1956). For large values $x_0 > b$, the function $h(x_0)$ is the horizontal line $h(x_0) = 0$, and the resulting Eq. (34) is identical to Eq. (33) for Case 3. For values of the bid-to-cost ratio, $a \le x_0 < b$, the term $1\text{-}F_Q(q)$ in Eq. (36) provides a smooth *S*-shaped curve that joins the horizontal line $h(x_0) = 1$ for small values of $x_0 < a$, to the horizontal line $h(x_0)=0$ for large values of $x_0 > b$. From a different perspective, the term $1\text{-}F_Q(q)$ makes the curve for Eq. (34), which at $x_0 = a$ equals Eq. (16), descend smoothly to Eq. (33) at $x_0 = b$. There are several distributions $f_Q(q)$ that could be used for this purpose, including the uniform, the triangular and the Beta distribution.

### Example: Reserve Price follows a Triangular Distribution

As an example, Case 4 will examine the case where contractor $A_0$ estimates that the owner's reserve-price-to-cost ratio $Q$ can be modeled by a triangular distribution $f_Q(q)$ with lower bound $a$, upper bound $b$, and mode $M$. The triangular distribution was chosen for this example because it is often used to model random variables for which no data are available.

The cumulative of the triangular distribution $F_Q(q)$ is given by the pair of quadratic functions below (Law 2014):

$$F_Q(q) = \begin{cases} 0 & , \quad q < a \\ \dfrac{(q-a)^2}{(b-a)(M-a)} & , \quad a \le q \le M \\ 1 - \dfrac{(b-q)^2}{(b-a)(b-M)} & , \quad M < q \le b \\ 1 & , \quad b < q \end{cases} \tag{37}$$

For this example, the probability that contractor $A_0$ will win against an unknown number of opponents is obtained by substituting Eq. (36) and Eq. (37) into Eq. (34):

$$P[A_0\text{Wins}\,|\,x_0, c_0] = \exp\left[-\lambda\left(1-F_S(\alpha-1)\right)\right] - e^{-\lambda} \times \begin{cases} 0 & , \quad x_0 < a \\ \dfrac{(x_0-a)^2}{(b-a)(M-a)} & , \quad a \le x_0 \le M \\ [1-\dfrac{(b-x_0)^2}{(b-a)(b-M)}] & , \quad M < x_0 \le b \\ 1 & , \quad b < x_0 \end{cases} \tag{38}$$

Eq. (38) is the probability that contractor $A_0$ would win the project when submitting a bid $b_0 = x_0 c_0$, against an unknown number of competitors $K$, when the probability $h(x_0)$ that contractor $A_0$ would be awarded the contract when no other bids have been submitted is given by Eq. (36), and the distribution of the owner's reserve-price-to-cost ratio $Q$ is given by Eq.(37).

## Modeling a Contractor's Bidding Strategy as Observed in Practice

Case 4 will also be used to model the situation described by one of the reviewers of this article as experienced in practice by a contractor that the reviewer knew personally:

> "A contractor got the impression in the pre-bid conference that the other potential bidders had lackluster interest, took a gamble by inflating his profit margin, became the only bidder, and was awarded the contract by the owner who really needed the project built."

To illustrate how to model this situation numerically, it will be assumed that the mode of $Q$ is $M$ =1.15, the lower bound $a$ = 1.10 and the upper bound $b$ = 1.20. This means that the contractor feels that the owner's reserve price is most likely at a markup of 15%, that the owner is certain to accept a bid that uses less than 10% markup, and that the owner will certainly reject a bid that uses more than 20% markup.

## Example Curves for the Probability of Winning

Fig. 5 shows graphs of the probability that contractor $A_0$ would win the contract given by Eq. (27) for Case 1, Eq. (30) for Case 2, Eq. (33) for Case 3, and Eq. (38) for Case 4. To allow comparison, the Poisson parameters $\lambda$, $\mu$, $\rho$, for these curves correspond to the same expected number of opponents, $E[K]$ = 2, 4, 6. The range for the bid-to-cost ratio, $x_0$, is deliberately quite wide, $0.6 < x_0 < 1.4$, to illustrate the asymptotic behavior of these curves for both small and large values of $x_0$. The example values for the Gamma distribution, $f_X(x_0)$, are $\alpha$ = 63 and $\beta$ = 1/60.

Fig. 5 uses the following line-types to indicate the curves that correspond to Cases 1-4:

Case 1 (dotted lines): $P[K=0] = 0$; $K$ follows a shifted Poisson with parameter $\mu$.

Case 2 (dashed lines): $P[K=0] = 0$; $K$ follows a truncated Poisson with parameter $\rho$.

Case 3 (dot-dashed lines): Contract not awarded to single bidder; $h(x_0) = 0$;

$K$ follows a Poisson with parameter $\lambda$.

Case 4 (double lines): $P$[Contract is awarded to $A_0 \mid K=0] = h(x_0) \geq 0$;

$f_Q(q)$ is triangular with parameters $a$=1.10, $M$=1.15, $b$=1.20;

$K$ follows a Poisson with parameter $\lambda$. Curve labels $\{\lambda, h \geq 0\}$ indicate the expected number of opponents, $E[K] = \lambda$, that match those for Cases 1, 2, 3.

A comparison of the graphs for Cases 1-4 in Fig. 5 indicates the following:

- Overall, the curves given by all cases for the probability of winning that correspond to the same expected number of opponents, $E[K] = 2, 4, 6$, are similar. Only the curves for Cases 3 and 4 when $E[K] = 2$ are substantially different.
- For the same expected number of opponents $E[K] = \lambda$, the curves for Case 3 and Case 4 are identical for values $x_0 > b$ =1.2 (as indicated above).
- For the same expected number of opponents $E[K] = \lambda$, and when $x_0 < a = 1.1$, the curve for Case 4 is identical in shape to the curve for Case 3 but is offset above it by a constant amount equal to the probability that contractor $A_0$ will face zero opponents, $e^{-\lambda}$.
- The largest vertical offset between the probability to win for Case 4 and Case 3 occurs when $E[K] = \lambda = 2$ (i.e., when $E[K]$ is smallest) and equals $e^{-2}$=13.5%.
- The curve for Case 4 for $E[K] = \lambda = 2$ descends by the offset 13.5% towards the curve for Case 3 by following an S-shaped transition curve that depends on $F_Q(q)$ and which starts at the lower bound $a$ = 1.1 and ends at the upper bound $b$ = 1.2.
- When $E[K] = \lambda = 2$, the probability of winning for Case 3 is small because there is a significant probability ($e^{-\lambda}$=13.5%) that contractor $A_0$ will be the only bidder and thus will not be awarded the contract. This is also true for Case 4 when $x_0 > b$ =1.2.

## Expected Profit

The expected profit for contractor $A_0$, when submitting a bid $b_0 = x_0 c_0$, against an unknown number $K$ of average competitors, is obtained by substituting Eq. (22) into Eq. (5):

$$\begin{aligned} E[V \mid x_0, c_0] &= P[A_0 \text{ Wins} \mid x_0, c_0](x_0 - 1)c_0 \\ &= \{g_K(0)h(x_0) + \sum_{k=1}^{\infty} g_K(k)P[A_0 \text{Wins} \mid x_0, c_0, k]\}(x_0 - 1)c_0 \end{aligned} \tag{39}$$

Eq. (39) is general and can be used with any bidding model that estimates the probability of winning against exactly $k$ opponents, such as (Friedman 1956, Gates 1967, Carr 1982).

For Friedman's model, the expected profit is obtained by substituting Eq. (23) in Eq. (5):

$$E[V\,|x_0, c_0] = \{g_K(0)h(x_0) + \sum_{k=1}^{\infty} g_K(k)[1-F_X(x_0)]^k\}(x_0-1)\,c_0 \tag{40}$$

For the example in (Friedman 1956) that uses the Poisson and Gamma distributions, the probability of winning is obtained by substituting Eq. (34) into Eq. (5):

$$E[V\,|x_0, c_0] = \{\exp\left[-\lambda\left(1-F_S(\alpha-1)\right)\right] - e^{-\lambda}[1-h(x_0)]\}(x_0-1)\,c_0 \tag{41}$$

## Example Curves for Expected Profit

Fig. 6 shows the expected profit, given by Eq. (41), divided by contractor $A_0$'s estimated cost, $E[V|x_0,c_0]/c_0$. Each of the three families of curves for expected profit in Fig. 6 corresponds to a different expected number of opponents $E[K] = \lambda = 2, 4, 6$, and includes four curves that correspond to Cases 1-4. The curves in Fig. 6 were obtained by multiplying $(x_0\text{-}1)$ by the

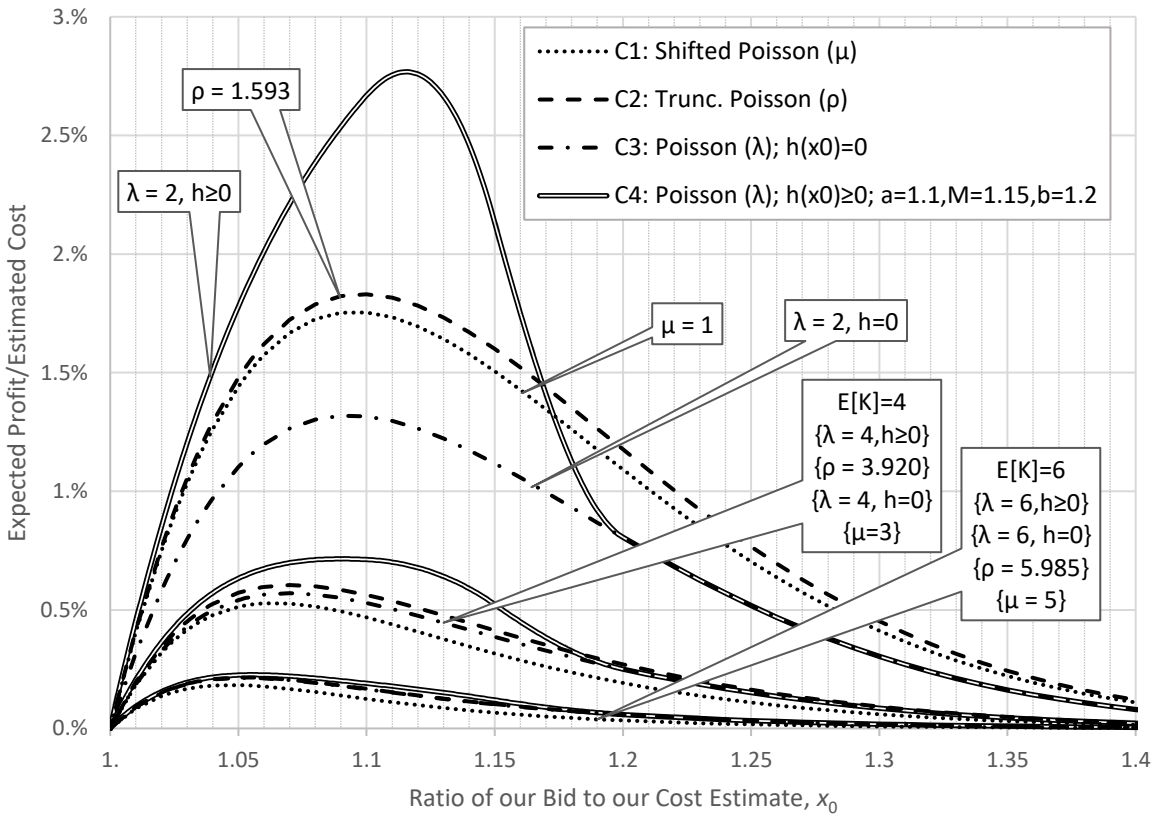



**Fig. 6.** Expected profit over estimated cost vs. bid-to-cost ratio, $x_0$, for Cases 1-4

corresponding probability of winning in Fig. 5 (i.e., Eq. (27) for Case 1, Eq. (30) for Case 2, Eq. (33) for Case 3, and Eq. (38) for Case 4).

## Optimal Bid-To-Cost Ratios—Numerical Examples

Table 1 shows the values of the optimal bid-to-cost ratios $x_0^*$ that maximize the example expected profit curves shown in Fig. 6. The data set parameters used for these examples are shown at the bottom of Table 1. The columns in Table 1 are organized into three sections that correspond to the three values for the expected number of opponents, $E[K] = 2, 4, 6$. Each section also shows the values of the associated Poisson parameters, $\mu$, $\rho$ and $\lambda$, used in Cases 1-4, along with three columns that show the values of the corresponding optimal bid-to-cost ratios, $x_0^*$, the probability of winning at the optimal bid-to-cost ratio, $P[W \mid x_0^*]$, and the corresponding maximum expected profit over estimated cost at the optimal bid-to-cost ratio, $E[V \mid x_0^*]/c_0$.

**Table 1.** Optimal Values of Bid-to-Cost Ratio, $x_0^*$, for Example Data Set(1)

| | $E[K] = 2$ ($\mu$=1, $\rho$=1.593, $\lambda$=2) | | | $E[K] = 4$ ($\mu$=3, $\rho$=3.920, $\lambda$=4) | | | $E[K] = 6$ ($\mu$=5, $\rho$=5.985, $\lambda$=6) | | |
|---|---|---|---|---|---|---|---|---|---|
| Case | $x_0^*$ | $P[W\|x_0^*]$ | $E[V\|x_0^*]/c_0$ | $x_0^*$ | $P[W\|x_0^*]$ | $E[V\|x_0^*]/c_0$ | $x_0^*$ | $P[W\|x_0^*]$ | $E[V\|x_0^*]/c_0$ |
| 1: $P[K=0]=0$, $\mu$ | 1.095 | 18.4% | 1.756% | 1.064 | 8.2% | 0.529% | 1.047 | 3.9% | 0.183% |
| 2: $P[K=0]=0$, $\rho$ | 1.098 | 18.6% | 1.831% | 1.071 | 8.5% | 0.604% | 1.052 | 4.1% | 0.215% |
| 3: $h(x_0)=0$, $\lambda$ | 1.093 | 14.2% | 1.319% | 1.070 | 8.2% | 0.570% | 1.052 | 4.1% | 0.213% |
| 4: $h(x_0)>0$, $\lambda$ | 1.115 | 24.1% | 2.768% | 1.090 | 7.9% | 0.715% | 1.056 | 4.0% | 0.226% |

(1) The apparent bid-to-cost ratio for the average opponent, $X$, follows a Gamma distribution with $\alpha = 63$, $\beta = 1/60$. The owner's reserve-price-to-cost ratio follows a triangular distribution with $a = 1.10$, $M = 1.15$, $b = 1.20$. The unknown number of opponents follows a form of a Poisson distribution with parameter $\mu$, $\rho$, $\lambda$, as shown.

The rows in Table 1 correspond to the four Cases 1-4 described above.

A comparison of the values $x_0^*$ in Table 1 shows that the optimal bid-to-cost ratios given by Cases 1-3 for the same expected number of opponents, $E[K]$, are very close. For example, when $E[K]=2$, the optimal bid-to-cost ratios $x_0^*$ that maximize the expected profit curves shown in Fig. 6 range only from 1.093 to 1.098, corresponding to markup percentages from 9.3% to 9.8%. The same is true for $E[K]=4$ and for $E[K]=6$.

## Optimal Bid-To-Cost Ratios and the Owner's Reserve-Price-to-Cost Ratio

The most striking results in Fig. 5, Fig. 6, and Table 1, are those for Case 4 where there is a chance that the owner will award the contract to contractor $A_0$ when no other bids have been submitted. When the expected number of opponents is small, e.g., $E[K] = \lambda = 2$, the probability

that $A_0$ will be the only bidder and be awarded the contract is significant. Thus, as shown in Fig. 6, the expected profit for Case 4 is more than double that of Case 3 and also significantly larger than for Cases 1 and 2. The substantial increase in expected profit from Case 3 to Case 4, when $E[K] = \lambda = 2$, is entirely due to the probability $h(x_0) \geq 0$ that contractor $A_0$ may be awarded the contract when $A_0$ is the only bidder.

Moreover, it is important to note the *range* in which the optimum bid-to-cost ratio $x_0^*$ occurs when the expected number of opponents $E[K] = \lambda$ is small. The expected profit for Case 4 for values of the bid-to-cost ratio $x_0 < a$ (the lower bound of the reserve-price-to-cost ratio, $Q$) is obtained by substituting the probability $h(x_0) = 1$ in Eq. (41):

$$\begin{aligned} E[V \,|\, x_0, c_0] &= \exp\left[-\lambda\left(1 - F_S(\alpha - 1)\right)\right](x_0 - 1)\,c_0 \\ &= \exp\left[-\lambda F_X(x_0)\right](x_0 - 1)c_0 \end{aligned} \qquad \text{for } x_0 < a \qquad (42)$$

When $\lambda < 3.73$, the slope of Eq. (42) is positive for all values $x_0 < a$. This means that when the expected number of opponents $E[K] = \lambda < 3.73$, the maximum expected profit for Case 4 given by Eq. (41) occurs at an optimum $x_0^*$ in the range $1.10 < x_0^* < 1.20$, where there is "a calculated risk" that the owner's reserve price $T = Qc_0$ will be less than the bid price $b_0^* = c_0 x_0^*$ and the bid will be rejected. This is illustrated by the example curve in Fig. 6 for Case 4 and $E[K] = \lambda = 2$ (< 3.73), for which the optimum bid-to-cost ratio is $x_0^* = 1.115$ (> $a$ =1.10), and $h(x_0^*) = 95.5\%$. Thus, there is a "calculated risk" of 4.5% that at the optimum markup of 11.5%, the owner's reserve price $T = Qc_0$ will be less than the bid price $b_0^* = 1.115c_0$ and the bid will be rejected.

Clearly, the optimal strategy for Case 4 when $E[K] = \lambda = 2$ is that followed by the real-life contractor described by one of the reviewers. When the expected number of opponents $E[K]$ is small, the strategy that maximizes the contractor's expected profit is to select a high $x_0^*$ and take a calculated risk that the submitted bid $b_0^* = c_0 x_0^*$ will be less than the owner's estimated reserve price $Q = qc_0$. When $E[K]$ is small, there is significant probability that no other contractor will submit a bid, in which case this strategy leads to a price close to the maximum that the owner would be willing to pay, and thus to the maximum expected profit.

Perhaps the most important point is that such an optimal strategy cannot be identified unless the bidding model in (Friedman 1956) is extended as above to include an explicit model for $A_0$'s estimated probability $h(x_0)$ that the owner would accept a bid $b_0 = c_0 x_0$ and award the contract to $A_0$ when no other bids are submitted.

When the expected number of opponents $E[K]$ increases, i.e., when $\lambda > 3.73$, the probability that $A_0$ will be the only bidder decreases towards zero and the strategy of submitting a bid that exceeds the lower bound for the owner's estimated reserve price, hoping that no other bid will be submitted, ceases to be optimal.

This is illustrated in Fig. 6 and Table 1 when $E[K] = \lambda = 4 > 3.73$. The maximum expected profit for Case 4 occurs when $x_0^* = 1.09$ ($< a = 1.1$), for which $h(x_0^*) = 1$. When $E[K] = \lambda = 6$, the optimum is even smaller, $x_0^* = 1.056$. Moreover, when $E[K] = \lambda = 6$, the expected profit curves for all four cases are almost identical because the probability of having no opponents approaches zero and has no significant effect on expected profit.

## Summary and Conclusions

Friedman's bidding model for the case where contractor $A_0$ is bidding against an unknown number of opponents contains errors that have a common underlying cause. It does not recognize that the case where the number of opponents is zero, and contractor $A_0$ is the only bidder, is fundamentally different and needs to be modeled separately from every other possibility where the number of opponents is an integer greater than zero.

Specifically, the mistakes in (Friedman 1956) stem from the application of Friedman's classic multiplicative model for the probability to win against $K$ opponents, which is applicable when the number of opponents is greater than or equal to one, $K \geq 1$, to the case where the number of opponents is zero, $K = 0$ (to which it does not apply).

The correct general probability that contractor $A_0$ would win when facing an unknown number of opponents $K$, is given by the new Eq. (22) which applies to all bidding models that estimate the probability of winning against exactly $k$ opponents, such as the models in (Friedman 1956, Gates 1967, and Carr 1982). The specific application of Eq. 24 to Friedman's model is given by Eq. (23) and has been illustrated by numerical examples in Fig. 5.

The new Eq. (22) and (23) for the probability to win are based on two sub-models. The first sub-model applies when the number of opponents is positive, $K \geq 1$, for which it is possible to employ Friedman's classic multiplicative model for the probability to win against $k$ opponents. The second sub-model is new and is the product of two probabilities, the probability $g_K(0)$ that contractor $A_0$ will have no opponents, and the conditional probability $h(x_0)$ that contractor $A_0$ would be awarded the contract when no other bids are submitted.

Cases 1-4 describe four new formulations for the second sub-model. Cases 1-3 illustrate three

possible models that could have been used in (Friedman 1956) to obtain correct results using only the assumptions and models already adopted in (Friedman 1956). Cases 1 and 2 modify the Poisson adopted by (Friedman 1956) such that $g_K(0)=0$. Case 3 represents a project where the contract will not be awarded to a single bidder and thus $h(x_0)$ is zero.

Case 4 presents for the first time the general model where both $g_K(0)$ and $h(x_0)$ may be non-zero. To apply this model, contractor $A_0$ needs to estimate the probability $h(x_0)$ that the contract would be awarded to $A_0$ at a bid price $b_0 = x_0 c_0$ when no other bids are submitted. The explicit consideration and modeling of the situation where $A_0$ is the only bidder and the assessment of the probability function $h(x_0)$ by all bidding models (and not just Friedman's) creates new possibilities for developing bidding strategies that could not be modeled before and which can lead to higher expected profit, especially when the expected number of opponents is small.

Such a new bidding strategy is illustrated by the example in Case 4 where contractor $A_0$ estimates that the owner's reserve-price-to-cost ratio follows a triangular distribution. This example shows that when the expected number of opponents is small, the optimal strategy is to bid high and close to (but hopefully below) the owner's reserve price and hope that no other opponents bid the project. Given the contractor's estimate of the owner's reserve-price distribution, this leads to the maximum expected profit that the owner would be willing to pay.

This strategy is new in that it cannot be identified by previous bidding models, such as (Friedman 1956, Gates 1967, Carr 1982), because they do not recognize and thus do not address the need for the conditional probability $h(x_0)$. The possibility of identifying such a strategy that leads to the maximum possible expected profit for the contractor is perhaps the most significant contribution of this article.

Modern computational tools make it possible to use the corrected Friedman bidding model in practice with any probability distributions that reflect both past experience and the specific information about the project at hand. For example, if a contractor faced exactly $k$ opponents in $n_k$ of the $N$ past projects, the probability of having $k$ opponents, $k = 0, 1, 2, \ldots$, can be estimated by the relative frequency, $g_K(k) = n_k / N$, and thus is not limited to a formal probability model such as the Poisson. Similarly, the apparent bid-to-cost ratio is not limited to the Gamma distribution and the probability that the owner would award the contract to a single bidder is not limited to the triangular distribution used as an example in Case 4. A contractor can draw upon Bayesian statistics and use suitable distributions supported by historical data and the information available about the project and about the project owner at the time of preparing the bid.

Finally, the above comments should not be viewed as criticism of Friedman's bidding model but rather as recognition of continued interest and of its impact on both theory and practice. Also, in Friedman's defense, the fact that the above errors in (Friedman 1956) went undetected for more than 60 years, even though Friedman's classic model has been cited hundreds of times, makes it clear that they were not easy to discern.

## Data Availability Statement

All data generated or analyzed during the study are included in the submitted article or supplemental materials files.

## Acknowledgements

The author wishes to recognize and thank the anonymous reviewers of this article for their insightful comments.

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# Average Bid Method — A Competitive Bidding Strategy

By Photios G. Ioannou,[1] Member, ASCE, and Sou-Sen Leu[2]

**Abstract:** The major drawback of the low bid method, often used for competitive bidding in the U.S. construction industry, is the possibility of awarding a construction contract to a contractor that submits either accidentally or deliberately an unrealistically low bid price. Often, such an occurrence works to the owner's and contractor's detriment by promoting disputes, increased costs, and schedule delays. To address this problem, other countries have adopted the average bid method and award the contract to the contractor whose price is closest to the average of all bids submitted. This paper presents a competitive bidding model for the average bid method and explores its merits relative to the low bid method. The bidding process is analyzed both mathematically and through Monte Carlo simulation. The final results of the average bid model, as well as Friedman's low bid model, are presented in four nomographs that can be used to analyze a competitive situation without the need for any mathematical or numerical manipulation. A comparison of the two methods reveals that the average bid method and its variations have the potential to improve contracting practices both for the owner and the contractor and deserve the industry's increased attention.

## Introduction

The competitive bidding process for awarding construction contracts in the U.S. is typically based on the low bid method and is probably as close to pure competition as possible. According to this method, the construction firm submitting the lowest bid receives the right to the construction contract. Its main advantage is that it forces contractors to continuously try to lower costs by adopting cost-saving technological and managerial innovations. These savings are then passed to the owner through the competitive process. When the number of bidders is large, however, as is the case in a slow economy, an owner runs a significant risk of selecting a contractor that has either accidentally or deliberately submitted an unrealistically low price (Grogan 1992). A contractor cannot adhere to such a price and at the same time expect to complete the project according to plans and specifications, and also make a reasonable profit. This often results in excessive claims and disputes during construction that lead to schedule delays, compromises in quality, and increased costs. Several countries, such as Italy and Taiwan, have developed variations of the average bid

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©1991 P.G. Ioannou. This manuscript was submitted to ASCE for review and possible publication on December 12, 1991. The final paper was published in the *Journal of Construction Engineering and Management*, ASCE, Vol. 199, No.1, March 1993, pp. 131-147.

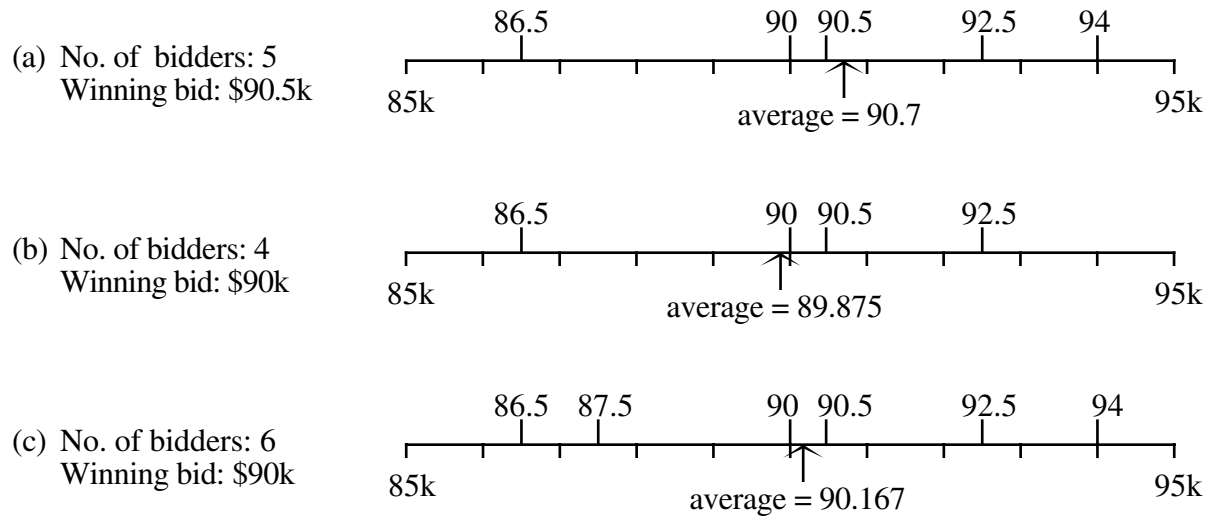



Figure 1: — **Example of the Average Bid Method**

method as an answer to these problems. In Taiwan, for example, the average bid method presented in this paper is rapidly becoming the preferred competitive bidding method.

The objective of this paper is to present a competitive bidding model for the average bid method and to explore its merits relative to the low bid method. The bidding process is analyzed both mathematically and through Monte Carlo simulation. Even though the former is more elegant, the latter is more powerful and can be easily adapted to other variations of the average bid method. The results of the bidding model are presented in four nomographs that can be used to analyze a competitive situation without the need for complicated mathematics. The same graphs present a similar solution for the low bid method based on Friedman's model (Friedman 1956, Ioannou 1988).

## The Average Bid Method

In general, the winner based on the average bid method is the contractor whose bid satisfies a certain relationship with the average of all bid prices. Different average bid methods use different procedures for calculating the average, or use different criteria for determining the winning bid. For example, some use an arithmetic average or a weighted average, while others use the average of the remaining bids after all bids that differ more than a certain percentage from the average of all other bids are eliminated. Similarly, the winner might be the contractor whose price is closest to the average, or the contractor whose bid is closest to, but less than the average. The former, for example, is used in Taiwan while the latter is used in Italy. The average bid method presented in this paper is the one used in Taiwan, where the winner is the bidder whose price is closest to arithmetic average of all submitted bids.

To illustrate the mechanism of the average bid method, as well as its differences from the low bid method, consider the example illustrated in Fig. 1. Fig. 1a shows five contractors that have submitted the following bids: 86.5k, 90k, 90.5k, 92.5k, and 94k. The average bid price is 90.7k. The bid of 90.5k is closest to the average and thus wins the project. Fig. 1b shows that if for some reason the bid of 94k had not been submitted, the average would drop to 89.875k, and the winning

bid would have been 90k. Similarly, Fig. 1c illustrates that if a sixth bid of 87.5k were submitted, the average would drop to 90.167k and the winning bid would again be 90k. Thus, even though neither of the two bids at 94k and 87.5k ever win, their presence influences which of the other bids does win. This clearly illustrates the interdependence of all bid prices in determining the winner. In contrast, under the low bid method the winning bid in all three cases would have been the same, 86.5k.

It is clear from this example that the chance of winning is affected by the number of bidders and all submitted bid prices. Consequently, the average bid method is significantly more difficult to model than the low bid method. For the low bid method, the main variable of interest, from a contractor's perspective, is simply the minimum of the bid prices submitted by the other competitors. The contractor's own bid price is not part of the minimum function but rather serves as a value to which the minimum of the opponents' bids is compared. The average bid price, on the other hand, depends on the contractor's own bid price. Furthermore, in order to determine the winner we need to determine the absolute value of the difference of every bid to the average and select the minimum. A change in any of the bid amounts, or the number of bidders, can have a profound effect on any particular bidder's chance of winning.

## Basic Definitions

The notation adopted in this paper is to indicate random variables and the identity of contractors using capitalized symbols; decision variables and the values of random variables are shown using lower-case symbols.

We shall examine the bidding process from the perspective of a particular contractor $A_0$ who is bidding on a new project against $n$ competitors $A_1, A_2, \ldots, A_n$. Let $C_i$ be the project cost estimate and $B_i$ the bid price for competitor $A_i$, $i = 0, 1, \ldots, n$.

Contractor $A_0$ has already determined a cost estimate $c_0$ and needs to decide on a bid price $b_0$. This is the decision variable of primary interest. Even though there is nothing uncertain about either of these quantities, the actual cost of the project, given that $A_0$ wins the project, is of course uncertain.

In preparing a cost estimate, contractor $A_0$ has to make a multitude of decisions: select crew compositions, estimate quantities, estimate production, etc. In fact, the entire estimating process is a series of decisions based on available information. A change in any of these will have a direct effect on the cost estimate $c_0$.

After the cost estimate is established, contractor $A_0$ must choose a markup $m_0$ to arrive at a bid price $b_0 = c_0 + m_0$. Equivalently, contractor $A_0$ may select a bid-to-cost ratio $x_0 = 1 + m_0/c_0$ and multiply it by his cost estimate $c_0$:

$$b_0 = x_0 c_0 \tag{1}$$

Obviously, contractor $A_0$ has complete control over the values of $c_0$, $m_0$, $x_0$, and $b_0$. All are decision variables, even if their values have not been determined yet.

$A_0$ does not know the competitors' cost estimates $C_i$ and bids $B_i$. Given this state of information, these are considered to be random variables, even though $C_i$ and $B_i$ are decision variables to the corresponding contractor $A_i$. Furthermore, $A_0$'s state of information, and hence his perspective, do not change depending on whether contractor $A_i$ has, or has not, already determined and thus fixed the values of $C_i$ and $B_i$.

The ratio of contractor $A_i$'s bid $B_i$ to contractor $A_0$'s cost estimate $c_0$ will be shown as $X_i$:

$$X_i = \frac{B_i}{c_0} \quad i = 1, 2, \ldots, n \tag{2}$$

It should be noted that for contractor $A_0$ the ratio $x_0 = b_0/c_0$ is a decision variable, whereas the $X_i$ ($i = 1, \ldots, n$) are random variables representing the apparent bid-to-cost ratios for each of the $n$ other competitors.

## Friedman's Propositions

To facilitate a direct comparison, the following model for the analysis of the average bid method is based on the same two assumptions as Friedman's model for the low bid method (Friedman 1956, Ioannou 1988):

1. In order to eliminate the effect of the project size, each opponent's bid $B_i$ ($i = 1, 2, \ldots, n$) is standardized by taking its ratio to contractor $A_0$'s cost estimate $c_0$. The resulting ratios $X_i$ ($i = 1, 2, \ldots, n$) are assumed to be mutually independent.

2. The probability distribution of each apparent bid-to-cost ratio $X_i$ ($i = 1, 2, \ldots, n$) does not depend on the values of $x_0$ and $c_0$ that $A_0$ chooses:

$$P[X_i < x_i | x_0, c_0] = P[X_i < x_i] \quad i = 1, 2, \ldots, n \tag{3}$$

The theoretical and practical ramifications of these assumptions are examined in detail elsewhere and are not presented here (Ioannou 1988).

## Mathematical Formulation

The formal definition for the event "Contractor $A_0$ wins" when using the average bid method can be stated as follows:

$$\{A_0 \text{ wins} | b_0\} = \{\bigcap_{i=1}^{n} (|b_0 - \overline{B}| < |B_i - \overline{B}|)\} \tag{4}$$

where $\overline{B} = (b_0 + B_1 + \ldots + B_n)/(n + 1)$ is the average of the bid prices. Notice that since $B_1, B_2, \ldots, B_n$ are random variables, so is $\overline{B}$. Furthermore, since $b_0$ is a decision variable, the value of $\overline{B}$ can be influenced by contractor $A_0$.

It should be noted here that the average bid method, as defined above, does not apply to the case of only two bidders, $A_0$ and $A_1$. The average $\overline{B} = (b_0 + B_1)/2$ is exactly midway between $b_0$ and $B_1$, and thus there is no way to determine the winner. The following analysis assumes that the competitive bidding process involves more than two competitors.

The general formula for the probability that contractor $A_0$ wins is:

$$P[A_0 \text{ wins} | b_0] = P[(|b_0 - \overline{B}| < |B_1 - \overline{B}|) \bigcap \ldots \bigcap (|b_0 - \overline{B}| < |B_n - \overline{B}|)] \tag{5}$$

$$= P[\bigcap_{i=1}^{n} (|b_0 - \overline{B}| < |B_i - \overline{B}|)] \tag{6}$$

$$= P[\bigcap_{i=1}^{n}(|x_0 - \overline{X}| < |X_i - \overline{X}|)] \quad (7)$$

in which $X_i = B_i/c_0$, and $\overline{X} = (x_0 + X_1 + \ldots + X_n)/(n+1)$.

The simplest application of this formula is the case where contractor $A_0$ faces only two opponents, $A_1$ and $A_2$. In order to win, $A_0$ must select a bid-to-cost ratio $x_0$ that satisfies the following two inequalities:

$$|x_0 - \overline{X}| < |X_1 - \overline{X}| \quad (8)$$
$$|x_0 - \overline{X}| < |X_2 - \overline{X}| \quad (9)$$

Replacing $\overline{X}$ with $(x_0 + X_1 + X_2)/3$ gives:

$$|(2x_0 - X_1 - X_2)| < |(2X_1 - x_0 - X_2)| \quad (10)$$
$$|(2x_0 - X_1 - X_2)| < |(2X_2 - x_0 - X_1)| \quad (11)$$

Squaring both sides of each inequality eliminates absolute values, and collecting terms:

$$3x_0^2 - 3X_1^2 < 6x_0X_2 - 6X_1X_2 \quad (12)$$
$$3x_0^2 - 3X_2^2 < 6x_0X_1 - 6X_1X_2 \quad (13)$$

Finally, extracting common factors produces the inequalities:

$$(x_0 - X_1)(x_0 + X_1) < 2X_2(x_0 - X_1) \quad (14)$$
$$(x_0 - X_2)(x_0 + X_2) < 2X_1(x_0 - X_2) \quad (15)$$

In order for contractor $A_0$ to win, the random variables $X_1$ and $X_2$ must take on values that satisfy both inequalities (14) and (15). Notice that the terms $(x_0 - X_1)$ and $(x_0 - X_2)$ appear on both sides of the above inequalities. They cannot be eliminated, however, because their sign determines the direction of inequality.

The region where the pairs $(x_1, x_2)$ satisfy inequalities (14) and (15) is indicated by the cross-hatched areas in Fig. 2. According to Friedman's assumptions, $X_1$ and $X_2$ are iid (independent and identically distributed) with a common probability density function, $f_X(x)$, and a common cumulative distribution function, $F_X(x)$. Integrating $f_X(x)$ over this discontinuous region produces the following probability of winning:

$$P[A_0 \text{ wins}|x_0] = \int_{-\infty}^{x_0} f_X(x_1)dx_1 \int_{x_0}^{+\infty} f_X(x_2)dx_2 + \int_{x_0}^{+\infty} f_X(x_1)dx_1 \int_{-\infty}^{x_0} f_X(x_2)dx_2 \quad (16)$$
$$= 2F_X(x_0)(1 - F_X(x_0)) \quad (17)$$

It can be easily shown that this result is the same as the probability that the value $x_0$ lies between $X_1$ and $X_2$. In fact, a simple examination shows that the pairs $(x_1, x_2)$ defined by the cross-hatched areas in Fig. 2 are indeed those that bracket $x_0$. This is as expected, since this is the necessary and sufficient condition for $x_0$ to win. If $x_0$ lies between $x_1$ and $x_2$ it will certainly be closest to the average.

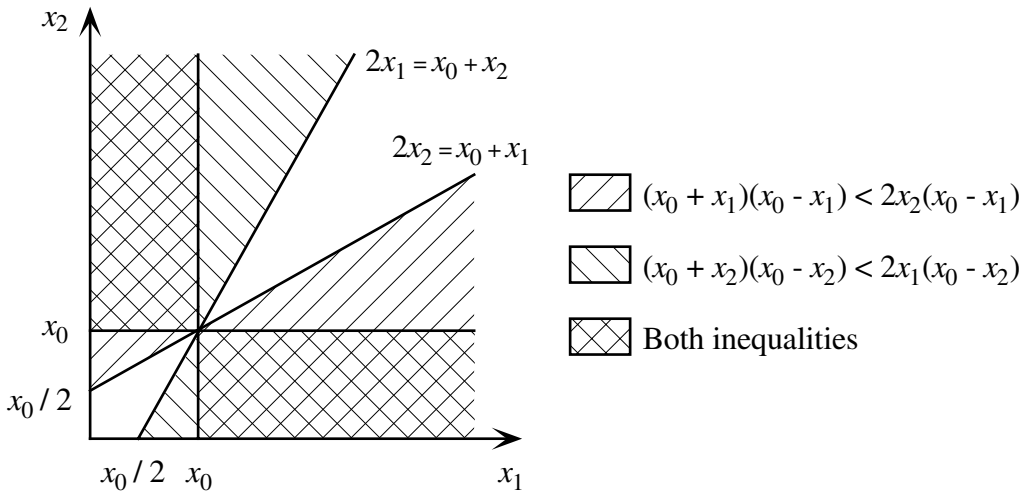



Figure 2: — **Region for Winning Against Two Opponents**

The estimated profit is simply given by $v_0 = b_0 - c_0$. Its expected value is given by:

$$\begin{aligned} E[V] &= P(A_0 \text{ wins})(b_0 - c_0) && (18)\\ &= 2F_X(x_0)(1 - F_X(x_0))(b_0 - c_0) && (19)\\ &= 2F_X(x_0)(1 - F_X(x_0))(x_0 - 1)c_0 && (20) \end{aligned}$$

The optimum bid-to-cost ratio $x_0$ can be determined by maximizing Eq. 20.

In general, the number of opponents facing contractor $A_0$ will be $n > 2$. In order to win the project, $A_0$'s bid-to-cost ratio $x_0$ must satisfy the following $n$ inequalities:

$$\begin{aligned} |x_0 - \overline{X}| &< |X_1 - \overline{X}| && (21)\\ |x_0 - \overline{X}| &< |X_2 - \overline{X}| && (22)\\ &\vdots \\ |x_0 - \overline{X}| &< |X_n - \overline{X}| && (23) \end{aligned}$$

Using the same algebra as in the case of two opponents, these can be simplified to:

$$\begin{aligned} (n-1)(x_0 - X_1)(x_0 + X_1) &< 2(x_0 - X_1)(X_2 + X_3 + \ldots + X_n) && (24)\\ (n-1)(x_0 - X_2)(x_0 + X_2) &< 2(x_0 - X_2)(X_1 + X_3 + \ldots + X_n) && (25)\\ &\vdots \\ (n-1)(x_0 - X_n)(x_0 + X_n) &< 2(x_0 - X_n)(X_1 + X_2 + \ldots + X_{n-1}) && (26) \end{aligned}$$

Notice again that even though the terms $(x_0 - X_i)$ appear on both sides of the above inequalities they cannot be eliminated because their sign determines the direction of inequality.

In order for contractor $A_0$ to win, the random variables $X_1, X_2, \ldots, X_n$ must take on values that satisfy inequalities (24)-(26). The region where points $(x_1, x_2, \ldots, x_n)$ satisfy inequalities (24)-(26) must be visualized in multidimensional coordinates. Since this is not practical, the probability of winning can be computed by using the Total Probability Theorem. This approach involves

the definition of a set of regions on the one-dimensional axis of real numbers that satisfy all the possibilities implied by the above set of inequalities. This is a very tedious task that leads to integral equations whose limits are linear functions of $x_0, x_1, \ldots, x_n$. These integrals, however, do not provide a closed-form solution for the general probability of winning. This probability can only be computed by assuming a particular probability distribution $f_X(x_i)$ and performing numerical integration. The same solution can be achieved much easier by using simulation as described below.

## Simulation Approach

Monte Carlo simulation can be used to determine the probability of winning and to select the optimum bid-to-cost ratio $x_0$ for a given distribution $f_X(x_i)$. We shall assume that the apparent bid-to-cost ratios, $X_1, X_2, \ldots, X_n$, are iid, following a Normal distribution with mean $m_X$, and variance $\sigma_X^2$.

In order to arrive at a numerical solution that is independent of $m_X$ and $\sigma_X^2$, the following standardized variables are defined:

$$m'_X = \frac{m_X - 1}{\sigma_X} \tag{27}$$

$$x'_0 = \frac{x_0 - m_X}{\sigma_X} \tag{28}$$

The main input to the simulation process consists of the number of competitors $n$, the number of simulation sets $s$, and the number of projects per simulation set $m$. The results presented here are based on $s = 100$, and $m = 1000$.

For each project $j = 1, \ldots, m$, in each simulation set $k = 1, \ldots, s$, we sample $n$ standardized apparent bid-to-cost ratios $x'_{ijk}$ $(i = 1, \ldots, n)$ from the standard Normal distribution N(0,1). Notice that the index $i$ represents a competitor and the index $j$ represents a project. Index $k$ represents the simulation set and may be ignored for the time being.

For a given project $(j, k)$ we compute the average $\bar{x}'_{jk}$ of the $n$ competitors' values $x'_{ijk}$ and a range around the average defined by a lower bound $l_{jk}$ and an upper bound $h_{jk}$. The range $(l_{jk}, h_{jk})$ defines the region within which $x'_{0jk}$ wins project $(j, k)$. Its bounds are determined by

$$h_{jk} = \min(\bar{x}'_{jk} + d^+, \bar{x}'_{jk} + \frac{n+1}{n-1} d^-) \tag{29}$$

$$l_{jk} = \max(\bar{x}'_{jk} - d^-, \bar{x}'_{jk} - \frac{n+1}{n-1} d^+) \tag{30}$$

where $d^+$ is the absolute value of the distance from the average $\bar{x}'_{jk}$ to the next higher $x'_{ijk}$, and $d^-$ is the absolute value of the distance from the average $\bar{x}'_{jk}$ to the next lower $x'_{ijk}$. The derivation of these bounds is shown in Appendix II.

Any value $x'_{0jk}$ in the region $(l_{jk}, h_{jk})$ wins project $(j, k)$. The best selection, however, is $x'_{0jk} = h_{jk}$ since it not only wins the project but also maximizes profit.

Repeating this process for a given simulation set $k$, results in $m$ ranges $(l_{jk}, h_{jk})$, one for each project. We now determine how many of the $m$ projects in the current simulation set would have been won by each of the $m$ candidate values $x'_{0jk} = h_{jk}$. This is determined by comparing each candidate value $h_{jk}$ to all ranges $(l_{qk}, h_{qk})$, where $q = 1, \ldots, m$. Project $q$ is considered won

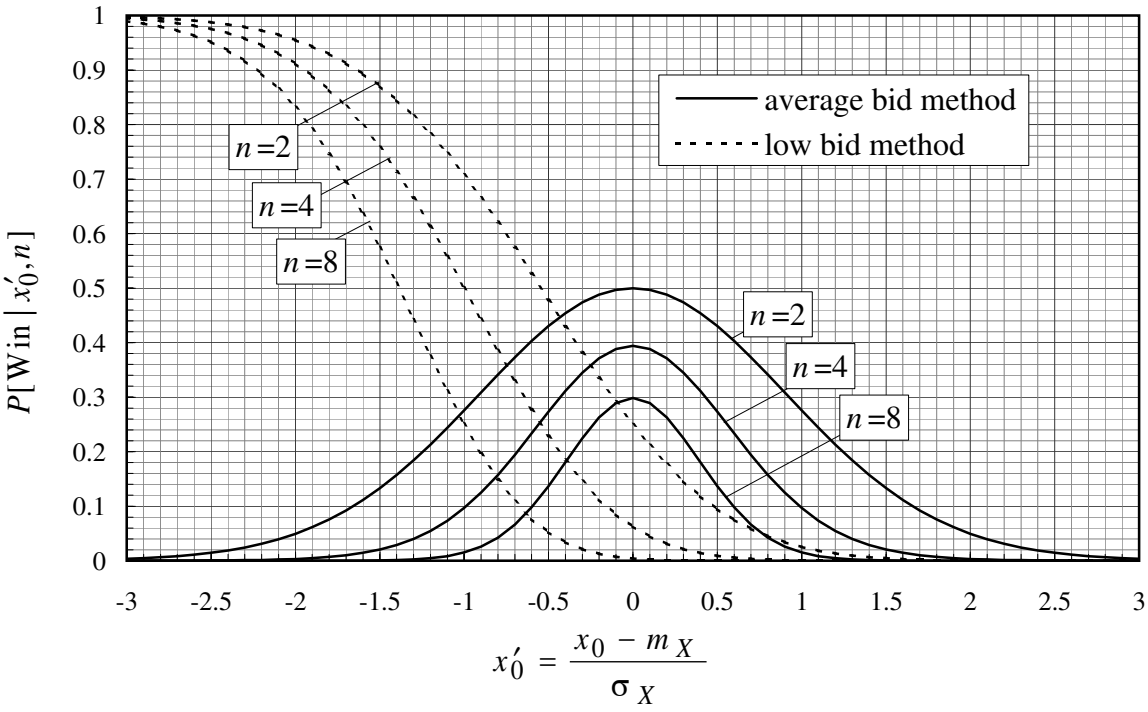



Figure 3: — **Probability of Winning as a function of $x'_0$, $n$**

if the selected $h_{jk}$ is within the project's corresponding range $(l_{qk}, h_{qk})$. Given that we use the standardized bid-to cost ratio $x'_{0jk} = h_{jk}$, the probability of winning any given project, $p_{jk}$, equals the number of projects won divided by $m$ (Fig. 3).

To determine the optimum bid-to-cost ratio we must first select a particular value for the standardized mean $m'_X$ and then compute the $m$ corresponding expected profits:

$$\frac{v_{jk}}{c_0\sigma_X} = p_{jk}\frac{x_{0jk}-1}{\sigma_X} \tag{31}$$

$$= p_{jk}(x'_{0jk} + m'_X) \tag{32}$$

$$= p_{jk}(h_{jk} + m'_X) \tag{33}$$

where $j = 1, \ldots, m$. Of the resulting $m$ profits the maximum defines the optimum quantity

$$\frac{x^*_{0k}-1}{\sigma_X} = (h^*_{jk} + m'_X) \tag{34}$$

that corresponds to the chosen value for the standardized mean $m'_X$. To create the graphs shown in Fig. 4 we select different values for $m'_X$ and compute the corresponding $(x^*_{0k} - 1)/\sigma_X$.

To improve accuracy, the actual $(x^*_0 - 1)/\sigma_X$ in Fig. 4 represent the average of the $s$ values $(x^*_{0k} - 1)/\sigma_X$ derived from all simulation sets for the same standardized mean $m'_X$. The same is true for $P[\text{Win}|x'_0, n]$ in Fig. 3, $E[V^*]/(c_0\sigma_X)$ in Fig. 5, and $P[\text{Win}|x'^*_0, m'_X, n]$ in Fig. 6.

As a check, the same results have also been obtained by using regression analysis to approximate the probability of winning $P[\text{Win}|x'_0, n]$ in Fig. 3 as a function of $x'_0$, and $n$. Once the probability of

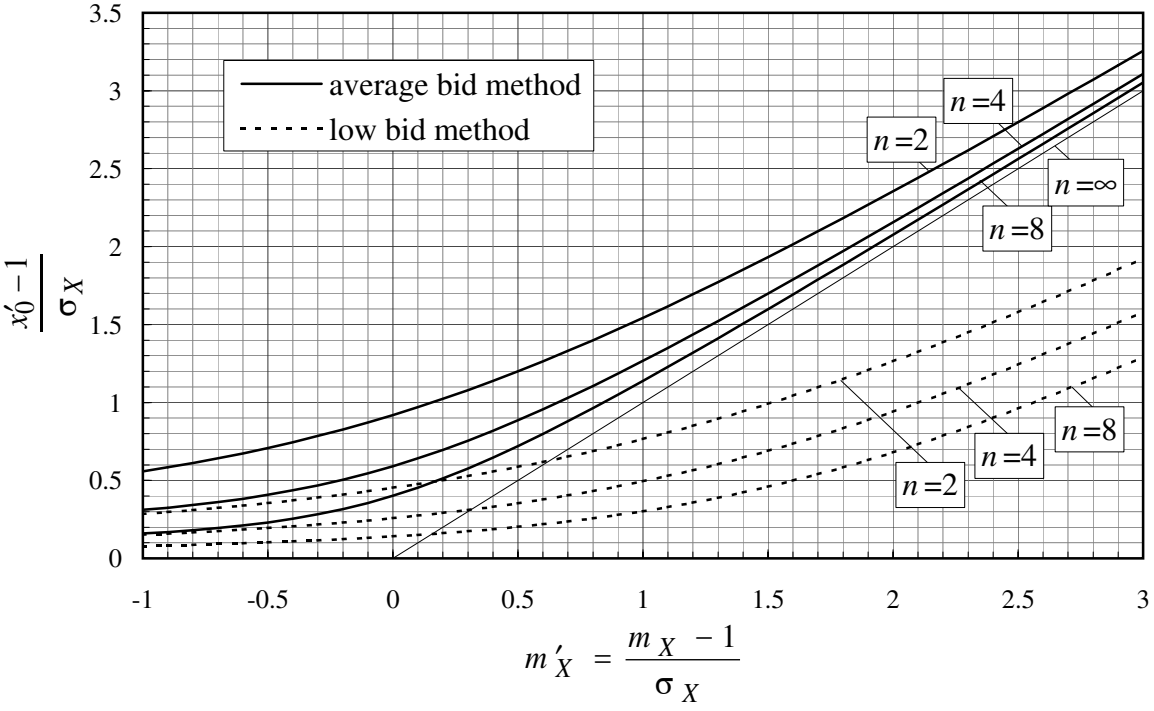



Figure 4: — **Optimum Markup as a function of** $m'_X$**,** $n$

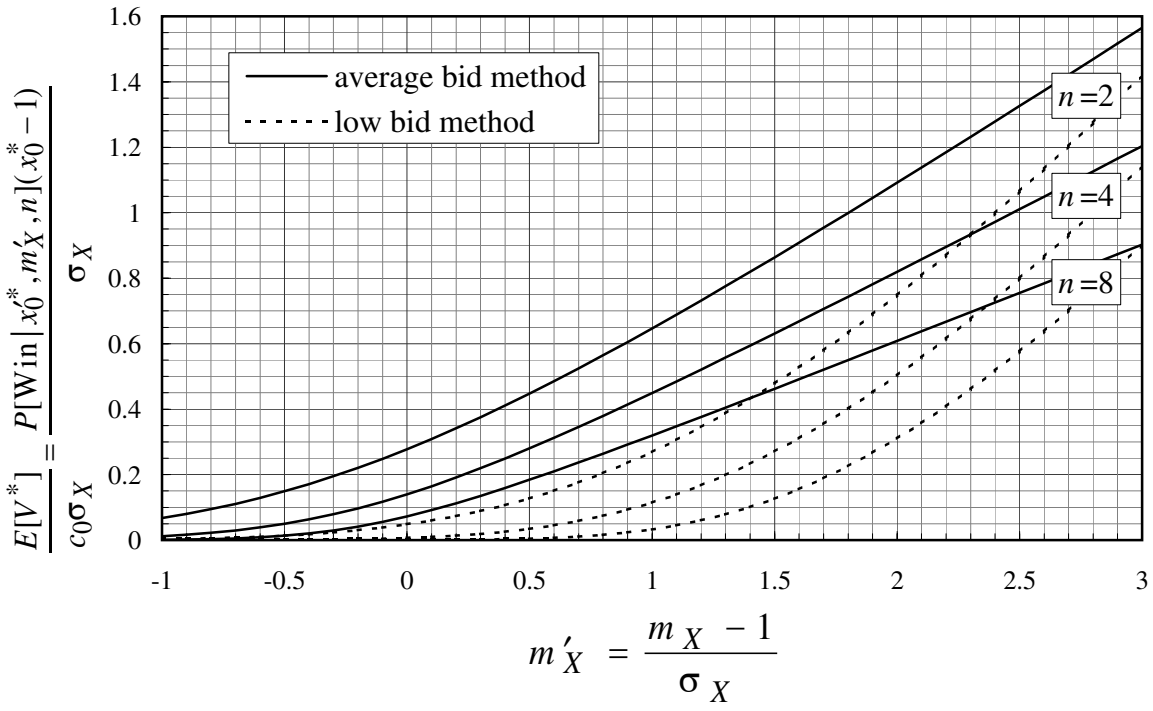



Figure 5: — **Optimum Expected Profit as a function of** $m'_X$**,** $n$

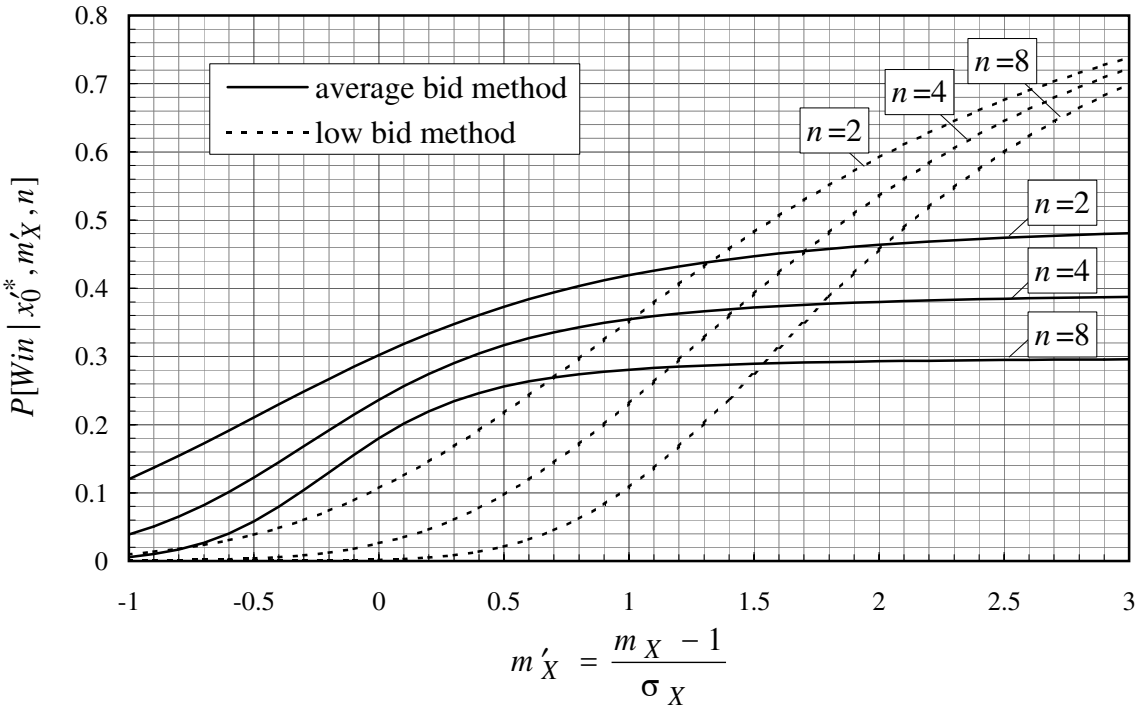



Figure 6: — **Probability of Winning as a function of $m'_X$, $n$**

winning is known analytically, the optimum bid-to-cost ratio can be determined by using calculus in a manner similar to that outlined in Appendix III. Appendix III presents the methodology for determining the optimum bid-to-cost ratio using Friedman's model for the low bid method. Both the simulation and analytical approaches produce the same results as shown in Fig. 3-6.

## Example Application

The graphs for either the average bid method or the low bid method, require that a contractor use past data to produce estimates $\hat{m}_X$ and $\hat{\sigma}_X^2$ for the Normal distribution's mean and variance:

$$\hat{m}_X = \frac{\sum_{j=1}^{N} \sum_{i=1}^{n_j} X_{ij}}{\sum_{j=1}^{N} n_j} \tag{35}$$

$$\hat{\sigma}_X^2 = \frac{\sum_{j=1}^{N} \sum_{i=1}^{n_j} (X_{ij} - \hat{m}_X)^2}{\sum_{j=1}^{N} n_j} \tag{36}$$

The historical bid-to-cost ratios are given by $X_{ij} = B_{ij}/c_{0j}$, where, for each project $j$, $B_{ij}$ is the bid submitted by the $i$th contractor, $c_{0j}$ is the estimated cost for contractor $A_0$, and $n_j$ is the number of competitors facing $A_0$. $N$ is the total number of projects for which the above data are available.

To illustrate the use of these figures for the average bid method, let us assume that contractor $A_0$ is bidding against $n = 4$ opponents, and that the mean and variance have been estimated to be $m_X = 1.1$ and $\sigma_X = 0.1$. The standardized mean is $m'_X = (1.1 - 1)/0.1 = 1$. From Fig. 4 we find

$(x_0^*-1)/\sigma_X = 1.27$, which leads to the optimum bid-to-cost ratio $x_0^* = 1.127$, or a markup of 12.7%. Fig. 5 shows that the expected profit for this markup is $E[V^*] = 0.45\sigma_X c_0 = 0.045c_0$, or 4.5% of the estimated cost. Finally, Fig. 6 provides us with the probability of winning at the optimum markup of 12.7%, $P[\text{Win}|x_0^*, m'_X, n] = 0.353 = 35.3\%$. Notice that the same probability can also be obtained from Fig. 3 for $x'_0 = (1.127 - 1.1)/0.1 = 0.27$. The use of Fig. 3-6 to determine the same quantities for the low bid method is completely analogous.

## The Average vs the Low Bid Method

It is apparent that the results obtained from the model for the average bid method are significantly different from those for the low bid method. These differences essentially fall into four groups.

The first concerns the probability of winning given a standardized bid-to-cost ratio $x'_0$, as shown in Fig. 3. It should be noted that the curves in this figure should not be confused with probability density functions, even though they appear to be very similar to the Normal distribution. The curves for the average bid method are symmetrical around $x'_0 = 0$. The maximum probability of winning always occurs at $x'_0 = 0$, irrespective of the number of opponents $n$. The maximum value for the probability of winning does not exceed 0.5 and this value occurs only when $n = 2$. In contrast, the curves for the low bid method are asymmetrical. The maximum value for the probability of winning approaches 1.0 at very low values of $x'_0$, no matter how many opponents a contractor expects. This implies that a contractor can guarantee winning the project by cutting down his bid, something that simply cannot be done when using the average bid method. This is probably the major reason why certain owners may prefer the average bid method. It provides a safeguard against very low bid prices, be they accidental or deliberate.

The second difference concerns the optimum standardized bid-to-cost ratio $(x_0^* - 1)/\sigma_X$ for a given standardized mean $m'_X$, as shown in Fig. 4. Given the same $m'_X$ and $n$, this ratio is always larger when using the average bid method. As the number of opponents $n$ becomes very large the curves for the average bid method approach a $45^o$ straight line through the origin. This indicates that the optimum bid-to-cost ratio $x_0^*$ approaches the mean $m_X$. The same trend is observed when $m'_X$ becomes large, irrespective of the number of bidders. This should be expected because a large value for $m'_X$ implies a very small variance $\sigma_X$. This means that the opponents' bids are closely clustered around the mean. The only way for a contractor to win is to select the mean bid-to-cost ratio no matter how many opponents he faces. In contrast, the curves for the low bid method do not merge but become parallel straight lines for large values of $m'_X$. As the number of opponents $n$ increases, these curves approach the $X$ axis, which implies that the contractor should bid the project at cost. Thus, for a large number of competitors, an owner using the average bid method should expect to sign a contract for a fair and reasonable price as established by the consensus of the industry, whereas an owner using the low bid method should expect an unrealistically low price that is bound to lead to problems later.

The third difference concerns the maximum expected profit given a particular $m'_X$, as shown in Fig. 5. For the same number of opponents $n$, this profit is always higher when using the average bid method. This explains why this method might be favored by contractors. As expected, when $n$ becomes very large, the optimum expected profit for either method is zero. Notice, however, that this does not mean that for both methods the owner should expect to sign a contract at cost. The reason the expected bid approaches zero for the average bid method is because the probability of

winning does so. The bid-to-cost ratio, however, approaches $m'_X$. For the low bid method, both the probability of winning and the optimum markup approach zero.

The final difference concerns the probability of winning at the optimum bid-to-cost ratio $(x_0^* - 1)/\sigma_X$ for a given $m'_X$, as shown in Fig. 6. For small values $m'_X$ this probability is larger for the average bid method, whereas for high values it is larger for the low bid method. This is again as expected, since a large variance favors winning under the average bid method, while a small variance favors winning when using the low bid method.

## Conclusion

The basic advantage of the average bid method, from an owner's perspective, is that it safeguards against signing a construction contract for an unrealistically low bid price that almost certainly will lead to adversarial relationships during construction (Grogan 1992). Similarly, contractors are protected from having to honor a bid containing a gross mistake or oversight. This is certainly the main drawback of the low bid method. Very low bid prices often lead to excessive claims and disputes, and result in cost increases and schedule delays.

The basic drawback of the average bid method as defined here is that it does not necessarily promote price competition that leads to lower costs for the owner. It is easy to see that a technological or managerial breakthrough that results in major cost savings will not necessarily be passed on to the owner in the form of lower prices, unless this breakthrough is known to be available to all bidders.

Contrary to first impressions, the average bid method does not discourage innovation, technology development and adaptation, and cost reduction. Any such savings, when not available to all bidders, may not give the contractor a competitive advantage, but do result in significantly higher profits in projects won. When such savings are available throughout the industry, however, bid prices should be expected to gradually fall and the savings will eventually be passed to the owner.

It is this mechanism that suggests more effective variations of the average bid method. For example, it would probably be more effective to award the project to the bid closest to, but less than the average, an approach used in Italy as well as in Taiwan. This approach would lead to optimum bid-to-cost ratios that are slightly less than the mean. The curves for the average bid method presented in Fig. 4 lie above the straight line for $n = \infty$ and thus imply that the optimum bid-to-cost ratio should be slightly higher than the mean. Under ideal conditions, where all bidders use the results presented here, this would slowly result in a price increase to the owner. The elimination of bids that exceed the mean, however, should negate this effect and produce curves that are on the lower side of $n = \infty$. Notice that for very small values of the mean $m'_X$, the curves for both methods in Fig. 4 call for bid-to-cost ratios that are greater than $m'_X$. This illustrates the counter effect of profitability. Contractors should not lower their prices to the point of losing money, even if doing so lowers the probability of winning.

Although attractive, the average bid method is not without its pitfalls. As is the case for the low bid method, collusion among the bidders and the absence of prequalification can negate its intent and produce undesirable results. For example, it has come to the authors' attention, that in a certain country contractors try to obtain a competitive advantage by forming several dummy construction companies that bid the same projects as their affiliated real contractor. These dummy companies and the contractor submit bid prices that are very close to each other and thus pull the overall average towards their own price. If a dummy company wins the project it simply passes the

entire project to the affiliated contractor.

In conclusion, it is evident from this discussion that the development of more efficient competitive bidding methods for the U.S. construction industry is a subject that deserves significant attention. A departure from the low bid method as practiced today has the potential to improve the longevity of construction firms, the elimination of accidentally or deliberately low bid prices, and the improvement of relationships between owner and contractor during construction. Furthermore, it should result in increased and fair profitability for contractors. Indeed, the low profitability of construction firms due to intense competition was recently cited by the president of a large U.S. construction company in his speech at Construction Congress '91 as the major reason why U.S. contractors cannot afford to invest in research and development at the levels enjoyed by Japanese firms.

## Appendix I. — References

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## Appendix II. — Upper and Lower Bounds Used in Simulation

Let contractor $A_0$ compete against $n$ opponents whose standardized bid-to-cost ratios are known to be $x_i'$, $i = 1, 2, \ldots, n$. Our objective is to define a region $(l, h)$ with upper and lower bounds $h$ and $l$ within which $A_0$'s standardized bid-to-cost ratio $x_0'$ wins the project.

Let $\bar{x}'$ be the average of the standardized bid-to-cost ratios submitted by the $n$ opponents:

$$\bar{x}' = \frac{1}{n}\sum_{i=1}^{n} x_i' \tag{37}$$

Notice that the average $\bar{x}'$ does not include $x_0'$. Let us define $\bar{x}_0'$ to be the average including $x_0'$:

$$\bar{x}_0' = \frac{1}{n+1}\sum_{i=0}^{n} x_i' \tag{38}$$

$$= \frac{n\bar{x}' + x_0'}{n+1} \tag{39}$$

Of the $n$ values $x_i'$, let $x_h'$ be the one that is higher than but closest to the average $\bar{x}'$. Similarly, let $x_l'$ be the one that is lower than but closest to the average $\bar{x}'$. Let $d^+$ be the absolute value of the distance from the average $\bar{x}'$ to $x_h'$, and $d^-$ be the absolute value of the distance from the average $\bar{x}'$ to $x_l'$.

First, we shall show that $x_0'$ cannot win the project if it is so high that $\bar{x}_0'$ is greater than $x_h'$. Let us determine the requirement for $x_0'$ so that $\bar{x}_0'$ is indeed larger than $x_h'$:

$$\begin{aligned} \bar{x}_0' &> x_h' && (40)\\ \frac{n\bar{x}' + x_0'}{n+1} &> x_h' && (41)\\ x_0' &> (n+1)x_h' - n\bar{x}' && (42) \end{aligned}$$

It is easy to show that because of inequality (42) $x_0'$ must also be greater than $\bar{x}_0'$. Thus, to win the project, $x_0'$ must satisfy:

$$\begin{aligned} x_0' - \bar{x}_0' &< \bar{x}_0' - x_h' && (43)\\ x_0' - \frac{n\bar{x}' + x_0'}{n+1} &< \frac{n\bar{x}' + x_0'}{n+1} - x_h' && (44)\\ nx_0' - n\bar{x}' &< n\bar{x}' + x_0' - (n+1)x_h' && (45)\\ (n-1)x_0' &< 2n\bar{x}' - (n+1)x_h' && (46)\\ x_0' &< \frac{2n}{n-1}\bar{x}' - \frac{n+1}{n-1}x_h' && (47) \end{aligned}$$

Combining inequalities (42) and (47) produces:

$$\begin{aligned} (n+1)x_h' - n\bar{x}' &< \frac{2n}{n-1}\bar{x}' - \frac{n+1}{n-1}x_h' && (48)\\ (n^2-1)x_h' - (n^2-n)\bar{x}' &< 2n\bar{x}' - (n+1)x_h' && (49)\\ (n^2 - 1 + n + 1)x_h' &< (n^2 - n + 2n)\bar{x}' && (50)\\ x_h' &< \bar{x}' && (51) \end{aligned}$$

which is impossible by the definition of $x_h'$. As a result, our original assumption is false, and in order for $x_0'$ to win the project the following must be true,

$$\bar{x}_0' < x_h' \tag{52}$$

which requires that:

$$\begin{aligned} x_0' &< (n+1)x_h' - n\bar{x}' && (53)\\ x_0' &< x_h' + nd^+ && (54) \end{aligned}$$

Furthermore, in order for $x_0'$ to win:

$$x_0' < x_h' < x_h' + nd^+ \tag{55}$$

Otherwise, $x_h'$ would be closer to the average $\bar{x}_0'$ and would win the project.

Similarly, we can show that the following two inequalities must also be true:

$$\begin{aligned} \bar{x}_0' &> x_l' && (56)\\ x_0' &> x_l' && (57) \end{aligned}$$

Thus, the following conditions are necessary for $x_0'$ to win:

$$x_l' < \bar{x}_0' < x_h' \tag{58}$$

$$x_l' < x_0' < x_h' \tag{59}$$

However, they are not sufficient. In order for $x_0'$ to win, it must also beat both $x_h'$ and $x_l'$, and thus be the closest to the average $\bar{x}_0'$.

For $x_0'$ to beat $x_l'$:

$$\begin{aligned} x_0' - \bar{x}_0' &< \bar{x}_0' - x_l' && (60)\\ x_0' - \frac{n\bar{x}' + x_0'}{n+1} &< \frac{n\bar{x}' + x_0'}{n+1} - x_l' && (61)\\ nx_0' - n\bar{x}' &< n\bar{x}' + x_0' - (n+1)x_l' && (62)\\ (n-1)x_0' &< 2n\bar{x}' - (n+1)(\bar{x}' - d^-) && (63)\\ x_0' &< \bar{x}' + \frac{n+1}{n-1}d^- && (64) \end{aligned}$$

$$\bar{x}' - d^- < x_0' < \bar{x}' + \frac{n+1}{n-1}d^- \tag{65}$$

Similarly, for $x_0'$ to beat $x_h'$:

$$\begin{aligned} \bar{x}_0' - x_0' &< x_h' - \bar{x}_0' && (66)\\ \frac{n\bar{x}' + x_0'}{n+1} - x_0' &< x_h' - \frac{n\bar{x}' + x_0'}{n+1} && (67)\\ n\bar{x}' - nx_0' &< (n+1)x_h' - n\bar{x}' - x_0' && (68)\\ 2n\bar{x}' - (n+1)(\bar{x}' + d^+) &< (n-1)x_0' && (69)\\ \bar{x}' - \frac{n+1}{n-1}d^+ &< x_0' && (70) \end{aligned}$$

$$\bar{x}' - \frac{n+1}{n-1}d^+ < x_0' < \bar{x}' + d^+ \tag{71}$$

The two lower bounds and two upper bounds given by inequalities (65) and (71) define a region as illustrated by the shaded area in Fig. 7. Given a value $\bar{x}'$, Fig. 7 shows the interval within which $x_0'$ must lie in order to win against both $x_l'$ and $x_h'$. Thus, the necessary and sufficient conditions for $A_0$'s standardized bid-to-cost ratio $x_0'$ to win are:

$$l < x_0' < h \tag{72}$$

where,

$$\begin{aligned} h &= \min(\bar{x}' + d^+, \bar{x}' + \frac{n+1}{n-1}d^-) && (73)\\ l &= \max(\bar{x}' - d^-, \bar{x}' - \frac{n+1}{n-1}d^+) && (74) \end{aligned}$$

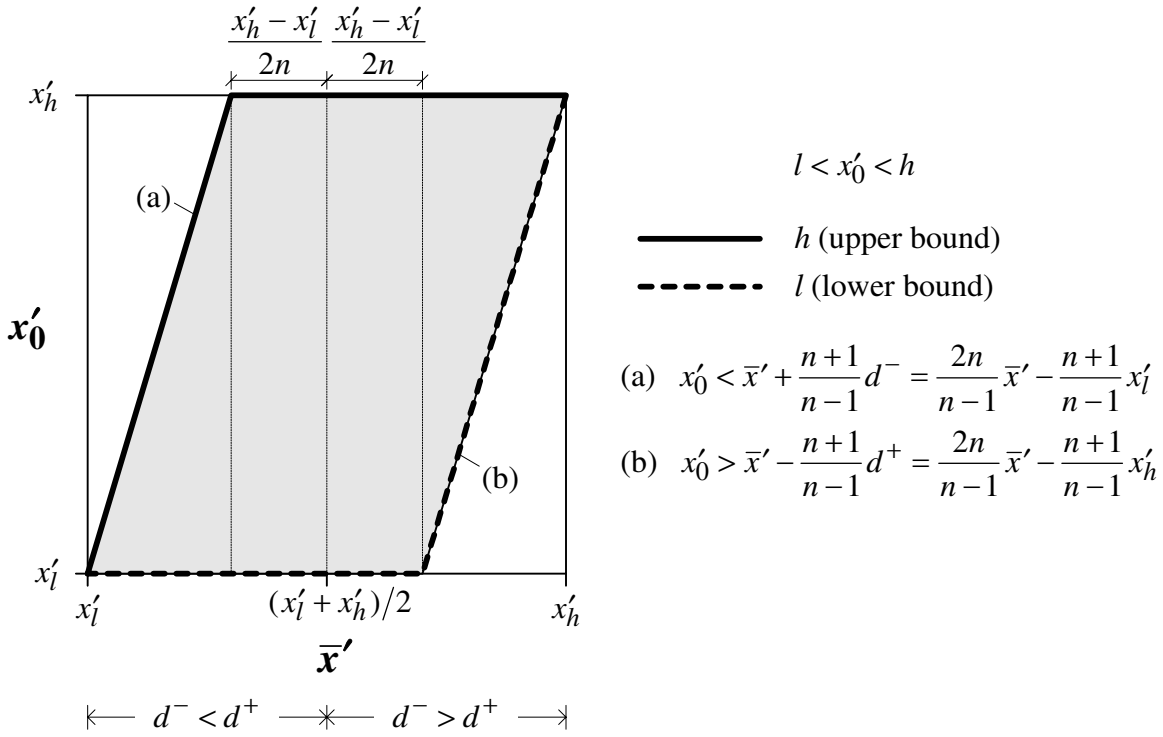



Figure 7: — **Winning region for $x'_0$ as a function of $\bar{x}'$**

## Appendix III — Optimum Markup by Direct Solution of Friedman's Model

The following illustrates how to directly solve Friedman's model for the optimum bid-to-cost ratio $x_0^*$:

$$E[V] = P[\text{Win}|x_0, n](x_0 - 1)c_0 \tag{75}$$

$$\frac{dE[V]}{dx_0} = c_0 P[\text{Win}|x_0, n] + c_0 \frac{dP[\text{Win}|x_0, n]}{dx_0}(x_0 - 1) \tag{76}$$

Setting $dE[V]/dx_0 = 0$, we get:

$$x_0^* - 1 = -\frac{P[\text{Win}|x_0^*, n]}{\frac{dP[\text{Win}|x_0, n]}{dx_0}|_{x_0^*}} \tag{77}$$

This is the basic equation for solving directly for $x_0^*$. To apply this equation let's use the simple version of Friedman's model:

$$P[\text{Win}|x_0, n] = [1 - F_U(x'_0)]^n \tag{78}$$

where $F_U(u)$ is the standard Normal distribution $N(0, 1)$; $x'_0 = (x_0 - m_X)/\sigma_X$; $m_X = E[X_i]$; $\sigma_X^2 = Var[X_i]$; and $n$ is the number of opponents. Differentiating Friedman's model yields:

$$\frac{dP[\text{Win}|x_0, n]}{dx_0} = -\frac{n}{\sigma_X}[1 - F_U(x'_0)]^{n-1} f_U(x'_0) \tag{79}$$

Substituting this into the basic equation gives:

$$x_0^* - 1 = \frac{[1 - F_U(x_0'^*)]^n}{\frac{n}{\sigma_X}[1 - F_U(x_0'^*)]^{n-1} f_U(x_0'^*)} \tag{80}$$

$$\frac{x_0^* - 1}{\sigma_X} = \frac{1 - F_U(x_0'^*)}{n f_U(x_0'^*)} \tag{81}$$

This can be simplified to:

$$x_0'^* + m'_X = \frac{1 - F_U(x_0'^*)}{n f_U(x_0'^*)} \tag{82}$$

$$x_0'^* = \frac{Q_U(x_0'^*)}{n f_U(x_0'^*)} - m'_X \tag{83}$$

where $m'_X = (m_X - 1)/\sigma_X$ and $Q(u) = 1 - F_U(u)$ is the complement of the standard Normal CDF. Given $m'_X$ and $n$, the above nonlinear equation can be solved by selecting an initial value $u_0$ and performing successive iterations:

$$u_{k+1} = \frac{Q_U(u_k)}{n f_U(u_k)} - m'_X \tag{84}$$

until we achieve the desired level of accuracy $\epsilon$:

$$|u_{k+1} - u_k| < \epsilon \tag{85}$$

At this point we simply set $x_0'^* = u_{k+1}$, and $x_0^* = m_X + \sigma_X u_{k+1}$.

## Appendix IV. — Notation

*The following symbols are used in this paper:*

$A_i$ = The $i$th competing contractor.

$b_0$ = The bid price of contractor $A_0$.

$B_i$ = The bid price of contractor $A_i$.

$\overline{B}$ = $(b_0 + B_1 + \ldots + B_n)/(n + 1)$.

$c_0$ = The project cost estimate of contractor $A_0$.

$C_i$ = The project cost estimate of contractor $A_i$.

$F_X(x)$ = $P[X \leq x]$. The cumulative distribution function of the random variable $X$ evaluated at $x$.

$f_X(x)$ = $P[x \leq X \leq x + dx]/dx$. The probability density function of the random variable $X$ evaluated at $x$.

$m_0$ = The markup of contractor $A_0$.

$m_X$ = The expected value of $X_i$.

$m'_X = (m_X - 1)/\sigma_X$.

$n$ = The number of competitors bidding against contractor $A_0$.

$\sigma_X^2$ = The variance of $X_i$.

$x_0 = b_0/c_0$. The bid-to-cost ratio of contractor $A_0$.

$x'_0 = (x_0 - m_X)/\sigma_X$.

$X_i = B_i/c_0$. Contractor $A_i$'s apparent bid-to-cost ratio (as viewed by $A_0$).

$\overline{X} = (x_0 + X_1 + \ldots + X_n)/(n+1)$.

# Below-Average Bidding Method

By Photios G. Ioannou,[1]M.ASCE, and Rita E. Awwad[2]

**Abstract:** The low-bid method, typically used for competitive bidding in the U.S., may result in a contract with a firm that submits either accidentally or deliberately an unrealistically low bid price. Such an occurrence hurts both the owner and the contractor by promoting disputes, increased costs, and schedule delays. To address this problem, other countries have adopted bidding methods based on the average of the bids submitted. One such approach is the below-average method where the winning bid is closest to, but below the average of all bids. A competitive bidding model for the below-average bidding method is presented and its merits relative to the average-bid method and the low-bid method are explored. The bidding process is investigated analytically and through Monte Carlo simulation. The bidding model results for the below-average, the average, and the low-bid methods are presented in four nomograms that can be used to determine the optimal lump-sum bid price without the need for complicated analysis. A comparison of the three methods reveals that competitive bidding methods based on the bid average do not hamper innovation and have the potential to improve contracting practices for both owners and contractors.

## Introduction

The majority of competitively bid construction contracts in the U.S. use the low-bid method where the contract is awarded to the firm submitting the lowest responsible bid. Competition on price encourages contractors to adopt cost-saving technological and managerial innovations and then pass these savings on to the owner who receives a project of specified quality at the lowest price. When the number of bidders is large, however, as is the case in a slow economy, an owner runs

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a significant risk of selecting a contractor that has either accidentally or deliberately submitted an unrealistically low price. A contractor cannot adhere to such a price and at the same time expect to complete the project according to plans and specifications, and also make a reasonable profit. This often results in excessive claims and disputes during construction that lead to schedule delays, compromises in quality, and increased costs (Grogan 1992).

In response to these problems, other countries have recognized that the lowest price may not be the best price for the owner and developed bidding procedures based on the average of the bids received. The rationale for these methods is that a price close to the average should improve quality, offer a fair price to the owner, and allow the contractor to perform the work at a reasonable profit.

In this paper we present various bidding methods based on the average bid and develop a competitive bidding model for the below-average bidding method. The below-average bidding model is investigated analytically and through Monte Carlo simulation and its results are presented in four nomograms that can be used to analyze a competitive situation without the need for complicated mathematics. We also present the results of bidding models for the average-bid method (Ioannou and Leu 1993) and the low-bid method (Friedman 1956, Ioannou 1988), which are used to compare the three models and explore their merits relative to each other. We conclude with a discussion and summary.

## Average Bid Methods

Over the years, several competitive bidding methods have been developed where the winner is determined by comparing the submitted bids to their average. Different average bid methods use different procedures for calculating the average, or use different criteria for determining the winning bid. For example, some use an arithmetic average or a weighted average, while others use the average of the remaining bids after all bids that differ more than a certain percentage from the average of all other bids are eliminated. Similarly, the winner might be the contractor whose price is closest to the average, or the contractor whose bid is closest to, but below the average. The former, for example, is used in Taiwan while the latter is used in Italy.

Other countries have adopted even more complex variations. Peru for example, uses a proce-

dure where if less than three bids are received, a bidding agency may award the contract to the lowest bidder. When three or more bids are received, the average of all bids and the base budget are calculated, and bids that lie 10 percent above and below this average are eliminated. A second average of the remaining bids and the base budget is calculated, and the bid closest to, but below the second average is the winner. If none of the bids lie below the second average, the winner is the bid closest to, but above the average.

In 1996, at the urging of the secretary of transportation, Florida passed a statute allowing use of alternative bidding techniques on $60 million out of the approximately $1.25 billion worth of annual FDOT contracts. As part of this effort, FDOT investigated several bidding alternatives on state-funded highway projects, including the bid-averaging method. According to this method, if there are five or more bidders, the high and low bids are discarded, the remaining bids are averaged, and the bidder closest to the numerical average is awarded the contract. If there are three or four bidders, no bids are discarded, and if one or two bids are received, the job is advertised again. The objectives of this approach were to encourage contractors to submit true and reasonable bids for the work that would result in higher quality projects with less monetary disputes. The bid averaging method was used primarily on typical low-risk projects, such as bridge rehabilitation and pavement resurfacing (ASCE 1998, AASHTO 2001).

## Average-Bid Bidding Model

The average bid method, where the winning bid is the one closest to the average of all submitted bids has been analyzed in detail in (Ioannou and Leu 1993). Though this method uses the simplest possible rule to determine the winner, it has proven very difficult to model analytically. As shown in (Ioannou and Leu 1993) a closed-form competitive bidding model can be formulated only in the simple case where a contractor faces two opponents. When the number of opponents is greater than two, the problem was solved through Monte Carlo simulation.

The final results of the bidding model for the average-bid method were presented in four nomograms that allow a contractor to select an optimal markup for a lump-sum bid without the need for complicated mathematics. These results are replicated here to allow comparison between the average bid method, the below-average bid method, and the low-bid method.

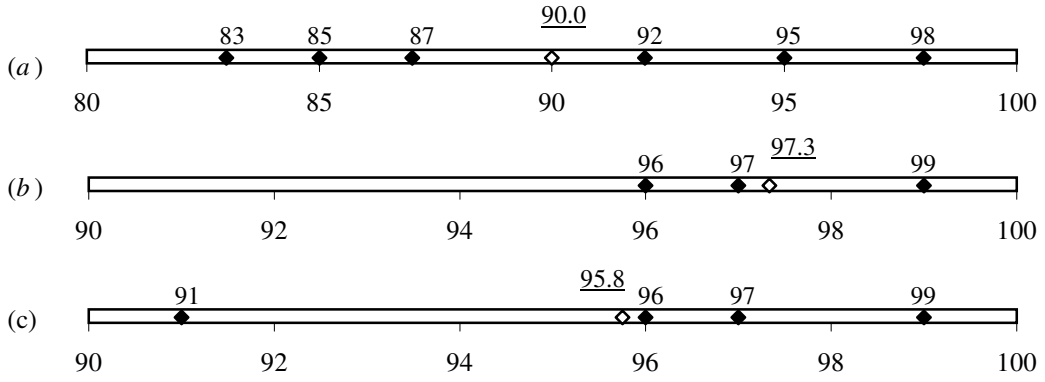



Figure 1: −− **Below-Average Bidding Method Examples**

A testament to the difficulty of modeling the average bid method is the fact that the competitive bidding model in (Ioannou and Leu 1993) was the first one to do so and as such has been used over the years in research in several domains outside construction and referenced in diverse publications. Yet, it is still the only model to address optimal bidding using the average bid method. Moreover, inquiries from several researchers about the availability of competitive models for other variants provided the impetus for the below-average model presented here.

# Below-Average Bid Method

In this paper we analyze the below-average bidding method where the winning bid is the one closest to but less than the arithmetic average of all submitted bids. We present a competitive bidding model for the below-average bidding method and compare it to bidding models for the average and the low-bid methods. We close with some remarks about all three methods.

We shall use the three examples shown in Fig. 1 to illustrate the below-average bidding method and compare it to the average and low-bid methods.

Fig. 1(a) shows six contractors that have submitted the following bid prices: 83, 85, 87, 92, 95, and 98. The average bid price is 90. The bid at 92 is closest to the average and would win the project under the average bid method. Under the below-average bid method, however, the winner would be the bid at 87. The low-bid winner is the bid at 83.

In this example, there is symmetry in the arrangement of bid prices: three bids are above the average and three below. This is the situation visualized by most people when they are introduced to the average bid method for the first time. Moreover, in order to distinguish between the average bid method and the below-average bid method, most people visualize an example where the average (90) is closer to the bid above (92) rather than the one below (87) to make it clear that the two methods can lead to different winnings bids.

Fig. 1(b) shows three contractors that have submitted the bid prices: 96, 97, and 99. The average bid price is 97.3. The bid at 97 is clearly the closest to and below the average and would win the project under both the below-average and the average bid methods.

Let us now consider a fourth contractor that is about to submit a bid. Clearly, if the fourth bid happens to be in the narrow range between 97 and 97.3, it will indeed be the winner. However, this is not the only possibility. Fig. 1(c) shows what happens if the fourth bid is 91. In this case, the average of the four bids drops to 95.8, which is below all three original bids. Hence, under the below-average bid method, the winner is actually *the low bid price* at 91.

This example illustrates a very important and not so obvious property of the below-average bidding method. A contractor can win by submitting a bid so low (91) that the resulting bid average (95.8) is below the minimum of its competitors' bid prices (96). This characteristic makes the below-average bidding method similar to the low-bid method. This and other possibilities for a contractor to win under the below average bid method are discussed in detail below.

It is clear from these examples that both average bid methods are significantly more complex and thus more difficult to model than the low-bid method. For the low-bid method, the main variable of interest, from a contractor's perspective, is the minimum of the bid prices submitted by the other competitors. The contractor's own bid price is the constant value to which the opponents' bids are compared. The average bid price, on the other hand, depends on the contractor's own bid price. Furthermore, in order to determine the winner we need to compare every bid to the average. A change in any of the bid amounts, or the number of bidders, can have a profound effect on any particular bidder's chance of winning.

Finally, it should be pointed out that an important difference between the average bid method and the below-average bid method is the fact that the former requires *the calculation of bid differences from the average* whereas the latter does not. The below-average bid method is actually an

ordering procedure where the bids are arranged in increasing order and their average is compared to each, from the largest to the smallest. The first bid found that is less than the average is the winner. Although a similar procedure can be followed for the average bid method, it is still necessary to calculate the difference between the average and the bids immediately above and below the average to determine which one is closer and thus the winner. This observation has important effects on the mathematical analysis below.

## Basic Definitions

The notation adopted in this paper is to indicate random variables and the identity of contractors using capitalized symbols; decision variables and the values of random variables are shown using lower-case symbols.

We shall examine the bidding process from the perspective of a particular contractor $A_0$ who is bidding on a new project against $n$ competitors $A_1, A_2, \ldots, A_n$. Let $C_i$ be the project cost estimate and $B_i$ the bid price for competitor $A_i$, $i = 0, 1, \ldots, n$.

We shall assume that contractor $A_0$ has already determined a cost estimate $c_0$ and needs to decide on a bid price $b_0$. To this end, $A_0$ must choose a markup $m_0$ to arrive at a bid price $b_0 = c_0 + m_0$. Equivalently, contractor $A_0$ may select a bid-to-cost ratio $x_0 = 1 + m_0/c_0$ and multiply it by its cost estimate $c_0$ to arrive at its bid price $b_0$:

$$b_0 = x_0 c_0 \tag{1}$$

For contractor $A_0$ the values $c_0$, $m_0$, $x_0$, and $b_0$ are decision variables, even if their values have not been determined yet. In contrast, $A_0$ does not know the competitors' cost estimates $C_i$ and bid prices $B_i$. Given this state of information, $A_0$ considers these to be random variables, even though $C_i$ and $B_i$ are clearly decision variables for the corresponding contractor $A_i$. Furthermore, $A_0$'s state of information, and hence its treatment of these variables, does not change depending on whether contractor $A_i$ has already determined and thus fixed the values of $C_i$ and $B_i$.

The ratio of contractor $A_i$'s bid $B_i$ to contractor $A_0$'s own cost estimate $c_0$ represents the apparent bid-to-cost ratio used by opponent $A_i$ and will be shown as $X_i$:

$$X_i = \frac{B_i}{c_0} \quad i = 1, 2, \ldots, n \tag{2}$$

It should be noted that for contractor $A_0$ the ratio $x_0 = b_0/c_0$ is a decision variable, whereas the $X_i$ $(i = 1, \ldots, n)$ are the main random variables for the bidding model analysis that follows.

## Standardization Assumptions

The bidding model for the below-average bid method is based on the same two assumptions as Friedman's low-bid model (Friedman 1956, Ioannou 1988) and the average-bid model (Ioannou and Leu 1993). This allows the comparison of all three bidding models on an equal basis. The necessary assumptions are as follows:

1. In order to eliminate the effect of the project size, each opponent's bid $B_i$ $(i = 1, 2, \ldots, n)$ is standardized by taking its ratio to contractor $A_0$'s cost estimate $c_0$. The resulting ratios $X_i$ $(i = 1, 2, \ldots, n)$ are assumed to be mutually independent.

2. The probability distribution of each apparent bid-to-cost ratio $X_i$ $(i = 1, 2, \ldots, n)$ does not depend on the values of $x_0$ and $c_0$ that $A_0$ chooses:

$$P[X_i < x_i | x_0, c_0] = P[X_i < x_i] \quad i = 1, 2, \ldots, n \tag{3}$$

The theoretical and practical ramifications of these assumptions are examined in detail in (Ioannou 1988) and are not presented here.

## Mathematical Formulation

The winning contractor under the below-average bid method is the one whose bid is closest to but less than the average of all bids submitted. Letting the average of all bid prices be $\overline{B} = (b_0 + B_1 + \ldots + B_n)/(n + 1)$, the formal definition for the event "Contractor $A_0$ wins" can be expressed in several different ways as follows:

$$\{A_0 \text{ wins} | b_0\} = \{\bigcap_{B_i < \overline{B}} (\overline{B} - b_0 < \overline{B} - B_i) \cap (b_0 < \overline{B})\} \tag{4}$$

$$= \{\bigcap_{B_i < \overline{B}} (B_i < b_0) \cap (b_0 < \overline{B})\} \tag{5}$$

$$= \{\bigcap_{i=1}^{n}[(B_i < b_0) \cup (\overline{B} < B_i)] \cap (b_0 < \overline{B})\} \quad (6)$$

$$= \{\bigcap_{i=1}^{n}(b_0 < \overline{B} < B_i) \cup (B_i < b_0 < \overline{B})\} \quad (7)$$

$$= \{\bigcap_{i=1}^{n}[(b_0 - B_i)(\overline{B} - B_i) > 0] \cap (b_0 < \overline{B})\} \quad (8)$$

Notice that the opposing bids $B_i$ in (4) and (5) are only those that satisfy: $B_i < \overline{B}$. The remaining expressions involve all opposing bids $B_i$. Moreover, since $B_1, B_2, \ldots, B_n$ are random variables, so is the average $\overline{B}$. At the same time, since $b_0$ is a decision variable, the average $\overline{B}$ is also influenced by contractor $A_0$.

An alternative definition for the event "Contractor $A_0$ wins" involves the use of order statistics to highlight the ranking nature of the process. Given the random variables $B_1, B_2, \ldots, B_n$, the order statistics $B_{(1)}, B_{(2)}, \ldots, B_{(n)}$ are also random variables, defined by sorting the values (realizations) of $B_1, B_2, \ldots, B_n$ in increasing order. Important order statistics, for example, are the low bid $B_{(1)} = \min(B_i)$, and the high bid $B_{(n)} = \max(B_i)$.

The conditional event that contractor $A_0$ wins, given that the average bid $\overline{B}$ is between order statistics $B_{(k)}$ and $B_{(k+1)}$ is:

$$\{A_0 \text{ wins}|b_0, (B_{(k)} < \overline{B} < B_{(k+1)})\} = \{(B_{(k)} < b_0 < \overline{B})|(B_{(k)} < \overline{B} < B_{(k+1)})\} \quad (9)$$

The use of order statistics will be particularly advantageous later on for determining the winning regions when using simulation.

In the trivial case of only two bidders, $A_0$ and $A_1$, the below-average bid method is equivalent to the low-bid method. The low bid is the only one below the average and wins the project. For this reason, the following analysis assumes contractor $A_0$ faces two or more competitors.

The above expressions for the event "Contractor $A_0$ wins" can also be rewritten in terms of bid-to-cost ratios by dividing all bids by the cost estimate $c_0$, i.e., by substituting the symbol $B$ with the symbol $X$. For example,

$$\{A_0 \text{ wins}|x_0, c_0\} = \{\bigcap_{X_i < \overline{X}} (\overline{X} - x_0 < \overline{X} - X_i) \cap (x_0 < \overline{X})\} \quad (10)$$

in which $X_i = B_i/c_0$, and $\overline{X} = (x_0 + X_1 + \ldots + X_n)/(n+1)$.

Similarly, the general formula for the probability that contractor $A_0$ wins can be expressed in a variety of ways, such as:

$$P[A_0 \text{ wins}|b_0] = P[\bigcap_{B_i<\overline{B}} \{B_i < b_0\} \cap (b_0 < \overline{B})] \tag{11}$$

$$P[A_0 \text{ wins}|x_0, c_0] = P[\bigcap_{X_i<\overline{X}} \{X_i < x_0\} \cap (x_0 < \overline{X})] \tag{12}$$

The conditional probability that contractor $A_0$ wins, given that the average bid $\overline{B}$ is between $B_{(k)}$ and $B_{(k+1)}$ is:

$$P[A_0 \text{ wins}|b_0, (B_{(k)} < \overline{B} < B_{(k+1)})] = P[(B_{(k)} < b_0 < \overline{B})|(B_{(k)} < \overline{B} < B_{(k+1)})] \tag{13}$$

$$P[A_0 \text{ wins}|x_0, c_0, (X_{(k)} < \overline{X} < X_{(k+1)})] = P[(X_{(k)} < x_0 < \overline{X})|(X_{(k)} < \overline{X} < X_{(k+1)})] \tag{14}$$

## The case of two opponents

The complexity of determining the probability that contractor $A_0$ wins when using the below-average bidding method is illustrated clearly even in the simplest case of facing only two opponents, $A_1$ and $A_2$. For $A_0$ to win, given that it has selected a bid-to-cost ratio $x_0$, one of the following four compound events (inequalities) must be true:

$$x_0 < \overline{X} < X_1 < X_2 \tag{15}$$

$$X_1 < x_0 < \overline{X} < X_2 \tag{16}$$

$$x_0 < \overline{X} < X_2 < X_1 \tag{17}$$

$$X_2 < x_0 < \overline{X} < X_1 \tag{18}$$

Replacing $\overline{X}$ with $(x_0 + X_1 + X_2)/3$ gives the equivalent four events:

$$(2x_0 < X_1 + X_2) \cap (x_0 + X_2 < 2X_1) \cap (X_1 < X_2) \tag{19}$$

$$(X_1 < x_0) \cap (2x_0 < X_1 + X_2) \cap (x_0 + X_1 < 2X_2) \tag{20}$$

$$(2x_0 < X_1 + X_2) \cap (x_0 + X_1 < 2X_2) \cap (X_2 < X_1) \tag{21}$$

$$(X_2 < x_0) \cap (2x_0 < X_1 + X_2) \cap (x_0 + X_2 < 2X_1) \tag{22}$$

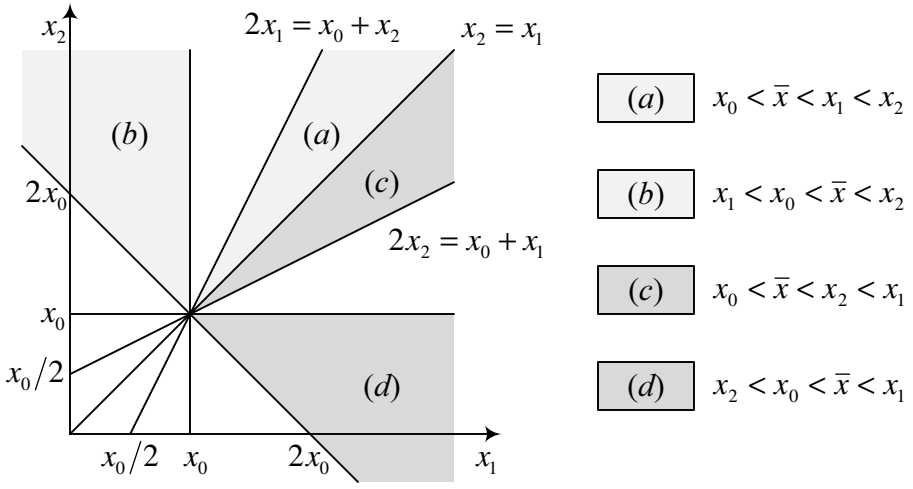



Figure 2: $--$ **Winning Regions Against Two Opponents**

The shaded areas in Fig. 2 show the values of $(x_1, x_2)$ defined by each of the four events (15), (16), (17), and (18), or equivalently, events (19), (20), (21), and (22).

Under Friedman's assumptions, $X_1$ and $X_2$ are iid (independent and identically distributed) with a common probability density function, $f_X(x)$, and a common cumulative distribution function, $F_X(x)$. We can take advantage of symmetry to integrate $f_X(x)$ over areas $(b)$ and $(a)$ to produce the following probability of winning:

$$P[A_0 \text{ wins}|x_0] = 2\int_{-\infty}^{x_0} f_X(x_1)[\int_{2x_0-x_1}^{+\infty} f_X(x_2)dx_2]dx_1 + 2\int_{x_0}^{+\infty} f_X(x_1)[\int_{x_1}^{2x_1-x_0} f_X(x_2)dx_2]dx_1 \tag{23}$$

$$= 2\int_{-\infty}^{x_0} f_X(x_1)[1 - F_X(2x_0 - x_1)]dx_1 + 2\int_{x_0}^{+\infty} f_X(x_1)[F_X(2x_1 - x_0) - F_X(x_1)]dx_1 \tag{24}$$

$$= 2F_X(x_0) - (F_X(x_0))^2 - 2\int_{-\infty}^{x_0} f_X(x)F_X(2x_0 - x)dx + 2\int_{x_0}^{+\infty} f_X(x)F_X(2x - x_0)dx \tag{25}$$

To proceed further it is necessary to make specific assumptions about $f_X(x)$ and evaluate the integrals using numerical techniques. The solution for any number of opponents $n$ can be achieved much easier using simulation as described below.

## Simulation Approach

Monte Carlo simulation can be used to determine the probability of winning and to select the optimum bid-to-cost ratio $x_0$ for a given distribution $f_X(x_i)$. We shall assume that the apparent bid-to-cost ratios, $X_1, X_2, \ldots, X_n$, are iid, following a normal distribution with mean $m_X$, and variance $\sigma_X^2$.

In order to arrive at a numerical solution that is independent of $m_X$ and $\sigma_X^2$, the following standardized variables are defined:

$$m'_X = \frac{m_X - 1}{\sigma_X} \tag{26}$$

$$x'_0 = \frac{x_0 - m_X}{\sigma_X} \tag{27}$$

The main input to the simulation process consists of the number of competitors $n$, the number of simulation sets $s$, and the number of projects per simulation set $m$. The results presented here are based on $n = 2, 4, 8$, $s = 1000$, and $m = 1000$.

For each project $j = 1, \ldots, m$, in each simulation set $k = 1, \ldots, s$, we sample $n$ standardized apparent bid-to-cost ratios $x'_{ijk}$ $(i = 1, \ldots, n)$ from the standard normal distribution N(0,1). Notice that the index $i$ represents a competitor and the index $j$ represents a project. Index $k$ represents the simulation set.

In the following discussion we shall concentrate first on a generic project $(j, k)$ and for simplicity we shall stop using indices $j$ and $k$. Thus, for the generic project we sample $n$ standardized apparent bid-to-cost ratios $x'_i$ $(i = 1, \ldots, n)$ and compute their average $\bar{x}'$:

$$\bar{x}' = \frac{1}{n}\sum_{i=1}^{n} x'_i = \frac{1}{n}((n+1)\bar{x_0}' - x'_0) \tag{28}$$

Notice that $\bar{x}'$ is different from $\bar{x_0}'$, the average of all standardized bid-to-cost ratios, including the currently selected value of $x'_0$:

$$\bar{x_0}' = \frac{1}{n+1}\sum_{i=0}^{n} x'_i = \frac{1}{n+1}(x'_0 + n\bar{x}') \tag{29}$$

## Winning regions

For a given sample of $n$ standardized apparent bid-to-cost ratios $x'_i$ $(i = 1, \ldots, n)$, we seek the conditions that must be satisfied for $x'_0$ to win the project. First, we will show that values of $x'_0$

greater than $\bar{x}'$ are also greater than the average of all bids $\bar{x_0}'$ and thus cannot win the project and can be ignored:

$$\bar{x}' < x'_0 \quad \Rightarrow \tag{30}$$

$$n\bar{x}' + x'_0 < nx'_0 + x'_0 \quad \Rightarrow \tag{31}$$

$$n\bar{x}' + x'_0 < (n+1)x'_0 \quad \Rightarrow \tag{32}$$

$$\bar{x_0}' < x'_0 \tag{33}$$

We now show that values of $x'_0$ greater than $\bar{x}'$ produce an average of all bids $\bar{x_0}'$ that is also greater than $\bar{x}'$:

$$\bar{x}' < x'_0 \quad \Rightarrow \tag{34}$$

$$n\bar{x}' + \bar{x}' < x'_0 + n\bar{x}' \quad \Rightarrow \tag{35}$$

$$(n+1)\bar{x}' < x'_0 + n\bar{x}' \quad \Rightarrow \tag{36}$$

$$\bar{x}' < \bar{x_0}' \tag{37}$$

Combining (33) and (37) gives the final result:

$$\bar{x}' < x'_0 \Rightarrow \bar{x}' < \bar{x_0}' < x'_0 \tag{38}$$

Thus, values of $x'_0$ greater than $\bar{x}'$ are of no interest since they are greater than the average of all standardized bid-to-cost ratios and hence cannot win. Reversing the direction of inequalities and following the same line of reasoning leads to the important result:

$$x'_0 < \bar{x}' \Rightarrow x'_0 < \bar{x_0}' < \bar{x}' \tag{39}$$

Condition (39) that $x'_0$ should be less than $\bar{x}'$, so that it may also be less than $\bar{x_0}'$, is necessary but not sufficient for $x'_0$ to win the project.

To find the sufficient conditions for $x'_0$ to win, it is expedient to sort the standardized bid-to-cost ratios $x'_i$ of the $n$ opponents in ascending order to form the order statistics $x'_{(1)}, x'_{(2)}, \ldots, x'_{(n)}$. Clearly, the value of the average $\bar{x}'$ must be greater than $x'_{(1)}$ and less than $x'_{(n)}$. In contrast, the average $\bar{x_0}'$ depends also on the selected $x_0$, and can have any value. It can be less than $x'_{(1)}$ and greater than $x'_{(n)}$.

For the sake of notation, let us assume that the average of the opponents' standardized bid-cost-ratios $\bar{x}'$ lies between order statistics $x'_{(h)}$ and $x'_{(h+1)}$:

$$x'_{(h)} < \bar{x}' < x'_{(h+1)} \tag{40}$$

It is clear that any selected value $x'_0$ between $x'_{(h)}$ and $\bar{x}'$ is certain to win the project because from (39) we have:

$$x'_{(h)} < x'_0 < \bar{x}' \qquad \Rightarrow \tag{41}$$

$$x'_{(h)} < x'_0 < \bar{x_0}' < \bar{x}' \tag{42}$$

Thus, $x'_0$ is closest and less than the average and wins over $x'_{(h)}$ which is the next value closest to and less than $\bar{x_0}'$. Most people, when encountering the below-average bidding method for the first time, consider the situation described by (41) as the only way $x'_0$ can win the project. A little reflection, however, indicates that this is far from true as shown below.

Let us now assume that $x'_0$ lies between order statistics $x'_{(r)}$ and $x'_{(r+1)}$ where $x'_{(r)} < x'_{(h)}$ (*i.e.*, $r < h$). For $x'_0$ to win, it is sufficient that:

$$x'_{(r)} < x'_0 < \bar{x_0}' < x'_{(r+1)} \tag{43}$$

For the rightmost inequality to be true we have:

$$\bar{x_0}' < x'_{(r+1)} \qquad \Rightarrow \tag{44}$$

$$(n+1)\bar{x_0}' < (n+1)x'_{(r+1)} \quad \Rightarrow \tag{45}$$

$$x'_0 + n\bar{x}' < x'_{(r+1)} + nx'_{(r+1)} \quad \Rightarrow \tag{46}$$

$$x'_0 < x'_{(r+1)} - n(\bar{x}' - x'_{(r+1)}) \tag{47}$$

Combining (43) and (47) gives the following sufficient condition for $x'_0$ to win the project for any $r < h$:

$$x'_{(r)} < x'_0 < x'_{(r+1)} - n(\bar{x}' - x'_{(r+1)}) \tag{48}$$

Notice that the upper bound of this range is $x'_{(r+1)}$ reduced by $n$ times its difference from the sample average $\bar{x}'$. Depending on the particular sample values $x'_i$, it is possible that the region defined by the two bounds in (48) may not exist because the upper bound may be less than the

lower bound, $x'_{(r)}$. In other words, not all intervals between successive order statistics $x'_{(r)}$ and $x'_{(r+1)}$ may include a subinterval of values where $x'_0$ is guaranteed to win the project.

In the final case where $x'_0$ is less than $x'_{(1)}$ the sufficient condition for $x'_0$ to win the project is:

$$x'_0 < x'_{(1)} - n(\bar{x}' - x'_{(1)}) \tag{49}$$

Thus, in summary, there are $h+1$ *potential* regions defined by inequalities (41), (48), and (49) that are sufficient for $x'_0$ to win the project. For example, when $n = 4$, the average $\bar{x}'$ may lie in any one of the 3 intervals between the 4 order statistics, *i.e.*, $h = 1, 2,$ or 3. If it so happens that $h = 2$, then there 3 potential regions for $x'_0$ defined by inequalities (41), (48), and (49) that are sufficient for $x'_0$ to win the project.

In conclusion, one of the most important observations we can make is that, for a given sample $x'_1, x'_2, \ldots, x'_n$, there may be several discontinuous regions, *i.e.*, up to $h+1$, that must be examined to find all values of $x'_0$ that can win the project.

## Probability of winning — simulation procedure

The simulation procedure for determining the probability of winning as a function of the standardized bid-to-cost ratio, $x'_0$, is based on the above winning regions and follows the steps below.

Step 1: Initialize the number of opponents $n$.

Step 2: Initialize the simulation set counter $k = 1$.

Step 3: In simulation set $k$, sample $m$ vectors $x'_{1jk}, x'_{2jk}, \ldots, x'_{njk}$ from the standard normal distribution. Each vector represents one of the $m$ projects in simulation set $k$. The $n$ values in each vector represent the standardized apparent bid-to-cost ratios of the $n$ opponents in the $j^{th}$ project.

Step 4: For each sampled vector $x'_1, x'_2, \ldots, x'_n$,

1. Compute its average $\bar{x}'$.
2. Sort the values $x'_1, x'_2, \ldots, x'_n$ in increasing order to form the corresponding order statistics $x'_{(1)}, x'_{(2)}, \ldots, x'_{(n)}$.

3. Compare $\bar{x}'$ to $x'_{(1)}, x'_{(2)}, \ldots, x'_{(n)}$ to identify $h$ such that $x'_{(h)} < \bar{x}' < x'_{(h+1)}$.
4. Use inequalities (41), (48), and (49) to identify all regions $(l_r, u_r)$, $(r = 0, 1, \ldots, h < n)$ where $x'_0$ could win the project. Each region is defined by a lower bound $l$ and an upper bound $u$ such that $l < u$. If inequalities (48) give $l > u$, then the corresponding region does not exist and is ignored. These regions are the only characteristic of each simulated project that is needed from now on.

Step 5: For $x'_0$ in the interval $(-3, 3)$ using a step of 0.001:

1. Initialize to zero the number of projects won at the current values $x'_0$ and $n$ for the current simulation set $k$, $w_k(x'_0, n) = 0$.
2. Consider the next project $j$ (out of the $m$ projects in simulation set $k$). If $x'_0$ belongs to one of the regions $(l_r, u_r)$, $(r = 0, 1, \ldots, h < n)$, then project $j$ is won: increase $w_k(x'_0, n)$ by 1; otherwise do nothing. Repeat this sub-step for all $m$ projects in the current simulation step $k$.
3. Compute the ratio $p_k(x'_0, n)$ of the total number of projects won $w_k(x'_0, n)$ over the total number of projects $m$ within simulation set $k$:

$$p_k(x'_0, n) = \frac{w_k(x'_0, n)}{m} = \frac{\text{number of projects won in simulation set } k}{\text{total number of projects in simulation set } k} \quad (50)$$

This is the $k^{th}$ estimate of the probability of winning at $x'_0$ and $n$.

Step 6: If the index $k$ equals the total number of simulation sets $s$, then go to step 7. Otherwise, increase $k$ by 1 and go back to Step 3.

Step 7: Calculate the final estimate for the probability of winning at $x'_0$ and $n$ by averaging $p_k(x'_0, n)$ over all simulation sets $s$:

$$P[\text{Win}|x'_0, n] = p(x'_0, n) = \frac{1}{s} \sum_{k=1}^{s} p_k(x'_0, n) \quad (51)$$

The results of this procedure for $n = 2, 4, 8$, $s = 1000$, and $m = 1000$ are shown in Fig. 3. For comparison, this figure also shows the probability of winning for the average bid method and the low-bid method.

At this point, an important observation concerning the above procedure must be made clear. The $m$ vectors (projects) within each simulation set $k$ must be generated first (Step 3) and the winning regions for each project should be stored (Step 4) *before* the loop through all the possible values of $x_0'$ is executed (Step 5). This is necessary so that in Step 5 the number of projects won at each value of $x_0'$ is determined *with all other things being equal*. In this case, what needs to remain equal are the winning regions for the $m$ projects. It would be inappropriate to reverse the order of the loops and thus to generate a new set of $m$ vectors for each new value of $x_0'$. This not only increases the computational effort and time significantly, it also produces estimates of the probability of winning $p_k(x_0', n)$ that jump up and down erratically as $x_0'$ increases from one value to the next. Even averaging the estimates of the probability of winning over a large number of simulation sets cannot easily rectify this erratic behavior.

## Optimization

An estimate of the expected profit $v$ (in dollars) when using a bid-to-cost $x_0$ is given by:

$$v = p(x_0', n)(x_0 - 1)(c_0) \tag{52}$$

This expression for expected profit cannot be evaluated directly because it depends on $c_0$ and requires the standardized value $x_0'$ for the probability to win. To proceed further we need to introduce the standardized expected profit $v'$ that uses only standardized values:

$$v' = \frac{v}{c_0 \sigma_X} = p(x_0', n)\frac{x_0 - m_X + m_X - 1}{\sigma_X} = p(x_0', n)(x_0' + m_X') \tag{53}$$

The optimum standardized bid-to-cost ratio $x_0'^*$ is that value of $x_0'$ that maximizes the standardized expected profit $v'$ for a given value of the standardized mean $m_X'$.

The results are shown in Fig. 4 for values $m_X'$ in the interval $(-1, 3)$ using a step of 0.01. For each value $m_X'$ we determined the value $x_0'^*$ that maximized $v'$. Given the number of opponents $n$, Fig. 4 shows $x_0'^* + m_X' = (x_0^* - 1)/\sigma_X$ for any standardized mean $m_X'$. Fig. 5 shows the corresponding standardized profit $E[V^*]/(c_0\sigma_X)$ and Fig. 6 shows the corresponding probability of winning $P[\text{Win}|x_0'^*, n]$. To allow comparison, all three figures also show the results for the average bid method and the low-bid method.

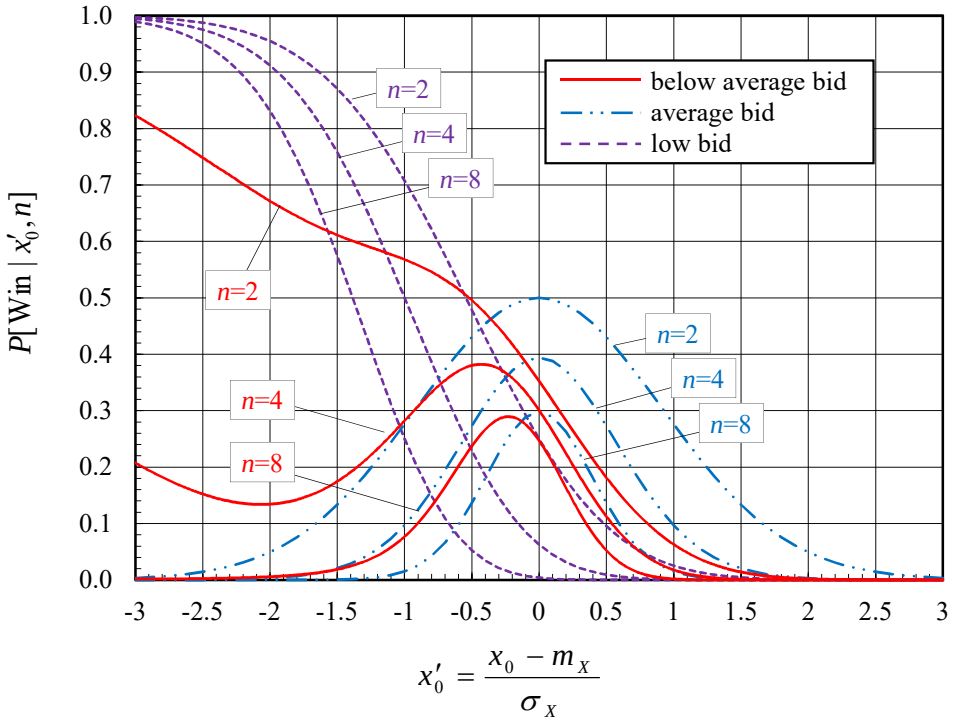



Figure 3: —— **Probability of Winning as a function of** $x'_0$**,** $n$

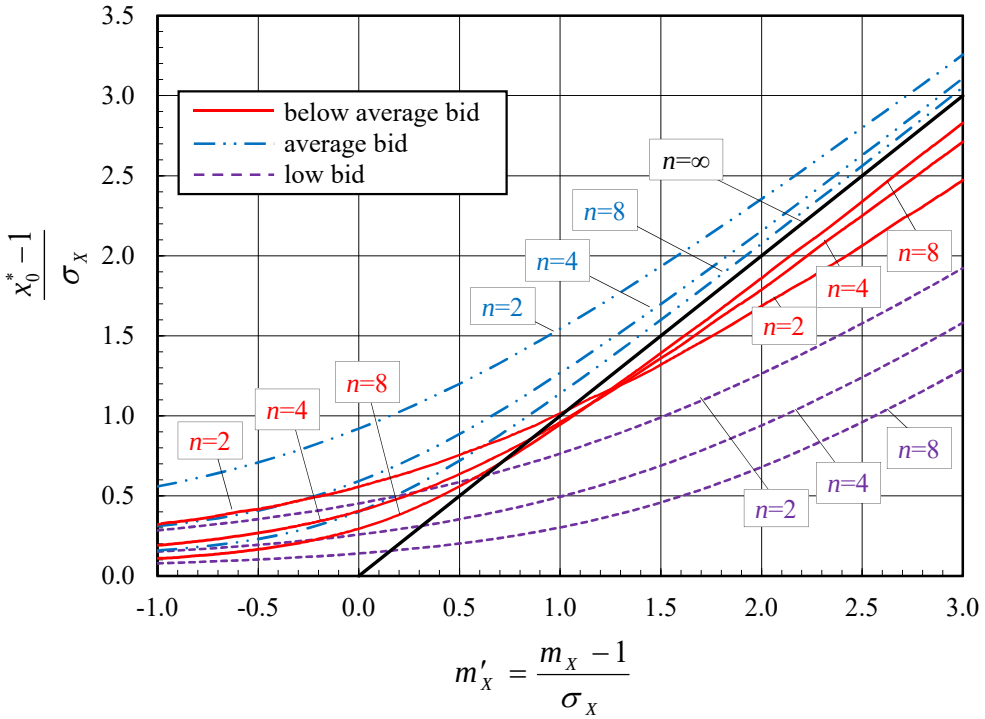



Figure 4: —— **Optimum Markup as a function of** $m'_X$**,** $n$

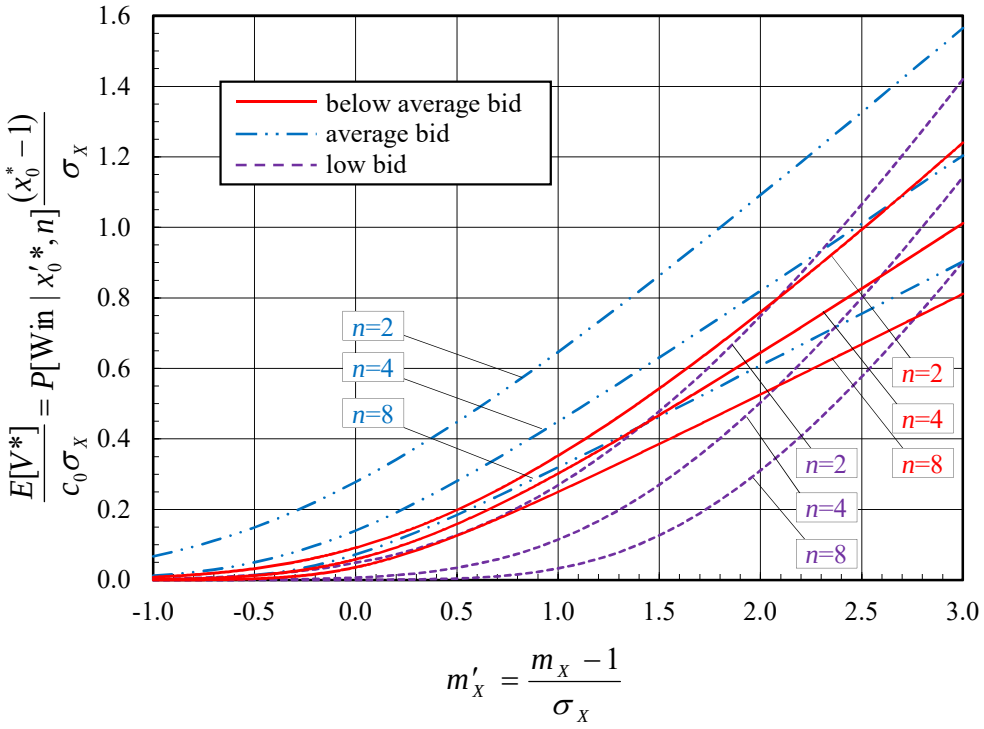



Figure 5: —— **Optimum Expected Profit as a function of** $m'_X$**,** $n$

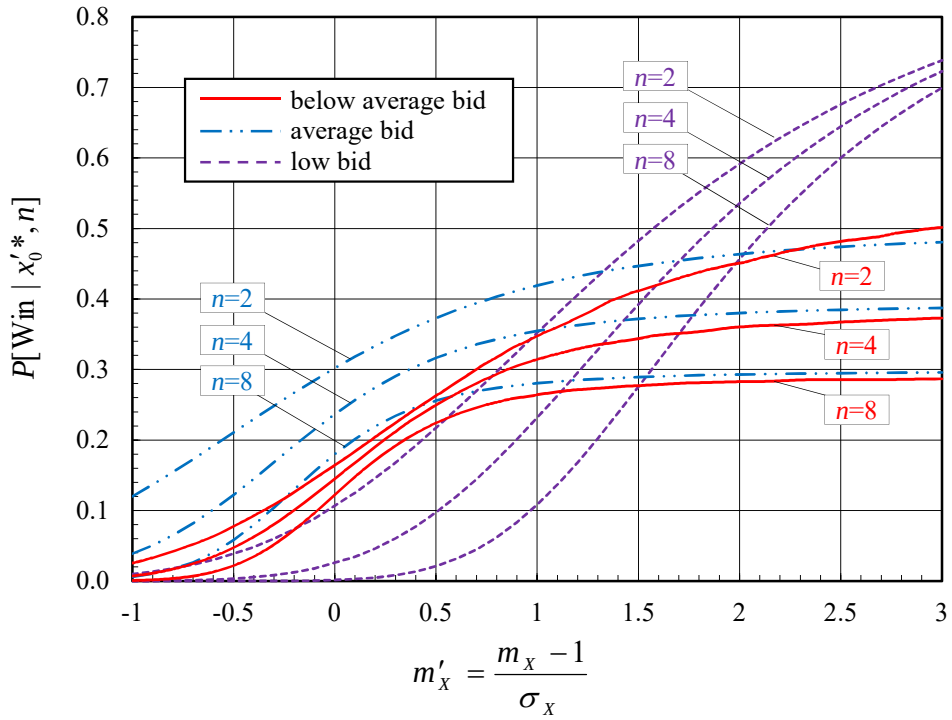



Figure 6: —— **Probability of Winning as a function of** $m'_X$**,** $n$

## Example Application

The nomograms for the below-average bid method, the average bid method, or the low-bid method, require that a contractor use past data to produce estimates $\hat{m}_X$ and $\hat{\sigma}_X^2$ for the normal distribution's mean and variance:

$$\hat{m}_X = \frac{\sum_{j=1}^{N} \sum_{i=1}^{n_j} X_{ij}}{\sum_{j=1}^{N} n_j} \tag{54}$$

$$\hat{\sigma}_X^2 = \frac{\sum_{j=1}^{N} \sum_{i=1}^{n_j} (X_{ij} - \hat{m}_X)^2}{\sum_{j=1}^{N} n_j} \tag{55}$$

The historical bid-to-cost ratios are given by $X_{ij} = B_{ij}/c_{0j}$, where, for each past project $j$, $B_{ij}$ is the bid submitted by the $i$th contractor, $c_{0j}$ is the estimated cost for contractor $A_0$, and $n_j$ is the number of competitors facing $A_0$. $N$ is the total number of past projects for which the above data are available.

To illustrate the use of these figures for the below-average bid method, let us assume that contractor $A_0$ is bidding against $n = 4$ opponents and that the mean and variance have been estimated to be $m_X = 1.05$ and $\sigma_X = 0.1$. We calculate the standardized mean $m'_X = (1.05 - 1)/0.1 = 0.5$ and use it to read the nomogram in Fig. 4. For $n = 4$ we find $(x_0^* - 1)/\sigma_X = 0.64$, which for $\sigma_X = 0.1$ gives the optimum bid-to-cost ratio $x_0^* = 1.064$, or a markup of 6.4%. Using again the standardized mean $m'_X = 0.5$ we see from Fig. 5 that the expected profit at the optimum markup is $E[V^*] = 0.16\sigma_X c_0 = 0.016c_0$, or 1.6% of the estimated cost. Finally, Fig. 6 shows that the probability of winning, when using the optimum markup of 6.4%, is $P[\text{Win}|x_0^*, m'_X, n] = 0.25 = 25\%$. Notice that the same probability can also be obtained from Fig. 3 for $x'_0 = (1.064 - 1.05)/0.1 = 0.14$. Figures 3-6 can be used in a similar manner to determine the same quantities for the average bid method or the low-bid method.

## Comparison of Bidding Methods

It is clear from the results shown in Figs. 3, 4, 5, and 6, that the bidding models for the below-average, average, and low-bid methods are significantly different from each other. We shall compare these results and explain their behavior and differences in the sections below.

## Probability of winning

The most striking difference between the three bidding methods concerns the probability of winning shown in Fig. 3.

As expected, the curves for the average bid method are symmetrical around $x_0' = 0$, with a maximum at $x_0' = 0$, irrespective of the number of opponents $n$. The maximum probability of winning does not exceed 50% and this value occurs only when $n = 2$. As the number of opponents, $n$, increases, the probability of winning decreases and in the limit becomes zero.

In contrast, the curves for the low-bid method are asymmetrical. As $x_0'$ becomes smaller, the probability of winning approaches 1.0 for any number of opponents. This is again as expected. A contractor can increase its chances of winning by cutting down its bid, something that simply cannot be done when using the average bid method. This is perhaps the most important reason why owners might prefer the average bid method. It provides a safeguard against very low bid prices, be they accidental or deliberate.

At first glance, the curves for the probability of winning when using the below-average bidding method appear to be surprising. As expected, the curves for 4 and 8 opponents are not far from symmetrical and bell-shaped like the curves for the average bid method. Also as expected, the curves for the same number of opponents are shifted to the left of the corresponding curves for the average bid method because now the winning bid cannot be greater than the average. What is surprising at first, is that the left tail of these curves rises as $x_0'$ decreases. Clearly, the tendency of the left tail to rise is most pronounced for $n = 2$ and becomes less so as the number of opponents increases to $n = 4$ and then to $n = 8$. In fact, the probability of winning for $n = 2$ increases monotonically as $x_0'$ decreases.

Yet, this left-tail behavior should be expected as it reflects a characteristic of the below-average bidding method that has already been pointed out. *I.e.*, that a contractor can win the project by submitting a bid that is so low, that it lowers the average of the $n$ opponents' bids to form a new average over $n+1$ values that is less than all the other $n$ competing bids. When a contractor submits such a low bid, it essentially pulls the average of the opponents' bids towards it, to form a new average that lies below all the other bids and wins the job. This is exactly the situation described by inequality (49). Moreover, the winning regions described by inequalities (48) contribute to this

behavior, too, and help raise the left tail of the probability of winning. Clearly, the effect of a single bid on the bid average is most pronounced when the number of opponents is small, i.e., when $n = 2$. It is obvious that this ability decreases as the number of opponents increases.

We can conclude with the observation that the probability of winning for the below-average bidding method combines the characteristics of the average bid method with those of the low-bid method. The characteristics of the average bid method dominate for large $n$ and middle and large values of $x_0'$, whereas the characteristics of the low-bid method come into play for small $n$ and small values of $x_0'$.

## Optimum bid-to-cost ratio

We shall now turn our attention to Fig. 4 and the optimum bid-to-cost ratio given by $(x_0^* - 1)/\sigma_X$ as a function of the standardized mean $m_X'$. As expected, the curves for all three methods rise monotonically as $m_X'$ increases. This is as expected, because, as the mean of the opponents' bids increases, so should the optimal markup for contractor $A_0$.

For the same number of opponents $n$, the curves for the average bid method are above those for the below-average bid method, which in turn are above the curves of the low-bid method. Again, this is as expected, because, other things being equal, the average bid method should lead to the largest optimal bid-to-cost ratios and the low-bid-method to the smallest. The below-average method produces optimal bid-to-cost ratios that are in between.

The curves for different values of the number of opponents $n$ for the average bid method do not cross each other. The same is true for the low-bid method. This means that, for both methods, the optimum bid-to-cost ratio $x_0^*$ for a given standardized mean $m_X'$, is always largest when facing the least number of opponents. As the number of competitors increases, contractor $A_0$ must lower its bid-to-cost ratio (hence its bid) to remain competitive.

Yet, this common-sense observation is not true for the below-average bid method. The curves for different number of opponents $n$ for this method cross each other at around $m_X' = 1.25$ and reverse their order. For values of the standardized mean $m_X' < 1.25$, the curves for the below-average bid method appear in the same order and behave like those for the other two methods. That is, for a given standardized mean $m_X'$, the optimum bid-to-cost ratio, $x_0^*$, decreases as the

number of opponents, $n$, increases. For values $m'_X > 1.25$, however, the opposite is true. The optimum bid-to-cost ratio, $x_0^*$, decreases as the number of opponents, $n$, decreases.

The reason for this surprising behavior is again rooted in the shape of the probability of winning for the below-average bid method shown in Fig. 3. A large value for the standardized mean $m'_X$ implies that the standard deviation of the opponents' apparent bid-to-cost ratios, $\sigma_X$, is small, and this implies that the opponents' bids are likely to be clustered closely around their mean $m_X$. The smaller the number of opponents, $n$, and the more their bids are clustered together, the easier it is for contractor $A_0$ to win by submitting a low bid that pulls the average below the bids of its opponents. As the number of opponents becomes large, it becomes difficult for a single bid to have a substantial effect and lower the average of all bids significantly. Thus, for large $n$, the best strategy for contractor $A_0$ is to select a higher bid-to-cost ratio, $x_0$, but still stay below the opponents' mean, $m_X$. This behavior is expected based on common sense and was predicted in (Ioannou and Leu 1993).

Fig. 4 also includes a $45^o$ straight line through the origin labeled $n = \infty$. For points on this line, $x_0^* = m_X$. Moreover, the vertical distance from this line to any of the curves in the figure gives the corresponding optimal standardized bid-to-cost ratio, $x_0'^*$, for the selected standardized mean $m'_X$. It is easy to see that the largest range for $x_0'^*$ occurs for $n = 2$, which extends from $x_0'^* = 0.56$ for $m'_X = 0$ to $x_0'^* = -0.54$ for $m'_X = 3$. Thus, for $m'_X > 0$, the probabilities of winning at the optimum $x_0'^*$, shown in Fig. 6, correspond to the relatively narrow region $-0.54 < x_0'^* < 0.56$ in Fig. 3. When $m'_X = 1.25$, we have $x_0'^* = -0.05$. This means that the reversed curves for the below-average bidding method in Fig. 4, *i.e.*, for $m'_X > 1.25$, correspond to probabilities of winning from the even narrower interval $-0.54 < x_0'^* < -0.05$ in Fig. 3.

For small values of the standardized mean $m'_X$, the curves for all three methods in Fig. 4, including the low-bid method, call for bid-to-cost ratios that are greater than the mean $m_X$. Small values of $m'_X$, and in particular $m'_X < 0$, imply that the mean of the opponents' bid-to-cost ratios must be $m_X < 1$. Thus, the expected value of the opposing bids is below the cost estimate of contractor $A_0$. In this case, it would be unwise for contractor $A_0$ to select a bid-to-cost ratio, $x_0 < 1$ (*i.e.*, bid below cost), and should select a bid-to-cost ratio $x_0^* > m_X$, or equivalently, should select a value $x_0'^* > 0$. This is the reason why, for small values of $m'_X$, the curves for the below-average bidding method have to be above the diagonal. Contractors should not lower their

prices to the point of losing money, even if doing so lowers the probability of winning.

In the limit, as the number of opponents, $n$, becomes very large, the curves for the average bid method approach the $45^o$ straight line from above. The curves for the below-average method behave similarly, except that for large values of $m'_X$ they approach the $45^o$ straight line from below. Thus, for both methods, as the number of opponents, $n$, becomes very large, a contractor should select as its optimum bid-to-cost ratio, $x_0^*$, a value close to the mean of its opponents' bid-to-cost ratios, $m_X$. The two methods, however, approach this limit from opposite directions. For the average bid method, the optimum bid-to-cost ratio should be a little above the mean, whereas for the below-average method it should be a little less.

## Maximum expected profit

The maximum expected standardized profit as a function of the standardized mean, $m'_X$, is shown in Fig. 5. As expected, for the same number of opponents, $n$, the profit is highest when using the average bid method and lowest when using the low-bid method. This explains why on the basis of profit, the average bid method might be favored by contractors whereas the low-bid method might be favored by owners.

For the same number of opponents, the expected profit for the below-average bid method is always less than that for the average bid method. This is not surprising because as we have seen in Figs. 3 and 4, the optimum bid-to-cost ratio $x_0^*$ for the average bid method is always greater than for the below-average bid method, as is the probability of winning for most of that region.

The expected profit for the below-average and the low-bid methods, however, behaves differently. For values $m'_X < 2$ the expected profit for the below-average bid method is greater than for the low-bid method, but for large values of $m'_X$ the situation is reversed. This is because large values of $m'_X$ imply a small standard deviation $\sigma_X$, which makes the opposing bids tend to be clustered together, and this results in high probability to win under the low-bid method. This behavior is shown clearly in Fig. 6 where for large values of $m'_X$ the probability of winning is greatest under the low-bid method.

As expected, when $n$ becomes very large, the optimum expected profit for all three bidding methods approaches zero. Notice, however, that this does not mean that the owner should expect

to sign a contract at cost. For the below-average and the average bid methods the expected profit approaches zero because the probability of winning does so. The optimum bid-to-cost ratio, $x_0$, however, approaches the mean $m_X$. Hence, the winning contractor should make a reasonable profit. For the low-bid method, however, both the probability of winning and the optimum markup approach zero and the winning contractor faces the difficult task of performing the work at a profit. Thus, the bidding methods based on the bid average have the advantage that as the number of competing bids increases, the owner is almost assured of a price that represents the consensus of the participating contractors.

## Probability of winning at optimum

Fig. 6 shows the probability of winning at the optimum bid-to-cost ratio $x_0^*$ as a function of $m'_X$. For small values $m'_X$ this probability is largest for the average bid method, whereas for high values $m'_X$ it is largest for the low-bid method. This is again as expected because small values of $m'_X$ imply a large variance, which favors winning under the average bid method, whereas large values of $m'_X$ imply a small variance, which makes it easy to underbid all opponents when using the low-bid method.

At the optimum, the probability of winning under the below-average bid method is almost always less than for the average bid method. The only exception is when $n = 2$ and $m'_X$ is large. This is as expected. From Fig. 3 we see that for the below-average bid method to have a higher probability of winning than the average bid method, $x'_0$ must be negative. This reduces the standardized profit $x'_0 + m'_X$ that would be realized if the project is won. For small (negative) values $x'_0$ to be optimal, the reduced $x'_0 + m'_X$ must be multiplied by a large probability of winning to produce an expected profit that is maximum and this happens only for $n = 2$ for values $x'_0$ around -0.5.

## Discussion

The basic advantage of the average bid method is that it safeguards the owner from signing a construction contract at an unrealistically low bid price that almost certainly will lead to adversarial relationships during construction (Grogan 1992). Similarly, contractors are protected from having

to honor a bid containing a gross mistake or oversight, which is in fact the main drawback of the low-bid method. Very low bid prices often lead to excessive claims and disputes, and result in cost increases and schedule delays.

On the other hand, the basic drawback of the average bid method is that it does not necessarily promote price competition that leads to lower costs for the owner. It is easy to see that a technological or managerial breakthrough that results in major cost savings will not necessarily be passed on to the owner in the form of lower prices, unless this breakthrough is known to be available to all bidders.

The below-average bid method shares some of the advantages and drawbacks of both the average and the low-bid method. It has the advantage that as the number of competitors increases, the owner should expect to sign a contract for a price a little below the average and be protected from an unrealistically low bid price. Moreover, since the optimum bid-to-cost ratio is below the mean, it induces competition based on price similar to the low-bid method, and thus encourages innovation and price reduction that passes some of the savings to the owner.

For projects with a relatively small number of competitors, the below-average bid method has the same drawback as the low-bid method in that it allows a deliberately or accidentally very low bid price to be successful. This possibility is even more probable for low-risk projects for which the bid-to-cost variance is small and thus a very low bid price cannot be justified. On the flip side, for projects with large bid-to-cost variance caused by significant differences in technological or managerial competence, the below-average bid method has the drawback of the average bid method in that it cannot recognize the superior ability of just a few of the competitors.

When one takes a broader view of the industry over time, it is easy to see that in the long-run, bidding methods based on the average do not discourage innovation, technology development, and cost reduction. Any such savings, when not available to all bidders, may not give the contractor a strong competitive advantage, but do result in significantly higher profits in projects won. When such savings are available throughout the industry, however, bid prices should be expected to gradually fall and the savings will eventually be passed to the owner. The difference is that the rate of price reduction will be slower to match the rate of industry adoption.

It is easy to show that contractors should be eager to adopt cost-saving innovations under all bidding methods. If we assume that a contractor has a consistent cost advantage over its competi-

tors, then its cost estimates $c_{0j}$ for past projects $j = 1, 2, \ldots, N$, would be consistently lower than its opponents, leading to higher historical apparent-bid-to-cost ratios $X_{ij}$ and thus to a higher estimate of their mean $m_X$. Other things being equal, a higher $m_X$ implies a higher standardized $m'_X$. From Figs. 4 and 5 we see that irrespective of the bidding method used, as $m'_X$ increases, so does the optimum bid-to-cost ratio and so does expected profit. Hence, contractors will seek and adopt cost-saving technological or managerial advances primarily because they wish to enhance their own profitability. Eventually, however, cost-saving innovations are adopted by the entire industry and part of the benefits are passed on to the owners.

Although attractive, both bidding methods based on the average are not without pitfalls. As is the case for the low-bid method, collusion among bidders and the absence of prequalification can negate their intent and produce undesirable results. For example, contractors in certain countries have tried to obtain a competitive advantage by forming several dummy construction companies that bid the same projects as their affiliated parent contractor. These dummy companies and the parent contractor bid in concert to achieve particular objectives. For example, they submit bid prices that are very close to each other and thus pull the overall average towards their own price. If a dummy company wins, it simply passes the entire project to the affiliated parent contractor. Or, if this is not possible, dummy companies submit very low and very high prices so that their average is close to the price of the parent contractor.

Clearly, such practices should be mitigated by using standard procedures, such as prequalifying all competitors, requiring a list of major subcontractors to be submitted along with each bid, and requiring that contractors supply performance and payment bonds.

Another alternative that is well suited for the below-average bid method is to eliminate the lowest and highest bids before computing the average of the remaining bids. The winner is the bid closest to, but below this average. The advantage of this approach is that it makes it impossible for a bid that is accidentally or deliberately very low to win. Similarly, it prevents a bid that is accidentally or deliberately very high to affect the outcome of the competitive process. The authors are currently developing a bidding model for this interesting variant of the below-average bidding process and expect the results to be availabe shortly.

## Conclusion

In this paper we have developed a bidding model for the below-average bidding method and presented the results in four nomograms. These are shown in Figs. 3-6 along with nomograms for the average-bid method and the low-bid method. Fig. 3 shows the probability that a particular contractor $A_0$ will win an upcoming project as a function of the selected standardized bid-to-cost ratio $x_0'$. This is the fundamental input to any bidding model. The remaining three nomograms provide information about optimal bidding. Their input is $m_X'$, the standardized mean of the opponents' apparent bid-to-cost ratio, which is estimated from historical data. Fig. 4 provides the optimal bid-to-cost ratio that $A_0$ should use, Fig. 5 gives the corresponding maximum expected profit, and Fig. 6 gives the probability of winning. These figures formed the basis for discussing extensively the behavior and differences of the three bidding methods.

In conclusion, it is evident that the development of fair and efficient competitive bidding methods is a subject of global interest and deserves significant attention from the U.S. construction industry. A departure from the low-bid method as practiced today has the potential to improve the longevity of construction firms, the elimination of accidentally or deliberately low bid prices, and the improvement of relationships between owner and contractor during construction. Furthermore, it should result in increased and fair profitability for contractors a necessary prerequisite for their technological and managerial success in the global arena.

## Appendix I. −− References

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