INFORMATION TO USERS

This material was produced from a microfilm copy of the original document. While the most advanced technological means to photograph and reproduce this document have been used, the quality is heavily dependent upon the quality of the original submitted.

The following explanation of techniques is provided to help you understand markings or patterns which may appear on this reproduction.

1. The sign or "target" for pages apparently lacking from the document photographed is "Missing Page(s)". If it was possible to obtain the missing page(s) or section, they are spliced into the film along with adjacent pages. This may have necessitated cutting thru an image and duplicating adjacent pages to insure you complete continuity.

2. When an image on the film is obliterated with a large round black mark, it is an indication that the photographer suspected that the copy may have moved during exposure and thus cause a blurred image. You will find a good image of the page in the adjacent frame.

3. When a map, drawing or chart, etc., was part of the material being photographed the photographer followed a definite method in "sectioning" the material. It is customary to begin photoing at the upper left hand corner of a large sheet and to continue photoing from left to right in equal sections with a small overlap. If necessary, sectioning is continued again — beginning below the first row and continuing on until complete.

4. The majority of users indicate that the textual content is of greatest value, however, a somewhat higher quality reproduction could be made from "photographs" if essential to the understanding of the dissertation. Silver prints of "photographs" may be ordered at additional charge by writing the Order Department, giving the catalog number, title, author and specific pages you wish reproduced.

5. PLEASE NOTE: Some pages may have indistinct print. Filmed as received.

Xerox University Microfilms
300 North Zeeb Road
Ann Arbor, Michigan 48106

76-9426

HWANG, Antonio Ng, 1942-
QUASI-REFLEXIVE AND REFLEXIVE OPERATORS.

The University of Michigan, Ph.D., 1975
Mathematics

Xerox University Microfilms, Ann Arbor, Michigan 48106

THIS DISSERTATION HAS BEEN MICROFILMED EXACTLY AS RECEIVED.

QUASI-REFLEXIVE AND REFLEXIVE OPERATORS

by
Antonio Ng Hwang

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
1975

Doctoral Committee:

Professor Allen L. Shields, Chairman
Associate Professor Larry K. Flanigan
Associate Professor Chung-Nim Lee
Professor Melapalayam S. Ramanujan
Professor B. Alan Taylor

To my wife, Lalana,

for making this work possible

ACKNOWLEDGMENTS

I wish to express my sincere appreciation to Professor Allen L. Shields for his guidance during the preparation of this paper and to Professor Melapalayam S. Ramanujan for his efforts as second reader.

In addition, I am indebted to my colleagues at the IBM Corporation (Mohansic, New York). Deserving particular mention are: Dr. M. Garth Notley, Manager, for making it possible for me to complete this paper, Mr. Charles M. N. Cree, for his encouragements, and Mrs. M. Celeste Puccini, for typing portions of this paper.

TABLE OF CONTENTS

DEDICATION ii

ACKNOWLEDGMENTS iii

CHAPTER

1. NOTATIONS AND DEFINITIONS 1

2. INTRODUCTION 4

3. ON FINITE-DIMENSIONAL SPACES 7

4. SOME GENERAL RESULTS 12

5. ON THE SPACE ω 17

6. ON FUNCTIONAL ALGEBRAS 30

7. ON FINITE DIRECT SUMS OF $L^p(\mu)$ 36

8. ON FINITE DIRECT SUMS OF H^p 52

REFERENCES 63

CHAPTER 1

NOTATIONS AND DEFINITIONS

This chapter will be devoted to establishing some of the notations and terms used in this paper. Terms used in this chapter but not defined can be found in Köthe [12].

Let V be a Fréchet space (i.e. a complete metrizable locally convex topological vector space) over the complex field $\mathbb{C}$. Let $\{x_\alpha\}$ be a net in V. Then $x_\alpha \text{ -----> } x$ denotes $\{x_\alpha\}$ converges to x.

The dual space of V, denoted by V^*, is the locally convex topological vector space (lctvs) which consists of the set of all continuous linear functionals on V with the strong dual topology (i.e. the topology of uniform convergence on bounded subsets of V). It is known that V^* is complete. Let $\phi \in V^*$. The value of ϕ at a vector x in V is denoted by $\phi(x)$ or (x,ϕ). A net $\{x_\alpha\}$ of vectors in V is said to converge weakly to the vector x, denoted by $x_\alpha \xrightarrow{W} x$, if $(x_\alpha,\phi) \text{ -----> } (x,\phi)$ for each ϕ in V^*.

The set of all bounded linear transformations (blt, or operators) on V is denoted by $\mathcal{L}(V)$. It is well known that a linear transformation on a Fréchet space is bounded if and only if it is continuous. We shall assume that every subalgebra of $\mathcal{L}(V)$ contains the identity. Two topologies will be defined on $\mathcal{L}(V)$. Let $\{T_\alpha\}$ be a net of blt on V. It is said to converge in the strong operator topology (SOT) to a blt T, denoted by $T_\alpha \xrightarrow{SOT} T$, if and only if

$T_\alpha x$ -----> Tx for each x in V. The net is said to converge in the weak operator topology (WOT) to T, denoted by $T_\alpha \xrightarrow{WOT} T$, if $T_\alpha x$ converges weakly to Tx for each x in V. Each of these topologies makes $\mathcal{L}(V)$ into a lctvs and they have the same continuous linear functionals. It follows that a vector subspace $\mathcal{A}$ of $\mathcal{L}(V)$ is closed in the weak operator topology (weakly closed) if and only if it is closed in the strong operator topology (strongly closed).

For each $2 \leq n \leq \infty$, the Cartesian product of n copies of V together with the product topology is a Fréchet space denoted by $V(n)$. Let $\{T_i\}_{i=1}^n$ be a (finite or infinite) sequence of blt on V. Then $T = \sum_{i=1}^n \oplus T_i$ is the blt on $V(n)$ defined by $(Tx)(i) = T_i x(i)$. For the case where the T_i's are all equal to a blt S we shall denote $\sum_1^n \oplus S$ by $S(n)$. Let $\mathcal{A}$ be a subalgebra of $\mathcal{L}(V)$. Then $\mathcal{A}(n)$ denotes the algebra of all $T(n)$ where T is in $\mathcal{A}$.

Given any two commuting blt A, B on V, $\mathcal{P}(A,B)$ denotes the set of all polynomials in A and B. If $B = 0$ then it is denoted by $\mathcal{P}(A)$. A subspace W of V is said to be invariant under A if $AW \subseteq W$. In this paper we restrict our attention to closed subspaces (referred to simply as subspaces). Let $\mathcal{A}$ be a subalgebra of $\mathcal{L}(V)$. Then Lat $\mathcal{A}$ denotes the set of all subspaces of V which are invariant under every blt in $\mathcal{A}$. If $\mathcal{A} = \mathcal{P}(A)$ or $\mathcal{P}(A,B)$ then Lat $\mathcal{A}$ is denoted by Lat A and Lat(A,B), respectively.

Let $\mathcal{A}$ be a subalgebra of $\mathcal{L}(V)$. Then Alg(Lat $\mathcal{A}$) denotes the algebra of all blt T which satisfy

Lat T $\supseteq$ Lat $\mathcal{A}$. It is easy to see that it is weakly closed. A weakly closed subalgebra $\mathcal{A}$ of $\mathcal{L}(V)$ is said to be reflexive if $\mathcal{A}$ = Alg(Lat $\mathcal{A}$). It is said to be n-reflexive (n a positive integer) if $\mathcal{A}(n)$ is reflexive.

A weakly closed subalgebra $\mathcal{A}$ of $\mathcal{L}(V)$ is quasi-reflexive if it contains every $T \in \mathcal{L}(V)$ which satisfies (i) T commutes with every blt in $\mathcal{A}$ and (ii) Lat T $\supseteq$ Lat $\mathcal{A}$.

Let $T \in \mathcal{L}(V)$. The weakly closed algebra generated by T and 1 is denoted by Alg(T). T is said to be reflexive (n-reflexive or quasi-reflexive) if Alg(T) is reflexive (n-reflexive or quasi-reflexive).

Let v be a vector in V and let $\mathcal{B}$ be a subset of $\mathcal{L}(V)$. The cyclic invariant subspace for $\mathcal{B}$ generated by v is the closure of $\mathcal{A}v$ where $\mathcal{A}$ is the smallest subalgebra of $\mathcal{L}(V)$ containing $\mathcal{B}$. If the closure of $\mathcal{A}v$ is V then v is said to be a cyclic vector for $\mathcal{B}$.

In the chapters that follow we shall use ====> to denote implication.

We shall also have occasions to use terms such as Banach space, Hilbert space, norm, semi-norm and inner product. These are standard terms and can be found in any functional analysis book.

We have restricted our notations and terms to Fréchet spaces. However, they may be used in the context of a lctvs.

CHAPTER 2

INTRODUCTION

The theory of reflexive algebras is relatively new. Most of the results are concerned with operator algebras on a Hilbert space. We shall mention some known results along this line. The oldest result states that every von Neumann algebra is reflexive. This is equivalent to the double commutant theorem (see [15]). More recently, Sarason [20] proved that every weakly closed commutative algebra of normal operators is reflexive. This algebra has many invariant subspaces. In a different direction, Radjavi and Rosenthal [16] proved that if a weakly closed algebra $\mathcal{A}$ of blt on a Hilbert space contains a maximal abelian self-adjoint algebra and if Lat $\mathcal{A}$ is totally ordered, then $\mathcal{A}$ is reflexive. The problem of determining all reflexive algebras remains unsolved.

Every reflexive algebra is quasi-reflexive. However, the converse is not true. Brickman and Fillmore [3] proved that every linear transformation on a finite-dimensional vector space over an arbitrary field is quasi-reflexive. It is not known if every blt on an infinite dimensional Hilbert space is quasi-reflexive.

Azoff [1] used the notion of k-reflexivity to measure how non-reflexive an algebra of linear transformations over a finite-dimensional complex vector space can be.

The purpose of this paper is to study these different

notions of reflexivity over Fréchet spaces.

Chapter 3 is concerned with an n-dimensional vector space V over an arbitrary field. It shows that if $n \leq 3$ and if A, B are commuting linear transformations on V then $\mathcal{P}(A,B)$ is quasi-reflexive. An example shows that this is not true for $n \geq 4$.

Some general results on reflexive and quasi-reflexive blt are established in Chapter 4. A consequence of these results is that if every blt of an infinite dimensional Hilbert space is quasi-reflexive then the countable direct sum of a blt with itself and the countable direct sum of reflexive blt are each reflexive.

Let ω be the Fréchet space of all complex-valued sequences with the pointwise convergence topology. The main result in Chapter 5 states that a blt on ω is quasi-reflexive if its spectrum is at most countable and is reflexive otherwise. This is established by exploiting the special properties of ω and its dual space, and the duality between them.

Chapters 6, 7 and 8 study the multiplication operators on functional algebras, L^p-spaces over a probability Borel measure on the complex plane having compact support and Hardy spaces H^p on the unit circle, respectively ($1 \leq p < \infty$). In each case the weak operator topology on the weakly closed algebra of all multiplication operators is identified with a 'nicer' topology. This is used to show that every weakly closed algebra of multiplication operators

is reflexive. The necessary and sufficient condition for a blt on a functional Fréchet space to be a multiplication operator is given in Chapter 6.

For $1 < n < \infty$, let $\mathcal{M}(p,n)$ ($\tilde{\mathcal{M}}(p,n)$) denote the algebra of all nxn matrices (T_{ij}) where T_{ij} is a multiplication operator on L^p (H^p). We show that $\mathcal{M}(p,n)$ ($\tilde{\mathcal{M}}(p,n)$) is a reflexive algebra of operators on the direct sum of n copies of L^p (H^p) and every weakly closed subalgebra of $\mathcal{M}(p,n)$ ($\tilde{\mathcal{M}}(p,n)$) is n-reflexive. Each $\mathcal{M}(p,n)$ has a non-reflexive commutative weakly closed subalgebra. In addition, a measure is constructed such that $\mathcal{M}(p,n)$ contains a non-reflexive blt. For $n \geq 4$, $\mathcal{M}(p,n)$ has a commutative weakly closed subalgebra which is not quasi-reflexive. The examples used in the preceding statements when adapted to $\tilde{\mathcal{M}}(p,n)$ are shown to be reflexive by using function theoretic methods.

CHAPTER 3

ON FINITE-DIMENSIONAL SPACES

In this chapter we shall let V be an n-dimensional vector space over an arbitrary field $\mathcal{F}$. We shall use the notations and definitions in Chapter 1 without the topological restrictions.

We state some known results for the case where $\mathcal{F} = \mathbb{C}$. Deddens and Fillmore [4] gave necessary and sufficient conditions for a linear transformation on V to be reflexive. In [1], Azoff showed that if $n \geq 4$ then (1) every subalgebra of $\mathcal{L}(V)$ is (n-1)-reflexive and (2) every commutative subalgebra of $\mathcal{L}(V)$ is [n/2]-reflexive, where [n/2] is the greatest integer in n/2. It was also shown that these results are best-possible.

Brickman and Fillmore in [3] proved that every linear transformation on V over an arbitrary field $\mathcal{F}$ is quasi-reflexive. This means that if A and B are commuting linear transformations on V, then $\mathrm{Lat}(A) \subseteq \mathrm{Lat}(B)$ if and only if B is a polynomial in A. We shall refer to this as the Brickman-Fillmore theorem. Fillmore suggested the following question (conveyed to me by Constantin Apostol):

> If A and B are commuting linear transformations on V then is $\mathcal{P}(A,B)$ quasi-reflexive?

This question is completely answered in the next theorem.

THEOREM 3.1: Let V be an n-dimensional vector space over an arbitrary field $\mathcal{F}$. Let A and B be commuting linear

transformations on V. Then

(1) for $n \leq 3$ the algebra $\mathcal{P}(A,B)$ is quasi-reflexive, and

(2) for $n \geq 4$ we can find A and B such that $\mathcal{P}(A,B)$ is not quasi-reflexive.

PROOF: (1) Let $C \in \mathcal{L}(V)$ which commutes with A and B, and Lat $C \supseteq$ Lat(A,B).

Suppose $n = 2$. If A has no nontrivial invariant subspace then, by the Brickman-Fillmore Theorem, C is a polynomial in A. If A is a scalar multiple of the identity then C is a polynomial in B. Similar statements can also be made for B. Finally, if A has a 1-dimensional eigenspace then A, B and C can be represented by upper triangular matrices relative to a fixed basis. By subtracting appropriate scalar multiples of the identity from A, B and C we may assume that

$$A = \begin{pmatrix} 0 & a_1 \\ 0 & a_2 \end{pmatrix} \quad B = \begin{pmatrix} 0 & b_1 \\ 0 & b_2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & c_1 \\ 0 & c_2 \end{pmatrix}$$

Since A and C commute we have $a_1c_2 = c_1a_2$. Thus (i) $a_1 \neq 0$ implies $C = (c_1/a_1)A$ (ii) $a_2 \neq 0$ implies $C = (c_2/a_2)A$ and (iii) $a_1 = a_2 = 0$ implies C is a polynomial in B. We have actually shown that C is a polynomial in either A or B.

Suppose $n = 3$. Applying the rational decomposition theorem to A we have $V = V_1 \oplus \ldots \oplus V_k$ where (i) each V_i is cyclic for A with cyclic vector e_i and (ii) if q_i is the A-annihilator of e_i then q_{i+1} divides q_i. We need to consider all possible cases for this representation.

Case (1) dim $V_1 = 1$. Then dim V_1 = dim V_2 = dim V_3 = 1 and hence $q_1 = q_2 = q_3 = x - a$. This implies that A is a scalar multiple of the identity matrix and hence C is a polynomial in B.

Case (2) dim $V_1 = 2$. Then dim $V_2 = 1$ and $q_2 = x - a$. Since q_2 divides q_1 we have $q_1 = (x - a)(x - b)$. We need to consider the two cases where $a \neq b$ and $a = b$.

Case (2a) $a \neq b$. Then A has a diagonal representation diag(a,a,b) relative to some fixed basis. By subtracting aI (where I is the 3x3 identity matrix) from A we may assume $a = 0$ and $b \neq 0$. Since B and C commute with A we have

$$B = \begin{pmatrix} B' & 0 \\ 0 & b \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} C' & 0 \\ 0 & c \end{pmatrix}$$

where B', C' are 2x2 matrices and $b, c \in \mathcal{F}$. It is clear that B' commutes with C' and Lat C' $\supseteq$ Lat B'. Thus C' is a polynomial in B' and hence C is the sum of a polynomial in B and a scalar multiple of A.

Case (2b) $a = b$. Then

$$A = \begin{pmatrix} a & 0 & 0 \\ 1 & a & 0 \\ 0 & 0 & a \end{pmatrix}$$

relative to some fixed basis. By subtracting aI from A and changing to a new basis we have

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Since B and C commute with A, after subtracting appropriate

multiples of I, we have

$$B = \begin{pmatrix} c & 0 & e \\ d & 0 & f \\ 0 & 0 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} c' & 0 & e' \\ d' & 0 & f' \\ 0 & 0 & 0 \end{pmatrix}$$

Since B and C commute we have $cd' = c'd$, $ce' = c'e$ and $de' = ed'$. These equations are essential for our discussion. If $c \neq 0$ then $(c'/c)B - C$ is a scalar multiple of A. If $c = 0$ then $c'd = 0$ and $c'e = 0$. If $c' \neq 0$ then $d = e = 0$ and hence C is a multiple of A. If $c' = 0$ then we need to consider the cases where $d \neq 0$ and $d = 0$. Suppose $d \neq 0$. Then $(d'/d)B - C$ is a multiple of A. On the other hand suppose $d = 0$. Since $A^2 = B^2 = AB = BA = 0$ and Lat $C \supseteq$ Lat (A,B) we have $d' = 0$ and $(e',f',0)$ is a linear combination of $(0,1,0)$ and $(e,f,0)$. Hence C is a linear combination of A and B.

Case (3) dim $V_1 = 3$. Then $V_1 = V$ and C is a polynomial in A.

(2) For n = 4 let

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

An easy computation shows that
$AB = BA = AC = CA = BC = CB = 0$ and $A^2 = B^2 = C^2 = 0$. Thus C cannot be expressed as a polynomial in A and B. We shall now show that Lat $C \supseteq$ Lat(A,B). It suffices to consider cyclic invariant subspaces for A and B. For any vector (w,x,y,z) we have $A(w,x,y,z) = (0,0,w,x)$, $B(w,x,y,z) = (0,0,0,w)$ and $C(w,x,y,z) = (0,0,w,w)$. It is

clear that the third vector is a linear combination of the first two vectors and we are done.

Examples for $n > 4$ can be constructed from the preceding example via direct sums.

This completes the proof of the theorem.

REMARKS: (1) Theorem 3.1 in its present form is based on an extension of its earlier version and due to the suggestions of an (anonymous) referee.

(2) The example in Theorem 3.1 can be extended to a vector space of arbitrary dimension (finite or infinite) via direct sums. This is connected to the Sarason-Rosenthal conjecture mentioned in the next chapter.

CHAPTER 4

SOME GENERAL RESULTS

In this chapter we prove some results which hold in a setting more general than specified in any of the following chapters. The following definition will be used in the next proposition as well as in the next chapter.

DEFINITION: A linear transformation T on a vector space V is said to be locally algebraic if for each vector x in V there exists a nonzero polynomial p_x such that $(p_x T)(x) = 0$. It is said to be algebraic if there exists a nonzero polynomial p such that $p(T) = 0$.

REMARK: Kaplansky in [11] showed that for blt in a Banach space the notions of locally algebraic and algebraic are equivalent; his proof actually works for a Fréchet space.

PROPOSITION 4.1: If T is a locally algebraic blt on a Fréchet space V then T is quasi-reflexive.

PROOF: Let U be a blt on V which commutes with T and let Lat U $\supseteq$ Lat T. Let Λ be the collection of all finite dimensional invariant subspaces of T partially ordered by set inclusion. Then Λ is a directed set. Since Lat U $\supseteq$ Lat T, we have $\Lambda \subseteq$ Lat U. Let $\lambda \in \Lambda$. Applying the Brickman-Fillmore Theorem we obtain a polynomial p_λ such that $p_\lambda(T) = U$ on λ. Since T is locally algebraic the union of Λ is V. It is now easy to verify that $p_\lambda(T)$ converges to U in the strong operator topology.

REMARK: This proposition also holds in a topological vector space.

PROPOSITION 4.2: Let V be a Fréchet space which satisfies:

(a) every blt on V is quasi-reflexive and

(b) for a fixed $2 \leq n \leq \infty$ there exists an invertible blt S_n from $V(n)$ onto V.

Then, for the fixed n in (b), we have

(i) $T(n)$ is reflexive on $V(n)$ for each blt T on V and

(ii) $\sum_{i=1}^{n} \oplus T_i$ is reflexive on $V(n)$ for any collection $\{T_i\}_{i=1}^{n}$ of reflexive blt on V.

PROOF: (i) Let $T \in \mathcal{L}(V)$, $A \in \mathcal{L}(V(n))$ with Lat $T(n) \subseteq$ Lat A $(2 \leq n \leq \infty)$. Then there exists $U \in \mathcal{L}(V)$ such that $A = U(n)$. We shall show that T and U commute.

For each $v \in V$ define y_v to be a vector in $V(n)$ whose first and second coordinates are v and Tv, respectively. Since Lat $U(n) \supseteq$ Lat $T(n)$ there exists a net $\{p_\alpha\}$ of polynomials such that $(p_\alpha(T(n)))y_v \text{ -----> } (U(n))y_v$. This implies that

$$(p_\alpha(T))v \text{ -----> } Uv \quad \ldots\ldots \quad (1)$$

and

$$(p_\alpha(T))Tv \text{ -----> } UTv \quad \ldots\ldots \quad (2)$$

It follows from (1) that $T(p_\alpha(T))v \text{ -----> } TUv$. Since $(p_\alpha(T))T = T(p_\alpha(T))$ we have, from (2), that $UTv = TUv$. Since v is arbitrary we have shown that T and U commute.

Consider the blt $T_1 = S_n T(n) S_n^{-1}$ and $U_1 = S_n U(n) S_n^{-1}$. Clearly, Lat $T_1 \subseteq$ Lat U_1. It follows from the preceding discussion that T_1 and U_1 commute. By hypothesis T_1 is

quasi-reflexive and hence U_1 is in $\mathrm{Alg}(T_1)$. Thus $U(n)$ is in $\mathrm{Alg}(T(n))$.

(ii) Let $U \in \mathcal{L}(V(n))$ with $\mathrm{Lat}\ U \supseteq \mathrm{Lat} \sum_{i=1}^{n} \oplus T_i$. Then $U = \sum_{i=1}^{n} \oplus U_i$ where $U_i \in \mathcal{L}(V)$. It is clear that $\mathrm{Lat}\ U_i \supseteq \mathrm{Lat}\ T_i$. Since T_i is reflexive, U_i is in $\mathrm{Alg}(T_i)$. In particular, U_i and T_i commute. Thus $\sum_{i=1}^{n} \oplus U_i$ commutes with $\sum_{i=1}^{n} \oplus T_i$. By a reasoning similar to that given in the preceding paragraph we can show that $\sum_{i=1}^{n} \oplus U_i$ is in $\mathrm{Alg}(\sum_{i=1}^{n} \oplus T_i)$.

REMARKS: (1) An example of a Fréchet space is given in the next chapter which satisfies the hypotheses of this propositon.

(2) D. Sarason and P. Rosenthal independently made the following conjecture (see [15]):

CONJECTURE: Every blt of a Hilbert space is quasi-reflexive.

An affirmative answer to this conjecture would answer the following open question.

QUESTION: Is $A \oplus A$ reflexive for every blt A on a Hilbert space?

(3) In view of this proposition we would like to propose the following question.

QUESTION: Is the direct sum of two reflexive blt on a Hilbert space reflexive?

A negative answer to this question would disprove the Sarason-Rosenthal conjecture.

(4) We should note that not every weakly closed

commutative algebra of blt on a Hilbert space $\mathcal{H}$, where dim $\mathcal{H} > 3$, is quasi-reflexive (see Theorem 3.1 and Theorem 7.10).

LEMMA 4.3: If T, W are blt on a Fréchet space V, and if Lat W(n) $\supseteq$ Lat T(n) $(n \geq 1)$, then W is in the SOT-closed algebra generated by T.

PROOF: Let $\mathcal{F}$ be the collection of all finite subsets of V partially ordered by set inclusion. Let $X = \{ x_1, \ldots, x_m \} \in \mathcal{F}$. Since Lat W(m) $\supseteq$ Lat T(m) there exists a polynomial p_X such that

$$\text{dist}(Wx_i, (p_X T)x_i) < \frac{1}{m} \qquad i = 1,2,\ldots,m.$$

It is now clear that the net $p_X(T)y$ converges to Wy for all y in V and hence $p_X T$ converges to W in the SOT.

REMARK: The proof for the preceding lemma is included for completeness. The special case when V is a Hilbert space is in Radjavi and Rosenthal [15]. Their proof also works for a Fréchet space.

PROPOSITION 4.4: If T is a blt on a Fréchet space V then $T' = T(\infty)$ is reflexive on $V' = V(\infty)$.

PROOF: Let W' be a blt on V' with Lat W' $\supseteq$ Lat T'. Since $T' = T(\infty)$ we have $W' = W(\infty)$ for some $W \in \mathcal{L}(V)$. For each positive integer n define $B_n : V'(n) \longrightarrow V'$ by

$$B_n((f_{11},f_{12},\ldots),(f_{21},f_{22},\ldots),\ldots,(f_{n1},f_{n2},\ldots))$$
$$= (f_{11},f_{21},\ldots,f_{n1},f_{12},f_{22},\ldots,f_{n2},\ldots) \qquad \text{where } f_{ij} \in V.$$

Then B_n is an invertible blt. A simple computation shows that $T'(n) = B_n^{-1} T' B_n$ and $W'(n) = B_n^{-1} W' B_n$. This implies that

Lat $W'(n) \supseteq$ Lat $T'(n)$. It follows from the preceding lemma that W' is in the SOT-closed algebra generated by T'. Hence T' is reflexive.

REMARK: Radjavi and Rosenthal in [15] proved this proposition in the special case when V is a Hilbert space. Their method involved Hilbert space theory. It should be noted that $V(\infty)$ is defined differently in this case. Our method provides a simpler proof in this special case.

CHAPTER 5

ON THE SPACE ω

Let Z be the set of all positive integers. Let ω be the vector space consisting of all the complex-valued functions (sequences) on Z. For each positive integer n define (a) e_n to be the function on Z which takes the value 1 on n and 0 everywhere else and (b) $\| \ \|_n$ to be the semi-norm on ω given by

$$\| f \|_n = \max \{ |f(i)| : 1 \leq i \leq n \}.$$

Then ω is a Fréchet space relative to the topology given by this family of semi-norms. In fact, this topology coincides with the metric topology given by

$$\rho(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f - g\|_n}{1 + \|f - g\|_n}$$

Our purpose is to study the reflexivity and quasi-reflexivity of a blt on ω. We need to mention some known facts most of which can be found in [10], [13] and [14].

On the dual space of ω:

Let ω_0 be the set of all terminating sequences.

Fact 1: $\omega^* = \omega_0$.

Fact 2: ω_0 with the strong dual topology is a complete lctvs.

Fact 3: The topological dual of ω_0 coincides with its algebraic dual. In particular, $\omega_0^* = \omega$.

Fact 4: Every linear subspace of ω_0 is closed.

Fact 5: ω is reflexive (i.e. $\omega_0^* = \omega$ and the strong dual topology on ω_0^* coincides with the topology of ω).

Fact 4 will play a very fundamental role in the proof of our main result -- Theorem 5.3. The duality between ω and ω_0 is given by:

$$(f,\phi) = \sum f(n)\phi(n) \qquad f \in \omega,\ \phi \in \omega_0.$$

On the topology of ω:

Fact 6: ω_0 is dense in ω.

Fact 7: The topology of ω coincides with the topology of pointwise convergence, i.e.,

$f_n \text{ -----> } f \iff f_n(i) \text{ -----> } f(i)$ for each $i \in \mathbb{Z}$.

In particular, it also coincides with the weak topology.

On $\mathcal{L}(\omega)$ and $\mathcal{L}(\omega_0)$:

Let $\mathcal{M} = \{ (a_{ij})_{i,j=1}^{\infty} : a_{ij} \in \mathbb{C} \}$ and let $\mathcal{M}_R$ be the set of all row finite matrices in $\mathcal{M}$.

Fact 8: Every blt on ω has closed range.

Fact 9: Every blt on ω has a nontrivial invariant subspace.

Fact 10: $\mathcal{L}(\omega) = \mathcal{M}_R$, i.e., every blt on ω can be represented as a matrix in $\mathcal{M}_R$ relative to the basis $\{e_n\}_{n=1}^{\infty}$.

Fact 11: The strong and weak operator topologies on $\mathcal{L}(\omega)$ coincide.

Fact 12: Let $\{ A_k = (a_{ij}^{(k)}) \}$ be a sequence in $\mathcal{M}_R$. Then

$$A_k \xrightarrow{\text{SOT}} 0 \iff a_{ij}^{(k)} \text{ -----> } 0 \text{ for each } i, j$$

and $\forall i,\ \exists j_i$ such that $a_{ij}^{(k)} = 0$ whenever $j \geq j_i$.

Fact 13: For each positive integer n and each bounded set $\mathcal{F} \subseteq \omega$ define a semi-norm $\| \ \|_{\mathcal{F},n}$ on $\mathcal{L}(\omega)$ by

$$\| T \|_{\mathcal{F},n} = \sup \{ \| Tf \|_n : f \in \mathcal{F} \}.$$

Then $\mathcal{L}(\omega)$ is complete in the topology induced by this family of semi-norms. It can be shown that this topology coincides with the strong operator topology defined in Chapter 1.

Fact 14: Every algebraic linear transformation on ω_0 is continuous.

Fact 15: Let $A \in \mathcal{M}_R$. Then the adjoint of A, denoted by A^*, is the blt on ω_0 with $(A^*\phi)f = \phi(Af)$ where $\phi \in \omega_0$ and $f \in \omega$. It can be shown that A^* is the transpose of the matrix A.

Fact 16: If $S, T \in \mathcal{L}(\omega)$ and Lat $S \subseteq$ Lat T then Lat $S^* \subseteq$ Lat T^*.

On the spectrum of a blt on ω and ω_0:

DEFINITION: The spectrum of a blt T on a lctvs V is the set $\sigma(T) = \{ \lambda \in \mathbb{C} : T - \lambda$ is not invertible in $\mathcal{L}(V) \}$. The resolvent $\rho(T)$ of T is the complement of $\sigma(T)$ in $\mathbb{C}$.

Körber, in [13], made an extensive study of the spectrum of a blt on ω and ω_0. Since our main result depends heavily on the facts mentioned below we shall provide proofs.

Fact 17: If $T \in \mathcal{L}(\omega)$ then $\sigma(T) = \sigma(T^*)$.

PROOF: Let $\lambda \in \mathbb{C}$. An easy computation shows that $T - \lambda$ is not one-to-one if and only if $T^* - \lambda$ is not onto. Using Fact 8 one can easily show that $T - \lambda$ is not onto if

and only if $T^* - \lambda$ is not one-to-one. We have thus shown that $T - \lambda$ is not invertible if and only if $T^* - \lambda$ is not one-to-one or onto. The desired result follows from Fact 14.

Fact 18: Let $A \in \mathcal{L}(\omega_0)$. Then (i) A is locally algebraic ====> $\sigma(A)$ is a nonempty countable set and (ii) A is not locally algebraic ====> $\rho(A)$ is countable.

PROOF: (i) Suppose A is locally algebraic. For each positive integer n let M_n be the smallest invariant subspace for A containing $e_1, \ldots, e_n$. Then M_n has finite dimension. Let A_n be the restriction of A to M_n. Since ω_0 is the union of the M_n's we have $\sigma(A) = \cup_1^\infty \sigma(A_n)$. Since each $\sigma(A_n)$ is a nonempty finite set we have $\sigma(A)$ is a nonempty countable set.

(ii) Suppose A is not locally algebraic. Then there exists $\phi_0 \in \omega_0$ such that for each polynomial p, if $(p(A))\phi_0 = 0$ then $p = 0$.

Suppose $\rho(A)$ is an uncountable set. Then $\{ (A - \lambda)^{-1}\phi_0 : \lambda \in \rho(A) \}$ is linearly dependent. Thus there exist distinct $\lambda_1, \ldots, \lambda_n \in \rho(A)$ and $\beta_1, \ldots, \beta_n \in \mathbb{C} \setminus \{0\}$ such that

$$\sum_{i=1}^{n} \beta_i (A - \lambda_i)^{-1}\phi_0 = 0$$

and hence

$$\beta_1(A - \lambda_2)\ldots(A - \lambda_n)\phi_0 + \ldots + \beta_n(A - \lambda_1)\ldots(A - \lambda_{n-1})\phi_0 = 0$$

This implies that

$$p(x) = \beta_1(x - \lambda_2)\ldots(x - \lambda_n) + \ldots + \beta_n(x - \lambda_1)\ldots(x - \lambda_{n-1}) = 0$$

which is a contradiction.

REMARKS: We have also shown that if $A \in \mathcal{L}(\omega_0)$ then (1) $\sigma(A)$ is countable <====> A is locally algebraic and (2) A is locally algebraic ====> $\sigma(A)$ consists of eigenvalues of A. In order to see (2) we only need to observe that in the proof of Fact 18 each $\sigma(A_n)$ consists of eigenvalues.

NOTATION: Let S be a subset of a Fréchet space V. Then sp S denotes the smallest subspace of V containing S.

Fact 19: Let $A = (a_{ij})_{i,j=1}^{\infty} \in \mathcal{M}_R$ be a lower triangular matrix. Then A is invertible <====> each of the diagonal entries $\{a_{ii}\}$ is nonzero. In particular, $\sigma(A) = \{a_{ii}\}$.

PROOF: Suppose each of the a_{ii}'s is nonzero. We first show that A is one-to-one. Let $f \in \omega$ with the first nonzero value at n. Then $Af = a_{nn}f(n)e_n + g$ where $g \in sp\{ e_{n+m} : m \geq 1 \}$. It is clear that Af is not equal to zero. We next show that A is onto. Let $\phi \in \omega_0$ with k the largest integer such that $\phi(k) \neq 0$. Then $\phi(Ae_k) = \phi(k)a_{kk} \neq 0$. Thus each nonzero bounded linear functional does not annihilate $sp\{ Ae_m : m \geq 1 \}$. This implies that A has dense range. By Fact 8, A is onto.

Suppose $a_{ii} = 0$ for some i. Then let P_i be the diagonal matrix in $\mathcal{M}_R$ whose first i diagonal elements are 1 and 0 otherwise. Then

$$P_i(sp\{ Ae_m : m \geq 1 \}) = P_i(sp\{ Ae_1, \ldots, Ae_{i-1} \})$$

and has dimension $\leq i - 1$. Thus there exists a nonzero ϕ in ω_0 with $\phi(j) = 0$ for $j > i$ and which annihilates $P_i(sp\{ Ae_1, \ldots, Ae_{i-1} \})$. It is clear that ϕ annihilates

$A\omega$. Thus A is not onto.

On the weighted shifts of ω:

DEFINITION: Let $\{w_n\}$ be an infinite sequence. The forward weighted shift induced by $\{w_n\}$ is the blt W on ω defined by

$$(Wf)(i) = \begin{cases} 0 & \text{if } i = 1 \\ w_{i-1}f(i-1) & \text{if } i > 1. \end{cases}$$

The backward weighted shift induced by $\{w_n\}$ is the blt $\tilde{W}$ on ω defined by

$$(\tilde{W}f)(i) = w_i f(i+1)$$

If the sequence is identically 1 then W and $\tilde{W}$ are called the unilateral forward and backward shifts, respectively.

DEFINITION: Let $T \in \mathcal{L}(\omega)$. The commutant of T, denoted by $\{T\}'$, is the set of all blt on ω which commute with T.

PROPOSITION 5.1: Let W be the forward weighted shift induced by the sequence $\{w_n\}$, $w_n \neq 0$ for each n. Then

(i) $\sigma(W) = \{0\}$

(ii) the only invariant subspaces for W are

$$M_n = \{ f \in \omega : f(1) = \ldots = f(n) = 0 \}$$

where n is a positive integer

(iii) W is not reflexive and

(iv) $\{W\}' = Alg(W) = \{ \sum_{n=0}^{\infty} a_n W^n : a_n \in \mathbb{C} \}$.

PROOF: (i) This follows from Fact 19.

(ii) Let $f \in \omega$. It suffices to show that if $f(1) \neq 0$ then $sp\{ W^n f : n \geq 0 \} = \omega$ (i.e., f is a cyclic vector for W). If not, then there exists a nonzero $\phi \in \omega_0$ which

annihilates sp { $W^n f : n \geq 0$ }. Let i be the largest integer such that $\phi(i) \neq 0$. Then

$$\phi(i)w_1 \ldots w_{i-1} f(1) = \phi W^{i-1} f = 0.$$

This is a contradiction.

(iii) This follows from the fact that each M_n in (ii) is an invariant subspace for any lower triangular matrix in $\mathcal{M}_R$ but (as we shall prove in (iv)) not every lower triangular matrix in $\mathcal{M}_R$ is in Alg(W).

(iv) Let $A = (a_{ij}) \in \mathcal{M}_R$. Then

$AW = WA \iff AWe_j = WAe_j$ for each j

$AWe_j = WAe_j \iff A(w_j e_{j+1}) = W(\sum_i a_{ij} e_i)$

$\iff \sum_i w_j a_{i,j+1} e_i = \sum_i a_{ij} We_i$

$\iff w_j a_{1,j+1} e_1 + \sum_i w_j a_{i+1,j+1} e_{i+1} = \sum_i w_i a_{ij} e_{i+1}$

$\iff w_j a_{1,j+1} = 0$ and $w_j a_{i+1,j+1} = w_i a_{ij}$ for all i

$\iff a_{1,j+1} = 0$ and $w_j a_{i+1,j+1} = w_i a_{ij}$ for all i

It is not hard to see that this implies $A = \sum_{n=0}^{\infty} a_n W^n$ for some $a_n \in \mathbb{C}$.

REMARK: The study of forward weighted shifts (induced by bounded nonzero sequences) on a separable infinite dimensional Hilbert space is an active area of research. The situation there is very different from parts (i), (ii) and (iii) of Proposition 5.1 (see Shields [22]).

PROPOSITION 5.2: Let $\tilde{W}$ be the backward weighted shift induced by $\{w_n\}$, $w_n \neq 0$ for each n. Then

(i) each $\alpha \in \mathbb{C}$ is an eigenvalue for $\tilde{W}$ and its corresponding eigenspace is generated by the vector

$$u_\alpha = (1, \frac{\alpha}{w_1}, \frac{\alpha^2}{w_1 w_2}, \frac{\alpha^3}{w_1 w_2 w_3}, \ldots)$$

(ii) sp $\{ u_\alpha : \alpha \in \mathbb{C} \} = \omega$

(iii) Alg(Lat $\tilde{W}$) = $\mathcal{P}(\tilde{W})$ and

(iv) $\{\tilde{W}\}' = \mathcal{P}(\tilde{W})$.

PROOF: (i) This statement is clearly true for $\alpha = 0$. Let $\alpha \neq 0$. Then

$$(\alpha - \tilde{W})f = (\alpha f(1) - w_1 f(2), \alpha f(2) - w_2 f(3), \ldots) = 0$$

implies that

$$f(2) = \frac{\alpha f(1)}{w_1}$$

$$f(3) = \frac{\alpha f(2)}{w_2} = \frac{\alpha^2 f(1)}{w_1 w_2}$$

$$\vdots$$

(ii) Let $\phi \in \omega_0$ such that $\phi u_\alpha = 0$ for each $\alpha \in \mathbb{C}$. Then

$$\phi(1) + \sum_{i=2}^{\infty} \frac{\phi(i)}{w_1 \cdots w_{i-1}} \alpha^{i-1} = 0 \quad \text{for each } \alpha \in \mathbb{C}.$$

Since ϕ is a finitely nonzero sequence, the left hand side of the preceding equation is actually a polynomial of finite degree in α. Thus $\phi = 0$.

(iii) Let $U = (u_{ij})$ be a blt on ω with Lat $U \supseteq$ Lat $\tilde{W}$.

Case (1): $u_{1n} = 0$ for each $n \in Z$.

Let $\alpha \in \mathbb{C}$. Since u_α is an eigenvector for $\tilde{W}$ and Lat $U \supseteq$ Lat $\tilde{W}$ we have that u_α is also an eigenvector for U. Since the first coordinates of Uu_α and u_α are 0 and 1, respectively, we have $Uu_\alpha = 0$. By (ii) we have that $U = 0$.

Case (2): $u_{1n} \neq 0$ for some n.

Let n_1 be the largest integer such that $u_{1n_1} \neq 0$. Let $\alpha_1, \ldots, \alpha_{n_1} \in \mathbb{C}$ be such that the first row of $U_1 = U - \sum_{j=1}^{n_1} \alpha_j \tilde{W}^{j-1}$ consists of zeros. It is clear that Lat $U_1 \supseteq$ Lat $\tilde{W}$. We can now use case (1) to conclude that $U_1 = 0$ and hence $U = \sum_{j=1}^{n_1} \alpha_j \tilde{W}^{j-1}$.

(iv) It is clear that $\mathcal{P}(\tilde{W}) \subseteq \{\tilde{W}\}'$. Let $T \in \{\tilde{W}\}'$. Let $a_0, a_1, \ldots, a_n \in \mathbb{C}$ such that $T_1 = T - \sum_{i=0}^{n} a_i \tilde{W}^i$ has first row consists entirely of zeros. Clearly, $T_1 \in \{\tilde{W}\}'$.

We note that each $\alpha \in \mathbb{C}$ is a simple eigenvalue for $\tilde{W}$. Since $\tilde{W}T_1 u_\alpha = T_1 \tilde{W} u_\alpha = \alpha T_1 u_\alpha$ we have $T_1 u_\alpha$ is a scalar multiple of u_α. But its first coordinate is 0 while the first coordinate of u_α is 1. Thus $T_1 u_\alpha = 0$. By (ii) we have $T_1 = 0$.

REMARKS: The definitions made in this discussion also make sense in ω_0. In particular, by Facts 15, 16 and the duality between ω and ω_0, Proposition 5.2 (iii) becomes:

Let W be the forward weighted shift on ω_0 induced by $\{w_n\}$, $w_n \neq 0$ for each n. Then Alg(Lat W) = $\mathcal{P}(W)$.

This plays a very important role in the next theorem. We should also refer to the remark after Proposition 5.1.

Reflexivity and quasi-reflexivity of a blt on ω:

THEOREM 5.3: Let $T \in \mathcal{L}(\omega)$.

(i) If $\sigma(T)$ is countable then T is quasi-reflexive. In fact, if $U \in \mathcal{L}(\omega)$, T commutes with U and Lat U $\supseteq$ Lat T then there exists a sequence $\{p_m\}_{m=1}^{\infty}$ of polynomials such that

$p_m(T) \xrightarrow{SOT} U$.

(ii) If $\sigma(T)$ is uncountable then $\mathrm{Alg}(\mathrm{Lat}\ T) = \mathcal{P}(T)$ and hence T is reflexive.

REMARK: It follows from (1) the duality between ω and ω_0 (2) Fact 16 and (3) our discussion on the spectrum of a blt on ω and ω_0, that Theorem 5.3 is equivalent to the next theorem.

THEOREM 5.3': Let $A \in \mathcal{L}(\omega_0)$.

(i) If A is locally algebraic then A is quasi-reflexive. In fact, if $B \in \mathcal{L}(\omega_0)$, B commutes with A and Lat $B \supseteq$ Lat A then there exists a sequence $\{p_n\}_{n=1}^{\infty}$ of polynomials such that $p_n(A) \xrightarrow{SOT} B$.

(ii) If A is not locally algebraic then $\mathrm{Alg}(\mathrm{Lat}\ A) = \mathcal{P}(A)$ and hence A is reflexive.

PROOF: (i) Suppose A is locally algebraic. Let B be a blt on ω_0 which commutes with A and Lat $A \subseteq$ Lat B. Let M_n be the invariant subspace for A generated by $e_1, e_2, \ldots, e_n$. Then M_n is finite dimensional. Since B commutes with A and Lat $B \supseteq$ Lat A we may apply the Brickman-Fillmore theorem to get a polynomial p_n such that $p_n(A) = B$ on M_n. It is clear that $p_n(A)$ converges to B in the SOT.

(ii) Suppose A is not locally algebraic and $B \in \mathcal{L}(\omega_0)$ with Lat $A \subseteq$ Lat B. Then there exists $\phi \in \omega_0$ such that $V = \mathrm{sp}\ \{ A^n\phi : n \geq 0 \}$ is infinite dimensional. Hence, for any polynomial p, $(p(A))\phi = 0$ implies that $p = 0$. The blt A restricted to V is similar to the unilateral forward shift S

on ω_0 (i.e. there exists an invertible blt T from V onto ω_0 such that $T^{-1}ST$ is equal to A on V). By Proposition 5.2, Alg(Lat S) = $\mathcal{P}(S)$. Thus there exists a polynomial p_ϕ such that

$$p_\phi(A) = B, \text{ on } V \quad \ldots\ldots \quad (1)$$

We shall show that $p_\phi(A) = B$ on ω_0. Let $\gamma \in \omega_0$ and $W = sp\{ A^n\gamma : n \geq 0 \}$. We shall consider two cases.

Case (1): Suppose W is finite dimensional. Then $V \cap W = 0$. Since Lat A $\subseteq$ Lat B we may use Fact 4 to get a polynomial p such that

$$(p(A))(\phi + \gamma) = B(\phi + \gamma)$$

and hence

$$(p(A))\phi - B\phi = B\gamma - (p(A))\gamma \quad \ldots\ldots \quad (2)$$

Since $V \cap W = 0$ we have $(p(A))\phi - B\phi = 0$. It follows from (1) that $(p(A))\phi - (p_\phi(A))\phi = 0$ and hence $p = p_\phi$. Combining this and the right hand side of (2) we get $B\gamma = (p_\phi(A))\gamma$.

Case (2): Suppose W is infinite dimensional. Then by an argument similar to the above we get a polynomial p_γ such that

$$p_\gamma(A) = B \quad \text{on } W \quad \ldots\ldots \quad (3)$$

As in case (1) we get a polynomial p such that

$$(p(A))(\phi + \gamma) = B(\phi + \gamma) \quad \ldots\ldots \quad (4)$$

From (1) and (3), $B(\phi + \gamma) = (p_\phi(A))\phi + (p_\gamma(A))\gamma$. Hence (4) becomes

$$((p_\gamma - p)(A))\gamma = ((p - p_\phi)(A))\phi.$$

Let $\phi_1 = ((p - p_\phi)(A))\phi = ((p_\gamma - p))(A)\gamma$. If $\phi_1 = 0$ then $p = p_\phi$ and $p = p_\gamma$. These equations combine with

(3) to give us $B\gamma = (p_\phi(A))\gamma$. If $\phi_1 \neq 0$ then for each polynomial q, if $(q(A))\phi_1 = 0$ then $q = 0$. By the definition of ϕ_1 and (1) we get

$$B\phi_1 = B((p - p_\phi)(A))\phi$$
$$= (p_\phi(A))((p - p_\phi)(A))\phi = (p_\phi(A))\phi_1.$$

Similarly, $B\phi_1 = (p_\gamma(A))\phi_1$. Thus $p_\phi = p_\gamma$ and hence it follows from (3) that $B\gamma = (p_\phi(A))\gamma$.

We have thus shown that $B = p_\phi(A)$ on ω_0.

COROLLARY 5.4:

(i) If $T \in \mathcal{L}(\omega)$ and $\sigma(T)$ is uncountable then $\mathrm{Alg}(T) = \mathcal{P}(T)$.

(ii) T(n) is a reflexive blt on $\omega(n)$ (or $\omega_0(n)$) for each blt T on ω (or ω_0) and each $2 \leq n \leq \infty$.

(iii) For each $2 \leq n \leq \infty$ and each collection $\{T_i\}_{i=1}^n$ of reflexive blt on ω (or ω_0), the blt $\sum_{i=1}^n \oplus T_i$ is reflexive on $\omega(n)$ (or $\omega_0(n)$).

PROOF: This follows from Theorem 5.3 and Proposition 4.2.

We now give an application of Theorem 5.3.

EXAMPLE: Let $A \in \mathcal{M}_R$. For each $i \in Z$ let n_i be the positive integer such that a_{in_i} is the last nonzero entry in the ith row. If n_i is increasing then it is easy to show that every complex number is an eigenvalue of A. From Theorem 5.3 we have $\mathrm{Alg}(\mathrm{Lat}\ A) = \mathcal{P}(A)$.

We shall make some observations about Theorem 5.3 (i) and Theorem 5.3' (i):

(1) We cannot in general replace the sequence of

polynomials by a polynomial. As an example consider the unilateral forward shift W on ω. It is clear that if $T \in \mathrm{Alg}(W)$ then $\mathrm{Lat}\, T \supseteq \mathrm{Lat}\, W$ but Proposition 5.1 gives us that $\mathrm{Alg}(W) = \{ \sum_{n=0}^{\infty} a_n W^n : a_n \in \mathbb{C} \}$.

(2) If $T \in \mathcal{L}(\omega)$ is algebraic then $\mathrm{Alg}(T) = \mathcal{P}(T)$. In this special case we don't need sequences.

(3) Let $T \in \mathcal{L}(\omega)$ be a diagonal blt. Then $\sigma(T)$ is countable. If $S \in \mathcal{L}(\omega)$ and $\mathrm{Lat}\, S \supseteq \mathrm{Lat}\, T$ then it is easy to show that S is diagonal. Thus S commutes with T and hence $S \in \mathrm{Alg}(T)$. It follows that T is reflexive -- a conclusion better than what the theorem gives.

(4) A locally algebraic blt on ω_0 is reflexive if it is reflexive on each finite dimensional invariant subspace. It is not known if the converse is true. See Deddens and Fillmore [4] for necessary and sufficient conditions for a linear transformation on a finite dimensional complex vector space to be reflexive.

REMARK: In view of Chapter 3 there exist two commuting blt A and B on ω such that the weakly closed subalgebra of $\mathcal{L}(\omega)$ generated by A and B is not quasi-reflexive.

CHAPTER 6

ON FUNCTIONAL ALGEBRAS

DEFINITIONS: Let X be a set. A Fréchet space $E(X)$ of complex-valued functions on X is said to be a functional Fréchet space if (1) the functions in $E(X)$ separate the points of X and (2) for each $x \in X$ the evaluation functional ϕ_x defined by

$$\phi_x f = f(x) \qquad \text{for each } f \text{ in } E(X)$$

is continuous. Let f be a complex-valued function on X. If $fE(X) \subseteq E(X)$ then the linear transformation M_f on $E(X)$ is defined by

$$M_f g = fg \quad \text{for each } g \in E(X).$$

M_f is called a multiplication operator on $E(X)$. A functional Fréchet space is said to be a functional algebra if it contains the constant functions and is closed under multiplication by each element of $E(X)$.

The Hardy spaces H^p, $1 \leq p < \infty$, are functional Fréchet spaces which are not closed under multiplication. A detailed study of these spaces is given in Chapter 8.

Examples of functional algebras:

1. The space of all continuous functions on a region $X \subseteq \mathbb{C}$ under the topology of uniform convergence on compact subsets of X.

2. The space of all analytic functions on a given region $X \subseteq \mathbb{C}$ under the topology of uniform convergence on compact subsets of X.

3. The space of all bounded analytic functions on a given region $X \subseteq \mathbb{C}$ under the supremum norm topology.

4. Let H be the set of all complex-valued functions f defined on [0,1] such that

$$f(x) = c + \int_0^x \phi(t)\, dt \quad \text{for some } \phi \in L^2(0,1),\ c \in \mathbb{C}$$

(i.e. f is absolutely continuous on [0,1] and $f' \in L^2(0,1)$). Define a norm on H by:

$$\| f \|^2 = |f(0)|^2 + \| f' \|^2_{L^2} .$$

Then H is a functional algebra. This example is also called a functional Hilbert space in [7; Chapter 6, Problem 54]. The inner product on H is given by:

$$(f,g) = f(0)\overline{g(0)} + \int_0^1 f'(x)\overline{g'(x)}\, dx.$$

5. For each $\alpha > 1$, let D_α be the set of analytic functions on the unit disk Δ with inner product given by:

$$(f,g) = \sum_{n=0}^{\infty} (n+1)^\alpha a_n \bar{b}_n$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$.

Then D_α is a Hilbert space. Taylor [23] showed that D_α is also a functional algebra.

It is the purpose of this chapter to show that if E(X) is a functional algebra then every weakly closed algebra of multiplication operators on E(X) is reflexive.

PROPOSITION 6.1: In a functional Fréchet space E(X) the linear transformation M_f is continuous.

PROOF: This is an easy consequence of the closed graph theorem and the fact that the evaluation functionals are continuous.

PROPOSITION 6.2: Let T be a blt on the functional Fréchet space E(X). Then $T = M_f$ for some complex-valued function f on X if and only if $\{\phi_x\}_{x\in X}$ are eigenvectors of T^* (T^* is the adjoint of T). If, in addition, E(X) is also a Banach space (functional Banach space) then f is bounded.

PROOF: Suppose $T = M_f$. Then $(T^*\phi_x)g = \phi_x(Tg) = \phi_x(fg) = f(x)g(x) = f(x)\phi_x(g)$. Thus $T^*\phi_x = f(x)\phi_x$.

Conversely, suppose for each $x \in X$ there exists α_x such that $T^*\phi_x = \alpha_x\phi_x$. Define a function f on X by $f(x) = \alpha_x$. Then $\phi_x(Tg) = (T^*\phi_x)g = f(x)\phi_x(g) = f(x)g(x)$. Thus $T = M_f$.

Finally, suppose $T = M_f$ and E(X) is also a Banach space. We shall show that f is bounded. We may without loss of generality assume that $\| T \| = 1$. For each $g \in E(X)$ we have $f(x)g(x) = \phi_x Tg$ and hence

$$|f(x)||g(x)| \leq \|\phi_x\| \|T\| \|g\| = \|\phi_x\| \|g\| .$$

Thus $|f(x)| \|\phi_x\| = \sup_{\|g\|=1} |f(x)||g(x)| \leq \|\phi_x\|$. If $\|\phi_x\| > 0$, then $|f(x)| \leq 1$. Since E(X) separates points, there is at most one point where $\|\phi_x\| = 0$.

REMARK: The last part of Proposition 6.2 occurs in [6].

PROPOSITION 6.3: If E(X) is a functional algebra then

$$M_{f_\alpha} \xrightarrow{WOT} 0 \iff f_\alpha \xrightarrow{W} 0.$$

PROOF: For each $f \in E(X)$ and for each $\phi \in (E(X))^*$

$$M_{f_\alpha} \xrightarrow{WOT} 0 \implies (M_{f_\alpha} f, \phi) \longrightarrow 0$$
$$\implies (f_\alpha f, \phi) \longrightarrow 0$$
$$\implies (f_\alpha, \phi) \longrightarrow 0$$
$$\implies f_\alpha \xrightarrow{W} 0.$$

The third implication uses the assumption that the constant function 1 is in E(X).

Suppose $f_\alpha \xrightarrow{W} 0$. Then for each $f \in E(X)$ and each $\phi \in (E(X))^*$

$$(M_{f_\alpha} f, \phi) = (M_f f_\alpha, \phi) = (f_\alpha, M_f^* \phi) \longrightarrow 0.$$

The first equality uses the assumption that E(X) is an algebra.

REMARK: If E(X) is a functional algebra then the weak topology on E(X) coincides with the weak operator topology on E(X) considered as multiplication operators on E(X).

PROPOSITION 6.4: Let E(X) be a functional Fréchet space. Then the set of multiplication operators on E(X) is closed in the weak operator topology.

PROOF: Let $T \in \mathcal{L}(E(X))$. Suppose there exists a net $\{f_\alpha\}$ of complex-valued functions on X such that $M_{f_\alpha} \xrightarrow{WOT} T$. Then for each f in E(X) and $x \in X$

$$M_{f_\alpha} \xrightarrow{WOT} T \implies (M_{f_\alpha} f, \phi_x) \longrightarrow (Tf, \phi_x)$$
$$\implies f_\alpha(x) f(x) \longrightarrow (Tf, \phi_x).$$

Since the functions in E(X) separate the points of X, there is at most one point $x_0 \in X$ at which every function in E(X) vanishes. Thus it follows from the preceding implications that $\lim f_\alpha(x)$ exists at each $x \neq x_0$. Define a complex-valued function g on X by $g(x_0) = 0$ and

$g(x) = \lim f_\alpha(x)$, $x \neq x_0$. Then

$$\phi_x(Tf) = (Tf, \phi_x) = \lim f_\alpha(x) f(x) = g(x) f(x).$$

Thus $T = M_g$.

THEOREM 6.5: Let $E(X)$ be a functional algebra. Then every weakly closed algebra $\mathcal{A}$ of multiplication operators on $E(X)$ is reflexive.

PROOF: Let $T \in \mathcal{L}(E(X))$ with Lat $T \supseteq$ Lat $\mathcal{A}$. For each x in X let ker ϕ_x be the set of all f in $E(X)$ with $f(x) = 0$. Note that $\{ \ker \phi_x : x \in X \} \subseteq$ Lat $\mathcal{A}$. Thus ker $\phi_x \in$ Lat T. Since the codimension of ker ϕ_x in $E(X)$ is 1 we have that ϕ_x is an eigenvector of T^*. By Proposition 6.2 $T = M_g$ for some complex-valued function g on X. Since $1 \in E(X)$ we have $g \in E(X)$.

Let $\tilde{\mathcal{A}} = \{ f : M_f \in \mathcal{A} \}$. Then $\tilde{\mathcal{A}}$ is an invariant subspace for $\mathcal{A}$ which contains 1. The fact that $\tilde{\mathcal{A}}$ is closed follows from Proposition 6.3. Since $T\tilde{\mathcal{A}} \subseteq \tilde{\mathcal{A}}$ we have $g \in \tilde{\mathcal{A}}$. Hence $T = M_g \in \mathcal{A}$.

The following question is open and seems to be difficult:

> QUESTION: Let $E(X)$ be a functional Fréchet space which contains the constant functions and let $\mathcal{M} = \{ f \in E(X) : f$ gives rise to a multiplication operator on $E(X) \}$. Suppose $\mathcal{M}$ is properly contained and dense in $E(X)$. Is every weakly closed subalgebra of $\mathcal{M}$ considered as multiplication operators on $E(X)$ reflexive?

The answer is affirmative for the Hardy spaces (see Chapter

8). The main difficulty in the general case seems to be that there is no easy way to identify the WOT of $\mathcal{M}$ with some 'nicer' topology such as in Proposition 6.3 and Theorem 8.3.

CHAPTER 7

ON FINITE DIRECT SUMS OF $L^p(\mu)$

We shall set up some measure theoretic notations following the terminologies in Rudin [18] before we motivate the contents of this chapter.

NOTATIONS: Let μ be a probability Borel measure on the complex plane $\mathbb{C}$ having compact support. For $1 \leq p < \infty$, $L^p(\mu)$ denotes the Banach space consisting of all complex-valued measurable functions f for which

$$\| f \|_p = \left(\int |f|^p \, d\mu \right)^{1/p} < \infty.$$

The Banach space $L^\infty(\mu)$ consists of all essentially bounded complex-valued measurable functions with the essential supremum norm $\| \ \|_\infty$. Let $\phi \in L^\infty(\mu)$ and $1 \leq p \leq \infty$. Then $M_{\phi,p}$ denotes the multiplication operator on $L^p(\mu)$ defined by

$$M_{\phi,p} g = \phi g \qquad g \in L^p(\mu).$$

It is well known that $M_{\phi,p}$ (sometimes denoted simply by ϕ) is a blt on $L^p(\mu)$. For each positive integer n and $1 \leq p \leq \infty$, $L(p,n)$ denotes the direct sum of n copies of $L^p(\mu)$ and $\mathcal{M}(p,n)$ denotes the subalgebra of $\mathcal{L}(L(p,n))$ whose elements are nxn matrices (T_{ij}) where T_{ij} is a multiplication operator on $L^p(\mu)$. In Hilbert space theory the algebra $\mathcal{M}(2,n)$ is known as a von Neumann algebra of type I_n and its elements are known as n-normal operators.

Some known facts about $M_{\phi,p}$ are:

(i) $\| M_{\phi,p} \| = \| \phi \|_\infty$;

(ii) $M_{\phi,p}$ is invertible $\Longleftrightarrow \frac{1}{\phi} \in L^\infty(\mu)$;

(iii) The spectrum of $M_{\phi,p}$ is the essential range of ϕ and

(iv) If the spectra of $M_{\phi,p}$ and $M_{\gamma,p}$ are disjoint then $M_{\phi-\gamma,p}$ is invertible.

Sarason [20] showed that if $\mathcal{A}$ is a commutative weakly closed algebra of normal operators on a Hilbert space which contains the identity then $\mathcal{A}$ is reflexive. His method used Hilbert space theory (such as the spectral theorem) and does not lend itself to generalization in other spaces. Radjavi and Rosenthal [16] pointed out that every algebra consisting of normal operators on a Hilbert space is commutative. Thus Sarason did not need commutativity as part of his hypotheses.

Recall that any maximal abelian von Neumann algebra of operators on a Hilbert space is representable as an algebra of multiplication operators on $L^2(m)$ for some suitable measure space with measure m (see [21]). Thus it is reasonable to study the reflexivity of multiplication operators on $L^p(\mu)$ where μ is as stated above and $1 \leq p < \infty$. Sarason [19] studied the case where $p = 2$. Our method is a variation of his but it is more transparent and can be used in the investigation of reflexivity of operators in $\mathcal{M}(p,n)$, $1 \leq p < \infty$.

On $L^p(\mu)$, $1 \leq p < \infty$

The next proposition is known. A proof is included for the sake of completeness.

PROPOSITION 7.1: Let T be a blt on $L^p(\mu)$. Then T commutes with $M_{z,p}$ and $M_{\bar{z},p}$ if and only if $T = M_{\phi,p}$ for some $\phi \in L^\infty(\mu)$.

PROOF: Suppose T commutes with $M_{z,p}$ and $M_{\bar{z},p}$. Let $\phi = T1$. Then $\phi \in L^p(\mu)$. We shall show that $\phi \in L^\infty(\mu)$ and $T = M_{\phi,p}$.

Let $\mathcal{P}$ be the set of all polynomials in z and $\bar{z}$. Then from Stone-Weierstrass Theorem and measure theory we know that $\mathcal{P}$ is dense in $L^p(\mu)$.

Let $r \in \mathcal{P}$. Then since T commutes with $M_{z,p}$ and $M_{\bar{z},p}$ we have $Tr = TM_{r,p}1 = M_{r,p}T1 = r\phi$. Let $f \in L^p(\mu)$. Then there exists a sequence $\{p_n\}$ in $\mathcal{P}$ such that $\| p_n - f \|_p \longrightarrow 0$ and $p_n \longrightarrow f$ pointwise a.e. By the continuity of T we have $Tp_n \longrightarrow Tf$. Since $p_n \longrightarrow f$ pointwise a.e. we have $\phi p_n \longrightarrow \phi f$ pointwise a.e. But $Tp_n = \phi p_n$. Thus $Tf = \phi f$.

Finally, suppose $\phi \notin L^\infty(\mu)$. Then for each positive integer n, the set $X_n = \{ x \in \mathbb{C} : |\phi(x)| \geq n \}$ has positive measure. Let χ_{X_n} be the characteristic function of X_n and let $f_n = \left[\frac{1}{\mu(X_n)}\right]^{1/p} \chi_{X_n}$. Then $\| f_n \|_p = 1$ and

$$\| Tf_n \|_p = \| \phi f_n \|_p = \left(\int |\phi f_n|^p \, d\mu\right)^{1/p}$$

$$\geq n\left(\int |f_n|^p \, d\mu\right)^{1/p} = n.$$

This contradicts the boundedness of T.

The converse of this proposition is obviously true.

COROLLARY 7.2: $\mathcal{M}(p,1)$ is a maximal abelian (and hence weakly closed) subalgebra of $\mathcal{L}(L^p(\mu))$.

Recall that $L^\infty(\mu)$ is the dual space of $L^1(\mu)$, i.e., every continuous linear functional Φ on $L^1(\mu)$ is of the form $\Phi(f) = \int \phi f \, d\mu$ for some $\phi \in L^\infty(\mu)$. This gives rise to a weak star (w^*) topology on $L^\infty(\mu)$ which is defined as follows:

Let $\{\phi_\lambda\}$ be a net in $L^\infty(\mu)$. Then

$$\phi_\lambda \xrightarrow{w^*} 0 \iff \int \phi_\lambda f \, d\mu \longrightarrow 0 \quad (f \in L^1(\mu)) \quad \ldots (1)$$

It is important to note that every linear functional F on $L^\infty(\mu)$ that is w^*-continuous has the form $F(\phi) = \int f\phi \, d\mu$ for some $f \in L^1(\mu)$. We may also consider the elements in $L^\infty(\mu)$ as multiplication operators on $L^p(\mu)$. Recall that every continuous linear functional Φ on $L^p(\mu)$ is of the form $\Phi(f) = \int \phi f \, d\mu$ for some $\phi \in L^q(\mu)$, $\frac{1}{p} + \frac{1}{q} = 1$. In this case $L^\infty(\mu)$ has a weak operator topology given by:

Let $\{\phi_\lambda\}$ be a net in $L^\infty(\mu)$. Then

$$\phi_\lambda \xrightarrow{WOT} 0 \iff \int \phi_\lambda fg \, d\mu \longrightarrow 0 \quad (f \in L^p, \, g \in L^q) \quad \ldots (2)$$

It follows from Hölder's inequality that (1) implies (2). To see that (2) implies (1) we note that if $f \in L^1(\mu)$ then $f = f_1 f_2$ with

$$f_1 = |f|^{1/p} \in L^p(\mu) \text{ and } f_2 = |f|^{1/q} \frac{f}{|f|} \in L^q(\mu).$$

This proves the next theorem (the special case when $p = 2$ is in Sarason [19]).

THEOREM 7.3: The w^*-topology of $L^\infty(\mu)$ is the same as the WOT of $L^\infty(\mu)$ considered as multiplication operators on $L^p(\mu)$.

REMARK: It is known that $L^\infty(\mu)$ is the smallest w^*-closed algebra in $L^\infty(\mu)$ containing z and $\bar{z}$.

THEOREM 7.4: Let $\mathcal{A}$ be a WOT closed subalgebra of $\mathcal{M}(p,1)$. Then $\mathcal{A}$ is reflexive.

PROOF: Let T be a blt on $L^p(\mu)$ with Lat T $\supseteq$ Lat $\mathcal{A}$. Then for each Borel measurable set S we have $\chi_S L^p(\mu)$ and $(1 - \chi_S)L^p(\mu)$, where χ_S is the characteristic function on S, are invariant subspaces for T. This implies that T commutes with $M_{\chi_S,p}$ for each Borel measurable set S. Hence T commutes with all multiplication operators induced by simple functions. It follows that T commutes with $M_{z,p}$ and $M_{\bar{z},p}$. By Proposition 7.1, $T = M_{\psi,p}$ for some $\psi \in L^\infty(\mu)$.

Let $\mathcal{B} = \{ \alpha \in L^\infty(\mu) : M_{\alpha,p} \in \mathcal{A} \}$. Suppose $T = M_{\psi,p} \notin \mathcal{A}$. Then, by Theorem 7.3, there exists $\theta \in L^1(\mu)$ such that $\int \psi\theta \, d\mu = 1$ and $\int \alpha\theta \, d\mu = 0$ for each $\alpha \in \mathcal{B}$. Let $\theta_1 = |\theta|^{1/p}$ and $\theta_2 = |\theta|^{1/q}\frac{\theta}{|\theta|}$. Then $\theta_1 \in L^p(\mu)$ and $\theta_2 \in (L^p(\mu))^* = L^q(\mu)$. Let V be the closure of $\mathcal{B}\theta_1 = \mathcal{A}\theta_1$ in $L^p(\mu)$. Then V is an invariant subspace for $\mathcal{A}$. Since $0 = \int \alpha\theta \, d\mu = \int (\alpha\theta_1)\theta_2 \, d\mu$ for each $\alpha \in \mathcal{B}$ we have that θ_2 annihilates V. On the other hand $\theta_1 \in V$ and $\int (\psi\theta_1)\theta_2 \, d\mu = \int \psi\theta \, d\mu = 1$. Thus $T\theta_1 = M_{\psi,p}\theta_1 = \psi\theta_1 \notin V$ and hence V is not invariant under T. This contradicts our assumption that Lat T $\supseteq$ Lat $\mathcal{A}$.

The following question remains unanswered:

QUESTION: Are Theorems 7.3 and 7.4 true for $p = \infty$?

On $L(p,n)$, $1 \leq p < \infty$, n an integer > 1

In the following discussion we shall denote $M_{\phi,p}$ by ϕ; the collection of all diagonal matrices in $\mathcal{M}(p,n)$ by $\mathcal{D}(p,n)$; and the matrices in $\mathcal{D}(p,n)$ by $\text{diag}(\phi_1,\ldots,\phi_n)$. Let S be a Borel measurable set. Then $L(p,n,E)$ denotes the set of all n-tuples in $L(p,n)$ in which each coordinate vanishes on E a.e.

It is an easy consequence of Proposition 7.1 that $\mathcal{M}(p,n)$ consists of all the blt on $L(p,n)$ which commute with $\text{diag}(z,\ldots,z)$ and $\text{diag}(\bar{z},\ldots,\bar{z})$. Thus $\mathcal{M}(p,n)$ is a weakly closed subalgebra of $\mathcal{L}(L(p,n))$.

PROPOSITION 7.5: Let T be a blt on $L(p,n)$ with $\text{Lat } T \supseteq \{ L(p,n,E) : E \text{ is a Borel measurable set} \}$. Then $T \in \mathcal{M}(p,n)$.

PROOF: Express T as an nxn matrix (T_{ij}) where $T_{ij} \in \mathcal{L}(L^p(\mu))$. For each Borel measurable set E since $L(p,n,E)$ and $L(p,n,\mathbb{C} \setminus E)$ are invariant subspaces for T we have $\chi_E L^p(\mu)$ and $(1 - \chi_E)L^p(\mu)$ are invariant subspaces for each T_{ij}. Then, as in the proof of Theorem 7.4, there exists $\phi_{ij} \in L^\infty(\mu)$ such that $T_{ij} = \phi_{ij}$. Hence $T \in \mathcal{M}(p,n)$.

COROLLARY 7.6: Let $\mathcal{A} \subseteq \mathcal{M}(p,n)$ and $T \in \mathcal{L}(L(p,n))$ with $\text{Lat } T \supseteq \text{Lat } \mathcal{A}$. Then $T \in \mathcal{M}(p,n)$. In particular, $\mathcal{M}(p,n)$ is reflexive.

PROOF: It is clear that Lat $\mathcal{A} \supseteq \{ L(p,n,E) : E$ is a Borel measurable set $\}$. The corollary follows immediately from the preceding proposition.

The algebra $\mathcal{M}(p,n)$ has a WOT. It also has a w^*-topology when considered as the dual space of $L(1,n^2)$. More precisely, if $\{ M_\alpha = (\phi_{ij}^\alpha) \}$ is a net in $\mathcal{M}(p,n)$ then

(i) $M_\alpha \xrightarrow{WOT} 0 \iff (M_\alpha(f_1,\ldots,f_n),(g_1,\ldots,g_n)) \longrightarrow 0$

where $f_i \in L^p(\mu)$, $g_i \in L^q(\mu)$ and $\frac{1}{p} + \frac{1}{q} = 1$

and

(ii) $M_\alpha \xrightarrow{w^*} 0 \iff \int \sum_{i,j} \phi_{ij}^\alpha f_{ij}\, d\mu \longrightarrow 0$

where $f_{ij} \in L^1(\mu)$.

It is clear that (ii) implies (i). To see that (i) implies (ii) we only need to note that

$$\int \phi_{ij}^\alpha f_{ij}\, d\mu =$$

$$(M_\alpha(0,\ldots,|f_{ij}|^{1/p},\ldots,0),(0,\ldots,|f_{ij}|^{1/q}\frac{f_{ij}}{|f_{ij}|},\ldots,0))$$

jth ith

Thus we have,

THEOREM 7.7: The WOT on $\mathcal{M}(p,n)$ coincides with its w^*-topology.

THEOREM 7.8: Let $\mathcal{A}$ be a weakly closed subalgebra of $\mathcal{M}(p,n)$. Then $\mathcal{A}$ is n-reflexive (i.e. if $\tilde{B} \in \mathcal{L}(\sum_1^n \oplus L(p,n))$ and Lat $\tilde{B} \supseteq$ Lat $\mathcal{A}(n)$ then $\tilde{B} \in \mathcal{A}(n)$).

PROOF: Let $\tilde{B} \in \mathcal{L}(\sum_1^n \oplus L(p,n))$ with Lat $\tilde{B} \supseteq$ Lat $\mathcal{A}(n)$. Then $\tilde{B} = \sum_1^n \oplus B$ where $B \in \mathcal{L}(L(p,n))$. Thus Lat $B \supseteq$ Lat $\mathcal{A}$. It follows from Corollary 7.6 that $B \in \mathcal{M}(p,n)$. Let $B = (\beta_{ij})$ where $\beta_{ij} \in L^\infty(\mu)$.

Suppose $B \notin \mathcal{A}$. By Theorem 7.7, there exists $\phi_{ij} \in L^1(\mu)$ such that $\sum_{i,j=1}^n \int \phi_{ij}\alpha_{ij}\, d\mu = 0$ for every $(\alpha_{ij}) \in \mathcal{A}$ and $\sum_{i,j=1}^n \int \phi_{ij}\beta_{ij}\, d\mu = 1$. Let $\phi = 1 + \sum_{i,j=1}^n |\phi_{ij}|$. Then $\phi \in L^1(\mu)$. Let δ_{ij} be the kronecker delta. For $i = 1,\ldots,n$ let

$$f_i = (\delta_{1i}\phi^{1/p}, \delta_{2i}\phi^{1/p}, \ldots, \delta_{ni}\phi^{1/p}) \in L(p,n)$$

and

$$g_i = (\phi^{1/q}\frac{\phi_{1i}}{\phi}, \phi^{1/q}\frac{\phi_{2i}}{\phi}, \ldots, \phi^{1/q}\frac{\phi_{ni}}{\phi}) \in L(q,n).$$

Then

$$([A(n)](f_1,\ldots,f_n),(g_1,\ldots,g_n)) = \sum_{i,j=1}^n \int \phi_{ij}\alpha_{ij}\, d\mu = 0$$

for every $A = (\alpha_{ij}) \in \mathcal{A}$ and

$$([B(n)](f_1,\ldots,f_n),(g_1,\ldots,g_n)) = \sum_{i,j=1}^n \int \phi_{ij}\beta_{ij}\, d\mu = 1.$$

This contradicts our assumption that $\mathrm{Lat}\, B(n) = \mathrm{Lat}\, \tilde{B} \supseteq \mathrm{Lat}\, \mathcal{A}(n)$. Thus $B \in \mathcal{A}$ and hence $\tilde{B} \in \mathcal{A}(n)$.

REMARK: A modification of the preceding proof shows that if $\mathcal{A}_i$ is a weakly closed subalgebra of $\mathcal{M}(p,n_i)$ for $i = 1, \ldots, k$, then $\sum_1^k \oplus \mathcal{A}_i$ is $\max\{n_1, \ldots, n_k\}$-reflexive.

THEOREM 7.9: Let $n \geq 2$. Then $\mathcal{M}(p,n)$ has a commutative weakly closed subalgebra which is not reflexive. Also, there exists a measure $\tilde{\mu}$ such that $\mathcal{M}(p,n)$ has a non-reflexive blt.

PROOF: We first consider the case when $n = 2$. Let $\mathcal{A}$ be the set of all 2x2 matrices

$$A(\phi,\phi_1) = \begin{pmatrix} \phi & \phi_1 \\ 0 & \phi \end{pmatrix} \qquad \text{where } \phi,\ \phi_1 \in L^\infty(\mu).$$

Then $\mathcal{A}$ is a commutative weakly closed subalgebra of $\mathcal{M}(p,2)$. Let

$$T_\psi = \begin{pmatrix} \psi & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{where } \psi \in L^\infty(\mu).$$

We shall show that Lat $\mathcal{A} \subseteq$ Lat T_ψ and hence $\mathcal{A}$ is not reflexive.

Let $(f,g) \in L(p,2)$. Let V be the closure of $\mathcal{A}(f,g)$. We must show that $T_\psi(f,g) = (\psi f,0) \in V$.

Let E be the support of g (i.e. $|g| > 0$ a.e. on E and $g = 0$ a.e. off E). Let χ_E be the characteristic function on E. Let $f_1 = f(1 - \chi_E)$ and $f_2 = f\chi_E$. Then $f = f_1 + f_2$ and $f_1,\ f_2 \in L^p(\mu)$. Thus

$$[A(\phi,\phi_1)](f,g) = [A(\phi,\phi_1)](f_1 + f_2,g) = (\phi f_1+\phi f_2+\phi_1 g,\phi g).$$

Let $\phi = 1 - \chi_E$ and $\phi_1 = 0$. Then $(f_1,0) \in V$ and hence $(\phi f_1,0) \in V$ for each $\phi \in L^\infty(\mu)$. In particular, $(\psi f_1,0) \in V$. It remains to show that $(\psi f_2,0) \in V$.

Since $(f_1,0) \in V$ we have $(f_2,g) \in V$. Thus $[A(1,\phi_1)](f_2,g) = (f_2 + \phi_1 g,g) \in V$ for each $\phi_1 \in L^\infty(\mu)$. We shall now show that $gL^\infty(\mu)$ is dense in $L^p(\mu)$ on E. Suppose this is not true. Then there exists $h \in L^q(\mu)$ such that $h \neq 0$ a.e. on E and $\int \gamma gh\, d\mu = 0$ for each $\gamma \in L^\infty(\mu)$. Since $gh \in L^1(\mu)$ we have $gh = 0$ a.e. on E. Hence $h = 0$ a.e. on E. This is a contradiction.

By the definition of f_2 it follows that $(0,g) \in V$. Hence $(f_2,g) - (0,g) = (f_2,0) \in V$. Thus $(\psi f_2,0) \in V$ and we are done.

For the last part of the theorem let $C(X)$ be the Banach space consisting of all complex-valued continuous functions on the compact set $X \subseteq \mathbb{C}$ under the supremum norm. If X is also totally disconnected and does not separate the plane then given $f, g \in C(X)$ there exists a sequence $\{p_j\}$ of polynomials such that p_j and its derivative p_j' converge uniformly on X to f and g, respectively (see [2]).

Define a sequence $\{K_j\}_{j=0}^{\infty}$ of closed subsets of $[0,1]$ as follows:

$$K_0 = [0,1]$$
$$K_1 = [0,1/3] \cup [2/3,1]$$
$$K_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$$
$$\cdots$$

The Cantor set K is defined to be the intersection of the K_j's. Next, define a sequence $\{\mu_j\}_{j=0}^{\infty}$ of probability Borel measure on $[0,1]$ by

μ_0 : Lebesgue measure on K_0

μ_1 : $3/2$ x Lebesgue measure on K_1

μ_2 : $(3/2)^2$ x Lebesgue measure on K_2

$\cdots$

It is clear that each μ_j can be considered as a bounded linear functional on $C([0,1])$ with norm 1. By Alaoglu's Theorem and Riesz Representation Theorem $\{\mu_j\}$ has a subsequence which converges in the w^* topology to a

probability Borel measure $\tilde{\mu}$ on [0,1]. In fact, one can show that $\{\mu_j\}$ is w^* convergent. It is easy to see that $\tilde{\mu}$ is supported on K. Thus $\tilde{\mu}$ can be considered as a probability Borel measure on $\mathbb{C}$ with support K.

Using the previous notations let T = A(z,1). Then for any polynomial p, p(T) = A(p(z),p'(z)). Since K is compact, totally disconnected and does not separate the plane it follows from our previous discussion that Alg(T) = $\mathcal{A}$, where $\mathcal{A}$ is as defined above.

Examples for the cases when $n > 2$ can be constructed from $\mathcal{A}$ via direct sums.

REMARKS: (1) The algebra $\mathcal{A}$ in Theorem 7.9 is quasi-reflexive. To see this let $T \in \mathcal{L}(L(p,2))$ commute with every blt in $\mathcal{A}$ and let Lat $T \supseteq$ Lat $\mathcal{A}$. Then

$$T = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \qquad \alpha, \beta, \gamma \in L^{\infty}(\mu).$$

Clearly, it suffices to show that $\beta = \gamma = 0$ implies $\alpha = 0$. Suppose $\beta = \gamma = 0$. Since T commutes with A(1,1), a simple computation shows that $\alpha = 0$ and hence $T \in \mathcal{A}$.

(2) Let m be the Lebesgue measure on [0,1] and $\tilde{\mu}$ be as defined in the theorem. It is possible to show that $L^p(m)$ is isometrically isomorphic to $L^p(\tilde{\mu})$ where $1 \leq p \leq \infty$. Thus the algebra $\mathcal{A}$ for the measure m is also singly generated, i.e., $\mathcal{A}$ = Alg(T) for some $T \in \mathcal{A}$.

THEOREM 7.10: For $n \geq 4$, $\mathcal{M}(p,n)$ has a commutative weakly closed subalgebra which is not quasi-reflexive.

PROOF: For n = 4 let $\mathcal{A}$ be the set of all 4x4 matrices

$$A(c,\phi,\gamma) = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ \phi & \gamma & c & 0 \\ 0 & \phi & 0 & c \end{pmatrix} \qquad \text{where } \phi, \gamma \in L^{\infty}(\mu) \text{ and } c \in \mathbb{C}.$$

Then $\mathcal{A}$ is a weakly closed commutative subalgebra of $\mathcal{M}(p,4)$. Let

$$T_{\psi} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \psi & 0 & 0 \end{pmatrix} \qquad \text{where } \psi \in L^{\infty}(\mu)$$

It is clear that T_{ψ} is not in $\mathcal{A}$ and commutes with each $A(c,\phi,\gamma)$. The proof for Lat $T_{\psi} \supseteq$ Lat $\mathcal{A}$ is similar to the proof of Theorem 7.9.

The examples for $n > 4$ can be constructed from $\mathcal{A}$ via direct sums.

REMARK: This is related to the Sarason-Rosenthal conjecture mentioned in Chapter 4 (see Remarks 2 and 4 after Proposition 4.2).

Theorem 7.9 shows that not every weakly closed subalgebra of $\mathcal{M}(p,n)$ is reflexive. We shall now exhibit some examples which are reflexive.

PROPOSITION 7.11:

(i) $\mathcal{D}(p,n)$ is a weakly closed subalgebra of $\mathcal{L}(L(p,n))$ and

(ii) every weakly closed subalgebra $\mathcal{A}$ of $\mathcal{D}(p,n)$ is reflexive.

PROOF: (i) It is clear that $\mathcal{D}(p,n)$ is a weakly closed subalgebra of $\mathcal{M}(p,n)$. Since $\mathcal{M}(p,n)$ is weakly closed in $\mathcal{L}(L(p,n))$ we are done.

(ii) Let $T \in \mathcal{L}(L(p,n))$ with Lat $T \supseteq$ Lat $\mathcal{A}$. It follows from Corollary 7.6 and $\mathcal{A} \subseteq \mathcal{D}(p,n)$ that $T \in \mathcal{D}(p,n)$. Let $T = \text{diag}(\beta_1,\ldots,\beta_n)$. If $T \notin \mathcal{A}$ then there exists $\phi_i \in L^1(\mu)$ such that $\sum_1^n \int \phi_i \alpha_i \, d\mu = 0$ for each $\text{diag}(\alpha_1,\ldots,\alpha_n) \in \mathcal{A}$ and $\sum_1^n \int \phi_i \beta_i \, d\mu = 1$. Let

$$f = (|\phi_1|^{1/p},\ldots,|\phi_n|^{1/p})$$

and

$$g = (|\phi_1|^{1/q}\frac{\phi_1}{|\phi_1|},\ldots,|\phi_n|^{1/q}\frac{\phi_n}{|\phi_n|}) \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1.$$

Then $f \in L(p,n)$ and $g \in L(q,n)$. Also, $(Af,g) = \sum_1^n \int \phi_i \alpha_i \, d\mu = 0$ for each $A = \text{diag}(\alpha_1,\ldots,\alpha_n) \in \mathcal{A}$ and $(Tf,g) = \sum_1^n \int \phi_i \beta_i \, d\mu = 1$. This contradicts the assumption that Lat $T \supseteq$ Lat $\mathcal{A}$.

In the next proposition let $\mathcal{S}(p,n)$ be the set of all nxn scalar matrices. It is clear that $\mathcal{S}(p,n)$ is a weakly closed subalgebra of $\mathcal{M}(p,n)$ and hence of $\mathcal{L}(L(p,n))$.

PROPOSITION 7.12: Let $\mathcal{A}$ be a weakly closed subalgebra of $\mathcal{S}(p,n)$. If the measure space $(\mathbb{C},\mu)$ satisfies the condition $\mathbb{C} = \cup_{i=1}^n X_i$ where $X_1, \ldots, X_n$ are pairwise disjoint sets each with positive measure then $\mathcal{A}$ is reflexive.

PROOF: Let $T \in \mathcal{L}(L(p,n))$ with Lat $T \supseteq$ Lat $\mathcal{A}$. By Corollary 7.6, $T \in \mathcal{M}(p,n)$. Since each blt in $\mathcal{A}$ takes an n-tuple of constant functions into another such n-tuple we have $T \in \mathcal{S}(p,n)$.

By Theorem 7.8, every subalgebra of $\mathcal{L}(\mathbb{C}^n)$ is n-reflexive. Consider $\mathcal{A}$ and T as blt on $\mathbb{C}^n$. The

hypothesis about the measure space gives us Lat $\mathcal{A}(n) \subseteq$ Lat T(n) in $\mathbb{C}^n$. Hence T $\in \mathcal{A}$.

REMARK: The hypothesis about the measure space in Proposition 7.12 is necessary. To illustrate this point we consider the space $\mathbb{C}^2$. Let $\mathcal{A}$ be the algebra of all matrices of the form

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \quad \text{where } a, b \in \mathbb{C}$$

considered as blt on $\mathbb{C}^2$. An easy computation shows that Lat $\mathcal{A} = \{ 0, \mathbb{C} \oplus 0, \mathbb{C}^2 \}$. Since $\mathcal{A}$ does not contain every diagonal matrix it fails to be reflexive.

For the next proposition let $T = (\phi_{ij}) \in \mathcal{M}(p,n)$ where each ϕ_{ij} is a simple function. Let $X_1, \ldots, X_m$ be pairwise disjoint measurable sets each with positive measure such that each ϕ_{ij} is a linear combination of the characteristic functions on $X_1, \ldots, X_m$.

PROPOSITION 7.13 Let T be as stated above. If each X_j can be expressed as the union of n pairwise disjoint sets each with positive measure then T is reflexive.

PROOF: Express T as a direct sum of m nxn scalar matrices. Proceed to show that T as a blt on $\mathbb{C}^{mn}$ is n-reflexive. The hypothesis on X_j implies that T is a reflexive blt on L(p,n).

PROPOSITION 7.14:

(i) Let $X \in \mathcal{L}(L^p(\mu))$. Then

$$S = \begin{pmatrix} 1 & X \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 0 \\ X & -1 \end{pmatrix}$$

are their own inverses on $L(p,2)$.

(ii) Let $\alpha, \beta, \gamma \in L^\infty(\mu)$. If $\frac{\beta}{\alpha - \gamma} \in L^\infty(\mu)$ then

$$A = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \alpha & 0 \\ \beta & \gamma \end{pmatrix}$$

are similar to $C = \operatorname{diag}(\alpha,\gamma)$ and hence are reflexive.

PROOF: (i) This is a trivial computation.

(ii) Let $X = \frac{\beta}{\alpha - \gamma}$ in part (i). Then SAS and TBT give us C.

Example of reflexive subalgebras of $\mathcal{M}(p,2)$:

Let $\mathcal{A}$ be a w^* closed subspace of $L^\infty(\mu)$. Let $\mathcal{B}$ be the set of all matrices of the form:

$$S(c,\alpha) = \begin{pmatrix} c & \alpha \\ 0 & c \end{pmatrix} \qquad c \in \mathbb{C},\ \alpha \in \mathcal{A}.$$

Then $\mathcal{B}$ is a weakly closed subalgebra of $\mathcal{L}(L(p,2))$.

Claim: $\mathcal{B}$ is reflexive (if μ is not a point mass).

Proof: Let $T \in \mathcal{L}(L(p,2))$ with Lat $T \supseteq$ Lat $\mathcal{B}$. Then

$$T = \begin{pmatrix} a & \phi \\ 0 & b \end{pmatrix} \qquad \text{where } a, b, \phi \in L^\infty(\mu).$$

It is easy to see that a and b are constant functions. We first show that $a = b$.

Suppose $a \neq b$. Then let

$$S = \begin{pmatrix} 1 & \frac{\phi}{a - b} \\ 0 & -1 \end{pmatrix}$$

Simple computations show that $S^2 = 1$, $\operatorname{diag}(a,b) = STS$ and $\mathcal{B} = S\mathcal{B}S$. Thus Lat $\operatorname{diag}(a,b) \supseteq$ Lat $\mathcal{B}$.

Let $f, g \in L^p(\mu)$. The invariant subspace of diag(a,b) generated by (f,g) is spanned by (f,0) and (0,g). The invariant subspace V of $\mathcal{B}$ generated by (f,g) is the closure of $\{ (cf + \alpha g, cg) : c \in \mathbb{C}, \alpha \in \mathcal{A} \}$. Take f = 1 and g to be the characteristic function on a set E with $0 < \mu(E) < 1$. It is easy to see that $(0,g) \notin V$ which is a contradiction. Thus a = b and we may assume that a = b = 0. We next show that $\phi \in \mathcal{A}$ and hence $T \in \mathcal{B}$.

Suppose $\phi \notin \mathcal{A}$. Then there exists $f \in L^1(\mu)$ which annihilates $\mathcal{A}$ but not ϕ. Let f = gh where $g \in L^p(\mu)$ and $h \in L^q(\mu)$. Then

$$([S(c,\alpha)](0,g),(h,0)) = \int \alpha f \, d\mu = 0$$

and

$$(T(0,g),(h,0)) = \int \phi f \, d\mu = 1.$$

This contradicts our hypothesis that Lat $T \supseteq$ Lat $\mathcal{B}$.

We conclude this chapter by posing the following questions:

QUESTION: If $n \geq 4$ then is every weakly closed subalgebra of $\mathcal{M}(p,n)$ (n-1)-reflexive?

This question is motivated by the fact that every subalgebra of $\mathcal{L}(\mathbb{C}^n)$, $n \geq 4$, is (n-1)-reflexive (see [1]).

QUESTION: Are there any reasonable necessary and sufficient conditions for a weakly closed subalgebra of $\mathcal{M}(p,n)$ to be (quasi-) reflexive?

QUESTION: Is every element in $\mathcal{M}(p,n)$ quasi-reflexive? (refer to Theorem 7.10)

QUESTION: What is the 'situation' in $\mathcal{M}(\infty,n)$?

CHAPTER 8

ON FINITE DIRECT SUMS OF H^p

NOTATIONS: Let $\mathbb{T}$ be the unit circle in the complex plane $\mathbb{C}$ and let θ be the normalized Lebesgue measure on $\mathbb{T}$. For each integer n let χ_n be the function on $\mathbb{T}$ defined by $\chi_n(e^{i\theta}) = e^{in\theta}$. Then for $1 \leq p \leq \infty$ the Hardy space $H^p(\mathbb{T})$ (usually denoted by H^p) is defined by:

$$H^p = \{ f \in L^p(\mathbb{T}) : \int_0^{2\pi} f(e^{i\theta})\chi_n(e^{i\theta})\, d\theta = 0,\ n \geq 1 \}$$

This is equivalent to defining H^p as the set of all analytic functions f on the open unit disc Δ such that the radial limit function $f(e^{i\theta}) = \lim_{r\to 1^-} f(re^{i\theta})$ is in $L^p(\mathbb{T})$ (or L^p).

It is clear that each H^p is a closed subspace of the corresponding Banach space L^p.

For each $\phi \in H^\infty$ and $1 \leq p \leq \infty$ we define the analytic Toeplitz operator $\tilde{M}_{\phi,p}$ to be the blt on H^p given by

$$\tilde{M}_{\phi,p} f = \phi f \qquad f \in H^p.$$

$\tilde{M}_{\phi,p}$ is invertible if and only if $\frac{1}{\phi} \in H^\infty$. Its norm is $\| \phi \|_\infty$ and its spectrum is the closure of $\phi(\Delta)$ in $\mathbb{C}$. For each positive integer n and $1 \leq p \leq \infty$, H(p,n) denotes the direct sum of n copies of H^p and $\tilde{\mathcal{M}}(p,n)$ denotes the subalgebra of $\mathcal{L}(H(p,n))$ whose elements are nxn matrices (T_{ij}) where each T_{ij} is an analytic Toeplitz operator on H^p.

Sarason [20] showed that each analytic Toeplitz operator on H^2 is reflexive by using Hilbert space techniques. We shall use function theoretic methods to

investigate the reflexivity of operators in $\tilde{\mathcal{M}}(p,n)$. Duren [5] and Hoffman [8] serve as standard references for this chapter.

On H^p, $1 \leq p < \infty$

Since H^p is a closed subspace of L^p we may, by the use of Hahn-Banach Theorem, relate the dual space of H^p to the space L^q, $\frac{1}{p} + \frac{1}{q} = 1$. More precisely, each bounded linear functional Φ on H^p is of the form $\Phi(f) = \int \phi f \, d\theta$ for some $\phi \in L^q$. This plays a very important role in our work.

PROPOSITION 8.1: H^p is a functional Banach space (see Chapter 6).

PROOF: We need only show that for each $|\xi| < 1$ the evaluation functional ϕ_ξ on H^p defined by $\phi_\xi f = f(\xi)$ is continuous. To see this define the function $f_\xi \in L^q$ by

$$f_\xi(z) = \frac{1}{1 - \xi\bar{z}} = 1 + \xi\bar{z} + (\xi\bar{z})^2 + \ldots$$

Let $f(z) = \sum a_n z^n \in H^p$. Then $\int f f_\xi \, d\theta = \sum a_n \xi^n = f(\xi) = \phi_\xi f$. The desired result follows from our discussion in the preceding paragraph.

PROPOSITION 8.2: If T is a blt on H^p with Lat $T \supseteq$ Lat $\tilde{M}_{z,p}$ then $T = \tilde{M}_{\phi,p}$ for some $\phi \in H^\infty$.

PROOF: For each $|\xi| < 1$ let $H^p_\xi = \{ f \in H^p : f(\xi) = 0 \}$. Then H^p_ξ is (i) the null space of the evaluation functional ϕ_ξ defined in Proposition 8.1 and (ii) an invariant subspace for $\tilde{M}_{z,p}$ and hence for T. Since the codimension of H^p_ξ in H^p is 1 the annihilator of H^p_ξ is 1-dimensional and hence it is spanned by ϕ_ξ. Since $(T^*\phi_\xi)f = \phi_\xi(Tf) = 0$ for each $f \in H^p_\xi$

we have $T^*\phi_\xi$ annihilates H^p_ξ and hence ϕ_ξ is an eigenvector for T^*. Since $1 \in H^p$ the desired result follows from Propositions 8.1 and 6.2.

H^∞, being a closed subspace of L^∞, has a w*-topology (see Chapter 7). It also has a WOT when its elements are considered as analytic Toeplitz operators on H^p. More precisely, let $\{\phi_\lambda\}$ be a net in H^∞. Then

(i) $\phi_\lambda \xrightarrow{w^*} 0 \iff \int \phi_\lambda f\, d\theta \longrightarrow 0 \qquad (f \in L^1)$

and

(ii) $\phi_\lambda \xrightarrow{WOT} 0 \iff \int \phi_\lambda gh\, d\theta \longrightarrow 0 \qquad (g \in H^p,\ h \in L^q)$

Since $H^pL^q \subseteq L^1$ we have (i) implies (ii). In order to prove that (ii) implies (i) we need the following known result.

> If $f \in L^1$ and $\int \log|f|\, d\theta > -\infty$ then there exists $g \in H^1$ such that $|g| = |f|$ a.e. and g is an outer function.

Let $f \in L^1$. Then $f = f_1 f_2$ where

$$f_1 = \begin{cases} |f| & \text{whenever } |f| \geq 1 \\ 1 & \text{otherwise} \end{cases}$$

and

$$f_2 = \begin{cases} \dfrac{f}{|f|} & \text{whenever } |f| \geq 1 \\ f & \text{otherwise} \end{cases}$$

It is clear that $f_1 \geq 1$, $f_1 \in L^1$, $\log f_1 \in L^1$ and $f_2 \in L^\infty$. By the result stated above there exists $F \in H^1$ such that $|F| = f_1$ a.e. and F is an outer function. In particular, F has no zeros in the open unit disc. Thus we have

$G = F^{1/p} \in H^p$. Let $G_1 = F^{1/q} \frac{|F|}{F} f_2 \in L^q$. Then $f = GG_1$ and hence (ii) implies (i). This proves the next theorem.

THEOREM 8.3: The w*-topology on H^∞ coincides with its WOT.

COROLLARY 8.4: $\tilde{\mathcal{M}}(p,1)$ is the weakly closed subalgebra of $\mathcal{L}(H^p)$ generated by $\tilde{M}_{z,p}$.

REMARK: We have actually shown that $L^1 = H^p L^q$.

THEOREM 8.5: If $\mathcal{A}$ is a WOT closed subalgebra of $\tilde{\mathcal{M}}(p,1)$ then $\mathcal{A}$ is reflexive.

PROOF: Let T be a blt on H^p with Lat T $\supseteq$ Lat $\mathcal{A}$. By Corollary 8.4, Lat $\mathcal{A} \supseteq$ Lat $\tilde{M}_{z,p}$. Hence Lat T $\supseteq$ Lat $\tilde{M}_{z,p}$. It follows from Proposition 8.2 that $T = \tilde{M}_{\phi,p}$ for some $\phi \in H^\infty$. The rest of the proof is similar to that of Theorem 7.4 and makes use of the preceding remark.

REMARK: From Theorem 6.5 we know that Theorem 8.5 holds for $p = \infty$.

On H(p,n), $1 \leq p < \infty$, n an integer > 1

As in Chapter 7 we shall denote $\tilde{M}_{\phi,p}$ by ϕ, the collection of all diagonal matrices in $\tilde{\mathcal{M}}(p,n)$ by $\mathcal{D}(p,n)$, the matrices in $\mathcal{D}(p,n)$ by $\text{diag}(\phi_1,\ldots,\phi_n)$ and the set of all nxn scalar matrices by $\mathcal{S}(p,n)$.

PROPOSITION 8.6: If $T \in \mathcal{L}(H(p,n))$ with Lat T $\supseteq$ { V(n) : V is a subspace of H^p invariant under multiplication by z } then $T \in \tilde{\mathcal{M}}(p,n)$.

PROOF: Express T as (T_{ij}) with $T_{ij} \in \mathcal{L}(H^p)$. Let V be a subspace of H^p with $zV \subseteq V$. Since $TV(n) \subseteq V(n)$ we have

$T(f,0,\ldots,0) = (T_{11}f,\ldots,T_{n1}f)$ is in $V(n)$ for each f in V. Thus $T_{i1}V \subseteq V$. In the same way one can show that $T_{ij}V \subseteq V$. The desired result follows from Proposition 8.2.

COROLLARY 8.7: If $\mathcal{A} \subseteq \tilde{\mathcal{M}}(p,n)$ and $T \in \mathcal{L}(H(p,n))$ with Lat $T \supseteq$ Lat $\mathcal{A}$ then $T \in \tilde{\mathcal{M}}(p,n)$. In particular, $\tilde{\mathcal{M}}(p,n)$ is reflexive and weakly closed.

THEOREM 8.8: The w^*-topology of $\tilde{\mathcal{M}}(p,n)$ coincides with its WOT.

PROOF: Same as Theorem 7.7 except we use $L^1 = H^p L^q$, $\frac{1}{p} + \frac{1}{q} = 1$.

THEOREM 8.9: Every weakly closed subalgebra of $\tilde{\mathcal{M}}(p,n)$ is n-reflexive.

PROOF: Similar to Theorem 7.8.

REMARK: If $\mathcal{A}_i$ is a weakly closed subalgebra of $\tilde{\mathcal{M}}(p,n_i)$ for $i = 1,\ldots,k$, then $\sum_1^k \oplus \mathcal{A}_i$ is $\max\{n_1,\ldots,n_k\}$-reflexive.

PROPOSITION 8.10:

(i) $\tilde{\mathcal{D}}(p,n)$ is a weakly closed subalgebra of $\mathcal{L}(H(p,n))$ and

(ii) every weakly closed subalgebra of $\tilde{\mathcal{D}}(p,n)$ is reflexive.

PROOF: Same as in Proposition 7.11.

PROPOSITION 8.11: $\tilde{\mathcal{J}}(p,n)$ is weakly closed and every weakly closed subalgebra $\mathcal{A}$ of $\tilde{\mathcal{J}}(p,n)$ is reflexive.

PROOF: Let $\mathcal{A}$ be a weakly closed subalgebra of $\tilde{\mathcal{J}}(p,n)$. Let $T \in \mathcal{L}(H(p,n))$ with Lat $\mathcal{A} \subseteq$ Lat T. It is easily shown that $T \in \tilde{\mathcal{J}}(p,n)$.

By Theorem 7.8, every subalgebra of $\mathcal{L}(\mathbb{C}^n)$ is n-reflexive. Thus we need only show that Lat $\mathcal{A}(n) \subseteq$ Lat $T(n)$ in $\mathbb{C}^n$. It suffices to show that if $v_i = (c_{1i},\ldots,c_{ni}) \in \mathbb{C}^n$, $i = 1,\ldots,n$, then $[T(n)](v_1,\ldots,v_n) \in [\mathcal{A}(n)](v_1,\ldots,v_n)$.

Let $x_1, \ldots, x_n$ be n distinct points in the open unit disc. Let $p_1, \ldots, p_n$ be polynomials such that $v_i = (p_1(x_i),\ldots,p_n(x_i))$. Since Lat $\mathcal{A} \subseteq$ Lat T, there exists a sequence $\{T_j\}$ in $\mathcal{A}$ such that

$$T_j(p_1,\ldots,p_n) \text{ -----> } T(p_1,\ldots,p_n) \quad \text{in } H(p,n).$$

This implies that

$$T_j(p_1(x_i),\ldots,p_n(x_i)) \text{ -----> } T(p_1(x_i),\ldots,p_n(x_i))$$

and hence

$$[T_j(n)](v_1,\ldots,v_n) \text{ -----> } [T(n)](v_1,\ldots,v_n)$$

in $\mathbb{C}^{nxn}$. The desired result follows.

PROPOSITION 8.12: Proposition 7.14 remains true when $L^p(\mu)$ is replaced by H^p, $L(p,2)$ is replaced by $H(p,2)$, and $L^\infty(\mu)$ is replaced by H^∞.

Examples of reflexive subalgebras of $\tilde{\mathcal{M}}(p,2)$:

1. Let $\mathcal{A}$ be the collection of all matrices in $\tilde{\mathcal{M}}(p,2)$ of the form

$$S(\phi,\gamma) = \begin{pmatrix} \phi & \gamma \\ 0 & \phi \end{pmatrix} \qquad \text{where } \phi, \gamma \in H^\infty.$$

Then $\mathcal{A}$ is a weakly closed subalgebra of $\tilde{\mathcal{M}}(p,2)$.

Claim: $\mathcal{A}$ is reflexive.

Proof: Let $T \in \mathcal{L}(H(p,2))$ with Lat $T \supseteq$ Lat $\mathcal{A}$. By Corollary 8.7 and the fact that each matrix in $\mathcal{A}$ is upper

triangular we get

$$T = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} \qquad \alpha,\ \beta,\ \delta \in H^\infty.$$

We want to show $T \in \mathcal{A}$. Let

$$T_1 = T - S(\delta,\beta) = \begin{pmatrix} \omega & 0 \\ 0 & 0 \end{pmatrix} \qquad \omega = \alpha - \delta \in H^\infty.$$

It suffices to show $T_1 \in \mathcal{A}$, i.e., $\omega = 0$.

Suppose $\omega \neq 0$. Then let ξ be a point in the open unit disc Δ such that $\omega(\xi) \neq 0$. Then $T_1(1, z - \xi) = (\omega, 0)$ and there exist ϕ_n, $\gamma_n \in H^\infty$ such that

$$[S(\phi_n,\gamma_n)](1, z - \xi)$$
$$= (\phi_n + (z - \xi)\gamma_n, (z - \xi)\phi_n) \text{ -----> } (\omega, 0).$$

This implies that

$$\| (z - \xi)\phi_n \|_p \text{ -----> } 0 \quad \ldots\ldots \quad (1)$$

and

$$\| \phi_n + (z - \xi)\gamma_n - \omega \|_p \text{ -----> } 0 \quad \ldots\ldots \quad (2).$$

Since $\| (z - \xi)\phi_n \|_p \geq \text{dist}(\xi, \mathcal{T}) \| \phi_n \|_p$ we get from (1) that $\| \phi_n \|_p$ -----> 0. Thus (2) becomes $\| (z - \xi)\gamma_n - \omega \|_p$ -----> 0. Hence $(z - \xi)\gamma_n$ ----> ω pointwise everywhere on Δ. This is impossible since each $(z - \xi)\gamma_n$ vanishes at ξ and $\omega(\xi) \neq 0$.

We should recall that this algebra with $H(p,2)$ replaced by $L(p,2)$ is not reflexive (see Theorem 7.9).

2. Let $\mathcal{B}$ be a w^* closed subspace of H^∞ and let $\mathcal{A}$ be the set of all matrices of the form $S(c,\alpha)$ (using $S(c,\alpha)$ as defined in Example 1) where $\alpha \in \mathcal{B}$, $c \in \mathcal{C}$. Then $\mathcal{A}$ is a weakly closed subalgebra of $\tilde{\mathcal{M}}(p,2)$.

Claim: $\mathcal{A}$ is reflexive.

Proof: Let $T \in \mathcal{L}(H(p,2))$ with Lat $T \supseteq$ Lat $\mathcal{A}$. By the preceding example we have $T = S(\beta,\gamma)$ where $\beta, \gamma \in H^\infty$. It is easy to see that β is a constant function. Thus it suffices to show that $T_1 = (0,\gamma)$ is in $\mathcal{A}$. Now proceed as in the corresponding example of Chapter 7.

3. Given $f, g \in H^\infty$. Let $\mathcal{A}(f,g)$ be the set of all matrices of the form $T(\psi) = S(\psi,\psi'\frac{g}{f'})$ where $\psi \in \text{Alg}(f)$ and $\psi'\frac{g}{f'} \in H^\infty$ ($S(\phi,\gamma)$ is defined in Example 1). It is easily seen that $\mathcal{A}(f,g)$ is an algebra.

Claim: $\mathcal{A}(f,g)$ is weakly closed.

Proof: Let $\{T(\psi_\alpha)\}$ be a net in $\mathcal{A}(f,g)$ such that $T(\psi_\alpha) \xrightarrow{WOT} T$. We want to show that $T \in \mathcal{A}(f,g)$. Since $\tilde{\mathcal{M}}(p,2)$ is weakly closed we have $T = S(\psi,\psi_1)$ where $\psi_\alpha \xrightarrow{w^*} \psi$ and $\psi_\alpha'\frac{g}{f'} \xrightarrow{w^*} \psi_1$. Thus $\psi \in \text{Alg}(f)$ and hence it suffices to show $\psi_1 = \psi'\frac{g}{f'}$. We need the following result in Rubel and Shields [17]:

> If $\{f_\alpha\}$ is a net in H^∞ and Δ is the open unit disc then
>
> $$f_\alpha \xrightarrow{w^*} f \iff \int f_\alpha \, d\mu \longrightarrow \int f \, d\mu$$
>
> for each finite Borel measure μ in Δ.

Choose $z_0 \in \Delta$ such that $z_0 \neq 0$ and $g(z_0) \neq 0$. Choose $a \in \Delta$ such that the segment $[a,z_0]$ contains no zeros of g. Then $\frac{f'}{g}\,dz$ is a finite Borel measure on $[a,z_0]$ and hence

$$\psi(z_0) - \psi(a) = \lim (\psi_\alpha(z_0) - \psi_\alpha(a)) = \lim \int_a^{z_0} \psi_\alpha' \, dz$$

$$= \lim \int_a^{z_0} (\psi_\alpha' \tfrac{g}{f'}) \tfrac{f'}{g} \, dz = \int_a^{z_0} \psi_1 \tfrac{f'}{g} \, dz.$$

Thus for fixed a the equation

$$\psi(z) - \psi(a) = \int_a^z \psi_1 \tfrac{f'}{g} \, dz$$

holds on an open set. This implies that $\psi' = \psi_1 \frac{f'}{g}$ or $\psi_1 = \psi' \frac{g}{f'}$. Therefore $T \in \mathcal{A}(f,g)$.

Claim: $\mathcal{A}(f,g)$ is reflexive.

Proof: Let $B \in \mathcal{L}(H(p,2))$ with Lat $B \supseteq$ Lat $\mathcal{A}(f,g)$. By Example 1 there exist ϕ, $\phi_1 \in H^\infty$ such that $B = S(\phi,\phi_1)$. Let $(h_1,h_2) \in H(p,2)$ and $\mathcal{M}$ be the invariant subspace for $\mathcal{A}(f,g)$ generated by (h_1,h_2). Then $\mathcal{M}$ is the closure of $\{ (\psi h_1 + \psi' \frac{g}{f'} h_2, \psi h_2) : \psi \in \text{Alg}(f) \text{ and } \psi' \frac{g}{f'} \in H^\infty \}$. Since $B(h_1,h_2) = (\phi h_1 + \phi_1 h_2, \phi h_2)$ and Lat $B \supseteq$ Lat $\mathcal{A}(f,g)$ there exists a sequence $\{\psi_j\}$ in Alg(f) such that

$$\psi_j' \tfrac{g}{f'} \in H^\infty \qquad \ldots (1)$$

$$\psi_j h_2 \text{ -----> } \phi h_2 \quad \text{in } H^p \qquad \ldots (2)$$

and

$$\psi_j h_1 + \psi_j' \tfrac{g}{f'} h_2 \text{ -----> } \phi h_1 + \phi_1 h_2 \quad \text{in } H^p \ldots (3)$$

From (2) $\| (\psi_j - \phi) h_2 \|_p \leq M$. It is known that if $h \in H^p$ then $|h(z)| \leq \frac{\| h \|_p}{(1 - |z|^2)^{1/p}}$. Fix $\varepsilon > 0$. Then $|(\psi_j(z) - \phi(z)) h_2(z)| \leq \frac{M}{\varepsilon}$ on $|z|^2 \leq 1 - \varepsilon^p$ and hence $(\psi_j - \phi) h_2$ is uniformly bounded on each compact subset of Δ.

Let $G = \{ z \in \Delta : h_2(z) \neq 0 \}$ and $G_\varepsilon = \{ z \in G : \text{dist}(z, \partial G) \geq \varepsilon \}$. Then G_ε is a compact set

and hence $|h_2| \geq \delta(\varepsilon) > 0$ on G_ε. Thus $|\psi_j - \phi| \leq M_\varepsilon$ on G_ε. Since ψ_j converges pointwise to ϕ on G, this implies that $\{\psi_j\}$ converges uniformly to ϕ on compact subsets of G. Hence $\psi_j' \dashrightarrow \phi'$ uniformly on compact subsets on G. It follows that $\psi_j h_1 + \psi_j' \frac{g}{f'} h_2 \dashrightarrow \phi h_1 + \phi' \frac{g}{f'} h_2$ uniformly on compact subsets of G. By (3) $\phi h_1 + \phi_1 h_2 = \phi h_1 + \phi' \frac{g}{f'} h_2$ on G. Therefore $\phi_1 = \phi' \frac{g}{f'}$ on G and hence on Δ.

Finally, restrict $\mathcal{A}(f,g)$ and B to $H^p \oplus 0$. By using the result that any weakly closed subalgebra of multiplication operators on H^p is reflexive we know $\phi \in \mathrm{Alg}(f)$ and hence $B \in \mathcal{A}(f,g)$.

This example poses the following question:

QUESTION: Is it true that the weakly closed algebra generated by S(f,g) is $\mathcal{A}(f,g)$ for any $f, g \in H^\infty$?

Here S(f,g) is defined as in Example 1. The answer is affirmative for the special case where $f(z) = z$ and $g(z) = 1$.

Example of reflexive subalgebras of $\tilde{\mathcal{M}}(p,4)$:

Let $\mathcal{A}$ be the set of all matrices of the form:

$$A(c,\phi,\gamma) = \begin{pmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ \phi & \gamma & c & 0 \\ 0 & \phi & 0 & c \end{pmatrix} \quad \text{where } c \in \mathbb{C} \text{ and } \phi, \gamma \in H^\infty.$$

Then $\mathcal{A}$ is a weakly closed subalgebra of $\tilde{\mathcal{M}}(p,4)$.

Claim: $\mathcal{A}$ is reflexive.

Proof: Let $T \in \mathcal{L}(H(p,4))$ with Lat $T \supseteq$ Lat $\mathcal{A}$. It is easy to show that T is of the form

$$\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ \alpha & \beta & a & 0 \\ 0 & \omega & 0 & a \end{pmatrix}$$ where $a \in \mathbb{C}$, and $\alpha, \beta, \omega \in H^\infty$.

Thus we only need to show $T - A(a,\omega,\beta) = 0$, i.e., $\alpha - \omega = 0$. The proof for this is similar to the example after Proposition 8.12. If H is replaced by L in this example then it is not quasi-reflexive (see Theorem 7.10).

This chapter leaves two questions unanswered.

QUESTION: Is every weakly closed subalgebra of $\tilde{\mathcal{M}}(p,n)$ reflexive (or even quasi-reflexive)? If not, are there any reasonable necessary and sufficient conditions for a weakly closed subalgebra of $\tilde{\mathcal{M}}(p,n)$ to be (quasi-) reflexive?

QUESTION: If $n \geq 4$ then is every weakly closed subalgebra of $\tilde{\mathcal{M}}(p,n)$ (n-1)-reflexive?

In view of the examples of reflexive subalgebras of $\tilde{\mathcal{M}}(p,2)$, these questions are probably quite difficult.

REFERENCES

1. E. A. Azoff, K-reflexivity in Finite-dimensional Spaces, Duke Math. J., to appear.

2. E. Bishop, Approximation by a Polynomial and its Derivatives on Certain Closed Sets, PAMS, 9(1958), 946-953.

3. L. Brickman and P. A. Fillmore, The Invariant Subspace Lattice of a Linear Transformation, Can. J. Math., 19(1967), 810-822.

4. J. A. Deddens and P. A. Fillmore, Reflexive Linear Transformations, J. Lin. Alg., to appear.

5. P. L. Duren, Theory of H^p Spaces, Academic Press, New York, 1970.

6. P. L. Duren, B. W. Romberg and A. L. Shields, Linear Functionals on H^p Spaces with $0 < p < 1$, J. Reine Angew. Math., 238(1969), 32-60.

7. P. R. Halmos, A Hilbert Space Problem Book, van Nostrand, Princeton, 1967.

8. K. Hoffman, Banach Spaces of Analytic Functions, Prentice Hall, Englewood Cliffs, 1962.

9. N. Jacobson, Lectures in Abstract Algebra Vol. II, van Nostrand, Princeton, 1953.

10. B. E. Johnson and A. L. Shields, Hyperinvariant Subspaces for Operators on the Space of Complex Sequences, Mich. Math. J., 19(1972), 189-191.

11. I. Kaplansky, Infinite Abelian Groups, The University of Michigan Press, Ann Arbor, 1969.

12. G. Köthe, Topological Vector Spaces I, Springer-Verlag, New York, 1969.

13. K.-H. Körber, Das Spektrum zeilenfiniter Matrizen, Math. Ann., 181(1969), 8-34.

14. K.-H. Körber, Die invarianten Teilräume der stetigen Endomorphismen von ω, Math. Ann., 182(1969), 95-103.

15. H. Radjavi and P. Rosenthal, Invariant Subspaces, Springer-Verlag, New York, 1973.

16. H. Radjavi and P. Rosenthal, On Invariant Subspaces and Reflexive Algebras, Amer. J. Math. 91(1969), 683-692.

17. L. A. Rubel and A. L. Shields, The Space of Bounded Analytic Functions on a Region, Ann. Inst. Fourier, Grenoble 16, 1(1966), 235-277.

18. W. Rudin, Real and Complex Analysis, McGraw-Hill, New York, 1966.

19. D. Sarason, Invariant Subspaces, Topics in Operator Theory, (Edited by C. Pearcy), American Mathematical Society, 1974.

20. D. Sarason, Invariant Subspaces and Unstarred Operator Algebras, Pacific J. Math. 17(1966), 511-517.

21. I. E. Segal, Decompositions of Operator Algebras, I, II. Memoirs Amer. Math. Soc. no. 9, 1951.

22. A. L. Shields, Weighted Shift Operators and Analytic Function Theory, Topics in Operator Theory, (Edited by C. Pearcy), American Mathematical Society, 1974.

23. G. D. Taylor, Contributions to the Theory of Multipliers, Univ. of Michigan Ph. D. Thesis, 1965.