CEE 631
CONSTRUCTION DECISIONS UNDER UNCERTAINTY

CLASS NOTES

Review of Time Value of Money

© 2024 Photios G. Ioannou, Ph.D., Dipl.Eng., F.ASCE
Professor of Civil and Environmental Engineering
Construction Engineering & Management
University of Michigan
Ann Arbor, Michigan 48109-2125

TIME-VALUE OF MONEY — ENGINEERING ECONOMY

Engineering decisions among alternatives are often made by comparing their economic value. Thus, the choice between alternative *A* and alternative *B* depends upon *A* delivering greater economic value than *B*. For example, alternative *A* may have a lower purchase price, or lower operation and maintenance costs, or returns higher profits, or a combination of these.

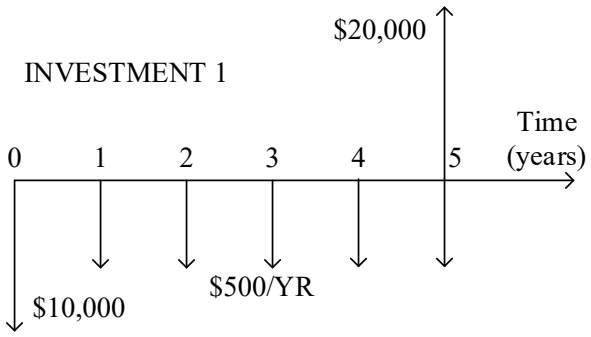
The two key elements that determine the economic values used in decision making are:

- The *amounts* of money in *cash* paid or received (i.e., the *amount* of *cash flows*).
- The *timing* of the monetary transactions involved (i.e., the *timing* of *cash flows*).

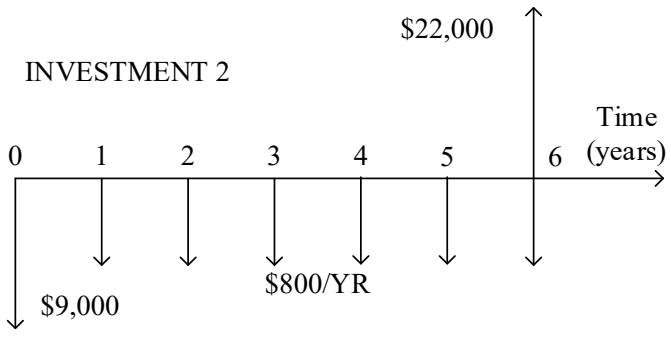
To make complex investments easier to understand, *we use cashflow* diagrams to show all cash transactions (cashflows) over time. Typically, all cashflows that occur *during a period in which interest is compounded* are shown to occur *at the end of that period*. Examples of cashflow diagrams *where the periods are years*, are shown below. Arrows pointing up are positive cash inflows (cash received); arrows pointing down are cash outflows (cash paid out).

Example Investments with Irregular Cashflows

In Investment 1 we pay an initial cash outflow of $10,000 and pay $500 cash at the end of each of 5 years. In return, we shall receive an inflow of $20,000 cash at the end of 5 years.



In Investment 2 the initial cash outflow is smaller but the annual cash outflow and final inflow are larger. Also, Investment 2 is for 6 years whereas Investment 1 is for only 5 years.



Clearly, the best choice among several alternatives, such as the choice between Investments 1 and 2, cannot be determined by simply comparing their cashflow diagrams.

Time Value of Money

The value of an amount of money depends upon its timing. We would prefer \$100 now to \$100 a year from now, especially when we may expect to earn at least \$4 on that money in a savings account in one year. In order to compare investment alternatives on a common base, different cash flows at different times must be converted to equivalent cash flows at a common point of time. This is done quite easily by using *present value* formulas (engineering economy relationships). As shown below, present value formulas are exactly the same as those used for compound interest calculations for savings in a bank or for loans from a bank.

The parameters that affect the present value of money are

- the *interest rate* per interest period, r, and
- the number of *interest periods*, or *discounting periods*, n.

The interest rate used for compounding has several equivalent *names* and *symbols*, such as:

- the i*nterest rate*, r or i
- the *discount rate, r*
- the *rate of return, r*
- the *rate of growth of money*, r
- the *opportunity cost of capital*, r
- the *cost of capital*, r

The compounding periods are also known by several other names, such as:

- *interest periods*, n
- *discounting periods*, n
- *compounding periods*, n

Money Grows by Compounding over Multiple Periods

At an interest rate of 5% per year, \$100 will grow to \$105 in a year. Thus, the two sums of money, \$100 now and \$105 a year from now, are equivalent to each other given a 5% interest rate compounded annually.

If we wish to know the value that \$100 today would have in 5 years (or at the end of n years) at a growth rate of 5% per year, we could do the following computations:

At the end of year:			Future value
1	$(100)\,(1 + 0.05) =$		\$105
2	$(105)\,(1 + 0.05) =$	$(100)\,(1 + 0.05)^2 =$	\$110.25
3	$(110.25)\,(1 + 0.05) =$	$(100)\,(1 + 0.05)^3 =$	\$115.76
5		$(100)\,(1 + 0.05)^5 =$	\$127.63
…	…	…	…
n		$(100)\,(1 + 0.05)^n$	

Thus, at interest rate r, the future value of \$100 after n interest periods = $\$100\,(1+r)^n$.

This exponential growth of money is called *compounding* and the factor $(1+r)^n$ is called the *single payment compound amount factor*. We can convert single sums of money or uniform series of money from one point in time to any other using this and other factors.

Present value of a future cash flow

The *present value* or *present worth* of a future stream of cash flows is determined based on *compound interest calculations* given a specified *discount rate* (also called *interest rate*, *rate of return, etc.*). Future cash flows are *discounted* at the discount rate, and the higher the discount rate, the lower the present value of the future cash flows. Generally, the higher the *risk* (i.e., the uncertainty) of *receiving* future cash flows, the higher the discount rate should be.

The present value formula shown below is *the fundamental formula for the time value of money*. Other discounting formulas (e.g., the annuity formula) are derived from this formula. The "present value, PV, at time 0 (or at time t)" of "a future value, FV, to be received (or paid) at the end of n compounding periods later", is:

$$PV = \frac{FV}{(1+r)^n} = FV \cdot (P \mid F, r, n)$$

- PV is the present value (equivalent cash now at time=0 or t) of the future cash flow F
- FV is the future value (future cash flow) which occurs at the end of n compounding periods later (i.e., at time = n or $n+t$)
- r = discount rate = rate of return = interest rate for compounding over *each* period
- n is the number of *compounding* periods between P and F
- $(P \mid F, r, n) = (1+r)^{-n}$ = present value factor = present worth factor = shorthand notation

The above formula allows cash (real money) to be *transferred from one point in time to another*.

- An amount of cash FV to be received at time n is equivalent to cash PV received today.
- An amount of cash FV received at time $t+n$ is equivalent to cash PV received at time t.
- An amount of cash PV to be received today is equivalent to cash FV received at time n.
- An amount of cash PV received at time t is equivalent to cash FV received at time $t+n$.

The total present value of several future cash flows FV_m, each received at a different time m, can be calculated by adding up their respective present values:

$$PV = \sum_{m=0}^{n} \frac{FV_m}{(1+r)^m} = \sum_{m=0}^{n} \{FV_m \cdot (P \mid F, r, m)\}$$

Note that this series can be summed for a given value of n, or when n is ∞. This is a very general formula. The formulas for a *Perpetuity* and an *Annuity* are special cases (see below).

Future value of a present cash flow amount

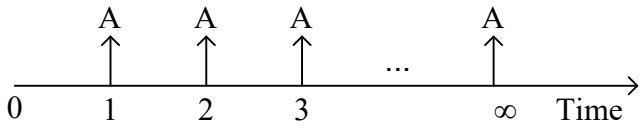
A cash amount P today (in general, cash at time t), is *equivalent* to a cash amount F at time n (in general, to cash at time $t+n$). The formula is the one above for present value, but solved for F:

$$F = P \cdot (1+r)^n = P \cdot (F \mid P, r, n)$$

$(F \mid P, r, n) = (1+r)^n$ = Future value factor = Future worth factor = Shorthand notation

Present value of a Perpetuity

A *perpetuity* is a financial arrangement where one receives (or pays) an infinite series of equal amounts of cash A at the ends of evenly spaced compounding periods, starting at time $t = 1$.



The formula for the present value of a perpetuity at time $t = 0$, given an interest rate or discount rate r, is very simple and is given by:

$$PV(P) = \frac{A}{r}$$

- $PV(P)$ is the present value of the perpetuity at time $t = 0$. It is the amount of cash today (i.e., at time $t = 0$) that is *equivalent* to the infinite stream of cashflows A.
- A is the cashflow amount at the end of each compounding period. This cashflow stream of A at the end of each compounding period continues forever.
- r is the interest (discount) rate used to compound (discount) money *each* period.

The above formula for the present value of a perpetuity can be verified quite easily by looking up the sum of an infinite declining geometric series in a math handbook.

$$PV(P) = \sum_{t=1}^{\infty} \frac{A}{(1+r)^t} = \frac{A}{r}$$

To understand and remember the perpetuity formula, however, a different approach is needed. Notice that a perpetuity can be equivalent to depositing (at time $t = 0$) an amount of money $PV(P)$ (the capital or principal) into a savings account earning a constant interest rate r over each compounding period. The interest earned at the end of the first compounding period is

$$A = r \cdot PV(P)$$

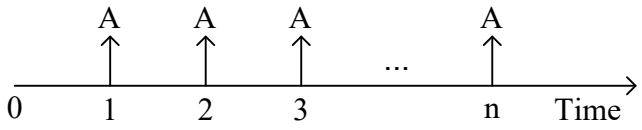
Suppose that at the end of the first, and each compounding period thereafter, we withdraw from the account (i.e., we receive as payment) an amount equal to the above earned interest, A. Thus, the remaining balance in the account will neither grow nor diminish and will always be $PV(P)$. Since the remaining balance in the account is always the same and equal to $PV(P)$, it will earn the same amount of interest $A = r \cdot PV(P)$ at the end of the next compounding period. At the end of that period, an amount $A = r \cdot PV(P)$ will again be withdrawn leaving a balance of $PV(P)$. Thus, this process can continue forever and produce an infinite stream of equal cash flow payments $A = r \cdot PV(P)$ at the end of each compounding period.

- Conclusion: Receiving $A = r \cdot PV(P)$ at the end of each compounding period is <u>*equivalent*</u> to the initial deposit $PV(P)$ today. I.e., the present value of a perpetuity is $PV(P) = A / r$.

Note: The present value of a perpetuity provides a convenient <u>approximation</u> for the present value of an annuity to be received over many periods. Moreover, as described below, it provides an easy way to compute the exact present value of any annuity.

Present value of an Annuity with *n* payment periods

An *annuity* is a series of equal cash flow amounts, A, that occur at the *end* of a *given number* of equal compounding periods, n. Examples are leases, rental payments, mortgages, car loans, etc.



Thus, an annuity is where one receives (or pays) n equal amounts of cash A at the end of each equal compounding period starting with period 1 and continuing for n periods. For a discount rate r, the formula for the present value of an annuity at time zero is given by:

$$PV(A) = \frac{A}{r} \cdot \left[1 - \frac{1}{(1+r)^n} \right] = A \cdot (P \mid A, r, n)$$

- $PV(A)$ is the present value of the annuity at time $t = 0$. It is the amount of cash today (at time $t = 0$) that is *equivalent* to n equally spaced future cash flow payments A.
- A is the future cash flow amount at the *end* of *each* of the n compounding periods.
- r is the interest rate that would be compounded for each time period.
- n = number of equal payments A = number of compounding periods.

When solved for the payment amount A, the formula for the present value of an annuity can be used to compute the (monthly) payments for a mortgage or a car loan:

$$A = PV(A) \cdot r \left[1 - \frac{1}{(1+r)^n} \right]^{-1} = PV(A) \cdot r \left[\frac{(1+r)^n}{(1+r)^n - 1} \right] = PV(A) \cdot (A \mid P, r, n)$$

It can also be solved for the number of periods n needed to repay a loan. However, solving for the interest rate r (given the other three variables) requires a numerical technique.

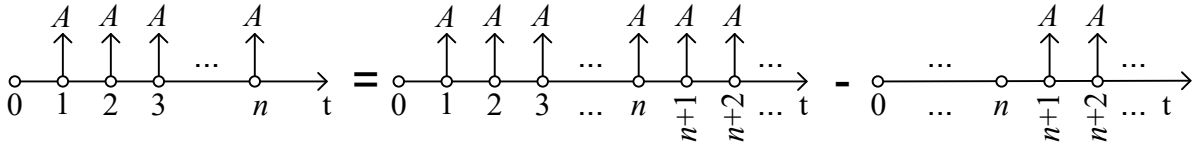
The formula for the present value of an annuity can be verified by looking up the sum of a finite geometric series with n terms in a math handbook.

$$PV(A) = \sum_{t=1}^{n} \frac{A}{(1+r)^t} = \frac{A}{r} \cdot \left[1 - \frac{1}{(1+r)^n} \right]$$

However, the easiest way to derive (and, most importantly, to *remember*) the formula for the present value of an annuity is to appreciate that:

- The cash flow of an annuity (i.e., receiving or paying an amount A at the end of each period for n periods) is the difference between two perpetuities *offset* by n time periods.
- For the first perpetuity P_1, payments A start at time 1 and continue forever at the end of each of the following compounding periods 2, 3, …

- For the second perpetuity P_2, payments A start at time $n+1$ and continue forever at the end of each succeeding compounding period, $n+2$, $n+3$, …
- Subtracting perpetuity P_2 from perpetuity P_1 has the effect that all payments from time $n+1$ onwards cancel each other $(A-A)$ and produce zero cash flows. This leaves only the payments A for the first n periods. Hence, the result $P_1 - P_2$ is an annuity.



The present value at time $t = 0$ of the first perpetuity P_1 is given by the perpetuity formula:

$$PV(P_1, t=0) = \frac{A}{r}$$

The key observation is that the present value **at time $t = n$** of the second perpetuity P_2 is also:

$$PV(P_2, t=n) = \frac{A}{r}$$

This present value at $t = n$ must be discounted by n periods to find its present value at $t = 0$:

$$PV(P_2, t=0) = \frac{A}{r} \cdot \frac{1}{(1+r)^n}$$

Thus, the present value **at time $t = 0$** of perpetuity P_1 minus the present value **at time $t = 0$** of perpetuity P_2 gives the final formula for the present value of an annuity:

$$PV(A, t=0) = PV(P_1, t=0) - PV(P_2, t=0) = \frac{A}{r} - \frac{A}{r} \cdot \frac{1}{(1+r)^n} = \frac{A}{r} \cdot \left[1 - \frac{1}{(1+r)^n} \right]$$

Future value of an annuity

The future value of an annuity is the future value of a stream of payments (annuity), assuming the payments are invested at a given rate of interest. The formula for the future value of an annuity, $F(A, t=n)$, has the same variables as the formula for the present value of an annuity:

$$FV(A, t=n) = A \cdot \frac{(1+r)^n - 1}{r} = A \cdot (F \mid A, r, n)$$

This is simply the future value (after n compounding periods) of the present value of an annuity:

$$FV(A, t=n) = PV(A, t=0) \cdot (1+r)^n = \frac{A}{r} \cdot \left[1 - \frac{1}{(1+r)^n} \right] \cdot (1+r)^n = \frac{A}{r} \cdot \left[(1+r)^n - 1 \right]$$

Using the Annuity Formula to calculate Bank Savings or Bank Loans

When dealing with bank loans or bank savings accounts it is important to understand the interest rate quoted and the interest rate and the compounding period that is actually being used.

Loans: When you borrow money from a bank, the interest rate is typically stated as an *APR* (Annual Percentage Rate) which is a nominal annual interest rate. This means that the *APR* is an interest rate in name only because it needs to be associated with a method of compounding to get a true effective interest rate.

For example, if interest on a loan amount is compounded monthly, then we must divide the *APR* by 12 (the number of compounding periods in a year) to find the effective monthly interest rate *APR*/12 which is then compounded monthly.

Loan Example: Suppose that you wish to borrow an amount of money *PV* from a bank to buy a car. You will repay this loan to the bank by making constant payments *A* at the end of each month for a certain number of years, *N*. The interest rate for such loans is typically quoted by the bank on a *yearly* basis as an *APR*. For example, $APR = 6\%$ per year.

Notice that for such loans, the *compounding period* is one month (and not one year).

As a result, to find the equivalent *monthly* interest rate you need to divide the *APR* by 12. For example, when $APR = 6\%$, we have:

$$r = \frac{APR}{12\,months\,/\,year} = \frac{6\%}{12} = 0.5\%\ \text{per month}$$

Suppose that you will need to repay the loan to the bank in $N = 5$ years. Again, because the *compounding period* is one month (and not one year) you will need to multiply the number of years *N* by 12 to find the equivalent monthly compounding periods *n*:

$$n = (N\ years) \times (12\,months\,/\,year) = 5 \times 12 = 60\ \text{months}$$

At this point, the loan has the structure of a monthly annuity. Your payment to the bank is a series of equal cash flow amounts, *A*, that occur at the *end* of each month, for a total of *n* months (where the compounding period is one month). The present value of this annuity equals the amount of money *PV* that you need to borrow from the bank at time $t = 0$.

$$PV(A) = \frac{A}{r} \cdot \left[1 - \frac{1}{(1+r)^n} \right] = A \cdot (P \mid A, r, n)$$

Given the amount of money *PV*(*A*) that you will need to borrow to pay for the car, you can use this annuity formula to calculate the monthly payment *A*.

To continue the above example, suppose that you need to borrow \$30,000 for $N = 5$ years at a yearly interest rate $APR = 6\%$. In this case, *PV*(*A*)= \$30,000, $r = 0.5\%$, and $n = 60$ months:

$$\$30000 = \frac{A}{0.005}\cdot\left[1-\frac{1}{(1+0.005)^{60}}\right] = A\times 51.72556$$
$$A = \$579.98\,/\,month$$

So, to repay the loan of $30,000 in 5 years at an *APR*=6%, you will need to pay the bank about $580 at the end of each month for a total of 60 months.

The same annuity formula can also be used to calculate the monthly payment on a loan to buy a house. Such a loan is called a *mortgage* and typically has a repayment period, N = 15 years (n = 180 months) or N = 30 years (n = 360 months). The *APR* for mortgages are also quoted on a *yearly* basis, so they must also be divided by 12 to obtain the required *monthly* interest rate r.

Savings: For a bank savings account the interest you earn on the money you save in that account is stated as an *APY* (Annual Percentage Yield) which is an effective annual interest rate.

Suppose that a bank savings account earns interest r compounded monthly. Then the *APY* for that account is found as follows:

$$1+APY = (1+r)^{12}$$

Savings Example:
Suppose that a bank offers a savings account with *APY* = 12%. We can calculate the effective monthly interest rate and the *APR* for that savings account as follows:

$$1+0.12 = (1+r)^{12}$$
$$r = \sqrt[12]{1.12}-1 = 0.009489 = 0.9489\% \text{ per month}$$
$$APR = 12*0.9489\% = 11.3866\% \quad (<12\%)$$

Note that if we start a savings account by depositing $1 at the start of the year at an interest rate of 0.9489% compounded monthly, then at the end of one year we will have $1.12, which represents an effective annual interest yield exactly equal to the *APY*=12%.

Also note that for a savings account that has an interest rate of 0.9489% compounded monthly the *APR* = 11.3866% is less than the *APY* = 12%. This explains why banks quote the annual interest rate for loans as an *APR* (to make the interest rate appear small) but quote the annual interest rate for savings accounts as an *APY* (to make the interest rate appear large).

Typically, *APY* > *APR*
For example, the *APY* for loan of $30,000 to be repaid in 5 years at an *APR*=6% is:

$$APY = \left(1+\frac{6\%}{12}\right)^{12}-1 = 6.1678\% \quad (>6\% = APR)$$

Thus, because of monthly compounding, the effective annual interest rate for this loan is *APY* = 6.1678% which is more than the *APR* = 6% quoted by the bank.

Annuity Functions in Excel

There are five annuity functions in Excel in which positive values for money indicate money that you receive and negative values for money indicate money that you pay:

- =**PMT**(rate, nper, pv, [fv], [type])
- =**PV**(rate, nper, pmt, [fv], [type])
- =**FV**(rate, nper, pmt, [pv], [type])
- =**NPER**(rate, pmt, pv, [fv], [type])
- =**RATE**(nper, pmt, pv, [fv], [type], [guess])

Annuity functions have the following arguments:

Rate The interest rate per period. For example, if you obtain a loan at a 12% annual interest rate, but make monthly payments, your interest rate per month is 12%/12, or 1%. You would enter 12%/12, or 1%, or 0.01, into the formula as the *rate*.

Nper The total number of payment periods in an annuity. For example, if you get a four-year loan and make monthly payments, your loan has 4*12 (or 48) periods. You would enter 48 into the formula for *nper*.

Pmt The payment made each period (constant over the life of the annuity). For example, the monthly payments on a $10,000, four-year car loan (*nper* = 48) at 12% per year (*rate* = 12%/12 = 1% per month) are $263.33 per month. You would enter a negative value *amt* = -263.33 into the formula to indicate that you pay this amount each month. For the PV and RATE functions, if *pmt* is omitted, you must include the *fv* value.

PV The present value, or the total amount that a series of future payments is worth now; also known as the principal. Note that if *fv* = 0, *pv* and *pmt* have opposite signs (+/-).

FV The future value, or a cash balance you want to attain after the last payment is made. If *fv* is omitted, it is assumed to be 0 (the future value of a loan, for example, is 0). For the PV and RATE functions, if *fv* is omitted, you must include the *pmt* argument.

Type (Optional) The number 0 or 1. A *type* value of 0 (or if *type* is omitted) indicates that payments are made at the end of each period. A *type* value of 1 indicates that payments are made at the start of each period.

Guess Your guess for what the *rate* will be. If you omit *guess*, it is assumed to be 10%.

Note: The *NPV* function is also available in Excel but it is not used often. Its arguments have strict requirements that make the practical use of this function difficult and prone to mistakes. See the Excel help for more details.

Where does the Discount Rate come from? — Cost of Capital

Cost of capital is the cost of funds used for financing a business. Cost of capital depends on the mode of financing used – it refers to the *cost of equity* if the business is financed solely through equity (shares), or to the *cost of debt* if it is financed solely through debt. Many companies use a combination of debt and equity to finance their businesses, and for such companies, their overall cost of capital is derived from a weighted average of all capital sources, widely known as the *weighted average cost of capital* (WACC).

- *Cost of capital* is the *hurdle rate* that a company must overcome to generate value.
- *Cost of capital* is extensively used in *capital budgeting* to determine whether a company should proceed with a project.

Components of the Cost of Capital

The cost of various capital sources varies from company to company, and depends on factors such as its operating history, profitability, credit worthiness, etc. In general, *newer* enterprises with limited operating histories will have higher costs of capital than established companies with a solid track record, because lenders and investors demand a higher risk premium (i.e., higher interest rates or rates of return) for newer (riskier) firms.

The cost of capital is a critical factor in deciding which financing track to follow – debt, equity or a combination of the two. Early-stage companies seldom have sizable assets to pledge as collateral for debt financing, so they rely primarily on equity financing.

Cost of debt is the *interest rate* paid by the company for debt. However, since interest expense is tax-deductible (assuming there are yearly profits), the after-tax cost of debt is calculated as:

(Debt Interest Rate) * (1 - T)

where T is the company's marginal tax rate.

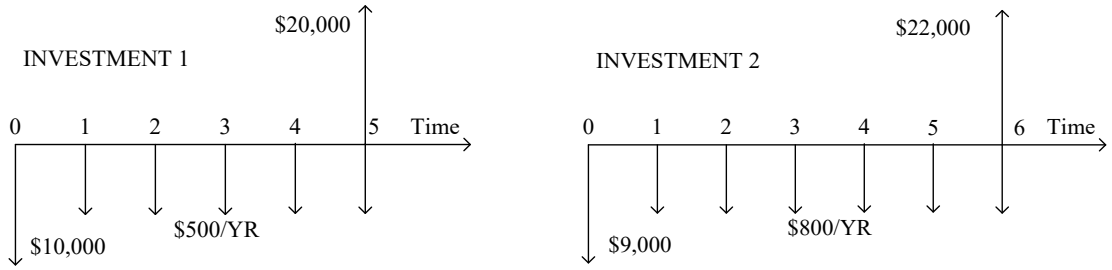
Cost of equity is more complicated because the rate of return demanded by equity investors (shareholders) is not as clearly defined as it is by lenders. Theoretically, the cost of equity is approximated by the *Capital Asset Pricing Model* (CAPM) which states that:

Expected return = Risk-free rate + (Company's Beta)*(Expected market return - Risk-free rate)

The firm's *overall cost of capital* is based on the *weighted average* of these costs. For example, consider an enterprise with a capital structure consisting of

- Equity = 70%
- Debt = 30%
- Cost of equity = 10% (= Return expected by investors)
- After-tax cost of debt = 7% (= After-tax interest rate paid to lenders)
- WACC = (0.7 x 10%) + (0.3 x 7%) = 9.1%.

Thus 9.1% would be the cost of capital that would be used as the *discount rate* to discount future cash flows from potential projects and other opportunities to estimate their Net Present Value (NPV) and ability to generate value.



We can now compute the present value of Investment 1 at time 0 for a discount rate of $r = 10\%$

$$PV(FV = \$20000, r = 10\%, n = 5) = \frac{20000}{(1+0.10)^5} = \$20000\left(P \mid F, 10\%, 5\right) = \$20000\left(0.6209\right) = \$12,418$$

$$PV(A = \$500, r = 10\%, n = 5) = \frac{500}{0.10}\cdot\left[1 - \frac{1}{(1+0.10)^5}\right] = \$500\left(P \mid A, 10\%, 5\right) = \$500\left(3.791\right) = \$1,895.5$$

The net present value at time zero of the entire cash flow at a discount rate r=10% is given by

$$NPV = -\$10,000 - 1,895.5 + 12,418 = \$522.50$$

The positive *NPV* at r=10% shows that Investment 1 *yields* over 10% per year. The *yield* of an investment is the discount rate that results in NPV=0, which in this case equals 11.06%.

Similarly, the net present value of Investment 2 at a discount rate of r=10%, is given by

$$\begin{aligned} NPV &= -9,000 - \frac{800}{0.10}\cdot\left[1 - \frac{1}{(1+0.10)^6}\right] + \frac{22,000}{(1+0.10)^6} \\ &= -9,000 - 800(P \mid A, 10\%, 6) + 22,000(P \mid F, 10\%, 6) \\ &= -9,000 - 800(4.355) + 22,000(0.5645) \\ &= -9,000 - 3,484 + 12,419 \\ &= -\$65 \end{aligned}$$

The negative *NPV* at 10% shows that the yield (or return) on Investment 2 is less than 10%.

At 9%, *NPV* = \$529.88. By using linear interpolation, the actual yield of the investment is approximately,

$$r = 9\% + (10\% - 9\%)\frac{529.88}{529.88 + 65.00} = 9.89\%$$

Investment 1 is preferred over Investment 2 because it has a higher *NPV* (\$522.50 > –\$65). Equivalently, Investment 1 is preferred over Investment 2 because it has a higher yield (11.06% > 9.89%).

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Engineering economy studies can easily include uncertainty in their estimates of outcomes. Instead of using directly the estimated cash flow as the outcome of each alternative, we use the expected value of the cash flow instead.

Expected Value of Net Present Value

Suppose that we must choose between two possible alternatives: expansion of facilities, *E*, and non-expansion, *N*. The two alternatives have the following estimated cash flows:

Alternative:	*E*		*N*
Outcome:	E_1	E_2	
P[Outcome]:	0.4	0.6	1.0
Year	Cash Flow		
0	–60,000	–60,000	0
1	+20,000	+25,000	0
2	+20,000	+20,000	0
3	+20,000	+15,000	0
4	+20,000	+10,000	0

Notice that the decision alternative "expand", *E*, leads to two possible outcomes, E_1 and E_2, each with a different cashflow. We can calculate the expected net present value, i.e., the expected equivalent value at time zero at a rate of return of 10%, of these two cashflows as follows,

$$E[NPV] = \sum_i PV[E_i]\,P[E_i] \;=\; 0.4\,PV[E_1] \;+\; 0.6\,PV[E_2]$$

$$PV[E_1] = -60,000 + 20,000(P \mid A,4,10\%) = -60,000 + 20,000(3.1699)$$
$$= \$3,398$$

$$PV[E_2] = -60,000 + 25,000(P \mid F,1,10\%) + 20,000(P \mid F,2,10\%)$$
$$+15,000(P \mid F,3,10\%) + 10,000(P \mid F,4,10\%)]$$
$$= -60,000 + 25,000(0.9091) + 20,000(0.8265) + 15,000(0.7513) + 10,000(0.6830)$$
$$= -\$2,643$$

$$E[NPV] = 0.4(\$3,398) + 0.6(-\$2,643)$$
$$= 1,359.2 - 1,585.8$$
$$= -\$226.6$$

Thus, at a discount rate of 10%, the alternative to invest in an expansion has a negative net present value. Equivalently, the expansion alternative has an expected yield less than 10%. As a result, the preferred alternative is not to expand.

Sometimes, the sample space may be 2-dimensional as in the following example. Consider the case where, at the preliminary design stage, the estimated costs of developing a piece of land, D, and of its final selling price, S, are in doubt (but independent).

Year	Development Costs ($k)		Selling Price ($k)		
	D_1	D_2	S_1	S_2	S_3
	0.6	0.4	0.2	0.5	0.3
0	-3,200	-3,200			
1	-800	-1,200			
2	-800	-1,600			
3			10,000	8,500	5,000

We can solve this problem by considering all possible combinations of sample points D_1, D_2, S_1, S_2, S_3, to obtain all 6 possible cashflows. However, it is easier to first find the present value of each event, and then use the probabilities associated with each event to determine the expected net present value. Assuming a minimum attractive rate of return of 20%, we have

$$E[NPV] = E[PV[D]] + E[PV[S]] = 0.6PV[D_1] + 0.4PV[D_2] + 0.2PV[S_1] + 0.5PV[S_2] + 0.3PV[S_3]$$

$$PV[D_1] = -3,200 - 800(P \mid A, 2, 20\%) = -3,200 - (800)(1.5278)$$
$$= -\$4,422^k$$

$$PV[D_2] = -3,200 - 1,200(P \mid F, 1, 20\%) - 1,600(P \mid F, 2, 20\%)$$
$$= -3,200 - 1,200(0.8333) - 1,600(0.6945)$$
$$= -\$5,311k$$

$$PV[S_1] = 10,000(P \mid F, 3, 20\%) = 10,000(0.5787)$$
$$= \$5,787k$$

$$PV[S_2] = 8,500(P \mid F, 3, 20\%) = 8,500(0.5787)$$
$$= \$4,919k$$

$$PV[S_3] = 5,000(P \mid F, 3, 20\%) = 5,000(0.5787)$$
$$= \$2,894k$$

$$E[PV[D]] = 0.6PV[D_1] + 0.4PV[D_2] = 0.6(-4,422) + 0.4(-5,311) = -2,653 - 2,124$$
$$= -\$4,778k$$

$$E[PV[S]] = 0.2PV[S_1] + 0.5PV[S_2] + 0.3PV[S_3] = 0.2(5,787) + 0.5(4,919) + 0.3(2,894) = 1,157 + 2,459 + 868$$
$$= \$4,485k$$

$$E[NPV] = E[PV[D]] + E[PV[S]] = -4,778 + 4,485$$
$$= -\$293k < 0$$

This investment has a negative net present value, or an expected yield of less than 20%.

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In the previous example we found first the present value of the cashflow that is associated with each event D_1, D_2, S_1, S_2, S_3. Then we found the expected value of the present values of the cashflows.

In the example below, we find first the expected value of the cashflow for each year t, $E[CF_t]$, and record it in the last column "Exp. Cash Flow" for that year.

Year	Development Costs ($k)		Selling Price ($k)			Exp Cash Flow ($k)
	D_1	D_2	S_1	S_2	S_3	$E[CF_t]$
	0.6	0.4	0.2	0.5	0.3	
0	-3,200	-3,200				-3,200
1	-800	-1,200				-960
2	-800	-1,600				-1,120
3			10,000	8,500	5,000	7,750

The expected value of the cash flow for each year is given by:

$$E[CF_t] = E[D_t] + E[S_t] = 0.6D_{1,t} + 0.4D_{2,t} + 0.2S_{1,t} + 0.5S_{2,t} + 0.3S_{3,t}$$

After the expected cashflows for each year t, $E[CF_t]$, are computed as above, we find the present value of the expected cashflows as follows assuming a rate of return of 20%:

$$PV[E[CF_0]] = -\$3,200^k$$

$$PV[E[CF_1]] = -960(P\,|\,F,1,20\%) = -960(0.8333) = -\$800^k$$

$$PV[E[CF_2]] = -1,120(P\,|\,F,2,20\%) = 1,120(0.6945) = -\$777.78k$$

$$PV[E[CF_3]] = 7,750(P\,|\,F,3,20\%) = 7,750(0.5787) = \$4,484.95k$$

$$NPV = PV[E[CF_0]] + PV[E[CF_1]] + PV[E[CF_2]] + PV[E[CF_3]]$$

$$-\$3,200^k - \$800^k - \$777.78k + \$4,484.95k$$

$$= -\$293k < 0$$

The NPV = -$293k is the same as in the previous example and is again negative. Thus, the investment is not desirable at a rate of return of 20%. Equivalently, its expected yield is less than 20%.

Example—Probabilistic Discounted Expense Allocation for Tractor

We shall now revisit the example presented earlier about the expense allocation for a purchased tractor by considering the time value of money. The first cost of the tractor is \$35,000. The joint PMF for the tractor's lifetime L in hours and its hourly operating costs H is shown in the table below. The expected salvage value at the end of the tractor's productive life is \$10,000. Insurance and taxes are \$2,000/year. Assume 2,000 hr/yr operation, and an interest rate, r = 20%.

$P[H=h, L=l]$	Operating Costs, h (\$/hr)			
Lifetime, l (hrs)	4.00	5.00	6.00	$P[L=l]$
4000	-	-	0.10	0.1
6000	0.02	0.06	0.12	0.2
8000	0.12	0.24	0.04	0.4
10000	0.21	0.09	-	0.3
$P[H=h]$	0.35	0.39	0.26	1.0

The expected operating cost per hour is the same as before,

$$E[H]=\sum_{\text{all } h} hP[H=h]=4\times 0.35+5\times 0.39+6\times 0.26=\$4.91/hr$$

We can calculate the ownership costs for the tractor for each possible lifetime in the following manner, assuming that costs are concentrated at the end of each year.

Let $Q_Y(n)$ = equivalent yearly cost (\$/yr), and n = number of years of life = L / 2000 hr,

$$Q_Y(n)=\$35,000(A\,|\,P,n,20\%)-\$10,000(A\,|\,F,n,20\%)+\$2000/yr$$

The tractor's hourly ownership cost $Q(n)$ as a function of its uncertain life n, in years, equals,

$$Q(n)=Q_Y(n)/2000\ hr=\$17.5(A\,|\,P,n,20\%)/hr-\$5(A\,|\,F,n,20\%)/hr+\$1/hr$$

and is evaluated in the following table:

Lifetime (hr)	n (yr)	$(A\|P, n, 20\%)$	$17.5(A\|P)$ (\$/hr)	$(A\|F, n, 20\%)$	$-5(A\|F)$ (\$/hr)	$Q(n)$ (\$/hr)
4000	2	0.6545	11.4545	0.4545	-2.2727	10.18
6000	3	0.4747	8.3077	0.2747	-1.3736	7.93
8000	4	0.3863	6.7601	0.1863	-0.9314	6.83
10,000	5	0.3344	5.8516	0.1344	-0.6719	6.18

$$E[Q]=\sum_{\text{all } n} q(n)P[n]=0.1(10.18)+0.2(7.93)+0.4(6.83)+0.3(6.18)=\$7.19/hr$$

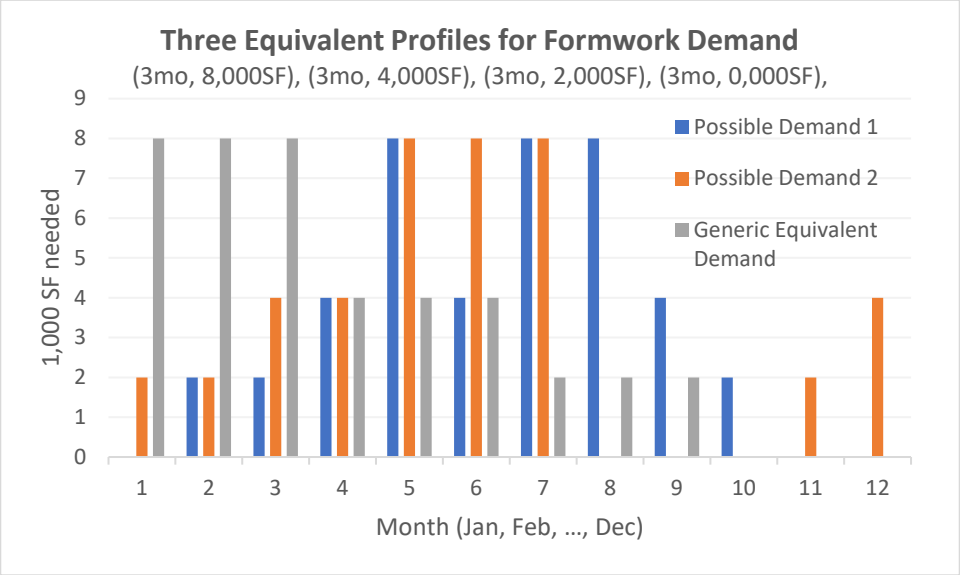
Thus, the expected value of the hourly total cost for this tractor is,

$$E[C]=E[Q]\ +\ E[H]=7.19+4.91=\$12.10/hr$$

Notice that when future dollar amounts are discounted, we have $E[C]$=\$12.10/*hr*, i.e., we should charge more than the $E[C]$=\$9.368/*hr* when future dollar amounts are not discounted.

Example—How Much Formwork to Buy

A contractor must decide upon the quantity of steel forms to purchase relative to its needs over the year. This is a challenging problem because there can be many profiles for formwork demand over the months of a year. As an example, the figure below shows three possible profiles.



The demand profiles "Possible Demand 1" and "Possible Demand 2" require exactly 8,000 SF for 3 months, exactly 4,000 SF for 3 months, exactly 2,000 SF for 3 months, and exactly 0 SF for 3 months. From the perspective of how much formwork to buy (where the remaining formwork that may be needed in each month will be rented) these two profiles are *equivalent*. The specific month is which each demand column appears in the above figure is not important. What matters is the total number of columns (months) that have each particular height (demand in SF).

Thus, each of the above two profiles is equivalent to the demand profile shown as "Generic Equivalent Demand" which requires exactly 8,000 SF over Jan-Feb-Mar (3 months), exactly 4,000 SF over Apr-May-Jun (3 months), exactly 2,000 SF over Jul-Aug-Sep (3 months), and exactly 0 SF over Oct-Nov-Dec (3 months).

To keep the number of decision alternatives small (and manageable), we can select a convenient size for the batches of formwork we can buy and use that batch size to discretize the above generic demand profile. For example, suppose that formwork can be purchased in batches of 2,000 SF. We can then discretize the above generic demand profile as shown below:

X_1 = Number of months/year the 1st 2000 SF of formwork are needed

X_2 = Number of months/year the 2nd 2000 SF of formwork are needed

X_3 = Number of months/year the 3rd 2000 SF of formwork are needed

X_4 = Number of months/year the 4th 2000 SF of formwork are needed

In this case, the number of choices for the SF of formwork to buy is only five (manageable):

0 SF, 2,000 SF, 4,000 SF, 6,000 SF, or 8,000 SF.

The above definition of the random variables X_1, X_2, X_3, and X_4, permits the assessment of their joint probability mass function (joint PMF) as shown in the table below.

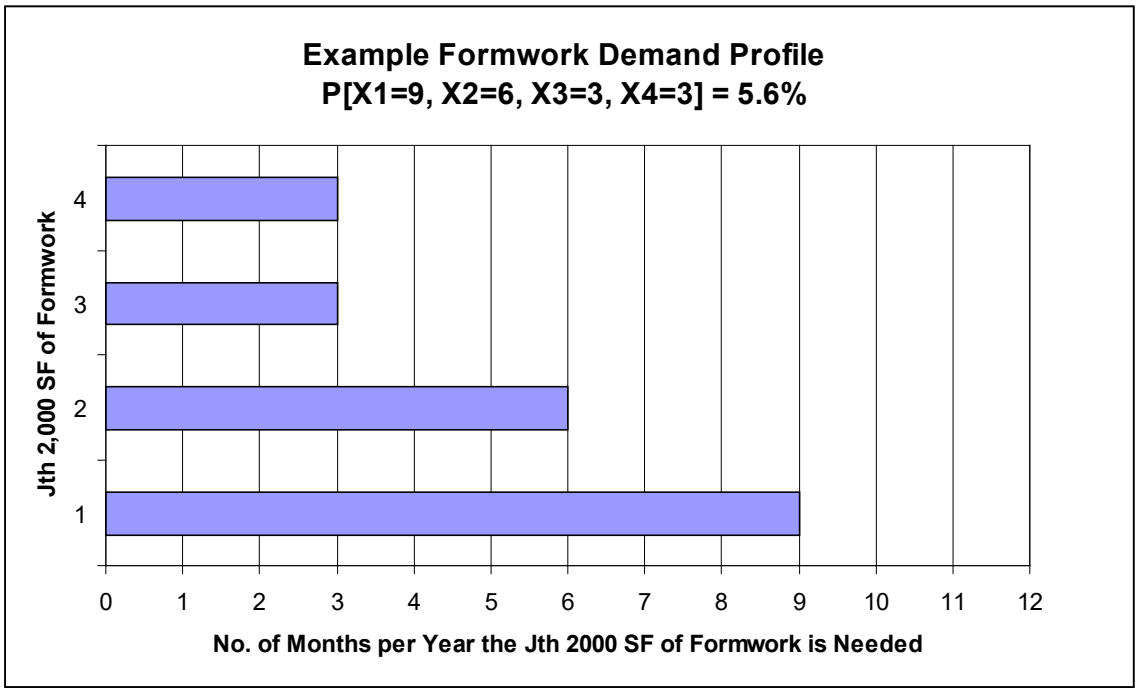
It should be stressed that the detailed assessment of the joint PMF of X_1, X_2, X_3, and X_4, is <u>not</u> necessary for finding *the best amount of formwork to purchase* — for this decision it is enough to estimate the marginal PMFs for each of the random variables X_1, X_2, X_3, and X_4.

Nevertheless, to show the relationship between the joint PMF of X_1, X_2, X_3, and X_4, and their marginal PMFs, the table below shows a joint PMF, $P[X_1=x_1, X_2=x_2, X_3=x_3, X_4=x_4]$.

x_1	x_2	x_3	x_4	*Joint PMF*
0	0	0	0	-
3	0	0	0	-
3	3	0	0	-
3	3	3	0	-
3	3	3	3	-
6	0	0	0	-
6	3	0	0	-
6	3	3	0	-
6	3	3	3	-
6	6	0	0	-
6	6	3	0	1.86%
6	6	3	3	8.14%
6	6	6	0	-
6	6	6	3	-
6	6	6	6	-
9	0	0	0	-
9	3	0	0	-
9	3	3	0	15.40%
9	3	3	3	4.60%
9	6	0	0	0.82%
9	6	3	0	5.96%
9	6	3	3	5.60%
9	6	6	0	3.95%
9	6	6	3	3.45%
9	6	6	6	-
9	9	0	0	12.02%
9	9	3	0	4.30%
9	9	3	3	2.60%
9	9	6	0	0.85%
9	9	6	3	0.45%
9	9	6	6	-
9	9	9	0	-
9	9	9	3	-
9	9	9	6	-
9	9	9	9	-

x_1	x_2	x_3	x_4	*Joint PMF*
12	0	0	0	-
12	3	0	0	-
12	3	3	0	-
12	3	3	3	-
12	6	0	0	7.92%
12	6	3	0	1.70%
12	6	3	3	0.60%
12	6	6	0	-
12	6	6	3	-
12	6	6	6	-
12	9	0	0	2.88%
12	9	3	0	3.00%
12	9	3	3	2.60%
12	9	6	0	0.85%
12	9	6	3	0.45%
12	9	6	6	-
12	9	9	0	-
12	9	9	3	-
12	9	9	6	-
12	9	9	9	-
12	12	0	0	6.36%
12	12	3	0	2.13%
12	12	3	3	1.51%
12	12	6	0	-
12	12	6	3	-
12	12	6	6	-
12	12	9	0	-
12	12	9	3	-
12	12	9	6	-
12	12	9	9	-
12	12	12	0	-
12	12	12	3	-
12	12	12	6	-
12	12	12	9	-
12	12	12	12	-

As an example, consider the "Generic Equivalent Demand" profile shown above. The same profile is also shown below using the random variables X_1, X_2, X_3, and X_4. The probability for this demand for formwork is shown in the joint PMF table as $P[X_1=9, X_2=6, X_3=3, X_4=3] = 5.6\%$:



It should be noted again that there are several possible profiles for formwork demand over the 12 months of a year that could result in the figure above where:

x_1 = demand for the 1st 2,000 SF of formwork = 9 months
x_2 = demand for the 2nd 2,000 SF of formwork = 6 months
x_3 = demand for the 3rd 2,000 SF of formwork = 3 months
x_4 = demand for the 4th 2,000 SF of formwork = 3 months

Clearly, these possible profiles must be such that for the 3rd 2,000 SF of formwork to be needed in May it is also necessary to need the 1st and 2nd 2,000 SF of formwork in May. This is reflected in the structure of the table for the joint PMF, as well as in the application of the total probability theorem to calculate the marginal PMFs $p_{X_i}(x)$ shown below:

$$p_{X_1}(x_1) = \sum_{x_2=0}^{x_1} \sum_{x_3=0}^{x_2} \sum_{x_4=0}^{x_3} p_{X_1,X_2,X_3,X_4}(x_1,x_2,x_3,x_4)$$

$$p_{X_2}(x_2) = \sum_{x_1=x_2}^{12} \sum_{x_3=0}^{x_2} \sum_{x_4=0}^{x_3} p_{X_1,X_2,X_3,X_4}(x_1,x_2,x_3,x_4)$$

$$p_{X_3}(x_3) = \sum_{x_1=x_3}^{12} \sum_{x_2=x_3}^{x_1} \sum_{x_4=0}^{x_3} p_{X_1,X_2,X_3,X_4}(x_1,x_2,x_3,x_4)$$

$$p_{X_4}(x_4) = \sum_{x_1=x_4}^{12} \sum_{x_2=x_4}^{x_1} \sum_{x_3=x_4}^{x_2} p_{X_1,X_2,X_3,X_4}(x_1,x_2,x_3,x_4)$$

The resulting marginal PMFs $p_{X_i}(x)$ and expected values $E[X_i]$ for X_1, X_2, X_3, X_4 are:

$x=$	0	3	6	9	12	$E[X_i]$
$p_{X_1}(x)=P[X_1=x]=$	-	-	0.1	0.6	0.3	9.6
$p_{X_2}(x)=P[X_2=x]=$	-	0.2	0.4	0.3	0.1	6.9
$p_{X_3}(x)=P[X_3=x]=$	0.3	0.6	0.1	-	-	2.4
$p_{X_4}(x)=P[X_4=x]=$	0.7	0.3	-	-	-	0.9

For example, there is a probability of $p_{X_3}(3)=P[X_3=3]=0.6$ that the 3rd 2000 SF of formwork will be needed for exactly 3 months in a year. This is the probability of the union of 20 simple events from the joint probability mass function as follows:

$$\begin{aligned}p_{X_3}(3)&=\sum_{x_1=3}^{12}\sum_{x_2=3}^{x_1}\sum_{x_4=0}^{3}p_{X_1,X_2,X_3,X_4}(x_1,x_2,3,x_4)\\&=p(3,3,3,0)+p(3,3,3,3)+\\&+p(6,3,3,0)+p(6,3,3,3)+p(6,6,3,0)+p(6,6,3,3)+\\&+p(9,3,3,0)+p(9,3,3,3)+p(9,6,3,0)+p(9,6,3,3)+p(9,9,3,0)+p(9,9,3,3)+\\&+p(12,3,3,0)+p(12,3,3,3)+p(12,6,3,0)+p(12,6,3,3)+p(12,9,3,0)+p(12,9,3,3)\\&\qquad+p(12,12,3,0)+p(12,12,3,3)\end{aligned}$$

Hence, $p_{X_3}(3)=P[X_3=3]=0.6$ is the probability that *at least* 6,000 SF will be needed for exactly 3 months/year. It equals the probability that 6,000 SF will be needed for exactly 3 months/year and the probability that 8,000 SF will be needed for exactly 3 months/year.

The definitions for X_1, X_2, X_3, X_4 allow combinations of events in a single year. For example, the probability of needing exactly 2000 SF for 6 months and exactly 6000 SF for (a non-overlapping) 3 months is $P[X_1=9,X_2=3,X_3=3,X_4=0]=0.154$. However, because the random variables X_1, X_2, X_3, X_4 *are not independent*, the probability of such joint events does not equal the product of the marginal probabilities. For example,

$$P[X_1=9,X_2=3,X_3=3,X_4=0]=0.154\neq 0.6\times 0.2\times 0.6\times 0.7=0.0504$$

Nevertheless, even though the random variables X_1, X_2, X_3, X_4 are not independent, we can compute their expected values, and the expected value of functions that involve linear combinations of X_1, X_2, X_3, X_4 as illustrated in the decision problem that follows.

Notice that this problem requires and makes use of the marginal probability mass functions only. It does not require the detailed joint probability mass function shown in the table above.

Suppose that forms can be purchased for \$7/SF, or may be rented for \$1/SF/month. Forms that are purchased also have a maintenance cost of \$1/SF/yr. The contractor expects a return of 20% on all invested capital. How many square feet should be purchased if the forms are expected to last 10 years? (Additional forms needed each year, beyond those purchased, will be rented.)

The annual cost (i.e., each year) that is equivalent to a purchase price of \$7/SF (at $t = 0$) and to a maintenance cost of \$1/SF each year is:

$$\begin{aligned}\text{Annual cost of owned forms (\$/SF/yr)} &= \$7(A|P, 10, 20\%) + \$1 \\ &= 7(0.2385) + 1 = 1.6695 + 1 \\ &= \$2.6695/\text{SF/yr}\end{aligned}$$

The annual rental cost, R_i, for each 2000 SF (if not purchased) is given by

$$R_i = (\$1/SF/month)(2{,}000\ SF)(X_i\ months) = 2{,}000X_i$$

The expected rental cost, E[R_i], in (\$/yr) for each 2000 SF (if not purchased) is given by:

i^{th} 2000 SF	E[R_i]	$\sum_{j=i}^{4} E[R_j]$
1st 2000 SF	$E[R_1] = 2{,}000E[X_1] = 2{,}000 \times 9.6 = \$19{,}200/yr$	\$39,600/yr
2nd 2000 SF	$E[R_2] = 2{,}000E[X_2] = 2{,}000 \times 6.9 = \$13{,}800/yr$	\$20,400/yr
3rd 2000 SF	$E[R_3] = 2{,}000E[X_3] = 2{,}000 \times 2.4 = \$\ 4{,}800/yr$	\$6,600/yr
4th 2000 SF	$E[R_4] = 2{,}000E[X_4] = 2{,}000 \times 0.9 = \$\ 1{,}800/yr$	\$1,800/yr

$$E[\text{Rental} \mid \text{none purchased}] = E[R] = \sum_i E[R_i] = \$39{,}600/yr$$

If N represents the square feet of forms purchased, then the expected total annual cost is the sum of the equivalent annual cost of purchasing N square feet of forms and the expected annual rental cost for the forms not purchased (8,000-N).

$$E[\text{Total Annual Cost} \mid N] = 2.6695N + E[\text{Rental Cost} \mid N]$$

The table below shows this computation for each alternative purchase decision, N:

Forms Purchased N (SF)	Annual Ownership Cost | N 2.6695 N (\$/yr)	E[Annual Rental Cost | N] (\$/yr)	E[Total Annual Cost | N] (\$/yr)
0	0	\$39,600	39,600
2000	5,339	20,400	25,739
4000	10,678	6,600	17,278
6000	16,017	1,800	17,817
8000	21,356	0	21,356

From this table, we see that the minimum expected annual cost is \$17,278/yr which occurs when the contractor purchases N^* = 4000 SF and rents the remainder of the formwork as needed.

Symbols. These symbols are used in the following explanation of interest formulas:

- i represents an interest rate per interest period.
- n represents a number of interest periods.
- P represents a present sum of money.
- F represents a sum of money at the end of n periods from the present date that is equivalent to P with interest i.
- A represents the end-of-period payment or receipt in a uniform series continuing for the coming n periods, the entire series equivalent to P at interest rate i.

Although a one-year interest period is used in most of the illustrations in this book, the formulas presented apply to interest periods of any length.

The various symbols are chosen so that each is an initial letter of a key word associated with the most common meaning of the symbol. Thus i applies to *interest*, n applies to *number* of periods, P applies to *present* worth, F applies to *future* worth, and A applies to *annual* payment or *annuity*.

Formulas. The fundamental interest formulas that express the relationship between P, F, and A in terms of i and n are as follows.

Given P, to find F. $$F = P(1+i)^n \quad (1)$$

Given F, to find P. $$P = F\left[\frac{1}{(1+i)^n}\right] \quad (2)$$

Given F, to find A. $$A = F\left[\frac{i}{(1+i)^n - 1}\right] \quad (3)$$

Given P, to find A. $$A = P\left[\frac{i(1+i)^n}{(1+i)^n - 1}\right] \quad (4)$$

or $$A = P\left[\frac{i}{(1+i)^n - 1} + i\right] \quad (4)$$

Given A, to find F. $$F = A\left[\frac{(1+i)^n - 1}{i}\right] \quad (5)$$

Given A, to find P. $$P = A\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right] \quad (6)$$

or $$P = A\left[\frac{1}{\frac{i}{(1+i)^n - 1} + i}\right] \quad (6)$$

FORMULAS FOR CALCULATING COMPOUND INTEREST FACTORS

Single Payment—Compound Amount Factor $(F/P, i, n)$	$(1+i)^n$
Single Payment—Present Worth Factor $(P/F, i, n)$	$\frac{1}{(1+i)^n}$
Sinking Fund Factor $(A/F, i, n)$	$\frac{i}{(1+i)^n - 1}$
Capital Recovery Factor $(A/P, i, n)$	$\frac{i(1+i)^n}{(1+i)^n - 1}$
Uniform Series—Compound Amount Factor $(F/A, i, n)$	$\frac{(1+i)^n - 1}{i}$
Uniform Series—Present Worth Factor $(P/A, i, n)$	$\frac{(1+i)^n - 1}{i(1+i)^n}$

SOURCE: PRINCIPLES OF ENGINEERING ECONOMY
E.L. GRANT & W.G. IRESON
5TH EDITION, RONALD PRESS, N.Y., 1970

TABLE E–16

10% Compound Interest Factors

	Single Payment		Uniform Series				
n	Compound Amount Factor F/P	Present Worth Factor P/F	Sinking Fund Factor A/F	Capital Recovery Factor A/P	Compound Amount Factor F/A	Present Worth Factor P/A	n
1	1.1000	0.9091	1.000 00	1.100 00	1.000	0.909	1
2	1.2100	0.8264	0.476 19	0.576 19	2.100	1.736	2
3	1.3310	0.7513	0.302 11	0.402 11	3.310	2.487	3
4	1.4641	0.6830	0.215 47	0.315 47	4.641	3.170	4
5	1.6105	0.6209	0.163 80	0.263 80	6.105	3.791	5
6	1.7716	0.5645	0.129 61	0.229 61	7.716	4.355	6
7	1.9487	0.5132	0.105 41	0.205 41	9.487	4.868	7
8	2.1436	0.4665	0.087 44	0.187 44	11.436	5.335	8
9	2.3579	0.4241	0.073 64	0.173 64	13.579	5.759	9
10	2.5937	0.3855	0.062 75	0.162 75	15.937	6.144	10
11	2.8531	0.3505	0.053 96	0.153 96	18.531	6.495	11
12	3.1384	0.3186	0.046 76	0.146 76	21.384	6.814	12
13	3.4523	0.2897	0.040 78	0.140 78	24.523	7.103	13
14	3.7975	0.2633	0.035 75	0.135 75	27.975	7.367	14
15	4.1772	0.2394	0.031 47	0.131 47	31.772	7.606	15
16	4.5950	0.2176	0.027 82	0.127 82	35.950	7.824	16
17	5.0545	0.1978	0.024 66	0.124 66	40.545	8.022	17
18	5.5599	0.1799	0.021 93	0.121 93	45.599	8.201	18
19	6.1159	0.1635	0.019 55	0.119 55	51.159	8.365	19
20	6.7275	0.1486	0.017 46	0.117 46	57.275	8.514	20
21	7.4002	0.1351	0.015 62	0.115 62	64.002	8.649	21
22	8.1403	0.1228	0.014 01	0.114 01	71.403	8.772	22
23	8.9543	0.1117	0.012 57	0.112 57	79.543	8.883	23
24	9.8497	0.1015	0.011 30	0.111 30	88.497	8.985	24
25	10.8347	0.0923	0.010 17	0.110 17	98.347	9.077	25
26	11.9182	0.0839	0.009 16	0.109 16	109.182	9.161	26
27	13.1100	0.0763	0.008 26	0.108 26	121.100	9.237	27
28	14.4210	0.0693	0.007 45	0.107 45	134.210	9.307	28
29	15.8631	0.0630	0.006 73	0.106 73	148.631	9.370	29
30	17.4494	0.0573	0.006 08	0.106 08	164.494	9.427	30
31	19.1943	0.0521	0.005 50	0.105 50	181.943	9.479	31
32	21.1138	0.0474	0.004 97	0.104 97	201.138	9.526	32
33	23.2252	0.0431	0.004 50	0.104 50	222.252	9.569	33
34	25.5477	0.0391	0.004 07	0.104 07	245.477	9.609	34
35	28.1024	0.0356	0.003 69	0.103 69	271.024	9.644	35
40	45.2593	0.0221	0.002 26	0.102 26	442.593	9.779	40
45	72.8905	0.0137	0.001 39	0.101 39	718.905	9.863	45
50	117.3909	0.0085	0.000 86	0.100 86	1 163.909	9.915	50
55	189.0591	0.0053	0.000 53	0.100 53	1 880.591	9.947	55
60	304.4816	0.0033	0.000 33	0.100 33	3 034.816	9.967	60
65	490.3707	0.0020	0.000 20	0.100 20	4 893.707	9.980	65
70	789.7470	0.0013	0.000 13	0.100 13	7 887.470	9.987	70
75	1 271.8952	0.0008	0.000 08	0.100 08	12 708.954	9.992	75
80	2 048.4002	0.0005	0.000 05	0.100 05	20 474.002	9.995	80
85	3 298.9690	0.0003	0.000 03	0.100 03	32 979.690	9.997	85
90	5 313.0226	0.0002	0.000 02	0.100 02	53 120.226	9.998	90
95	8 556.6760	0.0001	0.000 01	0.100 01	85 556.760	9.999	95
100	13 780.6123	0.0001	0.000 01	0.100 01	137 796.123	9.999	100

TABLE E–17

12% Compound Interest Factors

	Single Payment		Uniform Series				
n	Compound Amount Factor F/P	Present Worth Factor P/F	Sinking Fund Factor A/F	Capital Recovery Factor A/P	Compound Amount Factor F/A	Present Worth Factor P/A	n
1	1.1200	0.8929	1.000 00	1.120 00	1.000	0.893	1
2	1.2544	0.7972	0.471 70	0.591 70	2.120	1.690	2
3	1.4049	0.7118	0.296 35	0.416 35	3.374	2.402	3
4	1.5735	0.6355	0.209 23	0.329 23	4.779	3.037	4
5	1.7623	0.5674	0.157 41	0.277 41	6.353	3.605	5
6	1.9738	0.5066	0.123 23	0.243 23	8.115	4.111	6
7	2.2107	0.4523	0.099 12	0.219 12	10.089	4.564	7
8	2.4760	0.4039	0.081 30	0.201 30	12.300	4.968	8
9	2.7731	0.3606	0.067 68	0.187 68	14.776	5.328	9
10	3.1058	0.3220	0.056 98	0.176 98	17.549	5.650	10
11	3.4785	0.2875	0.048 42	0.168 42	20.655	5.938	11
12	3.8960	0.2567	0.041 44	0.161 44	24.133	6.194	12
13	4.3635	0.2292	0.035 68	0.155 68	28.029	6.424	13
14	4.8871	0.2046	0.030 87	0.150 87	32.393	6.628	14
15	5.4736	0.1827	0.026 82	0.146 82	37.280	6.811	15
16	6.1304	0.1631	0.023 39	0.143 39	42.753	6.974	16
17	6.8660	0.1456	0.020 46	0.140 46	48.884	7.120	17
18	7.6900	0.1300	0.017 94	0.137 94	55.750	7.250	18
19	8.6128	0.1161	0.015 76	0.135 76	63.440	7.366	19
20	9.6463	0.1037	0.013 88	0.133 88	72.052	7.469	20
21	10.8038	0.0926	0.012 24	0.132 24	81.699	7.562	21
22	12.1003	0.0826	0.010 81	0.130 81	92.503	7.645	22
23	13.5523	0.0738	0.009 56	0.129 56	104.603	7.718	23
24	15.1786	0.0659	0.008 46	0.128 46	118.155	7.784	24
25	17.0001	0.0588	0.007 50	0.127 50	133.334	7.843	25
26	19.0401	0.0525	0.006 65	0.126 65	150.334	7.896	26
27	21.3249	0.0469	0.005 90	0.125 90	169.374	7.943	27
28	23.8839	0.0419	0.005 24	0.125 24	190.699	7.984	28
29	26.7499	0.0374	0.004 66	0.124 66	214.583	8.022	29
30	29.9599	0.0334	0.004 14	0.124 14	241.333	8.055	30
31	33.5551	0.0298	0.003 69	0.123 69	271.292	8.085	31
32	37.5817	0.0266	0.003 28	0.123 28	304.847	8.112	32
33	42.0915	0.0238	0.002 92	0.122 92	342.429	8.135	33
34	47.1425	0.0212	0.002 60	0.122 60	384.520	8.157	34
35	52.7996	0.0189	0.002 32	0.122 32	431.663	8.176	35
40	93.0510	0.0107	0.001 30	0.121 30	767.091	8.244	40
45	163.9876	0.0061	0.000 74	0.120 74	1 358.230	8.283	45
50	289.0022	0.0035	0.000 42	0.120 42	2 400.018	8.305	50
∞				0.120 00		8.333	∞