A SYNOPSIS OF
THEORETICAL CONSIDERATIONS
CONCERNING THE CONTINUOUS
BETA RAY SPECTRA

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by

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A SYNOPSIS OF THEORETICAL CONSIDERATIONS CONCERNING THE CONTINUOUS β-RAY SPECTRA

CHAPTER I

INTRODUCTION

a. The Problem.[1] Of the three types of radiations (α, β, and γ) which are observed coming from radioactive atoms, the β-rays have so far been the least amenable to theoretical clarification. One very fundamental reason for this is made apparent by a study of the emission energies. The α-particles and the γ-rays are known to emerge with certain discrete values of the energy only. On the other hand, only a lesser portion of the β-particles form such homogeneous groups, and these have been definitely identified as photoelectrons produced by the absorption of γ-radiation in the K, L, M and other shells of the electronic cloud outside the nucleus. The greater part of the β-particles, the true disintegration electrons, emerge with velocities differing among themselves, so that all of them together form an energy spectrum which is continuous. Examples of such spectra are shown in Figures 1, 2, and 4. after b 16 Some of the artificially radioactive substances emit positrons, in place of electrons, which are distributed among the various energies in a similar manner, as Figure 3 demonstrates.

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[1] For discussions of this problem see Gamow, Phys. ZS. 35, 533 (1934); and reports by Beck and Ellis at International Conference of Nuclear Physics, London, 1934.

The continuous character of the β-spectra seems to be in conflict with the evidence given by the α and the γ-ray energies. The discreteness of the latter testifies to the existence of quantized states in atomic nuclei. Thus, in the sequence of disintegrating atoms:

$$\text{ThA} \xrightarrow{\alpha} \text{ThB} \xrightarrow{\beta} \text{ThC} \xrightarrow{\alpha} \text{ThC''}$$

the homogeneity of the α-particles emitted by ThA means that every atom of ThB so created is left in the same discrete energy state. Other homogeneous groups of α-particles from ThA may leave ThB with other values of the energy but they will always be discrete ones out of some level scheme such as: E_0, E_1, E_2, ...E_m. The sharpness of the frequency of any γ-rays which might be emitted shows that the corresponding energy transitions are between just such levels. Likewise, the homogeneity of α-particles and γ-rays from ThC show that this nucleus too existed in some discrete energy level, of the system: E_0', E_1', E_2', ...E_m' say, at the time of the emission. Therefore the β-process between ThB and ThC must have involved a discrete energy release, $\Delta E = E_K - E_L'$. Yet, the β-particles are observed to emerge with a continuous variety of velocities.

b. <u>The Solutions.</u> The possible ways out of this difficulty seem to be limited to three. One of these is based on the hypothesis that all the disintegration electrons really leave the nucleus with equal velocities, corresponding to the maximum energy of their spectrum, but that some of them lose a part of

this energy in secondary processes occurring outside the nucleus. Another possibility is that in a β-process two particles are emitted by a nucleus at the same time. The continuous energy distribution will result from a sharing of the total available energy by the two particles in varying proportions. A final way out of the difficulty is simply to assume that the principle of the conservation of energy is not valid for some nuclear processes.

The first of the three possibilities, besides being theoretically improbable, seems to be eliminated by an experimental test made by Ellis and Wooster.[2] The heating effect produced by the disintegration of RaE was measured in a calorimeter with walls thick enough to absorb all the products of the disintegration. RaE emits very few γ-rays, so that practically the entire heating effect is due to β-decay. The secondary energy losses of the β-particles are contributed to the heating effect so that if the electrons are all emitted with the same energy the quantity of heat developed per disintegration should be equivalent to the maximum energy of the RaE spectrum. Ellis and Wooster found it to be (.35 $\pm$ .04) mev., which agrees strikingly with the average energy of the spectrum, (.39 $\pm$ .06) mev., rather than the maximum energy, which is known to be greater than 1 mev. Thus, it is quite evident that the continuous character of the β-ray spectrum has its origin in the nuclear process itself.

If β-decay is to be regarded as a process in which two particles are emitted at the same time by a nucleus, the second

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[2] Proc. Roy. Soc. A117, 109 (1927). Also Meitner and Orthmann, ZS/Phys. 60, 143 (1930).

particle must have a zero charge and a mass of the same order of magnitude as the electron mass, since it is known that the residual nucleus has the same mass number as the parent, while its charge differs by one unit. The interpretation of the continuous character of the β-spectra as implying the existence of the hypothetical second particle, called the "neutrino", was first given by Pauli. It is not surprising that a particle with zero charge and small mass should not be perceptibly absorbed by the Ellis and Wooster calorimeter so that the part of the energy borne by it escapes detection.

Apparently the only alternative to accepting the existence of the neutrino is to give up the conservation laws. So far only the conservation of energy was considered, but there exist also difficulties concerning the conservation of angular momentum. Heisenberg[3] was successful in explaining many of the properties of nuclei by the assumption that they consist only of protons and neutrons, each type of particle having a spin equal to $\hbar/2$. The consequence that nuclei of even mass number should have spins equal to an integral number times $\hbar$ while nuclei of odd mass number have spins equal to half-integral multiples of $\hbar$ seems to be confirmed by experiment. Since nuclei which emit β-particles do not experience a change in mass number, their spins cannot differ otherwise than by an integral number times $\hbar$ before and after the disintegration.

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[3] ZS/Phys. 77, 1 (1932); 78, 156 (1932); 80, 587 (1933).

The electron is known to have a spin of $\hbar/2$ so one must account for a discrepancy in the angular momentum of at least $\hbar/2$. This cannot be helped by orbital angular momenta since they are always multiples of $\hbar$. If, therefore, the laws of conservation are to hold, a neutrino devoid of charge, small in mass and possessing a spin $\hbar/2$ appears to be necessary.

Mention must be made here of the attempts of Chadwick and Lea[4] and of Nahmias[5] to detect possible ionizations by the neutrino. It was hoped that the elusive particle would have a sufficiently great magnetic moment to cause a detectable amount of ionization. However, the results were negative and Nahmias was able to conclude that the neutrino does not produce as much as one pair of ions per 300,000 km. of path in air at N.T.P. Bethe[6] computed the amount of ionization that ought to be produced by a neutrino having a magnetic moment of n Bohr magnetous and found it to be 103 n^2 ion pairs per km. of path in air at N.T.P. (a value practically independent of the mass and the energy of the neutrino). Together with the data of Nahmias, Bethe's calculations put the upper limit on the value of the neutrino's magnetic moment at 1/5000 of a Bohr magneton.

There are arguments which perhaps render unlikely the nullification of the conservation laws for β-processes. An immediate one is often considered to be the experimental fact,

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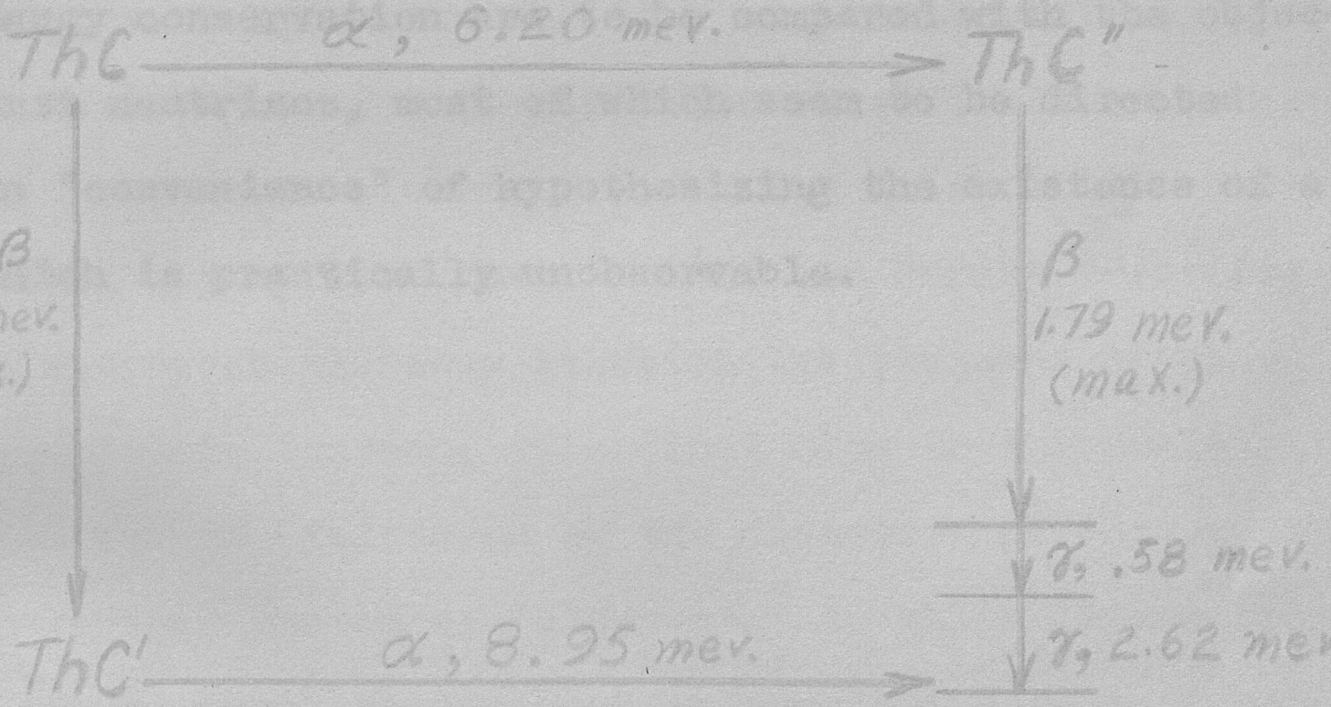
[4]Proc. Camb. Phil. Soc. 30, 59 (1934).
[5]Proc. Camb. Phil. Soc. 31, 99 (1935).
[6]Proc. Camb. Phil. Soc. 31, 105 (1935).

confirmed by Sargent, Henderson and others,[7] that there exists an upper limit for the energies of the β-rays. This suggests that the amount of energy available is strictly limited, a fact which seems hardly in keeping with nonconservation. A more concrete argument was offered by Ellis and Mott.[8] They pointed out that the energy released in both of the alternative chains of reactions by which ThC is transformed into ThD is the same, if it is assumed that the energy released in a β-process is equal to the energy of the fastest β-particles. The chain of reactions in question can be represented by:



ThC transforms into ThC" with the emission of α-particles having an energy of 6.20 mev. and the upper limit of the ThC".D β-spectrum is 1.79 mev., according to careful measurements

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[7] Sargent, Proc. Roy. Soc. A139, 659 (1933); Henderson, Proc. Roy. Soc. A 147, 572 (1934).

[8] International Conference on Nuclear Physics, London, 1934.

of Henderson.[9] For every electron emitted by ThC" there appears a γ-ray of energy .58 mev., and one of 2.62 mev., indicating that the β-emission leaves ThD in an excited state of .58+2.62 =3.20 mev. Accordingly, the energy released along the lower branch is 6.20+1.79+3.20=11.19 mev. The ThC.C' β-spectrum has an upper limit of 2.25 mev. and ThC' emits 8.95 mev. α-particles, making it a total energy release of 11.20 mev. along the upper branch, in good agreement with the value for the lower branch. This means that energy conservation must be valid for at least the fastest β-particles. All these objections against the denial of energy conservation are to be compared with the objections against neutrinos, most of which seem to be directed against the "convenience" of hypothesizing the existence of a particle which is practically unobservable.

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[9] Proc. Camb. Phil. Soc. 31, 285 (1935).

CHAPTER II

THE SOURCE OF THE β-ELECTRONS

The emission of electrons by nuclei has at one time been regarded as evidence that electrons exist within the nucleus. This point of view is fraught with difficulties. If an electron is confined within the dimensions which nuclei are known to have, the uncertainty principle predicts for it a kinetic energy great enough to make it much more considerably felt in the atomic weight than it actually seems to be. Furthermore, it has been very difficult to imagine a model of a nucleus containing protons, electrons and perhaps α-particles which could be stable. A first indication that electrons might not exist inside nuclei after all was given by the Gamow, Condon and Gurney[10] development of the theory of α-decay in which the presence of electrons was completely ignored. The final blow to the nuclear electron idea was administered by the discovery of the neutron and the subsequent theory of Heisenberg[3] in which it was proposed that the nucleus be regarded as consisting only of neutrons and protons. Now there arises the question: how can a nucleus which contains no electrons emit β-particles?

a. The Theory of Beck and Sitte.[11] One answer is given by the Beck and Sitte theory of β-decay, according to which the

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[10] Phys. Rev. 33, 127 (1929).

[11] Beck, ZS/Phys. 83, 496 (1933); 89, 811 (1933); Sitte, Phys. ZS. 34, 627 (1933); Beck and Sitte, ZS/Phys. 86, 105 (1933); Nature, 133, 722 (1934).

electrons arise from the well-established process of pair formation. The disintegration energy $\triangle E$ is converted into an electron-positron pair with kinetic energies W and W' related by:

$$\triangle E = E + E' + 2mc^2 \qquad (1)$$

In natural β-radioactivity, only the electron is observed and moreover the charge of the nucleus is raised by a unit. Therefore the positron must be absorbed by the nucleus in some manner not encompassed by present day knowledge. However, only the charge property of the positron (possibly also its mass) can be absorbed by the nucleus if it is to retain the uniqueness of energy and the spin character discussed in I. The energy of the positron, $E'+mc^2$ (perhaps only E'), as well as its spin, must somehow be lost. Accordingly, an essential point in the Beck and Sitte theory is that the conservation laws must be abandoned, at least for the mysterious positron absorption process. It might be argued that this shift in the responsibility for the lack of conservation from the β-process as a whole to the positron absorption permits the existence of an upper energy limit in the β-spectrum to which the conservation principle applies. Thus, the most concrete objection (the Thorium cycle argument) against the nullification of the conservation laws is avoided.

The extension of the theory to positron radioactivity uncovers an immediate weakness in it. Here, the <u>negative</u> charge of the electron-positron pair must be absorbed by the nucleus. This leads to the question of how it is determined whether the

electron or the positron member of the pair is emitted by a given element (perhaps also why sometimes one and sometimes the other are not emitted by the same nucleus, or why not both at the same time, occasionally). The answer of the theory seems to be a part of the uncertainty surrounding the charge absorption process. This weakness does not seem so glaring until one learns of the simple and natural way the question is answered by another theory, to be presented directly, which was constructed by Fermi in harmony with the Heisenberg model of the nucleus.

Although one should perhaps not inquire too closely into the matter at this stage, it is difficult to imagine how the conditions on the charge absorption demanded by the Beck and Sitte theory could be fulfilled. First of all, the heavy nuclei with their high positive charges are asked to absorb positrons more readily than the light nuclei, since positron emission is only found in the latter. Secondly, when the charged particle is admitted into the nucleus all the difficulties connected with the existence of electrons in nuclei are revived. Finally, there is required an odd divorcement of the charge property of the absorbed particle from its spin and its kinetic energy, perhaps even its mass. The only explanation of these peculiarities which the authors of the theory attempt to make is that the absorption of the positron (or electron) by the nucleus should not be taken too literally; that the process of pair formation was introduced only formally, in order to enable calculations to be made. However, it is doubtful whether

such calculations can lead to satisfactory results. Since the masses of the positron and the electron are equal, the distribution of the available energy between them must be symmetrical if the effect of the nuclear field is neglected. The influence of the positive field would be to slow down the electrons and accelerate the positrons. Accordingly, the electron spectra should have an amount of asymmetry toward low energies depending on the magnitude of the charge possessed by the emitting nucleus, while in the positron spectra there should be a preponderance of particles with high energies. Actually, it is found (see Figures 1, 2 and 3) that both the electron and the positron spectra, whether due to light or to heavy nuclei have about the same proportion of asymmetry toward low energies, making it rather unlikely that the nuclear field alone can be responsible for the type of asymmetry observed. It seems necessary that, if the continuous character of the β-ray spectrum is to be attributed to the emission of two particles, they must differ otherwise than merely in charge.

b. The Fermi Theory.[12] Another means by which a nucleus devoid of electrons could be provided with them for purposes of β-emission was evolved by Fermi, using the Heisenberg nuclear model as a starting point. The assumption that nuclei are made up of neutrons and protons only, in addition to avoiding the difficulties connected with nuclear electrons, led to

[12] ZS/Phys. 88, 161 (1934).

the satisfactory explanation of nuclear spins mentioned in Chapter I, section b, and also made possible a plausible picture of nuclear stability. Thus, the fact that the stable light nuclei have atomic numbers Z almost always equal to just one half their mass numbers M (i.e. (M - Z)= Z, where Z is the number of protons and (M - Z) the number of neutrons) is explained by assuming that the major forces keeping the nucleus together are short range interactions between neutrons and protons so that there is a maximum number of N - P bindings. The coulombian repulsion between the protons accounts for the increase in the ratio (M - Z)/Z for the heavier stable nuclei.

The value of (M - Z)/Z for the various nuclei is closely related to their stability toward β-radioactivity. All the electron emitting elements have larger (M -Z)/Z ratios than their neighbors in the periodic table; by emitting an electron they transform themselves into elements with one more proton and one less neutron, thus reducing their (M -Z)/Z values. The close connection between the transformation from neutron to proton and the electron emission caused Heisenberg to picture the neutron as consisting of a proton and an electron bound together in some unprecedentedly close manner. This picture, in turn, allowed him to treat the neutron-proton forces as interactions of the exchange type, such as are known to exist between the H and H^+ atoms of the once-ionized hydrogen molecule. Together with a slight modification by Majorana and with the use of the Pauli exclusion principle, the exchange forces of Heisenberg led to satisfactory reasons for

the comparatively great stability of the α-particle, for the observed increase of the nuclear radius with the mass number, and for the constant binding energy per particle (a typical saturation effect of the nuclear forces).

Fermi prefers to speak of the neutron and proton as alternative states of the heavy particle. A transition from the neutron into the proton stage involves the emission of an electron just as a transition between electronic states involves the emission of a photon. The electron is no more regarded as existing before a transition than is the light quantum. Putting the matter into these terms facilitates its extension to positron emission and the inclusion of the neutrino. Thus, the heavy particle undergoes a transition from the proton to the neutron state by absorbing an electron (thereby putting it out of existence) from one of the filled negative energy states of Dirac, the vacancy left in the negative spectrum being observed as a positron. The question as to whether a P→N or an N→P transition will take place in a given nucleus depends on the stability of its neutron to proton ratio; an excess of neutrons causes an electron emission, as has already been pointed out, and it is observed that the (M - Z)/Z ratio of the positron emitters is abnormally small. For example, Al^{28}_{13} emits electrons because the element into which it transforms, Si^{28}_{14}, has a greater number of N - P pairs. N^{13}_{7} emits positrons because the C^{13}_{6} nucleus is less subject to disruption by the interprotonic coulomb forces and so is more stable. Electron emitters are more

likely to be found among the naturally radioactive substances because the unstable nuclei occurring among these are due to α-decay. The latter process reduces the numbers of neutrons and protons equally, and since lighter elements are stable with smaller (M - Z)/Z ratios, the product nucleus is likely to have an excess of neutrons.[13]

Fermi requires a neutrino to accompany the emission of each β-particle so that the conservation laws can be kept. The fact that the existence of the neutrino is not required until after the transition perhaps makes this hypothesis more attractive. The neutrino is assumed to satisfy the only possible relativistically invariant wave equation[14] for a particle, which is similar to the Dirac equation in that it requires the existence of negative energy states for the neutrino. Accordingly, there is also the possibility of an "antineutrino" and the β-reactions can be written:

$$N \longrightarrow P + e^- + n \qquad (3)$$

$$P \longrightarrow N + e^+ + n' \qquad (4)$$

where either n or n' may be a neutrino or an antineutrino.[15] This point will be discussed in more detail later.

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[13] This argument is due to Bethe and Peierls, Nature, 133, 532 (1934).

[14] Carlsson and Oppenheimer, Phys. Rev. 41, 763 (1932).

[15] Wolfe and Uhlenbeck, Phys. Rev. 46, 237 (1934).

CHAPTER III

THE SPECTRA

a. The Statistical Factor. Some idea of the way in which the particles emitted in reactions (3) and (4) will divide the energy among themselves may be obtained from the following considerations. The heavy particle will absorb any amount of the momentum consistent with the conservation laws without acquiring an appreciable part of the energy so that impulse conservation and the directions of the momenta may be left out of further consideration. Assume now that the mechanism of the reaction does not cause either the β-particle or the neutrino to prefer any particular velocity. This means that when the β-particle is flung into its momentum space, the probability of its arriving in a given cell corresponding to a momentum between η and $\eta + d\eta$ is merely proportional to the size of the cell: $4\pi\eta^2 d\eta$. The probability that the β-particle have such a momentum and at the same time the neutrino have a momentum between p and $p + dp$ is given by:

$$P d\eta dp \sim \eta^2 d\eta \cdot p^2 dp, \qquad (5)$$

on which there is the restriction that the energy must be conserved,

$$W_0 = K + W \qquad (6)$$

where W_0 is the disintegration energy, W is the total energy

(including the rest energy) of the electron and K is the total energy of the neutrino. Expressing energy in units mc^2, impulse in units mc and mass in units m, $W = \sqrt{p^2 + 1}$ and $K = \sqrt{p^2 + \mu^2}$, μ being the ratio of the neutrino mass to the electron mass.[16] The impulse of the unobserved neutrino is eliminated from (5) by the use of (6), the result being:

$$P(p)dp \sim p^2 (W_0 - W)\sqrt{(W_0 - W)^2 - \mu^2}\, dp \qquad (7)$$

The probability that a β-particle be emitted with an energy between W and W + dW is:

$$P(W)dW \sim W\sqrt{W^2 - 1}\,(W_0 - W)\sqrt{(W_0 - W)^2 - \mu^2}\, dW \qquad (8)$$

which is zero at W = 1, rises to a maximum and finally drops to zero again at $W = W_0 - \mu$. If the neutrino mass were equal to the electron mass, i.e. $\mu = 1$, (8) would yield a distribution curve symmetrical about $W = W_0/2$; one should expect each value of the energy to be equally probable for particles of equal mass, since no other properties have so far been taken into account. For $\mu > 1$, the curve would have its maximum at greater values of W; for $\mu < 1$, the asymmetry is toward lower values of W, an extreme of this asymmetry being reached for $\mu = 0$. This extreme is compared with the observed energy distributions in Figures 1, 2, 3 and 4, from which it is seen that even for $\mu = 0$,

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[16] The units in use here, and in the following, are the rational relativistic units. The unit of length is $\hbar/mc$; of time, $\hbar/mc^2$; of mass, m. The usual expressions are put into these units by setting m, $\hbar$ and c equal to unity and e^2 to the fine structure constant, $\alpha = e^2/\hbar c$.

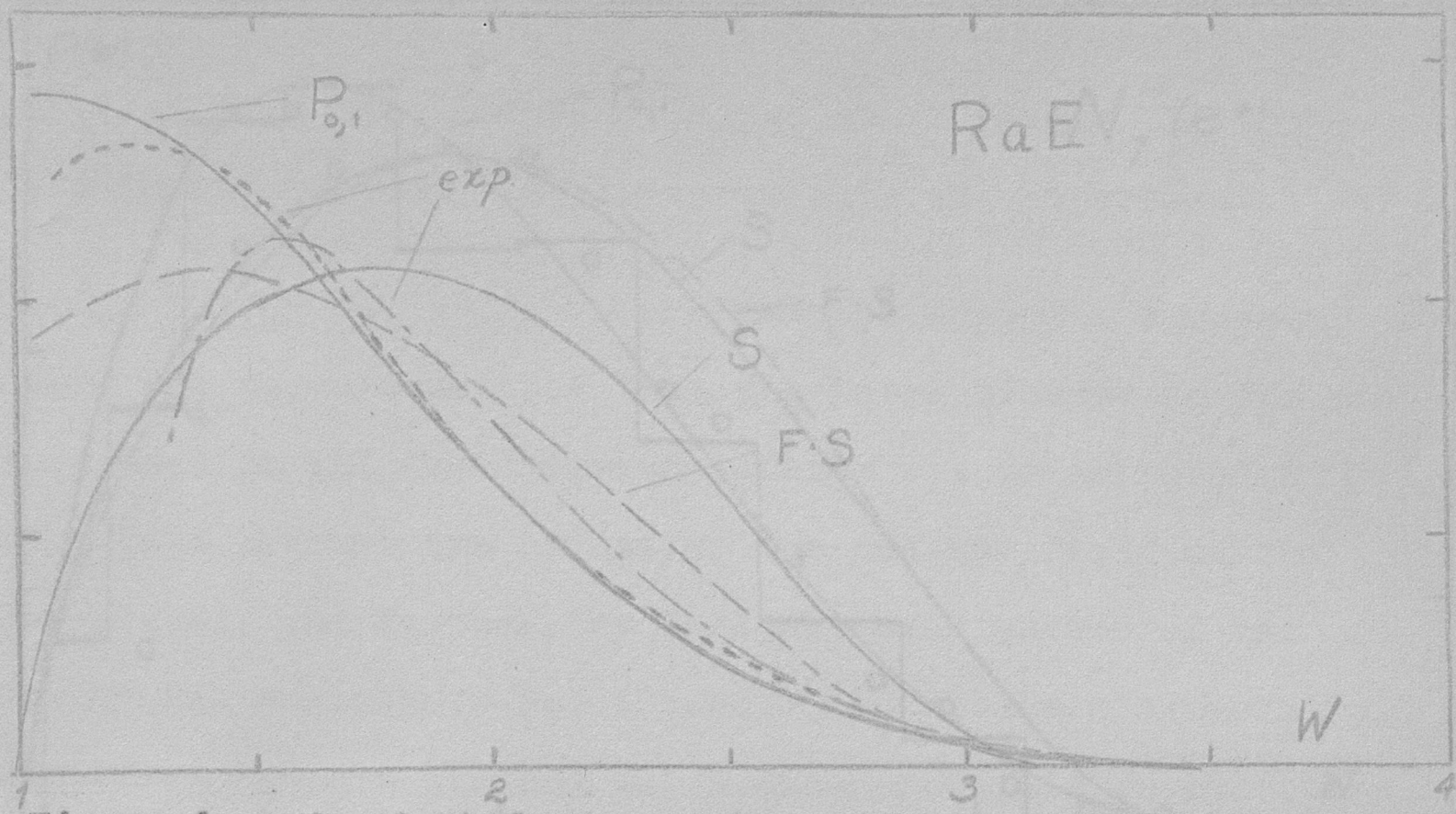

Figure 1. The dotted curve represents the experimental results of Alichanian, Alichanow, and Dzelepow which agree with those of Richardson. The dot-dash curve represents the older results. The areas under the theoretical curves are equal. For $P_{0,1}$, $W_0 = 3.7$; for S, $W_0 = 3.2$.

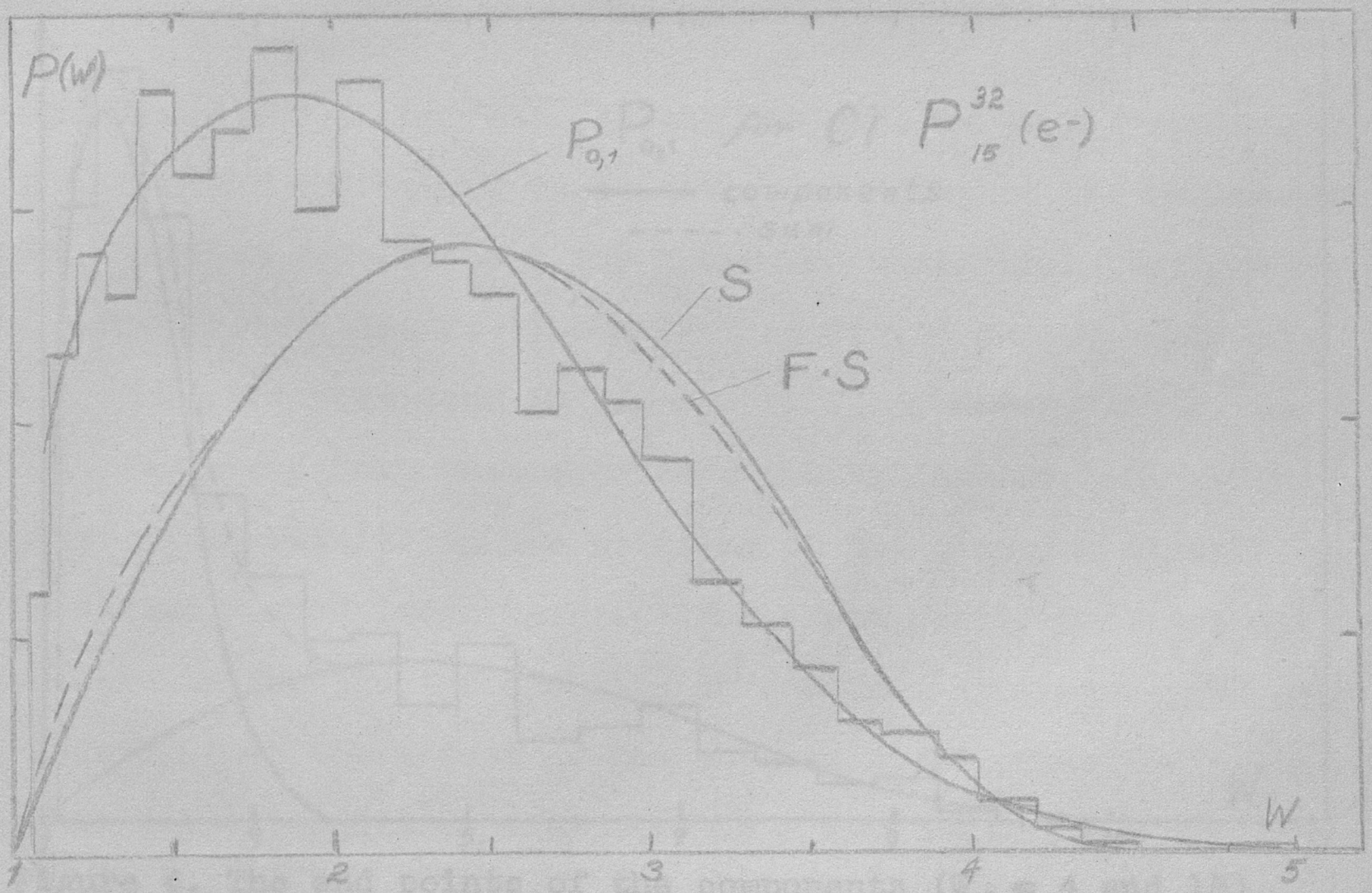

Figure 2. The experimental histogram is from the measurements of Kurie, Richardson and Paxton.

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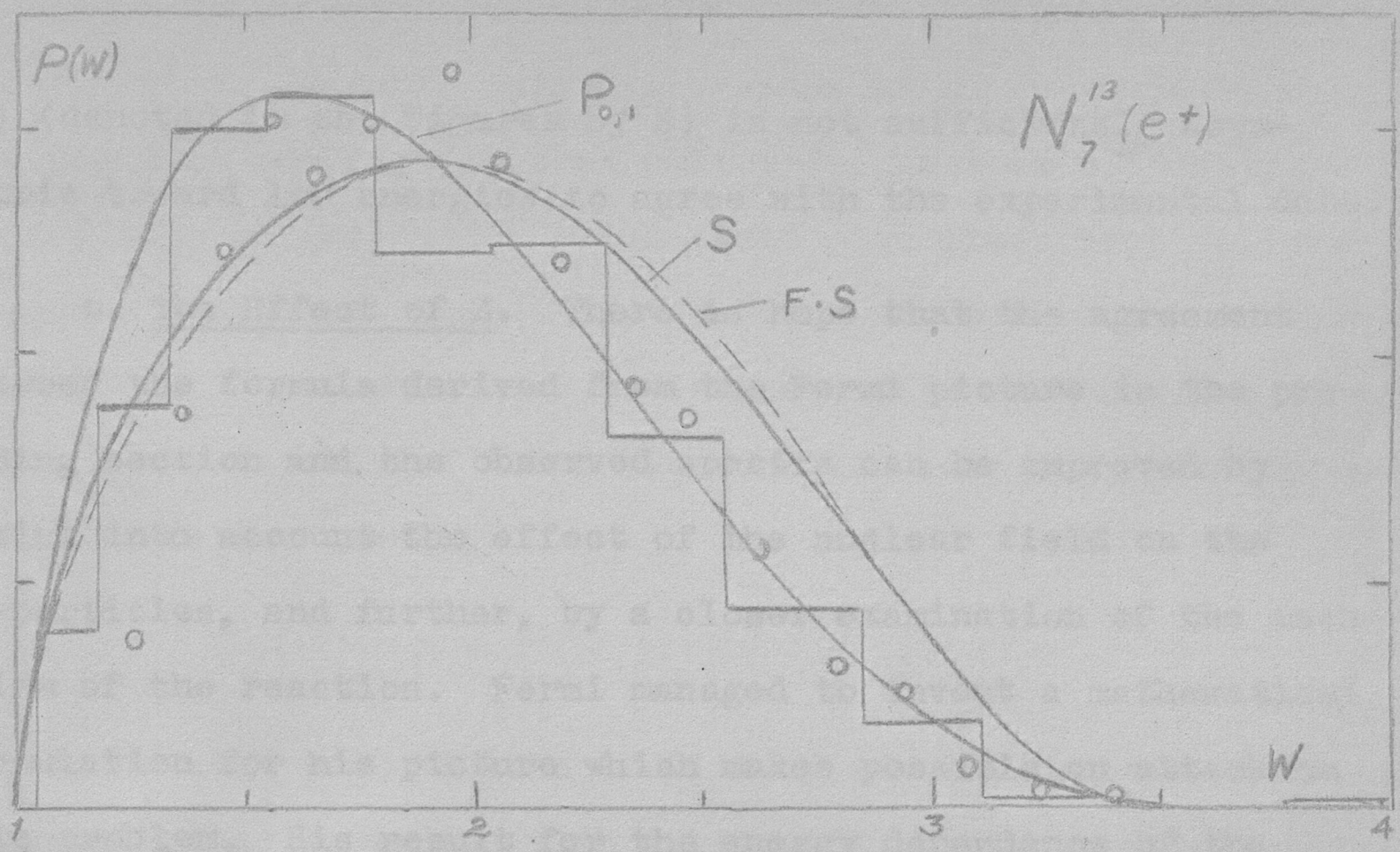

Figure 3. The experimental results of Kurie, Richardson and Paxton are given by the histogram and those of Fowler, Delsasso and Lauritsen by the circles (adjusted to equal areas). For $P_{0,1}$, $W_0 = 4$; for S, $W_0 = 3.5$.

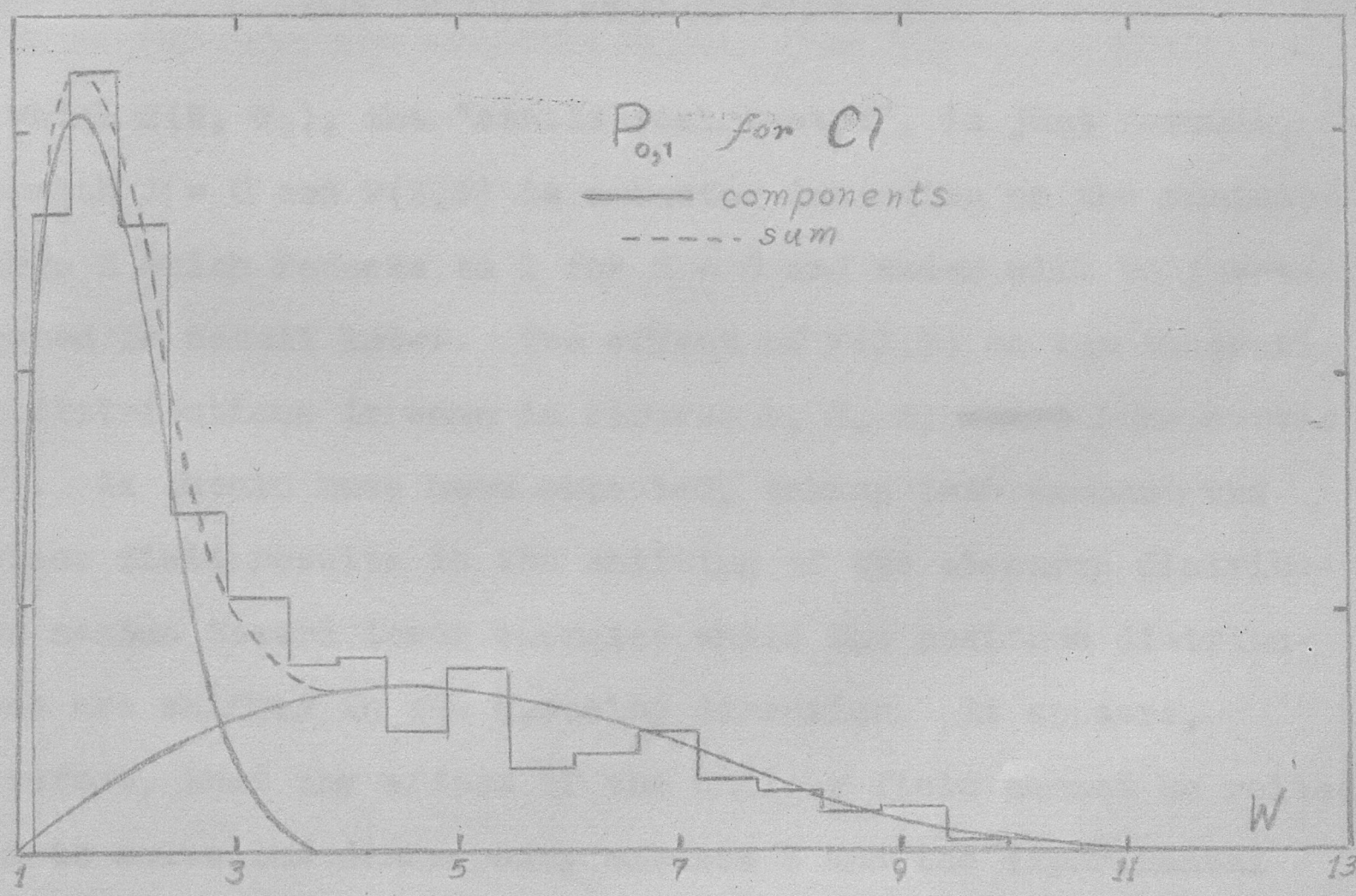

Figure 4. The end points of the components ($W_0 = 4$ and 15) were obtained from Figure 8. The histogram is from Kurie, Richardson and Paxton.

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(8) (denoted in the figures by S) is not sufficiently asymmetric toward low energies to agree with the experimental data.

b. The Effect of Z. There is hope that the agreement between the formula derived from the Fermi picture in the preceding section and the observed spectra can be improved by taking into account the effect of the nuclear field on the β-particles, and further, by a closer examination of the mechanism of the reaction. Fermi managed to invent a mathematical formulation for his picture which makes possible an attack on this problem. His result for the energy dependence of the number of β-particles may be represented by:[17]

$$P_{o,o}\,dW \sim F(Z,W)\cdot S(W,W_o) \qquad (9)$$

in which $S(W, W_o)$, the "statistical factor", is just formula (8) with $\mu = 0$ and $F(Z,W)$ is a factor depending on the nuclear charge Z which reduces to 1 for $Z = 0$ and which will be investigated in detail later. The effect of $F(Z,W)$ on the theoretical distributions is seen in Figures 1, 2, 3, ~~and 4~~ (the curves F·S). As should have been expected, taking into account the nuclear field results in the shifting of the electron distribution maxima toward lower energies while the positron distributions are shifted in the opposite direction. It appears, therefore, that the effect of the nuclear field cannot be relied upon to solve the discrepancy between S and the experimental

[17] The meaning of the indices 0,0 will become clear later.

distributions, because this discrepancy is of the same type for positron and for electron emitters. What is evidently needed is a weight factor which would give preference to lower electron energies. Such a one would be a power of the neutrino energy, $(W_o - W)$, which can be introduced as an empirical finding. It is capable of being incorporated into the Fermi formulation with the result:

$$P_{o,1}(W)\,dW \sim F(Z,W)(W_o-W)^2 S(W,W_o) \qquad (10)$$

As can be seen from Figures 1, 2, 3, 4 and 13, Formula (10) gives a better agreement with experiment than any of the other theoretical distributions.

It will be noted that the values of W_o required to fit (10) to the data are somewhat higher than the upper limits one would assign by inspection. This is due to the high order of contact of (10) with the energy axis, exceedingly few electrons being predicted beyond the experimental upper limit. An indication that the high theoretical W_o represents the true energy of disintegration is given by the measurements of Bonner, Delsasso, Fowler and Lauritsen[18] and of Cockroft and Lewis.[19] The reactions:

$$C^{12} + H^2 = C^{13} + H^1 \quad (+2.65 \text{ mev.})$$
$$C^{12} + H^2 = N^{13} + m^1 \quad (-0.37 \text{ mev.})$$
$$N^{13} = C^{13} + e^+ \quad (+W_o)$$

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[18] Berkeley Meeting, Am. Phys. Soc., Phys. Rev. 49, 203 (1936).
[19] Proc. Roy. Soc. A 154, 261 (1936).

can be used to relate W_o with the mass difference $(H' - n')$ as follows:

$$W_o = (H^1 - n') + 1.93 \text{ mev.} = 1.45 \text{ mev.},$$

using Feather's mass of the neutron. This is in better agreement with the theoretical end point (1.48 mev.) than the one obtained by inspection ($\sim$ 1.25 mev.). However, the current uncertainty about the true mass of the neutron makes this agreement questionable. These conclusions are also based on the assumption that the mass of the neutrino is negligible, for which there are also other grounds (see next section).

The greatest experimental difficulty in the measurement of β-spectra seems to be the scattering of the electrons, which uniformly results in too few slow particles being counted.[20] In almost every case the only systematic deviations of distribution (10) from experiment is at the low energy ends of the spectra, as is most clearly shown in Figures 5, 6, 7, and 8, in which the criterion for agreement with theory is that the experimental points lie on a straight line. From (10), and (7) with $\mu = 0$,

$$\sqrt[\epsilon]{\frac{P(\eta)}{\eta^2 F(Z,W)}} = W - W_o ,$$

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[20] The serious discrepancy in this respect between the theory and the older measurements of the RaE spectrum led Dr. M. E. Rose, of the Institute for Advanced Study at Princeton, to examine the effect of screening on the β-ray distributions. He found that it lowers the theoretical curves, not over 5% at the low energy end, for the heaviest β-emitters. I wish to take this opportunity to thank Dr. Rose for communications regarding these results, which are to appear in a forthcoming issue of the Physical Review.

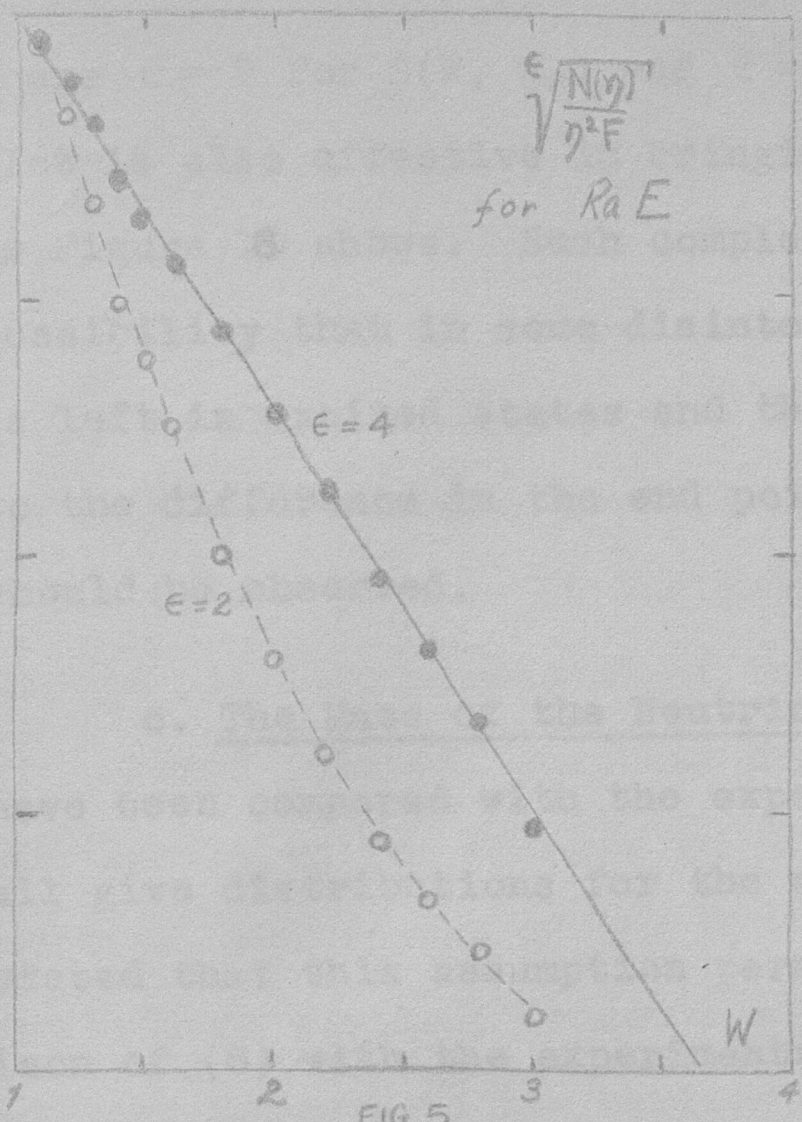

FIG. 5

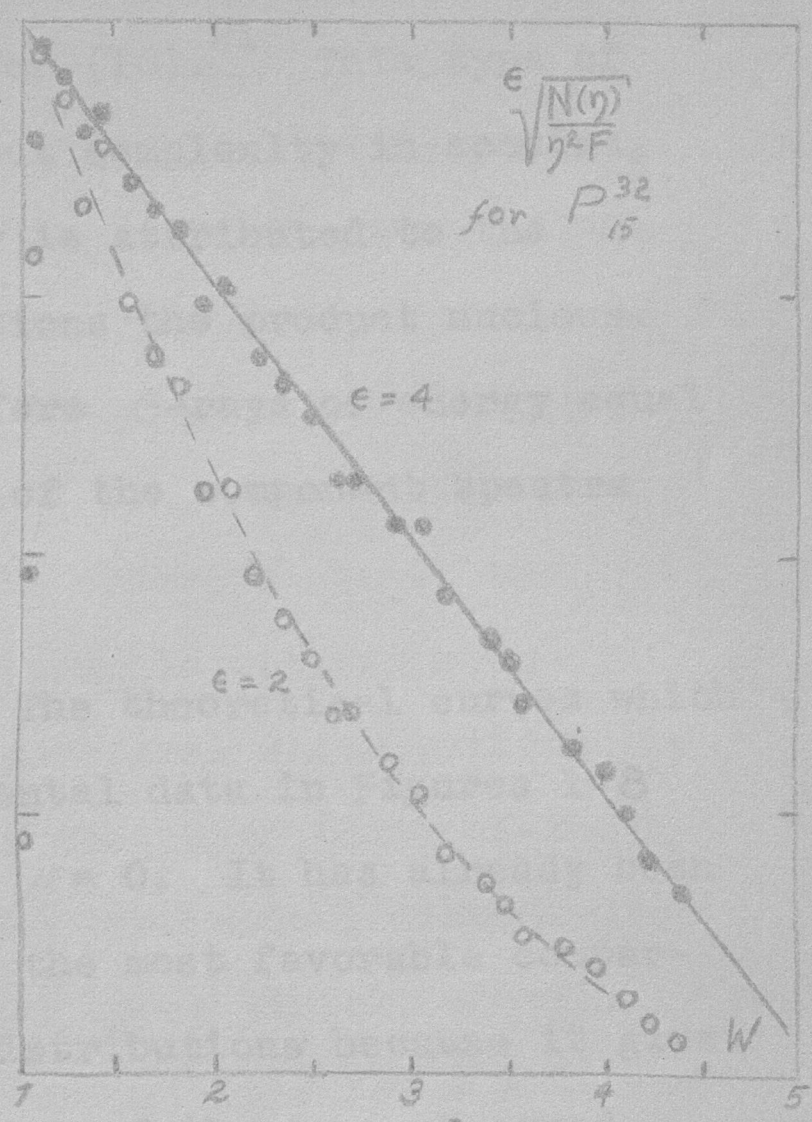

FIG. 6

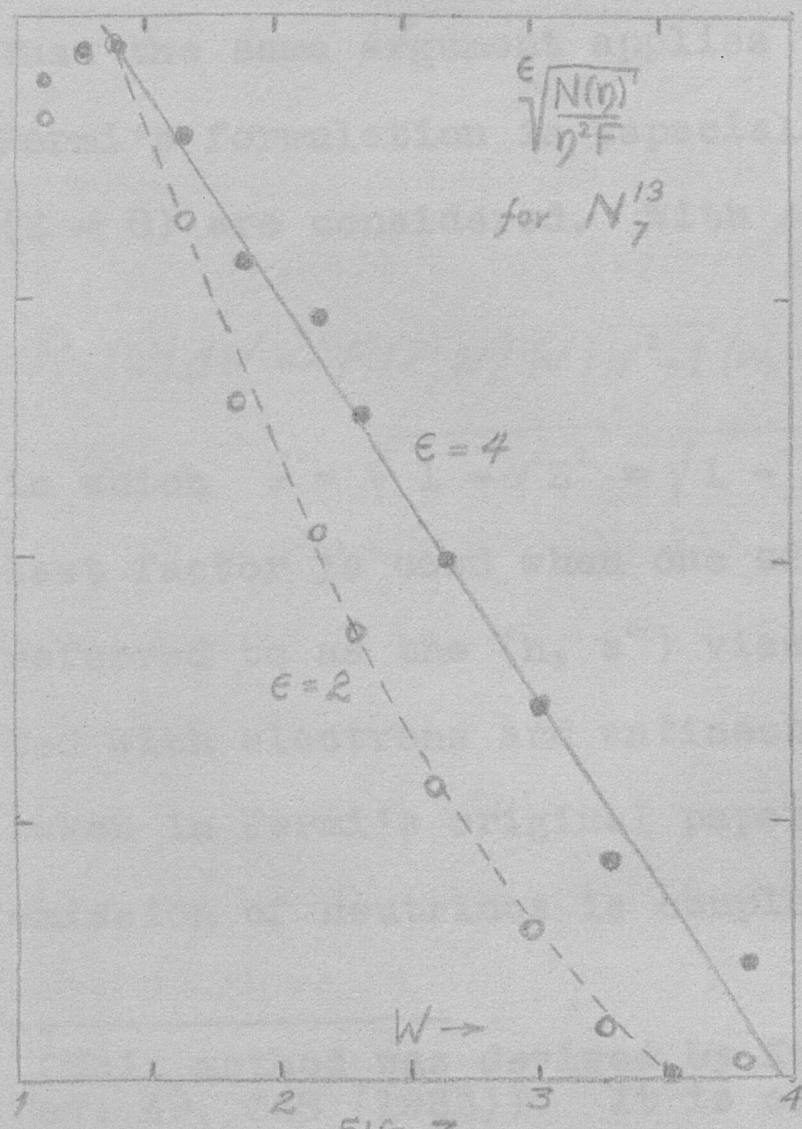

FIG. 7

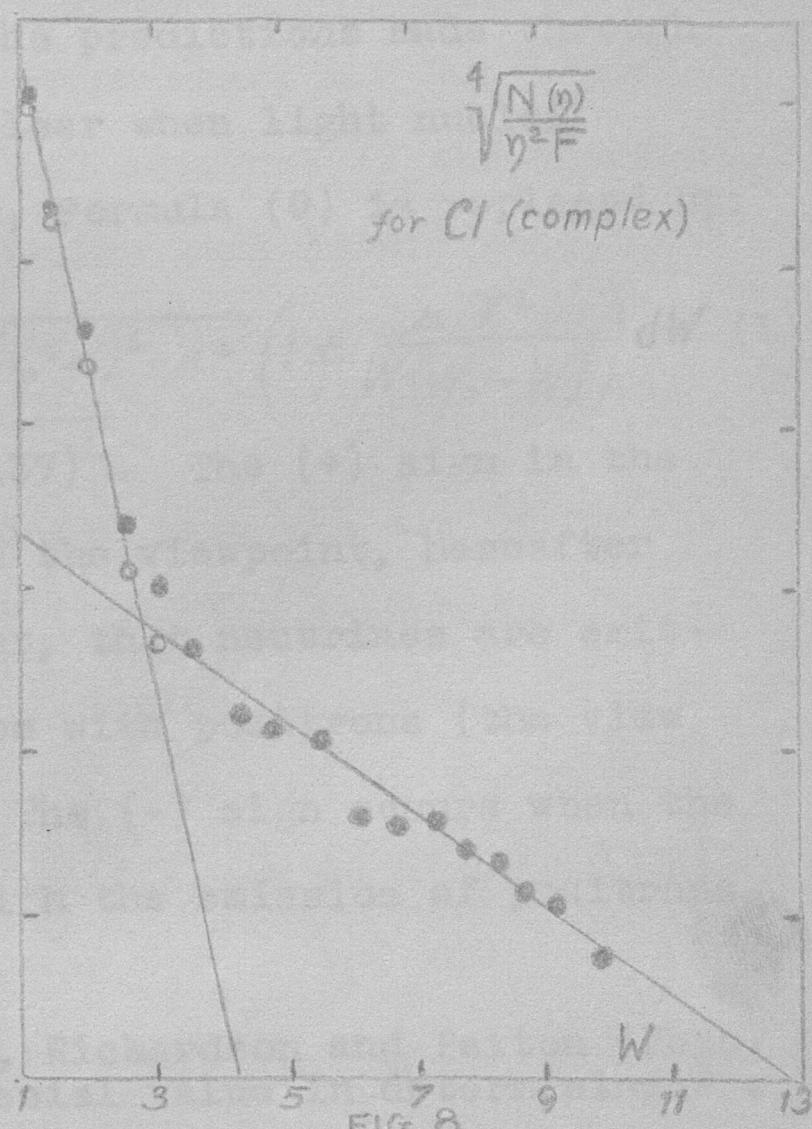

FIG. 8

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where $\epsilon = 2$ for $S(W, W_o)$ and $\epsilon = 4$ for (10).[21] This type of plot is also effective in bringing out complexity in spectra, as Figure 8 shows. Such complexity is attributed to the possibility that in some disintegrations the product nucleus is left in excited states and therefore β-rays of energy equal to the difference in the end points of the component spectra should be observed.

c. The Mass of the Neutrino. The theoretical curves which have been compared with the experimental data in Figures 1-8 all give distributions for the case $\mu = 0$. It has already been stated that this assumption permits the most favorable comparison of (8) with the experimental distributions because it gives to (8) the maximum amount of asymmetry of the type observed. That the same argument applies to the predictions made through Fermi's formulation is especially clear when light nuclei ($Z = 0$) are considered. With $\mu \neq 0$, Formula (9) is replaced by:

$$P_{o,o}(W)\,dW \sim F(Z,W)\,W\sqrt{W^2-1}\,(W_o-W)\sqrt{(W_o-W)^2-\mu^2}\left(1 \pm \frac{\mu\gamma^2}{W(W_o-W)}\right)dW \qquad (11)$$

in which $\gamma = \sqrt{1-\alpha^2Z^2} = \sqrt{1-(Z/137)^2}$. The (+) sign in the last factor is used when one adopts the viewpoint, hereafter referred to as the (n, e^-) viewpoint, that neutrinos are emitted with electrons and antineutrinos with positrons (the view taken in Fermi's original paper). The (-) sign occurs when the emission of neutrinos is coupled with the emission of positrons

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[21] This method was devised by Kurie, Richardson and Paxton (Phys. Rev. 49, 368 (1936)). It is of special value in determining W_o.

and of antineutrinos with electrons (the (n, e^+) viewpoint). In either case, the factor $(1 \pm \mu\gamma^2/W(W_o - W)$ has very little effect on the distributions even for the low energy spectra of the light elements (see Figure 9). Since equation (11) does not differ essentially from (8) for Z = 0, μ must be made zero in the former just as in the latter in order to obtain the closest possible agreement of these formulas with experiment.

Fermi pointed out that the theoretical distributions are most sensitive to changes in μ near the upper limit of the spectra. For values of $\mu \neq 0$, all the theoretical distribution curves have vertical tangents at their upper limits, whereas they are tangent to the horizontal (energy) axis for $\mu = 0$. As a result, (11) yields easily distinguishable types of behavior for $\mu = 0$ and μ as great as the electron mass, m. Henderson[22] made a careful investigation of the end portions of the ThC and the ThC" spectra which favored a neutrino mass considerably smaller than the electron mass, on the assumption again that (11) can represent the distributions.

When weight factors are introduced into the distribution formulae, as was done in the case of (10), the arguments of the preceding paragraphs for $\mu = 0$ are practically voided. With $\mu \neq 0$, Formula (10) is exactly (11) multiplied by the factor $(W_o - W)^2$. This new factor makes it difficult to distinguish the behaviors of the theoretical curves for $\mu = 0$ and $\mu = m$

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[22] Proc. Camb. Phil. Soc. 31, 285 (1935).

near their upper limits. For high energy spectra (such as those of B^{12} and Li^{8}, for which $W_o \sim 12$ mev.) the entire distribution given by P_{01}, with $\mu = m$ is found to nearly coincide with $P_{0,1}$ having $\mu = 0$. However, the two cases perhaps give sufficiently different forms for the low energy spectra to allow a decision to be made as to whether $\mu = m$ or not. The comparison of these for P^{32} and RaE in Figures 9 and 10 seems to favor $\mu < m$.

It may also be mentioned that Fowler, Delasasso and Lauritsen[18] have expressed the opinion that their data on the reactions given on page 18 are sufficiently accurate to indicate that the mass of the neutrino is at most about half as great as the mass of the electron.

There is one argument for a neutrino mass greater than zero, advanced by Bethe[23] to explain the occurrence of stable isobaric[24] pairs of nuclei differing by one in atomic number. Such pairs differ superficially only in that a neutron in one member takes the place of a proton in the other. It should be possible, therefore, for one to transform into the other with the emission of an electron or a positron, provided that the necessary energy is available. Accordingly, the criterion for the stability of such a pair is that its members must not have a mass difference, ΔM, of more than $(m + \mu)$. It is felt that

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[23] In a private discussion. See also Nature 133, 532 (1934).

[24] In the sense used here, two nuclei are isobars if their mass numbers are equal.

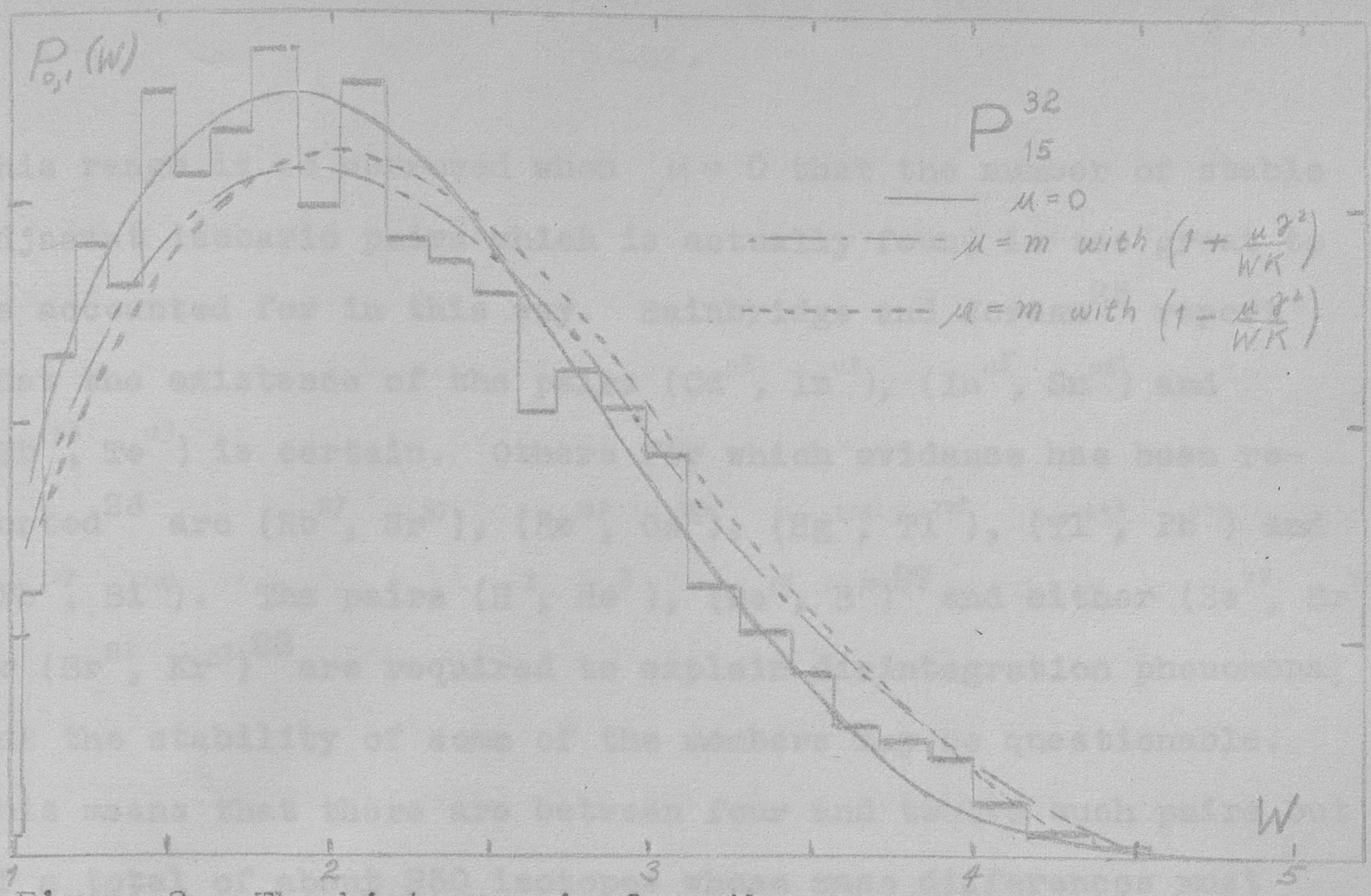

Figure 9. The histogram is from the measurements of Kurie, Richardson and Paxton.

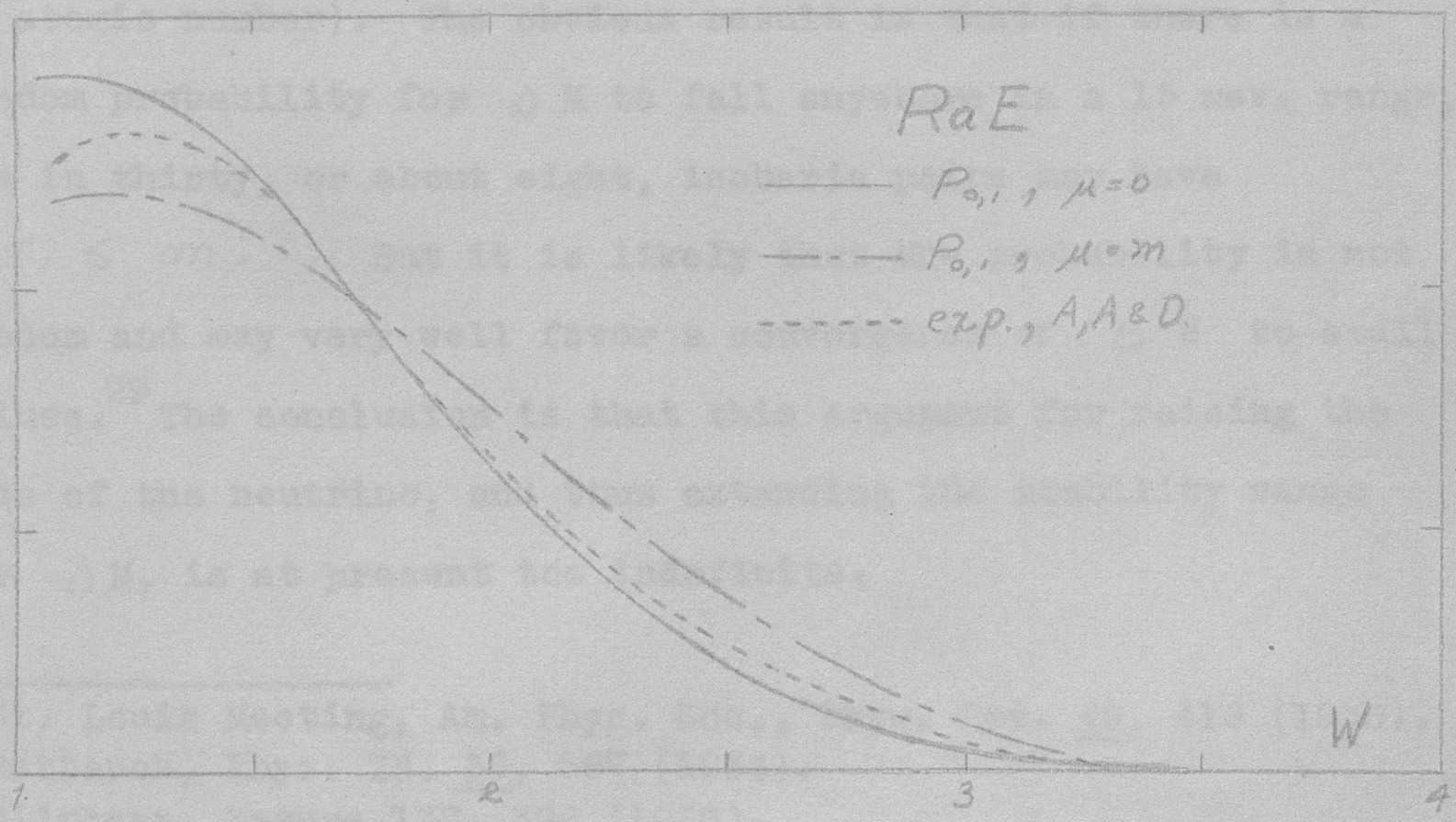

Figure 10. The experimental curve was given by Alichanow, Alichanian and Dzelepow.

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this range is so narrowed when $\mu = 0$ that the number of stable adjacent isobaric pairs which is actually found is too great to be accounted for in this way. Bainbridge and Jordan[25] report that the existence of the pairs (Cd^{113}, In^{113}), (In^{115}, Sn^{115}) and (Sb^{123}, Te^{123}) is certain. Others for which evidence has been reported[26] are (Rb^{87}, Sr^{87}), (Re^{187}, Os^{187}), (Hg^{203}, Tl^{203}), (Tl^{203}, Pb^{203}) and (Pb^{209}, Bi^{209}). The pairs (H^3, He^3), (Be^{10}, B^{10})[27] and either (Se^{77}, Br^{77}) or (Br^{83}, Kr^{83})[28] are required to explain disintegration phenomena, but the stability of some of the members may be questionable. This means that there are between four and twelve such pairs out of a total of about 250 isotopes whose mass differences must fall inside a range of .5 mev. when a $\triangle$ M as great as 15 mev. is observed in isobars of a more frequent type (differing by 2 in atomic number). The obvious result is that if there is a random probability for $\triangle$ M to fall anywhere in a 15 mev. range, one in thirty, or about eight, isobaric pairs may have $|\Delta M| \leq m$. But it is likely that the probability is not random and may very well favor a convergence of $\triangle$ M to small values.[29] The conclusion is that this argument for raising the mass of the neutrino, and thus extending the stability range for $\triangle$ M, is at present too indefinite.

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[25] St. Louis Meeting, Am. Phys. Soc., Phys. Rev. 49, 416 (1936).
[26] Mattauch, Phys. ZS. 35, 567 (1934).
[27] Oliphant, Nature 137, 396 (1936).
[28] Kurtchatov, C.R. 200, 1201 (1935).
[29] Curiously enough, in the only cases of those mentioned above in which the masses are known for both members of the pair, (H^3, He^3) and (Be^{10}, B^{10}), the mass difference is lately reported[27] to be nearly zero.

An example of an adjacent isobaric pair not mentioned above is the neutron and proton. According to the latest quotations on masses,[27] the neutron is heavier than the proton by about twice the electron mass and should therefore be instable. There is no evidence against this.

In view of the great uncertainties still attending the neutrino mass, it will not be given a definite value in what follows except when comparison with experiment is made. In that event μ will be set equal to zero; this will not invalidate the comparisons even for μ as great as m.

CHAPTER IV

THE MATHEMATICAL FORMULATION OF FERMI'S THEORY

a. The Hamiltonian. In setting up a formulation of the problem of a heavy particle emitting a pair of light particles, Fermi was guided by the analogous problem of an atom emitting a photon. In the latter case, the Hamilton equation can be written:[30]

$$H = H_{atom} + H_{field} + e\left[V + (\vec{\alpha} \cdot \vec{U})\right] \qquad (12)$$

in which the first term is the Hamilton function for the atomic system, the second represents the energy of the radiation field, and the last term gives the interaction of the atom with the radiation field. (U, iV) is the electromagnetic potential four-vector. $\vec{\alpha}$ is the vector with three matrices as components, introduced by Dirac in his theory of the electron; $e(\vec{\alpha} \cdot \vec{U})$ in this Dirac formulation replaces the more familiar $e/mc(\vec{p} \cdot \vec{U})$ of the Schrödinger theory, in which p is the impulse of the radiating electron. To this interaction term the responsibility for the emission and absorption of photons by the atom can be directly traced.

In a similar way, a β-emission is regarded as being the result of an interaction between the heavy particle and the "electron-neutrino field." This is represented by the last term

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[30] Fermi, Rev. of Modern Phys. 4, 87 (1932).

of the Hamiltonian:

$$H = H_{heavy} + H_{light} + \mathcal{H}, \qquad (13)$$

where the first two terms are the energy operators of the heavy particle and the light particles, respectively. $\mathcal{H}$ must be an operator of such a kind that when it causes the heavy particle to make a transition from the neutron to the proton state, there simultaneously appears a new electron and a neutrino in the light particle field. Likewise, the transition from the proton to the neutron state must entail the destruction of an electron and a neutrino. Since the destroyed electron may be one in a negative energy state, the positron emission process is provided for in these conditions on $\mathcal{H}$.

The neutron and the proton states of the heavy particle will be designated by the values +1 and -1, respectively, of the two-valued variable ρ. If $|\Psi_{+1}|^2$ is the probability of the heavy particle's being in the neutron state and $|\Psi_{-1}|^2$ is the probability of the proton state, then the normalized probability amplitude of the heavy particle can be represented by:

$$\Psi = \begin{vmatrix} \Psi_{+1} \\ \Psi_{-1} \end{vmatrix}$$

Now, $\mathcal{H}$ must contain operators Q and Q^* which cause P→N and N→P transitions respectively. Operators with the desired effect on Ψ, which defines the state of the heavy particle,

will be:

$$Q = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}, \qquad Q^* = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}.$$

Thus, if the heavy particle were originally in the neutron state, i.e.,

$$\Psi = \begin{vmatrix} \Psi \\ 0 \end{vmatrix},$$

then

$$Q\Psi = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \cdot \begin{vmatrix} \Psi \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} = 0 \qquad (14)$$

and

$$Q^*\Psi = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \cdot \begin{vmatrix} \Psi \\ 0 \end{vmatrix} = \begin{vmatrix} 0 \\ \Psi \end{vmatrix} \qquad (15)$$

In words, a term in the Hamilton operator (for a system with the heavy particle in the neutron state) containing Q will vanish as part of the interaction energy, whereas a term containing Q^* will cause the Ψ to define a proton state.

The creation and destruction of electrons and neutrinos as units requires the quantization of the electron-neutrino field for their formulation. Suppose, $\psi_1, \psi_2, \ldots \psi_s \ldots$ are normalized wave functions for the electron energy states: $W_1, W_2, \ldots, W_s, \ldots$ Then the state of the electron field will be defined by:

$$\psi = \sum_s \psi_s a_s \quad , \quad \psi^* = \sum_s \psi_s^* a_s^* \tag{16}$$

Since $|\psi_s|^2 = 1$, $a_s a_s^* = N_s$, the number of electrons in state s. According to the Pauli principle, N_s can have only the values 0 and 1. This restriction, together with the wave equation, can be shown to give a_s and a_s^* the following properties as operators:[31]

$$\begin{aligned} a_s \psi(N_1 N_2 \cdots N_s \cdots) &= (-)^{N_1+N_2+\cdots+N_{s-1}} N_s \, \psi(N_1 N_2 \cdots 1-N_s \cdots) \\ a_s^* \psi(N_1 N_2 \cdots N_s \cdots) &= (-)^{N_1+N_2+\cdots+N_{s-1}} (1-N_s) \, \psi(N_1 N_2 \cdots 1-N_s \cdots) \end{aligned} \tag{17}$$

Thus a_s is a "destruction" operator for state s, since it changes a state of the field for which $N_s = 1$ to one for which $N_s = 0$. Similarly, a_s^* is a "creation" operator.

If $\varphi_1, \varphi_2, \cdots, \varphi_\sigma, \cdots$ are the wave functions for the neutrino energy states: K_1, K_2, ..., K_σ, ... with the occupation numbers: M_1, M_2, ..., M_σ, ..., one has corresponding to (16):

$$\varphi = \sum_\sigma \varphi_\sigma b_\sigma \quad , \quad \varphi^* = \sum_\sigma \varphi_\sigma^* b_\sigma^* \tag{18}$$

Assuming that neutrinos obey the Pauli principle, $M_\sigma = b_\sigma b_\sigma^* = 0$ or 1. Therefore b_σ is a destruction operator and b_σ^* a creation operator for neutrinos in state σ just as a_s and a_s^* for electrons.

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[31] Pauli, Handbuch der Physik, Vol. 24/1, page 220.

$$b_\sigma \varphi(M_1 M_2 \cdots M_\sigma \cdots) = (-)^{M_1+M_2+\cdots+M_{s-1}} M_\sigma \varphi(M_1 M_2 \cdots 1-M_\sigma \cdots)$$
$$b_\sigma^* \varphi(M_1 M_2 \cdots M_\sigma \cdots) = (-)^{M_1+M_2+\cdots+M_{s-1}} (1-M_\sigma) \varphi(M_1 M_2 \cdots 1-M_\sigma \cdots) \qquad (19)$$

It is now possible to write down the form which $\mathcal{H}$ must have:

$$\mathcal{H} = G\left[Q \sum_{s\sigma} c_{s\sigma} a_s b_\sigma + Q^* \sum_{s\sigma} c_{s\sigma}^* a_s^* b_\sigma^*\right] \qquad (20)$$

Here, G is a constant which will determine the absolute magnitude of $\mathcal{H}$ and which, therefore, measures the strength of the coupling between the heavy particle and the electron-neutrino field. $C_{s\sigma}$ and $C_{s\sigma}^*$ are quantities still to be determined. The combination of operators, $Q\, a_s\, b_\sigma$ insures the destruction of an electron and neutrino in the P→N transition. It will not effect transitions in a system having the heavy particle in the neutron state, or in one lacking an electron in state s and a neutrino in state σ. $Q^* a_s^* b_\sigma^*$ plays the corresponding role in the N→P transitions.

b. <u>The Fermi Interaction "Ansatz"</u>. An expression for $\mathcal{H}$ consistent with (20) is:

$$\mathcal{H} = G[Q \psi \varphi + Q^* \psi^* \varphi^*] \qquad (21)$$

as is easily seen when (16) and (18) are recalled. According to this, $C_{s\sigma} = \psi_s \varphi_\sigma$ and $C_{s\sigma}^* = \psi_s^* \varphi_\sigma^*$. (21) is by no means the only possible choice for $\mathcal{H}$ but it seems to be the simplest that can be made with the formulation of the preceding paragraphs as a basis. Others which have been suggested are choices containing products of derivatives, rather than just the products, of

ψ and φ.

Much of the ensuing development, notably the positron emission, will rest on the use of the Dirac theory of the electron. This means that the state of the electron field will be specified by a four-fold wave function,

$$\psi = \left| \begin{array}{c} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{array} \right| .$$

It is sensible to assume that the neutrino also obeys a relativistic equation of the Dirac type so that:

$$\varphi = \left| \begin{array}{c} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \end{array} \right| .$$

There is now a multiplicity of products of electron and neutrino wave functions which can be introduced into (21). A criterion which can be used to reduce the arbitrariness of choosing any specific one may be established by further drawing on the analogy between $\mathcal{H}$ and the interaction term of (11). One can write:

$$\mathcal{H} = G\left[Q\{A_0 + (\vec{\alpha}_H \cdot \vec{A})\} + Q^*\{A_0^* + (\vec{\alpha}_H \cdot \vec{A})^*\}\right] \qquad (22)$$

where $(\vec{A}, iA_0)$ is a polar four-vector analogous to the four-vector potential, $(\vec{U}, iV)$ and $\vec{\alpha}_H$ is a "Dirac matrix" for the heavy particle. Thus a formal use is made of the Dirac formulation for the heavy particle for want of a relativistic theory of the latter. The use of (22) in the interaction term of a relativistically invariant equation of the heavy particle would not spoil the invariance property.

It can be shown with the use of spinor analysis that the only polar four vectors which can be constructed from bilinear products of ψ's and φ's as required by (21) are:[32]

$$A_o = \psi \delta \varphi \,, \quad \vec{A} = \psi \delta \vec{\alpha} \varphi \tag{23}$$

$$A'_o = \psi \varphi^* \,, \quad \vec{A}' = \psi \vec{\alpha}^* \varphi^* \,. \tag{24}$$

$\vec{\alpha}$ is the Dirac matrix vector previously mentioned, while δ is given by:

$$\delta = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

It pairs the ψ's and φ's for A_o as follows:

$$A_o = \psi \delta \varphi = -\psi_1 \varphi_2 + \psi_2 \varphi_1 + \psi_3 \varphi_4 - \psi_4 \varphi_3 \tag{25}$$

(23) is the four-vector adopted by Fermi for use in (22). (24) differs from it in that it replaces the combinations $Q\, a_s b_\sigma$ and $Q^* a_s^* b_\sigma^*$ by $Q\, a_s b_\sigma^*$ and $Q^* a_s^* b_\sigma$. According to the viewpoint represented by the use of (24), therefore, the anti-neutrino is emitted with the electron in the $N \rightarrow P$ transition while the neutrino accompanies a positron in its emission during a $P \rightarrow N$ transition. (This is the (n, e^+) point of view as distinguished from the (n, e^-) point of view represented by (23)). Just as

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[32] If the determinant multiplication rules for obtaining the products of matrices are strictly followed, the first of each pair of the matrices must be transposed. Transposition signs will be left off in this paper.

one should expect from physical considerations, the two viewpoints give identical results if the neutrino mass, μ, is taken as zero. If $\mu \neq 0$ then there occurs a sign change in Formula (11), as has already been mentioned, when (24) is used instead of (23). The difference between the two viewpoints is generally of such a character. It will be seen that of the two vectors, A' is somewhat simpler to handle when modifications are made in the interaction Ansatz, largely because of its similarity to the familiar current four-vector of Dirac theory.

c. The Matrix Element of the Perturbation. To find the effect of $\mathcal{H}$ on a given system, the perturbation method of the variation of constants will be used. The system with which one begins in the problem of ordinary β-decay contains a neutron ($\rho = +1$) in some nuclear energy state (n) and the electron-neutrino field with at least some of its states unfilled (s and σ, say). The probability amplitude of the initial state is at t = 0:

$$a_i = a_{+1, n, N_1, N_2, \cdots 0_s \cdots, M_1, M_2, \cdots 0_\sigma \cdots} \qquad (26)$$

while the probability amplitudes of all other possible states of the system under scrutiny are zero. The perturbation $\mathcal{H}$, as time goes on, will cause the growth of a new state:

$$a_f = a_{-1, m, N_1, N_2, \cdots 1_s \cdots M_1, M_2, \cdots 1_\sigma \cdots}$$

having the heavy particle in the proton state, and a newly created electron-neutrino pair in states s and σ. By the method

of variation of constants, a_f will grow according to:

$$|a_f|^2 = 4|\mathcal{H}_f^i|^2 \frac{\sin^2 t/2\,(E_i - E_f)}{(E_i - E_f)^2} . \qquad (27)$$

$\mathcal{H}_f^i = \int \Psi_f^* \mathcal{H} \Psi_i \, d\tau$, in which Ψ_i and Ψ_f are wave functions of the system in the initial and the final states, respectively. E_i and E_f are the corresponding values of the energy.

If the state of the electron-neutrino field is defined by: $\psi(N, N_2 \ldots) \cdot \varphi(M, M_2 \ldots)$, then:

$$\Psi_i = \psi(N, N_2 \cdots O_s \cdots) \cdot \varphi(M, M_2 \cdots O_\sigma \cdots) \cdot \begin{vmatrix} u_m \\ 0 \end{vmatrix} \qquad (28)$$

$$\Psi_f = \psi(N, N_2 \cdots 1_s \cdots) \cdot \varphi(M, M_2 \cdots 1_\sigma \cdots) \cdot \begin{vmatrix} 0 \\ v_m \end{vmatrix} \qquad (29)$$

In $\mathcal{H}_f^i$, $\mathcal{H}$ operates on Ψ_i. Of the terms of $\mathcal{H}$ as given by (22), only $G\, Q^*\, A_o^*$ will be kept. The first one is dropped because $Q\, \Psi_i = 0$. $(\vec{\alpha}_H \cdot \vec{A})^*$ is neglected in comparison with A_o^* because it is of the order v/c times smaller than A_o^*, v being the velocity of the heavy particle in the nucleus, usually adjudged to be about $c/10$.[33] (c is the velocity of light, used as the unit of velocity in most of this paper.) Taking Fermi's choice (23) for $A_o^* = \psi^* \delta \varphi^* = \sum_{s\sigma} \psi_s^* \delta \varphi_\sigma^* a_s^* b_\sigma^*$ and noticing that only one term of the $s\sigma$ summation need be

[33] This point will be discussed in more detail in Chapter VI.

kept in $\mathcal{H}_f^i$ after (17), (19) and the orthogonality conditions:

$$\int \psi(N_1 N_2 \cdots N_s \cdots)\, \psi(N_1' N_2' \cdots N_s' \cdots) = \delta^{N_1}_{N_1'}\, \delta^{N_2}_{N_2'} \cdots \delta^{N_s}_{N_s'} \cdots ,$$

$$\int \varphi(M_1 M_2 \cdots M_\sigma \cdots)\, \varphi(M_1' M_2' \cdots M_\sigma' \cdots) = \delta^{M_1}_{M_1'}\, \delta^{M_2}_{M_2'} \cdots \delta^{M_\sigma}_{M_\sigma'} \cdots ,$$

are applied, the matrix element becomes:

$$\begin{aligned} \mathcal{H}_f^i &= \pm G \int d\tau \left| v_m^{0*} \right| Q^* (\psi_s^* \delta \varphi_\sigma^*) \left| u_n^0 \right| \\ &= \pm G \int d\tau\, v_m^* (\psi_s^* \delta \varphi_\sigma^*) u_n . \end{aligned} \tag{30}$$

According to its introduction in (28), U_n is the wave function of a neutron in an energy state n of the nucleus which is parent to the β-decay. V_m is the wave function of the proton in a state m of the product nucleus. Very little is known about either U_n or V_m except that their values are zero outside the nuclear radius, r_0, which is known from scattering experiments to be at most 10^{-12} cm. The de Broglie wave lengths of the electrons and neutrinos in even the most energetic β-emissions known ($\sim 23\ mc^2$ for B^{12}) are much longer than r_0, so in a first approximation ψ_s and φ_σ will be regarded as constants over the very short range in which U_n and V_m are not zero. Accordingly,

$$\mathcal{H}_f^i = \pm G\, S(\psi_s^* \delta \varphi_\sigma^*) \int d\tau\, v_m^* u_n = G\, S(\psi_s^* \delta \varphi_\sigma^*) \cdot M , \tag{31}$$

in which S() denotes that summations over spins are to be made

on the wave functions contained in the brackets ( ).

d. The Energy Distribution, $Z \cong 0$. If the energy released in the β-disintegration (i.e., the difference in the energies of the parent nucleus with the neutron in state n and the product nucleus with the proton in state m) is W_0, then $E_i - E_f = W_0 - W_s - K_\sigma$. Using this and (31), the expression (27) becomes:

$$|a_f|^2 = 4G^2|M|^2|S(\psi_s \varphi_\sigma)|^2 \frac{\sin^2 t/2\,(W_0 - W_s - K_\sigma)}{(W_0 - W_s - K_\sigma)^2} . \qquad (32)$$

This is the probability that the β-process occurs with the emission of an electron having the specific value of the energy W_s and a neutrino with the specific energy K_σ. The sum of such terms as $|a_f|^2$ for all values of s and σ is the probability for the process regardless of the values of W_s and K_σ. As is customary in such problems, the electron and neutrino are at first confined to the volume Ω so that they have a discrete set of energies characterized by the impulses $\vec{p}_s$ and $\vec{p}_\sigma$. Eventually Ω is made infinite. As it becomes larger, the discrete levels draw very close together so that $\sum_s$ can be replaced in the usual way by $(\Omega/(2\pi)^3)\int d\vec{p}_s$ and $\sum_\sigma$ by $(\Omega/(2\pi)^3)\int d\vec{p}_\sigma$. Then,

$$\sum_{s\sigma} |a_f|^2 = 4G^2|M|^2 \frac{\Omega^2}{(2\pi)^6} \int d\vec{p}_s \int d\omega_\sigma \int_0^\infty dp_\sigma\, p_\sigma^2 |S(\psi_s \varphi_\sigma)|^2 \frac{\sin^2 t/2\; x}{x^2} , \qquad (33)$$

where $\int d\omega_\sigma$ denotes integration over the directions of $\vec{p}_\sigma$ and $x = W_0 - W_s - K_\sigma$. Changing the variable in the p_σ integration to

$$x = W_o - W_s - \sqrt{p_\sigma^2 + \mu^2},$$

$$\int_0^\infty p_\sigma^2 \, dp_\sigma \longrightarrow -\int_{W_o - W_s - \mu}^{-\infty} p_\sigma K_\sigma \, dx.$$

The values of $(\sin^2(tx/2))/x^2$ for $x \neq 0$ are negligibly small in comparison to its value at $x = 0$, so an integrand such as in

$$\int F(x) \frac{\sin^2 t/2\, x}{x^2} \, dx$$

is practically zero except in the neighborhood of $x = 0$. If the limits on this integral are above and below $x = 0$, respectively, no appreciable change is made by stretching them to $+\infty$ and $-\infty$. Moreover, of all the values of F(x) only F(0) is not "killed off" and so:

$$\int F(x) \frac{\sin^2 t/2\, x}{x^2} \, dx = F(0) \int_{-\infty}^{\infty} \frac{\sin^2 t/2\, x}{x^2} = F(0) \frac{\pi t}{2}$$

Applying this to (33),

$$\sum_{s\sigma} \frac{|a_f|^2}{t} = \frac{G^2 |M|^2}{(2\pi)^5} \int d\vec{p}_s \int d\omega_\sigma \, p_\sigma K_\sigma \left| S(\psi_s - \varphi_\sigma) \right|^2 \qquad (34)$$

subject to the conservation condition:

$$W_o = W_s + K_\sigma \, . \qquad (35)$$

Since the chargeless neutrino is scarcely affected by the presence of other particles, it is valid to use for φ_σ the solutions of the Dirac equation for a free particle (a plane wave in space Ω):

$$\varphi_\sigma = \frac{B_\sigma}{\sqrt{\Omega}} e^{i(p_\sigma \cdot r - K_\sigma t)} \tag{36}$$

where

$$B_\sigma^{(1)} = \sqrt{\frac{\mu + K_\sigma}{2K_\sigma}} \begin{vmatrix} -\frac{p_{\sigma z}}{\mu + K_\sigma} \\ -\frac{p_{\sigma x} + i p_{\sigma y}}{\mu + K_\sigma} \\ 1 \\ 0 \end{vmatrix}, \quad B_\sigma^{(2)} = \sqrt{\frac{\mu + K_\sigma}{2K_\sigma}} \begin{vmatrix} -\frac{p_{\sigma x} - i p_{\sigma y}}{\mu + K_\sigma} \\ \frac{p_{\sigma z}}{\mu + K_\sigma} \\ 0 \\ 1 \end{vmatrix} \tag{37}$$

The superscripts (1) and (2) denote a positive and a negative z-component of the spin, respectively.

Now it will be assumed that the disintegrating nucleus affects the electron wave function only negligibly ($Z \cong 0$) so that a plane wave can be used for ψ_s . Thus:

$$\psi_s = \frac{A_s}{\sqrt{\Omega}} e^{i(p_s \cdot r - W_s t)} \tag{38}$$

where A_s is exactly B_σ with μ replaced by 1, K_σ by W_s, and the components of $\vec{p}_\sigma$ by those of $\vec{p}_s$.

With the use of (25) it can be shown that:

$$|S(\psi_s \delta\varphi_\sigma)|^2 = \frac{1}{\Omega^2}\left\{|A_s^{(1)}\delta B_\sigma^{(1)}|^2 + |A_s^{(1)}\delta B_\sigma^{(2)}|^2 + |A_s^{(2)}\delta B_\sigma^{(1)}|^2 + |A_s^{(2)}\delta B_\sigma^{(2)}|^2\right\}$$

$$= \frac{1}{\Omega^2}\left\{1 + \frac{(p_s \cdot p_\sigma) + \mu}{W_s K_\sigma}\right\} \qquad (39)$$

Putting this into (34) and integrating over the directions of $\vec{p}_s$ and $\vec{p}_\sigma$,

$$\sum_{s\sigma}\frac{|a_f|^2}{t} = \frac{G^2|M|^2}{2\pi^3}\int_0^\infty dp_s\, p_s^2\, p_\sigma K_\sigma\left(1 + \frac{\mu}{W_s K_\sigma}\right). \qquad (40)$$

If the integration variable p_s is changed to $W_s = \sqrt{p_s^2 + 1}$, the differential of (40) will give the probability per second that an electron is emitted with an energy between W and W + dW. After applying condition (35), this becomes:

$$P_{o,\mu}(W)dW = dW\,\frac{G^2|M|^2}{2\pi^3}\,W\sqrt{W^2-1}\,(W_o - W)\sqrt{(W_o-W)^2-\mu^2}\left(1 + \frac{\mu}{W(W_o - W)}\right). \qquad (41)$$

This is just formula (11) for the case Z = 0. After this K will be written for $(W_o - W)$.

If, instead of A_o^*, A_1^* as given by (24) had been employed, the result would have been exactly the same except for a change

in sign in the last parenthesis of (41). Since this (n, e^+) viewpoint requires the destruction of a neutrino to accompany the creation of the electron, Ψ_i must contain a neutrino in the negative state σ and Ψ_f differs from (29) in that 0_σ replaces 1_σ. φ_σ would then be a wave function for a "hole" in the negative spectrum, analogous to the wave function of a positron, and:

$$B_\sigma^{(1)} = \sqrt{\frac{\mu + K_\sigma}{2K_\sigma}} \begin{vmatrix} 1 \\ 0 \\ -\frac{p_{\sigma z}}{\mu + K_\sigma} \\ -\frac{p_{\sigma x} + i p_{\sigma y}}{\mu + K_\sigma} \end{vmatrix}, \quad B_\sigma^{(2)} = \sqrt{\frac{\mu + K_\sigma}{2K_\sigma}} \begin{vmatrix} 0 \\ 1 \\ -\frac{p_{\sigma x} - i p_{\sigma y}}{\mu + K_\sigma} \\ \frac{p_{\sigma z}}{\mu + K_\sigma} \end{vmatrix}. \tag{42}$$

K_σ and p_σ are here the positive energy and impulse of the anti-neutrino. The calculation goes on as before except that $|S(\psi_s \varphi_\sigma^*)|^2$ replaces $|S(\psi_s \delta \varphi_\sigma)|^2$. (42) is used in $|S(\psi_s \varphi_\sigma^*)|^2$, resulting in:

$$|S(\psi_s \varphi_\sigma^*)|^2 = 1 + \frac{(p_s \cdot p_\sigma) - \mu}{W_s K_\sigma} \tag{43}$$

Between (43) and (39) there is the sign difference already mentioned.

In the positron emission problem, the initial wave function is the same as (29) with s and σ as <u>negative</u> energy states (from the (n, e^-) rather than the (n, e^+) point of view). Here,

G Q A$_{\sigma}$ is the important term of $\mathcal{H}$. For ψ_s, positron wave functions are used with amplitudes as given by (42) when $\mu = 1$, $K_\sigma = W_s$ and $\vec{p}_s = \vec{p}_\sigma$. The entire calculation is very little different from the electron emission case and the results are the same.

e. <u>The Energy Distribution, $Z \neq 0$.</u> For all the naturally radioactive substances, the nuclear charge Z is so large that its effect cannot be neglected. It is taken into account by using for ψ_s the solutions of the Dirac equation with a Coulomb potential:

$$\psi^{(a)}_{\ell,m} = \frac{1}{\sqrt{4\pi(\ell+m+1)!(\ell-m)!}} \begin{vmatrix} -i f_\ell P^m_{\ell+1} \\ -i f_\ell P^{m+1}_{\ell+1} \\ -(\ell+m+1) g_\ell P^m_\ell \\ (\ell-m) g_\ell P^{m+1}_\ell \end{vmatrix} \qquad (44)$$

(with $\ell \geq 0$; $-(\ell+1) \leq m \leq \ell$),

$$\psi^{(b)}_{\ell,m} = \frac{1}{\sqrt{4\pi(\ell+m)!(\ell-m-1)!}} \begin{vmatrix} -i(\ell+m) f_{-\ell-1} P^m_{\ell-1} \\ i(\ell-m-1) f_{-\ell-1} P^{m+1}_{\ell-1} \\ -g_{-\ell-1} P^m_\ell \\ -g_{-\ell-1} P^{m+1}_\ell \end{vmatrix} \qquad (45)$$

(with $\ell \geq 1$; $-\ell \leq m \leq \ell - 1$).

The superscripts (a) and (b) denote values of the total angular momentum, j = ℓ + 1/2 and j = ℓ - 1/2, respectively.

$$P_\ell^m(\vartheta,\varphi) = \frac{(\ell-m)!}{2^\ell \ell!} \sin^m\vartheta \left(\frac{d}{d(\cos\vartheta)}\right)^{\ell+m} (\cos^2\vartheta - 1)^\ell \, e^{im\varphi} \qquad (46)$$

$$f_K = \sqrt{1-W}\,(G_K - G_K^*) \;,\; g_K = \sqrt{1+W}\,(G_K + G_K^*)$$

(k = ℓ for type (a), k = - ℓ - 1 for type (b))

$$G_K = N_K (2r)^{\gamma_K - 1} e^{-ipr} e^{i\theta_K} (\gamma_K + i\lambda) F(\gamma_K + 1 + i\lambda, 2\gamma_K + 1; 2ipr)$$

$$F(a,b;z) = \sum_{m=0}^{\infty} \frac{\Gamma(a+m)}{\Gamma(b+m)} \cdot \frac{\Gamma(b)}{\Gamma(a)} \cdot \frac{z^m}{m!}$$

$$\gamma_K = \sqrt{(K+1)^2 - \alpha^2 Z^2} \;,\; \lambda = \frac{\alpha Z}{p} W$$

$$N_K = \frac{e^{\pi\lambda/2}\,|\Gamma(\gamma_K + i\lambda)|}{\sqrt{WR}\,\Gamma(2\gamma_K + 1)}\, p^{\gamma_K} \qquad (47)$$

$$e^{2i\theta_K} = \frac{\gamma_K - i\lambda}{K + 1 - i\alpha Z/p}$$

(R = radius of large sphere in which electron is confined).

Of these wave functions, $\psi^{(a)}_{0,m}$ and $\psi^{(b)}_{1,m}$ are infinite at r = 0 because $\gamma_0 = \gamma_{-2} < 1$. For small values of r($\sim r_0$), all the other eigenfunctions are comparatively small.

Since the Coulomb field breaks down for $r < r_0$, it is consistent with the approximation which led to (31) to take for ψ_s the wave functions $\psi^{(a)}_{0,m}$ and $\psi^{(b)}_{1,m}$ evaluated at $r = r_0$. Then,

$$|S(\psi_s \delta\varphi_\sigma)|^2 = \frac{1}{\Omega}\Big[|\psi^{(a)}_{0,-1}\delta B^{(1)}_\sigma|^2 + |\psi^{(a)}_{0,0}\delta B^{(1)}_\sigma|^2 + |\psi^{(a)}_{0,-1}\delta B^{(2)}_\sigma|^2 + |\psi^{(a)}_{0,0}\delta B^{(2)}_\sigma|^2$$

$$+ |\psi^{(b)}_{1,-1}\delta B^{(1)}_\sigma|^2 + |\psi^{(b)}_{1,0}\delta B^{(1)}_\sigma|^2 + |\psi^{(b)}_{1,-1}\delta B^{(2)}_\sigma|^2 + |\psi^{(b)}_{1,0}\delta B^{(2)}_\sigma|^2\Big]$$

$$= \frac{1}{4\pi\Omega K_\sigma}\Big[(K_\sigma - \mu)(f_0^2 + f_{-2}^2) + (K_\sigma + \mu)(g_0^2 + g_{-2}^2)\Big]$$

$$= \frac{2}{\pi R\Omega}\,\frac{e^{\pi\lambda}|\Gamma(\gamma + i\lambda)|^2}{[\Gamma(2\gamma+1)]^2}\,(2r_0)^{2\gamma-2}\,p^{2\gamma}\left(1 + \gamma^2\frac{\mu}{W_s K_\sigma}\right) \qquad (48)$$

In these calculations, $e^{\pm ipr}$ and the hypergeometric series are each set equal to unity, since r_0 is very small.

$\gamma = \sqrt{1 - \alpha^2 Z^2}$, both for k = 0 and k = -2.

(48) is put into (32) and summations over s and σ are made as in the Z = 0 case. This time, however, $\sum_s$ is replaced by $R/\pi \int_0^\infty dp_s$ because the summations over the indices (ℓ, m) of the spherical harmonics have taken the place of the angular integration of $\vec{p}_s$. One finally obtains:

$$P_{0,0}(W)\,dW = dW\,\frac{G^2|M|^2}{2\pi^3}\,F(Z,W)\,K\sqrt{K^2-\mu^2}\,W\sqrt{W^2-1}\left(1 + \frac{\mu\gamma^2}{WK}\right) \qquad (49)$$

where

$$F(Z,W) = 4\,\frac{e^{\pi\lambda}|\Gamma(\gamma + i\lambda)|^2}{[\Gamma(2\gamma+1)]^2}\,(2r_0 p)^{2\gamma-2}$$

(49) is just Formula (11) for the (n, e^-) viewpoint. The use of the (n, e^+) point of view would have changed the + sign of the last parenthesis in (49) into a - sign. In the positron emission problem, the results are exactly the same except that Z is replaced by -Z.

f. The Evaluation of G. This must be done empirically, from the absolute magnitude of the β-decay. According to the definition of the lifetime, τ, of a decaying substance:

$$1/\tau = \int_1^{W_0-\mu} P(W)\,dW \qquad (50)$$

$$= G^2 |M|^2 F(W_0) \qquad (51)$$

μ will be taken as zero both because of the fact that the evidence points to this value for it and because in any case it can be neglected in comparison to W_0 for order of magnitude calculations. $|M|^2$ is of the order of magnitude 1 except in special cases when u_m and v_m have such symmetry properties that $\int v_m^* u_n \, d\tau = 0$. This is the case for the "forbidden" transitions, to be discussed later. Thus, the order of magnitude of G can be obtained from a knowledge of the lifetime and the upper energy limit of a β-emitter. All the known β-emitters can be given a share in the evaluation of G by plotting their lifetimes against the function $F(W_0)$ of their upper limits obtained from:

$$F(W_0) = 1/2\pi^3 \int_1^{W_0} F(Z,W)\, W \sqrt{W^2-1}\, (W_0 - W)^2 \, dW \qquad (52)$$

This empirical plot is found to be very erratic and in consequence log τ was plotted against log $F(W_o)$. According to (51), with $|M|^2$ put equal to unity,

$$\log \tau + \log F(W_o) = - \log G^2 \qquad (53)$$

so that the log τ, log $F(W_o)$ graph should be a straight line with a 45 degree slope. G is straightforwardly calculable from the intercepts. In Figure 11 are shown the results, based on the values of τ and W_o given in Table I. The empirical points do not at all determine a straight line, but the region in which such a line should be (if the experimental data were better and $F(W_o)$ were correct) is enclosed by two 45 degree lines. The difference in their intercepts gives the uncertainty in G which itself comes out to be about 10^{-12} in magnitude.

An approximation evaluation of (52) for the heavy elements was made by Fermi. For lighter elements, an exact evaluation is not difficult.

It may be noticed in the figure that no attempt was made to include certain of the farther flung experimental points in the region from which G was evaluated. The reason for this will be made plain when Sargent's law is considered (Chapter VI).

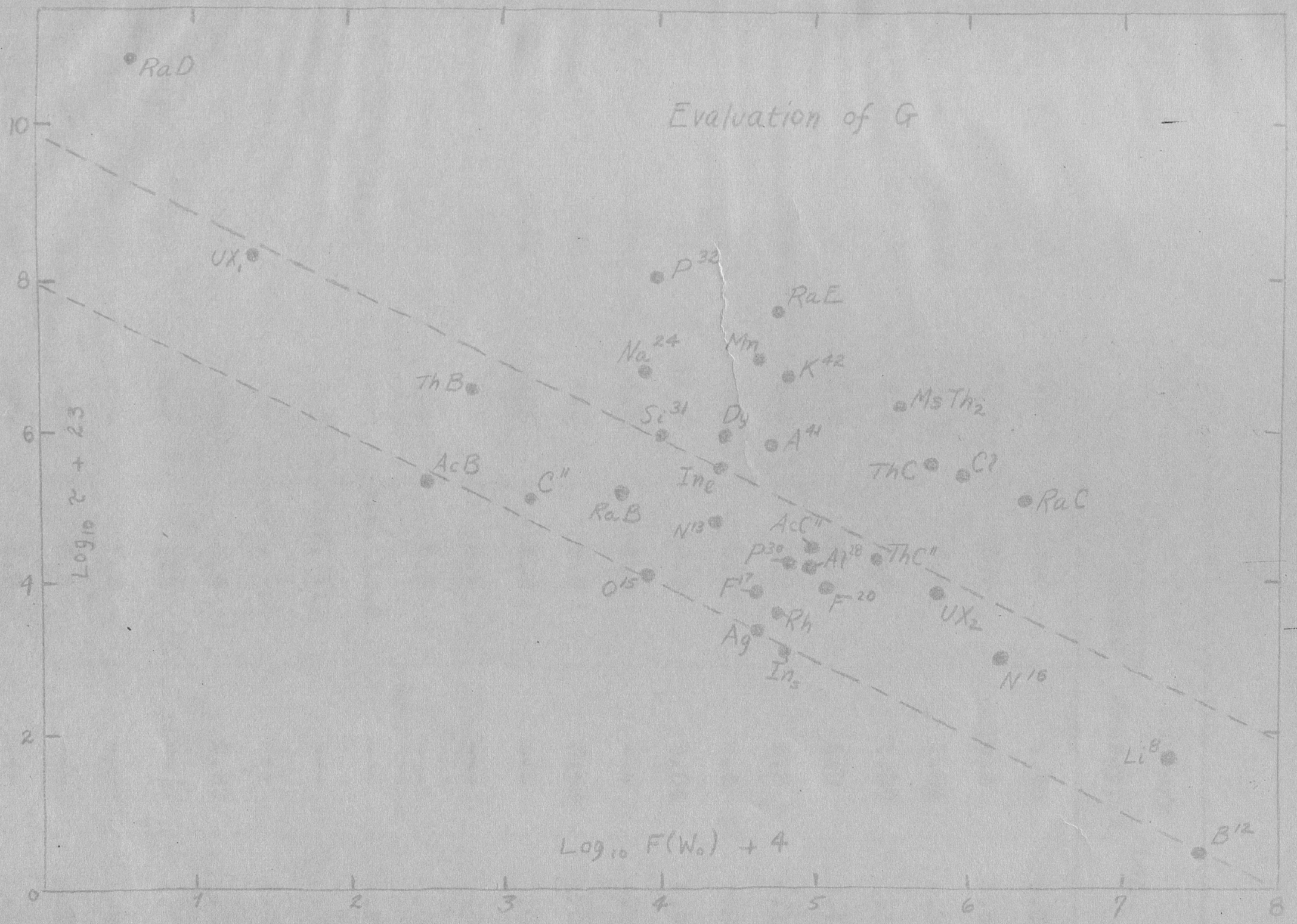

Figure 11. τ is measured in units $\hbar/mc^2$ for this diagram.

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TABLE I

Emitter	τ (sec.)	Upper Limit, W_o (mc^2)		Average Energy, $\bar{W}$	Log $F(W_o)$
		By Insp.	Mod. Theor.		
Li^8 (e^-)	.5	20.6	23.0	8.55	3.3
B^{12} (e^-)	.02	22.5	26.4	10.3	3.5
C^{11} (e^+)	12.60	3.25	3.54	1.84	-.32
N^{13} (e^+)	620	3.5	3.9	1.97	.35
N^{16} (e^-)	10	12.8	13.7	5.9	2.4
O^{15} (e^+)	126	4.3	4.9	2.2	-.15
F^{17} (e^+)	70	5.1	5.7	2.6	.25
F^{20} (e^-)	12	11.0	12.5	5.04	1.95
Na^{24} (e^-)	5.58×10^4	4.3	4.8	---	-.15
Al^{28} (e^-)	180	7.0	8.8	3.49	1.0
Si^{31} (e^-)	9000	4.5	5.0	---	.001
P^{30} (e^+)	195	6.5	8.0	---	.85
P^{32} (e^-)	1.2×10^6	4.5	5.2	---	-.001
Cl^- (e^-)	2340	10.4	3.9, 12.9	---	1.9
A^{41} (e^-)	6500	6.3	3.9, 11	---	.77
K^{42} (e^-)	5.75×10^4	7.9	3.7, 9.6	---	1.3
Mn (e^-)	9000	---	6.4	2.69	.6
Rh (e^-)	44	---	6.5	---	1.2
In_s (e^-)	13	---	7.3	---	1.45
In_ℓ (e^-)	3240	---	3.5	---	-.10
Ag (e^-)	22	---	5.3	---	.75

(continued)

TABLE I (cont'd)

Emitter	τ (sec.)	Upper Limit, $W_o(mc^2)$ By Insp.	Upper Limit, $W_o(mc^2)$ Mod. Theor.	Average Energy, $\bar{W}$	$\text{Log}_{10} F(W_o)$
Dy(e^-)	9000	---	3.7	---	.4
UX_1	3 x 10	1.26	---	---	-2.6
UX_2	99.6	5.64	---	2.64	1.8
RaB	2.3×10^3	2.27	---	1.46	-.2
RaC	1.7×10^3	7.16	7.16	2.28	2.35
RaD	1.15×10^9	1.07	---	---	-3.4
RaE	6.2×10^5	3.3	3.7	1.78	.78
$MsTh_2$	3.2×10^4	5.1	---	2.12	1.6
ThB	5.5×10^7	1.7	---	.28	-1.2
ThC	8.6×10^3	5.5	---	2.6	1.75
ThC"	274	4.7	---	2.16	1.4
AcB	3132	1.6	---	---	-1.5
AcC"	414	3.8	---	1.96	1.0

The data for this table was obtained from: Kurie, Richardson and Paxton (Phys. Rev. 49, 368 (1936)); Fowler, Delsasso and Lauritsen (Phys. Rev., in press); Gaerttner, Turin and Crane (to be published); Sargent (Proc. Roy. Soc. A 139, 659 (1933); Proc. Camb. Phil. Soc. 24, 538 (1932)); Alichanow, Alichanian and Dzelepow (ZS/Phys. 93, 350 (1935)).

CHAPTER V

THE PROPOSED MODIFICATIONS OF FERMI'S THEORY

a. The Vectors B and B'. The choice of (21) for H was made because it seemed to be the simplest consistent with (20). One can require that agreement with the experimental data be an added criterion for a choice. It was seen in Chapter III, section b, that the agreement of formulas obtained through (21) with the observed energy distributions could be greatly improved by the introduction of a power of the neutrino energy as a weight factor. By considering (36) it is evident that such a weight factor can be introduced by replacing in (21) φ by a derivative of φ.

There exist only two independent polar four vectors (besides arbitrary linear combinations) formed from products of ψ's and first order derivatives of φ's, in parallel with (23) and (24):

$$B_0 = \psi\beta\delta\frac{\partial\varphi}{\partial t}\ ,\quad \vec{B} = \psi\beta\delta\ \mathrm{grad}\,\varphi \tag{54}$$

$$B'_0 = \psi\beta\frac{\partial\varphi^*}{\partial t}\ ,\quad \vec{B}' = \psi\beta\ \mathrm{grad}\,\varphi^* \tag{55}$$

A complete spinor analytic proof of this is possible, but a briefer and less complete proof will be given here. It will be shown that B'^* is the only independent polar four-vector which can be formed from bilinear combinations of ψ^*'s and

first derivatives of φ's. It is not essential to make the corresponding proof for B. One always has a vector in the (n, e^-) viewpoint for every vector in the (n, e^+) viewpoint; one can obtain the former from the latter by substituting $\delta\varphi$ for φ^*.

For purposes of the proposed proof, it is necessary to introduce the Weyl notation as given by Pauli. $\psi^t = i\psi^*\beta$ takes the place of ψ^*, $\gamma^\kappa = -i\beta\alpha_\kappa$ replaces $\vec{\alpha}$, and β is written as γ^4. It is desired then to form polar four-vectors from bilinear combinations of ψ^t and $\partial\varphi/\partial x_i$, where $(x_1, x_2, x_3, x_4) = (x, y, z, ict)$. In this notation, a complete expression of the combinations can be made with the use of the γ's. Out of all the possible combinations of ψ^t and $\partial\varphi/\partial x_i$,

$$\psi^t \frac{\partial\varphi}{\partial x_i}, \quad \psi^t \gamma_i \frac{\partial\varphi}{\partial x_j}, \quad \psi^t \gamma_i \gamma_j \frac{\partial\varphi}{\partial x_j}, \quad \text{etc.},$$

the polar four-vectors are to be picked out. Of course, only combinations with one free index can be vectors.

A quantity V_i is a polar four-vector if in the coördinate transformation,

$$x_i' = a_{i\mu} x_\mu \quad (a_{\mu\kappa} a_{\mu\ell} = \delta_{\kappa\ell}), \tag{56}$$

it transforms like:

$$V_i' = a_{i\mu} V_\mu . \qquad (i = 1,2,3,4)$$

(Repeated Greek indices indicate summations; $\mu = 1,2,3,4$)

It is shown by Pauli that if the change in φ in this transformation is given by $\varphi' = S\varphi$ where S is a matrix, then $\psi^{\dagger\prime} = \psi^\dagger S^{-1}$. $(S^{-1}S = 1)$. The only condition on S is that:

$$S^{-1}\gamma^\kappa S = a_{\kappa\mu}\gamma^\mu \tag{57}$$

For future reference, it will be well to know that if

$$M_{ij} = \gamma^i\gamma^j \qquad (i \neq j)$$

$$K_{ijk} = \gamma^i\gamma^j\gamma^k \qquad (i \neq j \neq k)$$

then:

$$S^{-1}M_{ij}S = S^{-1}\gamma^i S S^{-1}\gamma^j S = a_{i\mu}a_{j\nu}M_{\mu\nu} \tag{58}$$

$$S^{-1}K_{ijk}S = a_{i\mu}a_{j\nu}a_{k\lambda}K_{\mu\nu\lambda} \tag{59}$$

(Accordingly, $\psi^\dagger M_{ij}\varphi$ and $\psi^\dagger K_{ijk}\varphi$ are tensors of the second and third ranks, respectively.)

Since $\partial/\partial x_i$ is a four-vector ($\partial/\partial x_i' = a_{i\mu}\partial/\partial x_\mu$) according to (56), it is immediately clear that:

$$\psi^\dagger \frac{\partial\varphi}{\partial x_i} \tag{60}$$

is a four-vector. As a matter of fact, it is just B'^*.

$$C_i = \psi^\dagger M_{i\alpha}\frac{\partial\varphi}{\partial x_\alpha}$$

is also a polar four-vector since,

$$C_i' = \psi^{t'} M_{i\alpha} \frac{\partial \varphi'}{\partial x_\alpha'} = \psi^t S^{-1} M_{i\alpha} S\, a_{\alpha\mu} \frac{\partial \varphi}{\partial x_\mu}$$

$$= a_{\alpha\mu} a_{i\nu} a_{\alpha\lambda} \psi^t M_{\nu\lambda} \frac{\partial \varphi}{\partial x_\mu} = a_{i\nu} C_\nu$$

(Using (58) and the fact that according to (56), $a_{\alpha\mu} a_{\alpha\lambda} = \delta_{\mu\lambda}$.) The Dirac equation for the neutrino,

$$\gamma^\alpha \frac{\partial \varphi}{\partial x_\alpha} + \mu \varphi = 0 \qquad (61)$$

can be used to reduce C_i. Since, according to the definition of $M_{i\alpha}$, α cannot equal i ,

$$C_i = \psi^t \gamma^i \left(-\gamma^i \frac{\partial \varphi}{\partial x_i} - \mu \varphi\right) = -B_i'^* - \mu A_i'^*$$

The fact was used here that $(\gamma^i)^2 = 1$ and that (24) can be written as $(\psi^t \gamma^i \varphi)^*$.

$$D_i = \psi^t \gamma^i \gamma^\alpha \frac{\partial \varphi}{\partial x_\alpha}$$

is also a four-vector but because of (61) it degenerates immediately into: $-\mu \psi^t \gamma^i \varphi = -\mu A_i'^*$. Any other four-vectors which can be written with ψ^t and $\partial\varphi/\partial x_i$ degenerate in the same way into A, B and linear combinations of A and B. Therefore, B is the only independent four-vector that can be constructed from ψ^t and $\partial\varphi/\partial x_i$ (in the (n, e^+) viewpoint).

b. The Modified Distribution Formula. The calculations here are very little different from those in Chapter IV. Formula (32) now contains $|S(\psi_s \beta \delta \partial\varphi_\sigma/\partial t)|^2$ in place of $|S(\psi_s \delta \varphi_\sigma)|^2$, if the (n, e⁻) viewpoint is followed. According to (36),

$$|S(\psi_s \beta \delta \frac{\partial \varphi_\sigma}{\partial t})|^2 = K_\sigma^2 |S(\psi_s \beta \delta \varphi_\sigma)|^2 \qquad (62)$$

For $Z \tilde{=} 0$, (38) may be used as well as (36) so:

$$|S(\psi_s \beta \delta \varphi_\sigma)|^2 = \frac{1}{\Omega^2}\left(1 - \frac{p_s \cdot p_\sigma - \mu}{W_s K_\sigma}\right). \qquad (63)$$

This differs from (37) only in a sign.[34] The final result is:

$$P(W)dW = \frac{G^2|M|^2}{2\pi^3} W\sqrt{W^2-1}\, K^3 \sqrt{K^2-\mu^2}\left(1+\frac{\mu}{WK}\right)dW. \qquad (64)$$

---

[34] It may be well to put the formulas of this type into a single list, because of their frequent use. For Z = 0:

$$|S(\psi_s \delta \varphi_\sigma)|^2 = \frac{1}{\Omega^2}\left(1 + \frac{p_s \cdot p_\sigma + \mu}{W_s K_\sigma}\right)$$

$$|S(\psi_s \varphi_\sigma^*)|^2 = \frac{1}{\Omega^2}\left(1 + \frac{p_s \cdot p_\sigma - \mu}{W_s K_\sigma}\right)$$

$$|S(\psi_s \beta \delta \varphi_\sigma)|^2 = \frac{1}{\Omega^2}\left(1 - \frac{p_s \cdot p_\sigma - \mu}{W_s K_\sigma}\right)$$

$$|S(\psi_s \beta \varphi_\sigma^*)|^2 = \frac{1}{\Omega^2}\left(1 - \frac{p_s \cdot p_\sigma + \mu}{W_s K_\sigma}\right)$$

Because, with $Z \neq 0$, $|S(\psi_s \beta \delta \varphi_\sigma)|^2 = |S(\psi_s \delta \varphi_\sigma)|^2$ as given by Formula (48):

$$P(W)dW = \frac{G^2|M|^2}{2\pi^3} F(Z,W) W\sqrt{W^2-1}\, K^3 \sqrt{K^2-\mu^2} \left(1 + \frac{u\gamma^2}{WK}\right) dW \qquad (65)$$

If B' had been employed in place of B, the factor $(1 + \mu/WK)$ of Formula (64) would be $(1 - \mu/WK)$. A like change would be made in (65).

If G is now calculated by the method of Chapter IV, section f, it will be found to be changed but little. It is now about 3×10^{-12}.

c. Ansätze with Higher Derivatives. It will be seen in Chapter VII that it is theoretically desirable to introduce more derivatives into $\mathcal{H}$ than was done in section a of this chapter. Therefore, polar four-vectors consisting of bilinear combinations of $\dfrac{\partial^m \psi t}{\partial x_a \partial x_b \cdots \partial x_\ell}$ and $\dfrac{\partial^n \varphi}{\partial x_a \partial x_b \cdots \partial x_\kappa}$ will be sought. These can be formed, as in section a, with the help of the gradient vector, the γ's, M_{ij}, and K_{ijk}.

Using only the gradient vector, $\partial / \partial x_i$, the most general four-vectors are

$$\frac{\partial}{\partial x_i} \cdot \frac{\partial^m \psi t}{\partial x_\alpha \partial x_\alpha \cdots \partial x_\mu \partial x_\mu \partial x_\nu \cdots \partial x_\rho} \cdot \frac{\partial^n \varphi}{\partial x_\beta \partial x_\beta \cdots \partial x_\nu \cdots \partial x_\rho}, \qquad (66)$$

$$\frac{\partial^m \psi t}{\partial x_\alpha \partial x_\alpha \cdots \partial x_\mu \partial x_\mu \partial x_\nu \cdots \partial x_\rho} \cdot \frac{\partial}{\partial x_i} \frac{\partial^n \varphi}{\partial x_\beta \partial x_\beta \cdots \partial x_\nu \partial x_\rho} \quad (67)$$

These immediately reduce to the types:

$$(m+1, m)_i = \frac{\partial}{\partial x_i} \frac{\partial^m \psi t}{\partial x_\alpha \partial x_\beta \cdots \partial x_\rho} \cdot \frac{\partial^m \varphi}{\partial x_\alpha \partial x_\beta \cdots \partial x_\rho} \quad (68)$$

$$(m, m+1)_i = \frac{\partial^m \psi t}{\partial x_\alpha \partial x_\beta \cdots \partial x_\rho} \cdot \frac{\partial}{\partial x_i} \frac{\partial^m \varphi}{\partial x_\alpha \partial x_\beta \cdots \partial x_\rho} \quad (69)$$

because factors like $\frac{\partial^2 \psi t}{\partial x_\alpha \partial x_\alpha}$ and $\frac{\partial^2 \varphi}{\partial x_\beta \partial x_\beta}$ can be replaced by ψt and $\mu^2 \varphi$, respectively, according to the second order Dirac equations; assuming that Z = 0:

$$\frac{\partial^2 \psi t}{\partial x_\alpha \partial x_\alpha} = \psi t \quad (70)$$

$$\frac{\partial^2 \varphi}{\partial x_\beta \partial x_\beta} = \mu^2 \varphi \quad (71)$$

Keeping (70) and (71) in mind, the most general four-vectors which can be written with the help of a single γ are the five:

$$(m, m)_i = \frac{\partial^m \psi t}{\partial x_\alpha \cdots \partial x_\rho} \gamma_i \frac{\partial^m \varphi}{\partial x_\alpha \cdots \partial x_\rho} \quad (72)$$

$$\frac{\partial}{\partial x_i}\,\frac{\partial}{\partial x_\beta}\,\frac{\partial^m \psi^\dagger}{\partial x_\alpha \cdots \partial x_\rho}\,\gamma^\beta\,\frac{\partial^m \varphi}{\partial x_\alpha \cdots \partial x_\rho} \tag{73}$$

$$\frac{\partial}{\partial x_i}\,\frac{\partial^m \psi^\dagger}{\partial x_\alpha \cdots \partial x_\rho}\,\gamma^\beta\,\frac{\partial}{\partial x_\beta}\,\frac{\partial^m \varphi}{\partial x_\alpha \cdots \partial x_\rho} \tag{74}$$

$$\frac{\partial}{\partial x_\beta}\,\frac{\partial^m \psi^\dagger}{\partial x_\alpha \cdots \partial x_\rho}\,\gamma^\beta\,\frac{\partial}{\partial x_i}\,\frac{\partial^m \varphi}{\partial x_\alpha \cdots \partial x_\rho} \tag{75}$$

$$\frac{\partial^m \psi^\dagger}{\partial x_\alpha \cdots \partial x_\rho}\,\gamma^\beta\,\frac{\partial}{\partial x_\beta}\,\frac{\partial}{\partial x_i}\,\frac{\partial^m \varphi}{\partial x_\alpha \cdots \partial x_\rho} \tag{76}$$

According to the first order Dirac equations (Z = 0, again), (61) and

$$\frac{\partial \psi^\dagger}{\partial x_\beta}\,\gamma^\beta - \psi^\dagger = 0 \tag{77}$$

(73) and (74) immediately reduce to the type (m + 1, m) given by (68); (75) and (76) to the type (m, m + 1).

Eight four-vectors can be written with the help of M_{ij}. They are (using the notation, $\nabla^m = \partial^m / \partial x_\alpha \cdots \partial x_\rho$):

$$\frac{\partial}{\partial x_\beta}\,\nabla^m \psi^\dagger\, M_{i\beta}\,\nabla^m \varphi \tag{78}$$

$$\nabla^m \psi^\dagger\, M_{i\beta}\,\frac{\partial}{\partial x_\beta}\,\nabla^m \varphi \tag{79}$$

$$\frac{\partial}{\partial x_i}\frac{\partial^2}{\partial x_\beta \partial x_\gamma}\nabla^m\psi^\dagger M_{\beta\gamma}\nabla^m\varphi \tag{80}$$

$$\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_\beta}\nabla^m\psi^\dagger M_{\beta\gamma}\frac{\partial}{\partial x_\gamma}\nabla^m\varphi \tag{81}$$

$$\frac{\partial}{\partial x_i}\nabla^m\psi^\dagger M_{\beta\gamma}\frac{\partial^2}{\partial x_\beta \partial x_\gamma}\nabla^m\varphi \tag{82}$$

$$\nabla^m\psi^\dagger M_{\beta\gamma}\frac{\partial}{\partial x_i}\frac{\partial^2}{\partial x_\beta \partial x_\gamma}\nabla^m\varphi \tag{83}$$

$$\frac{\partial}{\partial x_\beta}\nabla^m\psi^\dagger M_{\beta\gamma}\frac{\partial}{\partial x_\gamma}\frac{\partial}{\partial x_i}\nabla^m\varphi \tag{84}$$

$$\frac{\partial^2}{\partial x_\beta \partial x_\gamma}\nabla^m\psi^\dagger M_{\beta\gamma}\frac{\partial}{\partial x_i}\nabla^m\varphi \tag{85}$$

Using (77), the fourth component of (78), for example, becomes:

$$\nabla^m\left(\frac{\partial\psi^\dagger}{\partial x_4}\gamma^4-\psi^\dagger\right)\gamma^4\nabla^m\varphi = (m+1,m)_4 - (m,m)_4$$

In the same way, (79) = $-(m, m+1) - \mu(m, m)$. The rest of the eight vectors are not quite so obviously reducible to (m, m), $(m+1, m)$, $(m, m+1)$ and the linear combinations of

these. For example, when written out (80) is:

$$\frac{\partial}{\partial x_i}\nabla^m\Big[\frac{\partial}{\partial x_1}\Big(\frac{\partial\psi^\dagger}{\partial x_2}\gamma^2+\frac{\partial\psi^\dagger}{\partial x_3}\gamma^3+\frac{\partial\psi^\dagger}{\partial x_4}\gamma^4\Big)\gamma^1$$

$$+\frac{\partial}{\partial x_2}\Big(\frac{\partial\psi^\dagger}{\partial x_1}\gamma^1+\frac{\partial\psi^\dagger}{\partial x_3}\gamma^3+\frac{\partial\psi^\dagger}{\partial x_4}\gamma^4\Big)\gamma^2$$

$$+\frac{\partial}{\partial x_3}\Big(\frac{\partial\psi^\dagger}{\partial x_1}\gamma^1+\frac{\partial\psi^\dagger}{\partial x_2}\gamma^2+\frac{\partial\psi^\dagger}{\partial x_4}\gamma^4\Big)\gamma^3$$

$$+\frac{\partial}{\partial x_4}\Big(\frac{\partial\psi^\dagger}{\partial x_1}\gamma^1+\frac{\partial\psi^\dagger}{\partial x_2}\gamma^2+\frac{\partial\psi^\dagger}{\partial x_3}\gamma^3\Big)\gamma^4\Big]\nabla^m\varphi$$

With the help of (77) it reduces to:

$$\frac{\partial}{\partial x_i}\nabla^m\Big[\frac{\partial}{\partial x_1}\Big(\frac{\partial\psi^\dagger}{\partial x_1}\gamma^1-\psi^\dagger\Big)\gamma^1-\frac{\partial}{\partial x_2}\Big(\frac{\partial\psi^\dagger}{\partial x_2}\gamma^2-\psi^\dagger\Big)\gamma^2$$

$$-\frac{\partial}{\partial x_3}\Big(\frac{\partial\psi^\dagger}{\partial x_3}\gamma^3-\psi^\dagger\Big)\gamma^3-\frac{\partial}{\partial x_4}\Big(\frac{\partial\psi^\dagger}{\partial x_4}\gamma^4-\psi^\dagger\Big)\gamma^4\Big]\nabla^m\varphi$$

$$=\frac{\partial}{\partial x_i}\nabla^m\Big[-\frac{\partial^2\psi^\dagger}{\partial x_\alpha\partial x_\alpha}+\frac{\partial\psi^\dagger}{\partial x_\nu}\gamma^\nu\Big]\nabla^m\varphi$$

$$=-(m+1,\,m)_i+(m+1,\,m)_i\quad=0$$

The vectors (82), (83) and (85) also reduce to zero. (81) reduces to

$$-(m+2,\,m+1)-\mu\,(m+1,\,m)$$

and (84) to

$$-(m+1,\ m+2) - \mu\ (m,\ m+1).$$

Eleven vectors can be formed with K_{ijk}, of which six can be reduced to zero and the others to linear combinations of (68), (69) and (72) by the methods above.

It appears, therefore, that because of the requirement of the vector property the only types of interactions consistent with (20) which are allowed are the ones denoted by $(m,\ m)$, $(m,\ m+1)$ and $(m+1,\ m)$ together with their linear combinations. There are, of course, in parallel with these as always, the vectors in the $(n,\ e^-)$ viewpoint, obtained from the above by replacing φ^* with $\delta\varphi$.

d. The Distribution Formulae. The calculations of these will go as in Chapter IV with the vector A' except that $|S(\psi_s \varphi_\sigma^*)|^2$ is here replaced by

$$\left| S\left(\frac{\partial^m \psi_s}{\partial x_\alpha \cdots \partial x_\rho} \cdot \frac{\partial^m \varphi_\sigma^*}{\partial x_\alpha \cdots \partial x_\rho} \right) \right|^2 \tag{86}$$

for the $(m,\ m)$ type of vector ($\psi^\dagger = i\,\psi^* \beta$), by

$$\left| S\left(\frac{\partial}{\partial t} \frac{\partial^m \psi_s}{\partial x_\alpha \cdots \partial x_\rho} \beta \frac{\partial^m \varphi_\sigma^*}{\partial x_\alpha \cdots \partial x_\rho} \right) \right|^2 \tag{87}$$

for the $(m+1,\ m)$ type, and by

$$\left| S\left(\frac{\partial^m \psi_s}{\partial x_\alpha \cdots \partial x_\rho} \beta \frac{\partial}{\partial t} \frac{\partial^m \varphi_\sigma^*}{\partial x_\alpha \cdots \partial x_\rho} \right) \right|^2 \tag{88}$$

for the $(m,\ m+1)$ type. Only the case $Z = 0$ is treated here,

so that it is the β-spectra of the light elements which will test the results.

1°/ The (0, 0) Interaction. This is just the Ansatz adopted by Fermi; the results are given in Chapter IV.

2°/ The (0, 1) Interaction. This case was treated in Chapter V, section b.

3°/ The (1, 0) Interaction. Recalling (36) and (38),

$$\left|S\left(\frac{\partial}{\partial t}\psi_s \beta \varphi_\sigma^*\right)\right|^2 = W_s^2 \left|S(\psi_s \beta \varphi_\sigma^*)\right|^2 = \frac{W_s^2}{\Omega^2}\left(1 - \frac{p_s \cdot p_\sigma + \mu}{W_s K_\sigma}\right). \quad (89)$$

The distribution formula is therefore:

$$P_{1,0}(W)dW = dW \frac{G^2|M|^2}{2\pi^3} W^3\sqrt{W^2-1}\, K\sqrt{K^2-\mu^2}\left(1 - \frac{\mu}{WK}\right). \quad (90)$$

This gives greater weight to high electron energies than even the (0, 0) interaction. Since it was seen in Chapter III that the data require greater weight to be given to high neutrino energies, the (1, 0) interaction will be ruled out of further consideration. It is obvious that all the vectors of the type (m + 1, m) should be ruled out for the same reason.

4°/ The (1, 1) Interaction.

$$\left|S\left(\frac{\partial \psi_s}{\partial x_\alpha}\frac{\partial \varphi_\sigma^*}{\partial x_\alpha}\right)\right|^2 = \frac{1}{\Omega^2}\left(p_s \cdot p_\sigma - W_s K_\sigma\right)^2\left(1 + \frac{p_s \cdot p_\sigma - \mu}{W_s K_\sigma}\right). \quad (91)$$

Since terms having odd powers of $(p_s \cdot p_\sigma)$ drop out of the integrations over the directions of $\vec{p}_s$ and $\vec{p}_\sigma$, the only terms

of (91) which need be kept are:

$$\frac{1}{\Omega^2}\left\{\left[(p_s\cdot p_\sigma)^2+W_s^2K_\sigma^2\right]\left(1-\frac{\mu}{W_sK_\sigma}\right)-2(p_s\cdot p_\sigma)^2\right\}$$

$$=\frac{1}{\Omega^2}\left\{\left[W_s^2K_\sigma^2-(p_s\cdot p_\sigma)^2\right]-\frac{\mu}{W_sK_\sigma}\left[W_s^2K_\sigma^2+(p_s\cdot p_\sigma)^2\right]\right\}$$

Since $$\iint d\omega_s\, d\omega_\sigma (p_s\cdot p_\sigma)^2=\frac{16}{3}\pi^2 p_s^2 p_\sigma^2,$$

$$P_{b1}(W)\,dW = dW\,\frac{G^2|M|^2}{6\pi^3}\,W\sqrt{W^2-1}\,K\sqrt{K^2-\mu^2}\;\times$$

$$\left\{\left[K^2(2W^2+1)+\mu^2(W^2-1)\right]-\frac{\mu}{WK}\left[K^2(4W^2-1)-\mu^2(W^2-1)\right]\right\}$$

5°/ The (1, 2) Interaction.

$$\left|S\left(\frac{\partial\psi_s}{\partial x_\alpha}\beta\frac{\partial}{\partial t}\frac{\partial\varphi_\sigma^*}{\partial x_\alpha}\right)\right|^2=\frac{K_\sigma^2}{\Omega^2}(p_s\cdot p_\sigma-W_sK_\sigma)^2\left(1-\frac{p_s\cdot p_\sigma+\mu}{W_sK_\sigma}\right). \quad (93)$$

Dropping odd powers of ($p_s \cdot p_\sigma$), this expression becomes:

$$\frac{K_\sigma^2}{\Omega^2}\left\{\left[W_s^2K_\sigma^2+3(p_s\cdot p_\sigma)^2\right]-\frac{\mu}{W_sK_\sigma}\left[W_s^2K_\sigma^2+(p_s\cdot p_\sigma)^2\right]\right\}$$

It finally leads to:

$$P_{1,2}(W)dW = dW \frac{G^2|M|^2}{2\pi^3} W\sqrt{W^2-1}\, K^3 \sqrt{K^2-\mu^2} \times$$

$$\left\{\left[(2W^2-1)K^2-\mu^2(W^2-1)\right]-\frac{\mu}{3WK}\left[K^2(4W^2-1)-\mu^2(W^2-1)\right]\right\} . \quad (94)$$

6°/ The (2, 2) Interaction.

$$\left|S\left(\frac{\partial^2\psi_s}{\partial x_\alpha \partial x_\beta}\,\frac{\partial^2\varphi_\pi}{\partial x_\alpha \partial x_\beta}\right)\right|^2 = \frac{1}{\Omega^2}(p_s\cdot p_\sigma - W_s K_\sigma)^4\left(1+\frac{p_s\cdot p_\sigma-\mu}{W_s K_\sigma}\right). \quad (95)$$

The terms of this to be kept are:

$$\frac{1}{\Omega^2}\left\{\left[(p_s\cdot p_\sigma)^4+6(p_s\cdot p_\sigma)^2 W_s^2 K_\sigma^2+W_s^4 K_\sigma^4\right]\left(1-\frac{\mu}{W_s K_\sigma}\right)\right.$$

$$\left.-4\left[(p_s\cdot p_\sigma)^4+(p_s\cdot p_\sigma)^2 W_s^2 K_\sigma^2\right]\right\}$$

$$= \frac{1}{\Omega^2}\left\{\left[W_s^4 K_\sigma^4+2(p_s\cdot p_\sigma)^2 W_s^2 K_\sigma^2-3(p_s\cdot p_\sigma)^4\right]\right.$$

$$\left.-\frac{\mu}{W_s K_\sigma}\left[W_s^4 K_\sigma^4+6(p_s\cdot p_\sigma)^2 W_s^2 K_\sigma^2+(p_s\cdot p_\sigma)^4\right]\right\}$$

Since $$\iint d\omega_s\, d\omega_\sigma\,(p_s\cdot p_\sigma)^4 = \frac{16}{5}\pi^2 p_s^4 p_\sigma^4$$

$$P_{2,2}(W)dW = dW\,\frac{G^2|M|^2}{30\pi^3}\,W\sqrt{W^2-1}\;K\sqrt{K^2-\mu^2}$$

$$\times\left\{\left[K^4(16W^4+8W^2-9)+2\mu^2K^2(W^2-1)(4W^2-9)-9\mu^4(W^2-1)^2\right]\right.$$

$$\left.-\frac{\mu}{WK}\left[K^4\left((4W^2-1)^2-4W^2\right)-2\mu^2K^2(W^2-1)(6W^2-1)+\mu^4(W^2-1)^2\right]\right\}. \quad (96)$$

7°/ The (2, 3) Interaction.

$$\left|S\left(\frac{\partial^2\psi_s}{\partial x_\alpha\partial x_\beta}\,\beta\,\frac{\partial}{\partial t}\,\frac{\partial^2\varphi}{\partial x_\alpha\partial x_\beta}\right)\right|^2=\frac{K_\sigma^2}{\Omega^2}\left(p_s\cdot p_\sigma-W_sK_\sigma\right)^4\left(1-\frac{p_s\cdot p_\sigma+\mu}{W_sK_\sigma}\right) \quad (97)$$

becomes, upon dropping terms having odd powers of $(p_s \cdot p_\sigma)$:

$$\frac{K_\sigma^2}{\Omega^2}\left\{\left[5(p_s\cdot p_\sigma)^4+10(p_s\cdot p_\sigma)^2W_s^2K_\sigma^2+W_s^4K_\sigma^4\right]\right.$$

$$\left.-\frac{\mu}{W_sK_\sigma}\left[(p_s\cdot p_\sigma)^4+6(p_s\cdot p_\sigma)^2W_s^2K_\sigma^2+W_s^4K_\sigma^4\right]\right\}.$$

The final result is:

$$P_{2,3}(W)dW = dW\,\frac{G^2|M|^2}{6\pi^3}\,W\sqrt{W^2-1}\;K^3\sqrt{K^2-\mu^2}$$

$$\times\left\{\left[K^4(16W^4-16W^2+3)-2\mu^2K^2(W^2-1)(8W^2-3)+3\mu^4(W^2-1)^2\right]\right.$$

$$\left.-\frac{\mu}{W_sK_\sigma}\left[K^4(8W^4-8W^2+1)-2\mu^2K^2(W^2-1)(4W^2-1)+\mu^4(W^2-1)^2\right]\right\} \quad (98)$$

Of special interest is the case $\mu = 0$, for which the various interactions give the distributions:

$$P_{0,0}(W)dW = \frac{G^2|M|^2}{2\pi^3} W\sqrt{W^2-1}\,(W_0-W)^2 \qquad (99)$$

$$P_{0,1}(W)dW = \frac{G^2|M|^2}{2\pi^3} W\sqrt{W^2-1}\,(W_0-W)^4 \qquad (100)$$

$$P_{1,1}(W)dW = \frac{G^2|M|^2}{6\pi^3} W\sqrt{W^2-1}\,(W_0-W)^4(2W^2+1) \qquad (101)$$

$$P_{1,2}(W)dW = \frac{G^2|M|^2}{6\pi^3} W\sqrt{W^2-1}\,(W_0-W)^6(2W^2-1) \qquad (102)$$

$$P_{2,2}(W)dW = \frac{G^2|M|^2}{30\pi^3} W\sqrt{W^2-1}\,(W_0-W)^6\left[(4W^2+1)^2-10\right] \qquad (103)$$

$$P_{2,3}(W)dW = \frac{G^2|M|^2}{6\pi^3} W\sqrt{W^2-1}\,(W_0-W)^8\left[(4W^2-1)(4W^2+3)\right] \qquad (104)$$

These distributions can be compared with the experimental ones by means of their prediction of the average energy, which is defined by:

$$\overline{W} = \frac{\int_1^{W_0} W P(W)\,dW}{\int_1^{W_0} P(W)\,dW} \qquad (105)$$

It was calculated for the formulae (99) - (104) by means of Simpson's rule, and the results are plotted in Figure 14 which gives the ratio $W_o / \overline{W}$ as a function of $\overline{W}$. ($\overline{W}$ is taken as abscissa because it is better known experimentally than W_o). For large energies,

$$P_{m,m}(W)dW \sim W^{2m+2} (W_o - W)^{2m+2} dW$$

$$P_{m,m+1}(W)dW \sim W^{2m+2}(W_o - W)^{2m+3} dW$$

It can be shown by partial integrations that

$$\int_o^{W_o} W^x (W_o - W)^y dW = W_o^{x+y+1} \frac{y!}{(x+1)(x+2)\cdots(x+y+1)}$$

Using this,

$$\overline{W}_{m,m} \to \frac{\int_o^{W_o} W^{2m+3}(W_o - W)^{2m+2} dW}{\int_o^{W_o} W^{2m+2}(W_o - W)^{2m+2} dW} = \frac{W_o}{2} \quad \text{for large W,}$$

$$\overline{W}_{m,m+1} \to \frac{\int_o^{W_o} W^{2m+3}(W_o - W)^{2m+4} dW}{\int_o^{W_o} W^{2m+2}(W_o - W)^{2m+2} dW} = W_o \frac{2m+3}{4m+8} \quad \text{as W grows,}$$

Accordingly, $W_o / \overline{W}$ as a function of $\overline{W}$ will have asymptotes at $W_o / \overline{W} = 2$ for the (m, m) type and $W_o / \overline{W} = \frac{4m+8}{2m+3}$ for

the (m, m + 1) type. These asymptotes are valuable inasmuch as they set an upper limit on the amount of asymmetry that experimental curves can possess in order to obey the interaction rule they (the asymptotes) represent.

Figure 12 shows that the values of $W_o/\overline{W}$ for the high energy spectra (Li^8, B^{12}) seem to indicate that already the (2, 3) interaction does not give enough asymmetry. The form of the distributions for $m > 1$ are also quite unlike the experimental form as Figure 13 shows. Only the (m, m + 1) types are shown because only they can be expected to compare as favorably with the data as (0, 1). The formulae (99), (100), (102) and (104) are valid for this case because the effect of Z is quite surely negligible for the unusually high energy electrons emitted by the very light nucleus Li^8.

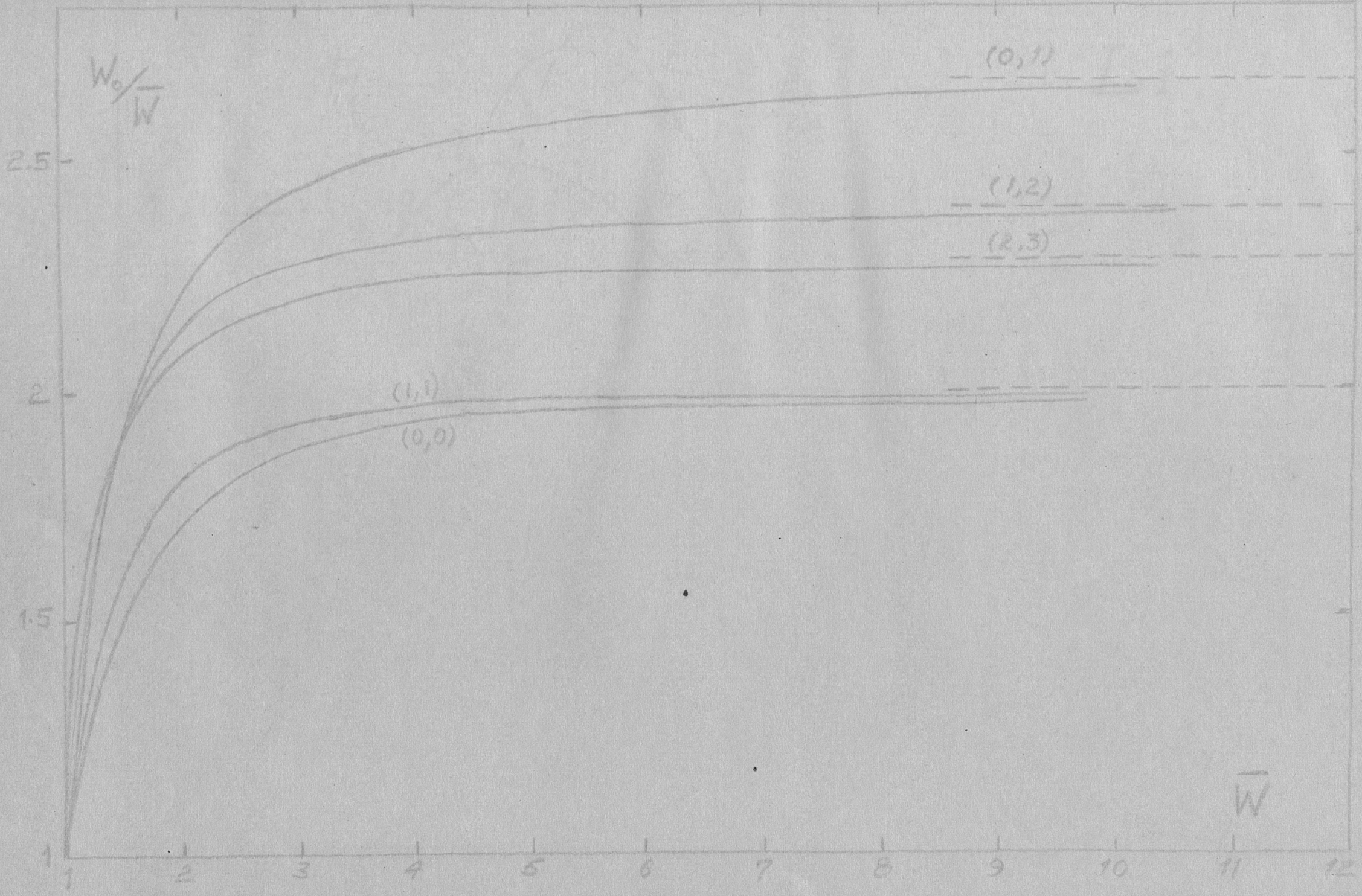

FIGURE 12.

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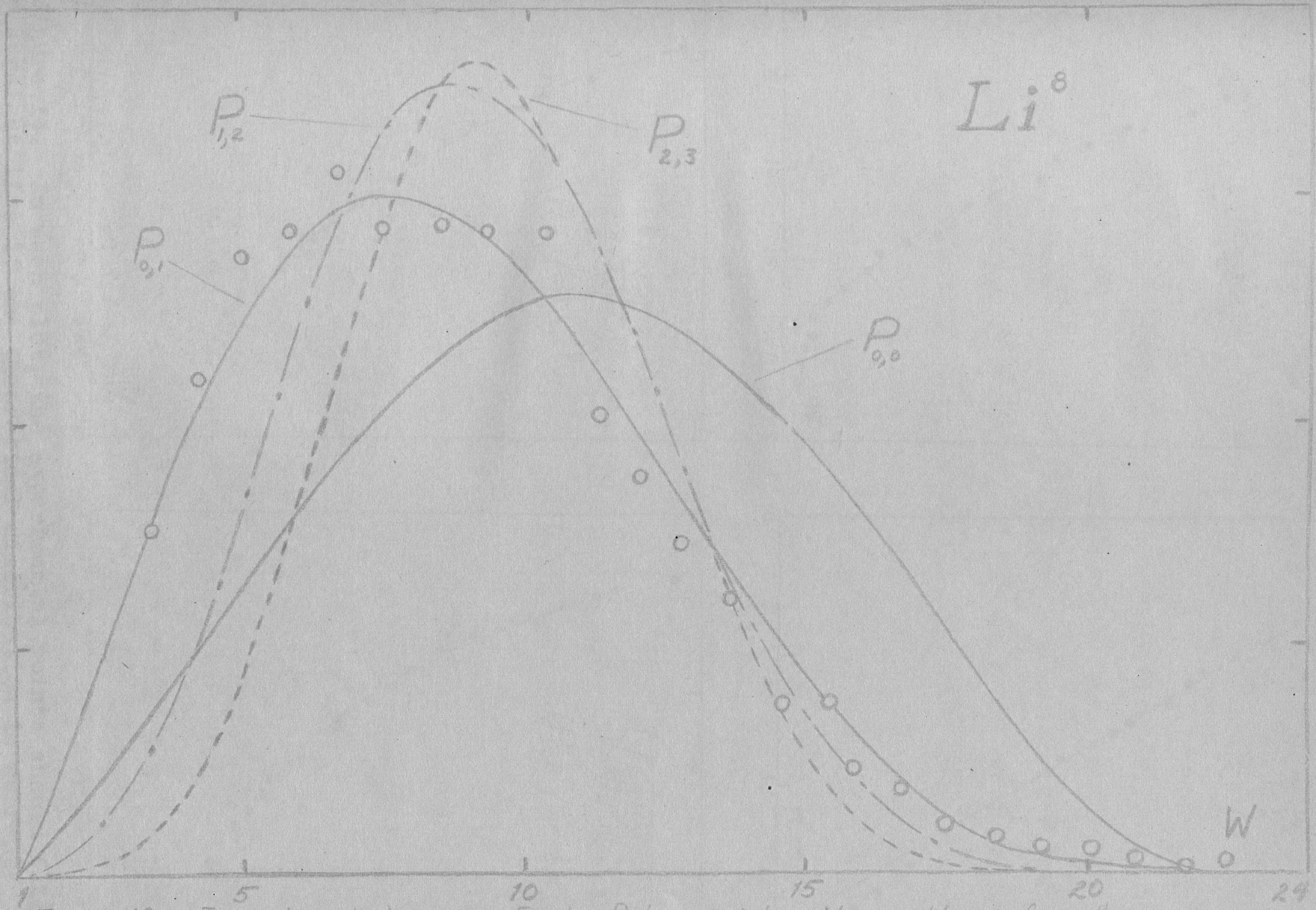

Figure 13. Experimental points: Fowler, Delsasso & Lauritsen. $W_0=23$ for all curves.

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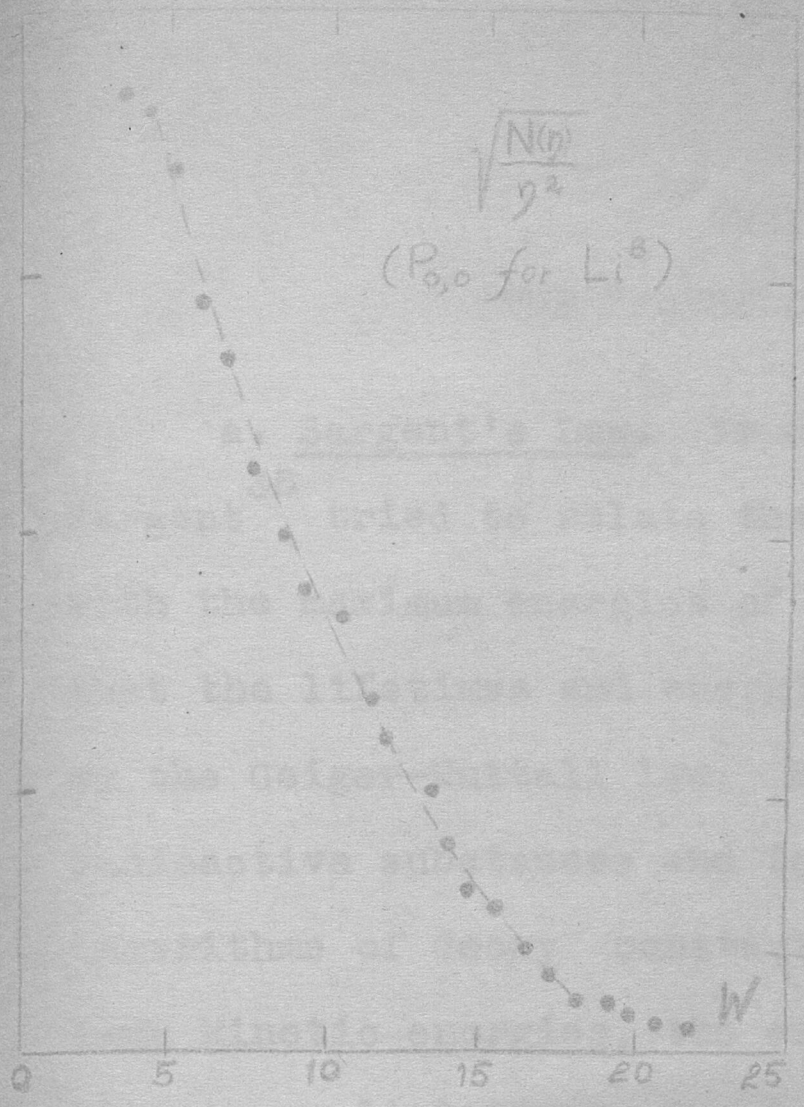

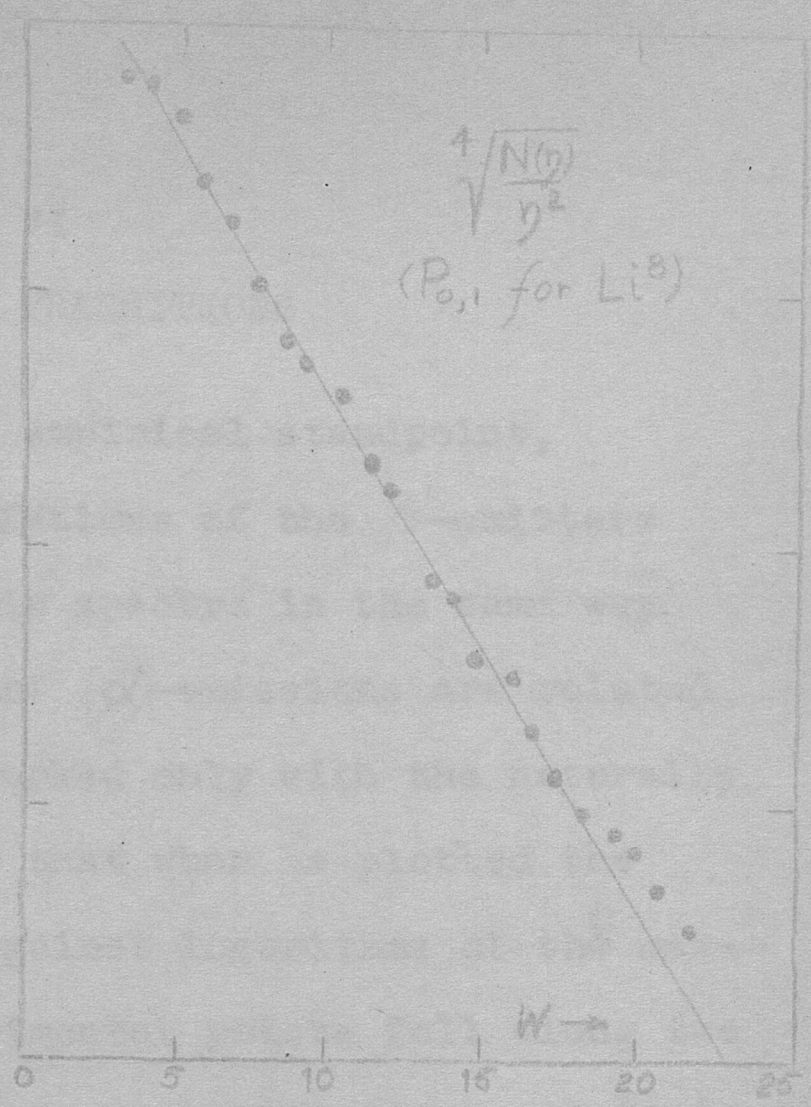

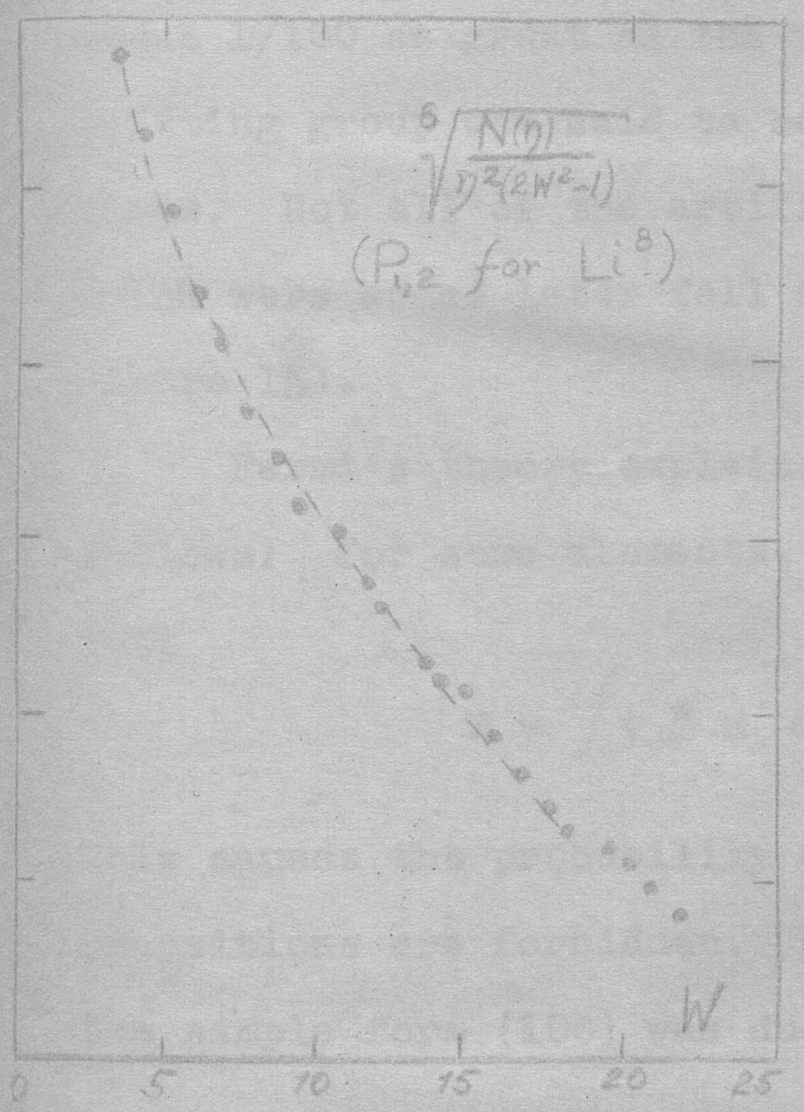

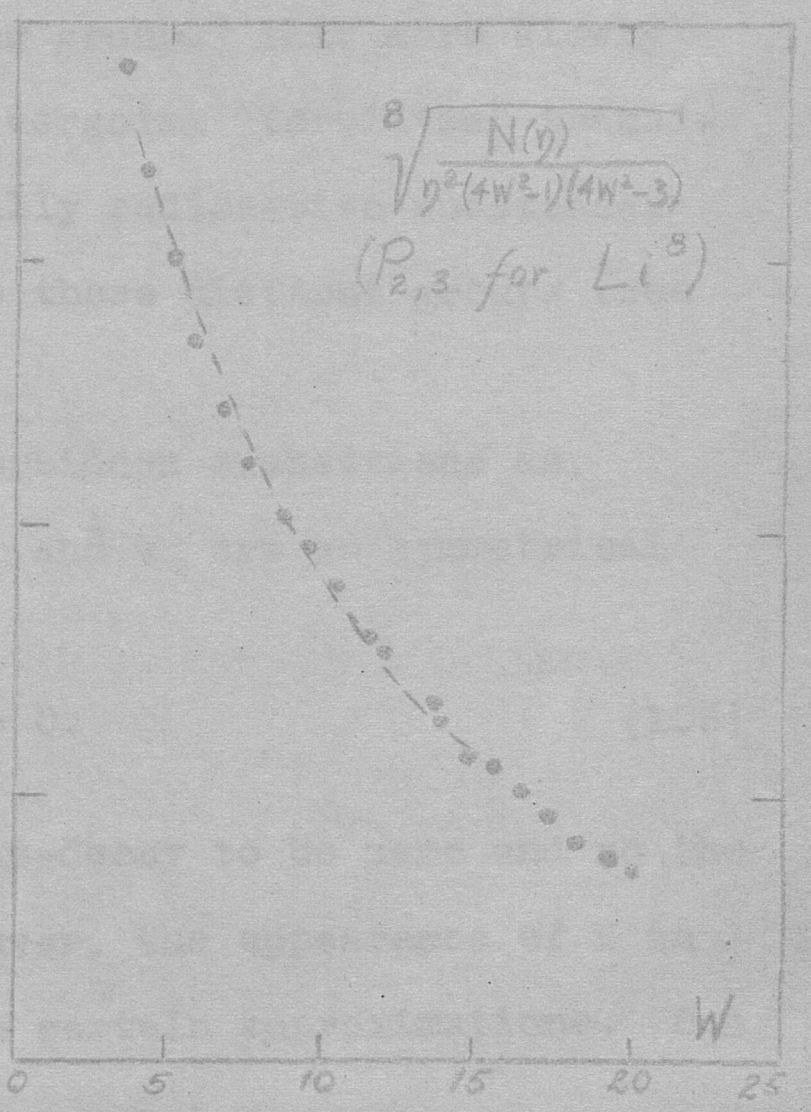

Figure 14. In each plot, the experimental points should fall on a straight line if the theoretical distribution is correct.

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CHAPTER VI

THE "FORBIDDEN" TRANSITIONS

a. Sargent's Law. From an empirical standpoint, Sargent[35] tried to relate the lifetimes of the β-emitters with the maximum energies of their spectra in the same way that the lifetimes and energies of α-emissions are related by the Geiger-Nuttall law. He worked only with the naturally radioactive substances and found that when he plotted the logarithms of decay constants against logarithms of the maximum kinetic energies, the experimental points fell along two almost parallel lines, the lower group having decay constants about 1/100 as great as the upper group. This more slowly emitting group was said to be undergoing "forbidden" transitions. Not all of the artificially radioactive substances which were added later fell into these distinct groups (see Figure 15).

Fermi's theory explains forbidden transitions as follows: for some elements, u_m and v_m are so symmetrical that

$$M = \int v_m^* u_n \, d\tau = 0. \qquad (106)$$

This causes the probability of β-decay to be zero and so the transitions are forbidden. However, the appearance of M in the simple form (106) was due to certain approximations. One

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[35] Proc. Roy. Soc. A 139, 659 (1933).

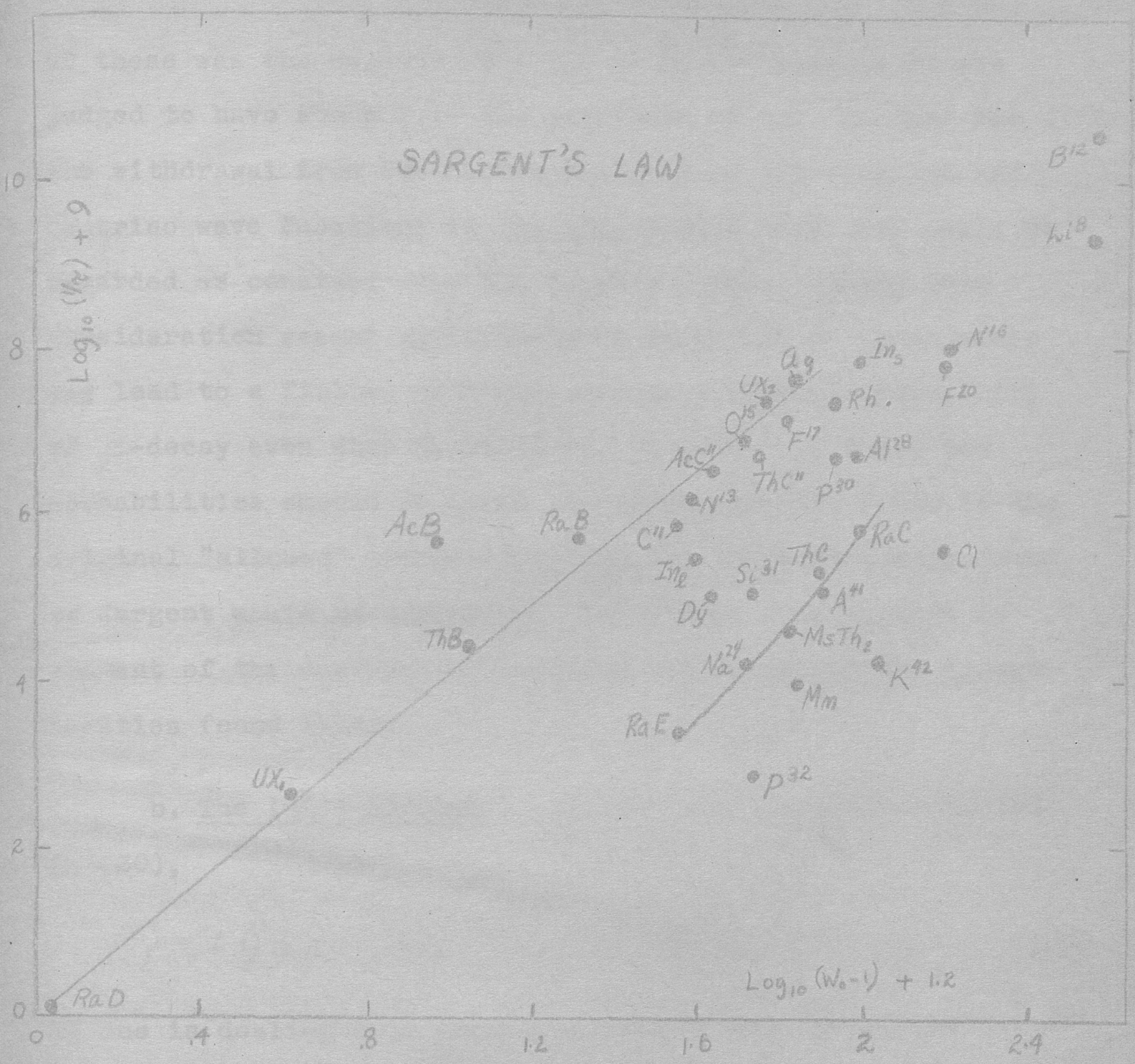

Figure 15. The lines are those originally drawn by Sargent.

of these was the neglect of $(\vec{\alpha}_H \cdot \vec{A})$ in $\mathcal{H}$ because it was judged to have about 1/10 the magnitude of A_o. Another was the withdrawal from the matrix integral of the electron and neutrino wave functions on the supposition that they could be regarded as constant over the nuclear range. Taking into consideration second approximations in either of these cases may lead to a finite, although perhaps a weaker, probability of β-decay even when M, as above, is zero. If these new probabilities should be found to bear a definite ratio to the original "allowed" transitions, the two distinct groups found by Sargent would be explained. Variations from element to element of the unevaluated matrices would explain the irregularities found later.

b. <u>The $(\vec{\alpha}_H \cdot \vec{A})$ Term</u> . If this had not been neglected in (30),

$$\mathcal{H}_f' = \pm g \int d\tau\, v_m^* [(\psi_s^* \delta \varphi_o^*) + (\psi_s^* \vec{\alpha}_H \cdot \vec{\tau}\, \delta \varphi_o^*)] u_n \qquad (107)$$

If one is dealing with forbidden transitions, it may be supposed that

$$\int v_m^* (\psi_s^* \delta \varphi_o^*) u_n d\tau = \delta(\psi_s^* \delta \varphi_o^*) \int d\tau\, v_m^* u_n = 0 \qquad (108)$$

Thus

$$\mathcal{H}_f' = \pm g \{ \delta(\psi_s^* \alpha_x \delta \varphi_o^*) M_x + \delta(\psi_s^* \alpha_y \delta \varphi_o^*) M_y + \delta(\psi_s^* \alpha_z \delta \varphi_o^*) M_z \}$$

in which $M_x = \int v_m^* \alpha_{Hx} u_n d\tau$, etc. For the case Z = 0,

(36) and (38) are employed and it is found that

$$|\mathcal{H}_f^i|^2 = \frac{G^2}{\Omega^2}\left(1 - \frac{(p_s \cdot p_\sigma) + \mu}{W_s K_\sigma}\right)\left\{|M_x|^2 + |M_y|^2 + |M_z|^2\right\}. \qquad (109)$$

With $Z \neq 0$,

$$|\mathcal{H}_f^i|^2 = \frac{G^2}{4\pi\Omega K_\sigma}\left[(K_\sigma + \mu)(f_0^2 + f_{-2}^2) + (K_\sigma - \mu)(g_0^2 + g_{-2}^2)\right]|\vec{M}|^2$$

$$= \frac{2G^2|\vec{M}|^2}{\pi R\Omega}\,\frac{e^{\pi\lambda}|\Gamma(\gamma + i\lambda)|^2}{[\Gamma(2\gamma+1)]^2}(2r_0)^{2\gamma-2}\,p^{2\gamma}\left(1 - \frac{g^2\mu}{W_s K_\sigma}\right). \qquad (110)$$

in which $|\vec{M}|^2 = |M_x|^2 + |M_y|^2 + |M_z|^2$. Comparison of (109) with (39) and (110) with (48) reveals that the "forbidden" distribution formulae as "allowed" by ($\vec{\alpha}_H \cdot A$) are essentially the same as the ordinary formulae except that in the former $|\vec{M}|^2$ appears instead of $|M|^2$. Of course, it is being assumed that the selection rules for $|\vec{M}|^2$ are such that it is not zero when $|M|^2 = 0$. Since $|\vec{M}|^2$ is about 1/100 as large as $|M|^2$ at its normal value, the decay in these forbidden transitions is expected to be as much weaker as Sargent's law suggests.

With the (0, 1) interaction, (107) is replaced by:

$$\mathcal{H}_f^i = \pm i g S(\psi_s^* \beta \delta \psi_\sigma^*)\left[K_\sigma M - (\vec{p}_\sigma \cdot \vec{M})\right]. \qquad (111)$$

Again, supposing that M = 0 for forbidden transitions,

$$|\mathcal{H}_f^i|^2 = \frac{G^2}{\Omega^2}\,p_\sigma^2|\vec{M}|^2\left(1 - \frac{p_s \cdot p_\sigma - \mu}{W_s K_\sigma}\right) \qquad (112)$$

when Z = 0 and

$$|\mathcal{H}_f^i|^2 = \frac{2G^2|\vec{M}|^2}{\pi R\Omega}\,\frac{e^{\pi\lambda}|\Gamma(\gamma+i\lambda)|^2}{[\Gamma(2\gamma+1)]^2}\,(2r_0)^{2\gamma-2}\,p^{2\gamma}\,p_\sigma^2\left(1-\frac{\gamma^2\mu}{N_s K_\sigma}\right) \qquad (113)$$

when $Z \neq 0$. Again the formulae are essentially the same as for ordinary transitions. Since p_σ^2 replaces K_σ^2, the order of contact with the axis near the upper limit is greater unless $\mu = 0$ ($p_\sigma = K_\sigma$), as is assumed in all comparisons with the data.

c. A Second Approximation in $\mathcal{H}_f^i$, Z = 0. The approximation represented by (30), is equivalent to putting for ($\psi_s\,\delta\,\varphi_\sigma$) the value ($A_s\,\delta\,B_\sigma$)/$\Omega$ (Z = 0) or for $e^{i(p_s+p_\sigma)\cdot r}$ unity. The approximation can perhaps be carried further by developing the exponential so that,

$$\mathcal{H}_f^i = \pm G\,S(A_s^*\,\delta\,B_\sigma^*)\int v_m^*\left[1 + i(p_s+p_\sigma)\cdot r + \cdots\right] u_n\,d\tau.$$

For a "forbidden" transition, one supposes that the term $\int v_m^*\,u_n\,d\tau$ vanishes, so that:

$$\mathcal{H}_f^i = \pm i\,\frac{G}{\Omega}\,S(A_s^*\delta B_\sigma^*)\left\{(p_{sx}+p_{\sigma x})\int v_m^* x\,u_n\,d\tau + \cdots + \cdots\right\}$$

$$|\mathcal{H}_f^i|^2 = \frac{G^2}{\Omega^2}\,|S(A_s\,\delta\,B_\sigma)|^2\left|(p_{sx}+p_{\sigma x})|X| + \cdots + \cdots\right|^2 \qquad (114)$$

In deriving the distribution formula, there are integrations

over the directions of $\vec{p}_s$ and $\vec{p}_\sigma$ to perform (see Chapter IV):

$$\iint d\omega_s d\omega_\sigma \left(1 + \frac{p_s \cdot p_\sigma + \mu}{W_s K_\sigma}\right)\Big\{(p_{sx}+p_{\sigma x})^2 |X|^2 + (p_{sy}+p_{\sigma y})^2 |Y|^2 + \cdots +$$

$$(p_{sx}+p_{\sigma x})(p_{sy}+p_{\sigma y})\left[|X||Y^*| + |X^*||Y|\right] + \cdots + \cdots\Big\}$$

$$= \frac{16\pi^2}{3}\left\{(p_s^2 + p_\sigma^2)\left(1 + \frac{\mu}{W_s K_\sigma}\right) + 2/3 \; \frac{p_s^2 p_\sigma^2}{W_s K_\sigma}\right\}$$

This finally leads to the formula:

$$P(W)dW = \frac{G^2}{2\pi^3}\left\{|X|^2 + |Y|^2 + |Z|^2\right\} W\sqrt{W^2-1}\; K\sqrt{K^2-\mu^2}\; \times$$

$$1/3\left\{(p_s^2 + p_\sigma^2)\left(1 + \frac{\mu}{WK}\right) + 2/3 \; \frac{p_s^2 p_\sigma^2}{W K}\right\} dW. \qquad (114)$$

With the (0, 1) interaction, besides the usual replacement of K by K^3, the sign of the last term inside the last bracket is changed because (see footnote 34)

$$|S(A_s \beta \delta B_\sigma)|^2 = 1 - \frac{(p_s \cdot p_\sigma) - \mu}{W_s K_\sigma}.$$

The curves for these forbidden transitions, both for the (0, 0) and (0, 1) interactions, are compared with the ordinary $P_{0,1}$ distribution for P^{32} in Figure 16. Although the Sargent plot (Figure 15) seems to definitely class P^{32} as an element undergoing forbidden transitions, the ordinary $P_{0,1}$ distribution is valid if it is the ($\vec{\alpha}_H \cdot A$) term (see preceding section) which is responsible for the transition.

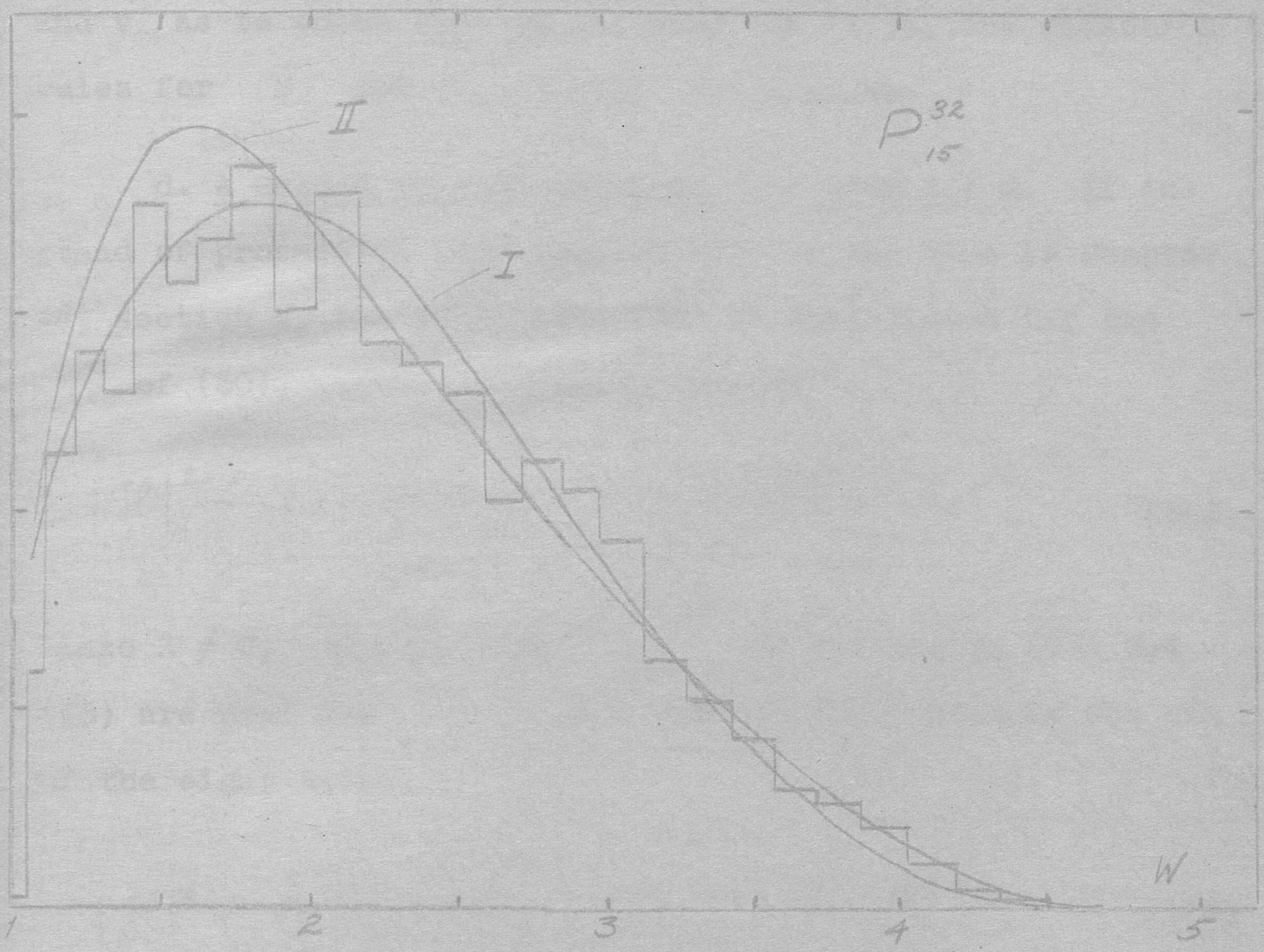

Figure 16. Forbidden transition curves with the (0, 1) interaction. I, when permitted by ($\vec{\alpha}_H \cdot \vec{A}$) term (same as ordinary transitions). II, when permitted by second term of expansion (see Chapter VI, section c).

No judgment can be made without more knowledge about u_m and v_m as to which type is correct. As it is, the selection rules for $|\vec{M}|$ and $|X|, |Y|, |Z|$ are unknown.

d. <u>A Second Approximation in $\mathcal{H}_f'$ with $Z \neq 0$.</u> If instead of proceeding from (30) to (31) as was done in Chapter IV, section e, the plane wave (36) is substituted for the ψ_σ of (30),

$$|\mathcal{H}|^2 = \frac{G^2}{\Omega} \left| \int d\tau\, v_m (\psi_s\, \delta B_\sigma)\, e^{i p_\sigma \cdot r}\, u_m^*\, d\tau \right|^2 . \qquad (115)$$

Since $Z \neq 0$, $\psi^{(a)}_{0,m}$ and $\psi^{(b)}_{1,m}$ as defined by (44) and (45) are used for ψ_s. As a result, (115) becomes the sum of the eight terms;

$$|\mathcal{H}|^2 = \frac{G^2}{\Omega} \{ |a|^2 + |\mathcal{B}|^2 + \cdots + |b|^2 \}, \qquad (116)$$

where

$$|a|^2 = \left| \int d\tau\, v_m (\psi^{(a)}_{0,-1}\, \delta B^{(1)}_\sigma)\, e^{i p_\sigma \cdot r}\, u_m^* \right|^2$$

$$|\mathcal{B}|^2 = \left| \int d\tau\, v_m (\psi^{(a)}_{0,0}\, \delta B^{(1)}_\sigma)\, e^{i p_\sigma \cdot r}\, u_m^* \right|^2$$

. . . . . .

$$|b|^2 = \left| \int d\tau\, v_m (\psi^{(b)}_{1,0}\, \delta B^{(2)}_\sigma)\, e^{i p_\sigma \cdot r}\, u_m^* \right|^2$$

Then, when (44), (45) and (37) are introduced:

$$a = \frac{1}{\sqrt{4\pi}}\sqrt{\frac{\mu+K_\sigma}{2K_\sigma}}\left\{ i\frac{p_{\sigma x}+ip_{\sigma y}}{\mu+K_\sigma}\int dz\, v_m f_0 e^{ip_\sigma\cdot r}\sin\vartheta\, e^{-i\varphi}u_m^* \right.$$

$$+ i\frac{p_{\sigma z}}{\mu+K_\sigma}\int dz\, v_m f_0 e^{ip_\sigma\cdot r}\cos\vartheta\, u_m^*$$

$$\left. - \int dz\, v_m g_0 u_m^* \right\}$$

$$= \frac{1}{\sqrt{4\pi}}\sqrt{\frac{\mu+K_\sigma}{2K_\sigma}}\left\{ a_1 + a_2 + a_3 \right\}$$

In introducing the expressions (46) and (47) for f and g, the exponentials $e^{\pm ip_s\cdot r}$, $e^{ip_\sigma\cdot r}$ and the hypergeometric functions are all set equal to unity because r is small in the region in which u_m and v_m are not zero and the first term of the developments of each of these functions is unity. Therefore:

$$a_1 = i\frac{p_{\sigma x}+ip_{\sigma y}}{\mu+K_\sigma}\sqrt{1-W}\,N_0(\omega-\omega^*)\int \Theta \sin\vartheta\, e^{-i\varphi}dz$$

where $\omega = e^{i\vartheta_0}(\gamma+i\lambda)$ and $\Theta = v_m u_m^* (zr)^{\gamma-1}$

$$a_2 = i\frac{p_{\sigma z}}{\mu+K_\sigma}\sqrt{1-W}\,N_0(\omega-\omega^*)\int \Theta \cos\vartheta\, dz$$

$$a_3 = -\sqrt{1+W}\,N_0(\omega+\omega^*)\int \Theta\, dz$$

In a similar way it will be found that

$$\mathcal{B} = \frac{1}{\sqrt{4\pi}} \sqrt{\frac{\mu + K_\sigma}{2K_\sigma}} \left\{ \mathcal{B}_1 + \mathcal{B}_2 \right\}$$

in which

$$\mathcal{B}_1 = -i \frac{P_{\sigma x} + i P_{\sigma y}}{\mu + K_\sigma} \sqrt{1 - W} N_0 (\omega - \omega^*) \int \Theta \cos\vartheta \, dz$$

$$\mathcal{B}_2 = i \frac{P_{\sigma z}}{\mu + K_\sigma} \sqrt{1 - W} N_0 (\omega - \omega^*) \int \Theta \sin\vartheta \, e^{i\varphi} dz$$

This procedure must be continued for $\mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{F}, \mathcal{G}, \mathcal{H}$ and $\mathcal{I}$. When one is finally ready to make substitutions into (116), one finds upon omitting terms linear in components of $\vec{p}_\sigma$ (they vanish in the subsequent angular integration):

$$|\mathcal{H}|^2 = \frac{G^2}{\Omega} \frac{N_0^2}{8\pi K_\sigma} \left\{ 2\left[(W+1)(K_\sigma + \mu)(\omega + \omega^*)^2 + (W-1)(K_\sigma - \mu)|\sigma - \sigma^*|^2 \right] \left| \int \Theta \, dz \right|^2 \right.$$

$$+ \left[(W-1)(K_\sigma - \mu)|\omega - \omega^*|^2 + (W+1)(K_\sigma + \mu)(\sigma + \sigma^*)^2 \right] \times$$

$$\left. \left[\left| \int \Theta \sin\vartheta \, e^{i\varphi} dz \right|^2 + \left| \int \Theta \sin\vartheta \, e^{-i\varphi} dz \right|^2 + 2 \left| \int \Theta \cos\vartheta \, dz \right|^2 \right] \right\}$$

in which $\sigma = e^{i\vartheta_2}(\gamma + i\lambda)$.

It is easy to show that:

$$\left|\int\Theta\sin\vartheta\, e^{i\varphi}\,d\tau\right|^2+\left|\int\Theta\sin\vartheta\, e^{-i\varphi}\,d\tau\right|^2$$

$$=2\left\{\left|\int\Theta\sin\vartheta\cos\varphi\,d\tau\right|^2+\left|\int\Theta\sin\vartheta\sin\varphi\,d\tau\right|^2\right\}$$

When $|\mathcal{H}|^2$ is simplified, it will lead to the following distribution formula:

$$P(W)dW=\frac{G^2}{2\pi^3}W\sqrt{W^2-1}\,K\sqrt{K^2-\mu^2}\,\frac{F(Z,W)}{(2r_0)^{2\gamma-2}}\times$$

$$\left\{(1+\gamma)\left(1+\frac{\mu\gamma}{WK}\right)\left|\int v_m u_n^*(2r)^{\gamma-1}\,d\tau\right|^2\right.$$

$$\left.+4(1-\gamma)\left(1-\frac{\mu\gamma}{WK}\right)\left[\left|\int v_m u_n^*(2r)^{\gamma-2}x\,d\tau\right|^2+\cdots+\cdots\right]\right\}. \quad (117)$$

Formula (49) is obtained from this by making the same approximation as in Chapter IV, section e, i.e., letting

$$4\left|\int v_m u_n^*(2r)^{\gamma-2}x\,d\tau\right|^2=\sin^2\vartheta\cos^2\varphi\,(2r_0)^{2\gamma-2}\left|\int v_m^* u_m\,d\tau\right|^2,$$

$$4\left|\int v_m u_n^*(2r)^{\gamma-2}y\,d\tau\right|^2=\sin^2\vartheta\cos^2\varphi\,(2r_0)^{2\gamma-2}\left|\int v_m^* u_m\,d\tau\right|^2,$$

$$4\left|\int v_m u_n^*(2r)^{\gamma-2}z\,d\tau\right|^2=\cos^2\,(2r_0)^{2\gamma-2}\left|\int v_m^* u_m\,d\tau\right|^2.$$

For forbidden transitions, $\left|\int v_m u_n^* (2r)^{\gamma-1} d\tau\right|^2 = 0$,

so

$$P(W)dW = \frac{4G^2}{\pi^3} W\sqrt{W^2-1}\,K\sqrt{K^2-\mu^2}\,\frac{p_s^{2\gamma-2} e^{\pi\lambda} |\Gamma(\gamma+i\lambda)|^2}{[\Gamma(2\gamma+1)]^2} \times$$

$$(1-\gamma)\left(1-\frac{\mu\gamma}{WK}\right)\left\{|\xi|^2+|\eta|^2+|\zeta|^2\right\} \qquad (118)$$

where $|\xi|^2 = \left|\int v_m^* u_m (2r)^{\gamma-2} \chi\, d\tau\right|^2$.

This is <u>not</u> the $Z \neq 0$ equivalent of Formula (114), for the latter was derived by developing the exponential through terms linear in r, whereas (117) was obtained by taking only the constant term of the exponentials and hypergeometric expansions. However, (118) represents a way in which transitions not obeying the same selection rules as $|M|$ may be accomplished. Due to the factor $(1-\gamma)$, this means of obtaining transitions is less and less important for $Z \to 0$ $(\gamma \to 1)$.

For the (0, 1) interaction, (118) is multiplied by K^2.

CHAPTER VII.

THE CONSEQUENCES OF FERMI'S THEORY

a. The Neutron-Proton Interaction. Nordsieck,[36] Tamm[37] and Iwanenko[37] pointed out almost simultaneously that because the Fermi theory provides for interactions of the neutron and the proton with an electron-neutrino field, it implies an interaction of the proton with the neutron through the electron-neutrino field. This might be visualized in the following way: a neutron N_1 oscillates between the two states represented in

$$N_1 \rightleftharpoons P_1 + e^- + n \qquad (119)$$

If a proton P_2 were present, it would be indistinguishable from P_1 and so could be the one to reabsorb the electron and neutrino in the leftward process of (119). As a result, there are states in which the second heavy particle is a neutron by virtue of absorbing the electron and neutrino emitted by N_1, i.e., the original neutron and proton have been exchanged.[38] In parallel to (119),

$$P_2 \rightleftharpoons N_2 + e^+ + n' \qquad (120)$$

offers another path by which neutrons and protons can be

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[36] Phys. Rev. 46, 234 (1934).

[37] Nature 133, 981 (1934).

[38] This fact will of course lead to an exchange force of the Heisenberg type, as opposed to the Majorana type in which the exchange is not complete (spins not exchanged).

interchanged.

The problem is to find the neutron-proton interaction function V as defined by the Hamiltonian:

$$H = H_N + H_P + V \tag{121}$$

on the assumption that the two heavy particles interact only through the electron-neutrino field. The effect of the field is to cause $N \rightleftarrows P$ exchanges as described in the preceding paragraph and if V is to represent this type of interaction it must be such as to create a state f in which the N_1 and P_2 of state i are replaced by P_1 and N_2. Accordingly,

$$V = JQ_1^* Q_2 \tag{122}$$

where Q_1^* and Q_2 are operators such as Q^* and Q of the preceding chapters, with the difference that Q_1^* operates only on heavy particle number 1 ($N_1 \rightarrow P_1$) while Q_2 operates on number 2 ($P_2 \rightarrow N_2$). The efficacy of (122) in causing the growth of the amplitude a_f of a state f is given by:

$$|a_f|^2 = 4|V_{fi}|^2 \frac{\sin^2 t/2 (E_i - E_f)^2}{(E_i - E_f)^2}, \tag{123}$$

where

$$V_{fi} = \iint d\vec{r}_1 d\vec{r}_2 \, \Psi_f^* V \Psi_i, \tag{124}$$

the components of $\vec{r}_1$ and $\vec{r}_2$ being the coördinates of the

two heavy particles.

$$\Psi_i = \begin{vmatrix} \Psi_{N_1}(r_1) \\ 0 \end{vmatrix} \begin{vmatrix} 0 \\ \Psi_{P_2}(r_2) \end{vmatrix}, \quad \Psi_f = \begin{vmatrix} 0 \\ \Psi_{P_1}(r_1) \end{vmatrix} \begin{vmatrix} \Psi_{N_2}(r_2) \\ 0 \end{vmatrix} \tag{125}$$

where Ψ_{N_1} is the solution of $H_{N_1}\Psi_{N_1} = E_{N_1}\Psi_{N_1}$, etc. With (125), V_{fi} becomes:

$$V_{fi} = \iint d\vec{r}_1 d\vec{r}_2 \, \Psi^*_{P_1}(r_1)\Psi^*_{N_2}(r_2) \, J \, \Psi_{N_1}(r_1)\Psi_{P_2}(r_2) \tag{126}$$

J can be obtained in terms of $\mathcal{H}$ (the interaction between the heavy particle and the electron-neutrino field) when one remembers that intermediate between states i and f there is a state b having P_2, P_1 an electron, and a neutrino according to (119) or a state b' having N_1, N_2, a positron, and an antineutrino according to (120). An earlier formula than (123) in the development of the variation of constants method is:

$$i\dot{a}_K = \sum_m a_m e^{i(E_K - E_m)t} \mathcal{J}_{Km} \tag{127}$$

in which $\mathcal{J}$ is the interaction responsible for the growth of state k out of states n. In the case concerned here, it is assumed that one starts with only the state heretofore denoted

by i ($a_i = 1$ at $t = 0$) and that to $\mathcal{H}$ is due the growth of state b (or b'). Therefore,

$$i\dot{a}_b = \mathcal{H}_{bi}\, e^{i(E_b - E_i)t}$$

or, integrating with respect to the time,

$$a_b = \frac{\mathcal{H}_{bi}}{E_i - E_b}\, e^{i(E_b - E_i)t}. \tag{128}$$

$\mathcal{H}$ is also responsible for the development of state f from b. According to (127),

$$i\dot{a}_f = \sum_b a_b\, e^{i(E_i - E_b)t}\, \mathcal{H}_{fb}$$

$$= \sum_b \frac{\mathcal{H}_{bi}\,\mathcal{H}_{fb}}{E_i - E_b}\, e^{i(E_f - E_i)t},$$

so that

$$a_f = \left(\sum_b \frac{\mathcal{H}_{bi}\,\mathcal{H}_{fb}}{E_i - E_b}\right) \frac{e^{i(E_f - E_i)t} - 1}{E_i - E_f} \tag{129}$$

the integration constant having been determined, from the fact that at $t = 0$, $a_f = 0$. Formula (123) follows directly from (129) if[39]

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[39] This is exactly the formula for the energy in ordinary second order perturbation theory.

$$V_{fi} = \sum_b \frac{\mathcal{H}_{bi}\,\mathcal{H}_{fb}}{E_i - E_b} \,. \tag{130}$$

The creation of state b from i is due to the term $GQ_1^* A_0^*$ of $\mathcal{H}$. Using the formulation of Chapter IV,

$$\mathcal{H}_{bi} = G \int d\vec{r}_1\, \Psi_{P_1}^*(r_1)\, C_{s\sigma}^*(r_1)\, \Psi_{N_1}(r_1) \,. \tag{131}$$

This is for a special state b in which the electron has energy W_s and the neutrino energy K_σ, and so $\sum_b$ is equivalent to $\sum_{s\sigma}$. What one uses for $C_{s\sigma}^*$ depends on the particular interaction of those discussed in Chapter V one selects. The growth of state f from b is due to the $GQ_2 A_0$ term of $\mathcal{H}$, so that:

$$\mathcal{H}_{fb} = G \int d\vec{r}_2\, \Psi_{N_2}^*(r_2)\, C_{s\sigma}(r_2)\, \Psi_{P_2}(r_2) \,. \tag{132}$$

Putting (131) and (132) into (130),

$$V_{fi} = 2 \sum_{s\sigma} \frac{G^2 \iint d\vec{r}_1\, d\vec{r}_2\, \Psi_{P_1}^*(r_1)\, \Psi_{N_2}^*(r_2)\, C_{s\sigma}^*(r_1)\, C_{s\sigma}(r_2)\, \Psi_{N_1}(r_1)\, \Psi_{P_2}(r_2)}{E_i - E_{s,\sigma}} \,. \tag{133}$$

The factor 2 is introduced because of the fact that the summation over the intermediate states should include the states represented in (120) and designated by b'. These states will develop due to the $GQ_2 A_0$ term of $\mathcal{H}$ and so

$$\mathcal{H}_{b'i} = G \int d\vec{r}_2 \, \Psi^*_{N_2}(r_2) \, C_{s\sigma}(r_2) \, \Psi_{P_2}(r_2).$$

Similarly,

$$\mathcal{H}_{fb'} = G \int d\vec{r}_1 \, \Psi^*_{P_1}(r_1) \, C^*_{s\sigma}(r_1) \, \Psi_{N_1}(r_1).$$

Since $\mathcal{H}_{b'i} \, \mathcal{H}_{fb'} = \mathcal{H}_{bi} \, \mathcal{H}_{fb}$, it is easy to see that V_{fi} is the same for both b and b'. Comparing (133) with (127) results in:

$$J = 2G^2 \sum_{s\sigma} \frac{C^*_{s\sigma}(r_1) \, C_{s\sigma}(r_2)}{E_i - E_{s\sigma}}. \qquad (134)$$

According to Chapter V, section d:

$$C_{s\sigma}(r) = \frac{\partial^m \psi_s}{\partial x_\alpha \cdots \partial x_\rho} \delta \frac{\partial^m \varphi_\sigma}{\partial x_\alpha \cdots \partial x_\rho} = \frac{S(A_s \delta B_\sigma)}{\Omega} (\beta_s \cdot p_\sigma - W_s K_\sigma)^m e^{i(\beta_s + p_\sigma)\cdot r}$$

for the (m, m) interactions in the (n, e^-) viewpoint. With the (m, m + 1) interactions:

$$C_{s\sigma}(r) = \frac{\partial^m \psi_s}{\partial x_\alpha \cdots \partial x_\rho} \beta\delta \frac{\partial}{\partial t} \frac{\partial^m \varphi_\sigma}{\partial x_\alpha \cdots \partial x_\rho} = \frac{S(A_s \beta\delta B_\sigma)}{\Omega} K_\sigma (\beta_s \cdot p_\sigma - W_s K_\sigma)^m e^{i(\beta_s + p_\sigma)\cdot r}.$$

When the expressions for $|S(A_s \delta E_\sigma)|^2$ and $|S(A_s \delta \beta B_\sigma)|^2$ given in footnote 34 are recalled, it is easily verified that for an (m, n) interaction (n = m or m + 1):

$$C^*_{s\sigma}(r_1) \, C_{s\sigma}(r_2) = \frac{|F_{m,n}(p_s, p_\sigma)|^2}{\Omega^2} e^{iP\cdot R} \qquad (135)$$

where $\vec{P} = \vec{p}_s + \vec{p}_\sigma$, $\vec{R} = \vec{r}_2 - \vec{r}_1$ and:

$$|F_{m,n}(p_s, p_\sigma)|^2 = K_\sigma^{2(n-m)} (p_s \cdot p_\sigma - W_s K_\sigma)^{2m} \left(1 + \frac{\mu + (-)^{m+n}(p_s \cdot p_\sigma)}{W_s K_\sigma}\right). \quad (136)$$

After $\sum_{s\sigma}$ is replaced as usual by $\Omega^2/(2\pi)^6 \int d\vec{p}_s \int d\vec{p}_\sigma$, (133) becomes:

$$J_{m,n}(R) = \frac{2G^2}{(2\pi)^6} \iint d\vec{p}_s d\vec{p}_\sigma \frac{|F_{m,n}|^2}{E_i - E_{s\sigma}} e^{i P \cdot R}. \quad (137)$$

The energy difference,

$$E_i - E_{s\sigma} = E_{N_1} + E_{p_2} - E_{p_1} - E_{N_2} - W_s - K_\sigma$$

is practically linear in p_s and p_σ so that one can see the integral in $J_{m,n}(R)$ will diverge, roughly as:

$$\iint_0^\infty p_s^2 dp_s p_\sigma^2 dp_\sigma \frac{p_\sigma^{2(n-m)} (p_s p_\sigma)^{2m}}{p_s + p_\sigma}$$

for any of the possible values of m and n (≥ 0). This means that the large values of p_s and p_σ are the most important and therefore it will be a good approximation to neglect $(E_{N_1} + E_{p_2} - E_{p_1} - E_{N_2})$ in comparison to $(W_s + K_\sigma)$ and moreover to replace W_s by p_s and K_σ by p_σ. When this is done, (137) becomes:

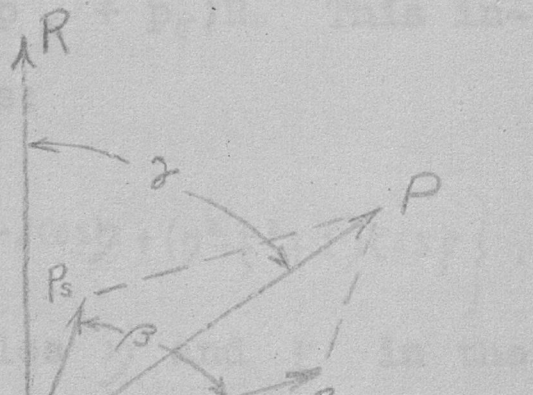
$$J_{m,n}(R) = -\frac{2G^2}{(2\pi)^6} \iint d\vec{p}_s d\vec{p}_\sigma \frac{p_s^{2m} p_\sigma^{2n}}{p_s + p_\sigma} (1 - \cos\beta)^{2m} (1 + (-)^{m+n} \cos\beta) e^{iPR\cos\gamma} \quad (138)$$

where β is the angle between $\vec{p}_s$ and $\vec{p}_\sigma$ and γ is the angle between $\vec{P}$ and $\vec{R}$.

1°/ <u>The (0, 0) Interaction</u>. With the Ansatz originally introduced by Fermi for the interaction between the heavy particle and the electron-neutrino field, (138) becomes:

$$J_{0,0}(R) = -\frac{2G^2}{(2\pi)^6}\iint d\vec{p}_s\, d\vec{p}_\sigma\, \frac{(1+\cos\beta)}{p_s + p_\sigma}\, e^{iPR\cos\gamma} \qquad (139)$$

In the angular integration, $\vec{p}_s$ and $\vec{p}_\sigma$ must be allowed to attain all possible directions. This will be accomplished if, first, an integration is performed over all directions of $\vec{P}$ while the rectangle it forms with $\vec{p}_s$ and $\vec{p}_\sigma$ is kept fixed relative to it (β, $|\vec{p}_s|$ and $|\vec{p}_\sigma|$ constant). To complete the integration, the rectangle is allowed to spin about $\vec{P}$ and β is allowed to go through all possible values, $|\vec{P}|$ varying so in the meanwhile that $|\vec{p}_s|$ and $|\vec{p}_\sigma|$ are constant. The variation of the azimuth of $\vec{P}$ and the spinning of the rectangle about P do not cause any changes in the integrand so their contribution to the result is just $4\pi^2$. Accordingly,



$$J_{0,0}(R) = -\frac{2G^2}{(2\pi)^4}\int_0^\infty \frac{p_s^2 dp_s\, p_\sigma^2 dp_\sigma}{p_s + p_\sigma}\int_0^\pi d\beta\, \sin\beta\,(1+\cos\beta)\int_0^\pi d\gamma\, \sin\gamma\, e^{iPR\cos\gamma}$$

$$= -\frac{2G^2}{(2\pi)^4}\iint_0^\infty \frac{p_s^2\,dp_s\,p_\sigma^2\,dp_\sigma}{p_s+p_\sigma}\int_0^\pi d\beta\,\sin\beta(1+\cos\beta)\frac{2\sin PR}{PR}.$$

When the change of the integration variable β to

$$P = \sqrt{p_s^2 + p_\sigma^2 + 2p_s p_\sigma \cos\beta}$$ is made,

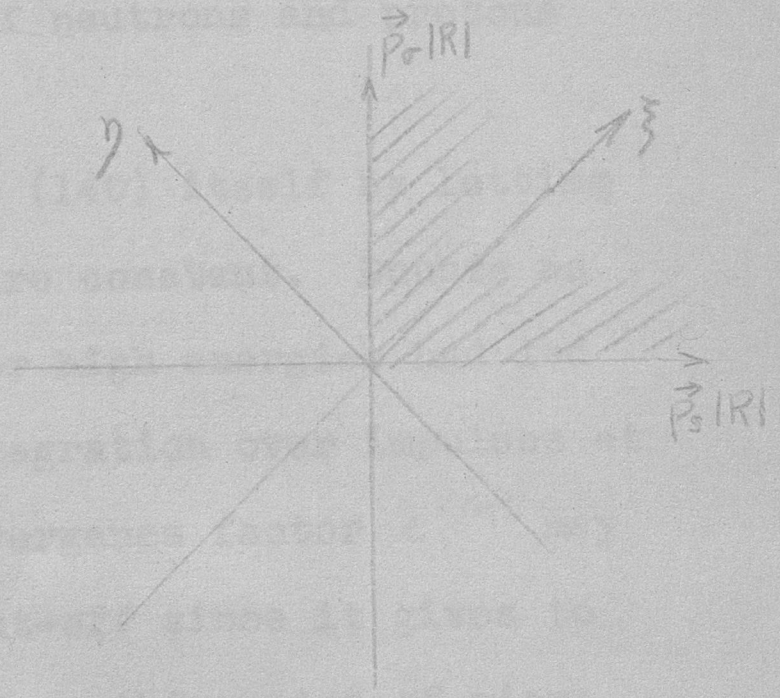
$$2\int_0^\pi d\beta\,\sin\beta(1+\cos\beta)\frac{\sin PR}{PR} = \frac{1}{p_s^2 p_\sigma^2 R}\int_{\eta/R}^{\xi/R} dP\,\sin PR\left(P^2-\frac{\eta^2}{R^2}\right),$$

in which $\eta = |p_s - p_\sigma|R$ and $\xi = (p_s + p_\sigma)R$. This integral is a simple one and the result is:

$$J_{0,0}(R) = -\frac{2G^2}{(2\pi)^4 R^3}\iint_0^\infty \frac{dp_s\,dp_\sigma}{\xi}\left\{2\xi\sin\xi - 2\eta\sin\eta - 2\cos\eta + (\eta^2-\xi^2+2)\cos\xi\right\}.$$

It is advantageous to employ the variables η and ξ in the remaining integrations.

$$R^2\,dp_s\,dp_\sigma = \pm\begin{vmatrix} \frac{\partial(p_\sigma R)}{\partial\xi} & \frac{\partial(p_\sigma R)}{\partial\eta} \\ \frac{\partial(p_s R)}{\partial\xi} & \frac{\partial(p_s R)}{\partial\eta}\end{vmatrix} d\xi\,d\eta = \tfrac{1}{2}\,d\xi\,d\eta$$



The integration is to cover the shaded quarter-plane of the diagram and so the limits for η are from $-\xi$ to $+\xi$ while ξ may vary from 0 to ∞. Then,

$$J_{0,0}(R) = -\frac{G^2}{(2\pi)^4 R^5}\int_0^\infty \frac{d\xi}{\xi}\int_{-\xi}^{\xi} d\eta \left\{2\xi\sin\xi - 2\eta\sin\eta - 2\cos\eta + (\eta^2 - \xi^2 + 2)\cos\xi\right\}$$

$$= -\frac{4G^2}{(2\pi)^4 R^5}\int_0^\infty d\xi \left\{-2\frac{\sin\xi}{\xi} + 2\cos\xi + \xi\sin\xi - \tfrac{1}{3}\xi^2\cos\xi\right\}.$$

The remaining integral is obviously divergent. Convergence may be forced by the frequently used method of introducing the factor $e^{-a\xi}$ and subsequently letting $a \to 0$. When this is done,

$$J_{0,0}(R) = -\frac{G^2}{2\pi^4 R^5}\left\{-\tan^{-1}1/a + \frac{a}{(a^2+1)} + \frac{a}{(a^2+1)^2} + \frac{4}{3}\frac{3-a^2}{(a^2+1)^3}\right\} \qquad (140)$$

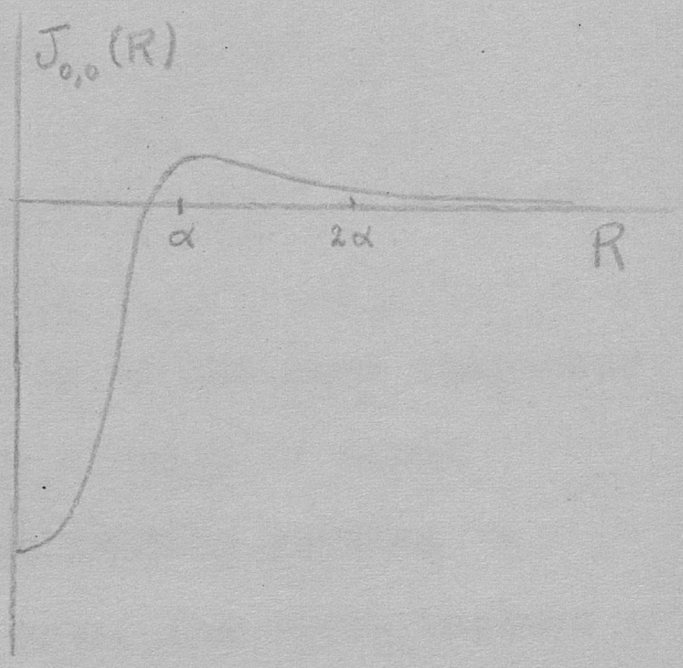
$$= G^2/4\pi^3 R^5 \quad \text{for} \quad a \to 0 \qquad (141)$$

This is an extremely weak repulsion ($\sim 10^{-26}\, mc^2$ with $G = 10^{-12}$) which cannot account for the amount of scattering of neutrons by protons which is observed and which, most certainly, cannot explain the binding of neutrons and protons into nuclei.

Some meaning might be given to (140) itself by letting $a = \alpha/R$ where α is the fine structure constant. Doubts as to the validity of quantum theory for high energies had led to the custom of cutting off the integration over impulses at mc/α. The introduction of the convergence factor e^{-ap_s} may be regarded as a method of such a cut-off since it gives to p_s a "mean-life" equal to mc/α. From this point of view:

$$J_{0,0}(R) = -\frac{G^2}{2\pi^4 R^5}\left\{-\tan^{-1}\left(\frac{R}{\alpha}\right) + \frac{\alpha R}{\alpha^2+R^2} + \frac{\alpha R^3}{(\alpha^2+R^2)^2} + \frac{\alpha R^3}{3}\,\frac{3R^2-\alpha^2}{(\alpha^2+R^2)^3}\right\}. \quad (142)$$

A plot of this is shown in the following figure. It has the appearance of the potential hole usually used to describe the close range interactions of neutrons with protons, as in the deuteron.[40] The depth of the hole is obtained by evaluating $J_{0,0}(R)$ for $R \to 0$, which yields: $-2G^2/5\pi^4\alpha^5$. This has the order of magnitude $2 \times 10^{-16}\ mc^2 = 10^{-16}$ mev., far smaller than the depth of the hole ($\sim 100\ mc^2$) required to explain the 2.2 mev. binding of the deuteron.[38]



It will be observed that in the diagram α is a rough measure of the radius of the hole. If, therefore, α is arbitrarily set equal to the radius deduced from scattering measurements ($\sim 10^{-13}$ cm. $\cong 3 \times 10^{-3}\ \hbar/mc$) the depth of the hole becomes $3 \times 10^{-14}\ mc^2$.

2°/ <u>The (0, 1) Interaction</u>. In this case,

$$J_{0,1}(R) = -\frac{2G^2}{(2\pi)^6}\iint d\vec{p}_s\, d\vec{p}_\sigma\, \frac{p_\sigma^2}{p_s + p_\sigma}(1-\cos\beta)\, e^{iPR\cos\gamma} \quad (143)$$

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[40] Bethe and Peierls, Proc. Roy. Soc. A <u>148</u>, 146 (1935).

The integration of this is exactly the same as in the (0, 0) case. The result is:

$$J_{0,1}(R) = -\frac{G^2}{4\pi^4 R^7}\left\{-5\tan^{-1}\left(\frac{R}{\alpha}\right) + 5\alpha\frac{R}{R^2+\alpha^2} + 6\alpha\frac{R^3}{(\alpha^2+R^2)^2}\right.$$

$$\left. - \tfrac{8}{3}\alpha R^3\frac{\alpha^2-3R^2}{(\alpha^2+R^2)^3} - 8\alpha R^5\frac{\alpha^2-R^2}{(\alpha^2+R^2)^4}\right\}.$$

At the limit $\alpha \to 0$ $(= R\to\infty)$, $J_{0,1} = 5G^2/8\pi^3 R^7$ which at $R = \hbar/mc$ is about $10^{-26}\,mc^2$. For $R\to 0$, $J_{0,1}(R) = -24\,G^2/7\pi^4\alpha^7$. $J_{0,1}(R)$ as a function of R is essentially of the same character as $J_{0,0}(R)$ but this time the hole has a depth of about $3 \times 10^{-11}\,mc^2$ for $\alpha = 1/137$. By arbitrarily putting α equal to the observed radius of the hole, the depth becomes $\sim 4 \times 10^{-8}\,mc^2$.

3°/ <u>The (1, 1) Interaction</u>. Here,

$$J_{1,1}(R) = -\frac{2G^2}{(2\pi)^6}\iint d\vec{p}_s\, d\vec{p}_\sigma\, \frac{p_s^2 p_\sigma^2}{p_s+p_\sigma}(1-\cos\beta)^2(1+\cos\beta)\, e^{iPR\cos\gamma} \quad (144)$$

Again, the integration proceeds just as in the (0, 0) case. The result is:

$$J_{1,1}(R) = -\frac{G^2}{\pi^4 R^9}\left\{-105\tan^{-1}\left(\frac{R}{\alpha}\right) + 105\frac{\alpha R}{\alpha^2+R^2} + 95\frac{\alpha R^3}{(\alpha^2+R^2)^2}\right.$$

$$\left. -25\alpha R^3\frac{\alpha^2-3R^2}{(\alpha^2+R^2)^3} - 96\alpha R^5\frac{\alpha^2-R^2}{(\alpha^2+R^2)^4} + 8\alpha R^5\frac{\alpha^4-10\alpha^2R^2+5R^4}{(\alpha^2+R^2)^5}\right\}.$$

For $\alpha \to 0\ (R \to \infty)$, $J_{1,1}(R) \to 105\, G^2/2\pi^3 R^9$. This is about $2 \times 10^{-24}\ mc^2$ at $R = \hbar/mc$. For $R \to 0$, $J_{1,1}(R) = -64\, G^2/3\pi^4 \alpha^9$. The depth of the hole in this case is therefore about $4 \times 10^{-6}\ mc^2$ with $\alpha = 1/137$. With $\alpha = 3 \times 10^{-3}\ \hbar/mc$ it is about $.04\ mc^2$.

4°/ <u>The (1, 2) Interaction</u>. This time one begins with

$$J_{1,2}(R) = -\frac{2G^2}{(2\pi)^6} \iint d\vec{p}_s\, d\vec{p}_\sigma\, \frac{p_s^2 p_\sigma^4}{p_s + p_\sigma} (1 - \cos\beta)^3 e^{iPR\cos\gamma}.$$

After the integration is completed,

$$J_{1,2}(R) = -\frac{G^2}{\pi^4 R^{11}} \left\{ -945 \tan^{-1}\left(\frac{R}{\alpha}\right) + 945 \frac{\alpha R}{\alpha^2 + R^2} + 945 \frac{\alpha R^3}{(\alpha^2 + R^2)^2} - 315 \alpha R^3 \frac{\alpha^2 - 3R^2}{(\alpha^2 + R^2)^3} \right.$$

$$\left. - 900 \alpha R^5 \frac{\alpha^2 - R^2}{(\alpha^2 + R^2)^4} + 144 \alpha R^5 \frac{\alpha^4 - 10\alpha^2 R^2 + 5R^4}{(\alpha^2 + R^2)^5} + 120 \alpha R^7 \frac{3\alpha^4 - 10\alpha^2 R^2 + 3R^4}{(\alpha^2 + R^2)^6} \right\}.$$

The appearance of the curve $J_{1,2}$ as a function of R is essentially the same as for $J_{0,0}$. For large values of R ($\alpha \to 0$), there is a repulsion $945\ G^2/2\pi^3 R^{11}$ which has the order of magnitude $1.5 \times 10^{-23}\ mc^2$ at $R = \hbar/mc$. The depth of the hole is given by $J_{1,2}(0) = -17{,}280\ G^2/11\pi^4 \alpha^{11}$ which is about equal to $5\ mc^2$ with $\alpha = 1/137$.

It can therefore be said that although the slowness of β-decay, which compéls making G so very small, does not seem reconcilable with the observed magnitude of the neutron-proton interaction on the basis of Fermi's original formulation, the

introduction of higher derivatives may make the reconciliation possible.[41]

b. Elastic Scattering of Neutrons by Protons. Far more directly observable than the law of interaction between neutrons and protons is the cross-section for elastic scattering of neutrons by protons. On the assumption that the heavy particles affect each other only through the medium of the electron-neutrino field, the alterations in their momenta which the particles suffer in the process of being scattered must be attributed to the neutron-proton exchanges which the field causes. One visualizes the neutron as being deflected when it emits an electron and a neutrino which are immediately absorbed by the proton.

One begins with a neutron at position $\vec{r}_1$ traveling with a momentum $\vec{P}_{N_1}$ and a proton at $\vec{r}_2$ having the impulse $\vec{P}_{P_2}$ (state i). At the end one has a proton at $\vec{r}_1$ with momentum $\vec{P}_{P_1}$ and a neutron at $\vec{r}_2$ with impulse $\vec{P}_{N_2}$ (state f). Intermediately there must have been a state b in which there were two protons, an electron and a neutrino, or a state b' with two neutrons, a positron, and an anti-neutrino. Because an intermediate state is concerned, Formula (129) applies and it follows that:

$$|a_f|^2 = 4\left|\sum_b \frac{\mathcal{H}_{bi}\mathcal{H}_{fb}}{E_i - E_b}\right|^2 \frac{\sin^2 t/2\,(E_i - E_f)}{(E_i - E_f)^2}. \qquad (145)$$

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[41] Heisenberg, Zeeman Jubilee Papers; Nijhoff, The Hague, 1935; p. 108.

Plane waves may be used for the wave functions of all the particles since the occurrent fields are small (Z = 0 or 1). With these, (131) and (132) become for the (m, n) interaction:

$$\mathcal{H}_{bi} = \frac{G}{\Omega^2} F^*_{m,n}(p_s, p_\sigma) \int d\vec{r}_1 \, e^{-i\vec{p}_{P_1}\cdot\vec{r}_1} e^{-i(\vec{p}_s+\vec{p}_\sigma)\cdot\vec{r}_1} e^{i\vec{p}_{N_1}\cdot\vec{r}_1}$$

$$= \frac{G}{\Omega} F^*_{m,n} \quad \text{for} \quad \vec{P}_{N_1} = \vec{P}_{P_1} + \vec{P}_s + \vec{P}_\sigma \,, \tag{146}$$

$$\mathcal{H}_{fb} = \frac{G}{\Omega^2} F_{m,n}(p_s, p_\sigma) \int d\vec{r}_2 \, e^{-i\vec{p}_{N_2}\cdot\vec{r}_2} e^{i(\vec{p}_s+\vec{p}_\sigma)\cdot\vec{r}_2} e^{i\vec{p}_{P_2}\cdot\vec{r}_2}$$

$$= \frac{G}{\Omega} F_{m,n} \quad \text{for} \quad \vec{P}_{P_2} + \vec{p}_s + \vec{p}_\sigma = \vec{P}_{N_2} \,. \tag{147}$$

No definite answers are given unless the impulse conversation conditions are fulfilled. From (146) and (147)

$$\vec{P}_{N_1} + \vec{P}_{P_2} = \vec{P}_{P_1} + \vec{P}_{N_2} \,. \tag{148}$$

In the further calculations of this problem, it is possible to ignore the difference between the neutron and proton masses, primarily because there are no mass differences between the initial and final states. It will also be convenient to introduce the relative coördinates,

$$\vec{R} = \vec{r}_1 - \vec{r}_2 \quad \text{and} \quad \vec{\xi} = 1/2\,(\vec{r}_1 + \vec{r}_2) \,.$$

The momenta conjugate to these are:

$$\vec{P} = 1/2\,(\vec{P}_{N_1} - \vec{P}_{P_2}) \quad \text{and} \quad \vec{\eta} = \vec{P}_{N_1} + \vec{P}_{P_2} \,.$$

This will yield to the following expression for the initial energy:

$$E_i = \frac{P_{N_1}^2 + P_{P_2}^2}{2M} = \frac{\eta^2}{4M} + \frac{P^2}{M} . \qquad (149)$$

Similarly, in the state b, which contains two protons, an electron, and an anti-neutrino, the energy is:

$$E_b = \frac{\eta'^2}{4M} + \frac{P'^2}{M} + W_s + K_\sigma . \qquad (150)$$

Where $\vec{\eta}' = \vec{P}_{P_1} + \vec{P}_{P_2}$ and $\vec{P}' = 1/2\ (\vec{P}_{P_1} - \vec{P}_{P_2})$.

$$E_f = \frac{\eta''^2}{4M} + \frac{P''^2}{M} , \qquad (151)$$

Where $\vec{\eta}'' = \vec{P}_P + \vec{P}_{N_2}$ and $\vec{P}'' = 1/2\ (\vec{P}_P - \vec{P}_{N_2})$. It is immediately obvious that η' and P' can be expressed in terms of η', η'', P and P". Thus,

$$\vec{\eta}' = 1/2\ (\vec{\eta}'' + \vec{\eta}) + (\vec{P}'' - \vec{P}),$$

$$\vec{P}' = 1/2\ (\vec{\eta}'' - \vec{\eta}) + (\vec{P}'' + \vec{P}). \qquad (152)$$

The condition (148) is now expressed by:

$$\vec{\eta}'' = \vec{\eta} \qquad (153)$$

If the expressions (146), (147), (149), (150) and (151) are substituted into (145), it is clear that this formula will give the probability that a neutron of initial momentum $\vec{P}_{N_1}$ is

deflected to momentum $\vec{P}_{N_2}$ by a proton of impulse $\vec{P}_{P_2}$. The final proton momentum, $\vec{P}_{P_1}$, is determined by (148). The conditions (146) and (147) show that in these circumstances $\vec{p} = \vec{p}_s + \vec{p}_\sigma$ must remain constant. In the language of the relative coördinates, for a given $\vec{\rho}$, $\vec{P}$ and $\vec{P}''$ it follows from (146) and (147) that:

$$\vec{p} = \vec{P} = \vec{P}''. \qquad (154)$$

The probability of the process with any final value $\vec{P}''$ of the relative impulse is:

$$\sum_{P''} |a_f|^2 = \frac{\Omega}{(2\pi)^3} \int dP'' \frac{4G^4}{\Omega^4} \left| \sum_{s\sigma} \frac{|F_{m,n}|^2}{E_i - E_{s\sigma}} \right|^2 \frac{\sin^2 t/2\left(\frac{P^2 - P''^2}{M}\right)}{\left(\frac{P^2 - P''^2}{M}\right)^2}, \qquad (155)$$

in which $(E_i = E_f)$ has been reduced by the application of (153). $\int dP'' = 2\int d\omega \int_0^\infty P''^2 dP''$, the factor 2 coming from the spin degeneracy. After making the substitution $x = (P^2 - P''^2)/M$ and integrating, it is easily seen that:

$$\sum_{P''} \frac{|a_f|^2}{t} = \frac{G^4}{\pi^3 \Omega^3} \int d\omega \left| \sum_{s\sigma} \frac{|F_{m,n}|^2}{E_i - E_{s\sigma}} \right|^2 \cdot \frac{MP''}{2} \cdot \pi/2 \qquad (156)$$

with the restriction: $|P| = |P''|$ (energy conservation). Remembering that $\vec{p} = \vec{p}_s + \vec{p}_\sigma$ is required to be a constant, $\sum_{s\sigma}$ is replaced by $\Omega/(2\pi)^3 \int d\vec{p}_s$, so that:

$$\sum_{P''} \frac{|a_f|^2}{t} = \frac{G^4 MP}{(2\pi)^8 \Omega} \int d\omega \left| \int d\vec{p}_s \frac{|F_{m,n}(\vec{p}_s, \vec{P} - \vec{p}_s)|^2}{W_s + K_\sigma} \right|^2. \qquad (157)$$

The probability of scattering per unit incoming current is obtained from this by multiplying with $\Omega M/2P$, since the incoming current is

$$\text{density x (relative) velocity} = \frac{1}{\Omega} \times \frac{2P}{M}$$

When this is done, (167) will have the dimensions of an area. Another factor 4 will be introduced because the $\sum_b$ should have included the states b' and it is easily verified that:

$$\mathcal{H}_{b'i}\,\mathcal{H}_{fb'} = \mathcal{H}_{bi}\,\mathcal{H}_{fb}\,.$$

One finally obtains for the elastic scattering cross-section:

$$\sigma^e = \frac{2G^4M^2}{(2\pi)^8}\int d\omega \left|\int d\vec{p}_s \frac{|F_{m,m}|^2}{W_s + K_\sigma}\right|^2. \qquad (158)$$

1°/ <u>The (0, 0) Interaction</u>. According to (136),

$$|F_{0,0}|^2 = 1 + \frac{p_s \cdot p_\sigma}{W_s K_\sigma}$$

and so the integral over $\vec{p}_s$ in (158) becomes:

$$\int_0^\infty \frac{dp_s\, p_s^2}{W_s}\int d\omega_s \frac{W_s\sqrt{P^2+p_s^2-2Pp_s\cos\beta'} + Pp_s\cos\beta - P^2}{\sqrt{P^2+p_s^2-2Pp_s\cos\beta'}\,(W_s+\sqrt{P^2+p_s^2-2Pp_s\cos\beta'})}$$

$$= \pi\int_0^\infty dp_s \frac{p_s}{W_s}\left\{6W_s - 2p_s + \frac{P^2-4p_s^2-3}{P}\log\frac{W_s+p_s+P}{W_s+p_s-P}\right\}$$

It is evident that this integral is divergent. The seriousness of the divergence will be investigated by expanding the integrand in inverse powers of p_s. Thus the first two terms become:

$$6p_s - \frac{2p_s^2}{W_s} = 6p_s - 2p_s\left(1+\frac{1}{p_s^2}\right)^{-1/2} = 4p_s + \frac{1}{p_s} - \frac{3}{4p_s^3} + \cdots$$

The third term is:

$$1/p\,(p^2 - 4p_s^2 - 3)\left(1 + 1/p_s^2\right)^{-1/2} \log \frac{1 + \dfrac{p}{W_s + p_s}}{1 - \dfrac{p}{W_s + p_s}}$$

$$= \frac{2}{p}(p^2 - 4p_s^2 - 3)\left(1 - \frac{1}{2p_s^2} + \frac{3}{8p_s^4} - \cdots\right)\left(\frac{p}{W_s + p_s} + 1/3\left(\frac{p}{W_s + p_s}\right)^3 + \cdots\right)$$

Since

$$\frac{p}{p_s + W_s} = p\,\frac{p_s - \sqrt{p_s^2+1}}{p_s^2 - (p_s^2+1)} = -p\left\{p_s - p_s\left(1 + \frac{1}{2p_s^2} - \frac{1}{8p_s^4} + \cdots\right)\right\},$$

the third term becomes:

$$-4p_s + 2/3\,\frac{p^2}{p_s} + \frac{c}{p_s^3} + \cdots$$

The integral diverges therefore as:

$$\pi\left(\tfrac{2}{3}P^2+1\right)\int^{\infty}\frac{dp_s}{p_s}\,.$$

According to (154), $\vec{p} = \vec{P} - \vec{P}''$ so $p^2 = 2P^2(1-\cos\theta)$ in which θ is the angle between $\vec{P}$ and $\vec{P}''$. (The fact that $|P| = |P''|$ was used.) From (158),

$$\sigma^{e}_{0,0} \cong \frac{2\pi^2 G^4 M^2}{(2\pi)^8}\int d\omega\left(\tfrac{2}{3}P^2+1\right)^2\left|\int^{\infty}\frac{dp_s}{p_s}\right|^2$$

$$\cong \frac{G^4M^2}{(2\pi)^5}\left[\frac{64}{27}P^4+\frac{8}{3}P^2+1\right]\left|\int^{\infty}\frac{dp_s}{p_s}\right|^2. \qquad (159)$$

If, as is customary, one sets the upper limit of p_s at mc/α = 137 mc,

$$\sigma^{e}_{0,0} \cong \frac{G^4M^2}{(2\pi)^5}\left[\frac{64}{27}P^4+\frac{8}{3}P^2+1\right](\log 1/\alpha)^2.$$

Even for the fastest neutrons, the expression in the square brackets cannot exceed $\sim 10^7$. Therefore the order of magnitude of $\sigma^{e}_{0,0}$ is about $10^{-37}\,h^2/m^2c^2$ or $2\times10^{-59}\,cm^2$, far below the limits of observation. Experimentally, neutron scattering is known to have a cross-section of at least $10^{-26}\,cm^2$.

2°/ <u>The (0, 1) Interaction</u>. In this case, the integral over p_s is:

$$\int_0^\infty dp_s \frac{p_s^2}{W_s} \int d\omega_s \frac{W_s K_\sigma - p_s \cdot p_\sigma}{W_s + K_\sigma} K_\sigma$$

By the methods of the preceding paragraphs, it is found to diverge as:

$$8\pi \int^\infty p_s^3 \, dp_s \, .$$

After cutting off at $p_s = mc/\alpha$ it leads to:

$$\sigma^e_{o,i} \cong \frac{G^4 M^2}{8\pi^5} \left(\frac{1}{\alpha}\right)^8 .$$

This has the order of magnitude, $\sigma^e_{o,i} \sim 2 \times 10^{-28} \hbar^2/m^2c^2 = 3 \times 10^{-51} cm^2$.

3°/ <u>The (1, 1) Interaction</u>. Here, there occurs the integral:

$$\int_0^\infty dp_s \frac{p_s^2}{W_s} \int d\omega_s \frac{(W_s K_\sigma + p_s \cdot p_\sigma)(W_s K_\sigma - p_s \cdot p_\sigma)^2}{K_\sigma (W_s + K_\sigma)} ,$$

which diverges as:

$$\pi \left(\frac{17}{3} p^2 + 1\right) \int^\infty dp_s \, p_s^3 \, .$$

After setting the upper limit at $1/\alpha$ and substituting into (158), one has:

$$\sigma^e_{1,1} \cong \frac{G^4 M^2}{(2\pi)^5} \left(\frac{4624}{27} p^4 + \frac{68}{3} p^2 + 1\right) \left(\frac{1}{\alpha}\right)^8 ,$$

of the order of magnitude $\sim 3 \times 10^{-20} \hbar^2/m^2c^2 = 5 \times 10^{-41} cm^2$.

4°/ The (1, 2) Interaction. The integral over p_s is in this case:

$$\int_0^\infty d\sigma_s \frac{p_s^2}{W_s} \int d\omega_s \frac{(W_s K_\sigma - P_s P_\sigma)^3}{W_s + K_\sigma} \cdot K_\tau .$$

It can be shown to diverge as:

$$\frac{21}{4} \pi \int^\infty p_s^7 \, dp_s .$$

Making the upper limit $1/\alpha$, one obtains:

$$\sigma_{12}^{e} = \frac{441}{16} \frac{G^4 M^2}{(2\pi)^5} \left(\frac{1}{\alpha}\right)^{16},$$

which has the order of magnitude: $2 \times 10^{-10} \hbar^2/m^2c^2$ or 3×10^{-21} cm^2.

This shows that one may hope also to account for the elastic scattering of neutrons by protons on the basis of Fermi's Theory with the introduction of higher derivatives.

c. Inelastic Scattering. It is possible that the intermediate states of the preceding section, there regarded as "virtual", might become permanent. For example, the emission of an electron and a neutrino might be observed when a neutron collides with a proton. Since the proton will not in this case be reabsorbing the light particles, as in the elastic scattering, it has no direct effect upon the process through the electron-neutrino field. The only part it may

play is as an absorber of momentum through its Coulombian influence on the newly created proton. For that reason, the problem may easily be generalized to the inelastic scattering of neutrons by an arbitrary nucleus of charge Z. The created proton will then have as wave function a plane wave distorted by the Coulomb field, $\propto Z/r$. According to ordinary perturbation theory, this is:

$$\frac{1}{\sqrt{\Omega}} e^{iP_p \cdot r} + \frac{2M_p}{\sqrt{\Omega}} \sum_{P_p'} \frac{e^{iP_p' r} V_{P_p' P_p}}{P_p^2 - P_p'^2} , \qquad (160)$$

where

$$V_{P_p' P_p} = \int \psi^*_{P_p'} \frac{\alpha Z}{r} \psi_{P_p} d\vec{r} = \frac{4\pi \alpha Z}{\Omega} \int_0^\infty r e^{i(P_p - P_p')r} dr$$

$$= \frac{4\pi \alpha Z}{|\vec{P}_p - \vec{P}_p'|^2 \Omega} . \qquad (161)$$

The matrix element of the electron-neutrino field perturbation ($\mathcal{H} = GQ^* c_{s\sigma}^* a_s^* b_\sigma^*$) of the neutron will be:

$$\mathcal{H}_{fi} = \frac{G}{\Omega} \int d\vec{r} \left[e^{-iP_p' r} + \frac{8\pi \alpha Z M_p}{\Omega} \sum_{P_p'} \frac{e^{-iP_p' \cdot r}}{|P_p - P_p'|^2 (P_p^2 - P_p'^2)} \right] C_{s\sigma}^* e^{iP_N \cdot r} . \qquad (162)$$

The first term of this,

$$G/\Omega \int d\vec{r}\, C^{*}_{s\sigma}\, e^{i(P_N - P_P)\cdot r} \quad , \tag{163}$$

is the matrix element of the perturbation in case there is no nucleus present (Z = 0). It will lead therefore to a calculation of the probability that a neutron disintegrate spontaneously, a problem which will be taken up in the next section (d). Discarding (163) for the present then, one obtains for the growth of the probability of neutron disintegration by collision the following formula:

$$|a_{P'}|^2 = \frac{256\pi^2}{\Omega^4} G^2 \alpha^2 Z^2 M_P^2 \left| \sum_{P'_P} \frac{\int d\vec{r}\, e^{i(P_N - P'_P)\cdot r}\, C^{*}_{s\sigma}}{|P_P - P'_P|^2 (P_P^2 - P_P'^2)} \right|^2 \frac{\sin^2 t/2\, X}{X^2} \quad , \tag{164}$$

where

$$\chi = \frac{P_N^2}{2M_N} - \frac{P_P^2}{2M_P} + (M_N - M_P) - W_s - K_\sigma \, . \tag{165}$$

(164) is summed over s and σ in order to give the probability of the neutron disintegration regardless of the energies of the election and neutrino which are emitted. When this $\sum_{s\sigma}$ is as usual replaced by $\Omega^2/(2\pi)^6 \int d\vec{P}_s \int d\vec{P}_\sigma$, it becomes evident that one part of the integration, say $\int dp_\sigma p_\sigma^2 \ldots$, can be performed immediately because of the special character of the function $\sin^2(t/2)\, x/X^2$. Using the same arguments as those given in Chapter IV, most of the integrand is put outside the integral:

$$\int_0^\infty dP_0 P_0^2 \frac{\sin^2 t/2\, x}{x^2} = P_0 K_\sigma \int_{-\infty}^{\infty} dx \frac{\sin^2 t/2\, x}{x^2} = P_0 K_\sigma \cdot \frac{\pi t}{2}, \qquad (166)$$

with the restriction, $x = 0$. Then,

$$\sum_{s\sigma} \frac{|a_s|^2}{t} = \frac{2}{\pi^3} \frac{G^2}{\Omega^2} \alpha^2 Z^2 M_P^2 \int dP_s \int d\omega_\sigma P_0 K_\sigma \left| \sum_{P'_P} \frac{F^*_{m,n}}{\Omega} \frac{\int dr\, e^{i(P_N - P'_P - P_s - P_0)\cdot r}}{|P_P - P'_P|^2 (P_P^2 - P'^2_P)} \right|^2,$$

where $F^*_{m,n}$ is defined by (136). The integral

$$\int dr\, e^{i(P_N - P'_P - P_s - P_0)\cdot r} = \Omega,$$

with the condition that P_P' can have only the single value:

$$\vec{P}_C = \vec{P}_N - \vec{p}_s - \vec{p}_\sigma \qquad (167)$$

so that the $\sum_{P'_P}$ reduces to one term and:

$$\sum_{s\sigma} \frac{|a_s|^2}{t} = \frac{2}{\pi^3} \frac{G^2}{\Omega^2} \alpha^2 Z^2 M_P^2 \int d\vec{P}_s \int d\omega_\sigma P_0 K_\sigma \frac{|F_{m,n}(P_s, P_0)|^2}{|P_P - P_C|^4 (P_P^2 - P_C^2)^2}. \qquad (168)$$

The implied restriction that the created proton must have the definite impulse $\vec{P}_P$ is removed by integrating over this quantity with the introduction of $2\Omega/(2\pi)^3 \int d\vec{P}_P$. Further, (168) is divided by $P_N/M\Omega$, the incoming neutron current, in order to make of it a cross-section for the process in the units of area $\hbar^2/m^2c^2$. One now has:

$$\sigma^{inc}_{m,n} = \frac{G^2 \alpha^2 Z^2 M_P^2 M_N}{2\pi^6 P_N} \int d\vec{P}_P \int d\vec{P}_s \int d\omega_\sigma P_0 K_\sigma \frac{|F_{m,n}(P_s, P_0)|^2}{|P_P - P_C|^4 (P_P^2 - P_C^2)^2}.$$

One may hope to obtain an approximate evaluation of these rather complex integrals, by making the simplification $P_C = P_N$, (see (167)). This means that the kinetic energy of the neutron must be large enough so that the energy obtained by the electron and neutrino from the mass difference $(M_N - M_p)$ is small in comparison. The case in which the kinetic energy of the neutron plays but a small part is that of spontaneous disintegration, which is treated in the next section. With $\vec{P}_C = \vec{P}_N$, the integration over the directions of $\vec{P}_p$ becomes:

$$\int d\omega \frac{1}{|\vec{P}_p - \vec{P}_N|^4} = \frac{\pi}{P_p P_N} \int_{(P_p - P_N)^2}^{(P_p + P_N)^2} \frac{dx}{x^2} = \frac{4\pi}{(P_p^2 - P_N^2)^2}$$

in which the substitution $x = |\vec{P}_p - \vec{P}_N|^2 = P_p^2 + P_N^2 - 2P_p P_N \cos\theta$ has been used. The cross-section now becomes:

$$\sigma_{m,n}^{inc} = \frac{2g^2 \alpha^2 Z^2 M_p^2 M_N}{\pi^5 P_N} \int dP_p \frac{P_p^2}{(P_p^2 - P_N^2)^4} \int dW_s p_s W_s p_\sigma K_\sigma \iint d\omega_s d\omega_\sigma |F_{m,n}|^2.$$

after the change of integration variable from p_s to $W_s = \sqrt{p_s^2 + 1}$ has been made. The condition $x = 0$ or, according to (165),

$$\frac{P_N^2}{2M_N} + M_N = \frac{P_p^2}{2M_p} + M_p + W_s + K_\sigma \qquad (169)$$

must now be recalled. First of all, it is used to eliminate p_σ and K_σ. Secondly, it sets an upper limit on the integration over W_s at:

$$\frac{P_N^2}{2M_N} - \frac{P_P^2}{2M_P} + M_N - M_P - \mu = \beta$$

since the maximum value which W_s can attain is measured by the amount of energy lost by the heavy particle minus the rest energy of the neutrino. Finally, (169) sets an upper limit on $P_P^2/2\,M_P$ at:

$$\frac{P_N^2}{2M_N} + M_N - M_P - 1 - \mu$$

(The "1" representing the mass energy of the electron in the units used here.) Then, of the integrals occurring in $\sigma^{inc}_{m,n}$, there is first,

$$\iint d\omega_s\, d\omega_\sigma\, |F_{m,n}(p_s, p_\sigma)|^2,$$

which, for various values of m and n has the character:

$$(0,0): \iint d\omega_s\, d\omega_\sigma \left(1 + \frac{p_s \cdot p_\sigma}{W_s K_\sigma}\right) = 16\pi^2$$

$$(0,1): \iint d\omega_s\, d\omega_\sigma\, K_\sigma^2 \left(1 - \frac{p_s \cdot p_\sigma}{W_s K_\sigma}\right) = 16\pi^2 K_\sigma^2$$

$$(1,1): \iint d\omega_s\, d\omega_\sigma \left(1 + \frac{p_s \cdot p_\sigma}{W_s K_\sigma}\right)(W_s K_\sigma - p_s \cdot p_\sigma)^2 = 16\pi^2 \left(W_s^2 K_\sigma^2 - \frac{p_s^2 p_\sigma^2}{3}\right)$$

$$(1,2):\ \iint d\omega_s d\omega_\sigma K_0^2\left(1-\frac{P_s P_\sigma}{W_s K_\sigma}\right)\left(W_s K_\sigma - P_s P_\sigma\right)^2 = 16\pi^4 K_0^2\left(W_s^2 K_\sigma^2 + P_s^2 P_\sigma^2\right)$$

Secondly, there is:

$$\int_1^\beta d W_s\, W_c\sqrt{W_s^2-1}\,(\beta+\mu-W_s)\sqrt{(\beta+\mu-W_s)^2-\mu^2}\iint d\omega_s d\omega_\sigma |F_{m,n}|^2 = f(\beta).$$

Finally, after a change of integration variable from P_P to β there is:

$$\int_1^\varepsilon d\beta\,\frac{M_P P_P}{(P_P^2 - P_N^2)^4}\,f(\beta)$$

in which

$$\varepsilon = \frac{P_N^2}{2M_N} + M_N + M_P - \mu.$$

There is no point in preserving the distinction between M_P and M_N where they still occur explicitly, for this order of magnitude calculation, so that:

$$\sigma_{m,n}^{ine} = \frac{G^2\alpha^2 Z^2}{8\pi^5}\int_1^\varepsilon d\beta\sqrt{\frac{\varepsilon-\beta}{\varepsilon}}\,\frac{f(\beta)}{\beta^4} = G^2\alpha^2 Z^2 f(\varepsilon). \quad (170)$$

This has the order of magnitude $10^{-19} Z^2 f(\varepsilon)$ cm^2., and since no strikingly great contribution can be expected from $f(\varepsilon)$, at least for reasonably small values of m and n, one can say that this process is below the present limits of observability.

The analagous problem for protons (with positron emission) is calculated in precisely the same way. One can see this from the fact that in this case (162) is merely replaced by its complex conjugate. The heavy particle is here subject to the Coulomb field before its conversion. The condition (169) is replaced by:

$$\frac{P_P^2}{2M_P} + M_P = \frac{P_N^2}{2M_N} + M_N + W_s + K_\sigma$$

and therefore the process will occur only for high enough P_P (since $M_N > M_P$).

d. Radioactivity of Neutrons. The matrix element of the electron-neutrino field perturbation of the neutron, responsible for its spontaneous disintegration, is, from (163):

$$\frac{G}{\Omega^2} F^*_{m,n} \int d\vec{r}\, e^{i(\vec{P}_N - \vec{P}_P - \vec{P}_s - \vec{P}_\sigma)\cdot r} = \frac{G}{\Omega} F^*_{m,n}$$

with the impulse conservation condition:

$$\vec{P}_N = \vec{P}_P + \vec{P}_s + \vec{P}_\sigma$$

Therefore,

$$\sum_{s\sigma} |a_f|^2 = \frac{\Omega^2}{(2\pi)^6} \iint d\vec{P}_s\, d\vec{P}_\sigma\, |F_{m,n}|^2 \frac{4G^2}{\Omega^2} \frac{\sin^2 c/2\, X}{X^2}$$

$$= \frac{4G^2}{(2\pi)^6} \int dP_s P_s^2\, P_\sigma K_\sigma \int d\omega_s\, d\omega_\sigma |F_{m,n}|^2 \cdot \frac{\pi t}{2}$$

This leads precisely to the formulas for β-decay derived in Chapters IV and V except that here $|M|^2 = 1$. For the (0, 0) interaction, the lifetime τ of neutrons would be given by $1/\tau = G^2 F(W_0)$. W_0, the neutron-proton mass difference,[42] is from 1 to 3 mc^2 so that τ has a lower limit of about 6×10^4 sec., or 17 hours. With the (0, 1) interaction,

$$\frac{1}{\tau} = \frac{G^2}{2\pi^3} \int_1^{W_0} W \sqrt{W^2 - 1}\, (W_0 - W)^4 dW.$$

Now the lifetime predicted has a lower limit of about 5 minutes (with $W_0 = 3\ mc^2$). Of course, if $\mu > 0$ then a longer lifetime is predicted.

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[42] Nature 136, 467 (1935); Oliphant, Nature 137, 396 (1936).

CHAPTER VIII

RESUMÉ AND CONCLUSION

An analysis of the problem in Chapter I has shown that the continuous character of the β-ray spectra requires for its explanation on the basis of present-day physical knowledge the introduction of a new, not directly observed, particle. The only alternative to its acceptance seems to be the abandonment of the conservation laws, the failure of which is made improbable by the apparent success of their application to the fastest β-particles. On the other hand, various features of the β-decay besides the continuous character of the spectra appear to be consistent with the existence of just such a particle as would be practically unobservable.

In Chapters II and IV an exposé of a coherent means of handling the problem, due to Fermi, was made. The essential assumptions were the following: first, the neutron and the proton are alternative states of the same "heavy particle"; second, a transition between the two states is necessarily accompanied by an emission of "light particles." The transitions of the heavy particle are related to the emission of the light particles by an interaction of the heavy particle with the light particle "field." An essential of the mathematical formulation is that the interaction function, H, contain the product $Qa_s b_\sigma$ (or $Qa_s b_\sigma^*$), the symbols representing operations

which are needed to insure the creation or destruction of an electron-neutrino pair simultaneously with transitions of the heavy particle. This essential is fulfilled by any interaction function which is bilinear in products of the electron and neutrino wave functions or their derivatives.

The great latitude in the choice of the interaction function is somewhat reduced by the requirement that it be relativistically invariant. The further condition is added that the invariant form be the scalar product of four vectors, in analogy to the scalar product of Dirac matrices and a vector-potential occurring in radiation theory. It is shown in Chapter V that the only interaction functions consistent with the above conditions contain four vectors of three types denoted by: (m, m), (m, m + 1) and (m + 1, m), together with their linear combinations. By comparing the form of the spectra predicted by the various interactions with the observed spectra, it is found that the agreement appears to be best with the (0, 1) interaction.

Besides giving the form of the spectra, a theory of β-decay must account for the relative speeds of decay of the various emitters. Sargent's law, which concerns itself with this feature, is found to be consistent with the above theory. The "forbidden transitions" which it predicts are dealt with in Chapter VI. It is found that the form of "forbidden" spectra may be the same as for the ordinary spectra if the former occur according to the selection rules needed to make the "vector" part of the interaction important. When,

however, second approximations are made with the "scalar" part, the form is somewhat changed (Figure 16).

An attractive feature of the β-decay theory as presented above is the possibility it presents for coördinating all of nuclear theory. By providing interactions between neutrons and protons as shown in Chapter VII, it leads to the explanation of the stability of nuclei and of scattering phenomena. Unfortunately, divergent integrals occur, a difficulty of essentially the "infinite self-energy" type to which all field theories seem to be subject. When evaluated by rather crude methods, the divergent integrals are shown to be able to yield correct orders of magnitude.

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