Geometrization in Algebraic and Arithmetic Geometry

by

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Моим родителям

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ABSTRACT

In the first part, we reconstruct the Cartier transform of Ogus and Vologodsky using the de Rham stack. Along the way, we observe that a weaker condition suffices for their equivalence: namely, to obtain the global Cartier transform for a family X/S, it is (necessary and) sufficient to have a lift of X to $W_2(S)$. We also obtain a decompleted version of the global Cartier transform. Finally, we apply it to obtain corollaries in the G-equivariant and logarithmic settings.

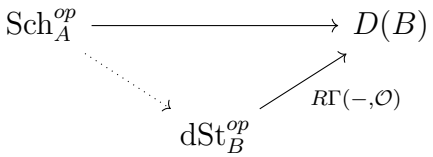
In the second part, we study the stack $W(k)^{syn}$ for a perfect field k of characteristic p. Namely, we prove that the subcategory $\mathrm{Perf}_{qc,[0,p-2]}(W(k)^{syn}) \hookrightarrow \mathrm{Perf}_{qc}(W(k)^{syn})$ of objects with Hodge–Tate weights in $[0, p-2]$ is equivalent to the derived category $\mathcal{D}^b(\mathcal{MF}_{[0,p-2]}(W(k)))$ of Fontaine–Laffaille modules with similar weight constraints.

CHAPTER I
Introduction

Suppose we have a cohomology theory which associates to every scheme X over A a complex $R\Gamma_?(X)$ of B-modules, i.e., we are given a functor

$$\mathrm{Sch}_A^{\mathrm{op}} \to \mathcal{D}(B).$$

The idea of geometrization amounts to constructing an intermediate functor $(-)^? : \mathrm{Sch}_A \to \mathrm{dSt}_B$ such that the triangle



commutes. In other words, it realizes the desired cohomology theory as quasi-coherent cohomology of an auxiliary geometric object. In particular, the category $\mathrm{Perf}(X^?)$ provides a notion of *coefficients* for the cohomology theory of interest. This idea goes back to Simpson. In [Sim97], he introduced, for any scheme X smooth over $\mathbb{Q}$, a certain stack denoted by X^{dR}. The defining property of this stack is that its quasi-coherent theory recovers the de Rham theory of X. Later, Bhatt–Lurie and Drinfeld constructed analogous stacks for de Rham cohomology in characteristic p, prismatic cohomology, and syntomic cohomology. Namely, for any p-adic formal scheme X, they attach certain stacks $X^{\mathbf{\Delta}}$, $X^{\mathcal{N}}$, and X^{syn}. This thesis is devoted both to studying the stacks that arise from this geometrization procedure and to exploiting the geometry they govern. The first part of the thesis is devoted to studying the Cartier transform functor introduced by Ogus and Vologodsky using the de Rham gerbe. The second part is concerned with understanding the stack $W(k)^{syn}$, where k is a perfect field of characteristic p, in more classical terms. Namely, we identify a certain subcategory of $\mathrm{Perf}_{qc}(W(k)^{syn})$ with the derived category of Fontaine–Laffaille modules. The latter part is based on joint work with Vadim Vologodsky

and Yujie Xu. We now discuss both parts in more detail.

I.0.1: Cartier Transform

For a compact Kähler manifold X, the non-abelian Hodge correspondence provides an equivalence between suitable categories of D-modules and Higgs modules. Despite being highly-transcendental, it admits a characteristic p counterpart established by Ogus and Vologodsky in their celebrated paper [OV07]. Let $X \to S$ be a smooth family of schemes over $\mathbf{F}_p$. One of the main tools in analyzing modules with integrable connection in characteristic p is the Azumaya property of the algebra of differential operators. Namely, there exists a map $\psi : ST_{X'/S} \to F_{X/S,*}D_{X/S}$ exhibiting the source as the center of the target. It is defined by $\psi(v) = \partial_v^p - \partial_{v^{[p]}}$ where $v^{[p]}$ is a vector field obtained by noticing that the p-fold composition of a derivation is again a derivation.

(1) Denote by $\mathrm{MIC}(X/S)$ the category of $F_{X/S,*}D_{X/S}$-modules. Then $\mathrm{MIC}_{\leq n}(X/S)$ is a full subcategory such that the action of $ST_{X'/S}$ factors through $S^{\leq n}T_{X'/S}$. Also, denote $\mathrm{MIC}^{\cdot}(X/S)$ to be the full subcategory of nilpotent D-modules, i.e., (M, ∇) such that every $m \in M$ is killed by $S^{\geq m}T_{X'/S}$ for some m.

(2) Denote $\mathrm{HIG}(X'/S) \simeq \mathrm{QCoh}(\mathbf{T}^*_{X'/S})$ to be the category of Higgs modules on X'/S. Then $\mathrm{HIG}_{\leq n}(X'/S)$ will stand for the full subcategory of those Higgs modules where the action of $ST_{X'/S}$ factors through $S^{\leq n}T_{X'/S}$, i.e., Higgs modules supported on m-th neighbourhood of the zero section. Similarly, $\mathrm{HIG}^{\cdot}(X'/S) = \mathrm{QCoh}(\widehat{\mathbf{T}^*_{X'/S}})$ are nilpotent Higgs modules.

One of the main results they prove, which is referred to as the *local Cartier equivalence*, is the following.

Theorem I.1. *([OV07, Theorem 2.11]) Let $X \to S$ be a smooth morphism. Any lift of the relative Frobenius $X \to X'$ to a flat lift $\tilde{S} \to \mathbf{Z}/p^2$ induces a symmetric monoidal equivalence* $\mathrm{HIG}^{\cdot}(X'/S) \simeq \mathrm{MIC}^{\cdot}(X/S)$.

Even when $S = \mathrm{Spec}(\mathbf{F}_p)$, having a lift of the relative Frobenius is extremely restrictive. To formulate a more flexible result that they prove, let us also denote

$$\mathrm{HIG}^{\cdot}_{\gamma}(X'/S) := \mathrm{QCoh}(\widehat{\mathbf{T}^{*,\gamma}_{X'/S}})$$

and $\mathrm{MIC}^{\cdot}_{\gamma}(X/S)$ to be the category of locally nilpotent modules over

$$F_{X/S,*}D_{X/S} \otimes_{ST_{X'/S}} \hat{\Gamma}T_{X'/S}.$$

We will refer to the following as *the global Cartier equivalence.*

Theorem I.2. *([OV07, Theorem 2.8]) Let $X \to S$ be a smooth morphism. Any flat lift of $X' \to S$ to $\mathbf{Z}/p^2$ induces a symmetric monoidal equivalence $\mathrm{MIC}^{\cdot}_{\gamma}(X/S) \simeq \mathrm{HIG}^{\cdot}_{\gamma}(X'/S)$. In particular, one obtains an equivalence $\mathrm{MIC}_{\leq p-1}(X/S) \simeq \mathrm{HIG}_{\leq p-1}(X'/S)$.*

Their proof is based on exploiting the Azumaya property of the algebra of differential operators $F_{X/S,*}D_{X/S}$. That is, they use the given data to produce a splitting module for $F_{X/S,*}D_{X/S}$ over an appropriate neighbourhood of the zero section in $\mathbf{T}^*_{X'/S}$. Namely, they observe that a lift $\tilde{X}'/\tilde{S}$ gives rise to a $F^*_{X/S}T_{X'/S}$-torsor $\mathcal{L}_{\mathcal{X}}$ on X with a flat connection[1] and, moreover, the dual of the ring of functions is a splitting module for $F_{X/S,*}D_{X/S}$ over $\widehat{\mathbf{T}}^{*,\gamma}_{X'/S}$, the completed PD-envelope of the zero section in $\mathbf{T}^*_{X'/S}$. Any lift of $F_{X/S}$ gives a section $X \to \mathcal{L}_{\mathcal{X}}$ and they observe that its PD-envelope is preserved by the connection and (the dual of) the functions on this PD-envelope define a splitting module for $F_{X/S,*}D_{X/S}$ over $\widehat{\mathbf{T}}_{X'/S}$.

Our approach can be seen as Koszul-dual. Instead of studying the algebra of differential operators $F_{X/S,*}D_{X/S}$, we focus on studying the de Rham complex $F_{X/S,*}\,\mathrm{dR}_{X/S} \in \mathcal{D}_{qc}(X')$. One advantage of this approach is that the de Rham complex gives rise to a geometric object, called the *de Rham stack*, and denoted by $(X/S)^{dR}$. One of the key structures possessed by this stack is a map $\nu_{X/S} : (X/S)^{dR} \to X'$ which, when X/S is smooth, exhibits the source as a $T^{\sharp}_{X'/S}$-gerbe over the target and which can be viewed as an avatar of the Azumaya property of $F_{X/S,*}D_{X/S}$. Using the map $T^{\sharp}_{X'/S} \to T_{X'/S}$, we construct a $T_{X'/S}$-gerbe over X' which we denote by $(X/S)^{dR,\gamma}$. One of the main results is the following:

Theorem I.3. *Let X/S be a representable morphism of algebraic stacks over $\mathbf{F}_p$. The gerbe of splittings of $\nu^{\gamma}_{X/S} : (X/S)^{dR,\gamma} \to X'$ is identified with the gerbe of liftings of X/S to $W_2(S)$.*

In particular, any flat lift of X/S to $W_2(S)$ gives rise to a global Cartier equivalence.

Corollary I.4. *Any flat lift $\tilde{X}/W_2(S)$ gives rise to an equivalence $(X/S)^{dR,\gamma} \simeq B_{X'}T_{X'/S}$ of X'-stacks. One can identify quasi-coherent sheaves over $(X/S)^{dR,\gamma}$ with locally nilpotent modules over $F_{X/S,*}D_{X/S} \otimes_{ST_{X'/S}} \hat{\Gamma}T_{X'/S}$ and Cartier duality identifies $\mathrm{QCoh}(B_{X'}T_{X'/S}) \simeq \mathrm{QCoh}(\widehat{\mathbf{T}^{*,\gamma}_{X'/S}})$ implying $\mathrm{MIC}^{\cdot}_{\gamma}(X/S) \simeq \mathrm{HIG}^{\cdot}_{\gamma}(X'/S)$.*

Moreover, any flat lift $\tilde{S} \to \mathbf{Z}/p^2$ identifies the gerbe of flat lifts of X/S to $W_2(S)$ with the gerbe of flat lifts of X'/S to $\tilde{S}$. In particular, it makes sense to compare the equivalence from Theorem I.4 with the global Cartier equivalence Theorem I.2 in the presence of $\tilde{S}$.

[1] Note $F^*_{X/S}T_{X'/S}$ has a canonical connection with the vanishing p-curvature.

In Theorem II.103 we prove that they are equivalent. By construction, $(X/S)^{dR,\gamma}$ comes equipped with a map $(X/S)^{dR} \to (X/S)^{dR,\gamma}$. By Theorem I.3, any lift $\tilde{X}/W_2(S)$ defines a splitting $(X/S)^{dR,\gamma} \simeq B_{X'}T_{X'/S}$, thus we get a map $(X/S)^{dR} \to (X/S)^{dR,\gamma} \simeq B_{X'}T_{X'/S}$. This defines a $F^*_{X/S}T_{X'/S}$-torsor on $(X/S)^{dR}$ which, by descent along $X \to (X/S)^{dR}$, can be viewed as a $F^*_{X/S}T_{X'/S}$-torsor with a flat connection on X. We identify the underlying torsor on X with the torsor of strong lifts of $F_{X/S}$ to $\tilde{X}$, see Theorem II.50. Moreover, we prove that any flat lift $\tilde{S}$ of S to $\mathbf{Z}/p^2$ identifies this torsor on $(X/S)^{dR}$ with the torsor constructed by Ogus and Vologodsky for $\tilde{X}' := \tilde{X} \times_{W_2(S)} \tilde{S}$, where $\tilde{S} \to W_2(S)$ is the canonical map.

I.0.2: Local Cartier transform

Unfortunately, we do not have a statement similar to Theorem I.3 in the case of $\nu_{X/S} : (X/S)^{dR} \to X'$. One source of splittings of $\nu_{X/S}$ comes from the fact that $\nu_{X/S}$ is an affine map. Namely, it was observed by Bhatt that lifts of X and S to $\mathbf{Z}/p^2$ with a compatible lift of Frobenii give rise to an isomorphism

$$F_{X/S,*}\,\mathrm{dR}_{X/S} \simeq (\bigoplus_i \Omega^i_{X'/S}[-i], 0)$$

of derived $\mathcal{O}_{X'}$-algebras. In particular, this data defines a splitting of the $T^\sharp_{X'/S}$-gerbe $\nu_{X/S} : (X/S)^{dR} \to X'$. In the absence of a lift of S, and assuming only a lift of X to $W_2(S)$ with a strong Frobenius lift, we adapt Bhatt's argument to prove the following.

Theorem I.5. *Given a flat lift $\tilde{X}/W_2(S)$ of X/S and a map $f : W_2(X) \to \tilde{X}$, which reduces to the identity map on X, via the natural embedding $X \hookrightarrow W_2(X)$ and $X \hookrightarrow \tilde{X}$, we obtain a splitting of $\nu_{X/S} : (X/S)^{dR} \to X'$.*

Another way to construct the splitting of $\nu_{X/S}$ is based on the following observation. Let E be a locally free sheaf on $(X/S)^{dR}$ and $f : (X/S)^{dR} \to B(E)$ be a map classifying an E-torsor on $(X/S)^{dR}$. Denote $\bar{f} : X \xrightarrow{\pi_{X/S}} (X/S)^{dR} \xrightarrow{f} B(E)$ to be the underlying E-torsor $Y \to X$ on X. Applying $(-/S)^{dR}$ gives $(Y/S)^{dR} \to (X/S)^{dR}$ an $(\mathbf{V}(E)/S)^{dR}$-torsor and we claim it is classified by the composition

$$(X/S)^{dR} \to B(E) \to B(E/E^\sharp) = B((\mathbf{V}(E)/S)^{dR}).$$

Therefore, any trivialization of $\bar{f} : X \to B(E)$ gives rise to a map

$$(X/S)^{dR} \to \mathrm{fib}(B(E) \to B((\mathbf{V}(E)/S)^{dR})) \simeq B(E^\sharp)$$

and we describe this $E^\sharp$-torsor on X with its flat connection explicitly. Namely, if $s : X \to Y$ is a section provided by the trivialization of $\bar{f}$, then its PD-envelope $D_X(Y)$ has a natural connection compatible with the connection on Y. This is motivated by [OV07, Remark 2.4].
Now, a lift $\tilde{X}/W_2(S)$ gives rise to

$$(X/S)^{dR} \to (X/S)^{dR,\gamma} \simeq B_{X'}(T_{X'/S}),$$

a $F^*_{X/S}T_{X'/S}$-torsor on X equipped with a flat connection. We observe that this torsor can be split in the presence of $f : W_2(X) \to \tilde{X}$ which reduces to the identity map on X. Applying the observation above, we obtain a map $(X/S)^{dR} \to B_{X'}(T^\sharp_{X'/S})$ which we check to be an isomorphism of $T^\sharp_{X'/S}$-gerbes.

Corollary I.6. *Let X/S be a smooth morphism. Let $\tilde{X}$ be a lift to $W_2(S)$ equipped with a strong Frobenius lift. One obtains a symmetric monoidal equivalence $C_f : \mathrm{MIC}^\cdot(X/S) \simeq \mathrm{HIG}^\cdot(X'/S)$. If, moreover, we are given a flat lift $\tilde{S} \to \mathbf{Z}/p^2$ of S and a lift $\tilde{X}_S$ of X to $\tilde{S}$, then f gives rise to a map $\tilde{F}_f : \tilde{X}_S \to \tilde{X}' := \tilde{X} \times_{W_2(S)} \tilde{S}$ lifting the relative Frobenius $F_{X/S} : X \to X'$. Then C_f is equivalent to the local Cartier transform of Ogus-Vologodsky which is a symmetric monoidal equivalence $C_{\tilde{F}_f} : \mathrm{MIC}^\cdot(X/S) \simeq \mathrm{HIG}^\cdot(X'/S)$ constructed from $\tilde{F}_f : \tilde{X}_S \to \tilde{X}'$, see [OV07, Theorem 2.11].*

We also verify that the two splittings of the de Rham gerbe constructed from a strong Frobenius lift agree.

As a corollary, we obtain the local Cartier equivalence for all toric fibrations $X \to S$ since they satisfy the conditions from Theorem I.5.

I.0.3: Powers of Frobenii

For a smooth family $X \to S$ equipped with a lift $\tilde{X}/W_2(S)$, the global Cartier equivalence yields an equivalence between certain categories of 'complete' modules. In order to get a decompleted version of this equivalence, we study the $\hat{T}_{X'/S}$-gerbe over X' obtained from the $T^\sharp_{X'/S}$-gerbe $(X/S)^{dR} \to X'$ by pushing out along $T^\sharp_{X'/S} \to \hat{T}_{X'/S}$, we denote it by $\widehat{\nu}_{X/S} : \widehat{(X/S)^{dR}} \to X'$. In pursuit of this, we prove the following.

Theorem I.7. *Let X/S be a smooth family of schemes over $\mathbf{F}_p$.*

- *Given a flat lift $\tilde{X}/W_2(S)$ of X/S together with a lift of some power of $F_{X/S}$, we obtain a splitting of $\widehat{\nu}_{X/S} : \widehat{(X/S)^{dR}} \to X'$.*

- $\mathcal{D}_{qc}(\widehat{(X/S)^{dR}})$ *is identified with the derived category of modules over the algebra* $D^{\gamma}_{X/S} := F_{X/S,*}D_{X/S} \otimes_{ST_{X'/S}} \Gamma T_{X'/S}$.

The first part of the theorem is a corollary of a refined statement. Namely, by pushing out the de Rham stack along $T^{\sharp}_{X'/S} \to T_{X'/S} \otimes_{\mathbf{G}_a} \alpha_{p^n}$ we get an fppf-torsor for $T_{X'/S} \otimes_{\mathbf{G}_a} \alpha_{p^n}$ which we denote by $(X/S)^{dR,n} \to X'$. This gerbe is equivalent to the gerbe of lifts $F^n_{X/S}$ to $W_2(S)$. We denote the category of quasi-coherent sheaves on a trivial $\hat{T}_{X'/S}$-gerbe by $\mathrm{HIG}_{\gamma}(X'/S)$. Via the Cartier duality it is identified with $\mathrm{QCoh}(\mathbf{T}^{*,\gamma}_{X'/S})$ equipped with the convolution monoidal structure. Combining both parts in Theorem I.7 gives the following.

Corollary I.8. *For smooth X/S with a flat lift $\tilde{X}/W_2(S)$ together with a lift of $F^n_{X/S}$, one obtains a symmetric monoidal equivalence* $\mathrm{MIC}_{\gamma}(X/S) \simeq \mathrm{HIG}_{\gamma}(X'/S)$.

I.0.4: Applications.

The advantage of allowing families $X \to S$ whose inputs are (reasonable) stacks is that one can leverage the torsor property to obtain useful results about schemes. The two cases we consider are the following.

(1) If (X, D) is an snc pair with smooth $X/\mathbf{F}_p$, one obtains $f : X \to (\mathbf{A}^1/\mathbf{G}_m)^n =: S_n$ classifying D. Contemplating various versions of the de Rham stacks, one obtains versions of non-abelian Hodge correspondence related to the logarithmic geometry of (X, D).

(2) If G is an affine algebraic group acting on X, one can contemplate either various de Rham stacks of X/G, or of the family $X/G \to BG$. This gives versions of the non-abelian Hodge correspondence for weakly and strongly G-equivariant D-modules on X.

A version of logarithmic non-abelian Hodge theory in characteristic p was developed by Schepler in [Sch05]. Our results appear to be of a different nature: on the de Rham side, we do not impose any nilpotence condition on the residues, but allow all logarithmic connections with nilpotent p-curvature. On the Higgs side, however, we consider Higgs modules in $1/p$-parabolic sheaves relative to the divisor defining the logarithmic structure.

I.0.5: Fontaine-Laffaille modules and prismatic F-gauges

Fix a perfect field k of characteristic p, and let $W(k)$ be the ring of Witt vectors. For a smooth p-adic formal scheme[2] X over $W(k)$ of dimension $\leq p-1$, we have a canonical

[2]Recall that a p-adic formal scheme over $W(k)$ is a functor from the category of p-nilpotent $W(k)$-algebras to sets such that, for every positive integer n, its restriction to $W_n(k)$-algebras is represented by a

isomorphism in the derived category of Zariski sheaves on X:

$$(\Omega_X^\bullet, p \cdot d_{\mathrm{dR}}) \xrightarrow{\sim} (\Omega_X^\bullet, d_{\mathrm{dR}}). \tag{I.0.1}$$

Its composition with the inclusion of complexes $(\Omega_X^\bullet, d_{\mathrm{dR}}) \hookrightarrow (\Omega_X^\bullet, p \cdot d_{\mathrm{dR}})$ given by $\Omega_X^i \xrightarrow{p^i \,\mathrm{Id}} \Omega_X^i$ in degree i, is the crystalline Frobenius morphism. A proof of this fundamental and beautiful result, as well as a more general logarithmic version, appears in [Ogu21, Theorem 6.10]. The idea behind the construction of the map (I.0.1) can be traced back to [Maz73][3]. One infers from (I.0.1) that the crystalline Frobenius φ restricted to the i-th term of the "filtration bête" is canonically divisible by p^i:

$$\frac{\varphi}{p^i} : (\Omega_X, d)^{\geq i} \to (\Omega_X, d). \tag{I.0.2}$$

More generally, in [Maz73], Mazur proved that, for a smooth p-adic formal scheme X over $W(k)$ of any dimension and any integer $i \geq 0$, the crystalline Frobenius φ restricted to $(\Omega_X, d)^{\geq i}$ is canonically divisible by $p^{[i]}$, where $[i] := \min_{m \geq i} \mathrm{ord} \frac{p^m}{m!}$; we refer the reader to Remark III.35 for an interpretation of this result. See [Kat87, AMMN22] for a construction of (I.0.2) via quasi-syntomic descent.

The reduction of (I.0.1) modulo p recovers the Deligne–Illusie isomorphism from [DI87], provided that X has dimension $\leq p - 2$.[4]

Inspired by [FL82], we describe the linear-algebraic structure on the de Rham cohomology $\mathrm{R}\Gamma_{\mathrm{dR}}(X)$ induced by the isomorphism (I.0.1) as follows: we view the complex $\mathrm{R}\Gamma_{\mathrm{dR}}(X)$, equipped with the Hodge filtration $F^\cdot$, as an object $\mathcal{M}$ of the derived category $\mathcal{D}_{qc}(\mathbf{A}^1/\mathbf{G}_m)$ of quasi-coherent sheaves on $\mathbf{A}^1/\mathbf{G}_m$, where $\mathbf{A}^1$ is considered as a p-adic formal scheme over $W(k)$ with the standard $\mathbf{G}_m$-action. The isomorphism (I.0.1) can be rewritten as[5]

$$F^*(\mathcal{M}_{v=p}) \xrightarrow{\sim} \mathcal{M}_{v=1}. \tag{I.0.3}$$

Here $\mathcal{M}_{v=a}$ denotes the (derived) fibre of $\mathcal{M}$ at $\mathrm{Spf}\, W(k) \xrightarrow{a} \mathbf{A}^1/\mathbf{G}_m$ and $F : W(k) \to W(k)$ is the Frobenius. We define and denote by $\mathscr{DMF}^{\mathrm{big}}(W(k))$ the stable ∞-category consisting of $\mathcal{M} \in \mathcal{D}_{qc}(\mathbf{A}^1/\mathbf{G}_m)$ equipped with an isomorphism (I.0.3). We refer to it as the category of *big Fontaine-Laffaille modules* (see Definition III.6).

scheme X_n over $W_n(k)$. Thus, $X = \mathrm{colim}_n X_n$, where the colimit is taken in the category of functors from p-nilpotent $W(k)$-algebras to sets.

[3] We refer the reader to Remark III.36 for a sketch of the Mazur-Ogus argument.

[4] In dimension $p - 1$, the reduction of (I.0.1) differs from the Deligne–Illusie isomorphism; see Remark III.59.

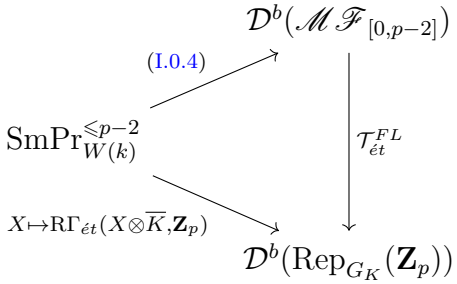
[5] Later in the text, we will write $\mathbf{A}^1_-$ with coordinate given by v_-, instead of simply v.

For any integer $n \geq 0$, we denote by $\mathscr{DMF}^{\mathrm{big}}_{[0,n]}(W(k)) \subset \mathscr{DMF}^{\mathrm{big}}(W(k))$ the full subcategory consisting of objects whose Hodge filtration ranges from 0 to n[6]. The preceding construction describes a contravariant functor from the category $\widehat{\mathrm{Sm}}^{\leqslant p-1}_{W(k)}$ of smooth formal schemes of dimension $\leq p-1$ to $\mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(W(k))$:

$$\widetilde{\Phi}_{\mathrm{Maz}} : \widehat{\mathrm{Sm}}^{\leqslant p-1}_{W(k)} \to \mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(W(k)). \tag{I.0.4}$$

The subcategory $\mathscr{MF}(W(k))$ of $\mathscr{DMF}^{\mathrm{big}}(W(k))$ with $\mathcal{M} \in \mathrm{Coh}(\mathbf{A}^1/\mathbf{G}_m)$ is the category of finitely-generated ("small") Fontaine-Laffaille modules studied in [FL82, §1.5] and in [Win84, §1.3]. In particular, it is proven in *loc. cit.* that $\mathscr{MF}(W(k))$ is an abelian category. The original work [FL82] does not use the language of sheaves on stacks; see Remark III.5 for a comparison of the original definition in [FL82] and our geometric reinterpretation. In Proposition III.1.2 we show that the bounded derived category $\mathcal{D}^b(\mathscr{MF}(W(k)))$ of $\mathscr{MF}(W(k))$ is a full subcategory of $\mathscr{DMF}^{\mathrm{big}}(W(k))$ consisting of bounded complexes with coherent cohomology.

Functor (I.0.4) is an input for various comparison theorems in p-adic Hodge theory. In particular, for $p > 2$, the subcategory $\mathscr{MF}^{\mathrm{tors}}_{[0,1]}(W(k))$ of p-power-torsion objects in $\mathscr{MF}_{[0,1]}(W(k))$ is equivalent to the category of commutative finite flat p-group schemes over $W(k)$ (see [FL82])[7]. In [FL82], an exact functor $\mathcal{T}^{FL}_{\acute{e}t}$ was constructed from $\mathscr{MF}_{[0,p-2]}(W(k))$ to the category of $\mathbf{Z}_p$-linear representations of the absolute Galois group G_K of $K := \mathrm{Frac}(W(k))$. A central result of [Fal89, Theorem 5.3] is the following commutative diagram involving the category $\mathrm{SmPr}^{\leqslant p-2}_{W(k)}$ of smooth proper schemes over $W(k)$ of dimension $\leq p-2$,



I.0.6: Prismatic F-gauges

Given a bounded prism (A, I, δ) equipped with a map $W(k) \to A/I$, Bhatt and Scholze constructed in [BS22] a new cohomology theory

[6] *i.e.*, $F^m \to F^{m-1}$ are quasi-isomorphisms for $m \leq 0$ and F^ℓ are acyclic for $\ell > n$.

[7] This statement is not true for $p = 2$. In fact, the category of commutative finite flat 2-group-schemes over $\mathbf{Z}_2$ is not an abelian category: for example, the nontrivial homomorphism $\mathbf{Z}/2\mathbf{Z} \to \mu_2$ has trivial kernel and cokernel, but it is not an isomorphism.

$X \mapsto R\Gamma_{\mathbf{\Delta}}(X/A)$ for p-adic formal

schemes over $W(k)$ with values in the derived category $\mathcal{D}(A)$ of A-modules. Bhatt–Lurie [BL22a] and Drinfeld [Dri24] advanced this theory further by organizing all prismatic cohomology theories, equipped with the prismatic Nygaard filtrations and the Frobenii, into a single contravariant functor

$$\mathcal{H}_{syn} : \widehat{\mathrm{Sch}}_{W(k)} \to \mathcal{D}_{qc}(W(k)^{syn}) \tag{I.0.5}$$

from the category $\widehat{\mathrm{Sch}}_{W(k)}$ of p-adic formal schemes over $W(k)$ to a stable ∞-category $\mathcal{D}_{qc}(W(k)^{syn})$. Moreover, the latter category can be interpreted as the derived category of quasi-coherent sheaves on a stack $W(k)^{syn}$, called the *syntomification* of $W(k)$. The prismatic cohomology theory attached to (A, I, δ) arises from (I.0.5) via the pullback along a map $\mathrm{Spf}\, A \to W(k)^{syn}$ constructed using (I, δ). The syntomification is defined in [Dri24], [Bha23] for any (nice) p-adic formal scheme X; the functor (I.0.5) carries a p-adic formal scheme X over $W(k)$ to the derived direct image of the structure sheaf under the morphism of stacks

$$X^{syn} \to W(k)^{syn} \tag{I.0.6}$$

constructed by functoriality.

In this article, we address the following two questions: (notations as in (I.0.4))

1. Given a smooth scheme X of dimension $\leq p-1$, can one recover the (big) Fontaine-Laffaille module $\widetilde{\Phi}_{\mathrm{Maz}}(X)$ from $\mathcal{H}_{syn}(X)$?

2. Does $\mathcal{H}_{syn}(X)$ contain more information than $\widetilde{\Phi}_{\mathrm{Maz}}(X)$?

I.0.7: Main results

For a non-negative integer n, we denote by $\mathcal{D}_{qc,[0,n]}(W(k)^{syn})$ the full subcategory of $\mathcal{D}_{qc}(W(k)^{syn})$ spanned by objects with Hodge-Tate weights (see Definition III.10) in the interval $[0, n]$. In particular, the category $\mathcal{D}_{qc,[0,n]}(W(k)^{syn})$ contains objects of the form $\mathcal{H}_{syn}(X)$ for a smooth p-adic formal scheme X of dimension $\leq n$. The main result (Theorem III.39) of our article is the following:

Theorem I.9. *There is a functor*

$$\Phi_{\mathrm{Maz}} : \mathcal{D}_{qc,[0,p-1]}(W(k)^{syn}) \to \mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(W(k)) \tag{I.0.7}$$

that induces an equivalence of ∞-categories

$$\mathcal{D}_{qc,[0,p-2]}(W(k)^{syn}) \xrightarrow{\sim} \mathscr{DMF}^{\mathrm{big}}_{[0,p-2]}(W(k)). \tag{I.0.8}$$

In fact, the composition of Φ_{Maz} with $\mathcal{H}_{syn}$ is isomorphic to $\widetilde{\Phi}_{\mathrm{Maz}}$ (see Remark III.36). The functor Φ_{Maz} (see (III.3.4)) restricts to an equivalence

$$\mathrm{Perf}_{[0,p-2]}(W(k)^{syn}) \xrightarrow{\sim} \mathcal{D}^b\left(\mathscr{MF}_{[0,p-2]}(W(k))\right), \tag{I.0.9}$$

where the left-hand side in (I.0.9) is the full subcategory $\mathrm{Perf}_{[0,p-2]}(W(k)^{syn})$ of the category of perfect complexes $\mathrm{Perf}(W(k)^{syn})$ formed of objects of Hodge-Tate weights in $[0, p-2]$. Moreover, the functor (I.0.9) is t-exact, *i.e.*, it restricts to an equivalence between abelian subcategories[8]

$$\mathrm{Coh}_{[0,p-2]}(W(k)^{syn}) \xrightarrow{\sim} \mathscr{MF}_{[0,p-2]}(W(k)). \tag{I.0.10}$$

We also show that p-torsion objects in $\mathrm{Coh}_{[0,p-2]}(W(k)^{syn})$ are vector bundles on $W(k)^{syn} \otimes \mathbf{F}_p$ (see Remark III.58). We remark that Φ_{Maz} is not an equivalence (see Remark III.57).

Let us now describe some of the earlier works related to our main theorem.

Firstly, in [CL19], Cais and Liu constructed a functor from the category of p-torsionfree Fontaine–Laffaille modules with Hodge-Tate weights in $[0, p-2]$ to the category of Breuil–Kisin modules. The abelian category of coherent prismatic F-gauges also admits a functor to the category of Breuil–Kisin modules. We expect that our functors agree.

Secondly, using results from [BS23], Bhatt and Lurie established an equivalence of categories

$$\mathcal{T}^{BL}_{\acute{e}t} : \mathrm{Coh}^{\mathrm{refl}}(W(k)^{syn}) \xrightarrow{\sim} \mathrm{Rep}^{crys}_{G_K}(\mathbf{Z}_p)$$

between the category of *reflexive* coherent sheaves on $W(k)^{syn}$ and the category of $\mathbf{Z}_p$-lattices in $\mathbb{Q}_p$-linear crystalline representations of the absolute Galois group G_K of $K := \mathrm{Frac}(W(k))$ (see [Bha23, Theorem 6.6.13]). On the other hand, Fontaine and Laffaille ([FL82, §7.14]) constructed an equivalence of categories $\mathcal{T}^{FL}_{\acute{e}t}$ between the full subcategory $\mathscr{MF}^{f}_{[0,p-2]}(W(k)) \subset \mathscr{MF}_{[0,p-2]}(W(k))$ of p-torsionfree Fontaine–Laffaille modules and the full subcategory $\mathrm{Rep}^{crys}_{G_K,[0,p-2]}(\mathbf{Z}_p) \subset \mathrm{Rep}^{crys}_{G_K}(\mathbf{Z}_p)$, whose objects are lattices in crystalline representations with Hodge-Tate weights lying in $[0, p-2]$[9]. In Remark III.61 we prove

[8] By definition, the category $\mathrm{Coh}_{[0,p-2]}(W(k)^{syn})$ is a full subcategory of the category $\mathrm{Coh}(W(k)^{syn})$ of coherent sheaves on $W(k)^{syn}$ that consists of objects lying in $\mathcal{D}_{qc,[0,p-2]}(W(k)^{syn})$.

[9] The essential surjectivity of $\mathcal{T}^{FL}_{\acute{e}t}$ is not stated explicitly in [FL82]. The observation that the results of *loc. cit.* imply the essential surjectivity is due to Breuil [Bre99, Proposition 3].

that the Bhatt–Lurie and Fontaine–Laffaille functors agree via (I.0.10):

$$\mathcal{T}_{\acute{e}t}^{FL} \xrightarrow{\sim} \mathcal{T}_{\acute{e}t}^{BL} \circ \Phi_{\mathrm{Maz}}^{-1} : \mathscr{MF}^{f}_{[0,p-2]}(W(k)) \to \mathrm{Rep}^{crys}_{G_K,[0,p-2]}(\mathbf{Z}_p). \tag{I.0.11}$$

In particular, the Bhatt–Lurie equivalence carries every object of $\mathrm{Rep}^{crys}_{G_K,[0,p-2]}(\mathbf{Z}_p)$ to a *vector bundle* over $W(k)^{syn}$, and not just a reflexive sheaf.

Thirdly, Theorem F (2) from [AMMN22] suggested that the category of Fontaine–Laffaille modules with small Hodge-Tate weights is a derived full subcategory of the category of prismatic F-gauges. The latter result implies, in particular, that for a smooth p-adic formal scheme X of dimension $\leq p-2$, the syntomic cohomology $R\Gamma(W(k)^{syn}, \mathcal{H}_{syn}(X)\{i\})$ is isomorphic, for $i \leq p-2$, to $\mathrm{RHom}(W(k), \widetilde{\Phi}_{\mathrm{Maz}}(X)(i))$ computed in the the category of Fontaine–Laffaille modules. The original proof from [AMMN22] uses methods from algebraic topology. An algebraic proof of the mod p version is explained in [Bha23, Remark 6.5.15]. Our argument is closer in spirit to the latter. We refer the reader to Remark III.60 for further details and a generalization.

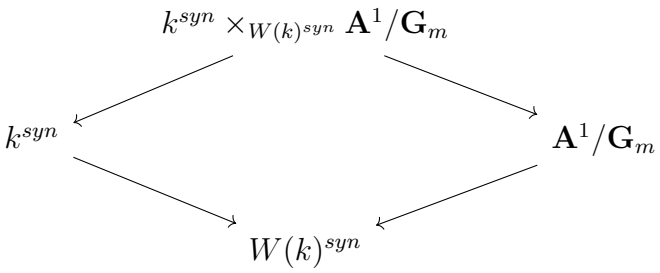
Lastly, a version of (I.0.9) with *rational coefficients* holds in all Hodge-Tate weights. Indeed, using the results from [Bha23, §6], one can prove [Bha24] that the category $\mathrm{Perf}(W(k)^{syn}) \otimes \mathbb{Q}_p$ of perfect prismatic F-gauges up to isogeny is equivalent to the bounded derived category of crystalline representations (with arbitrary Hodge-Tate weights). On the other hand, combining [FL82, Proposition 7.8] with [CF00, Theorem A], the category of crystalline representations is equivalent to the category of Fontaine–Laffaille modules up to isogeny. This yields an equivalence

$$\mathrm{Perf}(W(k)^{syn}) \otimes \mathbb{Q}_p \xrightarrow{\sim} \mathcal{D}^b\left(\mathscr{MF}(W(k))\right) \otimes \mathbb{Q}_p. \tag{I.0.12}$$

I.0.8: Overview of the proof

Recall from [Bha23] that the construction of syntomification can be applied to any p-adic formal scheme over $\mathbf{Z}_p$. In particular, we can apply it to Spec$k$; by functoriality, we have a morphism $\mathfrak{p}_{\mathrm{cris}} : k^{syn} \to W(k)^{syn}$, and thus the pullback functor $\mathfrak{p}_{\mathrm{cris}}^* : \mathcal{D}_{qc}(W(k)^{syn}) \to \mathcal{D}_{qc}(k^{syn})$. The target category has a concrete linear-algebraic description due to [Eke86, FJ21, Bha23]. In geometric context, $\mathfrak{p}_{\mathrm{cris}}^* \mathcal{H}_{syn}(X)$ recovers the crystalline cohomology of $X \otimes \mathbf{F}_p$, equipped with the Nygaard filtration and divided Frobenii. In addition, one has a morphism $\mathfrak{p}_{\bar{\mathrm{dR}}} : \mathbf{A}^1/\mathbf{G}_m \to W(k)^{syn}$, which in geometric context recovers the de Rham

cohomology equipped with the Hodge filtration. We consider the Cartesian square

(I.0.13)


and the induced functor

$$\mathcal{D}_{qc}(W(k)^{syn}) \to \mathcal{D}_{qc}(k^{syn}) \times_{\mathcal{D}_{qc}(\mathfrak{D})} \mathcal{D}_{qc}(\mathbf{A}^1/\mathbf{G}_m), \tag{I.0.14}$$

where $\mathfrak{D}$ is the fibre product in diagram (I.0.13). A key input for our construction of the functor Φ_{Maz} is an explicit description of the fibre product $\mathfrak{D}$ communicated to us by Drinfeld (see §III.2.2[10]).

To describe $\mathfrak{D}$, recall from the end of §III.1.3 that k^{syn} is obtained from[11] $k^{\mathcal{N}} = \operatorname{Spf} A/\mathbf{G}_m$, where $A = W(k)[v_-, v_+]/(v_-v_+ - p)$, by identifying two open points $j_\pm : \operatorname{Spf} W(k) \rightrightarrows k^{\mathcal{N}}$. Let $B^\flat$ be the PD-envelope of the ideal (v_+) in A. In Lemma III.26 and Proposition III.2.2 we show that $\mathfrak{D}$ is isomorphic to $\operatorname{Spf} B^\flat/\mathbf{G}_m$ such that the map $\mathfrak{D} \to k^{syn} \times \mathbf{A}^1/\mathbf{G}_m$ is given by the natural algebra homomorphisms $A \to B^\flat$ and $W(k)[v] \xrightarrow{v \mapsto v_-} B^\flat$. We also consider a "baby" version B of $B^\flat$ defined to be the quotient of $B^\flat$ by the p-power-torsion elements (see the beginning of §III.2.1 for a more concrete description of B). Our main geometric input is the functor

$$\mathcal{D}_{qc,[0,\infty]}(W(k)^{syn}) \to \mathcal{D}_{qc,[0,\infty]}(k^{syn}) \times_{\mathcal{D}_{qc}(\operatorname{Spf} B/\mathbf{G}_m)} \mathcal{D}_{qc,[0,\infty]}(\mathbf{A}^1/\mathbf{G}_m). \tag{I.0.15}$$

obtained from (I.0.14) by applying $\mathcal{D}_{qc}(\mathfrak{D}) \to \mathcal{D}_{qc}(\operatorname{Spf} B/\mathbf{G}_m)$. The target category has a concrete linear-algebraic description. In geometric context, functor (I.0.14) extracts from the F-gauge $\mathcal{H}_{syn}(X)$ the information about the Nygaard filtered crystalline cohomology with its divided Frobenii, the Hodge filtered de Rham cohomology (*i.e.*, the cohomology theories known before the invention of prismatic cohomology), together with a comparison isomorphism relating Nygaard and Hodge filtrations (that, in particular, recovers (I.0.2)). See Remark III.33 for more details. We check (see Remark III.34) that the functor from $\mathcal{D}_{qc,[0,p-1]}(k^{syn}) \times_{\mathcal{D}_{qc}(k^{\mathcal{N}})} \mathcal{D}_{qc,[0,p-1]}(\mathbf{A}^1/\mathbf{G}_m)$ to $\mathcal{D}_{qc,[0,p-1]}(k^{syn}) \times_{\mathcal{D}_{qc}(\operatorname{Spf} B/\mathbf{G}_m)}$

[10] §III.2.1 contains a description of a certain closed substack of $\mathfrak{D}$ which is sufficient for all our applications.

[11] Note that A is not p-complete. By our convention, we write $\operatorname{Spf} A$ for the p-adic formal scheme $\operatorname{colim}_n \operatorname{Spec} A/p^n$.

$\mathcal{D}_{qc,[0,p-1]}(\mathbf{A}^1/\mathbf{G}_m)$, induced by the map $\operatorname{Spf} B/\mathbf{G}_m \to k^{\mathcal{N}}$, is an equivalence and identifies the left-hand side with $\mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(W(k))$.

This defines Φ_{Maz} from (I.0.7). The proof of equivalence (I.0.8) uses a concrete description of the reduced locus $W(k)^{syn}_{red}$ of $W(k)^{syn}$ from [Dri24] that leads to an equivalence $\mathcal{D}_{qc,[0,p-1]}(W(k)^{syn}_{red}) \xrightarrow{\sim} \mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(W(k)) \otimes \mathbf{F}_p$. Finally, via a deformation theory argument (suggested to us by Bhatt), we prove that the pullback along $W(k)^{syn}_{red} \hookrightarrow W(k)^{syn} \otimes \mathbf{F}_p$ induces an equivalence

$$\mathcal{D}_{qc,[0,p-2]}(W(k)^{syn} \otimes \mathbf{F}_p) \xrightarrow{\sim} \mathcal{D}_{qc,[0,p-2]}(W(k)^{syn}_{red}). \tag{I.0.16}$$

The last step involves cohomology computations from [Bha23, §6]. We infer that (I.0.8) induces an equivalence after taking the tensor product with $\mathbf{F}_p$, which, by p-completeness, completes the proof.

Though functor (I.0.15) is not an equivalence in large weights, it can be used to construct some interesting invariants of F-gauges. Following a suggestion of Drinfeld's, we define in Remark III.35 a functor from the right-hand side of (I.0.15) to a category of Mazur modules $\mathrm{Mod}_{\mathrm{Maz}}(W(k))$, giving

$$\mathbf{\Phi}_{\mathrm{Maz}} : \mathcal{D}_{qc,[0,\infty]}(W(k)^{syn}) \to \mathrm{Mod}_{\mathrm{Maz}}(W(k)).$$

We refer the reader to Remark III.36 for an interpretation of this functor in geometric context.

As another application of Drinfeld's description of $\mathfrak{D}$, we construct in Remark III.37 a refinement of the crystalline Frobenius functor on the category of crystals on a smooth p-adic formal scheme X over $\operatorname{Spf} W(k)$. This construction is related to the Cartier transform studied in [OV07] and especially to its p-adic lift constructed in [Xu19].

CHAPTER II
Ogus-Vologodsky Equivalence via Stacks

II.1: Preliminaries

II.1.1: Animated rings

Even though the main results concern schemes, it will be useful for us to work with derived stacks. For this, recall the ∞-category $\mathrm{CAlg}^{\mathrm{an}}_{\mathbf{Z}}$ of animated commutative rings, which can be defined by localizing the category of simplicial rings at weak equivalences. The resulting ∞-category inherits many good properties. For example, it can be shown that it is the universal ∞-category containing the 1-category $\mathrm{CAlg}_{\mathbf{Z}}$ and closed under sifted colimits. This procedure (*i.e.*, free generation under sifted colimits) is called animation, and we refer to [ČS24, §5.1.4]. Using it, one defines the category of derived schemes $\mathrm{dSch}_{\mathbf{Z}}$ and derived stacks $\mathrm{dSt}_{\mathbf{Z}}$. For example, for us a derived stack will be an accessible fppf sheaf $\mathrm{CAlg}_{\mathbf{Z}} \to \mathrm{Spc}$. For an animated ring A, one also has the category of animated A-algebras $\mathrm{CAlg}^{\mathrm{an}}_{A}$, defined as the slice category $(\mathrm{CAlg}^{\mathrm{an}}_{\mathbf{Z}})_{/A}$.

Let us recall the cotangent complex for derived stacks. Let X/A be a derived stack over an animated ring A. Its cotangent complex $L_{X/A}$, when it exists, satisfies the following universal property: for any $x \in X(R)$ one has an equivalence $\mathrm{Map}_R(x^*L_{X/A}, M) \simeq \mathrm{Der}_x(X/A, M)$ where the right-hand side is the space of A-linear derivations of $X \to \mathrm{Spec}(A)$ at x valued in an R-animated module M, see [ČS24]. In particular, let $R' \to R$ be a square zero extension of animated A-algebras with the fiber $N \in \mathcal{D}(R)$ (see [ČS24, §5.1.9]). The fiber of $X(R') \to X(R)$ over $x \in X(R)$, if non-empty, is a torsor for the animated abelian group $\mathrm{Map}_R(x^*L_{X/A}, N)$.

Mostly, we will be working with $\mathrm{CAlg}^{\mathrm{an}}_{\mathbf{F}_p}$ which can also be defined via animation of $\mathrm{CAlg}_{\mathbf{F}_p}$. In particular, every animated $\mathbf{F}_p$-algebra A carries the Frobenius endomorphism $F_A : A \to A$. Let us collect some facts about animated $\mathbf{F}_p$-algebras.

Lemma II.1. *Let A be a 1-truncated animated $\mathbf{F}_p$-algebra. Then there exists an essentially unique map of $\mathbf{F}_p$-algebras $F'_A : \pi_0(A) \to A$ such that the composition $A \to \pi_0(A) \xrightarrow{F_{A'}} A$ is*

homotopic to F_A. In particular, this map is characterized by the property that $\pi_0(A) \xrightarrow{F'_A} A \to \pi_0(A)$ is equal to $F_{\pi_0(A)}$.

Proof. Uniqueness follows from a more general statement: let $f : A \to B$ be a map of 1-truncated animated rings such that $\pi_1(f) = 0$, then the fiber of $\mathrm{Map}(\pi_0(A), B) \to \mathrm{Map}(A, B)$ over f, if non-empty, is a torsor for $\mathrm{Map}(L_{\pi_0(A)/A}, \pi_1(B)[1])$ and the latter space is contractible since $L_{\pi_0(A)/A}$ is 2-connective. In particular, such map f is uniquely determined by $\pi_0(f)$. For the existence part: given $f : A \to B$ a map of 1-truncated animated algebras, the obstruction of deforming the map $\pi_0(A) \to \pi_0(B)$ of A-algebras to $\pi_0(A) \to B$ lies in $\mathrm{Map}_A(L_{\pi_0(A)/A}, \pi_1(B)[2])$. Since $\pi_2 L_{\pi_0(A)/A} = \pi_1(A)$, the latter space is discrete and equivalent to $\mathrm{Hom}_{\pi_0(A)}(\pi_1(A), \pi_1(B))$. Under this identification, the obstruction class corresponds to $\pi_1(f)$. To finish the proof note that the Frobenius map for animated $\mathbf{F}_p$-algebras induces 0 on higher homotopy groups. □

II.1.2: Ring stacks

For any smooth scheme $X/\mathbb{Q}$, Simpson introduces a stack X^{dR} which *geometrizes* de Rham cohomology in the sense that $\mathrm{Vect}(X^{dR})$ is identified with the category of vector bundles on X endowed with a flat connection. Moreover, for any $(E, \nabla) \in \mathrm{Vect}(X^{dR})$, one has

$$R\Gamma(X^{dR}, (E, \nabla)) \simeq R\Gamma(X, E \xrightarrow{\nabla} E \otimes \Omega^1_X \to \cdots).$$

Following this idea, Drinfeld and Bhatt-Lurie constructed, for any p-adic formal scheme X, certain stacks that *geometrize* the theory of prismatic cohomology. The crucial notion for this construction is the notion of a *ring stack*. Namely, given an fppf sheaf on A-algebras

$$\mathcal{R} : \mathrm{CAlg}_A \to \mathrm{CAlg}^{\mathrm{an}}_B$$

valued in animated B-algebras, one defines a cohomology theory on the category of B-schemes.

Construction II.2. (transmutation) Let $\mathcal{R}$ be an A-algebra stack. Define a prestack $\mathrm{dSt}_B \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{an}}_A, \mathrm{Spc})$ which takes X/B to $X^{\mathcal{R}}$ defined by $X^{\mathcal{R}}(R) := \mathrm{Map}_B(\mathrm{Spec}(\mathcal{R}(R)), X)$.

The ring stacks we will need are 1-truncated. These admit an explicit model discovered by Drinfeld in [Dri21]. Namely, he constructed a $(2,1)$-category whose objects are morphisms of R-modules $d\colon I \to R$, called quasi-ideals, where R is a commutative ring, I is an R-module, and such that for any $x, y \in I$ one has $d(x)y = xd(y)$. He also explicitly

describes the mapping groupoids, but we do not need the full description. Let us only mention that an obvious commutative square $(I \to R) \to (J \to S)$ of quasi-ideals, where $f\colon R \to S$ is a map of rings and $I \to f_*J$ is a map of R-modules, gives a point of the mapping groupoid. This notion also makes sense when working with schemes, i.e., if R is a ring scheme and I is an R-module, then $d\colon I \to R$ is a quasi-ideal in schemes if $d(x)y = xd(y)$ for any test points x, y of I. To give an example, recall the Witt vector ring scheme W. It can be extended to all animated rings; see [BL22b, Appendix A].

Definition II.3. Denote $\mathbf{G}_a^{dR}$ to be the W-algebra stack $\mathrm{Cone}(F_*W \xrightarrow{p} F_*W)$.

Remark II.4. For any ring R we have a commutative diagram

$$\begin{array}{ccc} F_*W(R) & \xrightarrow{\quad V \quad} & W(R) \\ \| & & \downarrow F \\ F_*W(R) & \xrightarrow{\quad p \quad} & F_*W(R). \end{array}$$

It is a map of quasi-ideals and thus defines $\pi : \mathbf{G}_a \to \mathbf{G}_a^{dR}$. In particular, $\mathbf{G}_a^{dR}$ is an $\mathbf{F}_p$-algebra stack and one has its Frobenius endomorphism $F : \mathbf{G}_a^{dR} \to \mathbf{G}_a^{dR}$. Let R be an $\mathbf{F}_p$-algebra, then we have two endomorphisms of $\mathbf{G}_a^{dR}(R)$: namely, $\mathbf{G}_a^{dR}(F_R)$ and $F(R)$. We note that they are naturally homotopic.[1] See [Bha23, Remark 5.1.10] or use Theorem II.1. In particular, for any $X/\mathbf{F}_p$ one has a natural homotopy between $F_{X^{dR}}, F_X^{dR} : X^{dR} \to X^{dR}$.

The map $\pi : \mathbf{G}_a \to \mathbf{G}_a^{dR}$ from Theorem II.4 gives rise to a map of commutative animated rings $\pi_A : A \to W(A)/p$. In particular, if $A = W(R)/p$, there is another natural map $W(\pi_R)/p : A \to W(A)/p$ obtained by reducing $W(\pi_R)$ modulo p. We learned the following argument from Bhatt.

Lemma II.5. *Let R be an animated $\mathbf{F}_p$-algebra. For $A = W(R)/p$, the two maps π_A and $W(\pi_R)/p$ are naturally homotopic.*

Proof. Step 1. Let A be a δ-ring. Let $u_A : A \to W(A/p)$ be a δ-lift of $A \xrightarrow{\phi_A} A \to A/p$. The following diagram is commutative

$$\begin{array}{ccc} A & \xrightarrow{\quad u \quad} & W(A/p) \\ \downarrow & & \downarrow \\ A/p & \xrightarrow{\quad e_{A/p} \quad} & W(A/p)/p. \end{array}$$

Indeed, it is enough to check it for a free δ-ring. Then all rings involved are discrete and one can check commutativity on elements. The composition $A \to A/p \xrightarrow{e_{A/p}} W(A/p)/p$

[1] We warn the reader it is not automatic for an $\mathbf{F}_p$-algebra stack. For example, it fails for $\mathbf{G}_a \otimes^L_{\mathbf{Z}} \mathbf{F}_p$.

sends $x \to \bar{x} \to [\bar{x}]^p$. Note the delta lift of $A \to A/p$ carries x to $[x] + V(z)$ for some z, applying Frobenius and reducing modulo p gives the desired commutativity.
Step 2. Now let $A = W(R)$. Note u_A coincides with $W(e_R)$. Indeed, both maps are δ-maps so it is enough to check it after composing with $W(W(R)/p) \xrightarrow{R} W(R)/p$. Then the claim amounts to observing that $W(R) \xrightarrow{F} W(R) \to W(R)/p$ is homotopic to $W(R) \xrightarrow{R} R \xrightarrow{u_R} W(R)/p$ which is essentially the definition of u_R.
Step 3. Therefore, the diagram

$$\begin{array}{ccc} W(R) & \xrightarrow{W(e_R)} & W(W(R)/p) \\ \downarrow & & \downarrow \\ W(R)/p & \xrightarrow{e_{W(R)/p}} & W(W(R)/p)/p \end{array}$$

is commutative. This finishes the proof as $e_{W(R)/p}$ is a morphism of $\mathbf{F}_p$-algebras. □

For a ring stack $\mathcal{R}$, we will want to use descent properties of $X^{\mathcal{R}}$. The following lemma is essentially proven in [Pet25, Lemma 3.3], but we include the proof for the reader's convenience.

Lemma II.6. *Let $\mathcal{R}$ be a 1-truncated A-algebra stack. Suppose $\pi_0(\mathcal{R}), \pi_1(\mathcal{R})$ satisfy étale descent. For a stack X/A assume that $X^{\pi_0(\mathcal{R})}$ satisfies étale descent. Then $X^{\mathcal{R}}$ also satisfies étale descent.*

Proof. Given $R \to S$ an étale cover in CAlg^{an}, we have to prove that $X^{\mathcal{R}}(R) \to \lim X^{\mathcal{R}}(S^{\otimes_R n})$ is an equivalence. Consider

$$\begin{array}{ccc} X(\mathcal{R}(R)) & \longrightarrow & \lim X(\mathcal{R}(S^{\otimes_R n})) \\ \downarrow & & \downarrow \\ X(\pi_0(\mathcal{R})(R)) & \longrightarrow & \lim X(\pi_0(\mathcal{R})(S^{\otimes_R n})) \end{array}$$

By assumption the lower horizontal arrow is an equivalence. Thus, it is enough to check it fiberwise. For $x \in X(\pi_0(\mathcal{R})(R))$ the fiber of the left vertical arrow is $\mathrm{Map}(x^* L_X, \pi_1(\mathcal{R})[1])$. Fiber on the right over x is $\lim \mathrm{Map}_{\pi_0(\mathcal{R})(S^{\otimes_R n})}(x^* L_X, \pi_1(\mathcal{R})(S^{\otimes_R n})[1])$. □

Remark II.7. For a ring stack $\mathcal{R}$, the presheaf sending a derived stack X to the prestack $X^{\mathcal{R}}$ commutes with arbitrary limits. Indeed, for any diagram X_i of derived stacks one has $(\lim X_i)(A) = \lim X_i(A)$ for any animated ring A.

Lemma II.8. *Let $\mathcal{R}$ be a $\mathbf{F}_p$-algebra stack with $\pi_0(\mathcal{R}) = \mathbf{G}_a$ and $\pi_1(\mathcal{R}) := G$ representable by an affine group scheme such that semiperfect algebras are $\pi_1(\mathcal{R})$-acyclic. Assume that $\pi_0(\mathcal{R}(S)) = S$ for any semiperfect algebra S. If $X/\mathbf{F}_p$ is lci, then the natural map $\nu_X : X^{\mathcal{R}} \to X$ realizes the source as a quasi-syntomic $G \otimes_{\mathbf{G}_a} L_X^\vee[1]$-torsor over the target.*

Proof. This is essentially proven in [Bha23, Footnote 18] but we recall the idea. The action of $\mathrm{Map}(L_X, G[1])$ on $X^{\mathcal{R}} \to X$ comes from the derived deformation theory since $\nu : \mathcal{R} \to \mathbf{G}_a$ is a square-zero extension by $G[1]$. Namely, for a point $x : \mathrm{Spec}(R) \to X$ the fiber of $X^{\mathcal{R}}(R) = X(\mathcal{R}(R)) \to X(R)$ over x, if nonempty, is a torsor for $\mathrm{Map}_R(x^*L_X, G[1](R))$. When X is lci over $\mathbf{F}_p$ we get

$$\mathrm{Map}_R(x^*L_X, G[1](R)) \simeq x^*L_X^\vee \otimes_R G[1](R) \in \mathcal{D}_{qc}^{\leq 0}(R)$$

due to dualizability of L_X. To finish the proof it is enough to construct a section of $X^{\mathcal{R}} \to X$ fppf-locally. Passing to open subschemes of X, we can assume $X = \mathrm{Spec}(R)$ where R is quasi-syntomic, then giving a section of ν_X is equivalent to giving a section of $\nu(R) : \mathcal{R}(R) \to R$. Moreover, we can find a faithfully flat map $f : R \to S$ where S is quasi-regular semiperfect and $\pi_1 L_S$ is a finite free S-module by [BMS19, Remark 4.29]. Thus, we assume $X = \mathrm{Spec}(S)$ a qrsp ring with $L_X = M[1]$ where M is a finite free S-module of finite rank. In that case, the map $\mathcal{R}(S) \to S$ exhibits the source as a square-zero extension with the fiber $G[1](S)$. Therefore, we get the obstruction class $\nu^*L_S \to G[2](S)$ whose vanishing is sufficient for existence of a section. Since $L_S = M[1]$ with M being finite free, the claim follows from the assumption that G is acyclic on semiperfect rings.

□

II.1.3: Derived rings and affine stacks

We extensively use the notion of affine stack which is based on a notion of a derived ring, discovered by Mathew. This notion is supposed to extend animated rings to the nonconnective setting. A typical example is the complex $R\Gamma(X, \mathcal{O}_X) \in \mathcal{D}(k)$ for a scheme X/k. More generally, any cosimplicial commutative ring gives rise to an example. Moreover, coconnective derived rings are exactly cosimplicial commutative rings. For us, the main example of a derived ring will be derived de Rham cohomology $\mathrm{dR}_{B/A}$ of a map $A \to B$ between commutative rings. In general, it is not represented by a cosimplicial commutative ring since $H^i(\mathrm{dR}_{B/A})$ might not vanish for $i < 0$. For an animated algebra A we denote the category of derived A-algebras by DAlg_A, see [Rak20, §4.2]. Any derived A-algebra $R \in \mathrm{DAlg}_{\mathbf{F}_p}$ gives rise to a derived stack $\mathrm{Spec}(R)$ whose functor of points takes $A \in \mathrm{CAlg}_A^{\mathrm{an}}$

to the space $(\mathrm{Spec}(R))(A) = \mathrm{Map}_{\mathrm{DAlg}_{\mathbf{F}_p}}(R, A)$. Analogously, one defines affine stack in the relative setting.

Definition II.9. ([MM25, Remark 4.5]) A morphism $X \to S$ of derived k-stacks is said to be affine if for every derived affine scheme $\mathrm{Spec}(A)$ with a map $\mathrm{Spec}(A) \to S$, the fiber product $X \times_S \mathrm{Spec}(A)$ is an affine derived stack.

Remark II.10. It is clear from the definition that affine S-stacks are closed under limits. Similarly, if $X \to Y$ is affine, then for any $Y' \to Y$, the map $X \times_Y Y' \to Y'$ is affine. Therefore, if $f : X \to Y$ is an S-map with affine $X \to S$, then f is affine. Indeed, for any $\mathrm{Spec}(A) \to Y$ one has $\mathrm{Spec}(A) \times_Y X = (X \times_S Y) \times_Y \mathrm{Spec}(A)$.

For a scheme X with a derived $\mathcal{O}_X$-algebra $\mathcal{A} \in \mathrm{DAlg}(\mathcal{D}(X))$ one defines an affine map $\underline{\mathrm{Spec}}_X(\mathcal{A}) \to X$. The main object of interest in this paper is the de Rham stack X^{dR} for a scheme $X/\mathbf{F}_p$ and its relative version $(X/S)^{dR}$ for a morphism $X \to S$ of $\mathbf{F}_p$-schemes. We will use that $(X/S)^{dR}$ is an affine $X \times^L_{S,F_S} S := X'$-stack represented by $F_{X/S,*}\,\mathrm{dR}_{X/S} \in \mathcal{D}_{qc}(X')$.

Remark II.11. For a vector bundle E on a scheme X, a stack $B^n_X E^\sharp$ is affine over X for any $n \geq 0$. Indeed, Zariski localising X we can assume $E = \mathbf{G}_a^n$ and then it follows since $B^n \mathbf{G}_a^\sharp$ is an affine stack, see [MM25, Proposition 4.14, 4.19]. Moreover, if $a : E_1 \to E_0$ is a map of vector bundles, then $B^n_X E^\sharp \to X$ is affine for any n where $E := \mathrm{cofib}(a)$. Indeed, note $B^n_X E^\sharp = \mathrm{fib}(B^{n+1} E_0^\sharp \to B^{n+1} E_1^\sharp)$. In particular, if $f : Y \to X$ is a torsor for $E^\sharp$, then f is affine. Indeed, if $X \to B_X E^\sharp$ is the map classifying Y, then f is obtained as the pullback of the canonical map $X \to B_X E^\sharp$ along f.

II.2: Relative de Rham stack

II.2.1: Preliminaries on de Rham stack

Recall from Theorem II.3 the $\mathbf{G}_a$-algebra stack $\mathbf{G}_a^{dR} := \mathrm{Cone}(F_* W \xrightarrow{p} F_* W)$ on the category of animated $\mathbf{F}_p$-algebras.

Remark II.12. Recall $\pi_0(\mathbf{G}_a^{dR}) = F_* \mathbf{G}_a$ and $\pi_1(\mathbf{G}_a^{dR}) = F_* \mathbf{G}_a^\sharp$ as W-modules. In particular, there is a map of ring stacks $\nu : \mathbf{G}_a^{dR} \to \mathbf{G}_a$ that is a square-zero extension with the fiber $\mathbf{G}_a^\sharp[1]$. There is also a map $\pi : \mathbf{G}_a \to \mathbf{G}_a^{dR}$ such that $\nu \circ \pi = F_{\mathbf{G}_a}$.

Definition II.13. For a stack $X/\mathbf{F}_p$ we define a derived prestack X^{dR} by the rule

$$X^{\mathrm{dR}}(R) := \mathrm{Map}_{\mathrm{dSt}_{\mathbf{F}_p}}(\mathrm{Spec}(\mathbf{G}_a^{dR}(R)), X)$$

.

Remark II.14. Transmuting π, ν from Theorem II.12 we get $\pi_X : X \to X^{dR}$ and $\nu : X^{dR} \to X$ for any derived stack $X/\mathbf{F}_p$.

Remark II.15. Recall the natural transformation $\pi_X : X \to X^{dR}$ from Theorem II.14. It gives rise to two natural transformations $(\pi_X)^{dR}, \pi_{X^{dR}} : X^{dR} \to (X^{dR})^{dR}$. They are naturally homotopic, see Theorem II.5

We will need a version of the de Rham stack for any morphism $X \to S$.

Definition II.16. Given a morphism $f : X \to S$ of $\mathbf{F}_p$-stacks, one defines $(X/S)^{dR}$ to be the fiber product $X^{dR} \times_{S^{dR}} S$ in the ∞-category of derived prestacks, where the maps are f^{dR} and $\pi_S : S \to S^{dR}$. Sometimes, we will also denote it by $(X \xrightarrow{f} S)^{dR}$.

Remark II.17. The relative de Rham stack $(X/S)^{dR}$ is an S-stack via the projection map $(X/S)^{dR} \to S$. Explicitly, it represents the functor which takes $\mathrm{Spec}(R) \to S$ to $\mathrm{Map}_S(\mathrm{Spec}(F_*W(R) \otimes^L \mathbf{F}_p), X)$; here $\mathrm{Spec}(F_*W(R) \otimes^L \mathbf{F}_p)$ is a stack over S via

$$\mathrm{Spec}(F_*W(R) \otimes^L \mathbf{F}_p) \to \mathrm{Spec}(R) \to S$$

where the first map is from Theorem II.12. In particular, this definition coincides with the one in [Bha23, Definition 2.5.3].

Remark II.18. Recall that for an ∞-category $\mathcal{C}$ the arrow category $\mathcal{C}^{\Delta^1}$ defined as $\mathrm{Fun}(\Delta^1, \mathcal{C})$. The relative de Rham stack defines a functor $(-/-)^{dR} : \mathrm{dStk}_{\mathbf{F}_p}^{\Delta^1} \to \mathrm{dStk}_{\mathbf{F}_p}$.

Remark II.19. The relative de Rham stack functor $(-/-)^{dR} : \mathrm{dStk}_{\mathbf{F}_p}^{\Delta^1} \to \mathrm{dStk}_{\mathbf{F}_p}$ commutes with all limits. Indeed, it is true for the functor $(-)^{dR} : \mathrm{dStk}_{\mathbf{F}_p} \to \mathrm{dStk}_{\mathbf{F}_p}$ by Theorem II.7. It formally implies the statement as $(X/S)^{dR} = X^{dR} \times_{S^{dR}} S$ and fiber products commute with arbitrary limits.

Corollary II.20. *The restriction of* $(-/-)^{dR} : \mathrm{dStk}_{\mathbf{F}_p}^{\Delta^1} \to \mathrm{dStk}_{\mathbf{F}_p}$ *to the subcategory* $\mathrm{Sch}_{\mathbf{F}_p}^{\Delta^1}$ *commutes with Tor-independent limits.*

Remark II.21. Let $f : X \to S$. There is a map $X \to (X/S)^{dR} := S \times_{S^{dR}} X^{dR}$ given by (f, π_X) with the homotopy between two compositions provided by naturality of π_-. We will denote it by $\pi_{X/S}$.

Lemma II.22. *If* $f : X \to S$ *is smooth, then* $\pi_{X/S} : X \to (X/S)^{dR}$ *is surjective fpqc locally.*

Proof. Let $V \to S$ be an affine open subscheme. It follows from Theorem II.19 that both

squares in the following diagram are cartesian.

$$\begin{array}{ccc} X \times_S V & \longrightarrow & X \\ \downarrow & & \downarrow \\ (X \times_S V/V)^{dR} & \longrightarrow & (X/S)^{dR} \\ \downarrow & & \downarrow \\ V & \longrightarrow & S \end{array}$$

Therefore, we can assume that S is affine. In this case we refer to the proof of [Bha23, Theorem 2.5.6] where the statement is proven.

□

Remark II.23. For any derived stack X one obtains $\nu_X : X^{dR} \to X$ by transmuting ν from Theorem II.12. When $X/\mathbf{F}_p$ is an lci scheme, the map $X^{dR} \to X$ realizes the source as a quasi-syntomic torsor for the group stack $L_X^\vee[1] \otimes_{\mathbf{G}_a} \mathbf{G}_a^\sharp$. This follows from Theorem II.8 since $\pi_1(\mathbf{G}_a^{dR}) = F_*\mathbf{G}_a^\sharp$ as sheaves of W-modules for fppf topology and so semiperfect algebras are $\pi_1(\mathbf{G}_a^{dR})$-acyclic by [Bha23, Corollary 2.6.4]. If, moreover, $X/\mathbf{F}_p$ is smooth, then $X^{dR} \to X$ realizes the source as a quasi-syntomic $T_X^\sharp$-gerbe over the target.

If X is any derived stack over $\mathbf{F}_p$ and $s\colon X \to X^{dR}$ is any section of $\nu_X\colon X^{dR} \to X$, we obtain two points in $X^{dR}(X)$, namely $s \circ F_X$ and π_X. In the next lemma, we observe that they are canonically homotopic.

Lemma II.24. *Let $s : X \to X^{dR}$ be any map satisfying $\nu_X \circ s \simeq \mathrm{Id}_X$. Then there exists a canonical homotopy $F_X \circ s \simeq \pi_X$.*

Proof. Indeed, one has a canonical isomorphism $s \circ F_X \simeq F_{X^{dR}} \circ s$ since Frobenius commutes with any morphism of derived $\mathbf{F}_p$-stacks. By Theorem II.1 one has a canonical identification $F_X^{dR} \simeq \pi_X \circ \nu_X$ and since $F_X^{dR} \simeq F_{X^{dR}}$ canonically by Theorem II.4, we get $F_{X^{dR}} \circ s \simeq \pi_X \circ \nu_X \circ s \simeq \pi_X$. □

Lemma II.25. *Let $f : Y \to X$ be a map over $\mathbf{F}_p$. Then $f^{dR} : Y^{dR} \to X^{dR} \times_{X'} Y'$ is equivariant for the map $\mathrm{Map}(f^*L_X, \mathbf{G}_a^\sharp[1]) \to \mathrm{Map}(L_Y, \mathbf{G}_a^\sharp[1])$ of Y-group stacks.*

Proof. This follows since the action comes from derived deformation theory. Let $A' \to A$ be a square-zero extension with the ideal $I \in \mathcal{D}^{\leq 0}(A)$. Fix $x \in X(A)$ and $y \in Y(A)$ compatibly with respect to f. The fiber of $X(A') \to X(A)$ over x, if non-empty, is a torsor for $\mathrm{Map}_A(x^*L_X, I)$ and similarly for y. Pulling back the map $df : f^*L_X \to L_Y$ along x gives $\mathrm{Map}_A(y^*L_Y, I) \to \mathrm{Map}_A(x^*L_X, I)$ and the map $Y(A') \to X(A')$ is equivariant with respect to this map. □

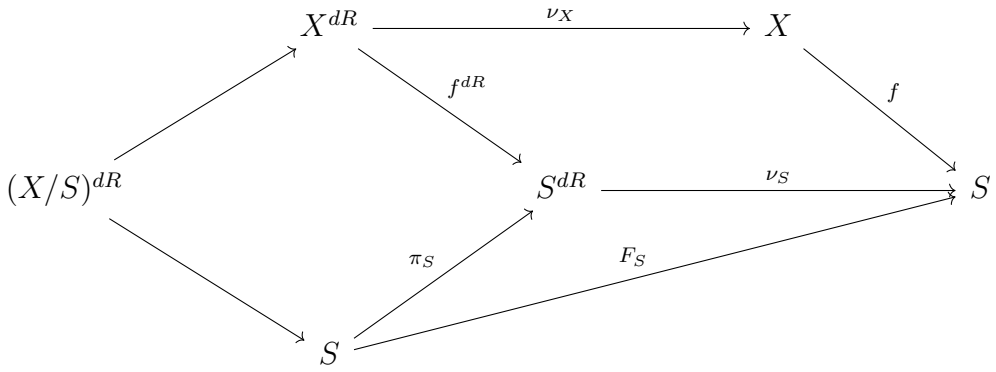
Remark II.26. Let $\tilde{X}$ be a flat $\mathbf{Z}_p$-scheme with a δ-structure. Let us show how that splits $X^{dR} \to X$, it is due to [BL22b, Remark 5.13]. First note

$$X(R) = \mathrm{Map}_{\mathbf{F}_p}(\mathrm{Spec}(R), X) = \mathrm{Map}_{\mathbf{Z}_p}(\mathrm{Spec}(R), \tilde{X}) = \mathrm{Map}_{\delta-\mathbf{Z}_p}(\mathrm{Spec}(W(R)), \tilde{X}).$$

Using the natural map $\mathrm{Map}_{\delta-\mathbf{Z}_p}(\mathrm{Spec}(W(R)), \tilde{X}) \to \mathrm{Map}_{\mathbf{F}_p}(\mathrm{Spec}(W(R)) \otimes^L \mathbf{F}_p, X)$, one obtains a splitting $X \to X^{dR}$. It was observed by Bhatt and later by Petrov and Vologodsky, that it is enough to have a flat lift of X to $\mathbf{Z}/p^2$ with a lift of Frobenius to split $X^{dR} \to X$. One proof is based on exploiting affinness of $X^{dR} \to X$, we refer to [Pet23, Proposition 5.14] for the argument. Moreover, it was observed by Petrov and Vologodsky that the splitting coming from a flat δ-scheme $\tilde{X}$ only depends on $\tilde{X} \times \mathrm{Spec}(\mathbf{Z}/p^2)$ with the Frobenius on it. Moreover, the splitting is isomorphic to the splitting constructed using affinness of the de Rham stack.

Lemma II.27. *There exists a map $\nu_{X/S} : (X/S)^{dR} \to X' := X \times_{S,F_S} S$ such that for an lci X/S it exhibits the source as a quasi-syntomic torsor for $L^{\vee}_{X'/S}[1] \otimes_{\mathbf{G}_a} \mathbf{G}_a^\sharp$.*

Proof. The map $(X/S)^{dR} \to X'$ comes from the following commutative diagram



For a scheme Y/S denote $\mathrm{Hom}(L_{Y/S}, \mathbf{G}_a^\sharp)$ to be the group stack on Y sending $\eta : \mathrm{Spec}(R) \to Y$ to $\tau^{\leq 0}\mathrm{Hom}_R(\eta^* L_{Y/S}, R^\sharp[1])$ computed in $\mathcal{D}(R)$. Theorem II.8 implies that the fiber product

(II.2.1)
$$\begin{array}{ccc} \mathrm{Hom}(L_{X'/S}, \mathbf{G}_a^\sharp[1]) & \longrightarrow & \mathrm{Hom}(L_X, \mathbf{G}_a^\sharp[1]) \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{Hom}(L_S, \mathbf{G}_a^\sharp[1]) \end{array}$$

of X'-group stacks acts on $(X/S)^{dR}$. Indeed, it is enough to check that the fiber of $\mathrm{Hom}(L_X, \mathbf{G}_a^\sharp[1]) \to \mathrm{Hom}(L_S, \mathbf{G}_a^\sharp[1])$ is identified with $\mathrm{Hom}(L_{X'/S}, \mathbf{G}_a^\sharp[1])$. This amounts to showing that $(f \circ \rho)^* L_X \to \rho^* L_X \to L_{X'/S}$ is a fiber sequence where $\rho : X' \to X$ is the canonical projection. This follows from the base change $L_{X'/S} = \rho^* L_{X/S}$ and the transitivity triangle $f^* L_S \to L_X \to L_{X/S}$. To construct a local splitting of $\nu_{X/S} : (X/S)^{dR} \to X'$ we proceed as in Theorem II.8. First, we can assume that $S = \mathrm{Spec}(R)$ and $X = \mathrm{Spec}(A)$. Then write

$$\mathrm{Map}_R(F_R^* A, W(F_R^* A) \otimes^L \mathbf{F}_p) = (X/S)^{dR}(X') \to X'(X') = \mathrm{Map}_R(F_R^* A, F_R^* A).$$

In particular, finding a section of $\nu_{X/S}$ is equivalent to finding a section of $R : W(F_R^* A) \otimes^L \mathbf{F}_p \to F_R^* A$. By [BMS19, Variant 4.33] we can find a quasisyntomic cover $A \to B$ with B being a quasiregular semiperfect R-algebra such that $\pi_1(L_{B/R})$ is a finite free B-module. In this case the map $W(F_R^* B) \xrightarrow{p} W(F_R^* B)$ is surjective and thus $W(F_R^* B) \otimes^L \mathbf{F}_p \to F_R^* B$ is a square-zero extension with fiber $\mathbf{G}_a^\sharp[1](F_R^* B)$. Thus, we obtain a map $L_{F_R^* B/R} \to \mathbf{G}_a^\sharp[2]$ whose vanishing is sufficient for splitting this square-zero extension in R-algebras. Since B is semiperfect, the claim follows. □

Remark II.28. Let $X \to Y$ be a map of S-stacks. Then the induced map

$$(X/S)^{dR} \to (Y/S)^{dR} \times_{Y'} X'$$

is equivariant with respect to the map

$$\mathrm{Map}(L_{X/S}, F_* \mathbf{G}_a^\sharp[1]) \to \mathrm{Map}(f^* L_{Y/S}, F_* \mathbf{G}_a^\sharp[1])$$

of Y'-group stacks. This follows from Theorem II.25 and the definition of the action as in diagram (II.2.1).

Remark II.29. Note X' is a classical scheme when X/S is lci. This amounts to observing that for $f : A \to B = A/(f_1, \cdots, f_n)$, the A-modules $F_* A$ and B are tor-independent. See [Bha12, Lemma 3.41] for the argument.

Remark II.30. Let $f : X \to Y$ be a map of S-stacks. Recall $(X/Y)^{dR}$ is defined as $X^{dR} \times_{Y^{dR}} Y$. However, it is naturally isomorphic to $(X/S)^{dR} \times_{(Y/S)^{dR}} Y$. It is enough to observe that $(X/S)^{dR} \simeq X^{dR} \times_{Y^{dR}} (Y/S)^{dR}$ where the map $(Y/S)^{dR} \to Y^{dR}$ is the canonical projection. This follows since

$$X^{dR} \times_{Y^{dR}} (Y/S)^{dR} \simeq X^{dR} \times_{Y^{dR}} (Y^{dR} \times_{S^{dR}} S) \simeq X^{dR} \times_{S^{dR}} S.$$

Lemma II.31. *If $f : X \to Y$ is an étale surjection of Noetherian algebraic stacks with a quasi-affine diagonal, then $f^{dR} : X^{dR} \to Y^{dR}$ is an étale surjection.*

Proof. Fix $y \in Y^{dR}(R)$. Endow $W(R)/p$ with its induced V-adic filtration. Then

$$Y^{dR}(R) = \mathrm{Map}(\mathrm{Spec}(W(R)/p), X) \simeq \mathrm{Map}(\mathrm{Spf}(W(R)/p), Y)$$

by [BHL17, Corollary 1.5]. Consider $\bar{y} : \mathrm{Spec}(R/p) \to \mathrm{Spf}(W(R)/p) \to Y$ and choose an étale $\bar{g} : R/p \to \bar{S}$ such that $\bar{g}(\bar{y}) : \mathrm{Spec}(\bar{S}) \to Y$ comes from $\bar{x} \in X(S/p)$. Note $\bar{S}$-corresponds to a unique étale algebra $g : R \to S$ and under the equivalence $\mathrm{Spf}(W(R)/p)_{\acute{e}t} \simeq \mathrm{Spec}(R)_{\acute{e}t}$ sends g to $\mathrm{Spf}(W(S)/p) \to \mathrm{Spf}(W(R)/p)$ which completes the proof. □

Lemma II.32. *Let G be an affine group scheme over $\mathbf{F}_p$. Then $(B_{\acute{e}t}G)^{dR} \simeq B_{\acute{e}t}G^{dR}$. Moreover, this equivalence sends $\pi_{B_{\acute{e}t}G} : B_{\acute{e}t}G \to (B_{\acute{e}t}G)^{dR}$ to $B(\pi_G) : B_{\acute{e}t}G \to B_{\acute{e}t}G^{dR}$.*

Proof. Consider an étale covering $f : \mathrm{Spec}(\mathbf{F}_p) \to B_{\acute{e}t}G$. Then $f^{dR} : \mathrm{Spec}(\mathbf{F}_p) \to (B_{\acute{e}t}G)^{dR}$ is an étale covering by Theorem II.31. Its Cech nerve is obtained from the Cech nerve $G^\bullet$ of f by applying $(-)^{dR}$, since $(-)^{dR}$ preserves products. Thus, it is identified with $(G^{dR})^\bullet$ the Cech nerve of $\mathrm{Spec}(\mathbf{F}_p) \to B_{\acute{e}t}G^{dR}$. The map $B(\pi_G)$, by definition, is represented by a map of simplicial stacks $G^\bullet \xrightarrow{(\pi_G^{dR})^\bullet} (G^{dR})^\bullet$. To finish the proof, it is enough to observe that the map $B_{\acute{e}t}G \to (B_{\acute{e}t}G)^{dR} \simeq B_{\acute{e}t}G^{dR}$ is pointed and the induced map $G \to G^{dR}$ is π_G. □

Lemma II.33. *Let $X \to S$ be a representable quasi-syntomic map of algebraic stacks over $\mathbf{F}_p$. Then $(X/S)^{dR} \to X'$ is a quasi-syntomic torsor for $L^\vee_{X'/S} \otimes_{\mathbf{G}_a} \mathbf{G}_a^\sharp$*

Proof. It is enough to show local non-emptiness. For this, choose a smooth cover $f : T \to S$ where $T/\mathbf{F}_p$ is a scheme. Also, choose a scheme Y/S with a smooth cover $g : Y \to X_T := X \times_S T$ of S-schemes. By Theorem II.27 we can choose an $h : U \to Y$ quasi-syntomic cover such that $\nu_{U/S} : (U/S)^{dR} \to U'$ admits a section s. Define $U' \to (X/S)^{dR} \times_{X'} U'$ to be equal to the identity on the second component, and the first component is

$$U' \xrightarrow{s} (U/T)^{dR} \xrightarrow{h^{dR}} (Y/T)^{dR} \xrightarrow{g^{dR}} (X_T/T)^{dR} \simeq (X/S)^{dR} \times_S T \to (X/S)^{dR}.$$

□

II.2.2: Affineness

Lemma II.34. *For any scheme $X/\mathbf{F}_p$ the derived stack X^{dR} is relatively affine over X and, moreover, is isomorphic to $\underline{\mathrm{Spec}}_X(F_{X,*}\,\mathrm{dR}_X)$.*

Proof. First, let us observe that $X^{dR} \to X$ is affine. Note $\mathbf{G}_a^{dR}$ is affine by [LM24, Proposition 2.34]. Thus, it is true for $X = \mathbf{A}^n$ since affine stacks are closed under products. Thus, we know it for any affine X since affine stacks are closed under cosifted limits. To finish the proof, note that for an arbitrary X and open subscheme $U \xrightarrow{j} X$, the natural map $U^{dR} \xrightarrow{(j^{dR}, \nu_U)} X^{dR} \times_X U$ is an isomorphism.

Now, it is enough to show $\nu_{X,*}\mathcal{O}_{X^{dR}} \simeq F_{X,*}\,\mathrm{dR}_X$ for any X. By [Bha23, Corollary 2.7.2 (3)] or [Pet25, Proposition 3.6] we have this identification for any smooth X. This implies the statement for any affine X since both functors commute with limits in X. Finally, both functors satisfy Zariski descent and thus the statement holds for any X. □

Lemma II.35. *Let E be a vector bundle over X. The X-group stack $(\mathbf{V}(E)/X)^{dR}$ is naturally identified with the quotient* $\mathrm{Cone}(\mathbf{V}(E)^\sharp \to \mathbf{V}(E))$.

Proof. There is a natural map of X-group stacks $\pi_{\mathbf{V}(E)/X} : \mathbf{V}(E) \to (\mathbf{V}(E)/X)^{dR} = \mathbf{V}(E)^{dR} \times_{X^{dR}} X$ given by $(\pi_{\mathbf{V}(E),h}, \mathrm{pr})$ where $\pi_{\mathbf{V}(E)} : \mathbf{V}(E) \to \mathbf{V}(E)^{dR}$ and h is the natural homotopy $\pi_X \circ \mathrm{pr} \simeq \mathrm{pr}^{dR} \circ \pi_{\mathbf{V}(E)}$. It is enough to conclude that $\mathrm{fib}(\pi_{\mathbf{V}(E)/X}) \simeq \mathbf{V}(E)^\sharp$. For this we assume $X = \mathrm{Spec}(A)$ and M is the A-module corresponding to E. Then the map $\mathbf{V}(E)(R) \to (\mathbf{V}(E)/X)^{dR}(R)$ is identified with $M \otimes_A R \xrightarrow{\mathrm{Id} \otimes \pi_R} M \otimes_A \mathbf{G}_a^{dR}(R)$, the claim follows since $\mathbf{V}(E)^\sharp(R) = M \otimes_A \mathbf{G}_a^\sharp(R)$. □

Remark II.36. Let E be a vector bundle on S. Then $B_S(E^\sharp)$ is identified with the relative de Rham stack of $S \to B_S(E)$. Indeed, by definition, the relative de Rham stack is given by the fiber product of $S \simeq (S/S)^{dR} \to B_S(E)^{dR}$ and $B_S(E) \xrightarrow{\pi_{B_S E/S}} (B_S(E)/S)^{dR}$. Thus, it is isomorphic to $\mathrm{fib}(\pi_{B_S E/S}) \simeq B_S(E^\sharp)$ by Theorem II.35

Remark II.37. Let X be a smooth scheme with an action of a smooth group scheme G. Then $\nu_{X/G} : (X/G)^{dR} \to X/G$ is affine by Theorem II.11. Indeed, $T_{X/G}$ is represented by a G-equivariant complex $g \boxtimes \mathcal{O}_X \to T_X$ in $\mathrm{Perf}(X)$, where T_X is placed in degree 0.

Remark II.38. The derived stack X^{dR} is not classical in general, see [BL22b, Warning 7.4]. By Theorem II.23 one sees that the derived stack X^{dR} is classical for lci X. It also follows from [BL22b, Corollary 8.13].

Lemma II.39. *For a map of schemes $X \to S$, the map $\nu_{X/S} : (X/S)^{dR} \to X'$ from Theorem II.27 is an affine map and exhibits the source as* $\underline{\mathrm{Spec}}_{X'}(F_{X/S,*}\,\mathrm{dR}_{X/S})$.

Proof. Step 1. Reduction to the case of affine S. It is enough to check that $(X/S)^{dR} \to X'$ is affine after changing the base along $X' \times_S V \to X'$ for any affine open $V \subset S$. One computes $(X/S)^{dR} \times_{X'} (X' \times_S V) \simeq (X/S)^{dR} \times_S V \simeq (X \times_S V/V)^{dR}$. Thus, we can assume that S is affine.

Step 2. Reduction to the case of affine X. For an open immersion $U \to X$ the map $U^{dR} \to X^{dR}$ is a representable open immersion by [Bha23, Theorem 2.5.6] and one gets that the diagram

$$\begin{array}{ccc} U & \longrightarrow & U^{dR} \\ \downarrow & & \downarrow \\ X & \longrightarrow & X^{dR} \end{array}$$

is cartesian. To prove surjectivity $X \to (X/S)^{dR}$ it is enough to do this Zariski locally on $(X/S)^{dR}$, for an open immersion $U \to X$ we get the following diagram

$$\begin{array}{ccccc} U & \longrightarrow & (U/S)^{dR} & \longrightarrow & U^{dR} \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & (X/S)^{dR} & \longrightarrow & X^{dR} \\ & & \downarrow & & \downarrow \\ & & S & \longrightarrow & S^{dR} \end{array}$$

where the vertical upper right arrow is an open immersion which implies that $(U/S)^{dR} \to (X/S)^{dR}$ is also an open immersion. Moreover, right two squares are cartesian which implies that the left upper square is also cartesian. Moreover, if X is written as a colimit of a diagram $U^\bullet$ consisting of open affine subschemes of X, then the colimit of $(U/S)^{\bullet,dR}$ is $(X/S)^{dR}$, it also follows from the proof of [Bha23, Theorem 2.5.6] since the analogous statement holds for X^{dR} combining with the fact that the upper right square is cartesian. Thus, we can assume that X is affine. In this case $(X/S)^{dR} = X^{dR} \times_{S^{dR}} S$ is an affine stack since all terms are affine.

Step 3. Computing $\nu_* \mathcal{O}_{(X/S)^{dR}}$. The argument is similar to the one in Theorem II.34. By Zariski descent it is enough to prove it for affine X, S and, moreover, it is enough to assume that the map $X \to S$ is smooth. In that case we refer to [Bha23, Corollary 2.7.1. (3)].

Remark II.40. Let $X \to S$ be a representable quasi-syntomic map of $\mathbf{F}_p$-algebraic stacks. Then $\nu_{X/S} : (X/S)^{dR} \to X'$ is an affine map. Indeed, by Theorem II.33 it is a pullback of $X' \to B_{X'}(T^\sharp_{X'/S})$ which is affine by Theorem II.11. Moreover, $\nu_{X/S,*} \mathcal{O}_{(X/S)^{dR}}$ is identified with $F_{X/S,*}\, \mathrm{dR}_{X/S}$ by the computation for smooth schemes.

Remark II.41. Note that $(X/S)^{dR}$ is classical for lci X/S since it admits a structure of a $L^{\vee}_{X'/S}[1] \otimes_{\mathbf{G}_a} \mathbf{G}_a^{\sharp}$-torsor over X'. Indeed, fppf locally on X' one can split $\nu_{X/S}$ by Theorem II.27. Since $L^{\vee}_{X'/S}[1] \otimes_{\mathbf{G}_a} \mathbf{G}_a^{\sharp}$ is classical, the statement follows.

□

Lemma II.42. *Let $i : Z \to X$ be a regular closed immersion. Then $(Z/X)^{dR}$ is identified with the PD-envelope $D_Z(X)$.*

Proof. First, assume that $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(A/I)$ where I is generated by a regular sequence $(f_1, \cdots, f_n)$. Then [Bha12, Corollary 3.40] provides a functorial isomorphism of $\mathbb{E}_\infty - A/\phi(I)$-algebras $\mathrm{dR}_{(A/I)/A} \simeq D_I(A)$. By Zariski descent and Theorem II.39 it gives rise to $(Z/X)^{dR} \simeq D_Z(X)$. □

Remark II.43. Given an lci map $X \to S$, then Theorem II.27 provides us with a torsor structure on the relative de Rham stack. In particular, one can take $X \to S$ to be a regular closed immersion. In this case, by Theorem II.42 the relative de Rham is identified with the PD-envelope $D_X(S)$. The twist X' is identified with the Frobenius neighbourhood of X in S which we denote by X^{ϕ}. The group stack $BT^{\sharp}_{X'/S}$ is identified with the PD-envelope of the zero section in the normal bundle $\mathbf{N}^{\sharp}_{X^{\phi}/S}$. Indeed, the tangent complex of a regular immersion is given by $(I/I^2)^{\vee}[-1]$ where I is the ideal of $X \to S$. Therefore, one obtains an action of $\mathbf{N}^{\sharp}_{X^{\phi}/S}$ on $D_X(S) \to X^{\phi}$. Let us compare this action with the one coming from D-module theory which was discovered by Ogus and explained to us by Vologodsky. Denote $f : D_X(S) \to X^{\phi}$. Then $f_*\mathcal{O}_{D_X(S)}$ is naturally a D-module on X^{ϕ}. Then the desired action is given by the integration of the p-curvature of $f_*\mathcal{O}_{D_X(S)}$. More explicitly, to define an action of $\mathbf{N}^{\sharp}_{X^{\phi}/S}$ on $D_X(S) \to X^{\phi}$, it is enough to define an action of the Lie algebra of $\mathbb{N}_{X^{\phi}/S}$. In local coordinates the action is given by $\partial \cdot \gamma_n(x) = c'(\partial)\gamma_n(x)$ where $c' : F^*_{X^{\phi}} T_{X^{\phi'}} \to D_{X^{\phi}}$ is the p-curvature map. It is easy to see that it factors through the quotient $T_{X^{\phi}} \to N_{X^{\phi}/S}$.

Lemma II.44. *For a regular immersion $X \to S$ of $\mathbf{F}_p$-schemes, the two actions of $\mathbb{N}^{\sharp}_{X^{\phi}/S}$ on $D_X(S)$ constructed in Theorem II.43 coincide.*

Proof. The statement is local and we can assume that $X \to S$ comes from a surjection $A \to A/I$ with the regular $I = (f_1, f_2, \cdots, f_n)$. In this case

$$A/I^p = A/f_1^p \otimes_A \cdots \otimes_A A/f_n^p \to D_I(A) = \otimes_i D_{f_i} A$$

is a torsor for $\mathbb{N}_{A/I^p}(I/I^2)^{\vee}$ and it is a product of $\mathbb{N}_{A/f_i^p}(f_i/f_i^2)^{\vee}$-torsors. Thus, it is enough to prove it in the case of $I = (f)$. Consider the map $\mathbf{F}_p[x] \xrightarrow{x \to f} A$, then $A/f \simeq A \otimes_{\mathbf{F}_p[x]} \mathbf{F}_p$

and by the base change we can assume that $A = \mathbf{F}_p[x]$ and $f = x$. Thus, we are reduced to prove it for $X = \mathrm{Spec}(\mathbf{F}_p) \to \mathrm{Spec}(\mathbf{F}_p[x]) = S$. In this case the map $(X/S)^{dR} \to X'$ is identified with the canonical map $\mathbf{G}_a^\sharp \to \alpha_p$. Indeed,

$$(\mathrm{Spec}(\mathbf{F}_p)/\mathbf{G}_a^{dR}) = \mathrm{Spec}(\mathbf{F}_p) \times_{\mathbf{G}_a} \mathbf{G}_a^{dR} \simeq \mathrm{fib}(\mathbf{G}_a \to \mathbf{G}_a^{dR}) \simeq \mathbf{G}_a^\sharp$$

and $X' = \mathrm{Spec}(\mathbf{F}_p) \times_{\mathbf{G}_a, F_{\mathbf{G}_a}} \mathbf{G}_a \simeq \mathrm{fib}(\mathbf{G}_a \xrightarrow{F_{\mathbf{G}_a}} \mathbf{G}_a) \simeq \alpha_p$. Recall that the action of $\mathrm{Hom}(L_{X'/S}, \mathbf{G}_a^\sharp[1])$ comes from the diagram (II.2.1). Note that in our case $\mathrm{Hom}(L_X, \mathbf{G}_a^\sharp[1]) = \mathrm{Spec}(\mathbf{F}_p)$ as a stack since $L_X = 0$. In particular, we learn that $\mathrm{Hom}(L_{X'/S}, \mathbf{G}_a^\sharp[1]) \simeq F_* \mathbf{G}_a^\sharp$. Unwinding the definitions, the action is given by $F_* \mathbf{G}_a^\sharp \times \mathbf{G}_a^\sharp \to \mathbf{G}_a^\sharp$ sending (t, x) to $V(t) + x$. In particular, if $R = \mathbf{F}_p[\varepsilon]$, then for any $x \in \mathbf{F}_p[\varepsilon]^\sharp$ one has $\gamma_n(\varepsilon \cdot x) = \gamma_{n-p}(x)\varepsilon$ for $n \geq p$ and 0 otherwise. It is easy to see the action coming from p-curvature is given by the same formula since $\partial_x^p \gamma_n(x) = \gamma_{n-p}(x)$ where x is a coordinate on $\mathbf{G}_a$ and ∂_x is the associated vector field. □

Remark II.45. Another case where the action of $B_{X'}(T^\sharp_{X'/S})$ on $(X/S)^{dR}$ can be explicitly understood is when $X \to S$ is smooth. Denote $a : B_{X'}(T^\sharp_{X'/S}) \times_{X'} (X/S)^{dR} \to (X/S)^{dR}$ to be the action morphism. Note $(X/S)^{dR} \times_{X'} B_{X'}(T^\sharp_{X'/S}) \simeq B_{(X/S)^{dR}}(F^*_{X/S} T^\sharp_{X'/S})$ and by Cartier duality the derived category of quasi-coherent sheaves on it is identified with the ∞-category of pairs (M, θ) where $M \in \mathcal{D}_{qc}((X/S)^{dR})$ and $\theta : M \to M \otimes_{\mathcal{O}_X} F^*_{X/S} \Omega_{X'/S}$ is a map in $\mathcal{D}_{qc}((X/S)^{dR})$ such that θ acts locally nilpotently on $\mathcal{H}^i(M)$ for any i. We claim that $a^*(M) = (M \xrightarrow{\theta_M} M \otimes F^*_{X/S} \Omega^1_{X'/S})$ where θ_M is the p-curvature map of M. Equivalently, it is map $F^*_{X/S} T_{X'/S} \to \underline{\mathrm{End}}(M)$ and for a local section ∂ the endomorphism $\theta_M(\partial)$ is multiplication by the corresponding central element of $F_{X/S,*} D_{X/S}$. The statement is local and by standard reductions we can assume that $X = \mathbf{A}^1$. In this case the computation is the same as in Theorem II.44. Indeed, the pullback along $\mathbf{G}_a^{dR} \to B_{\mathbf{G}_a^{dR}}(F_* \mathbf{G}_a^\sharp)$ takes (M, θ_M) to M. In particular, the functor a^* does not change the underlying crystal. Explicitly, a is given by $\mathrm{Cech}(\mathbf{G}_a \to B_{\mathbf{G}_a^{dR}}(\pi^* F_* \mathbf{G}_a^\sharp)) \to \mathrm{Cech}(\mathbf{G}_a \to \mathbf{G}_a^{dR})$. In particular, we have a map $a_1 : \mathbf{G}_a \times \mathbf{G}_a^\sharp \times F_* \mathbf{G}_a^\sharp \to \mathbf{G}_a \times \mathbf{G}_a^\sharp$ given by $a_1(x, a, b) = (x, a + V(b))$. Note that the map induced by $V : F_* \mathbf{G}_a^\sharp \to \mathbf{G}_a^\sharp$ on Lie algebras takes the standard derivation ∂_x to $\partial_x^{[p]}$. Thus, the nilpotent operator θ_M on $a^* M$ is given by the action of the central element ∂^p.

In [OV07, Remark 1.8], it is explained how to compute the p-curvature under the inverse image functor. In the next remark, we note that this can be understood via the interpretation of the p-curvature from Theorem II.45 in terms of the torsor action, together with the fact that any S-map $X \to Y$ induces a map $(X/S)^{dR} \to (Y/S)^{dR}$ which is equivariant with respect to this action; see Theorem II.28.

Remark II.46. Let $h : X \to Y$ be a map of smooth S-schemes. Consider the map

$$H : (X/S)^{dR} \times_{X'} B_{X'}(T^\sharp_{X'/S}) \to (Y/S)^{dR} \times_{Y'} B_{Y'}(T^\sharp_{Y'/S})$$

over $h' : X' \to Y'$. The derived category of quasi-coherent sheaves on the target is equivalent to the ∞-category of pairs (M, ψ_M) where $M \in \mathcal{D}_{qc}((Y/S)^{dR})$ and

$$\psi_M : M \to M \otimes F^*_{Y/S}\Omega^1_{Y'/S}$$

is a horizontal map. Then H^* sends $M \xrightarrow{\psi_M} M \otimes F^*_{Y/S}\Omega^1_{Y'/S}$ to the composition

$$h^*M \xrightarrow{h^*\psi_M} h^*M \otimes h^*F^*_{Y/S}\Omega^1_{Y'/S} \to h^*M \otimes F^*_{X/S}\Omega^1_{X'/S} \tag{II.2.2}$$

where the last map is induced by $h^*F^*_{Y/S}\Omega^1_{Y'/S} \simeq F^*_{X/S}h'^*\Omega^1_{Y'/S} \xrightarrow{dh'} F^*_{X/S}\Omega^1_{X'/S}$. Thus, for $E \in \mathrm{MIC}^{\cdot}(X/S)$ the p-curvature is given by (II.2.2). Indeed, the diagram

$$\begin{array}{ccc} (X/S)^{dR} \times B_{X'}(T^\sharp_{X'/S}) & \xrightarrow{a_{X/S}} & (X/S)^{dR} \\ \downarrow{\scriptstyle h^{dR}\times dh} & & \downarrow{\scriptstyle h^{dR}} \\ (Y/S)^{dR} \times_{Y'} B_{X'}(h^*T^\sharp_{Y'/S}) & \xrightarrow[a_{Y/S}]{} & (Y/S)^{dR} \end{array}$$

is commutative by Theorem II.28, and the pullback along the action maps sends a D-module to its p-curvature by Theorem II.45.

Corollary II.47. *Let $(E, \nabla) \in \mathrm{MIC}^{\cdot}(X/S)$ for a representable smooth map X/S of algebraic $\mathbf{F}_p$-stacks. Then the Frobenius pullback of the complex $F_{X/S,*}\,\mathrm{dR}(E, \nabla) \in \mathcal{D}_{qc}(X')$ is isomorphic to*

$$E \xrightarrow{\psi_E} E \otimes F_{X/S}\Omega^1_{X'/S} \xrightarrow{\psi_E} E \otimes F_{X/S}\Omega^2_{X'/S} \to \cdots \tag{II.2.3}$$

where ψ_E is the p-curvature operator. Indeed, recall from Theorem II.45 that the pullback of $(E, \nabla) \in \mathrm{MIC}^{\cdot}(X/S)$ under the action map

$$a : B_{X'}(T^\sharp_{X'/S}) \times_{X'} (X/S)^{dR} \to (X/S)^{dR}$$

is a pair (E, ψ_E) where $\psi_E : E \to E \otimes F^*_{X/S}\Omega^1_{X'/S}$ is a map of crystals given by the p-curvature. Pulling back further to $B_{X'}(T^\sharp_{X'/S}) \times_{X'} X$ and pushing forward to X gives the complex (II.2.3). On the other hand, by the base change this complex is isomorphic to

$F^*_{X/S}\nu_{X/S,*}(E,\nabla)$ *where* $\nu_{X/S}:(X/S)^{dR}\to X'$*, it completes the proof.*

Remark II.48. Let $\pi: X\to S$ be smooth. Consider the multiplication

$$m: B_X(T^\sharp_{X/S})\times_X B_X(T^\sharp_{X/S})\to B_X(T^\sharp_{X/S}).$$

Note the left-hand side is the classifying stack of $\pi^*T^\sharp_{X/S}$ on $B_X(T^\sharp_{X/S})$ where $\pi: B_X(T^\sharp_{X/S})\to X$ is the projection. Therefore, its category of quasicoherent sheaves is the ∞-category of pairs (E,θ_E) where $E\in \mathrm{HIG}^{\cdot}(X/S)$ and $\theta_E: E\to E\otimes\Omega^1_{X/S}$ is a nilpotent map of Higgs modules. Analogously to Theorem II.45, we learn that m^* takes a Higgs module (E,θ_E) to the same pair (E,θ_E) where θ_E is emphasized to be a map of Higgs modules and not merely an $\mathcal{O}_X$-linear map.

II.2.3: Splittings of the relative de Rham gerbe

It was observed by Bhatt in [Bha12] that a compatible lift of Frobenii on X and S gives rise to the formality of the de Rham complex which gives rise to a section of $\nu_{X/S}: X'\to (X/S)^{dR}$.

Remark II.49. Let X/S be smooth. If we are given $\tilde{X},\tilde{S}$ with lifts of F_X and F_S, then [Bha12, Proposition 3.17] provides an isomorphism $F_{X/S,*}\,\mathrm{dR}_{X/S}\simeq(\bigoplus\Omega^i_{X'/S},0)$ of $\mathbb{E}_\infty$-algebras in $\mathcal{D}_{qc}(X')$. The proof uses a lift of $F_{X/S}$ to $\tilde{X}\to\tilde{X}\times_{\tilde{S},\tilde{F}_S}\tilde{S}$ to construct a map $L_{X'/S}[-1]\to F_{X/S,*}\,\mathrm{dR}_{X/S}$ which splits the conjugate filtration in degree 1. Let us note that the proof only uses a lift of F_S through a lift of $F_{X/S}$; i.e., it is enough to have a lift $\tilde{S}$ with flat lifts $\tilde{X},\tilde{X}'$ and a lift $\tilde{F}_{X/S}:\tilde{X}\to\tilde{X}'$. In the next lemma, we explain how to split the de Rham gerbe in the absence of a lift of S.

If X/S is a family with a lift $\tilde{X}/W_2(S)$, then the correct replacement of a lift of the relative Frobenius is given in the following definition.

Definition II.50. Let $X\to S$ be a map over $\mathbf{F}_p$. Let $\tilde{X}/W_2(S)$ be a lift. A strong Frobenius lift is a $W_2(S)$-map $W_2(X)\to\tilde{X}$ which reduces to the identity map on X, via the natural embeddings $X\hookrightarrow W_2(X)$ and $X\hookrightarrow\tilde{X}$.

Remark II.51. Let X/S be a family admitting a lift $\widetilde{X}/W_2(S)$. Using the Frobenius on $W_2(S)$ we define

$$\widetilde{X}':=\widetilde{X}\times_{W_2(S),F_{W_2(S)}}W_2(S).$$

One can then consider lifts of the relative Frobenius $F_{X/S}\colon X\to X'$ to a $W_2(S)$-morphism $\widetilde{F}\colon\widetilde{X}\to\widetilde{X}'$.

We warn the reader that this is weaker than a strong Frobenius lift. Namely, for a local section a of $\mathcal{O}_{\widetilde{X}}$, an arbitrary Frobenius lift satisfies $\widetilde{F}(a) = a^p + i_a$ with $i_a \in \mathrm{Ker}(\mathcal{O}_{\widetilde{X}} \to \mathcal{O}_X)$, whereas a strong Frobenius lift requires $\widetilde{F}(a) = a^p + p\delta_a$ with $\delta_a \in \mathcal{O}_{X'}$.

Lemma II.52. *Let $R \to A$ be a map of commutative rings. Let $W_2(R) \to \tilde{A}$ a be flat lift of $R \to A$. Assume we are given a strong Frobenius lift, i.e., a map $f : \tilde{A} \to W_2(A)$ such that $\tilde{A} \xrightarrow{f} W_2(A) \to A$ is the natural map $\tilde{A} \to A$. Then f gives rise to an isomorphism $\mathrm{dR}_{A/R} \simeq \bigoplus_k \Lambda^k L_{A'/R}[-k]$ of derived algebras.*

Proof. The proof is essentially given in [Bha12, Proposition 3.17]. Choose a free simplicial $W_2(A)$-algebra resolution $\tilde{P}_\bullet \to \tilde{B}$, then we get $\tilde{h} : \tilde{P}_\bullet \to W_2(P_\bullet)$ and therefore $\tilde{F} : \tilde{P}_\bullet \to \tilde{P}_\bullet$ as well as its linearized version $\tilde{F}_{\tilde{P}/W_2(A)} : \tilde{P'}_\bullet \to \tilde{P}_\bullet$. Note $\tilde{F}(x)$ can be canonically written as $x^p + p\delta_x$ for some $\delta_x \in P_\bullet$.[2] This gives rise to a well-defined map

$$\frac{\Omega^1(\tilde{F})}{p} : \Omega^1_{\tilde{P'}_\bullet/W_2(A)} \to \Omega^1_{\tilde{P}_\bullet/W_2(A)}.$$

Analogously, one obtains

$$\frac{\Omega^k(\tilde{F})}{p} : \Omega^k_{\tilde{P'}_\bullet/W_2(A)} \to \Omega^k_{\tilde{P}_\bullet/W_2(A)}.$$

These maps assemble into a map of bicomplexes $\Omega^1_{P'_\bullet/A}[-1] \to \Omega^\bullet_{P_\bullet/A}$. This totalizes to a map $L_{A'/R}[-1] \to \mathrm{dR}_{A/R}$ splitting the first term of the conjugate filtration. The standard argument shows it gives rise to the desired isomorphism. □

Corollary II.53. *Let $X \to S$ be a quasi-syntomic map of $\mathbf{F}_p$-schemes. Let $\tilde{X}/W_2(S)$ be a flat lift of X and let $f : \tilde{X} \to W_2(X)$ be a strong Frobenius lift. Then f gives rise to a splitting $s_f : X' \to (X/S)^{dR}$ of the de Rham torsor.*

Let $s : X' \to (X/S)^{dR}$ be a section of the de Rham torsor as in Theorem II.53. Although when $S = \mathrm{Spec}(\mathbf{F}_p)$ one obtains a canonical homotopy $s \circ F_{X/S} \simeq \pi_{X/S}$, it is not clear whether the same holds for arbitrary S. In the next remark, we observe that in this particular case it does hold.

Remark II.54. Let X/S be a smooth map over $\mathbf{F}_p$. Let $C : (\Omega^\bullet_{X'/S}, 0) \simeq \mathrm{dR}_{X/S}$ be the chain quasi-isomorphism from Theorem II.52. The splitting $s_f : X' \to (X/S)^{dR}$ is given by the composition

$$\mathrm{dR}_{X/S} \xrightarrow{C^{-1}} (\Omega^\bullet_{X'/S}, 0) \to \mathcal{O}_{X'}$$

[2] It is wrong for an arbitrary Frobenius lift to $W_2(R)$. It is crucial to have a map $\tilde{P}_\bullet \to W_2(P_\bullet)$.

where the second map is the projection onto the first component. Note one has a canonical homotopy $\pi_{X/S} \simeq s \circ F_{X/S}$. Indeed, it amounts to observe that the chain map C is given by the Frobenius $\mathcal{O}_{X'} \to F_{X/S,*}\mathcal{O}_X$ in degree 0.

Generalizing Theorem II.26, we obtain a relative version.

Lemma II.55. *Let X/S be a smooth family. Let $\tilde{X}/W(S)$ be a δ-lift. It gives rise to a section $X' \to (X/S)^{dR}$.*

Proof. Let us define a map $s : \tilde{X}' := \tilde{X} \times_{W(S),F_{W(S)}} W(S) \to \tilde{X}$. Namely, we define

$$\tilde{X}'(R) = \mathrm{Map}_{W(S)}(\mathrm{Spec}(F_*R), \tilde{X}) \to \mathrm{Map}_{W(S)}(\mathrm{Spec}(\mathbf{G}_a^{dR}(R), \tilde{X}) = (\tilde{X}/W(S))^{dR}(R)$$

as follows. Let $x : \mathrm{Spec}(F_*R) \to \tilde{X}$. Then δ-structure on $\tilde{X}$ gives rise to a map $\mathrm{Spec}(F_*W(R)) \to \tilde{X}$ and the composition $\mathrm{Spec}(W(F_*R)/p) \hookrightarrow \mathrm{Spec}(W(F_*R)) \to \tilde{X}$ uniquely factors through some map $\mathrm{Spec}(W(F_*R)/p) \to X$ since $\mathrm{Spec}(W(F_*(R))/p)$ has a natural S-scheme structure. □

Remark II.56. In Theorem II.55 we observed that a δ-lift $\tilde{X}/W(S)$ gives rise to a map $s : \tilde{X}' \to (\tilde{X}/W(S))^{dR}$. Let us note that $s \circ \tilde{F} \simeq \pi_{\tilde{X}/W(S)}$ where $\tilde{F} : \tilde{X} \to \tilde{X}'$ is the Frobenius defined by the δ-structure. Indeed, if $S = \mathrm{Spec}(R)$ and $\tilde{X} = \mathrm{Spec}(\tilde{A})$, then this point $\pi_{X/S} \in (\tilde{X}/W(S))^{dR}$ is given by a map $\tilde{A} \to \mathbf{G}_a^{dR}(\tilde{A})$ which is isomorphic to the map

$$\tilde{A} \xrightarrow{w_{\tilde{A}}} W(\tilde{A}) \xrightarrow{F_{W(\tilde{A})}} W(\tilde{A}) \to W(\tilde{A})/p. \tag{II.2.4}$$

Indeed, $w_{\tilde{A}}(a) = [a] + Vx$ and $F_{W(\tilde{A})}([a] + Vx) = [a^p] + px$. Then $s : \tilde{X}' \to (\tilde{X}/W(S))^{dR}$ sends a point $\tilde{F} : \tilde{A} \to \tilde{A}$ to the composition

$$\tilde{A} \xrightarrow{\tilde{F}} \tilde{A} \xrightarrow{w_{\tilde{A}}} W(\tilde{A}) \to W(\tilde{A})/p$$

which is naturally isomorphic to (II.2.4). Indeed, the naturality of $w_{\tilde{A}}$ gives $w_{\tilde{A}} \circ W(\tilde{F}) = \tilde{F} \circ w_{\tilde{A}}$ and it can be seen that the left-hand side is equal to $w_{\tilde{A}} \circ F_{W(\tilde{A})}$ which can be checked on ghost coordinates.

Lemma II.57. *Let $\tilde{X}/W(S)$ be a δ-lift of smooth X/S with $\tilde{F} : \tilde{X} \to \tilde{X}'$ the corresponding Frobenius lift. Let s_1, s_2 be the maps $\tilde{X}' \to (\tilde{X}/W(S))^{dR}$ that reduce to sections of the de Rham stack after the base change along $S \to W(S)$. Assume there exist isomorphisms $h_i : s_i \circ \tilde{F} \simeq \pi_{\tilde{X}/W(S)}$ for $i = 1, 2$. If S is reduced, then there exists $s_1 \simeq s_2$.*

Proof. Note h_i realizes s_i as a map of simplicial objects

$$\mathrm{Cech}(\tilde{X} \to \tilde{X}') \to \mathrm{Cech}(\tilde{X} \to (\tilde{X}/W(S))^{dR})$$

where the map on 1-simplices is $D^{\phi}(\tilde{X} \times_{W(S)} \tilde{X}) \to D_{\Delta}(\tilde{X} \times_{W(S)} \tilde{X})$ splitting the natural projection. Such a map is equivalent to endowing $D^{\phi}(\tilde{X} \times_{W(S)} \tilde{X})$ with a PD-structure on the ideal of the diagonal. Since it is flat over $W(S)$, it is also flat over $\mathbf{Z}_p$ because S is reduced. Thus, this PD-structure is unique. □

Lemma II.58. *Let $\tilde{X}/W(S)$ be a δ-scheme. There is a natural map $\tilde{X}' \to (\tilde{X}/W(S))^{dR}$ deforming the section from Theorem II.53.*

Proof. Step 1. Consider the category of maps $(Y, W(T), \pi_{Y,T})$ where $\pi_{Y,T} : Y \to W(T)$ is a flat map over the formal scheme $W(T)$, i.e., $\pi_{Y,T}$ is equivalent to a set of compatible flat maps $\pi^n_{Y,T} : Y_n \to W_n(T)$. Formally, it is defined as the fiber product of

$$\mathrm{Fun}^{fl}(\Delta^1, \mathrm{FSch}_{\mathbf{Z}_p}) \xrightarrow{t} \mathrm{FSch}_{\mathbf{Z}_p} \xleftarrow{W} \mathrm{Sch}_{\mathbf{F}_p}$$

where $\mathrm{Fun}^{fl}(\Delta, \mathrm{FSch})$ is the full subcategory of $\mathrm{Fun}(\Delta, \mathrm{FSch})$ consisting of objects $X \to Y$ where the map is flat. Denote this category to be $\mathcal{C}$. The relative de Rham stack restricts to a functor $(-/-)^{dR} : \mathcal{C} \to \mathrm{dSt}_{\mathbf{Z}_p}$. Also, there is a functor $T : \mathcal{C} \to \mathrm{Sch}_{\mathbf{Z}_p}$ sending $Y \to W(T)$ to $Y' := Y \times_{W(T), F_{W(T)}} W(T)$. We want to construct a natural transformation $T \to (-/-)^{dR}$. Both functors are Zariski sheaves. Let $\mathcal{C}_0$ be a full subcategory of $\mathcal{C}$ where T and Y are affine formal schemes. Now, if F, G are Zariski sheaves on $\mathcal{C}$, then any natural transformations of their restriction to $\mathcal{C}_0$ uniquely extends to a natural transformation between F and G. Thus, we restrict ourselves to $\mathcal{C}_0$.

Step 2. Assume $Y \to W(T)$ is an object of $\mathcal{C}_0$ with $T = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(A)$. We construct $Y' \to (Y/W(T))^{dR}$ in two steps. First, we use the Deligne-Illusie chain map $(\Omega^{\bullet}_{A/R}, pd) \to (\Omega^{\bullet}_{A/R}, d)$ which is in degree n given as follows: the map $F^* : \Omega^n_{A'/W(R)} \to \Omega^n_{A/W(R)}$ is divisible by p^n and thus defines

$$F^*/p^n : \Omega^n_{A'/W(R)} \to \Omega^n_{A/W(R)}$$

and by construction it gives rise to a map of complexes

$$C_{A/W(R)} : (\Omega^{\bullet}_{A/W(R)}, pd) \to (\Omega^{\bullet}_{A/W(R)}, d)$$

which is a quasi-isomorphism. The chain map $C_{A/W(R)}$ is natural in pairs $W(R) \to A$ with A being a $\delta - W(R)$-algebra. The second step is to consider the composition

$$F^0 \circ C^{-1}_{A/W(R)} : (\Omega^{\bullet}_{A/W(R)}, d) \to (\Omega^{\bullet}_{A'/W(R)}, pd) \to A'$$

which is natural as a composition of two natural transformations. This gives rise to the

desired section $X' \to (X/S)^{dR}$. □

Remark II.59. One can observe that in the second step of the proof Theorem II.58 we produce, for a flat $\delta - W(S)$-scheme $\tilde{X}$, an isomorphism of stacks $(\tilde{X}/W(S))^{dR} \simeq (\tilde{X}/W(S))^{dR,p}$ where the latter is defined as the fiber of the Hodge-filtered de Rham stack

$$(\tilde{X}/W(S))^{dR,+} \to \mathbf{A}^1/\mathbf{G}_m$$

at $t = p$. To complete the argument, one has to check that for affine S and $\tilde{X}$ the stack $(\tilde{X}/W(S))^{dR,p}$ is affine and represented by $(\Omega^{\bullet}_{\tilde{X}/W(S)}, pd)$. Since we will not need it, we do not include the details. We note however that the category of quasi-coherent sheaves on $(\tilde{X}/W(S))^{dR,p}$ is identified with the category of (topologically quasi-nilpotent) p-connections. In particular, when $S = \mathrm{Spec}(\mathbf{Z}_p)$, this isomorphism of stacks is closely related to the work of Shiho on the deformation of the local Ogus-Vologodsky equivalence to $\mathbf{Z}/p^n$, see [Shi15].

Corollary II.60. *Let X/S be a smooth family and $\tilde{X}/W(S)$ its δ-lift. The two maps $s_1, s_2 : \tilde{X}' \to (\tilde{X}/W(S))^{dR}$ obtained from Theorem II.55 and Theorem II.58 are isomorphic as points of $(\tilde{X}/W(S))^{dR}(\tilde{X}')$. Indeed, they are both equal to $\pi_{\tilde{X}/W(S)}$ after pre-composing with $\tilde{F}$ and Theorem II.57 concludes the statement.*

II.2.4: Quasi-coherent sheaves on de Rham stack

In this subsection, we recall that for smooth $X/\mathbf{F}_p$ the category $\mathcal{D}_{qc}(X^{dR})$ is identified with the ∞-category $\mathrm{Crys}(X/\mathbf{F}_p)$ of crystals of quasi-coherent complexes on the p-torsion crystalline site $(X/\mathbf{F}_p)_{\mathrm{crys}}$. For example, this follows from the more general result [BL22b, Theorem 6.5] by taking the crystalline prism. For the reader's convenience, we give a (much less general) more elementary proof, which works in the case of a de Rham stack. We also explain that this equivalence holds for an lci X/S.

Remark II.61. Let $X \to S$ be a smooth map. Then by Theorem II.22 we get an fpqc covering $\pi_{X/S} : X \to (X/S)^{dR}$. Moreover, by Theorem II.19 one identifies n-th term of the Cech nerve $X^{\times_{(X/S)^{dR}} n}$ with $(X/X^{\times_S n})^{dR}$. Moreover, by Theorem II.42 the stack $(X/X^{\times_S n})^{dR}$ is represented by the PD-envelope $D_{I_\Delta}(X^{\times_S n})$ of the diagonal map $X \to X^{\times_S n}$. In particular, one gets $D_{qc}((X/S)^{dR}) \simeq \mathrm{Crys}(X/S)$ since $\mathrm{Cech}(X \to (X/S)^{dR})$ is identified with the simplicial scheme $D^{\bullet/S}_{I_\Delta}$.

Theorem II.62. *Let $X \to S$ be an lci map. Then $D_{qc}((X/S)^{dR}) \simeq \mathrm{Crys}(X/S)$.*

Proof. Recall from [BO78b, Th. 6.6] that for any closed immersion $Y \to Z$ with smooth Z/S, the category $\mathrm{Crys}(Y/S)$ is identified with the category of modules over $D_Y(Z)$ with an HPD-stratification compatible with the natural stratification on $D_Y(Z)$. Moreover, the latter category is identified with the category of sheaves on the Cech nerve of the covering $D_Y(Z) \to Y$. Both $D_{qc}((-/-)^{dR})$ and $\mathrm{Crys}(-/-)$ are functors $\mathrm{Fun}(\Delta^1, \mathrm{Sch}^{op}) \to \mathrm{Pr}^L_{st}$. Moreover, they both satisfy Zariski descent and it is enough to prove that their restrictions to $\mathrm{Fun}(\Delta^1, \mathrm{Ring})$ are isomorphic. Assuming $X = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(A)$ we can use the canonical projection $A[x_b | b \in B] := P \to B$ to identify $\mathrm{Crys}(B/A) = D_{qc}(\mathrm{Cech}(D \to B))$ where $D := D_P(J)$ for $J = \ker(P \to B)$ and the Cech nerve is computed in $(B/A)_{crys}$, i.e. its n-term is $D(n) := D_{P \otimes_A \cdots \otimes_A P}(J(n))$ for $J(n) := \ker(P \otimes_A \cdots \otimes_A P \to B)$ and both tensor products contain $n+1$ term. Note that $A[B]$ is functorial in B, therefore $\mathrm{Crys}(B/A)$ is functorially identified with $D_{qc}(\mathrm{Cech}(D \to B))$. Let us now check that $D_{qc}((B/A)^{dR})$ is also functorially identified with $D_{qc}(\mathrm{Cech}(D \to B))$. We have a commutative diagram

$$\begin{array}{ccc} D \simeq (B/P)^{dR} & \longrightarrow & (B/A)^{dR} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(P) & \longrightarrow & (P/A)^{dR}. \end{array}$$

Note that the bottom horizontal arrow is a faithfully flat cover by Theorem II.22 since $A \to P$ is smooth, therefore $(B/P)^{dR} \to (B/A)^{dR}$ is a faithfully flat covering. Identify $(B/P)^{dR} \simeq D$ by Theorem II.42. Moreover,

$$\begin{aligned} &(B/P)^{dR} \times_{(B/A)^{dR}} \cdots \times_{(B/A)^{dR}} (B/P)^{dR} \\ &\quad \simeq (B \otimes_B \cdots \otimes_B B/P \otimes_A \cdots \otimes_A P)^{dR} \\ &\quad \simeq (B/P \otimes_A \cdots \otimes_A P)^{dR}. \end{aligned}$$

This is isomorphic to $D(n)$ by Theorem II.19 and Theorem II.42; all tensor products contain $n+1$ terms. Using this covering we identify $D_{qc}((B/A)^{dR})$ with $D_{qc}(\mathrm{Cech}(D \to B))$ and this identification is functorial. This completes the proof. □

Let us now recall that for smooth X/S the category $\mathcal{D}_{qc}((X/S)^{dR})$ is identified with certain D-modules. Denote

$$\hat{D}_{X/S} := F_{X/S,*} D_{X/S} \otimes_{ST_{X'/S}} \hat{S}T_{X'/S} \in \mathrm{QCoh}(X').$$

Let $\mathrm{Mod}^{\mathrm{ln}}(\hat{D}_{X/S})$ be the full subcategory of $\mathrm{Mod}_{X'}(\hat{D}_{X/S})$ consisting of those left $\hat{D}_{X/S}$-modules such that every local section is killed by $S^{\geq n}T_{X'/S} \subset \hat{D}_{X/S}$ for some $n > 0$. Denote by $\mathcal{D}^{\mathrm{ln}}(\hat{D}_{X/S})$ the full subcategory of $\mathcal{D}(\hat{D}_{X/S})$ consisting of complexes E such that $\mathcal{H}^i(E) \in \mathrm{Mod}^{\mathrm{ln}}(\hat{D}_{X/S})$.

Lemma II.63. *Let X/S be smooth. Then $\mathcal{D}^{ln}(\hat{D}_{X/S}) \simeq \mathcal{D}_{qc}((X/S)^{dR})$.*

Proof. Note the left $F_{X/S,*}D_{X/S}$-module $F_{X/S,*}\mathcal{O}_X$ is compact. Moreover, the minimal $\mathcal{D}_{qc}(X')$-linear stable subcategory of $\mathcal{D}(D_{X/S})$ containing $F_{X/S,*}\mathcal{O}_X$ coincides with $\mathcal{D}^{ln}(D_{X/S})$. Thus, $\mathcal{D}^{ln}(D_{X/S})$ is identified with the category of $\mathrm{End}_{D_{X/S}}(F_{X/S,*}\mathcal{O}_X)$-modules in $\mathcal{D}_{qc}(X')$. Moreover, the Spencer resolution identifies this endomorphism algebra with $F_{X/S,*}\,\mathrm{dR}_{X/S}$. Recall from Theorem II.39 that $\nu_{X/S} : (X/S)^{dR} \to X'$ is affine, thus $\mathcal{D}_{qc}((X/S)^{dR})$ is identified with the left completion of $\mathrm{Mod}_{\mathcal{D}_{qc}(X')}(F_{X/S,*}\,\mathrm{dR}_{X/S})$ by [MM25, Theorem 1.4]. It completes the proof since this category is already left-complete. □

II.3: Lifts and Cartier transform

This section is devoted to a proof of [OV07, Theorem 2.8] as well as some variants of it. Note this theorem only uses a flat lift $\tilde{X}'/\tilde{S}$ of X'/S which is not enough to split $(X/S)^{dR} \to X'$. However, one can define a modification of the de Rham gerbe denoted by $(X/S)^{dR,\gamma}$ which is split by the data of a flat lift of X'/S and which can be used to deduce [OV07, Theorem 2.8]. The main result of this section is Theorem II.90 where the gerbe of splittings of $(X/S)^{dR,\gamma}$ is identified with the gerbe of lifts of X to $W_2(S)$.

Definition II.64. Denote $\mathbf{G}_a^{dR,\gamma}$ to be the $\mathbf{G}_a$-algebra stack $F_*\mathbf{G}_a \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p$ with $\mathbf{G}_a$-algebra structure coming from a map of quasi-ideals

$$\begin{array}{ccc} F_*W & \xrightarrow{\quad V \quad} & W \\ {\scriptstyle R}\downarrow & & \downarrow{\scriptstyle F\circ R} \\ F_*\mathbf{G}_a & \xrightarrow{\quad 0 \quad} & F_*\mathbf{G}_a. \end{array}$$

We denote $\pi^{\gamma}_{\mathbf{G}_a} : \mathbf{G}_a \to \mathbf{G}_a^{dR,\gamma}$.

Remark II.65. If R is an $\mathbf{F}_p$-algebra, then $\mathbf{G}_a^{dR,\gamma}(R)$ is also an $\mathbf{F}_p$-algebra since $\mathbf{G}_a^{dR,\gamma}$ is a $\mathbf{G}_a$-algebra stack. Explicitly, $\mathbf{G}_a^{dR,\gamma}(R) = R \otimes^L_{\mathbf{Z}_p} \mathbf{F}_p$ has two natural $\mathbf{F}_p$-algebra structures and the $\mathbf{F}_p$-algebra structure coming from the second factor is the one provided by Theorem II.64.

Remark II.66. It is easy to see $\pi_0(\mathbf{G}_a^{dR,\gamma}) = F_*\mathbf{G}_a$ and $\pi_1(\mathbf{G}_a^{dR,\gamma}) = F_*\mathbf{G}_a$ as sheaves of $\mathbf{G}_a$-modules. In particular, we have a map $\nu^\gamma : \mathbf{G}_a^{dR,\gamma} \to F_*\mathbf{G}_a$. We warn the reader that the section of ν^γ given by a map of quasi-ideals

$$\begin{array}{ccc} 0 & \longrightarrow & F_*\mathbf{G}_a \\ \downarrow & & \| \\ F_*\mathbf{G}_a & \xrightarrow{0} & F_*\mathbf{G}_a \end{array}$$

is not $\mathbf{F}_p$-linear but only is $\mathbf{Z}/p^2$-linear as a simple diagram chasing shows; we refer to Theorem II.70 for this diagram chasing in a more general setting. In particular, ν^γ admits a $\mathbf{Z}/p^2$-linear section and $\mathbf{G}_a^{dR,\gamma} \simeq \mathbf{G}_a \oplus B\mathbf{G}_a$ as a $\mathbf{Z}/p^2$-algebra stack.

Remark II.67. Let $A/\mathbf{F}_p$ be a commutative algebra. In Theorem II.66 we contemplate two maps $c_A, \pi_a^\gamma : A \to \mathbf{G}_a^\gamma(A)$ of $\mathbf{Z}/p^2$-algebras. Let us note that their fiber product $A \times_{\mathbf{G}_a^\gamma(A)} A$ is canonically identified with $W_2(A)$, see [BS22, Remark 2.5].

Definition II.68. For an $\mathbf{F}_p$-scheme X we use $X^{dR,\gamma}$ to denote the prestack on $\mathbf{F}_p$-algebras obtained by the transmutation of an $\mathbf{F}_p$-algebra stack $\mathbf{G}_a^{dR,\gamma}$. For X/S denote $(X/S)^{dR,\gamma} = X^{dR,\gamma} \times_{S^{dR,\gamma}} S$ where $\pi_S^\gamma : S \to S^{dR,\gamma}$ is the transmutation of π^γ from Theorem II.64. We denote $\pi_{X/S}^\gamma : X \to (X/S)^{dR,\gamma}$ to be the map induced by $\pi_X^\gamma : X \to X^{dR,\gamma}$.

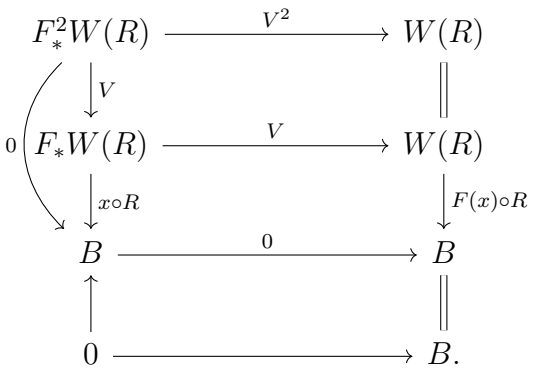
Remark II.69. Let $S = \mathrm{Spec}(R), X = \mathrm{Spec}(A)$. Then the functor of points of the R-stack $(X/S)^{dR,\gamma}$ sends $x : R \to B$ to $(X/S)^{dR,\gamma}(B) = \mathrm{Map}_R(A, B \otimes_{\mathbf{Z}} \mathbf{F}_p)$ where the R-algebra structure on $B \otimes \mathbf{F}_p$ is given by

$$\begin{array}{ccc} F_*W(R) & \xrightarrow{V} & W(R) \\ \downarrow{\scriptstyle x\circ R} & & \downarrow{\scriptstyle F(x)\circ R} \\ B & \xrightarrow{0} & B. \end{array}$$

Note it is a map of quasi-ideals, thus defines a map of rings $R \to B \otimes F_p$. To specify this R-algebra structure on $B \otimes \mathbf{F}_p$, we denote it by $\mathbf{G}_a^{dR,\gamma}(B)$.

Remark II.70. For $R \to B$, recall the R-algebra stack $\mathbf{G}_a^{dR,\gamma}(B)$. We warn the reader that the canonical map of rings $B \to \mathbf{G}_a^{dR,\gamma}(B)$ is not R-linear but is $W_2(R)$-linear. This can be

seen from the following diagram:



Remark II.71. If X/S is quasi-syntomic, then $\nu_{X/S} : (X/S)^{dR} \to X'$ is an fppf $L^{\vee}_{X'/S}[1] \otimes_{\mathbf{G}_a} \mathbf{G}_a^{\sharp}$-torsor and its pushout along the map induced by $\mathbf{G}_a^{\sharp} \to \mathbf{G}_a$ is identified with $(X/S)^{dR,\gamma}$. In particular, we get a map $\nu^{\gamma}_{X/S} : (X/S)^{dR,\gamma} \to X'$ exhibiting the source as $L^{\vee}_{X'/S}[1]$-torsor over the target. It also follows from Theorem II.8 via an argument as in Theorem II.23.

II.3.1: Partial Crystalline Site

In this subsection, we introduce a partial crystalline site which plays the same role for $(X/S)^{dR,\gamma}$ as the crystalline site does for $(X/S)^{dR}$. We prove that for lci X/S the category $\mathcal{D}_{qc}((X/S)^{dR,\gamma})$ is equivalent to the category of partial crystals; see Theorem II.83. We also interpret this category as (continuous) modules over a suitable algebra of differential operators; see Theorem II.84.

Definition II.72. Given a pair (A, I), a partial PD-structure on it is a map $\gamma_p : I \to A$ such that $\gamma_p(x)p! = x^p$. We denote the category of partial PD-rings by PDAlg^{γ}.

Remark II.73. If (A, I, γ_p) is a partial PD-ring and $pA = 0$, then $x^p = 0$ for any $x \in I$.

Definition II.74. Given a scheme X with a sheaf of ideals $\mathcal{I}$ we say that $\gamma_p : \mathcal{I} \to \mathcal{I}$ is a partial PD-structure on $\mathcal{I}$ if for every affine Zariski open $U = \mathrm{Spec}(A) \subset X$ the triple $(A, \mathcal{I}(U), \gamma_p(U))$ is a partial PD-ring.

Definition II.75. Let $(S, \mathcal{J}, \gamma_p)$ be a partial PD-scheme. Define a big partial crystalline site $\mathrm{CRIS}^{\gamma}(\mathrm{X}/((\mathrm{S}, \mathcal{J}, \gamma_{\mathrm{p}})))$ for any $X \to V(\mathcal{J})$ on which p acts in a locally nilpotent way. The underlying category consists of partial PD-schemes $(T, \mathcal{I}, \delta_p)$ over $(S, \mathcal{J}, \gamma_p)$ endowed with a map $V(\mathcal{I}) \to X$.

Definition II.76. For X as in (II.75) define the category of partial crystals $\mathrm{Crys}^{\gamma}(X/((S, \mathcal{J}, \gamma_p)))$ by $\lim_{(T,I,\gamma_p)\in\mathcal{C}_X} \mathcal{D}(T)$ where $\mathcal{C}_X$ is the category of affine objects

in $\mathrm{CRIS}^{\gamma}(\mathrm{X}/((\mathrm{S},\mathcal{J},\gamma_{\mathrm{p}})))$, i.e. the category of pairs $(B,J) \in \mathrm{PDAlg}^{\gamma}_{/S}$ and a map $\mathrm{Spec}(B/J) \to X$.

Remark II.77. When X is a scheme over a perfect field k of characteristic p, we note that $(W(k),p)$ has a canonical partial PD-structure given by $\gamma_p(p) = p!/p$ and we will use $\mathrm{Crys}^{\gamma}(X/W(k))$ and $\mathrm{CRIS}^{\gamma}(X/W(k))$ instead of $\mathrm{Crys}^{\gamma}(X/((W(k),(p),\mathrm{can}))$ and instead of $\mathrm{CRIS}^{\gamma}(X/((W(k),(p),\mathrm{can})))$ respectively.

Lemma II.78. *Let (A,I,γ_p) be a partial PD-ring. Let $(A,I) \to (B,J)$ be a map of pairs, then there exists $(A,I,\gamma_p) \to (D,\bar{J},\bar{\gamma}_p)$ satisfying $Hom_{(A,I,\gamma_p)}((D,\bar{J},\bar{\gamma}_p),(C,K,\delta_p)) = Hom_{(A,I)}((B,J),(C,K))$.*

Proof. The proof is the same as in [Aut19, Tag 07H7]. □

Definition II.79. Let (A,I,γ_p) be a partial PD-ring and $(A,I) \to (B,J)$ a map of pairs. The partial PD-algebra over (A,I,γ_p) provided by Theorem II.78 will be denoted by $D^{\gamma}_B(J)$ and referred to as the partial PD-envelope of J (or B/J) in B.

Corollary II.80. *Let $X \to Y$ be a closed embedding of S-schemes with Y/S being smooth. Then $\mathrm{Crys}^{\gamma}(X/S)$ identifies with modules over a cosimplicial scheme $\mathrm{Cech}(D^{\gamma}_X(Y) \to X)$.*

Proof. Indeed, $D^{\gamma}_X(Y)$ is weakly final in $\mathrm{CRIS}^{\gamma}(\mathrm{X}/((\mathrm{S},\mathcal{J},\gamma_{\mathrm{p}})))$. After enlarging it we can assume it has a final object $*$ by [Aut19, TAG 03CI]. Thus, since $D^{\gamma}_X(Y) \to *$ is a surjective map of presheaves, it can be used to compute $\mathrm{Crys}^{\gamma}(X/S)$ which gives the result. □

Lemma II.81. *Given a closed immersion $Y \to X$, the stack $(Y/X)^{dR,\gamma}$ is represented by $D^{\gamma}_Y(X)$ the partial PD-envelope of Y in X.*

Proof. By definition $(Y/X)^{dR,\gamma}$ is the quotient of $(Y/X)^{dR} \times_{Y^{\phi}} \mathbb{N}_{Y^{\phi}}(X) \simeq D_Y(X) \times_{Y^{\phi}} \mathbb{N}_{Y^{\phi}}(X)$ by the anti-diagonal action of $\mathbb{N}_{Y^{\phi}}(X)^{\sharp}$. Consider the map $D_Y(X) \to D^{\gamma}_Y(X)$, it is enough to observe that it is equivariant for $\mathbb{N}_{Y^{\phi}}(X)^{\sharp} \to \mathbb{N}_{Y^{\phi}}(X)$. □

Corollary II.82. *For smooth $X \to S$ the Cech nerve of $X \to (X/S)^{dR,\gamma}$ is represented by a simplicial scheme $D_{I_n}(X^{\bullet/S})$ where I_n is the ideal of the diagonal $X \to X^{\times n/S}$. In particular, the category $\mathcal{D}_{qc}((X/S)^{dR,\gamma})$ is identified with the ∞-category of crystals in quasi-coherent complexes on $(X/S)^{\gamma}_{crys}$.*

Corollary II.83. *Combining Theorem II.82 and Theorem II.81 one observes that for an lci morphism $X \to S$ we obtain the equivalence $\mathcal{D}_{qc}((X/S)^{dR,\gamma}) \simeq \mathrm{Crys}^{\gamma}(X/S)$ by the same argument as in Theorem II.62.*

Recall from [OV07, §2.3] the category $\mathrm{MIC}^{\cdot}_{\gamma}(X/S)$ which is defined as the full subcategory of the abelian category of $F_{X/S,*}D_{X/S} \otimes_{ST_{X'/S}} \hat{\Gamma}T_{X'/S}$-modules, consisting of locally nilpotent objects.

Theorem II.84. *Let X be a smooth scheme over S. Then $\mathrm{MIC}^{\cdot}_{\gamma}(X/S)$ is equivalent to the category of partial crystals in quasi-coherent sheaves on X relative to S.*

Proof. Consider an fppf-covering

$$C_1 := (X/S)^{dR} \times_{X'} B_{X'}(T_{X'/S}) \to (X/S)^{dR,\gamma} =: C_0.$$

A quasi-coherent sheaf on $(X/S)^{dR,\gamma}$ amounts to $E \in \mathrm{QCoh}(B_{(X/S)^{dR}}(F^*_{X/S}T_{X'/S}))$ with $p_1^*E \simeq p_2^*E$ satisfying the cocycle condition, where $p_1, p_2 : C_1 \times_{C_0} C_1 \rightrightarrows C_1$ are the projections. By Cartier duality E amounts to a quasicoherent sheaf $\mathcal{E} \in \mathrm{QCoh}((X/S)^{dR})$ together with a continuous[3] horizontal map $\psi^\gamma : \Gamma_{\mathcal{O}_X}(F^*_{X/S}T_{X'/S}) \to \mathcal{E}nd(\mathcal{E})$, i.e., every local section of $\mathcal{E}$ is annihilated by $\Gamma^n_{\mathcal{O}_X}(F^*_{X/S}T_{X'/S})$ for some n. Note

$$C_1 \times_{C_0} C_1 \xrightarrow{\simeq} B_{X'}T_{X'/S} \times_{X'} B_{X'}T^\sharp_{X'/S} \times_{X'} (X/S)^{dR} \underset{(id,a)}{\overset{(a',\mathrm{id})}{\rightrightarrows}} B_{X'}(T_{X'/S}) \times_{X'} (X/S)^{dR}$$

where a is the action map $B_{X'}(T^\sharp_{X'/S}) \times_{X'} (X/S)^{dR} \to (X/S)^{dR}$ and similarly a' is the action of $B_{X'}(T^\sharp_{X'/S})$ on $B_{X'}(T_{X'/S})$. Moreover, these maps (a', id) and (id, a) correspond to the projections $p_1, p_2 : C_1 \times_{C_0} C_1 \to C_1$ under this identification. Note the category of quasi-coherent sheaves on $C_1 \times_{C_0} C_1$ is the category $(\mathcal{M}, \xi^\gamma, \xi)$ where $\mathcal{M}$ is a crystal, $\xi^\gamma : \Gamma_{\mathcal{O}_X}(F^*_{X/S}T_{X'/S}) \to \mathcal{E}nd(\mathcal{M})$ and $\xi : S_{\mathcal{O}_X}(F^*_{X/S}T_{X'/S}) \to \mathcal{E}nd(\mathcal{M})$ are horizontal maps. Under this identification, the object $(a', \mathrm{id})^*(\mathcal{E}, \psi^\gamma)$ corresponds to $(\mathcal{E}, \psi^\gamma, \psi)$ where

$$\psi : S_{\mathcal{O}_X}(F^*_{X/S}T_{X'/S}) \to \Gamma_{\mathcal{O}_X}(F^*_{X/S}T_{X'/S}) \xrightarrow{\psi^\gamma} \mathcal{E}nd(\mathcal{M}).$$

Similarly, the object $(\mathrm{id}, a)^*(\mathcal{E}, \psi^\gamma) = (\mathcal{E}, \psi^\gamma, \psi_\mathcal{E})$ where $\psi_\mathcal{E}$ is the p-curvature of $\mathcal{M}$. In particular, supplying an isomorphism $\alpha : p_1^*(\mathcal{E}, \psi^\gamma) \simeq p_2^*(\mathcal{E}, \psi^\gamma)$ amounts to a D-module automorphism α_0 of $\mathcal{E}$ commuting with two actions of $S_{\mathcal{O}_X}(F^*_{X/S}T_{X'/S})$. In particular, the two actions agree $\psi = \psi_\mathcal{E}$ and α_0 gives rise to an isomorphism of this descent datum to the one with the identity isomorphism between two pullbacks, which completes the proof. □

[3] With respect to the PD-filtration on the source.

II.3.2: Gerbe of liftings

Recall [OV07, Theorem 2.8] uses a lift of X'/S in order to construct the global Cartier transform. Let us first show that a lift of X' to $\mathbf{Z}/p^2$ is enough.

Corollary II.85. *For X/S any flat lift of X' to $\mathbf{Z}/p^2$ produces a section of $(X/S)^{dR,\gamma} \to X'$.*

Proof. By definition, $X'^{dR,\gamma} = X^{dR,\gamma} \times_{S^{dR,\gamma}, F_{S^{dR,\gamma}}} S^{dR,\gamma}$ and by Theorem II.1 we get a factorization of $F_{S^{dR,\gamma}} : S^{dR,\gamma} \to S \xrightarrow{F'_{S^{dR,\gamma}}} S^{dR,\gamma}$ which gives $X'^{dR,\gamma} \simeq (X/S)^{dR,\gamma} \times_S S^{dR,\gamma}$. In particular, we obtain a map $q : X^{dR,\gamma} \to (X/S)^{dR,\gamma}$. Moreover, by the construction this map is over X'. In particular, since a lift of X' is equivalent to giving a section s of $X'^{dR,\gamma} \to X'$, the map $q \circ s$ defines a section of $(X/S)^{dR,\gamma}$. □

Remark II.86. The proof of Theorem II.85 shows that there is a map of groupoids

$$\{\text{Flat lifts of } X' \text{ to } \mathbf{Z}/p^2\ \} \to \{\text{Splittings of } (X/S)^{dR,\gamma}\}.$$

Let us explain how to see this map using elementary methods.[4] The transitivity triangle for $f' : X' \to S \to \mathrm{Spec}(\mathbf{F}_p)$ gives $f'^* L_S \to L_{X'} \to L_{X'/S}$. Let us check that the connecting map $o : L_{X'/S} \to f'^* L_S[1]$ is canonically homotopic to 0. Note $L_{X'/S} = \nu^* L_{X/S}$ and by adjunction o corresponds to a map $L_{X/S} \to \nu_* f'^* L_{X/S}$. Since $\nu_* f'^* L_S = f^* F_{S,*} L_S$, where $\nu : X' \to X$ is the canonical map, we learn that o corresponds to $L_{X/S} \to f^* F_{S,*} L_S[1]$. By functoriality this map factors as $L_{X/S} \to f^* L_S[1] \to f^* F_{S,*} L_S[1]$, where the first map is the connecting homomorphism in the transitivity triangle for $f : X \to S$ and the second map is obtained by applying f^* to $dF_S[1] : L_S[1] \to F_{S,*} L_S[1]$, which is canonically 0. Therefore, we obtain a section of $L_{X'} \to L_{X'/S}$.

Remark II.87. Given a flat lift $\tilde{X}/W_2(S)$ of X/S, any flat lift of S provides a splitting of $(X/S)^{dR,\gamma}$. Indeed, for $\tilde{S}/\mathbf{Z}/p^2$ there is a natural map $\tilde{S} \to W_2(S)$ and pullback $\tilde{X} \times_{W_2(S)} \tilde{S}$ provides a flat lift of X'/S.

Let $S/\mathbf{F}_p$ be a scheme. Recall the closed immersion $S \to W_2(S)$ is a square-zero deformation with an ideal $F_* \mathcal{O}_S$. Therefore, for $f : X \to S$ the groupoid of flat lifts to $W_2(S)$ is a quasi-torsor for $\mathrm{Map}_{\mathcal{O}_X}(L_{X/S}, f^* F_{S,*} \mathcal{O}_S[1]) = \mathrm{Map}_{\mathcal{O}_X}(L_{X/S}, \nu_* \mathcal{O}_{X'}[1])$ where $\nu : X' = X \times_{S,F_S} S \to X$ is the projection map. This space is $\mathrm{Map}_{\mathcal{O}_{X'}}(L_{X'/S}, \mathcal{O}_{X'}[1])$ by adjunction and base change for cotangent complex. In particular, the obstruction class $\mathrm{ob}_{X/W_2(S)}$ of deforming X to $W_2(S)$ defines a point in $\mathrm{Map}_{\mathcal{O}_{X'}}(L_{X'/S}, \mathcal{O}_{X'}[2])$.

[4] If $X = \mathrm{Spec}(B), S = \mathrm{Spec}(A)$ are smooth, then $f^* T_S \to T_X$ sends an A-derivation ∂ to a derivation of $B \otimes_A F_* A$ defined by $\partial(b \otimes a) = b \otimes \partial a$. This gives the splitting $T_{X'} = T_{X'/S} \oplus f^* T_S$.

Remark II.88. Let X/S be lci. Consider the Zariski sheaf on X that assigns to $U \subset X$ a groupoid of lifts $\tilde{U} \to W_2(S)$ of U/S to $W_2(S)$. Recall the obstruction $\mathrm{ob}_{U/W_2(S)} : L_{U/S} \to f^* L_{S/W_2(S)}[1] \simeq \nu_* \mathcal{O}_{U'}[2]$ where $\nu : U' \to U$ is the canonical map. Assume U is affine, then $\mathrm{ob}_{U/W_2(S)} = 0$. Indeed, U' is affine and $\mathcal{H}^i(T_{U'/S}) = 0$ for $i > 1$, thus $H^2(U', T_{U'/S}) = 0$.

Lemma II.89. *For any flat lift $\tilde{S} \to \mathbf{Z}/p^2$ of S, the obstruction class $ob_{X'/\tilde{S}}$ is equal to $ob_{X/W_2(S)}$.*

Proof. Recall obstruction class $\mathrm{ob}_{X'/\tilde{S}} : L_{X'/S} \to f'^* L_{S/\tilde{S}}[1]$, which corresponds to $L_{X'/S} \to \nu_* f'^* L_{S/\tilde{S}}[1] \simeq f^* F_{S,*} L_{S/\tilde{S}}[1]$ by adjunction for $\nu : X' \to X$. Note the canonical map $\tilde{S} \to W_2(S)$ induces $dF_{S,\tilde{S}/W_2(S)} : L_{S/W_2(S)} \to F_{S,*} L_{S/\tilde{S}}$, the local computation shows it is an isomorphism on $\mathcal{H}^{-1}$. Therefore, composing $\mathrm{ob}_{X/W_2(S)} : L_{X/S} \to f^* L_{S/W_2(S)}[1]$ with $dF_{S,\tilde{S}/W_2(S)}[1]$ we obtain $\mathrm{ob}_{X'/S}$. □

The main result of this subsection is the following.

Lemma II.90. *Let X/S be a representable quasi-syntomic map of algebraic stacks over $\mathbf{F}_p$. The fppf $T_{X'/S}[1]$-torsor $\nu^{\gamma}_{X/S} : (X/S)^{dR,\gamma} \to X'$ is equivalent to the torsor of liftings of X to $W_2(S)$.*

Proof. Given $\tilde{X}/W_2(S)$, our goal is to produce

$$X'(R) = \mathrm{Map}_S(\mathrm{Spec}(F_* R), X) \xrightarrow{s_{\tilde{X}}} \mathrm{Map}_S(\mathrm{Spec}(\mathbf{G}_a^{\gamma}(R)), X) = (X/S)^{dR,\gamma}(R).$$

Note $\mathrm{Map}_S(\mathrm{Spec}(\mathbf{G}_a^{\gamma}(R)), X) = \mathrm{Map}_{W_2(S)}(\mathrm{Spec}(\mathbf{G}_a^{\gamma}(R)), \tilde{X})$ and $\mathrm{Spec}(\mathbf{G}_a^{\gamma}(R))$, when viewed as $W_2(S)$-scheme, is the split square-zero deformation of $\mathrm{Spec}(F_* R)$ with the ideal $F_* R[1]$. Thus, $\tilde{X}/W_2(S)$ identifies the space $(X/S)^{dR}(R)$ with $\mathrm{Map}_S(\mathrm{Spec}(F_* R \oplus F_* R[1]), X)$ and we declare $s_{\tilde{X}}$ to be the map sending $x : \mathrm{Spec}(F_* R) \to X$ to $\pi \circ x$ where $\pi : \mathrm{Spec}(F_* R \oplus F_* R[1]) \to \mathrm{Spec}(F_* R)$ is the canonical map. It is clearly the section since $F_* R \xrightarrow{\pi} F_* R \oplus F_* R[1] \to F_* R$ is the identity. □

Corollary II.91. *Let X/S be a smooth morphism of algebraic stacks over $\mathbf{F}_p$. Splitting of $\mathcal{O}_X \to \tau^{\leq 1} \mathrm{dR}_{X/S} \to \Omega^1_{X'/S}[-1]$ in $\mathcal{D}_{qc}(X')$ is equivalent to lifting X/S to $W_2(S)$.*

Proof. Indeed, from Theorem II.90 we learn that a groupoid of lifts of X/S to $W_2(S)$ is equivalent to the groupoid of splitting of $(X/S)^{dR} \to X'$. The groupoid of splittings of $\tau^{\leq 1} \mathrm{dR}_{X/S}$, when non-empty, is a torsor for $T_{X'/S}$, any splitting of $\nu^{\gamma}_{X/S}$ gives rise to a splitting $\Omega^1_{X'/S}[-1] \to \tau^{\leq 1} \mathrm{dR}_{X/S}$ and thus gives rise to an equivalence of groupoids. □

Now Theorem II.90 allows us to construct an $F^*_{X/S} T_{X'/S}$-torsor on X with a flat connection, which plays a central role.

Construction II.92. Let X/S be an lci family and $\tilde{X}/W_2(S)$ a lift of X/S. Then

$$X \xrightarrow{\pi^{\gamma}_{X/S}} (X/S)^{dR,\gamma} \simeq B_{X'}(T_{X'/S})$$

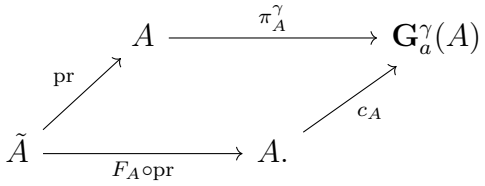
defines a $F^*_{X/S}T_{X'/S}$-torsor on X. We will denote it by $\mathcal{L}_{\tilde{X}}$. Note it comes equipped with a flat connection since $\pi^{\gamma}_{X/S}$ naturally factors as $X \xrightarrow{\pi_{X/S}} (X/S)^{dR} \to (X/S)^{dR,\gamma}$. We denote by $\mathcal{L}^c_{\tilde{X}}$ the corresponding $F^*_{X/S}T_{X'/S}$-torsor on $(X/S)^{dR}$.

First, we identify the underlying torsor of $\mathcal{L}_{\tilde{X}}$ with a torsor of Frobenius lifts in a suitable sense, and then compute its connection.

Remark II.93. Let $f : X \to S$ be an lci family. A lift $\tilde{X}/W_2(S)$ is equivalent to a map $c_{\tilde{X}} : L_{X/W_2(S)} \to \nu_*\mathcal{O}_{X'}[1]$ where $\nu : X \to X'$ is the canonical map. Then $F_{X/S} : X \to X'$ induces $\nu_*\mathcal{O}_{X'} \to F_{X,*}\mathcal{O}_X$ and we get $L_{X/W_2(X)} \xrightarrow{c_{\tilde{X}}} \nu_*\mathcal{O}_{X'}[1] \to F_{X,*}\mathcal{O}_X[1]$. Note the square-zero extension $X \to W_2(X)$ has an ideal $F_{X,*}\mathcal{O}_X$ and thus defines $c_{W_2(X)} : L_{X/W_2(S)} \to F_{X,*}\mathcal{O}_X[1]$. Note $c_{\tilde{X}} - c_{W_2(X)} : L_{X/W_2(S)} \to F_{X,*}\mathcal{O}_X[1]$ naturally factors through $L_{X/W_2(S)} \to L_{X/S}$. Thus, we obtain $F^*_{X/S}T_{X'/S}$-torsor which measures the difference between two square zero extensions. In other words, it is the torsor of deformations of the identity map on X to a $W_2(S)$-linear map $W_2(X) \to \tilde{X}$.

Lemma II.94. *Let X/S be an lci family of $\mathbf{F}_p$-schemes. A lift $\tilde{X}/W_2(S)$ defines a section $s_{\tilde{X}} : X' \to (X/S)^{dR,\gamma}$ which defines a $F^*_{X/S}T_{X'/S}$-torsor on X. This torsor is identified with the torsor from Theorem II.93.*

Proof. Recall from Theorem II.92 the $F^*_{X/S}T_{X'/S}$-torsor $\mathcal{L}_{\tilde{X}} \simeq X \times_{(X/S)^{dR,\gamma}} X'$. In particular, the space of splittings is identified with the path space between $\pi^{\gamma}_{X/S}$ and $s_{\tilde{X}} \circ F_{X/S}$ in $(X/S)^{dR}(X)$. Let us now identify it with the set of maps $\tilde{X} \to W_2(X)$ deforming the identity. Let $S = \mathrm{Spec}(R), X = \mathrm{Spec}(A)$. Let $s : X' \to (X/S)^{dR,\gamma}$ be a section. Explicitly, $(X/S)^{dR}(X') = \mathrm{Map}_R(A, \mathbf{G}^{\gamma}_a(A'))$, i.e., s corresponds to a map $A \to \mathbf{G}^{\gamma}_a(A')$ of R-algebras whose post-composition with $\mathbf{G}^{\gamma}_a(A') \to F_*A'$ is the canonical map $A \to F_*A'$. Recall from Theorem II.67 that a map $\tilde{A} \to W_2(A)$ is the same as a homotopy filling in the square



Using the factorization $F_A : A \xrightarrow{\nu_A} A' \xrightarrow{F_{A/R}} A$, we obtain $c_A \circ F_A \circ \mathrm{pr} \simeq \mathbf{G}^{\gamma}_a(F_{A/R}) \circ c_{A'} \circ \nu_A \circ \mathrm{pr}$. Recall $\tilde{A}$ comes equipped with a canonical homotopy $s \circ \mathrm{pr} \simeq c_{A'} \circ \nu_A \circ \mathrm{pr}$. Thus, we learn

that giving a map $\tilde{A} \to W_2(A)$ is equivalent to giving a homotopy $\pi_A^\gamma \circ \mathrm{pr} \simeq \mathbf{G}_a^\gamma(F_{A/R}) \circ s \circ \mathrm{pr}$. Since $\mathrm{Map}_{W_2(R)}(\tilde{A}, \mathbf{G}_a^\gamma(A)) \simeq \mathrm{Map}_R(A, \mathbf{G}_a^\gamma(A))$, the claim follows. □

We would like to compare the torsor from Theorem II.92 with the torsor of Ogus–Vologodsky constructed in [OV07, Theorem 1.1], denoted by $\mathcal{L}_{\mathcal{X}}$. First, they depend on different data: $\mathcal{L}_{\tilde{X}}$ only depends on a lift $\tilde{X}$ of X to $W_2(S)$. The torsor $\mathcal{L}_{\mathcal{X}}$ depends on a lift $\tilde{S}$ of S to $\mathbf{Z}/p^2$ and a further lift of X' to $\tilde{S}$. However, any such lift $\tilde{S}$ identifies the groupoids

$$\{\text{Lifts of } X \text{ to } W_2(S)\} \dashleftarrow\dashrightarrow \{\text{Lifts of } X' \text{ to } \tilde{S}\}$$

since both are torsors for $F_{X/S}^* T_{X'/S}$, and there is a natural map which sends a lift $\tilde{X}/W_2(S)$ to $\tilde{X}' := \tilde{X} \times_{\tilde{S}} W_2(S)$, where $\tilde{S} \to W_2(S)$ is the natural map.

Thus, one may ask whether the torsors $\mathcal{L}_{\tilde{X}}$ and $\mathcal{L}_{\mathcal{X}}$ are isomorphic. The next remark explains that the underlying torsors are isomorphic. Later, we will explain that they are isomorphic as torsors with flat connections; see Theorem II.100.

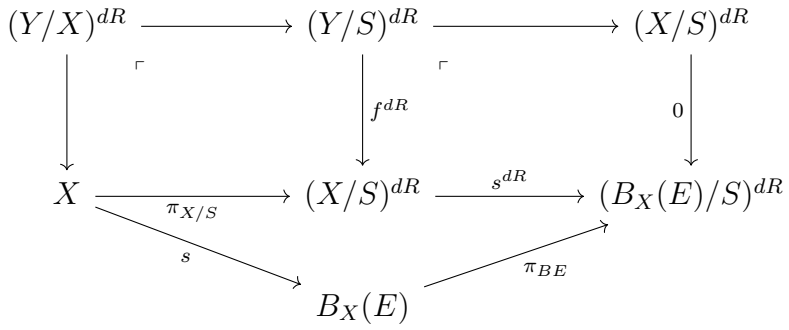
Remark II.95. Let X/S be a representable quasi-syntomic map of algebraic $\mathbf{F}_p$-stacks. Let $\tilde{X}/W_2(S)$ be a lift of X. In particular, we obtain a $F_{X/S}^* T_{X'/S}$-torsor on $(X/S)^{dR}$ and we denote its underlying torsor by $\mathcal{L}_{\tilde{X}}$. Assume we are given $\tilde{S} \to \mathbf{Z}/p^2$ a lift of S. This defines $\tilde{X}' := \tilde{X} \times_{W_2(S)} \tilde{S}$ a lift of X'/S to $\tilde{S}$. Let $\mathcal{L}_{\mathcal{X}}$ be the $F_{X/S}^* T_{X'/S}$-torsor associated with $\tilde{X}'/\tilde{S}$ as in [OV07, Theorem 1.1]. Assume we are given a lift $\tilde{X}_S \to \tilde{S}$ of X. Then $\mathcal{L}_{\mathcal{X}}$ is identified with the torsor of lifts of $F_{X/S} : X \to X'$ to a map $\tilde{X}_S \to \tilde{X}' := \tilde{X} \times_{W_2(S)} \tilde{S}$, see [OV07, Theorem 1.1]. Note there is a canonical map $\mathcal{L}_{\tilde{X}} \to \mathcal{L}_{\mathcal{X}}$ of $F_{X/S}^* T_{X'/S}$-torsors. In particular, any such lift $\tilde{S}$ identifies $\mathcal{L}_{\tilde{X}}$ with $\mathcal{L}_{\mathcal{X}}$.

First, we compute the p-curvature of the torsor from Theorem II.92. We need the following auxiliary lemma.

Lemma II.96. *Let $s : X \to B_X(E)$ be a map classifying an E-torsor $q : Y \to X$. Denote $t : B_X(E^\sharp) \to B_X(E)$ to be the map induced by the canonical map $E^\sharp \to E$. Then $t^* s_* \mathcal{O}_X \in \mathcal{D}_{qc}(B_X(E^\sharp))$ is identified with the pair[5] $(f_* \mathcal{O}_Y, d)$ where $d : f_* \mathcal{O}_Y \to f_* \mathcal{O}_Y \otimes E^\vee$ is the de Rham differential under $f_* \mathcal{O}_Y \otimes E^\vee \simeq f_* \Omega^1_{Y/X}$.*

[5]We recall that $\mathcal{D}_{qc}(B_X E^\sharp)$ is identified with the ∞-category of pairs: $(\mathcal{M}, \theta)$ where $\mathcal{M} \in \mathcal{D}_{qc}(X)$ and $\theta : \mathcal{M} \to \mathcal{M} \otimes E^\vee$ a locally nilpotent map.

Proof. Consider the following diagram



where both squares are cartesian. Namely, the right square is cartesian because $(-/S)^{dR}$ preserves limits and the left square is cartesian by definition. Using the factorization of the bottom composition, we learn that

$$(Y/X)^{dR} \simeq X \times_{B_X(E)} B_X(E^\sharp),$$

where the maps are s and t. Indeed, the fiber product of π_{BE} and 0 is $B_X(E^\sharp)$, with the map $B_X(E^\sharp) \to B_X(E)$ being the canonical map t. In particular, $t^* s_* \mathcal{O}_X$ is identified with $g_* \mathcal{O}_{(Y/X)^{dR}}$ for $g\colon (Y/X)^{dR} \to B_X(E^\sharp)$. Note that the $\mathbf{V}(E^\sharp)$-torsor classified by g is $\pi_{Y/X}\colon Y \to (Y/X)^{dR}$. In particular, it follows that the underlying $\mathcal{O}_X$-module of $g_* \mathcal{O}_Y$ is $f_* \mathcal{O}_X$. By descent along $X \to B_X(E^\sharp)$, the locally nilpotent endomorphism on $f_* \mathcal{O}_X$ comes from the $\mathbf{V}(E)^\sharp$-action on Y, and it is the canonical action coming from the $\mathbf{V}(E)$-action. Since $D_\Delta(Y \times_X Y) \simeq Y \times \mathbf{V}(E)^\sharp$, exchanging the groupoid action on the left with the canonical action on the right, we obtain $[Y/\mathbf{V}(E)^\sharp] \simeq (Y/X)^{dR}$, which is also the desired result. □

A crucial input to the theory of Ogus–Vologodsky is the fact that the crystal of Frobenius lifts has a rich p-curvature, see [OV07, Proposition 1.5]. It corresponds to the fact that the map $(X/S)^{dR} \to B_{X'}(T_{X'/S})$, which is obtained from the lift $\tilde{X}/W_2(S)$, is equivariant with respect to the action of $B_{X'}(T^\sharp_{X'/S})$, which we make precise in the following lemma.

Lemma II.97. *Let X/S be a representable map of $\mathbf{F}_p$-stacks. Let $\tilde{X}/W_2(S)$ be a lift. It gives rise to a $F^*_{X/S} T_{X'/S}$-torsor with a flat connection $\mathcal{L}^c_{\tilde{X}} \to (X/S)^{dR}$ coming from $(X/S)^{dR} \to (X/S)^{dR,\gamma} \simeq B_{X'}(T_{X'/S})$. The p-curvature of $q_* \mathcal{O}_{\mathcal{L}^c_{\tilde{X}}}$ is given by the formula $q_* \mathcal{O}_{\mathcal{L}_{\tilde{X}}/X} \xrightarrow{d} q_* \Omega^1_{\mathcal{L}_{\tilde{X}}/X} \simeq q_* q^* F^*_{X'/S} \Omega^1_{X'/S} \simeq q_* \mathcal{O}_{\mathcal{L}_{\tilde{X}}} \otimes F^*_{X/S} \Omega^1_{X'/S}$.*

Proof. By Theorem II.45 it is enough to compute $a^* q_* \mathcal{O}_{\mathcal{L}_{\tilde{X}^c}}$ where $a : (X/S)^{dR} \times$

$B_{X'}(T^\sharp_{X'/S}) \to (X/S)^{dR}$ is the action morphism. Consider the composition

$$B_{(X/S)^{dR}}(F^*_{X'/S}T^\sharp_{X'/S}) \simeq (X/S)^{dR} \times_{X'} B_{X'}(T^\sharp_{X'/S}) \xrightarrow{a} (X/S)^{dR} \to (X/S)^{dR,\gamma} \simeq B_{X'}(T_{X'/S})$$

and note the resulting map $B_{(X/S)^{dR}}(F^*_{X'/S}T^\sharp_{X'/S}) \to B_{X'}(T_{X'/S})$ lives over $\nu_{X/S} : (X/S)^{dR} \to X'$ and the map $F^*_{X/S}T^\sharp_{X'/S} \to F^*_{X/S}T_{X'/S}$ is the canonical map since the map $(X/S)^{dR} \to B_{X'}(T_{X'/S})$ is equivariant for $B_{X'}(T^\sharp_{X'/S})$. The claim follows from Theorem II.96. □

Now we will compute the connection of the torsor from Theorem II.92. First, we recall what it means.

Remark II.98. Let $E \in \mathrm{Vect}((X/S)^{dR})$ and $f : (X/S)^{dR} \to B_{(X/S)^{dR}}E$ an E-torsor on $(X/S)^{dR}$. Specifying f is equivalent to specifying a map $\mathcal{O}_{(X/S)^{dR}} \to E[1]$ in $\mathcal{D}_{qc}((X/S)^{dR}) \simeq \mathrm{Crys}(X/S)$. Discussion in [OV07, §1.1, (1.0.9)] explains that this extension is equivalent to giving an E-torsor $\mathcal{L}$ over X together with $\nabla_{\mathcal{L}} : \mathcal{L} \to E \otimes \Omega^1_{X/S}$ such that $\nabla_{\mathcal{L}}(l+e) = \nabla_{\mathcal{L}}(l) + \nabla_E(e)$ for all local sections $l \in \mathcal{L}, e \in E$ and $\nabla_E \circ \nabla_{\mathcal{L}} = 0$.

Remark II.99. Let E be a locally free sheaf on $(X/S)^{dR}$. Let $(X/S)^{dR} \xrightarrow{f} B_{(X/S)^{dR}}(E)$ be an E-torsor on X with a flat connection. This is equivalent to giving an E-torsor $\mathcal{L}$ over X together with $\nabla_{\mathcal{L}} : \mathcal{L} \to E \otimes \Omega^1_{X/S}$ satisfying some properties. In this remark we explain how to compute $\nabla_{\mathcal{L}}(s)$ for a local section $s \in \mathcal{L}(X)$. Note that s provides a commutativity datum for the diagram

$$\begin{array}{ccc} (X/S)^{dR} & \xrightarrow{\quad f \quad} & B_{(X/S)^{dR}}(E) \\ {\scriptstyle \pi_{X/S}}\uparrow & & \uparrow{\scriptstyle 0} \\ X & = & X \end{array}$$

which gives rise to a map of Čech nerves $\mathrm{Cech}(X \to (X/S)^{dR}) \to \mathrm{Cech}(X \to B_{(X/S)^{dR}}(E))$. The map on 1-simplices is some map $f : D_\Delta(X^2) \to \mathbf{V}(E)$. By construction, this map restricts to $X \xrightarrow{\mathrm{Id}} X$ with $X \hookrightarrow D_\Delta(X^2)$ and $X \hookrightarrow \mathbf{V}(E)$. Equivalently, the map $E^\vee \to \mathcal{O}(D_\Delta(X^{\times_S 2}))$ lands in the ideal of the diagonal I_Δ. In particular, we obtain a map $E^\vee \to I_\Delta/I_\Delta^2 \simeq \Omega^1_{X/S}$, which is the same as a section of $E \otimes \Omega^1_{X/S}$, and by construction it is $\nabla_{\mathcal{L}}(s)$.

Now we are ready to compute the connection on $\mathcal{L}_{\tilde{X}}$.

Lemma II.100. *Let X/S be a representable map of $\mathbf{F}_p$-stacks. Let $\tilde{X}/W_2(S)$ be a flat lift. One obtains a $F^*_{X/S}T_{X'/S}$-torsor on X with a flat connection corresponding to $(X/S)^{dR} \to (X/S)^{dR,\gamma} \simeq B_{X'}(T_{X'/S})$. The corresponding connection $\mathcal{L}_{\tilde{X}} \to F^*_{X/S}T_{X'/S} \otimes_{\mathcal{O}_X} \Omega^1_{X/S}$ sends*

a local section $f : W_2(X) \to \tilde{X}$ to $d\tilde{F}/p \in \mathcal{H}om(F_{X/S}^* \Omega^1_{X'/S}, \Omega^1_{X/S})$ where $\tilde{F}$ is the Frobenius lift defined by f.

Proof. The claim is Zariski local. We assume $X = \mathrm{Spec}(A)$, $S = \mathrm{Spec}(R)$, and $f \in \mathcal{L}_{\tilde{X}}(X)$ a local section. Recall f is equivalent to a homotopy $h_f : \pi^{\gamma}_{X/S} \simeq s_{\tilde{X}} \circ F_{X/S}$. In particular, h_f can be used to represent a map $(X/S)^{dR} \to (X/S)^{dR,\gamma}$ using their Cech nerves along $\pi_{X/S} : X \to (X/S)^{dR}$ and $s_{\tilde{X}} : X' \to (X/S)^{dR,\gamma}$. The map on 1-simplices is

$$r_2 : D_\Delta(X \times_S X) \simeq X \times_{(X/S)^{dR}} X \to X' \times_{(X/S)^{dR,\gamma}} X' \simeq \mathbf{T}_{X'/S}.$$

It is a map of X'-schemes and corresponds to a Frobenius-linear derivation $A \to D_\Delta(A^{\otimes_R 2})$. We claim it is given by the formula

$$\theta_f(a) := \frac{1 \otimes \tilde{F}(\tilde{a}) - \tilde{F}(\tilde{a}) \otimes 1}{p} \mod p$$

where $\tilde{a}$ is any lift of a along $\tilde{A} \to A$.

Step 1. Identifying the derivation. Let (x, y, h_γ) be a B-point of $X \times_{(X/S)^{dR}} X$ where B is a test R-algebra; here $x, y : A \to B$ and h_γ is a homotopy between two compositions in $\mathrm{Map}_R(A, \mathbf{G}_a^{dR}(B))$. Then $X \times_{(X/S)^{dR}} X \to X' \times_{(X/S)^{dR,\gamma}} X'$ sends this B-point to $(F(x), F(y), h)$. Here, h is a homotopy obtained from h_γ by using $h_f : s_{\tilde{X}} \circ F_{X/S} \simeq \pi^{\gamma}_{X/S}$. Let us make h_f explicit to compute the map. Note the points $s_{\tilde{X}} \circ F_{X/S}, \pi^{\gamma}_{X/S} \in (X/S)^{dR,\gamma}(X) = \mathrm{Map}_R(A, F_*A/p)$ are represented by

$$\begin{array}{ccc} F_*A' & \xrightarrow{i} & \tilde{A} \\ \downarrow & & \downarrow{\scriptstyle \nu \circ pr} \\ F_*A' & \xrightarrow{0} & F_*A' \\ {\scriptstyle F_{A/R}}\downarrow & & \downarrow{\scriptstyle F_{A/R}} \\ F_*A & \longrightarrow & F_*A \end{array} \qquad \begin{array}{ccc} F_*A & \xrightarrow{V} & W_2(A) \\ \| & & \downarrow{\scriptstyle F} \\ F_*A & \xrightarrow[0]{} & F_*A \end{array}$$

respectively. The first diagram defines a map $A \to F_*A/p$ which is also obtained from

$$\tilde{A} \xrightarrow{f} W_2(A) \to F_*\tilde{A} \to \mathrm{Cone}(F_*\tilde{A} \xrightarrow{p} F_*\tilde{A})$$

via the base change along $W_2(R) \to R$. Similarly, the second diagram is obtained from a similar map $\mathrm{Cone}(F_*\tilde{A} \xrightarrow{V} W_2(A)) \to \mathrm{Cone}(F_*\tilde{A} \xrightarrow{p} F_*\tilde{A})$. Now we note that f gives rise to

a homotopy between these two points in

$$(\tilde{X}/W_2(S))^{dR,\gamma}(\tilde{X}) \simeq \mathrm{Map}_{W_2(R)}(\tilde{A}, \mathrm{Cone}(F_*\tilde{A} \xrightarrow{p} F_*\tilde{A})).$$

Explicitly, under $\pi_{\tilde{X}/W_2(S)}$ an elements $\tilde{a}$ is sent to $\tilde{a}^p$. Under the other map, an element $\tilde{a}$ is sent to $\tilde{a}^p + p\delta_{\tilde{a}}$. By construction, these two elements are canonically homotopic and $\delta_{\tilde{a}}$ is the desired homotopy.

Step 2. Take $B = D_\Delta(A^{\otimes_R 2})$ and (x, y, h_γ) to be the canonical B-point of $X \times_{(X/S)^{dR}} X$, i.e., $x, y : A \to D_\Delta(A^{\otimes_R 2})$ the two canonical maps and h_γ is the homotopy between two compositions

$$A \to D_\Delta(A^{\otimes_R 2}) \to \mathbf{G}_a^{dR}(D_\Delta(A^{\otimes_R 2}))$$

coming from divided powers on the ideal $\mathrm{Ker}(D_\Delta(A^{\otimes_R 2}) \to A)$. The map $X \times_{(X/S)^{dR}} X \to X' \times_{(X/S)^{dR,\gamma}} X'$ sends this point to some map $\mathrm{Sym}\,\Omega^1_{A'/R} \to D_\Delta(A^{\otimes_R 2})$ which we can compute using the first step. Namely, it is the homotopy between two compositions

$$A \to D_\Delta(A^{\otimes_R 2}) \xrightarrow{F} F_*D_\Delta(A^{\otimes_R 2}) \xrightarrow{\pi} \mathbf{G}_a^{dR,\gamma}(D_\Delta(A^{\otimes_R 2}))$$

which comes from a homotopy between two compositions

$$\tilde{A} \to D_\Delta(\tilde{A}^{\otimes_{W_2(R)} 2}) \xrightarrow{\tilde{F}} \tilde{F}_*D_\Delta(\tilde{A}^{\otimes_{W_2(R)} 2}) \xrightarrow{s} \mathbf{G}_a^{dR,\gamma}(D_\Delta(\tilde{A}^{\otimes_{W_2(R)} 2})).$$

Explicitly, for any $\tilde{a} \in \tilde{A}$, this homotopy provides a path between $1 \otimes \tilde{F}(a)$ and $\tilde{F}(a) \otimes 1$ in $\mathbf{G}_a^{dR,\gamma}(D_\Delta(\tilde{A}^{\otimes_{W_2(R)} 2}))$. Thus, the resulting $D_\Delta(A^{\otimes_R 2})$-derivation of A is given by the desired formula.

Step 3. Note the image of the map $\theta_f : \Omega^1_{A'} \to D(A^{\otimes_R 2})$ lands in the ideal of the diagonal I_Δ by construction. Therefore, one obtains $\Omega^1_{A'/R} \to I_\Delta/I_\Delta^2 \simeq \Omega^1_{A/R}$ which is equal to $\zeta_{\tilde{F}} = p^{-1}d\tilde{F}$ from the explicit formula of θ_f. □

Corollary II.101. *Let X/S be a smooth family of $\mathbf{F}_p$-schemes. Let $\tilde{X}/W_2(S)$ be a flat lift of X and let $\tilde{S} \to \mathbf{Z}/p^2$ be a flat lift of S. Denote $\tilde{X}' := \tilde{X} \times_{W_2(S)} \tilde{S}$ be the corresponding lift of X' to $\tilde{S}$ induced by $\tilde{X}$. The torsor $\mathcal{L}_{\tilde{X}}$ associated with $\tilde{X}$ is isomorphic to the torsor $\mathcal{L}_{\mathcal{X}}$ studied by Ogus and Vologodsky associated to $\tilde{X}'/\tilde{S}$ compatibly with the connections.*

We need the following auxiliary remark in order to compare the splitting of $F_{X/S}D_{X/S}$ we obtain from the splitting of the de Rham gerbe with the splitting of Ogus-Vologodsky.

Remark II.102. Let $s : X \to (X/S)^{dR,\gamma}$ be a section of $\pi^\gamma_{X/S} : (X/S)^{dR,\gamma} \to X'$. It extends

to an isomorphism $B_{X'}T_{X'/S} \simeq (X/S)^{dR,\gamma}$ of $T_{X'/S}$-gerbes. In particular, one obtains

$$\mathcal{E}nd_{(X/S)^{dR,\gamma}}(s_*\mathcal{O}_{X'}, s_*\mathcal{O}_{X'}) \simeq \mathcal{E}nd_{(B_{X'}(T_{X'/S}))}(i_*\mathcal{O}_{X'}, i_*\mathcal{O}_{X'}) \simeq \hat{\Gamma}_{X'}T_{X'/S} \tag{II.3.1}$$

where $i : X' \to B'_X(T_{X'/S})$ is the zero-section. Note $s_*\mathcal{O}_{X'}$, as a $D^\gamma_{X/S}$-module, is $q_*\mathcal{O}_{\mathcal{L}_{\tilde{X}}}$ endowed with its $D^\gamma_{X/S}$-module structure. Under the equivalence $\mathrm{QCoh}((X/S)^{dR,\gamma}) \simeq \mathrm{MIC}^{\cdot}(X/S)$, the left term in (II.3.1) corresponds to $\mathcal{E}nd_{D^\gamma_{X/S}}(s_*\mathcal{O}^*_{X'}, s_*\mathcal{O}^*_{X'})$ where $s_*\mathcal{O}^*_{X'}$ is the $\mathcal{O}_X$-linear dual of $q_*\mathcal{O}_{\mathcal{L}_{\tilde{X}}}$ with its flat connection. In particular, the equivalence induced by s is the same as the one induced by the splitting of the Azumaya algebra $D^\gamma_{X/S}$ given by the $D^\gamma_{X/S}$-module $\mathcal{H}om_{\mathcal{O}_X}(q_*\mathcal{O}_{\mathcal{L}_{\tilde{X}}}, \mathcal{O}_X)$.

Now we are ready to recover the Cartier transform.

Theorem II.103. *Let X/S be a representable quasi-syntomic morphism of algebraic $\mathbf{F}_p$-xstacks. Any lift $\tilde{X}/W_2(S)$ gives rise to a symmetric monoidal equivalence $C^{-1}_{\tilde{X}} : \mathrm{HIG}^{\cdot}_\gamma(X/S) \to \mathrm{MIC}^{\cdot}_\gamma(X/S)$. Moreover, if X/S is a smooth map of $\mathbf{F}_p$-schemes and we are given a lift $\tilde{S} \to \mathbf{Z}/p^2$ of S, then $C^{-1}_{\tilde{X}}$ is isomorphic to the global Cartier transform from [OV07, Theorem 2.8] applied to the data of a flat lift $\tilde{X} \times_{W_2(S)} \tilde{S} \to \tilde{S}$ of X'/S.*

Proof. Step 0. A lift $\tilde{X}/W_2(S)$ provides a splitting of $\nu^\gamma_{X/S} : (X/S)^{dR,\gamma} \to X'$ by Theorem II.90. Therefore, it defines a symmetric monoidal equivalence of categories $\mathrm{MIC}^{\cdot}_\gamma(X/S) \simeq \mathrm{HIG}^{\cdot}_\gamma(X/S)$.

Step 1. Now assume $X \to S$ is a smooth map of $\mathbf{F}_p$-schemes. Let us compare the equivalences in the presence of $\tilde{S}$. Since $\tilde{X}' := \tilde{X} \times_{W_2(S)} \tilde{S}$ is a flat lift of X'/S to $\tilde{S}$, thus [OV07, Theorem 2.8] provides us with a symmetric monoidal equivalence $C^{-1}_{\mathcal{X}} : \mathrm{MIC}^{\cdot}_\gamma(X/S) \to \mathrm{HIG}^{\cdot}_\gamma(X/S)$. Let us prove that it is isomorphic to $C^{-1}_{\tilde{X}}$. The composition $(X/S)^{dR} \to (X/S)^{dR,\gamma} \to B_{X'}(T_{X'/S})$ defines a $F^*_{X/S}T_{X'/S}$-torsor on X with a flat connection. By Theorem II.102 it is enough to prove that this torsor is isomorphic to the torsor from [OV07, Theorem 1.1]. Recall the torsor $\mathcal{L}_{\tilde{X}'}$ from [OV07, Theorem 2.8] is the torsor of liftings of $F_{X/S}$ to X'/S with the connection as in [OV07, Theorem 1.1]. By Theorem II.95 the underlying torsor of $(X/S)^{dR} \to (X/S)^{dR,\gamma} \simeq B_{X'}(T_{X'/S})$ is isomorphic to $\mathcal{L}_{\mathcal{X}}$. Thus, it is enough to compare connections. We recall that the connection of Ogus-Vologodsky sends a local section $\tilde{F}$ to a map $d\tilde{F}/p : F^*_{X/S}\Omega^1_{X'/S} \to \Omega^1_{X/S}$. Then Theorem II.100 identifies the two connections. This finishes the proof. □

CHAPTER III
Prismatic F-Gauges and Fontaine-Laffaille Modules

III.0.1: Notations

Fix a prime number p. For any stack X, we denote by $\mathcal{D}_{qc}(X)$ the stable ∞-category of quasi-coherent sheaves on X. For a stack X over $W(k)$, denote by $X^\wedge$ the corresponding p-adic formal stack, *i.e.*, for any ring R, we have that $X^\wedge(R) = X(R)$ if p is nilpotent in R, and $X^\wedge(R) = \varnothing$ otherwise. In other words, $X^\wedge = X \times \operatorname{Spf} \mathbf{Z}_p$. In particular, $\mathcal{D}_{qc}(X^\wedge) = \varprojlim_n \mathcal{D}_{qc}(X \otimes \mathbf{Z}/p^n\mathbf{Z})$.

For a commutative ring R, we will write $\operatorname{Spf} R$ for the formal spectrum of the p-completion of R with respect to the ideal (p), *i.e.*, $\operatorname{Spf} R = (\operatorname{Spec} R)^\wedge$. For a finitely generated ideal $I \subset R$, we shall write $\widehat{\mathcal{D}}(R)$ for the full subcategory of $\mathcal{D}(R)$ spanned by I-complete objects (see for example [BL22a, §1.9]).

III.1: Preliminaries

III.1.1: Quasi-coherent sheaves on $\mathbf{A}^1/\mathbf{G}_m$

We recall the standard dictionary between the derived category of quasi-coherent sheaves on $\mathbf{A}^1/\mathbf{G}_m$ and the filtered derived category of graded $W(k)$-modules. See [Bha23, §2.2] for additional details.

Let z denote the coordinate function on $\mathbf{G}_m$. Consider the standard action of $\mathbf{G}_m$ on $\mathbf{A}^1 = \operatorname{Spec} W(k)[v_+]$ given by $\mathbf{G}_m \times \mathbf{A}^1 \to \mathbf{A}^1, v_+ \mapsto z \otimes v_+$. We shall denote the p-adic formal scheme associated to $\mathbf{A}^1$ with the above $\mathbf{G}_m$-action by $\mathbf{A}^1_+$, *i.e.*, $\mathbf{A}^1_+ := \operatorname{Spf} W(k)[v_+] := (\operatorname{Spec} W(k)[v_+])^\wedge$. Similarly, we define an action of $\mathbf{G}_m$ on $\mathbf{A}^1_- := \operatorname{Spf} W(k)[v_-]$ given by $v_- \mapsto z^{-1} \otimes v_-$. We write $\mathbf{A}^1_\pm/\mathbf{G}_m$ for the quotient p-adic formal stacks over $B\mathbf{G}_m$. Thus, for a p-nilpotent $W(k)$-algebra R, the groupoid of R-points of $\mathbf{A}^1_+/\mathbf{G}_m$ (resp. $\mathbf{A}^1_-/\mathbf{G}_m$) classifies pairs (L, v_+) (resp. (L, v_-)), where L is a line bundle over $S := \operatorname{Spec} R$ and $v_+ : \mathcal{O}_S \to L$ is a section of L (resp. $v_- : L \to \mathcal{O}_S$ is a section of L^*). We denote by $(\mathcal{O}(1), v_+)$ (resp. $(\mathcal{O}(1), v_-)$) the universal pair over $\mathbf{A}^1_+/\mathbf{G}_m$ (resp. $\mathbf{A}^1_-/\mathbf{G}_m$). We have

a map $\mathbf{A}^1_\pm/\mathbf{G}_m \to B\mathbf{G}_m$ classifying the line bundle $\mathcal{O}(1)$ [1]. We identify the derived category of quasi-coherent sheaves on $\mathbf{A}^1_\pm/\mathbf{G}_m$ with the derived category of p-complete graded modules over $W(k)[v_\pm]$[2]:

$$\mathcal{D}_{qc}(\mathbf{A}^1_\pm/\mathbf{G}_m) \simeq \widehat{\mathcal{D}}_{gr}(W(k)[v_\pm]). \tag{III.1.1}$$

The functor $\mathcal{D}_{qc}(\mathbf{A}^1_-/\mathbf{G}_m) \to \widehat{\mathcal{D}}_{gr}(W(k)[v_-])$ is given by

$$\mathcal{M} \mapsto (\oplus F^i, F^i \xrightarrow{v_-} F^{i-1}, \deg F^i = i), \tag{III.1.2}$$

where $F^i := \mathrm{R}\Gamma(\mathbf{A}^1_-/\mathbf{G}_m, \mathcal{M}\otimes\mathcal{O}(i))$ with $\mathcal{O}(i)$ the line bundle pulled back from $B\mathbf{G}_m$. The map $v_- : F^i \to F^{i-1}$ comes from the global section $v_- \in \Gamma(\mathbf{A}^1_-/\mathbf{G}_m, \mathcal{O}(-1))$. Similarly, the functor $\mathcal{D}_{qc}(\mathbf{A}^1_+/\mathbf{G}_m) \to \widehat{\mathcal{D}}_{gr}(W(k)[v_+])$ is given by

$$\mathcal{M} \mapsto (\oplus G_i, G_i \xrightarrow{v_+} G_{i+1}, \deg G_i = i) \tag{III.1.3}$$

where $G_i := \mathrm{R}\Gamma(\mathbf{A}^1_+/\mathbf{G}_m, \mathcal{M} \otimes \mathcal{O}(i))$.

Definition III.1. For $a, b \in \mathbf{Z} \cup \{+\infty, -\infty\}$ such that $a \leq b$, let $\mathcal{D}_{qc,[a,b]}(\mathbf{A}^1_\pm/\mathbf{G}_m)$ be a full subcategory of $\mathcal{D}_{qc}(\mathbf{A}^1_\pm/\mathbf{G}_m)$ formed of objects $\mathcal{M}$ such that, under the identification (III.1.3) (resp. (III.1.2)), G_j (resp. F^j) is acyclic for $j < a$ (resp. $j > b$) and $v_+ : G_i \to G_{i+1}$ (resp. $v_- : F^i \to F^{i-1}$) is a quasi-isomorphism for $i \geq b$ (resp. $i \leq a$). Similarly, one defines a full subcategory $\mathrm{Coh}_{[a,b]}(\mathbf{A}^1_\pm/\mathbf{G}_m) \subset \mathrm{Coh}(\mathbf{A}^1_\pm/\mathbf{G}_m)$.

Remark III.2. Consider the map $B\mathbf{G}_m \to \mathbf{A}^1_\pm/\mathbf{G}_m$ given by inclusion of the origin into $\mathbf{A}^1_\pm$. We identify $\mathcal{D}_{qc}(B\mathbf{G}_m)$ with the derived category of p-complete graded complexes of $W(k)$-modules. For any $\mathcal{M} \in \mathcal{D}_{qc,[a,b]}(\mathbf{A}^1_\pm/\mathbf{G}_m)$, its restriction to $B\mathbf{G}_m$ is acyclic in grading degrees outside of $[a, b]$. Thus we have a well-defined functor

$$\mathcal{D}_{qc,[a,b]}(\mathbf{A}^1_\pm/\mathbf{G}_m) \to \mathcal{D}_{qc,[a,b]}(B\mathbf{G}_m).$$

Definition III.3. A sheaf $\mathcal{M} \in \mathcal{D}_{qc}(\mathbf{A}^1_\pm/\mathbf{G}_m)$ is *effective* if $\mathcal{M} \in \mathcal{D}_{qc,[0,+\infty]}(\mathbf{A}^1_\pm/\mathbf{G}_m)$.

III.1.2: Fontaine-Laffaille modules

Denote by $F : W(k) \to W(k)$ the Frobenius endomorphism. For $a \in W(k)$, denote by $i_a : \mathrm{Spf}\, W(k) \hookrightarrow \mathbf{A}^1_- \twoheadrightarrow \mathbf{A}^1_-/\mathbf{G}_m$ the point given by equation $v_- = a$. Given $\mathcal{M} \in$

[1] Note that $\mathbf{A}^1_+/\mathbf{G}_m$ is isomorphic to $\mathbf{A}^1_-/\mathbf{G}_m$ as abstract stacks, but not as stacks over $B\mathbf{G}_m$.

[2] A graded complex $\oplus_i F^i$ of $W(k)[v_-]$-modules is said to be p-complete if each F^i is p-complete.

$\mathcal{D}_{qc}(\mathbf{A}^1_-/\mathbf{G}_m)$, we denote by $\mathcal{M}_{v_-=a} := i_a^*\mathcal{M} \in \mathcal{D}_{p\text{-}comp}(W(k))$ the *derived* pullback; for $\mathcal{M} \in \mathrm{Coh}(\mathbf{A}^1_-/\mathbf{G}_m)$, we write $L_0 i_a^*\mathcal{M} \in \mathrm{Mod}(W(k))$ for the *non-derived* fiber.

Definition III.4. (cf. [FL82, §1.5], [Win84, §1.3]) A finitely generated ("small") Fontaine–Laffaille module over $W(k)$ is $\mathcal{M} \in \mathrm{Coh}(\mathbf{A}^1_-/\mathbf{G}_m)$ together with an isomorphism between the fibres

$$\varphi : F^*(L_0 i_p^*\mathcal{M}) \xrightarrow{\sim} L_0 i_1^*\mathcal{M}.$$

Morphisms between Fontaine–Laffaille modules are morphisms between coherent sheaves on $\mathbf{A}^1_-/\mathbf{G}_m$ compatible with the isomorphisms φ. We denote the category of Fontaine–Laffaille modules over $W(k)$ by $\mathscr{MF}(W(k))$.

Remark III.5. The full subcategory $\mathscr{MF}^{\mathrm{tor}}(W(k))$ of $\mathscr{MF}(W(k))$ spanned by p-power-torsion objects is equivalent to the category $\underline{MF}^f_{tor}$ introduced in [FL82, §1.5]. However, the definition in [FL82] does not use the language of stacks. To compare Definition III.4 with the one given in [FL82], observe an equivalence between $\mathrm{Coh}(\mathbf{A}^1_-/\mathbf{G}_m)$ and the category of finitely generated graded modules $\bigoplus F^i$ over $W(k)[v_-]$ as explained in §III.1.1. Under this equivalence, the fibre $L_0 i_1^*\mathcal{M}$ of $\mathcal{M}$ over the point $v_- = 1$ is sent to $F^{-\infty} := \varinjlim_{v_-} F^i$. Giving a morphism $\varphi : F^*(L_0 i_p^*\mathcal{M}) \to L_0 i_1^*\mathcal{M}$ is equivalent to giving a collection of F-linear maps $\varphi_i : F^i \to F^{-\infty}$ such that $\varphi_{i-1} v_- = p\varphi_i$. Assume that $\mathcal{M}$ is of p-power torsion. Then by [Fal89, Theorem 2.1] (see also Proposition III.1.2 below for the implication in the forward direction), the fact that φ is an isomorphism is equivalent to the combination of the following two statements:

1. Each $v_- : F^i \to F^{i-1}$ is a *split* injection; and
2. $F^{-\infty} = \sum_i \mathrm{im}(\varphi_i)$.

These two are precisely the conditions given in [FL82, §1.5] except for the adjective *split* in (1) which is missing in *loc. cit.* However, using that the $W(k)$-module $F^{-\infty}$ has finite length, one shows (see [Win84, Proposition 1.4.1 (ii)] or [Fal89, Theorem 2.1]) that the splitting condition for $v_- : F^i \to F^{i-1}$ is redundant.

The whole category $\mathscr{MF}(W(k))$ can be described similarly: an object of $\mathscr{MF}(W(k))$ consists of a finitely generated $W(k)$-module $F^{-\infty}$ equipped with an exhaustive filtration by $W(k)$-submodules $F^\bullet$ together with F-linear maps $\varphi_i : F^i \to F^{-\infty}$, $\varphi_{i-1} v_- = p\varphi_i$, such that the conditions (1) and (2) above hold[3]. This category was introduced in [Win84, §1.3]. The category of strongly divisible lattices that appeared in [FL82, §7.11] is equivalent to the full subcategory of p-torsionfree objects of $\mathscr{MF}(W(k))$.

[3] In the nontorsion case the adjective *split* in (1) is essential.

Definition III.6. Let $\mathscr{D}\mathscr{M}\mathscr{F}^{\mathrm{big}}(W(k))$ be the stable $\mathbf{Z}_p$-linear ∞-category formed of objects $\mathcal{M} \in \mathcal{D}_{qc}(\mathbf{A}^1_-/\mathbf{G}_m)$ together with an isomorphism $\varphi : F^*(\mathcal{M}_{v_-=p}) \xrightarrow{\sim} \mathcal{M}_{v_-=1}$, *i.e.*, we have

$$\mathscr{D}\mathscr{M}\mathscr{F}^{\mathrm{big}}(W(k)) := \mathrm{Eq}\left(\mathcal{D}_{qc}(\mathbf{A}^1_-/\mathbf{G}_m) \underset{F^*\circ i_p^*}{\overset{i_1^*}{\rightrightarrows}} \mathcal{D}_{qc}(\mathrm{Spf}\, W(k))\right),$$

where i_1^* denotes the pullback to the point $v_- = 1$ and $F^* \circ i_p^*$ denotes the pullback to the point $v_- = p$ post-composed with the Frobenius.

Part (1) of the following Proposition, with $\mathscr{M}\mathscr{F}(W(k))$ replaced by $\mathscr{M}\mathscr{F}^{\mathrm{tor}}(W(k))$, is proven in [FL82, §1.8], and in [Win84, Proposition 1.4] in general. Part (2) is proven in [Fal89, Theorem 2.1]. pr

1. For every $(\mathcal{M}, \varphi) \in \mathscr{M}\mathscr{F}(W(k))$ and every point $i_a : \mathrm{Spf}\, W(k) \to \mathbf{A}^1_-/\mathbf{G}_m$, the derived pullback $Li_a^*\mathcal{M}$ is concentrated in cohomological degree 0. In particular, for every $j \in \mathbf{Z}$, the map $F^j \xrightarrow{v_-} F^{j-1}$ is injective[4].

2. The category $\mathscr{M}\mathscr{F}(W(k))$ is abelian. Moreover, the functor $\mathscr{M}\mathscr{F}(W(k)) \to \mathrm{Coh}(\mathbf{A}^1_-/\mathbf{G}_m)$ given by $(\mathcal{M}, \varphi) \mapsto \mathcal{M}$ is exact.

3. Let $\mathcal{D}^b(\mathscr{M}\mathscr{F}(W(k)))$ be the bounded derived category of $\mathscr{M}\mathscr{F}(W(k))$. Then the functor $\mathcal{D}^b(\mathscr{M}\mathscr{F}(W(k))) \to \mathscr{D}\mathscr{M}\mathscr{F}^{\mathrm{big}}(W(k))$ is fully faithful, and its essential image is formed of pairs $(\mathcal{M}, \varphi)$ with $\mathcal{M} \in \mathcal{D}^b(\mathrm{Coh}(\mathbf{A}^1_-/\mathbf{G}_m))$.

Proof. Though the first two assertions are known, for the reader's convenience, we include a proof.

To prove (1) we first verify, by induction on n, that if $(\mathcal{M}, \varphi)$ is a Fontaine-Laffaille module with $p^n\mathcal{M} = 0$, then $L_1 i_a^*\mathcal{M} = 0$. For $n = 1$, we check that the pullback of $\mathcal{M}$ to $\mathbf{A}^1_- \otimes k$ is a vector bundle; this would imply the assertion. Indeed, by $\mathbf{G}_m$-equivariance it suffices to show $\dim L_0 i_p^*\mathcal{M} = \dim L_0 i_1^*\mathcal{M}$[5]. But this follows from the existence of φ. For the induction step, set $\mathcal{M}' = \mathrm{Ker}(\mathcal{M} \to \mathcal{M}/p)$ and consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^*L_0 i_p^*\mathcal{M}' & \longrightarrow & F^*L_0 i_p^*\mathcal{M} & \longrightarrow & F^*L_0 i_p^*(\mathcal{M}/p) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_{\mathcal{M}'}} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} & & \\ 0 & \longrightarrow & L_0 i_1^*\mathcal{M}' & \longrightarrow & L_0 i_1^*\mathcal{M} & \longrightarrow & L_0 i_1^*(\mathcal{M}/p) & \longrightarrow & 0 \end{array} \tag{III.1.4}$$

[4]Indeed, $i_0^*\mathcal{M}$ is isomorphic to $\bigoplus_j \mathrm{Cone}(F^j \xrightarrow{v_-} F^{j-1})$.

[5]Indeed, a coherent sheaf $\mathcal{F}$ on $\mathbf{A}^1_- \otimes k$ is a vector bundle if, for every closed point a of $\mathbf{A}^1_- \otimes k$, the dimension of its fiber over a is equal to the dimension of its fiber over the generic point. If $\mathcal{F}$ is $\mathbf{G}_m$-equivariant, then $\mathcal{F}$ restricted to $(\mathbf{A}^1_- \otimes k)\backslash\{0\}$ is automatically a vector bundle. Thus $\mathcal{F}$ is a vector bundle if $\dim L_0 i_0^*\mathcal{M} = \dim L_0 i_p^*\mathcal{M}$ is equal to $\dim L_0 i_1^*\mathcal{M}$.

Using that $L_1 i_p^*(\mathcal{M}/p) = 0$, the rows are exact. It follows that the left downward arrow is an isomorphism. Thus, $(\mathcal{M}', \varphi_{\mathcal{M}'})$ is a Fontaine-Laffaille module. Applying the induction hypothesis to $\mathcal{M}'$ and $\mathcal{M}/p$ we infer the assertion for $\mathcal{M}$.

To complete the proof of (1), let us check that $L_1 i_a^* \mathcal{M} = 0$, for *any* Fontaine-Laffaille module. Indeed, let M denote the space of global sections of $\mathcal{M}$ pulled back to $\mathbf{A}^1_-$. Since $\mathcal{M}$ is coherent, M is a finitely generated module over $W(k)[v_-]^\wedge$. In particular, M is p-complete in the classical (non-derived) sense, $M \xrightarrow{\sim} R^0 \varprojlim M/p^n \xrightarrow{\sim} R \varprojlim M/p^n$. The derived fiber $i_a^* \mathcal{M}$ is isomorphic to $\mathrm{Cone}(M \xrightarrow{v_- - a} M)$. We have $\mathrm{Cone}(M \xrightarrow{v_- - a} M) = R \varprojlim \mathrm{Cone}(M/p^n \xrightarrow{v_- - a} M/p^n)$. Since $\mathcal{M}/p^n$ has a structure of Fontaine-Laffaille module, $\mathrm{Cone}(M/p^n \xrightarrow{v_- - a} M/p^n)$ is supported in cohomological degree 0, as was shown in the previous paragraph. Thus, $L_n i_a^* \mathcal{M} = 0$, for $n > 0$.

To prove (2), we start with a preliminary construction. Let $\mathscr{MF}'(W(k))$ be the category formed of objects $\mathcal{M} \in \mathrm{Coh}(\mathbf{A}^1_-/\mathbf{G}_m)$ together with a morphism $\varphi : F^* L_0 i_p^* \mathcal{M} \to L_0 i_1^* \mathcal{M}$. Since the functor $L_0 i_1^* \colon \mathrm{Coh}(\mathbf{A}^1_-/\mathbf{G}_m) \to \mathrm{Coh}(\mathrm{Spf}\, W(k))$ is exact and $L_0 i_p^*$ is right exact, the category $\mathscr{MF}'(W(k))$ is abelian, and the forgetful functor $\mathscr{MF}'(W(k)) \to \mathrm{Coh}(\mathbf{A}^1_-/\mathbf{G}_m)$ is exact and conservative. The category $\mathscr{MF}(W(k))$ is a full subcategory of the abelian category $\mathscr{MF}'(W(k))$. Thus to show that $\mathscr{MF}(W(k))$ is abelian, it suffices to show that, for any morphism f in $\mathscr{MF}(W(k))$, its kernel and cokernel computed in $\mathscr{MF}'(W(k))$ lie in $\mathscr{MF}(W(k))$. The assertion on cokernel is clear from the right exactness properties of $L_0 i_p^*$ and $L_0 i_1^*$. Let us check that $\mathrm{Ker}\, f \in \mathscr{MF}(W(k))$. Indeed, using the vanishing of $L_1 i_p^*$ for Fontaine-Laffaille modules we conclude that $L_0 i_p^* \mathrm{Ker}(\mathcal{M} \xrightarrow{f} \mathcal{M}') \xrightarrow{\sim} \mathrm{Ker}(L_0 i_p^* \mathcal{M} \xrightarrow{f} L_0 i_p^* \mathcal{M}')$ which implies that $\phi : F^* L_0 i_p^* \mathrm{Ker}(\mathcal{M} \xrightarrow{f} \mathcal{M}') \to L_0 i_0^* \mathrm{Ker}(\mathcal{M} \xrightarrow{f} \mathcal{M}')$ is an isomorphism.

To prove (3): by part (1), for any $\mathcal{M} \in \mathscr{MF}(W(k))$, we have a canonical isomorphism $F^* i_p^* \mathcal{M} \simeq F^* i_p^* M$, where the pullbacks are *derived*. This defines an exact functor $\mathscr{MF}(W(k)) \to \mathscr{DMF}^{\mathrm{big}}(W(k))$ that extends uniquely to an exact functor

$$\mathcal{D}^b(\mathscr{MF}(W(k))) \to \mathscr{DMF}^{\mathrm{big}}(W(k)). \tag{III.1.5}$$

Let us check that (III.1.5) is fully faithful. It suffices to show that, for any two objects $(\mathcal{M}, \varphi_{\mathcal{M}}), (\mathcal{N}, \varphi_{\mathcal{N}}) \in \mathscr{MF}(W(k))$, the map of mapping spectra

$$\mathrm{Hom}_{\mathcal{D}^b(\mathscr{MF}(W(k)))}(\mathcal{M}, \mathcal{N}) \to \mathrm{Hom}_{\mathscr{DMF}^{\mathrm{big}}(W(k))}(\mathcal{M}, \mathcal{N}) \tag{III.1.6}$$

is an equivalence. First, we check this for a p-torsionfree $\mathcal{M}$. Using part (1), this condition is equivalent to $\mathcal{M}$ being a locally free $\mathcal{O}$-module. By definition of the equalizer category,

the spectrum $\mathrm{Hom}_{\mathscr{DMF}^{\mathrm{big}}(W(k))}(\mathcal{M},\mathcal{N})$ is given by

$$\mathrm{Fib}\left(\mathrm{Hom}_{\mathcal{D}_{qc}(\mathbf{A}^1_-/\mathbf{G}_m)}(\mathcal{M},\mathcal{N}) \to \mathrm{Hom}_{\mathcal{D}_{qc}(W(k))}(F^*i_p^*\mathcal{M}, i_1^*\mathcal{N})\right), \tag{III.1.7}$$

where the map carries $f \in \mathrm{Hom}_{\mathcal{D}_{qc}(\mathbf{A}^1_-/\mathbf{G}_m)}(\mathcal{M},\mathcal{N})$ to $i_1^*(f)\circ\varphi_{\mathcal{M}} - \varphi_{\mathcal{N}}\circ F^*i_p^*(f)$. In particular, if $\mathcal{M}$ is a locally free $\mathcal{O}$-module, $\pi_i\mathrm{Hom}_{\mathscr{DMF}^{\mathrm{big}}(W(k))}(\mathcal{M},\mathcal{N}) = 0$ unless $i = 0$ or -1: each mapping spectrum appearing in (III.1.7) is isomorphic to an abelian group. Using [BK90, Lemma 4.4], the same is true for $\mathrm{Hom}_{\mathcal{D}^b(\mathscr{MF}(W(k)))}(\mathcal{M},\mathcal{N})$. Since $\mathscr{MF}(W(k)) \to \mathscr{DMF}^{\mathrm{big}}(W(k))$ is fully faithful and its essential image is closed under extensions (that is, for any morphism $\mathcal{N} \to \mathcal{M}[1]$ in $\mathscr{DMF}^{\mathrm{big}}(W(k))$, its fiber is isomorphic to an object of $\mathscr{MF}(W(k))$) it follows that (III.1.6) is an equivalence for torsion-free $\mathcal{M}$. In general, for any $\mathcal{M}$, we have a fiber sequence $\mathcal{M}_{tor} \hookrightarrow \mathcal{M} \twoheadrightarrow \mathcal{M}/\mathcal{M}_{tor}$, where $\mathcal{M}_{tor}$ is a torsion Fontaine-Laffaile module and $\mathcal{M}/\mathcal{M}_{tor}$ is torsion-free. Thus, it remains to prove that (III.1.6) is an equivalence for $\mathcal{M} = \mathcal{M}_{tor}$. By dévissage, we may assume that $p\mathcal{M} = 0$. In this case the assertion reduces to the torsion-free case using the following observation.

Lemma III.7. *For any p-torsion Fontaine–Laffaille module $(\mathcal{M},\varphi)$, there exists a torsion-free Fontaine–Laffaille module $(\tilde{\mathcal{M}},\tilde{\varphi})$ with $(\tilde{\mathcal{M}},\tilde{\varphi})\otimes\mathbf{F}_p \xrightarrow{\sim} (\mathcal{M},\varphi)$.*

Proof. Any vector bundle over $\mathbf{A}^1_-/\mathbf{G}_m\otimes\mathbf{F}_p$ is isomorphic to a direct sum of line bundles of the form $\mathcal{O}(i)$, $i\in\mathbf{Z}$. In particular, it lifts to a vector bundle over $\mathbf{A}^1_-/\mathbf{G}_m$. Pick a vector bundle $\tilde{\mathcal{M}}$ over $\mathbf{A}^1_-/\mathbf{G}_m$ that lifts $\mathcal{M}$ and then choose any $\tilde{\varphi} : F^*(\tilde{\mathcal{M}}_{v_-=p}) \xrightarrow{\sim} \tilde{\mathcal{M}}_{v_-=1}$ that lifts $\varphi : F^*(\tilde{\mathcal{M}}_{v_-=p})\otimes\mathbf{F}_p \xrightarrow{\sim} \tilde{\mathcal{M}}_{v_-=1}\otimes\mathbf{F}_p$. □

It remains to prove that the essential image of the functor (III.1.5) consists of all objects $(\mathcal{M},\varphi)$ whose underlying complex $\mathcal{M}\in\mathcal{D}_{qc}(\mathbf{A}^1_-/\mathbf{G}_m)$ is bounded and has coherent cohomology. We induct on length of the complex. If $\mathcal{M}$ is supported in a single cohomological degree the assertion is clear. For the induction step, we assume that $\mathcal{M}$ is connective and $H^0(\mathcal{M}) \neq 0$. Note that the isomorphism $\varphi : F^*i_p^*\mathcal{M} \xrightarrow{\sim} i_1^*\mathcal{M}$ induces $H^0(\varphi) : F^*L_0i_p^*H^0(\mathcal{M}) \xrightarrow{\sim} L_0i_1^*H^0(\mathcal{M})$. Thus $(H^0(\mathcal{M}), H^0(\varphi)) \in \mathscr{MF}(W(k))$. By part (1) of the Proposition $L_ji_p^*H^0(\mathcal{M}) = L_ji_1^*H^0(\mathcal{M}) = 0$ for $j > 0$. It follows that the map $\mathcal{M} \to H^0(\mathcal{M})$ in $\mathcal{D}_{qc}(\mathbf{A}^1_-/\mathbf{G}_m)$ lifts to a morphism in $\mathscr{DMF}^{\mathrm{big}}(W(k))$. Applying the induction assumption to its fiber we complete the proof. □

Let $\mathscr{MF}_{[a,b]}(W(k))$ (resp. $\mathscr{DMF}^{\mathrm{big}}_{[a,b]}(W(k))$) be a full subcategory of $\mathscr{MF}(W(k))$ (resp. $\mathscr{DMF}^{\mathrm{big}}(W(k))$) formed of objects $(\mathcal{M},\varphi)\in\mathscr{MF}(W(k))$ (resp. $\mathscr{DMF}^{\mathrm{big}}(W(k))$) such that $\mathcal{M}\in\mathcal{D}_{qc,[a,b]}(\mathbf{A}^1_-/\mathbf{G}_m)$.

III.1.3: Syntomification via filtered Cartier-Witt divisors

Recall from [Dri24], [Bha23, §5.3] definitions of the stacks $W(k)^{\mathbf{\Delta}}$, $W(k)^{\mathcal{N}}$, $W(k)^{syn}$, and X^{syn}.

Throughout this paper we let W be the p-typical Witt ring scheme pulled back to $\mathrm{Spec}\mathbf{Z}_{(p)}$. For a p-nilpotent $W(k)$-algebra R, set $W_R := W \times \mathrm{Spec}R$. A *$W_R$-module* M is an affine group scheme over R with an action of the ring scheme W_R. We say that M is invertible if, Zariski locally on $\mathrm{Spec}R$, M is isomorphic to the free W_R-module W_R. Denote by $W(k)^{\mathbf{\Delta}}(R)$ the groupoid whose objects are pairs (M, ξ), where M is an invertible W_R-module and $\xi : M \to W_R$ is a morphism of W_R-modules such that Zariski locally on $\mathrm{Spec}R$ the pair (M, ξ) is isomorphic to $(W_R, w\,\mathrm{Id})$, $w = (w_1, w_2, \cdots) \in W(R)$, with $w_1 \in R$ nilpotent and $w_2 \in R^*$.

For a W_R-module M, precomposing the W_R-action on M with the Frobenius $F : W_R \to W_R$ we obtain a new W_R-module F_*M whose underlying group scheme is that of M. Applying this to the free W_R-module W_R, we have an exact sequence

$$0 \to \mathbf{G}_{a,R}^{\sharp} \longrightarrow W_R \xrightarrow{F} F_*W_R \to 0 \tag{III.1.8}$$

of W_R-modules[6]. Note that the action of W_R on $\mathbf{G}_{a,R}^{\sharp}$ factors through the "first coordinate" homomorphism $W_R \to \mathbf{G}_{a,R}$. In particular, we have a natural action of the group $\mathbf{G}_{m,R}$ on the W_R-module $\mathbf{G}_{a,R}^{\sharp}$. Consequently, given an invertible R-module L we can twist $\mathbf{G}_{a,R}^{\sharp}$ by L; the resulting W_R-module is denoted by $\mathbf{V}(L)^{\sharp}$. A W_R-module M is called *admissible* if M fits into a sequence of W_R-modules of the form

$$0 \to \mathbf{V}(L)^{\sharp} \to M \to F_*M' \to 0, \tag{III.1.9}$$

where $L \in \mathrm{Pic}(R)$ and M' is an invertible W_R-module. One checks (see [Dri24, Lemma 3.12.7]) that L, M', and the sequence (III.1.9) are functorial in M. In particular, every morphism $\xi : M \to W_R$ of W_R-modules extends uniquely to a morphism between short exact sequences (III.1.9) and (III.1.8). Consider the category $W(k)^{\mathcal{N},c}(R)$ consisting of pairs (M, ξ), called filtered Cartier-Witt divisors, where M is an admissible W_R-module and $\xi : M \to W_R$ is a morphism of W_R-modules such that the pair $(M', \bar{\xi} : M' \to W_R)$, constructed from the morphism of short exact sequences, is an object $W(k)^{\mathbf{\Delta}}(R)$. Morphisms between (M, ξ) and (M', ξ') in $W(k)^{\mathcal{N},c}(R)$ are given by morphisms $f : M \to M'$ of W_R-modules

[6] To justify our notation for the kernel of F recall a fundamental observation of Drinfeld [Dri24, Lemma 3.2.6] and Bhatt-Lurie [BL22a] that the (nonunital) ring scheme $W_R^{(F)} := \ker(W_R \xrightarrow{F} W_R)$ is isomorphic to the PD-hull $\mathbf{G}_{a,R}^{\sharp}$ of $\mathbf{G}_{a,R}$ at 0.

such that $\xi' \circ f = \xi$. The groupoid $W(k)^{\mathcal{N}}(R)$ is obtained from $W(k)^{\mathcal{N},c}(R)$ by discarding all morphisms which are not isomorphisms. By definition, the groupoid of R-points of the stack $W(k)^{\mathcal{N}}$ is $W(k)^{\mathcal{N}}(R)$. The functor $W(k)^{\mathcal{N},c}$ from the category of p-nilpotent $W(k)$-algebras to the 2-category of categories that carries an algebra R to $W(k)^{\mathcal{N},c}(R)$ is an example of what Drinfeld calls in [Dri24] a *c-stack*. We will not use this notion in the main body of the paper except for Remark III.25.

Given an object (M, ξ) of $W(k)^{\mathcal{N}}(R)$, the morphism of W_R-modules $\mathbf{V}(L)^\sharp \to \mathbf{G}_{a,R}^\sharp$ derived from ξ lifts uniquely to a morphism of R-modules $L \to R$ (see [Dri24, Lemma 3.12.4, (ii)]). This defines a morphism of stacks called the Rees map:

$$t_{W(k)} : W(k)^{\mathcal{N}} \to \mathbf{A}^1_-/\mathbf{G}_m \,. \tag{III.1.10}$$

Sending $(M, \xi) \in W(k)^{\mathcal{N}}(R)$ to $(M', \bar{\xi} : M' \to W_R) \in W(k)^{\mathbf{\Delta}}(R)$ determines a morphism $\pi : W(k)^{\mathcal{N}} \to W(k)^{\mathbf{\Delta}}$ called the structure morphism. Every invertible W_R-module is admissible and, moreover, every point $(M, \xi) \in W(k)^{\mathbf{\Delta}}(R)$ is also a point of $W(k)^{\mathcal{N}}(R)$ (see [Dri24, §5.3]). This defines a map $j_+ : W(k)^{\mathbf{\Delta}} \to W(k)^{\mathcal{N}}$ which exhibits $W(k)^{\mathbf{\Delta}}$ as an open substack of $W(k)^{\mathcal{N}}$ (see [Dri24, Lemma 5.3.1]). As in [Dri24, §5.6], we define another open embedding $j_- : W(k)^{\mathbf{\Delta}} \to W(k)^{\mathcal{N}}$ as follows. For $(M, \theta) \in W(k)^{\mathbf{\Delta}}(R)$, define a filtered Cartier-Witt divisor $j_-(M, \theta) := (N, \xi) \in W(k)^{\mathcal{N}}(R)$ by the pullback diagram

$$\begin{array}{ccc} N & \longrightarrow & F_*M \\ \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle F_*(\theta)} \\ W_R & \xrightarrow{F} & F_*W_R \end{array} , \tag{III.1.11}$$

One can check that open substacks defined by j_- and j_+ do not intersect (see [Dri24, Lemma 5.6.3]). The stack $W(k)^{syn}$ is obtained from $W(k)^{\mathcal{N}}$ by gluing two copies of $W(k)^{\mathbf{\Delta}}$ using the maps j_- and j_+.

For every $(M, \xi) \in W(k)^{\mathcal{N}}(R)$ one considers the $W(k)$-algebra stack $\mathrm{Cone}(M \xrightarrow{\xi} W_R)$ (we refer the reader to [Dri24, §1.3-1.4] and the references therein for a discussion of the notion of ring stack). As a group stack, this is simply the quotient stack $[W_R/M]$. The algebra structure on the quotient comes the natural DG-algebra structure on $M \xrightarrow{\xi} W_R$ given by the W_R-module structure on M (see [Dri24, §1.3.3]). By "*transmutation*" ([Bha23, Remark 2.3.8], [Dri24, §1.4.2]), this defines a *Nygaardization* functor $X \mapsto X^{\mathcal{N}}$ from the category of bounded p-adic formal schemes over $\mathrm{Spf}\, W(k)$ to stacks: for a p-nilpotent $W(k)$-algebra R, an R-point of the groupoid $X^{\mathcal{N}}(R)$ consists of $(M, \xi) \in W(k)^{\mathcal{N}}(R)$ together with a morphism $\mathrm{Spec}(\mathrm{Cone}(M \xrightarrow{\xi} W_R)(R)) \to X$ of derived schemes over $W(k)$. For every $(M, \theta) \in W(k)^{\mathbf{\Delta}}(R)$, setting $j_+(M, \theta) = (N_+, \xi_+)$, $j_-(M, \theta) = (N_-, \xi_-)$, one constructs

(see [Dri24, §5.8])) a natural isomorphism $\mathrm{Cone}(N_- \xrightarrow{\xi_-} W_R) \xrightarrow{\sim} \mathrm{Cone}(N_+ \xrightarrow{\xi_+} W_R)$ of $W(k)$-algebra stacks (the map is given by the rightward arrows in diagram (III.1.1.11)). Consequently, the fibers of $X^{\mathcal{N}} \to W(k)^{\mathcal{N}}$ over the two copies of $W(k)^{\mathbf{\Delta}}$ inside $W(k)^{\mathcal{N}}$ are naturally identified (and denoted by[7] $j_+, j_- : X^{\mathbf{\Delta}} \hookrightarrow X^{\mathcal{N}}$). The stack X^{syn} is obtained from $X^{\mathcal{N}}$ by gluing these two open substacks. This procedure defines a functor $X \mapsto X^{syn}$ from bounded p-adic formal schemes over $\mathrm{Spf}\, W(k)$ to stacks. In particular, if $f : X \to \mathrm{Spf}\, W(k)$ is such a scheme, we have by functoriality a map $f^{syn} : X^{syn} \to W(k)^{syn}$. We refer to $\mathcal{H}_{syn}(X) := Rf^{syn}_*(\mathcal{O}_{X^{syn}}) \in \mathcal{D}_{qc}(W(k)^{syn})$ as the F-gauge associated to X.

Recall from [Dri24, Lemma 5.13.4 (i)] that we have a morphism

$$\mathfrak{p}_{\bar{\mathrm{dR}}} : \mathbf{A}^1_-/\mathbf{G}_m \to W(k)^{\mathcal{N}} \tag{III.1.12}$$

constructed as follows. Recall that an R-point of $\mathbf{A}^1_-/\mathbf{G}_m$ is a pair (L, v_-), where L is an invertible R-module equipped with an R-linear morphism $v_- : L \to R$. The map $\mathfrak{p}_{\bar{\mathrm{dR}}}$ is given by $\mathfrak{p}_{\bar{\mathrm{dR}}}(L, v_-) = (M_{\mathfrak{p}_{\bar{\mathrm{dR}},R}}, \xi)$, where $M_{\mathfrak{p}_{\bar{\mathrm{dR}},R}} = \mathbf{V}(L)^\sharp \oplus F_* W_R$, and

$$\xi : M_{\mathfrak{p}_{\bar{\mathrm{dR}},R}} = \mathbf{V}(L)^\sharp \oplus F_* W_R \to W_R \tag{III.1.13}$$

is given by $\xi := (v_-^\sharp, V)$. Here V is the Verschiebung and $v_-^\sharp : \mathbf{V}(L)^\sharp \to W_R$ is the composition of the map $\mathbf{V}(L)^\sharp \to \mathbf{G}^\sharp_{a,R}$ induced by v_- and the embedding $\mathbf{G}^\sharp_{a,R} \hookrightarrow W_R$ from (III.1.1.8). One can verify (see [Bha23, Theorem 2.5.6]) that, for a smooth p-adic formal scheme X over $\mathrm{Spf}\, W(k)$, the complex $\mathfrak{p}^*_{\bar{\mathrm{dR}}}(\mathcal{H}_{syn}(X)) \in \mathcal{D}_{qc}(\mathbf{A}^1_-/\mathbf{G}_m)$ recovers the Hodge filtered de Rham cohomology of X.

The stack $k^{\mathcal{N}}$ can be described explicitly as follows. Set

$$A := W(k)[v_+, v_-]/(v_+ v_- = p). \tag{III.1.14}$$

We endow A with a grading such that $\deg v_+ = 1$ and $\deg v_- = -1$, and consider the corresponding action of $\mathbf{G}_m$ on $\mathrm{Spf}\, A$. Then $k^{\mathcal{N}}$ is identified with $\mathrm{Spf}\, A/\mathbf{G}_m$. We shall just explain a construction of the map $\mathrm{Spf}\, A/\mathbf{G}_m \to k^{\mathcal{N}}$, referring the reader to [Bha23, §5.4] for a proof of the isomorphism property. First, we describe the composite map $\mathfrak{p}_{\mathrm{cris}} : \mathrm{Spf}\, A/\mathbf{G}_m \xrightarrow{\sim} k^{\mathcal{N}} \to W(k)^{\mathcal{N}}$ explicitly. Here the map $k^{\mathcal{N}} \to W(k)^{\mathcal{N}}$ comes from the morphism $\mathrm{Spec} k \to \mathrm{Spf}\, W(k)$ by functoriality. An R-point of $\mathrm{Spf}\, A/\mathbf{G}_m$ is given by $R \xrightarrow{v_+} L \xrightarrow{v_-} R$, where L is an invertible R-module and $v_+ v_- = p$. The map $\mathfrak{p}_{\mathrm{cris}}$ takes $(R \xrightarrow{v_+} L \xrightarrow{v_-} R)$ and sends it to $(M_{\mathfrak{p}_{\mathrm{cris}},R}, \xi)$, where $M_{\mathfrak{p}_{\mathrm{cris}},R} := \mathrm{coker}(\mathbf{G}^\sharp_{a,R} \xrightarrow{(v_+^\sharp, -can)} \mathbf{V}(L)^\sharp \oplus W_R)$,

[7] Bhatt's notations for j_+ and j_- are j_{HT} and j_{dR}, respectively.

and ξ is induced by the map

$$\mathbf{V}(L)^\sharp \oplus W_R \xrightarrow{(can \circ v_-^\sharp, \cdot p)} W_R. \tag{III.1.15}$$

The notation "can" stands for the canonical embedding $\mathbf{G}_{a,R}^\sharp \hookrightarrow W_R$, and $\cdot p$ stands for multiplication by p. This defines the map $\mathfrak{p}_{\mathrm{cris}} : \operatorname{Spf} A/\mathbf{G}_m \to W(k)^{\mathcal{N}}$. To lift this map to $k^{\mathcal{N}}$ we need to endow $\mathrm{Cone}(M_{\mathfrak{p}_{\mathrm{cris}},R} \xrightarrow{\xi} W_R)$ with the structure of a k-algebra stack. In fact, we have a map of quasi-ideal pairs $(W_R \xrightarrow{\cdot p} W_R) \to (M_{\mathfrak{p}_{\mathrm{cris}},R} \xrightarrow{\xi} W_R)$ given by $W_R \xrightarrow{(0,\mathrm{Id})} \mathbf{V}(L)^\sharp \oplus W_R \to M_{\mathfrak{p}_{\mathrm{cris}},R}$ on the source and by Id on the target and, for any $W(k)$-algebra D, the animated ring $\mathrm{Cone}(D \xrightarrow{\cdot p} D)$ receives a map from $k \xrightarrow{\sim} \mathrm{Cone}(W(k) \xrightarrow{\cdot p} W(k))$. Under the isomorphism $\operatorname{Spf} A/\mathbf{G}_m \xrightarrow{\sim} k^{\mathcal{N}}$, the open embedding $j_+ : k^{\mathbf{\Delta}} \hookrightarrow k^{\mathcal{N}}$ (resp. $j_- : k^{\mathbf{\Delta}} \hookrightarrow k^{\mathcal{N}}$) is identified with $\operatorname{Spf} W(k) \xrightarrow{F} \operatorname{Spf} W(k) \xrightarrow{v_+=1, v_-=p} \operatorname{Spf} A \to \operatorname{Spf} A/\mathbf{G}_m$ (resp. $\operatorname{Spf} W(k) \xrightarrow{v_+=p, v_-=1} \operatorname{Spf} A \to \operatorname{Spf} A/\mathbf{G}_m$). Let us just construct an isomorphism of k-*algebra* stacks obtained by pulling back $\mathbf{G}_{a,k}^{\mathcal{N}}$ along $j_\pm : \operatorname{Spf} W(k) \to \operatorname{Spf} A/\mathbf{G}_m$. The restriction of $\mathbf{G}_{a,k}^{\mathcal{N}}$ to $\operatorname{Spf} W(k) \xrightarrow{v_+=1, v_-=p} \operatorname{Spf} A/\mathbf{G}_m$ is given by $\mathrm{Cone}(W \xrightarrow{\cdot p} W)$; its restriction along $\operatorname{Spf} W(k) \xrightarrow{v_+=p, v_-=1} \operatorname{Spf} A/\mathbf{G}_m$ is $\mathrm{Cone}(F_* W \xrightarrow{\cdot p} F_* W)$. As a k-algebra stack the latter is obtained from the former by precomposing the action of k with the Frobenius. Equivalently, the k-algebra stack $\mathrm{Cone}(F_* W \xrightarrow{\cdot p} F_* W)$ is isomorphic to the pullback of $\mathrm{Cone}(W \xrightarrow{\cdot p} W)$ along the Frobenius on $\operatorname{Spf} W(k)$. We refer the reader to [Bha23, §3.3] and especially diagram (3.3.2) in *loc. cit.* for details.

For a smooth p-adic formal scheme X over $\operatorname{Spf} W(k)$, $\mathfrak{p}_{\mathrm{cris}}^*(\mathcal{H}_{syn}(X))$ recovers the crystalline cohomology of $X \otimes \mathbf{F}_p$ equipped with the Nygaard filtration (see [Bha23, §3.3]).

III.1.4: The reduced locus of $W(k)^{\mathcal{N}}$

Let $C_2 := \mathrm{Spec} k[v_+^p, v_-]/(v_+^p v_-)$. We endow C_2 with a $\mathbf{G}_m$-action given by the grading: $\deg v_+^p = p$ and $\deg v_- = -1$. The inclusion of $k[v_+^p, v_-]/(v_+^p v_-) \hookrightarrow k[v_+, v_-]/(v_+ v_-)$ gives the morphism $k^{\mathcal{N}} \otimes \mathbf{F}_p \to C_2/\mathbf{G}_m$. Recall from [Dri24, § 5.16.10] a factorization of $\mathfrak{p}_{\mathrm{cris}}$:

$$k^{\mathcal{N}} \otimes \mathbf{F}_p \to C_2/\mathbf{G}_m \xrightarrow{\widetilde{\mathfrak{p}}_{\mathrm{cris}}} W(k)^{\mathcal{N}}_{red} \to W(k)^{\mathcal{N}} \otimes \mathbf{F}_p. \tag{III.1.16}$$

Let us just explain a construction of the composite

$$C_2 \twoheadrightarrow C_2/\mathbf{G}_m \xrightarrow{\widetilde{\mathfrak{p}}_{\mathrm{cris}}} W(k)^{\mathcal{N}}, \tag{III.1.17}$$

given by a pair $(M_{C_2}, \alpha : M_{C_2} \to W_{C_2})$, where M_{C_2} is an admissible W_{C_2}-module and α is a W_{C_2}-module homomorphism. Explicitly, M_{C_2} is given by the pullback diagram

$$\begin{array}{ccc} M_{C_2} & \longrightarrow & F_*W_{C_2} \\ \downarrow & & \downarrow{\scriptstyle [v_+^p]} \\ W_{C_2} & \xrightarrow{F} & F_*W_{C_2} \end{array}, \tag{III.1.18}$$

where the right downward arrow $[v_+^p]$ takes a Witt vector $w \in W_{C_2}$ to the product $[v_+^p]w$. The morphism α is defined by $M_{C_2} = W_{C_2} \times_{F_*W_{C_2}} F_*W_{C_2} \xrightarrow{([v_-],V)} W_{C_2}$. The isomorphism between the composition

$$C := \mathrm{Spec} k[v_+, v_-]/(v_+v_-) \twoheadrightarrow C/\mathbf{G}_m = k^{\mathcal{N}} \otimes \mathbf{F}_p \xrightarrow{\mathfrak{p}_{\mathrm{cris}}} W(k)^{\mathcal{N}}$$

and $C \to C_2 \xrightarrow{\text{(III.1.17)}} W(k)^{\mathcal{N}}$ is given by an isomorphism of W_C-modules

$$\mathrm{coker}\left(W_C^{(F)} \xrightarrow{[v_+], -\mathrm{Id}} W_C^{(F)} \times W_C\right) \xrightarrow{\sim} W_C \times_{F_*W_C} F_*W_C =: C \times_{C_2} M_{C_2} \tag{III.1.19}$$

defined by the matrix $\begin{pmatrix} \mathrm{Id} & [v_+] \\ 0 & F \end{pmatrix}$. In [Dri24, Corollary 7.5.1], Drinfeld refines the morphism $\widetilde{\mathfrak{p}}_{\mathrm{cris}} : C_2/\mathbf{G}_m \to W(k)^{\mathcal{N}}_{red}$ to an isomorphism

$$BH \xrightarrow{\sim} W(k)^{\mathcal{N}}_{red}, \tag{III.1.20}$$

where H is an affine group scheme over $C_2/\mathbf{G}_m$ constructed as follows. Let $\widetilde{H} := W^{(F)} \times C_2$. For $(w,t), (w',t) \in \widetilde{H}$, the formula $(w,t) * (w',t) = (w + w' - [v_-^p(t)]ww', t)$ defines a map $\widetilde{H} \times \widetilde{H} \to \widetilde{H}$, making $\widetilde{H}$ a group scheme over C_2. The restriction of $\widetilde{H}$ to the open subscheme of C_2 given by $v_- \neq 0$ is $\mathbf{G}_m^\sharp$, and its restriction to the closed subscheme given by $v_- = 0$ is $\mathbf{G}_a^\sharp$. Define a $\mathbf{G}_m$-equivariant structure on $W \times C_2$ by

$$\mathbf{G}_m \times W \times C_2 \to W \times C_2, \quad (\lambda, w, t) \mapsto ([\lambda^p]w, \lambda t). \tag{III.1.21}$$

This restricts to a $\mathbf{G}_m$-equivariant structure on $\widetilde{H} \hookrightarrow W \times C_2$ compatible with the group structure. This defines the promised group scheme H over $C_2/\mathbf{G}_m$.

We write $(C_2)_{v_+^p=0}$, $(C_2)_{v_-=0}$ for the closed subschemes given by the respective equations. Denote by D_{HT} (resp. D_{dR}) the restrictions of $C_2/\mathbf{G}_m$-stack BH to $(C_2)_{v_-=0}/\mathbf{G}_m$ (resp. $(C_2)_{v_+^p=0}/\mathbf{G}_m$). Thus, via (III.1.20), one has $W(k)^{\mathcal{N}}_{red} = D_{\mathrm{HT}} \cup D_{\mathrm{dR}}$. We refer to D_{HT} (resp. D_{dR}) as the *Hodge-Tate* (resp. *de Rham*) component of $W(k)^{\mathcal{N}}_{red}$. We denote by

D_{Hod} the fibre of BH over $(C_2)_{v_+^p=v_-=0}/\mathbf{G}_m = B\mathbf{G}_m$. Geometrically, one pictures $W(k)^{\mathcal{N}}_{red}$ as a union of two components (*i.e.*, D_{HT} and D_{dR}) meeting *transversally* at D_{Hod}. The restriction gives a functor

$$\mathcal{R}^* : \mathcal{D}_{qc}(W(k)^{\mathcal{N}}_{red}) \to \mathcal{D}_{qc}(D_{\mathrm{HT}}) \times_{\mathcal{D}_{qc}(D_{Hod})} \mathcal{D}_{qc}(D_{\mathrm{dR}}). \tag{III.1.22}$$

The following result is explained in [Bha23, §6] as a consequence of a general statement from [Lur18b, Theorem 16.2.0.2].

Lemma III.8. *Functor* (III.1.22) *is fully faithful and admits right adjoint* $\mathcal{R}_*$ *given by*

$$(\mathcal{F}_{\mathrm{HT}}, \mathcal{F}_{\mathrm{dR}}, \alpha : \mathcal{F}_{\mathrm{HT}}|_{D_{Hod}} \simeq \mathcal{F}_{\mathrm{dR}}|_{D_{Hod}}) \mapsto \mathrm{Fib}(i_{HT*}\mathcal{F}_{\mathrm{HT}} \oplus i_{\mathrm{dR}*}\mathcal{F}_{\mathrm{dR}} \to i_{Hod*}(\mathcal{F}_{\mathrm{HT}}|_{D_{Hod}})),$$

where $i_?$ *is the closed embedding of* $D_?$ *into* $W(k)^{\mathcal{N}}_{red}$.

The following result describes the essential image of $\mathcal{R}^*$ in (III.1.22). Let $\mathcal{F} := (\mathcal{F}_{\mathrm{HT}}, \mathcal{F}_{\mathrm{dR}}, \alpha : \mathcal{F}_{\mathrm{HT}}|_{D_{Hod}} \simeq \mathcal{F}_{\mathrm{dR}}|_{D_{Hod}})$ be an object of the right-hand side of (III.1.22).

Lemma III.9. *Then* $\mathcal{F}$ *lies in the essential image of* $\mathcal{R}^*$ *if and only if its pullback to*

$$\mathcal{D}_{qc}\left((C_2)_{v_-=0}/\mathbf{G}_m\right) \times_{\mathcal{D}_{qc}\left((C_2)_{v_+^p=v_-=0}/\mathbf{G}_m\right)} \mathcal{D}_{qc}\left((C_2)_{v_+^p=0}/\mathbf{G}_m\right) \tag{III.1.23}$$

comes from an object of $\mathcal{D}_{qc}(C_2/\mathbf{G}_m)$.

Proof. $\mathcal{F}$ lies in the essential image if and only if the canonical morphism $\mathcal{R}^*\mathcal{R}_*\mathcal{F} \to \mathcal{F}$ is an isomorphism. Since the map $\widetilde{\mathfrak{p}}_{\mathrm{cris}} : C_2/\mathbf{G}_m \to W(k)^{\mathcal{N}}_{red}$ is faithfully flat, it suffices to check this after pulling back along $\widetilde{\mathfrak{p}}_{\mathrm{cris}}$ to $C_2/\mathbf{G}_m$. □

In [Bha23, §6.2], Bhatt and Lurie give a convenient description of D_{HT}, *i.e.*, $D_{\mathrm{HT}} \simeq (\mathbf{A}^{1,\mathrm{dR}}_+/\mathbf{G}_m) \otimes \mathbf{F}_p$. To reconcile this isomorphism with the above description, we recall that the Frobenius morphism $\mathbf{A}^1_+ \otimes \mathbf{F}_p \to \mathrm{Spec} k[v_+^p] = (C_2)_{v_-=0}$ factors through $(\mathbf{A}^1_+ \otimes \mathbf{F}_p)^{\mathrm{dR}}$. Moreover, a choice of δ-structure on $\mathbf{A}^1_+$, such that $\delta(v_+) = 0$, gives a $\mathbf{G}_m$-equivariant isomorphism $(\mathbf{A}^1_+ \otimes \mathbf{F}_p)^{\mathrm{dR}} = B\mathbf{G}_a^\sharp \times (C_2)_{v_-=0}$, where the action of $\mathbf{G}_m$ on $\mathbf{G}_a^\sharp$ is given by $\lambda * z = \lambda^p z$. Using the identification of the Cartier dual to $\mathbf{G}_a^\sharp$ and $\widehat{\mathbf{G}}_a = \mathrm{Spf}\, k[[D^p]]$, we obtain an equivalence $\mathcal{D}_{qc}(D_{\mathrm{HT}}) \simeq \mathcal{D}_{gr,D^p\text{-nilp}}(k[v_+^p, D^p])$, where the right hand side is a full subcategory of the derived category $\mathcal{D}_{gr}(k[D^p, v_+^p])$ of graded modules over the polynomial algebra $k[D^p, v_+^p]$ with $\deg v_+^p = -\deg D^p = p$. Objects of this full subcategory $\mathcal{D}_{gr,D^p\text{-nilp}}(k[D^p, v_+^p])$ consist of complexes $\mathcal{M}$ such that the action of D^p on $\bigoplus_i H^i(\mathcal{M})$ is locally nilpotent.

III.1.5: Coherent sheaves and vector bundles over X^{syn}

Let Q be a p-complete Noetherian regular local ring. In [Bha23, Remark 5.5.19], it is shown that the category $\mathrm{Perf}(Q^{syn})$ of perfect complexes on Q^{syn} has a unique t-structure whose category of connective objects $\mathrm{Perf}^{\leq 0}(Q^{syn})$ consists of all $\mathcal{F} \in \mathrm{Perf}(Q^{syn})$ such that, for any p-nilpotent ring R and a map $f : \mathrm{Spec}R \to Q^{syn}$, the pullback $f^*\mathcal{F}$ is in $D^{\leq 0}(R)$. We refer to the heart of this t-structure as the category $\mathrm{Coh}(Q^{syn})$ of coherent sheaves on Q^{syn}. The latter has a more concrete description. Let $(\tilde{Q}, (d), \delta)$ be a p-torsion prism with $\tilde{Q}/(d) = Q$. Then there exists a canonical faithfully flat map

$$\mathrm{Spf}\, \tilde{Q}[v_-, v_+]/(v_-v_+ - d) \to Q^{syn}, \tag{III.1.24}$$

where $\mathrm{Spf}\, \tilde{Q}[v_-, v_+]/(v_-v_+ - d)$ is the (p, d)-adic formal scheme $\mathrm{colim}_N \mathrm{Spec}\tilde{Q}[v_-, v_+]/(v_-v_+ - d, d^N, p^N)$; see [Bha23, Remark 5.5.19], *cf.* the end of §III.1.3. Then $\mathcal{F} \in \mathcal{D}_{qc}(Q^{syn})$ belongs to $\mathrm{Coh}(Q^{syn})$ if and only of its pullback along (III.1.24) is coherent, that is, a finite module over[8] $\tilde{Q}[v_-, v_+]/(v_-v_+ - d)$.

Recall that, for any stack $\mathcal{X}$, the category $\mathrm{Vect}(\mathcal{X})$ of vector bundles over $\mathcal{X}$ is defined to be $\lim_{\mathrm{Spec}R \to \mathcal{X}} \mathrm{Vect}(\mathrm{Spec}R)$, where the limit is taken over the category all affine schemes over $\mathcal{X}$. An object $\mathcal{F} \in \mathcal{D}_{qc}(Q^{syn})$ is a vector bundle if and only if its pullback along (III.1.24) is a finite projective module over $\tilde{Q}[v_-, v_+]/(v_-v_+ - d)$.

III.1.6: Hodge-Tate weights

Recall that $k^{\mathcal{N}}$ can be identified with $\mathrm{Spf}\, A/\mathbf{G}_m$ [Bha23, §5.4]. Consequently, the category $\mathcal{D}_{qc}(k^{\mathcal{N}})$ is equivalent to the subcategory $\widehat{\mathcal{D}}_{gr}(A)$ of the derived category $\mathcal{D}_{gr}(A)$ of graded A-modules spanned by p-complete objects. Given a gauge $\mathcal{F} \in \mathcal{D}_{qc}(k^{\mathcal{N}})$, we will denote by

$$(\mathcal{N}^{\cdot} = \bigoplus_i \mathcal{N}^i, v_- : \mathcal{N}^i \to \mathcal{N}^{i-1}, v_+ : \mathcal{N}^i \to \mathcal{N}^{i+1}, v_-v_+ = v_+v_- = p) \tag{III.1.25}$$

the corresponding object in $\widehat{\mathcal{D}}_{gr}(A)$.

Definition III.10. We say that a gauge $\mathcal{F} \in \mathcal{D}_{qc}(k^{\mathcal{N}})$ has Hodge-Tate weights $\geq a$ if, under the identification (III.1.25), the maps v_- in $\mathcal{N}^a \xrightarrow{v_-} \mathcal{N}^{a-1} \xrightarrow{v_-} \mathcal{N}^{a-2} \xrightarrow{v_-} \cdots$ are all quasi-isomorphisms.

A gauge $\mathcal{F}$ is said to be *effective* if it has Hodge-Tate weights ≥ 0. We say that a gauge $\mathcal{F}$ has Hodge-Tate weights $\leq b$ if under the identification (III.1.25), the maps v_+ in $\cdots \mathcal{N}^{b+3} \xleftarrow[v_+]{} \mathcal{N}^{b+2} \xleftarrow[v_+]{} \mathcal{N}^{b+1} \xleftarrow[v_+]{} \mathcal{N}^b$ are all quasi-isomorphisms.

[8] Note that since the ring is regular, every finite module over it is perfect.

We say that a gauge $\mathcal{F} \in \mathcal{D}_{qc}(W(k)^{\mathcal{N}})$ has Hodge-Tate weights $\geq a$ (resp. $\leq b$) if its pullback to $k^{\mathcal{N}}$ has the corresponding property. Likewise, we say that an F-gauge $\mathcal{F} \in \mathcal{D}_{qc}(W(k)^{syn})$ has Hogde-Tate weights $\geq a$ (resp. $\leq b$) if its pullback to $k^{\mathcal{N}}$ has the corresponding property. We denote by $\mathcal{D}_{qc,[a,b]}(k^{\mathcal{N}})$ the full subcategory of $\mathcal{D}_{qc}(k^{\mathcal{N}})$ that consists of objects with Hodge-Tate weights in $[a,b]$. Similarly, we define $\mathcal{D}_{qc,[a,b]}(W(k)^{\mathcal{N}})$.

Remark III.11. We observe that $\mathcal{F} \in \mathcal{D}_{qc}(k^{\mathcal{N}})$ has Hodge-Tate weights in $[a,b]$ if and only if the pullbacks of $\mathcal{F}$ along both composites $\mathrm{Spec} k[v_{\pm}]/\mathbf{G}_{m,k} \to k^{\mathcal{N}} \otimes \mathbf{F}_p \to k^{\mathcal{N}}$ lie in $\mathcal{D}_{qc,[a,b]}((\mathbf{A}^1_{\pm}/\mathbf{G}_m) \otimes \mathbf{F}_p)$ in the sense of Definition III.1.

Lemma III.12. *Suppose $\mathcal{F} \in \mathcal{D}_{qc,[a,b]}(\mathbf{A}^1_-/\mathbf{G}_m)$, then $t^*\mathcal{F} \in \mathcal{D}_{qc,[a,b]}(k^{\mathcal{N}})$.*

Proof. The map $i_{\pm} : (\mathbf{A}^1_{\pm}/\mathbf{G}_m) \otimes \mathbf{F}_p = \mathrm{Spec} k[v_{\pm}]/\mathbf{G}_{m,k} \hookrightarrow k^{\mathcal{N}}$ given by the equation $v_{\mp} = 0$ is a closed embedding. Since v_+ is a nonzero divisor in $W(k)[v_+, v_-]/(v_+v_- - p)$, the restriction functor along the above embedding corresponds algebraically to taking the cone of v_+ on the corresponding graded module. Then saying that $\mathcal{G} \in \mathcal{D}_{qc}(k^{\mathcal{N}})$ has weights $\leq b$ is equivalent to saying that the restriction $i_-^*\mathcal{G}$ has weights $\leq b$. The composition $t \circ i$ is the embedding $\mathrm{Spec} k[v_-]/\mathbf{G}_m \to \mathrm{Spf}\, W(k)[v_-]/\mathbf{G}_m$ given by the equation $p = 0$. Since, for any $\mathcal{F} \in \mathcal{D}_{qc,(-\infty,b]}(\mathbf{A}^1_-/\mathbf{G}_m)$, its restriction to the special fiber $(\mathbf{A}^1_-/\mathbf{G}_m) \otimes \mathbf{F}_p$ has weights $\leq b$, we conclude that $t^*\mathcal{F} \in \mathcal{D}_{qc,(-\infty,b]}(k^{\mathcal{N}})$. Saying that $\mathcal{G} \in \mathcal{D}_{qc}(k^{\mathcal{N}})$ has weights $\geq a$ is equivalent to saying that the restriction $i_+^*\mathcal{G}$ has weights $\geq a$. The composition $t \circ i_+$ is isomorphic to $(\mathbf{A}^1_+/\mathbf{G}_m) \otimes \mathbf{F}_p \to B\mathbf{G}_m \otimes \mathbf{F}_p \to \mathbf{A}^1_-/\mathbf{G}_m$ and the lemma follows. □

Lemma III.13. *Suppose $\mathcal{F} \in \mathcal{D}_{qc,(-\infty,b]}(W(k)^{\mathcal{N}})$, then $\mathfrak{p}_{\mathrm{d\bar{R}}}^*\mathcal{F} \in \mathcal{D}_{qc,(-\infty,b]}(\mathbf{A}^1_-/\mathbf{G}_m)$.*

Proof. By p-completeness, it suffices to show $\mathfrak{p}_{\mathrm{d\bar{R}}}^*\mathcal{F} \otimes \mathbf{F}_p \in \mathcal{D}_{qc,(-\infty,b]}\left((\mathbf{A}^1_-/\mathbf{G}_m) \otimes \mathbf{F}_p\right)$. It remains to observe that the composition $(\mathbf{A}^1_-/\mathbf{G}_m) \otimes \mathbf{F}_p \to \mathbf{A}^1_-/\mathbf{G}_m \xrightarrow{\mathfrak{p}_{\mathrm{d\bar{R}}}} W(k)^{\mathcal{N}}$ is isomorphic to $(\mathbf{A}^1_-/\mathbf{G}_m) \otimes \mathbf{F}_p \xrightarrow{i_-} k^{\mathcal{N}} \xrightarrow{\mathfrak{p}_{\mathrm{cris}}} W(k)^{\mathcal{N}}$, where i_- is defined in the first line of the proof of Lemma III.12; see, for example, [Bha23, §2.8.3]. □

Remark III.14. If $\mathcal{F} \in \mathcal{D}_{qc}(k^{\mathcal{N}})$ is effective, then its restriction along the map $B\mathbf{G}_{m,k} \hookrightarrow k^{\mathcal{N}}$ is given by $B\mathbf{G}_{m,k} \overset{v_-=0}{\hookrightarrow} (\mathbf{A}^1_-/\mathbf{G}_m) \otimes \mathbf{F}_p \overset{v_+=0}{\hookrightarrow} k^{\mathcal{N}}$, viewed as a $\mathbf{Z}$-graded object $M^\bullet$ of $\mathcal{D}_{qc}(k)$, has weights ≥ 0, *i.e.*, M^i is acyclic for every $i < 0$. If $\mathcal{F}$ is perfect, then the converse is true.

Lemma III.15. *The embedding $e : \mathcal{D}_{qc,(-\infty,b]}(k^{\mathcal{N}}) \hookrightarrow \mathcal{D}_{qc}(k^{\mathcal{N}})$ admits a right adjoint functor $w_{\leq b} : \mathcal{D}_{qc}(k^{\mathcal{N}}) \to \mathcal{D}_{qc,(-\infty,b]}(k^{\mathcal{N}})$.*

Proof. We define $w_{\leq b}$ by sending an object $(M^\bullet, v_-, v_+) \in \mathcal{D}_{qc}(k^{\mathcal{N}})$ to the object $(M'^\bullet, v'_-, v'_+) \in \mathcal{D}_{qc,(-\infty,b]}(k^{\mathcal{N}})$, where $M'^i = M^i$ for $i < b$ and $M'^i = M^b$ otherwise. Here

$v'_+ : M'^i \to M'^{i+1}$ is equal to v_+ for $i < b$ and equal to Id otherwise; here $v'_- : M'^i \to M'^{i-1}$ is equal to v_- for $i \leq b$ and equal to $p \cdot \mathrm{Id}$ otherwise. Indeed, we have a canonical map $f(\bullet)$ sending $(M'^\bullet, v'_-, v'_+)$ to $(M^\bullet, v_-, v_+)$ given by $f(i) = \mathrm{Id}$ for $i \leq b$ and $f(i) = v_+^{i-b}$ otherwise. This defines a natural transformation of functors $e \circ w_{\leq b} \to \mathrm{Id}$. On the other hand, we have an isomorphism $w_{\leq b} \circ e \xrightarrow{\sim} \mathrm{Id}$. □

III.2: The fiber product $k^{\mathcal{N}} \times_{W(k)^{\mathcal{N}}} \mathbf{A}^1_-/\mathbf{G}_m$

III.2.1: Drinfeld Lemma

Recall from (III.1.1.14) that $A := W(k)[v_+, v_-]/(v_+v_- = p)$ with a grading such that $\deg v_+ = 1$ and $\deg v_- = -1$. Let A^i be the subgroup of elements of degree i. Let $B \subset A \otimes \mathrm{Frac}(W(k))$ be the $W(k)$-subalgebra generated by v_- and $\frac{v_+^n}{n!}$ for $n \geq 0$. The grading on A induces a grading on B and we write $B = \bigoplus_i B^i$.

Definition III.16. For $n \in \mathbb{N}_{>0}$, its associated *Mazur number* is $[n] := \min_{m \geq n} \mathrm{ord} \frac{p^m}{m!}$.

Note that $[n] = n$ for $n < p$, and $[n+1] - [n] = 1$ or 0 for every $n \in \mathbb{N}_{>0}$. For every positive integer n, the ideal $(p^{[n]}) \subset \mathbf{Z}_p$ is the n-th divided power of the ideal (p).

Lemma III.17. *For every $i \geq 0$, B^i is the free $W(k)$-module on $\frac{p^{[i]} v_+^i}{p^i}$. For every $i \leq 0$, B^i is the free $W(k)$-module generated by v_-^i.*

Corollary III.18. *The embedding $A \hookrightarrow B$ is an isomorphism in degrees $< p$. In particular, B as a graded $W(k)[v_-]$-module is effective in the sense of Definition III.3.*

Remark III.19. Lemma III.17 shows that B is the Rees algebra of the filtration on $W(k)$ given by divided powers of the ideal $(p) \subset W(k)$.

Remark III.20. Recall from [Bha23, §3.3] that the stable ∞-category $\mathcal{D}_{qc}(k^{\mathcal{N}}) = \widehat{\mathcal{D}}_{gr}(A)$ of graded p-complete A-modules can be identified with the derived category of filtered p-complete modules over the filtered algebra $W(k)$, where the latter is equipped with the filtration by powers of the ideal $(p) \subset W(k)$. Likewise, the category $\widehat{\mathcal{D}}_{gr}(B)$ of graded p-complete B-modules can be identified with the derived category of filtered p-complete modules over the filtered algebra $(W(k), (p)^{[\cdot]})$, where $(p)^{[\cdot]}$ denotes the filtration by *divided* powers of the ideal (p).

Recall the maps $\mathfrak{p}_{\mathrm{d\bar{R}}}$ and $\mathfrak{p}_{\mathrm{cris}}$ from §III.1.3. Consider the following diagram

(III.2.1)
$$\begin{array}{ccc} & \operatorname{Spf} B/\mathbf{G}_m & \\ & \downarrow{\scriptstyle a} & \\ & k^{\mathcal{N}} = \operatorname{Spf} A/\mathbf{G}_m & \\ {\scriptstyle \mathfrak{p}_{\mathrm{cris}}}\swarrow & & \searrow{\scriptstyle t} \\ W(k)^{\mathcal{N}} & \xleftarrow{\quad\mathfrak{p}_{\mathrm{d\bar{R}}}\quad} & \mathbf{A}^1_-/\mathbf{G}_m \end{array}$$

where a and t are induced by homomorphisms of graded $W(k)$-algebras: the map t is induced by the map $W(k)[v_-] \to A$ sending $v_- \mapsto v_-$, and the map a is induced by the obvious embedding $A \hookrightarrow B$. Warning: $\mathfrak{p}_{\mathrm{cris}} \not\simeq \mathfrak{p}_{\mathrm{d\bar{R}}} \circ t$.

Theorem III.21 (Drinfeld)**.** *There exists a unique isomorphism* $\Psi_{\mathrm{Maz}} : \mathfrak{p}_{\mathrm{cris}} \circ a \simeq \mathfrak{p}_{\mathrm{d\bar{R}}} \circ t \circ a$.

We will need the following lemma in the proof of the Theorem.

Lemma III.22. *The element* $[v_+^p]$ *in* $W(B)$ *is uniquely divisible by* p.

Proof. Uniqueness is obvious because p is not a zero-divisor in B (hence in $W(B)$). It is enough to show that $[v_+]^p$ is divisible by p in the ring of big Witt vectors. Recall that the additive group of the big Witt vectors is $(1+xB[[x]])^*$ and $[v_+]^p$ corresponds to $1-v_+^p x$ and to show that it is divisible by p we need to show that there exists a series $g \in (1+B[[x]])^*$ such that $g^p = 1 - v_+^p x$. Explicitly, $g = \exp(\frac{1}{p}\log(1-v_+^p x))$. However, we need to explain why this formula makes sense. Note that $\log(1-v_+^p x) = -\sum_{n\geq 1} \dfrac{v_+^{pn} x^n}{n}$ makes sense since v_+^{pn} is divisible by $(pn)!$ and, moreover, this series is divisible by p since all the summands are. To make sense of $\exp(\frac{1}{p}\log(1-v_+^p x)) = \exp(-\sum_{n\geq 1} \dfrac{v_+^{pn} x^n}{pn})$ we need to explain why this element which a priori lives in $(1+x\mathbb{Q}[[x]])^*$ actually lives in $(1+xB[[x]])^*$. That is, we need to explain why $\left(\sum_{n\geq 1} \dfrac{v_+^{pn} x^n}{pn}\right)^m$ is divisible by $m!$ and it is enough to show that $\left(\dfrac{v_+^{pn} x^n}{pn}\right)^m = \dfrac{v_+^{pnm} x^{nm}}{(pn)^m}$ is divisible by $m!$. Recall that v_+^{pnm} is divisible by $(pnm)!$ so we are reduced to show that $\dfrac{(pnm)!}{(pn)^m m!}$ is a p-adic integer. This follows from a known fact that $\dfrac{(nm)!}{(n)^m m!} \in \mathbf{Z}$. □

Remark III.23. Let us sketch another proof of Lemma III.22. There exists a unique element $y' \in \mathbf{G}_a^\sharp(B)$ whose image in $\mathbf{G}_a(B)$ is v_+. Let $y \in W^{(F)}(B)$ be the image of y' under the isomorphism $\mathbf{G}_a^\sharp \xrightarrow{\sim} W^{(F)}$ (see [Dri24, Lemma 3.2.6]). Then the first ghost coordinate of

$[v_+] - y$ is 0. Thus, there exists a unique $z \in W(B)$ with $V(z) = [v_+] - y$. We claim that $p \cdot z = [v_+^p]$. Indeed, $p \cdot z = FV(z) = F([v_+] - y) = [v_+^p]$.

Remark III.24. The Witt vector $\frac{[v_+^p]}{p} \in W(B)$ determines a map $\mathrm{Spec}B \xrightarrow{d} W$. We claim that the following diagram is commutative

$$\begin{array}{ccc} \mathbf{G}_m \times \mathrm{Spec}B & \xrightarrow{\mathrm{Id} \times d} & \mathbf{G}_m \times W \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ \mathrm{Spec}B & \xrightarrow{\quad d \quad} & W\,. \end{array} \tag{III.2.2}$$

Here the action α of $\mathbf{G}_m$ on $\mathrm{Spec}B$ is given by the grading and the action of $\mathbf{G}_m$ on W is given by the formula $\beta(\lambda, w) = [\lambda^p]w$, where $\lambda \in \mathbf{G}_m, w \in W$, and $[\cdot]$ refers to the Teichmüller representative. Indeed, since $\mathbf{G}_m \times \mathrm{Spec}B$ is flat over $\mathbf{Z}_p$ it suffices to check that $d \circ \alpha = \beta \circ (\mathrm{Id} \times d)$ after post-composing with $W \xrightarrow{p} W$ which is clear.

Using (III.2.2) the map d descends to a map

$$\bar{d} : \mathrm{Spec}B/\mathbf{G}_m \to W/\mathbf{G}_m$$

of the quotient stacks.

Proof of Theorem III.21. Fix an S-point $\mathcal{P}$ of $\mathrm{Spf}\, B/\mathbf{G}_m$. Using the morphism a, we have an S-point of $\mathrm{Spf}\, A/\mathbf{G}_m$, *i.e.*, $\mathcal{O}_S \xrightarrow{v_+} L \xrightarrow{v_-} \mathcal{O}_S$, where L is a line bundle over S and $v_+v_- = p$. Define a morphism of W_S-modules $w : W_S \to \mathbf{V}(L)^\sharp \hookrightarrow [L]$, where the invertible W_S-module $[L]$ is the Teichmüller lift of L (see [Dri24, 3.11]). By Lemma III.22, the homomorphism $[v_+^p] : W_S \to [L^{\otimes p}]$ of W_S-modules is canonically divisible by p. Indeed, we first define

$$\frac{[v_+^p]}{p} : W_S \to [L^{\otimes p}] \tag{III.2.3}$$

locally on S assuming a lift $\tilde{\mathcal{P}} : S \to \mathrm{Spf}\, B$ of $\mathcal{P}$. The choice of a lift trivializes L and $[L]$. Composing $\tilde{\mathcal{P}}$ with $\mathrm{Spec}B \xrightarrow{d} W$ from Remark III.24 we obtain a Witt vector $\frac{[v_+^p]}{p} \in W(S)$. The multiplication by $\frac{[v_+^p]}{p}$ defines the desired map $\frac{[v_+^p]}{p} : W_S \to W_S = [L^{\otimes p}]$. Diagram (III.2.2) shows that (III.2.3) does not depend on the choice of a trivialization[9] and, thus, it is well-defined globally. Applying the Verschiebung, we obtain a homomorphism $V(\frac{[v_+^p]}{p}) : W_S \to [L]$ of W_S-modules. Set $w := [v_+] - V(\frac{[v_+^p]}{p}) : W_S \to [L]$. We claim that w lands in $\mathbf{V}(L)^\sharp$. Indeed, the composition $F \circ w : W_S \to [L^{\otimes p}]$ is zero.

[9]Indeed, consider the map $\bar{d} : \mathrm{Spec}B/\mathbf{G}_m \to W/\mathbf{G}_m$ from Remark III.24. Note that an S-point of $W/\mathbf{G}_m$ is a line bundle L' over S together with a map W_S-modules $W_S \to [L'^{\otimes p}]$. The image of $\mathcal{P}$ under $\bar{d}$ is our (III.2.3).

We want to construct an isomorphism $\mathfrak{p}_{\mathrm{cris}} \circ a(\mathcal{P}) \simeq \mathfrak{p}_{\mathrm{d\bar{R}}} \circ t \circ a(\mathcal{P})$. Define a morphism $\Psi_{\mathrm{Maz}} : M_{\mathfrak{p}_{\mathrm{cris}},S} := \mathrm{coker}\left(\mathbf{G}^{\sharp}_{a,S} \xrightarrow{(v_+^{\sharp},-can)} \mathbf{V}(L)^{\sharp} \oplus W_S\right) \longrightarrow \mathbf{V}(L)^{\sharp} \oplus F_* W_S := M_{\mathfrak{p}_{\mathrm{d\bar{R}},S}}$ as follows: let the map $\mathbf{V}(L)^{\sharp} \oplus W_S \to \mathbf{V}(L)^{\sharp} \oplus F_* W_S$ be given by the matrix $\begin{pmatrix} 1 & w \\ 0 & F \end{pmatrix}$. We claim that $\Psi_{\mathrm{Maz}} \circ (v_+^{\sharp}, -can)$ is zero. Indeed, $F \circ can$ is zero. We also have that $v_+^{\sharp} - w \circ can : \mathbf{G}^{\sharp}_{a,S} \to \mathbf{V}(L)^{\sharp}$ is zero because $V(\frac{[v_+^p]}{p}) \circ can = 0$[10]. This defines Ψ_{Maz}. The following commutative diagram shows that Ψ_{Maz} is an isomorphism.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{V}(L)^{\sharp} & \longrightarrow & M_{\mathfrak{p}_{\mathrm{cris}},S} & \xrightarrow{(0,F)} & F_* W_S & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \Psi_{\mathrm{Maz}}} & & \| & & \\ 0 & \longrightarrow & \mathbf{V}(L)^{\sharp} & \longrightarrow & M_{\mathfrak{p}_{\mathrm{d\bar{R}}},S} & \xrightarrow{(0,\mathrm{Id})} & F_* W_S & \longrightarrow & 0 \end{array}$$

Finally, one verifies that Ψ_{Maz} commutes with ξ given by (III.1.13) and (III.1.15). This proves the existence of the isomorphism in Theorem III.21.

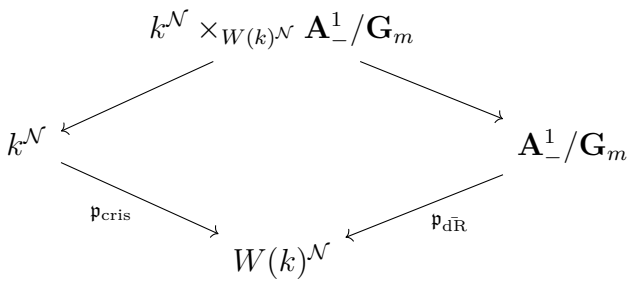
For the uniqueness, it suffices to show that the point $\mathfrak{p}_{\mathrm{d\bar{R}}} \circ t \circ a : \mathrm{Spf}\, B/\mathbf{G}_m \to W(k)^{\mathcal{N}}$ has no nontrivial automorphisms. Since $\mathrm{Spf}\, B \to \mathrm{Spf}\, B/\mathbf{G}_m$ is a faithfully flat cover, it suffices to show this for the corresponding point $s : \mathrm{Spf}\, B \to W(k)^{\mathcal{N}}$. Explicitly, s is given by the morphism $\mathbf{G}^{\sharp}_a \oplus F_* W_B \xrightarrow{(v_-,V)} W_B$ of W_B-modules. Using [Bha23, Proposition 5.2.1], any automorphism α of W_B-module $\mathbf{G}^{\sharp}_a \oplus F_* W_B$ has the form $\begin{bmatrix} t & x \\ 0 & y \end{bmatrix}$ for $t \in \mathbf{G}_a(B), x \in \ker(\mathbf{G}^{\sharp}_a(B) \to \mathbf{G}_a(B)) = 0, y \in F_* W(B)$. The group of automorphisms of s consists of α satisfying $(v_-, V) \circ \alpha = (v_-, V)$. In particular, for any $(a, b) \in \mathbf{G}^{\sharp}_a(B) \oplus F_* W(B)$, we must have $v_- ta + V(by) = v_- a + V(b)$. Substitutions $a = 0, b = 1$ and $a = [v_+], b = 0$ give $t = y = 1$ *i.e.*, $\alpha = \mathrm{Id}$. □

[10] We have that $V(yF(x)) = xV(y)$, for any x, y in the ring of Witt vectors. In particular, if x is in Frobenius kernel then $V(y)x = 0$.

III.2.2: A description of $k^{\mathcal{N}} \times_{W(k)^{\mathcal{N}}} \mathbf{A}^1_-/\mathbf{G}_m$

Consider the Cartesian square

(III.2.4)



Using Theorem III.21, we have a morphism

$$(a, t \circ a) : \operatorname{Spf} B/\mathbf{G}_m \to k^{\mathcal{N}} \times_{W(k)^{\mathcal{N}}} \mathbf{A}^1_-/\mathbf{G}_m := \mathfrak{D}. \tag{III.2.5}$$

Note that the composition $\mathbf{A}^1_-/\mathbf{G}_m \xrightarrow{\mathfrak{p}_{\mathrm{d\bar{R}}}} W(k)^{\mathcal{N}} \xrightarrow{t_{W(k)}} \mathbf{A}^1_-/\mathbf{G}_m$ is the identity. Indeed, recall that $\mathfrak{p}_{\mathrm{d\bar{R}}}$ takes $x = (L, v_-) \in \mathbf{A}^1_-/\mathbf{G}_m(R)$ to an admissible module $(v_-^\sharp, V) : \mathbf{V}(L)^\sharp \oplus F_* W_R \to W_R$ as in (III.1.13). This gives rise to a morphism of admissible sequences (III.1.9) where the left vertical map is $v_-^\sharp : \mathbf{V}(L)^\sharp \to \mathbf{G}_{a,R}^\sharp$. In particular, the Rees map sends $\mathfrak{p}_{\mathrm{d\bar{R}}}(x)$ to x. Now, giving a point an S-point of $\mathfrak{D}$ is equivalent to giving $y \in \mathbf{A}^1_-/\mathbf{G}_m(S)$ and $x \in k^{\mathcal{N}}(S)$ together with $\mathfrak{p}_{\mathrm{d\bar{R}}}(y) \simeq \mathfrak{p}_{cris}(x)$. Applying to it $t_{W(k)}$ we get $t(x) \simeq y$. Thus, giving an S-point of $\mathfrak{D}$ is equivalent to giving an S-point x of $k^{\mathcal{N}}$ together with an isomorphism $\mathfrak{p}_{\mathrm{d\bar{R}}} \circ t(x) \simeq \mathfrak{p}_{\mathrm{cris}}(x)$.

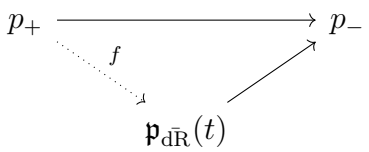
Let us define a closed substack of $\mathfrak{D}$. Note that both $\mathfrak{p}_{\mathrm{d\bar{R}}}$ and $\mathfrak{p}_{\mathrm{cris}}$, precomposed with the structure map $\pi : W(k)^{\mathcal{N}} \to W(k)^{\mathbf{\Delta}}$, factor through the de Rham point $\mathfrak{p}_{\mathrm{dR}} : \operatorname{Spf} W(k) \to W(k)^{\mathbf{\Delta}}$, classifying the Cartier-Witt divisor given by $W \xrightarrow{p} W$. This defines a map $\tilde{\pi} : \mathfrak{D} \to \operatorname{Spf} W(k) \times_{W(k)^{\mathbf{\Delta}}} \operatorname{Spf} W(k)$. The stack on the right has a canonical point given by the diagonal map $\operatorname{Spf} W(k) \to \operatorname{Spf} W(k) \times_{W(k)^{\mathbf{\Delta}}} \operatorname{Spf} W(k)$. Consider the fiber over this point of the morphism $\tilde{\pi}$ and denote it by $\mathfrak{D}_0$.

The following remark will not be used in the remaining part of the paper.

Remark III.25. Drinfeld suggested to us another interpretation of $\mathfrak{D}$ as well as its substack $\mathfrak{D}_0$ that makes use of the c-stack enhancement of $W(k)^{\mathcal{N}}$ introduced in [Dri24] and reviewed in §III.1.3. For a scheme S, consider the category $\mathfrak{D}'(S)$ consisting of pairs (t, f), where $t : S \to \mathbf{A}^1_-/\mathbf{G}_m$ and $f : p_+ \to \mathfrak{p}_{\mathrm{d\bar{R}}}(t)$ is a morphism in the category $W(k)^{\mathcal{N},c}(S)$ from §III.1.3. Here p_+ is defined as the composition $S \to k^{\mathcal{N}} \xrightarrow{\mathfrak{p}_{cris}} W(k)^{\mathcal{N}}$, where the first morphism is given by $A \to H^0(S, \mathcal{O}_S)$ sending v_- to p and v_+ to 1. We claim that the underlying groupoid of this category is equivalent to $\mathfrak{D}(S)$. This follows from the fact

([Dri24, §5.5.3]) that the Rees maps $t_{W(k)} : W(k)^{\mathcal{N},c} \to (\mathbf{A}^1_-/\mathbf{G}_m)^c$, $k^{\mathcal{N},c} \to (\mathbf{A}^1_-/\mathbf{G}_m)^c$ are left fibrations of c-stacks. Indeed, since $W(k)^{\mathcal{N},c} \to (\mathbf{A}^1_-/\mathbf{G}_m)^c$ is a left fibration, the category $W(k)^{\mathcal{N},c}(S)_{p_+/}$ is identified with $(\mathbf{A}^1_-/\mathbf{G}_m)^c(S)_{p/} \simeq k^{\mathcal{N},c}(S)$.[11] Under this identification, the natural functor $W(k)^{\mathcal{N},c}(S)_{p_+/} \to W(k)^{\mathcal{N},c}$ is identified with $\mathfrak{p}_{cris}$. The category $\mathfrak{D}'(S)$ is, by definition, equivalent to $W(k)^{\mathcal{N},c}(S)_{p_+/} \times_{W(k)^{\mathcal{N},c}(S)} (\mathbf{A}^1_-/\mathbf{G}_m)^c(S)$. Its underlying groupoid is $\mathfrak{D}(S)$ by the definition of the latter (III.2.4). To describe the substack $\mathfrak{D}_0$, note that any object (t, f) of this category provides us with a triangle in $W(S)^{\mathcal{N},c}$

(III.2.6)



where p_- is defined by $A \to H^0(S, \mathcal{O}_S)$, $v_- \mapsto 1$, $v_+ \mapsto p$, and the solid arrows are the canonical morphisms (constructed by observing that, for any S, p_+ (resp. p_-) is the initial (resp. final) object of the category $k^{\mathcal{N},c}(S)$ and p_- is the final object of $(\mathbf{A}^1_-/\mathbf{G}_m)^c(S)$). The triangle is not commutative in general; points that correspond to commutative triangles are classified by a substack of $\mathfrak{D}$; this substack is equivalent to $\mathfrak{D}_0$.

Recall from [Dri24, §1.8] that every *effective* gauge $\mathcal{F} \in \mathcal{D}_{qc,[0,+\infty]}(W(k)^{\mathcal{N}})$ gives rise to a complex of contravariant $\mathcal{O}$-modules on $W(k)^{\mathcal{N},c}$. In particular, for every S-point of $\mathfrak{D}_0$ the canonical morphism $\mathcal{F}|_{p_-} \to \mathcal{F}|_{p_+}$ factors through $\mathcal{F}|_{\mathfrak{p}_{\mathrm{d\bar{R}}}(t)}$. We also note that, for an effective F-gauge $\mathcal{F}$, we have an isomorphism $F^*(\mathcal{F}|_{p_+}) \simeq \mathcal{F}|_{p_-}$ whose composition with $\mathcal{F}|_{p_-} \to \mathcal{F}|_{p_+}$ is the crystalline Frobenius.

In this subsection, we give an explicit description of $\mathfrak{D}$, which was explained to us by Drinfeld and Lurie. The authors of this paper are responsible for any possible mistakes in the exposition of their results.

We endow W with the $\mathbf{G}_m$-action given by the formula: $\lambda * w = [\lambda]w$, where $[\cdot]$ refers to the Teichmüller representative. This gives an action of $\mathbf{G}_m$ on $\ker(W \xrightarrow{F} W) =: W^{(F)} \hookrightarrow W$. We consider the fibre product $\operatorname{Spf} A \times_{\mathbf{A}^1_+} W^{(F)} =: \widetilde{\mathfrak{D}}'$, where $\operatorname{Spf} A$ is viewed as a scheme over $\mathbf{A}^1_+$ via the natural embedding $W(k)[v_+] \hookrightarrow A$, and $W^{(F)} \to \mathbf{A}^1_+$ is given by the first Witt vector coordinate[12]. We endow $\operatorname{Spf} A \times_{\mathbf{A}^1_+} W^{(F)}$ with the diagonal $\mathbf{G}_m$-action. Let

[11] An object of $(\mathbf{A}^1_-/\mathbf{G}_m)^c(S)_{p/}$ consists of $(L \xrightarrow{v_-} \mathcal{O}_S) \in (\mathbf{A}^1_-/\mathbf{G}_m)^c(S)$ together with a morphism $f : (\mathcal{O}_S \xrightarrow{p} \mathcal{O}_S) \to (L \xrightarrow{v_-} \mathcal{O}_S)$ in $(\mathbf{A}^1_-/\mathbf{G}_m)^c(S)$. Giving such an f is equivalent to specifying a map $v_+ : \mathcal{O}_S \to L$ with $v_- v_+ = p$.

[12] *i.e.*, the map sends a Witt vector $w = \sum_{i \geq 0} V_i[w_i]$ to w_0.

$\widetilde{\mathfrak{D}}'_0 \hookrightarrow \operatorname{Spf} A \times_{\mathbf{A}^1_+} W^{(F)}$ be the closed subscheme given by equation[13]

$$[v_-]w = p - V(1). \tag{III.2.7}$$

Recall from [Dri24, Lemma 3.2.6] the unique isomorphism $W^{(F)} \simeq \mathbf{G}_a^\sharp$ characterized by the commutative diagram

$$\begin{array}{ccc} W^{(F)} & \xrightarrow{\simeq} & \mathbf{G}_a^\sharp \\ \downarrow & \swarrow & \\ \mathbf{G}_a & & \end{array} \tag{III.2.8}$$

where the vertical map is induced by the projection $W \to \mathbf{G}_a$ to the first coordinate. Thus under the isomorphism $W^{(F)}(\mathbf{Z}_p) \simeq \mathbf{G}_a^\sharp(\mathbf{Z}_p) = p\mathbf{Z}_p \subset \mathbf{Z}_p$, the element $p - V(1)$ is sent to the number p. Consequently, under the identification $\operatorname{Spf} A \times_{\mathbf{A}^1_+} W^{(F)} \simeq \operatorname{Spf} A \times_{\mathbf{A}^1_+} \mathbf{G}_a^\sharp$, the equation (III.2.7) has the form $v_- \cdot z = p$, where z is a point of $\mathbf{G}_a^\sharp$, and the product on the left-hand side refers to the $\mathbf{G}_a$-module structure on $\mathbf{G}_a^\sharp$. This yields a concrete description of $\widetilde{\mathfrak{D}}'_0$ as the formal spectrum of the $W(k)$-algebra generated by v_- and $\gamma_{p^n}(v_+)$, $n \geq 0$, subject to the following relations:

$$\frac{p^n!}{(p^{n-1}!)^p}\gamma_{p^n}(v_+) = (\gamma_{p^{n-1}}(v_+))^p, \quad v_-^{p^n}\gamma_{p^n}(v_+) = \frac{p^{p^n}}{p^n!}. \tag{III.2.9}$$

Lemma III.26. *1. Let $B^\flat$ be the PD-envelope of the ideal (v_+) in A, and let $W(k)\langle v_+\rangle[v_-]$ be the PD-envelope of the ideal $(v_+) \subset W(k)[v_+, v_-]$. Then $B^\flat \xrightarrow{\sim} W(k)\langle v_+\rangle[v_-]/(v_+v_- - p)$ and $\widetilde{\mathfrak{D}}' \xrightarrow{\sim} \operatorname{Spf} B^\flat$.*

2. Let $B^\dagger$ be the PD-envelope of the ideal (v_+) in A relative to $(\mathbf{Z}_p, (p))$ [Sta20, 07H7]. Then $\widetilde{\mathfrak{D}}'_0 \xrightarrow{\sim} \operatorname{Spf} B^\dagger$.

Proof. To prove the first part, set $\widetilde{A}^\sharp := W(k)\langle v_+\rangle[v_-]$, $\widetilde{A} := W(k)[v_+, v_-]$, and let $J \subset \widetilde{A}^\sharp$ be the PD ideal generated by $\frac{v_+^n}{n!}$, $n \geq 1$. Set $I := (v_-v_+ - p) \subset \widetilde{A}^\sharp$. Then[14] $I \cap J = I \cdot J$. In particular, $I \cap J \subset J$ is a sub-PD-ideal. Consequently, there exists a unique PD structure on the image $\bar{J}$ of J under the projection $\widetilde{A}^\sharp \to \widetilde{A}^\sharp/I$. We claim that the map $(A, (v_+)) \to (\widetilde{A}^\sharp/I, \bar{J})$ exhibits the right-hand side as the PD-envelope $B^\flat$ of $(v_+) \subset A$. Indeed, by

[13] Set $p - V(1) = \sum_{i\geq 0} V_i[a_i]$, $w = \sum_{i\geq 0} V_i[w_i]$. Then the equation (III.2.7) reads as follows: $v_-^{p^i} w_i = a_i$, for every $i \geq 0$.

[14] Indeed, given any $f = (v_-v_+ - p)g \in I \cap J$, we conclude, using that $v_-v_+ - p$ and v_+ are coprime in $\tilde{A} \otimes \mathbb{Q}$, that g is divisible by v_+ in $\tilde{A} \otimes \mathbb{Q}$. Hence, $g \in J$.

the universal property we have a PD-homomorphism $B^\flat \to \widetilde{A}^\sharp/I$. On the other hand, the projection $\widetilde{A} \to A$ gives a PD map $\widetilde{A}^\sharp \to B^\flat$ sending I to 0. By looking at generators, one readily checks that these are mutually inverse homomorphisms. Finally, we note that $\widetilde{A}^\sharp/I = A \otimes_{W(k)[v_+]} W(k)\langle v_+\rangle$. Using the isomorphism $\mathbf{G}_a^\sharp \xrightarrow{\sim} W^{(F)}$, this implies the claim.

To prove the second assertion, observe that $B^\dagger$ is identified with the tensor product $B^\dagger \otimes_{D_{\mathbf{Z}_p}((p))} \mathbf{Z}_p$, where $D_{\mathbf{Z}_p}((p))$ is the PD envelope of $(p) \subset \mathbf{Z}_p$ and the homomorphism $D_{\mathbf{Z}_p}((p)) \to \mathbf{Z}_p$ takes each $\gamma_n(p) \in D_{\mathbf{Z}_p}((p))$ to $\frac{p^n}{n!}$. Using (III.2.9) the claim follows. □

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There is a $\mathbf{G}_m$-equivariant isomorphism $\widetilde{\mathfrak{D}} := \mathfrak{D} \times_{\mathbf{A}^1_-/\mathbf{G}_m} \mathbf{A}^1_- \xrightarrow{\sim} \widetilde{\mathfrak{D}}'$ that identifies the substack $\widetilde{\mathfrak{D}}_0 := \mathfrak{D}_0 \times_{\mathbf{A}^1_-/\mathbf{G}_m} \mathbf{A}^1_-$ with $\widetilde{\mathfrak{D}}'_0$. Here the action of $\mathbf{G}_m$ on $\widetilde{\mathfrak{D}}$ comes from the $\mathbf{G}_m$-action on $\mathbf{A}^1_-$.

Proof. We start with a preliminary observation. The line bundle $\mathcal{O}_{\mathbf{A}^1_-/\mathbf{G}_m}(-1)$ determines via the Rees map $t_{W(k)} : W(k)^{\mathcal{N}} \to \mathbf{A}^1_-/\mathbf{G}_m$ a $\mathbf{G}_m$-torsor $W(k)^{\mathcal{N},r} \to W(k)^{\mathcal{N}}$. Thus an S-point of $W(k)^{\mathcal{N},r}$ is a pair consisting of an S-point of $W(k)^{\mathcal{N}}$ together with a trivialization of the pullback of $\mathcal{O}_{\mathbf{A}^1_-/\mathbf{G}_m}(-1)$ to S. Observe $\mathbf{G}_m$-equivariant isomorphisms $W(k)^{\mathcal{N},r} \times_{W(k)^{\mathcal{N}}} k^{\mathcal{N}} \xrightarrow{\sim} \operatorname{Spf} A$ and $W(k)^{\mathcal{N},r} \times_{W(k)^{\mathcal{N}}} \mathbf{A}^1_-/\mathbf{G}_m \xrightarrow{\sim} \mathbf{A}^1_-$. Consequently, we obtain an isomorphism $\operatorname{Spf} A \times_{W(k)^{\mathcal{N},r}} \mathbf{A}^1_- \xrightarrow{\sim} \widetilde{\mathfrak{D}}$ of $\mathbf{G}_m$-torsors over $\mathfrak{D}$. In particular, we conclude that the projection $\mathfrak{D} \to \operatorname{Spf} A/\mathbf{G}_m$ lifts to a map $\widetilde{\mathfrak{D}} \to \operatorname{Spf} A$.

Now we prove the proposition. Let R be a p-nilpotent ring. Recall from (III.2.4) that a point in $\mathfrak{D}(R)$ consists of a point $x \in k^{\mathcal{N}}(R)$ together with $\mathfrak{p}_{cris}(x) \simeq \mathfrak{p}_{\mathrm{d\bar{R}}} \circ t(x)$. Since $\widetilde{\mathfrak{D}} := \mathfrak{D} \times_{\mathbf{A}^1_-/\mathbf{G}_m} \mathbf{A}^1_-$, we see that a point of $\widetilde{\mathfrak{D}}(R)$ consists of a homomorphism $g : A \to R$ together with an isomorphism β of W_R-modules

$$\text{(III.2.10)} \qquad \operatorname{coker}(\mathbf{G}^\sharp_{a,R} \xrightarrow{(g(v_+),-\mathrm{can})} \mathbf{G}^\sharp_{a,R} \oplus W_R) := M_{\mathfrak{p}_{\mathrm{cris},R}} \underset{\simeq}{\overset{\beta}{\to}} M_{\mathfrak{p}_{\mathrm{d\bar{R}},R}} := \mathbf{G}^\sharp_{a,R} \oplus F_* W_R$$

such that the following two diagrams commute

$$\text{(III.2.11)} \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{G}^\sharp_{a,R} & \xrightarrow{(\mathrm{Id},0)} & M_{\mathfrak{p}_{\mathrm{cris},R}} & \xrightarrow{(0,F)} & F_* W_R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \nu} & & \\ 0 & \longrightarrow & \mathbf{G}^\sharp_{a,R} & \xrightarrow{(\mathrm{Id},0)} & M_{\mathfrak{p}_{\mathrm{d\bar{R}},R}} & \xrightarrow{(0,\mathrm{Id})} & F_* W_R & \longrightarrow & 0 \end{array}$$

$$\text{(III.2.12)} \qquad \begin{array}{ccc} M_{\mathfrak{p}_{\mathrm{cris},R}} & \xrightarrow{\beta} & M_{\mathfrak{p}_{\mathrm{d\bar{R}},R}} \\ {\scriptstyle (g(v_-)\circ\,\mathrm{can},p)}\downarrow & & \downarrow{\scriptstyle (g(v_-)\circ\,\mathrm{can},V)} \\ W_R & \xrightarrow{\mathrm{Id}} & W_R, \end{array}$$

for some $\nu \in \mathrm{Hom}_{W_R}(W_R, W_R) = W(R)$. Explicitly, $\beta' : \mathbf{G}^{\sharp}_{a,R} \oplus W_R \to M_{\mathfrak{p}_{cris,R}} \xrightarrow{\beta} \mathbf{G}^{\sharp}_{a,R} \oplus F_* W_R$ is given by $\begin{pmatrix} 1 & w \\ 0 & \nu \circ F \end{pmatrix}$, where $w = (w_1, w_2, \cdots) \in \mathrm{Hom}_{W_R}(W_R, \mathbf{G}^{\sharp}_{a,R}) = W^{(F)}(R)$ is subject to the equation $w_1 = g(v_+)$ and $[g(v_-)]w = p - V(\nu)$.

Sending a point of $\widetilde{\mathfrak{D}}(R)$ to $(g \in (\mathrm{Spf}\, A)(R), w \in W^{(F)}(R))$ defines a functorial map

$$\widetilde{\mathfrak{D}}(R) \to (\mathrm{Spf}\, A \times_{\mathbf{A}^1_+} W^{(F)})(R). \tag{III.2.13}$$

We next show that (III.2.13) is an isomorphism. To do this we shall construct its inverse.

Given a point $(g, w) \in (\mathrm{Spf}\, A \times_{\mathbf{A}^1_+} W^{(F)})(R)$, observe that since $w_1 = g(v_+)$ the first coordinate of $p - [g(v_-)]w$ is 0. Therefore this element can be uniquely written as $V(\nu)$, for some $\nu \in W(R)$. Then the matrix $\begin{pmatrix} 1 & w \\ 0 & \nu \circ F \end{pmatrix}$ defines a homomorphism $M_{\mathfrak{p}_{\mathrm{cris},R}} \xrightarrow{\beta} M_{\mathfrak{p}_{\mathrm{d\bar{R}},R}}$ making the diagrams (III.2.11) and (III.2.12) commutative. It remains to check that β is an isomorphism. Using the diagram (III.2.11) it is enough to check that $\nu \in W(R)$ is invertible. Let ν_1 be the first ghost coordinate of ν. If ν is not invertible we can find a maximal ideal $\mathfrak{m} \subset R$ which contains ν_1. Since $W^{(F)}(R/\mathfrak{m}) = 0$ and $w \in W^{(F)}(R)$, we conclude from the equation $[g(v_-)]w = p - V(\nu)$ that $\nu_1 = 1$ in $R/\mathfrak{m}$ contradicting to $\nu_1 \in \mathfrak{m}$. Thus β is an isomorphism. This defines a functorial map

$$(\mathrm{Spf}\, A \times_{\mathbf{A}^1_+} W^{(F)})(R) \to \widetilde{\mathfrak{D}}(R)$$

which is inverse to (III.2.13).

For the second part, note that $\beta = \begin{pmatrix} 1 & w \\ 0 & \nu \circ F \end{pmatrix} \in \widetilde{\mathfrak{D}}(R)$ lands in $\widetilde{\mathfrak{D}}_0$ if and only if $\nu = 1$ and the result follows. □

Remark III.27. Another way to compute the fiber product $k^{\mathcal{N}} \times_{W(k)^{\mathcal{N}}} \mathbf{A}^1_-$ relies on the computation of p-adically completed derived de Rham cohomology $\widehat{\mathrm{dR}}(k/W(k))$ by Bhatt [Bha12, Proposition 8.5], namely $\widehat{\mathrm{dR}}(k/W(k)) = \widehat{W(k)\langle x\rangle}/(x-p)$, where $\widehat{W(k)\langle x\rangle}$ stands for the p-completed PD-envelope of $(x) \subset W(k)[x]$. Moreover, he defines a descending filtration on $\widehat{W(k)\langle x\rangle}$ whose n-th term is topologically generated by elements of the form $\gamma_i(x)$ with $i \geq n$. The induced filtration $\mathrm{Fil}^\bullet$ on $\widehat{W(k)\langle x\rangle}/(x-p)$ matches the derived Hodge filtration $\mathrm{Hod}^\bullet$ under this isomorphism. Using the above result we shall construct a $\mathbf{G}_m$-equivariant isomorphism between the derived fiber product $k^{\mathcal{N}} \times^{\mathbf{L}}_{W(k)^{\mathcal{N}}} \mathbf{A}^1_-$ and the derived p-adic formal stack represented by the derived p-completion of $B^\dagger$. Recall that, for any derived p-adic formal stack X over $W(k)$, the Hodge filtered derived de Rham stack $(X/W(k))^{\mathrm{dR},+}$ is identified with the fiber product $X^{\mathcal{N}} \times^{\mathbf{L}}_{W(k)^{\mathcal{N}}} \mathbf{A}^1_-/\mathbf{G}_m$. We apply this to

$X = \mathrm{Spec} k$. To complete the proof observe that $(k/W(k))^{\mathrm{dR},+}$ is affine over $\mathbf{A}^1_-/\mathbf{G}_m$ in the sense of [BL22b, §7.3] and, hence, $(k/W(k))^{\mathrm{dR},+} \times^{\mathbf{L}}_{\mathbf{A}^1_-/\mathbf{G}_m} \mathbf{A}^1_-$ is represented by the algebra $R\Gamma\Big((k/W(k))^{\mathrm{dR},+} \times^{\mathbf{L}}_{\mathbf{A}^1_-/\mathbf{G}_m} \mathbf{A}^1_-, \mathcal{O}\Big)$. The latter is the derived p-completion of the Rees algebra $\mathrm{Rees}_{\mathrm{Hod}^\bullet} \widehat{\mathrm{dR}}(k/W(k))$ which is identified with the derived p-completion of $B^\dagger$ using Bhatt's result explained above.

Note that the derived p-completion of $B^\dagger$ is the quotient of the p-completion of $W(k)\langle v_+\rangle[v_-]$ by the ideal $(v_+v_- - p)$. Using [BMS18, Remark 6.16] it follows that this quotient is not p-adically separated and, in particular, it is not p-complete. Thus, the derived stack $k^{\mathcal{N}} \times^{\mathbf{L}}_{W(k)^{\mathcal{N}}} \mathbf{A}^1_-$ is not classical.

Remark III.28. Proposition III.2.2 gives another proof of Theorem III.21. Indeed, since B is p-torsionfree and v_+ has divided powers in B, the embedding $A \hookrightarrow B$ extends uniquely to a homomorphism of graded $W(k)$-algebras $B^\dagger \to B$. This defines a $\mathbf{G}_m$-equivariant map $\mathrm{Spf}\, B \to \widetilde{\mathfrak{D}}'_0 \xrightarrow{\sim} \widetilde{\mathfrak{D}}_0$. In particular, we obtain a morphism $\mathrm{Spf}\, B/\mathbf{G}_m \to k^{\mathcal{N}} \times_{W(k)^{\mathcal{N}}} \mathbf{A}^1_-/\mathbf{G}_m$ whose composition with the projection to $k^{\mathcal{N}}$ is given by $A \hookrightarrow B$. This is equivalent to Theorem III.21.

Remark III.29. Observe that the morphism

$$\mathfrak{D} = k^{\mathcal{N}} \times_{W(k)^{\mathcal{N}}} \mathbf{A}^1_-/\mathbf{G}_m \to k^{syn} \times_{W(k)^{syn}} \mathbf{A}^1_-/\mathbf{G}_m \tag{III.2.14}$$

induced by maps $k^{\mathcal{N}} \to k^{syn}$ and $W(k)^{\mathcal{N}} \to W(k)^{syn}$ is an isomorphism. Indeed, since the maps $k^{\mathcal{N}} \to k^{syn}$, $W(k)^{\mathcal{N}} \to W(k)^{syn}$ are étale, the morphism (III.2.14) is also étale. Thus it suffices to check that (III.2.14) induces an equivalence on points with values in every algebraically closed field K of characteristic p. This follows from [Dri24, Propositions 5.16.5 and 8.10.4 (iv)] which assert that the maps $k^{\mathcal{N}} \to W(k)^{\mathcal{N}}$, $k^{syn} \to W(k)^{syn}$ induce equivalences on K-points.

III.3: Construction of the functor $\mathcal{D}_{qc,[0,p-1]}(W(k)^{syn}) \to \mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(W(k))$

III.3.1: Corollary of Drinfeld's Lemma

The isomorphism $\mathfrak{p}_{\mathrm{cris}} \circ a \simeq \mathfrak{p}_{\mathrm{d\bar{R}}} \circ t \circ a$ from Theorem III.21 gives an isomorphism of functors $\mathcal{D}_{qc}(W(k)^{\mathcal{N}}) \to \mathcal{D}_{qc}(\mathrm{Spf}\, B/\mathbf{G}_m)$

$$a^* t^* \mathfrak{p}^*_{\mathrm{d\bar{R}}} \simeq a^* \mathfrak{p}^*_{\mathrm{cris}}. \tag{III.3.1}$$

We shall see that after restriction to $\mathcal{D}_{qc,[0,p-1]}(W(k)^{\mathcal{N}})$ a stronger statement holds.

Theorem III.30. *There exists a unique isomorphism* $t^*\mathfrak{p}^*_{\mathrm{d}\bar{\mathrm{R}}} \simeq \mathfrak{p}^*_{\mathrm{cris}}$ *of functors* $\mathcal{D}_{qc,[0,p-1]}(W(k)^{\mathcal{N}}) \to \mathcal{D}_{qc,[0,p-1]}(k^{\mathcal{N}})$ *whose pullbacks recover* (III.3.1).

Proof. Observe that indeed the functors $t^*\mathfrak{p}^*_{\mathrm{d}\bar{\mathrm{R}}}$ and $\mathfrak{p}^*_{\mathrm{cris}}$ carry $\mathcal{D}_{qc,[0,p-1]}(W(k)^{\mathcal{N}})$ to the subcategory $\mathcal{D}_{qc,[0,p-1]}(k^{\mathcal{N}}) \subset \mathcal{D}_{qc}(k^{\mathcal{N}})$: the assertion about $t^*\mathfrak{p}^*_{\mathrm{d}\bar{\mathrm{R}}}$ follows from Lemmas III.12 and III.13 combined with Remark III.14; the assertion about $\mathfrak{p}^*_{\mathrm{cris}}$ is true by definition of weights.

We will prove the theorem by showing that

$$a^* : \mathcal{D}_{qc,[0,p-1]}(k^{\mathcal{N}}) \to \mathcal{D}_{qc}(\mathrm{Spf}\, B/\mathbf{G}_m) \tag{III.3.2}$$

is fully faithful. We start with the following.

Lemma III.31. *Let* $\cdots \mathcal{N}^i \underset{v_+}{\overset{v_-}{\rightleftarrows}} \mathcal{N}^{i-1} \underset{v_+}{\overset{v_-}{\rightleftarrows}} \cdots$ *be an effective object of* $\mathcal{D}_{qc}(k^{\mathcal{N}})$. *Then the map*[15] $\mathcal{N}^i \to \left((\mathcal{N}^\bullet \overset{\mathbb{L}}{\otimes}_A B)^\wedge\right)^i$ *induced by the homomorphism of graded algebras* $A \to B$ *is an isomorphism for* $i < p$.

Proof. Equivalently, it suffices to show that $\left((\mathcal{N}^\bullet \overset{\mathbb{L}}{\otimes}_A B/A)^\wedge\right)^i$ is acyclic for $0 \le i \le p-1$. Moreover, by derived p-adic completeness, it suffices to prove that $(\mathcal{N}^\bullet \overset{\mathbb{L}}{\otimes}_A (B/A \overset{\mathbb{L}}{\otimes}_{W(k)} k))^i$ is acyclic for $0 \le i \le p-1$. The complex $B/A \overset{\mathbb{L}}{\otimes}_{W(k)} k$ is supported in two cohomological degrees: 0 and -1. Using Lemma III.17, the $A \otimes k$-module $B/A \otimes_{W(k)} k$ has grading degrees greater than or equal to p and is supported at the origin, *i.e.*, every element $x \in B/A \otimes k$ is killed by a power of the maximal ideal $\mathfrak{m} \subset A \otimes k$. Thus, $B/A \otimes k$ admits an exhaustive filtration $L_0 \subset L_1 \subset \cdots \subset B/A \otimes k$ by $A \otimes k$-submodules with each L_i/L_{i-1} of the form $k\{j\}$ with $j \ge p$. That $\mathcal{N}^\bullet$ is effective implies, using Remark III.14, that $(\mathcal{N}^\bullet \overset{\mathbb{L}}{\otimes}_A k)^i$ is acyclic for $i < 0$. Equivalently, $(\mathcal{N} \overset{\mathbb{L}}{\otimes} k\{j\})^i$ is acyclic for $i < j$. Thus $(\mathcal{N}^\bullet \overset{\mathbb{L}}{\otimes}_A (B/A \otimes_{W(k)} k))^i$ is acyclic for $i < p$.

To compute $\mathrm{Tor}_1^{W(k)}(B/A, k) = \ker(B/A \xrightarrow{p} B/A)$: by Lemma III.17 again, one has $\ker(B^i/A^i \xrightarrow{p} B^i/A^i) = \frac{1}{p} v_+^i k$ for $i \ge p$ and 0 otherwise. Thus $\mathrm{Tor}_1^{W(k)}(B/A, k) = (A/(v_-))\{p\}$ as an A-module. We have that

$$(\mathcal{N}^\bullet \overset{\mathbb{L}}{\otimes}_A \mathrm{Tor}_1^{W(k)}(B/A, k))^i \xrightarrow{\sim} \mathrm{Cone}(\mathcal{N}^{i-p+1} \xrightarrow{v_-} \mathcal{N}^{i-p}).$$

Using effectivity, the cone above is acyclic for $i < p$. □

[15] For a complex M of abelian groups, we denote by $M^\wedge$ its derived p-adic completion.

Corollary III.32. *For every $M \in \mathcal{D}_{qc,[0,+\infty]}(k^{\mathcal{N}})$, the natural transformation* $\mathrm{Id} \to a_* a^*$ *induces an isomorphism* $w_{\leq p-1} M \xrightarrow{\sim} w_{\leq p-1} a_* a^* M$.

Now we can prove that (III.3.2) is fully faithful. For any $M, M' \in \mathcal{D}_{qc,[0,p-1]}(k^{\mathcal{N}})$,

$$\begin{aligned} \mathrm{RHom}(M, M') &= \mathrm{RHom}(M, w_{\leq p-1} M') \simeq \mathrm{RHom}(M, w_{\leq p-1} a_* a^* M') \\ &\simeq \mathrm{RHom}(M, a_* a^* M') \simeq \mathrm{RHom}(a^* M, a^* M'), \end{aligned}$$

where the second and fourth isomorphisms follow by adjunction, and the third isomorphism follows by Corollary III.32. This proves fully faithfulness and thus completes the proof of the theorem. □

Let us record a generalization of Theorem III.30. pr There exists an isomorphism $w_{\leq p-1} t^* \mathfrak{p}^*_{\overline{\mathrm{dR}}} \simeq w_{\leq p-1} \mathfrak{p}^*_{\mathrm{cris}}$ of functors $\mathcal{D}_{qc,[0,+\infty]}(W(k)^{\mathcal{N}}) \to \mathcal{D}_{qc,[0,p-1]}(k^{\mathcal{N}})$.

Proof. Using Theorem III.21, we have $a_* a^* t^* \mathfrak{p}^*_{\overline{\mathrm{dR}}} \mathcal{F} \simeq a_* a^* \mathfrak{p}^*_{\mathrm{cris}} \mathcal{F}$. Applying $w_{\leq p-1}$ and using Corollary III.32 we obtain the result. □

Remark III.33. In geometric context, *i.e.*, for the prismatic gauge $\mathcal{H}_{\mathcal{N}}(X) \in \mathcal{D}_{qc}(W(k)^{\mathcal{N}})$ associated to a smooth p-adic formal scheme X over $W(k)$, (III.3.1) gives

$$a^* t^* \mathfrak{p}^*_{\overline{\mathrm{dR}}}(\mathcal{H}_{\mathcal{N}}(X)) \simeq a^* \mathfrak{p}^*_{\mathrm{cris}}(\mathcal{H}_{\mathcal{N}}(X)), \tag{III.3.3}$$

which expresses a relation between the Nygaard filtration on $R\Gamma_{\mathrm{cris}}(X_k)$, where $X_k := X \times \mathrm{Spec} k$, and the Hodge filtration on $R\Gamma_{\mathrm{dR}}(X)$. In fact, $a^* t^* \mathfrak{p}^*_{\overline{\mathrm{dR}}}(\mathcal{H}_{\mathcal{N}}(X))$, viewed as a complex of filtered modules over the filtered algebra $(W(k), (p)^{[\cdot]})$ (see Remark III.20), can be identified, using [BO78a, Theorem 7.2] with the crystalline Hodge filtration on $R\Gamma_{\mathrm{cris}}(X_k)$. Thus, (III.3.3) says that the crystalline Hodge filtration can be obtained from the Nygaard filtration by "downgrading" using the functor a^*.

III.3.2: Construction of the functor

Define a functor

$$\Phi_{\mathrm{Maz}} : \mathcal{D}_{qc,[0,p-1]}(W(k)^{syn}) \to \mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(W(k)) \tag{III.3.4}$$

as follows. Let $\mathcal{F} \in \mathcal{D}_{qc,[0,p-1]}(W(k)^{syn})$ be a prismatic F-gauge with Hodge-Tate weights in $[0, p-1]$. We have the following diagram ([Bha23, Remark 1.4.2] or §III.1.3)

$$\begin{array}{ccccccc} & & & & & & \mathbf{A}^1_-/\mathbf{G}_m \\ & & & & & \overset{t}{\nearrow} & \downarrow \mathfrak{p}_{\mathrm{d\bar{R}}} \\ \operatorname{Spf} W(k) & \underset{j_+\circ F^{-1}}{\overset{j_-}{\rightrightarrows}} & \operatorname{Spf} A/\mathbf{G}_m & \xrightarrow{\simeq} & k^{\mathcal{N}} & \xrightarrow{\mathfrak{p}_{\mathrm{cris}}} & W(k)^{\mathcal{N}} \\ & & & & \downarrow \bar{j}_{\mathcal{N}} & & \downarrow j_{\mathcal{N}} \\ & & & & k^{syn} & \longrightarrow & W(k)^{syn} \end{array} \tag{III.3.5}$$

where j_- (resp. $j_+ \circ F^{-1}$) is an open embedding onto the substack given by $v_- \neq 0$ (resp. $v_+ \neq 0$), $\bar{j}_{\mathcal{N}} : k^{\mathcal{N}} \to k^{syn}$ exhibits the target as the coequalizer of $j_\pm : \operatorname{Spf} W(k) = k^{\mathbf{\Delta}} \rightrightarrows k^{\mathcal{N}}$ (see, [Bha23, Def. 6.1.1]), and the square is commutative. The morphism $\operatorname{Spf} W(k) \xrightarrow{v_-=1} \mathbf{A}^1_-/\mathbf{G}_m$ is isomorphic to $t \circ j_-$ and $\operatorname{Spf} W(k) \xrightarrow{v_-=p} \mathbf{A}^1_-/\mathbf{G}_m$ is isomorphic to $t \circ j_+ \circ F^{-1}$; see the end of §III.1.3. By definition of $\bar{j}_{\mathcal{N}}$

$$\bar{j}_{\mathcal{N}} \circ j_+ \simeq \bar{j}_{\mathcal{N}} \circ j_-. \tag{III.3.6}$$

Let $\mathcal{M} := \mathfrak{p}^*_{\mathrm{d\bar{R}}} \circ j^*_{\mathcal{N}} \mathcal{F}$. Note that $F^*(\mathcal{M}|_{v_-=p}) \simeq j^*_+ \circ t^* \mathcal{M}$ and $\mathcal{M}|_{v_-=1} \simeq j^*_- \circ t^* \mathcal{M}$. Then using Theorem III.30, one has that $\mathfrak{p}^*_{\mathrm{cris}} \circ j^*_{\mathcal{N}} \mathcal{F} \simeq t^* \mathcal{M}$. Hence we obtain $F^*(\mathcal{M}|_{v_-=p}) \simeq j^*_+ \circ \mathfrak{p}^*_{\mathrm{cris}} \circ j^*_{\mathcal{N}} \mathcal{F}$ and $\mathcal{M}|_{v_-=1} \simeq j^*_- \circ \mathfrak{p}^*_{\mathrm{cris}} \circ j^*_{\mathcal{N}} \mathcal{F}$. By (III.3.6), we get an isomorphism $F^*(\mathcal{M}_{v_-=p}) \simeq \mathcal{M}_{v_-=1}$, which we denote by φ. Define the functor Φ_{Maz} in (III.3.4) by sending $\mathcal{F}$ to $(\mathcal{M}, \varphi)$.

Remark III.34. Observe that Ψ_{Maz} from Theorem III.21 gives a functor

$$\mathcal{D}_{qc}(W(k)^{syn}) \to \mathcal{D}_{qc}(k^{syn}) \times_{\mathcal{D}_{qc}(\operatorname{Spf} B/\mathbf{G}_m)} \mathcal{D}_{qc}(\mathbf{A}^1_-/\mathbf{G}_m) \tag{III.3.7}$$

sending $\mathcal{D}_{qc,[a,b]}(W(k)^{syn})$ to $\mathcal{D}_{qc,[a,b]}(k^{syn}) \times_{\mathcal{D}_{qc}(\operatorname{Spf} B/\mathbf{G}_m)} \mathcal{D}_{qc,[a,b]}(\mathbf{A}^1_-/\mathbf{G}_m)$. We assert that there is a functor

$$\mathcal{D}_{qc,[0,p-1]}(k^{syn}) \times_{\mathcal{D}_{qc}(\operatorname{Spf} B/\mathbf{G}_m)} \mathcal{D}_{qc,[0,p-1]}(\mathbf{A}^1_-/\mathbf{G}_m) \to \mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(W(k)) \tag{III.3.8}$$

whose precomposition with (III.3.7) is Φ_{Maz}. To see this, observe that the functor

$$\begin{aligned} &\mathcal{D}_{qc,[0,p-1]}(k^{syn}) \underset{\mathcal{D}_{qc,[0,p-1]}(k^{\mathcal{N}})}{\times} \mathcal{D}_{qc,[0,p-1]}(\mathbf{A}^1_-/\mathbf{G}_m) \to \\ &\mathcal{D}_{qc,[0,p-1]}(k^{syn}) \underset{\mathcal{D}_{qc}(\operatorname{Spf} B/\mathbf{G}_m)}{\times} \mathcal{D}_{qc,[0,p-1]}(\mathbf{A}^1_-/\mathbf{G}_m) \end{aligned}$$

induced by the fully faithful embedding (III.3.2) is an equivalence[16]. We define

$$\mathcal{D}_{qc,[0,p-1]}(k^{syn}) \underset{\mathcal{D}_{qc,[0,p-1]}(k^{\mathcal{N}})}{\times} \mathcal{D}_{qc,[0,p-1]}(\mathbf{A}^1_-/\mathbf{G}_m) \to \mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(W(k))$$

sending an object $(\mathcal{F}_k, \mathcal{M}, v : \bar{j}^*_{\mathcal{N}}\mathcal{F}_k \xrightarrow{\sim} t^*\mathcal{M})$ of the fiber product to $(\mathcal{M}, \varphi)$, where $\varphi : F^*(\mathcal{M}_{v_-=p}) \simeq \mathcal{M}_{v_-=1}$ is constructed from diagram (III.3.5) and equation (III.3.6). This defines (III.3.8). The fact that its precomposition with (III.3.7) is Φ_{Maz} is straightforward.

The following remarks will not be used in the remainder of the paper. However, we believe that they are important in their own right.

Remark III.35 (A refinement of the crystalline Frobenius). For every object $N \in \mathcal{D}_{qc,[0,+\infty]}(k^{\mathcal{N}}) \subset \widehat{\mathcal{D}}_{gr}(A)$, we have a canonical morphism

$$N \to \pi^* \circ (j_+ \circ F^{-1})^* N, \tag{III.3.9}$$

where $\pi : k^{\mathcal{N}} \to \operatorname{Spf} W(k)$ is the structure map. One can define (III.3.9) by observing that π^* is the right adjoint to $(j_+ \circ F^{-1})^* \colon \mathcal{D}_{qc,[0,+\infty]}(k^{\mathcal{N}}) \to \mathcal{D}_{qc}(\operatorname{Spf} W(k))$. Alternatively, (III.3.9) can be constructed using Drinfeld's observation that every effective gauge $\mathcal{F}$ extends to a contravariant $\mathcal{O}$-module on the c-stack $k^{\mathcal{N},c}$ and that, for any scheme S, the category $k^{\mathcal{N},c}(S)$ has initial object given by p_+ (see Remark III.25).

Given an effective F-gauge $\mathcal{F} \in \mathcal{D}_{qc,[0,+\infty]}(W(k)^{syn})$, we have functorial morphisms

(III.3.10)

$$a^*t^*\mathfrak{p}^*_{\mathrm{d\bar{R}}}(\mathcal{F}) \underset{\text{(III.3.1)}}{\xrightarrow{\sim}} a^*\mathfrak{p}^*_{\mathrm{cris}}(\mathcal{F}) \underset{\text{(III.3.9)}}{\to} a^*\pi^*(j_+ \circ F^{-1})^*\mathfrak{p}^*_{\mathrm{cris}}(\mathcal{F}) \underset{\text{(III.3.6)}}{\xrightarrow{\sim}} a^*\pi^*F_*j^*_-\mathfrak{p}^*_{\mathrm{cris}}(\mathcal{F}).$$

To rewrite it more suggestively, let $F^\bullet = \oplus F^i \in \widehat{\mathcal{D}}_{gr}(W(k)[v_-])$ be the complex of graded $W(k)[v_-]$-modules corresponding to $\mathfrak{p}^*_{\mathrm{d\bar{R}}}(\mathcal{F})$ under the equivalence (III.1.1). Then the composition (III.3.10) yields a homomorphism in $\widehat{\mathcal{D}}_{gr}(W(k)[v_-])$

$$\varphi_\bullet : F^\bullet \to F_*M \otimes_{W(k)} B, \tag{III.3.11}$$

where $M := F^0$ and the action of $W(k)[v_-]$ on $F_*F^0 \otimes_{W(k)} B$ comes from the homomorphism $W(k)[v_-] \hookrightarrow B$. Explicitly, (III.3.11) amounts to specifying $W(k)$-linear maps $\varphi_i : F^i \to F_*M$ for each $i \geq 0$ such that $p^{[i+1]-[i]}\varphi_{i+1} = \varphi_i v_-$ (see Definition III.16). We claim that $\varphi_0 : M \to F_*M$ coincides with the crystalline Frobenius. Indeed, taking $S = \operatorname{Spf} B/\mathbf{G}_m$

[16] In fact, for any ∞-categories $\mathcal{C}_i$, $i = 1, 2, 3, 4$, and functors $\mathcal{C}_1 \to \mathcal{C}_3$, $\mathcal{C}_2 \to \mathcal{C}_3$, $F : \mathcal{C}_3 \hookrightarrow \mathcal{C}_4$ such that F is fully faithful, the functor $\tilde{F} : \mathcal{C}_1 \times_{\mathcal{C}_3} \mathcal{C}_2 \to \mathcal{C}_1 \times_{\mathcal{C}_4} \mathcal{C}_2$, induced by F, is an equivalence of categories. Indeed, the description of the mapping spaces in the fiber product category [Lur18a, tag 046R] implies that $\tilde{F}$ is fully faithful. The essential surjectivity is obvious.

in Remark III.25, the composition (III.3.10) is obtained by taking fibers of $\mathcal{F}$ along $p_+ \to \mathfrak{p}_{\mathrm{d\bar{R}}}(t)$ and using (III.3.6). The map $\mathrm{Spf}\, B/\mathbf{G}_m \to \mathfrak{D}$ from (III.2.5) is induced by the natural map $B^\flat \to B$ from the PD envelope of $(v_+) \subset A$ to B; see Lemma III.26. Since B has no p-torsion, this map factors through the PD envelope $B^\dagger$ of $(v_+) \subset A$ *relative* to $(\mathbf{Z}_p, (p))$[17]. It follows that the map $\mathrm{Spf}\, B/\mathbf{G}_m \to \mathfrak{D}$ factors through the closed substack $\mathfrak{D}_0$. Now our claim follows from the commutativity of diagram (III.2.6)[18].

Denote by $\mathrm{Mod}_{\mathrm{Maz}}(W(k))$ the ∞-category formed of pairs $(F^\bullet, \varphi)$, where $F^\bullet \in \widehat{\mathcal{D}}_{gr}(W(k)[v_-])$ is effective and $\varphi : F^\bullet \to F_* F^0 \otimes_{W(k)} B$ is a map in $\widehat{\mathcal{D}}^{\mathrm{eff}}_{gr}(W(k)[v_-])$. More formally, it is defined as the lax equalizer $\mathrm{LEq}(\mathrm{Id}, F_* F^0 \otimes B)$ as in ([NS18, Def. II.1.4]). Here by $F_* F^0 \otimes_{W(k)} B$ we mean the endofunctor of $\widehat{\mathcal{D}}^{\mathrm{eff}}_{gr}(W(k)[v_-])$ taking $F^\bullet$ to $F_* F^0 \otimes_{W(k)} B$. We refer to objects of $\mathrm{Mod}_{\mathrm{Maz}}(W(k))$ as Mazur modules. The above construction determines a functor

$$\mathbf{\Phi}_{\mathrm{Maz}} : \mathcal{D}_{qc,[0,\infty]}(W(k)^{syn}) \to \mathrm{Mod}_{\mathrm{Maz}}(W(k)), \quad \mathcal{F} \mapsto (\mathfrak{p}^*_{\mathrm{d\bar{R}}}(\mathcal{F}), \varphi_\bullet). \tag{III.3.12}$$

Next, we shall construct a functor

$$\mathscr{DMF}^{\mathrm{big}}_{[0,\infty]}(W(k)) \to \mathrm{Mod}_{\mathrm{Maz}}(W(k)), \tag{III.3.13}$$

whose precomposition with (III.3.4) is isomorphic to (III.3.12). To do this, observe that the ∞-category $\mathcal{C}_1$ formed of pairs $(F^\bullet, \varphi')$, where $F^\bullet \in \widehat{\mathcal{D}}^{\mathrm{eff}}_{gr}(W(k)[v_-])$ and $\varphi' : F^\bullet \to F_* F^0 \otimes_{W(k)} A$ is a map in $\widehat{\mathcal{D}}^{\mathrm{eff}}_{gr}(W(k)[v_-])$, is equivalent to the lax equalizer

$$\mathcal{C}_2 := \mathrm{LEq}\left(\mathcal{D}_{qc,[0,\infty]}(\mathbf{A}^1_-/\mathbf{G}_m) \underset{F^* \circ i_p^*}{\overset{i_1^*}{\rightrightarrows}} \mathcal{D}_{qc}(\mathrm{Spf}\, W(k)) \right).$$

The latter category consists of $\mathcal{M} \in \mathcal{D}_{qc,[0,\infty]}(\mathbf{A}^1_-/\mathbf{G}_m)$ together with a morphism $\varphi' : F^*(i_p^* \mathcal{M}) \to i_1^* \mathcal{M}$ in $\mathcal{D}_{qc}(\mathrm{Spf}\, W(k))$ (*cf.* Definition III.6). Note that, for every $\mathcal{M} \in \mathcal{D}_{qc,[0,\infty]}(\mathbf{A}^1_-/\mathbf{G}_m)$, we have

$$\mathrm{Hom}(F^* i_p^* \mathcal{M}, i_1^* \mathcal{M}) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{M}, i_{p*} F_* i_1^* \mathcal{M}) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{M}, w_{\geq 0} i_{p*} F_* i_1^* \mathcal{M}),$$

where $w_{\geq 0}$ is the right adjoint to the embedding $\mathcal{D}_{qc,[0,\infty]}(\mathbf{A}^1_-/\mathbf{G}_m) \hookrightarrow \mathcal{D}_{qc}(\mathbf{A}^1_-/\mathbf{G}_m)$ (*cf.* Lemma III.15). The right-hand side of the above is naturally isomorphic to

[17] Indeed, the map from the PD-envelope of $(p) \subset \mathbf{Z}_p$ to $\mathbf{Z}_p$ is an isomorphism after inverting p.

[18] More generally, we infer that $\varphi_\bullet$ lifts to a homomorphism of $W(k)[v_-]$-modules $F^\bullet \to (F_* M \otimes_{W(k)} B^\flat)^\wedge$, where the superscript $\wedge$ stands for the graded p-completion. In degree zero, the composition $F^\bullet \to (F_* M \otimes_{W(k)} B^\dagger)^\wedge$ of the latter with $(F_* M \otimes_{W(k)} B^\flat)^\wedge \to (F_* M \otimes_{W(k)} B^\dagger)^\wedge$ is homotopic to the crystalline Frobenius.

$\mathrm{Hom}_{\widehat{\mathcal{D}}^{\mathrm{eff}}_{gr}(W(k)[v_-])}(F^\bullet, F_*F^0 \otimes_{W(k)} A)$, where $F^\bullet$ is the graded module corresponding to $\mathcal{M}$[19]. This gives $\mathcal{C}_1 \xrightarrow{\sim} \mathcal{C}_2$.

The category $\mathcal{C}_2$ receives an obvious functor $(\mathcal{M}, \varphi') \mapsto (\mathcal{M}, \varphi')$ from $\mathscr{DMF}^{\mathrm{big}}_{[0,\infty]}(W(k))$, see Definition III.6. This functor is fully faithful by [NS18, Proposition II.1.5]. To complete the construction of (III.3.13), note that the morphism $A \to B$ induces a functor $\mathcal{C}_1 \to \mathrm{Mod}_{\mathrm{Maz}}(W(k))$ sending $(F^\bullet, \varphi')$ to $(F^\bullet, \varphi)$, where φ is the composition $F^\bullet \xrightarrow{\varphi'} F_*F^0 \otimes A \to F_*F^0 \otimes B$. Thus, in the notation of §III.1.2, $\varphi_i = p^{i-[i]}\varphi'_i$.

Note that the restriction of (III.3.13) to $\mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(W(k)) \to \mathrm{Mod}_{\mathrm{Maz}}(W(k))$ is fully faithful. Indeed, using [NS18, Proposition II.1.5], it is enough to show that, for $M^\bullet, N^\bullet \in \mathcal{D}_{qc,[0,p-1]}(\mathbf{A}^1_-/\mathbf{G}_m)$, the natural map $\mathrm{Hom}_{\widehat{\mathcal{D}}^{\mathrm{eff}}_{gr}(W(k)[v_-])}(M^\bullet, F_*N_0 \otimes A) \to \mathrm{Hom}_{\widehat{\mathcal{D}}^{\mathrm{eff}}_{gr}(W(k)[v_-])}(N^\bullet, F_*N_0 \otimes B)$, is an equivalence. This follows since $\mathcal{D}_{qc,(-\infty,p-1]}(\mathbf{A}^1_-/\mathbf{G}_m) \hookrightarrow \mathcal{D}_{qc}(\mathbf{A}^1_-/\mathbf{G}_m)$ admits a right adjoint $w_{\le p-1}$ (*cf.* Lemma III.15) and $F_*N^0 \otimes A \to F_*N^0 \otimes B$ is an equivalence after applying $w_{\le p-1}$, see Corollary III.18.

Remark III.36 (Relation to the Mazur-Ogus construction). In the geometric context, (III.3.11) encodes divisibility properties of the crystalline Frobenius discovered by Mazur. Namely, following the idea of Mazur [Maz73], we define a functor

$$\widetilde{\mathbf{\Phi}}_{\mathrm{Maz}} : \widehat{\mathrm{Sm}}^{\mathrm{op}}_{W(k)} \to \mathrm{Mod}_{\mathrm{Maz}}(W(k)) \tag{III.3.14}$$

as follows. Let X be a smooth p-adic formal scheme over $W(k)$. First, assume that there exists a closed embedding $X \hookrightarrow Y$, where Y is a smooth p-adic formal scheme over $W(k)$ equipped with a Frobenius lift $\tilde{F}_Y$ (that is not required to preserve X). Denote by $\mathcal{D}_X(Y)$ the p-completed PD-hull of X in Y and by J the sheaf of ideals of the closed embedding $X \hookrightarrow \mathcal{D}_X(Y)$. Note $\mathcal{D}_X(Y)$ can be also viewed as the p-completed PD-hull of X_k in Y; in particular, it follows that $\tilde{F}_Y$ extends uniquely to an endomorphism $\mathcal{D}_X(Y)$. Set $\Omega^j_{\mathcal{D}_X(Y)}$ to be the tensor product $\mathcal{O}_{\mathcal{D}_X(Y)} \otimes_{\mathcal{O}_Y} \Omega^j_Y$. For each i, define a subcomplex $\mathrm{Fil}^i(\Omega^\bullet_{\mathcal{D}_X(Y)}, d_{\mathrm{dR}})$ of $(\Omega^\bullet_{\mathcal{D}_X(Y)}, d_{\mathrm{dR}})$ to be

$$J^{[i]} \to J^{[i-1]}\Omega^1_{\mathcal{D}_X(Y)} \to \cdots \to \Omega^i_{\mathcal{D}_X(Y)} \to \Omega^{i+1}_{\mathcal{D}_X(Y)} \to \cdots,$$

where $J^{[m]}$ stands for the m-th divided power of J. The restriction map $(\Omega^\bullet_{\mathcal{D}_X(Y)}, d_{\mathrm{dR}}) \to (\Omega^\bullet_X, d_{\mathrm{dR}})$ extends to a map of filtered complexes

$$\mathrm{Fil}^\cdot(\Omega^\bullet_{\mathcal{D}_X(Y)}, d_{\mathrm{dR}}) \to (\Omega^\bullet_X, d_{\mathrm{dR}})^{\ge \cdot} \tag{III.3.15}$$

[19]Indeed, for every $\mathcal{V} \in \mathcal{D}_{qc}(\mathrm{Spf}\, W(k))$, the sheaf $w_{\ge 0}i_{p*}\mathcal{V}$ corresponds, under the equivalence $\mathcal{D}_{qc,[0,\infty]}(\mathbf{A}^1_-/\mathbf{G}_m) \xrightarrow{\sim} \widehat{\mathcal{D}}^{\mathrm{eff}}_{gr}(W(k)[v_-])$, to $\mathcal{V} \otimes A$.

and the filtered PD Poincaré Lemma of Berthelot ([Maz73, p. 61]) asserts this is an isomorphism in the filtered derived category. A key observation ([Maz73, p. 63]) is that, for every i, the chain map $(\Omega^\bullet_{\mathcal{D}_X(Y)}, d_{\mathrm{dR}}) \xrightarrow{\tilde{F}_Y^*} (\Omega^\bullet_{\mathcal{D}_X(Y)}, d_{\mathrm{dR}})$ restricted to $\mathrm{Fil}^i(\Omega^\bullet_{\mathcal{D}_X(Y)}, d_{\mathrm{dR}})$ is (uniquely) divisible by $p^{[i]}$. Combining this with (III.3.15), we obtain $W(k)$-linear maps $\varphi_i : F^i R\Gamma_{\mathrm{dR}}(X) \to F_* R\Gamma_{\mathrm{dR}}(X)$ together with homotopies $p^{[i+1]-[i]}\varphi_{i+1} \simeq \varphi_i v_-$, where $v_- : F^{i+1}R\Gamma_{\mathrm{dR}}(X) \to F^i R\Gamma_{\mathrm{dR}}(X)$ denotes the usual "inclusion" map. Note that, by construction, the map φ_0 is the crystalline Frobenius.

The above construction determines a contravariant functor from the category $\mathcal{C}$ of tuples $(s : X \hookrightarrow Y, \tilde{F}_Y)$ to $\mathrm{Mod}_{\mathrm{Maz}}(W(k))$ that carries every morphism in $\mathcal{C}$, which is an isomorphism on X's, to an isomorphism in $\mathrm{Mod}_{\mathrm{Maz}}(W(k))$. A standard argument shows that $\mathcal{C}^{\mathrm{op}} \to \mathrm{Mod}_{\mathrm{Maz}}(W(k))$ descends uniquely to (III.3.14)[20] (*cf.* [BO78a, §8, p. 23]).

For the sake of completeness, let us remark that, for every $n < p$, the maps φ_i, $(0 \leq i \leq n)$, can be assembled to a morphism in the derived category of sheaves

$$\frac{\varphi}{p^n} : (p^n\mathcal{O}_X \xrightarrow{d_{\mathrm{dR}}} p^{n-1}\Omega^1_X \xrightarrow{d_{\mathrm{dR}}} \cdots \xrightarrow{d_{\mathrm{dR}}} \Omega^n_X \xrightarrow{d_{\mathrm{dR}}} \cdots) \longrightarrow (\Omega^\bullet_X, d_{\mathrm{dR}}),$$

such that $p^n\frac{\varphi}{p^n}$ is the crystalline Frobenius precomposed with the embedding of the source into $(\Omega^\bullet_X, d_{\mathrm{dR}})$. In particular, we obtain morphism (I.0.1). The above construction of (I.0.1) is due to Ogus; we refer the reader to [Ogu21, Theorem 6.8] for a detailed exposition and a proof of the isomorphism property of (I.0.1).

We claim that

$$\widetilde{\mathbf{\Phi}}_{\mathrm{Maz}} \xrightarrow{\sim} \mathbf{\Phi}_{\mathrm{Maz}} \circ \mathcal{H}_{syn}. \tag{III.3.16}$$

In fact, any functor $\widehat{\mathrm{Sm}}^{\mathrm{op}}_{W(k)} \to \mathrm{Mod}_{\mathrm{Maz}}(W(k))$ equipped with an isomorphism between its post-composition with the forgetful functor $\mathrm{Mod}_{\mathrm{Maz}}(W(k)) \to \widehat{\mathcal{D}}_{gr}(W(k)[v_-])$ and the Hodge filtered de Rham functor such that φ_0 is homotopic to the crystalline Frobenius is isomorphic to $\mathbf{\Phi}_{\mathrm{Maz}} \circ \mathcal{H}_{syn}$. To see this, consider the functor $F^i : \widehat{\mathrm{Sm}}^{\mathrm{op}}_{W(k)} \to \hat{\mathcal{D}}(W(k))$ sending a smooth p-adic formal scheme X to the i-th term of the Hodge filtration on $R\Gamma_{\mathrm{dR}}(X)$. It suffices to show that, for every i, the space $\mathrm{Map}(F^i, F^0)$ of morphisms between the functors is discrete, $\pi_0\,\mathrm{Map}(F^i, F^0)$ is p-torsionfree, and the map $\pi_0\,\mathrm{Map}(F^i, F^0) \to \pi_0\,\mathrm{Map}(F^{i+1}, F^0)$ induced by the canonical morphism $v_- : F^{i+1} \to F^i$ is injective. Using that F^i is a sheaf for the Zariski topology, the space $\mathrm{Map}(F^i, F^0)$ does not change if $\widehat{\mathrm{Sm}}_{W(k)}$ is replaced by the subcategory $\widehat{\mathrm{SmAff}}_{W(k)}$ of smooth affine formal schemes. Let

[20]Here is the argument: the fiber of $\mathcal{C}^{\mathrm{op}} \to \widehat{\mathrm{Sm}}^{\mathrm{op}}_{W(k)}$ over each affine X is non-empty and admits finite products. Therefore, the nerve of the fiber is contractible. This proves the independence of the Mazur module structure on $R\Gamma_{\mathrm{dR}}(X)$ on the choice of $(s : X \hookrightarrow Y, \tilde{F}_Y)$.

$\mathrm{QSyn}_{W(k)}$ be the category of quasi-syntomic affine formal schemes over $W(k)$, and let $LF^i : \mathrm{QSyn}^{\mathrm{op}}_{W(k)} \to \hat{\mathcal{D}}(W(k))$ be the p-complete left Kan extension along the embedding $\mathrm{SmAlg}^{\mathrm{op}}_{W(k)} \hookrightarrow \mathrm{QSyn}^{\mathrm{op}}_{W(k)}$. Then by the universal property of the left Kan extension, we have a homotopy equivalence $\mathrm{Map}(F^i, F^0) \xrightarrow{\sim} \mathrm{Map}(LF^i, LF^0)$ [Lur09, Proposition 4.3.2.17.]. Our claim follows from the following facts: LF^i is a sheaf for the quasi-syntomic topology on $\mathrm{QSyn}_{W(k)}$ ([BMS19, Theorem 3.1]); the inclusion $\mathrm{QRSPerf}_{W(k)} \hookrightarrow \mathrm{QSyn}_{W(k)}$ of the category of quasi-regular semi-perfectoid affine formal schemes into the category of all quasi-syntomic affine formal schemes induces an equivalence between the corresponding categories of sheaves ([BMS19, Proposition 4.31]); for any $X \in \mathrm{QRSPerf}_{W(k)}$, the complex $F^i(X)$ is concentrated in degree zero, $H^0(F^i(X))$ is p-torsionfree and the map $v_- : H^0(F^{i+1}(X)) \to H^0(F^i(X))$ is injective ([AMMN22, Construction 6.6]).

Using (III.3.13), the category $\mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(W(k))$ is a full subcategory of $\mathrm{Mod}_{\mathrm{Maz}}(W(k))$. In particular, we derive from (III.3.16) that the functors

$$\widetilde{\Phi}_{\mathrm{Maz}}, \Phi_{\mathrm{Maz}} \circ \mathcal{H}_{syn} : \widehat{\mathrm{Sm}}^{\leq p-1}_{W(k)} \to \mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(W(k))$$

are isomorphic.

Remark III.37 (A "non-abelian" refinement of the crystalline Frobenius). Let X be a p-adic formal scheme over $W(k)$. Let $X_k := X \times_{\mathrm{Spf}\, W(k)} \mathrm{Spec} k$ be its special fibre. By Theorem III.21, we obtain a canonical isomorphism

$$X_k^{\mathcal{N}} \times_{k^{\mathcal{N}}} \mathrm{Spf}\, B/\mathbf{G}_m \simeq X^{\mathcal{N}} \times_{W(k)^{\mathcal{N}}} \mathrm{Spf}\, B/\mathbf{G}_m \simeq X^{\mathrm{dR},+} \times_{\mathbf{A}^1_-/\mathbf{G}_m} \mathrm{Spf}\, B/\mathbf{G}_m \tag{III.3.17}$$

of stacks over $\mathrm{Spf}\, B/\mathbf{G}_m$. Recall from [Bha23, Remark 3.3.4] a canonical morphism of stacks over $k^{\mathcal{N}}$:

$$X_k^{\mathrm{cris}} \times k^{\mathcal{N}} = X^{\mathrm{dR}} \times k^{\mathcal{N}} \to X_k^{\mathcal{N}}, \tag{III.3.18}$$

which is an isomorphism over the open substack $\mathrm{Spf}\, W(k) \xrightarrow{j_+} k^{\mathcal{N}}$ given by $v_+ \neq 0$. Likewise, we have a map of stacks over $\mathbf{A}^1_-/\mathbf{G}_m$

$$X^{\mathrm{dR},+} \to X^{\mathrm{dR}} \times \mathbf{A}^1_-/\mathbf{G}_m, \tag{III.3.19}$$

which is an isomorphism for $v_- \neq 0$. The map can be constructed by restricting the structure map $X^{\mathcal{N}} \to X^{\mathbf{\Delta}} \times W(k)^{\mathcal{N}}$ (see [Bha23, Definition 5.3.10 (3)]) to $\mathbf{A}^1_-/\mathbf{G}_m \xrightarrow{\mathfrak{p}_{\mathrm{dR}}}$

$W(k)^{\mathcal{N}}$. Composing (III.3.17) with (III.3.18), we obtain

$$F_{\mathrm{ref}} : X^{\mathrm{dR}} \times \mathrm{Spf}\, B/\mathbf{G}_m \to X^{\mathrm{dR},+} \times_{\mathbf{A}^1_-/\mathbf{G}_m} \mathrm{Spf}\, B/\mathbf{G}_m, \tag{III.3.20}$$

whose postcomposition with the map (III.3.19), pulled back to $\mathrm{Spf}\, B/\mathbf{G}_m$, is the crystalline Frobenius.

Remark III.38 (The category of Fontaine-Laffaille modules over a scheme). Let X be a smooth p-adic formal scheme over $W(k)$. We shall explain how the refined Frobenius morphism F_{ref} in (III.3.20) can be used to define a certain stable ∞-category $\mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(X)$. By definition, an object of $\mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(X)$ is a pair $(\mathcal{M}, \varphi)$, where $\mathcal{M} \in \mathcal{D}_{qc,[0,p-1]}(X^{\mathrm{dR},+})$ and $\varphi : (F^*_{\mathrm{ref}} c^* \mathcal{M})^{p-1} \xrightarrow{\sim} \mathcal{M}_{v_-=1}$ is an isomorphism in $\mathcal{D}_{qc}(X^{\mathrm{dR}})$. Here $c : X^{\mathrm{dR},+} \times_{\mathbf{A}^1_-/\mathbf{G}_m} \mathrm{Spf}\, B/\mathbf{G}_m \to X^{\mathrm{dR},+}$ is the projection to the first component, and F^*_{ref} is the pullback along (III.3.20). The superscript $(\cdot)^i : \mathcal{M} \mapsto \mathcal{M}^i$ refers to the functor $\mathcal{D}_{qc}(X^{\mathrm{dR}} \times \mathrm{Spf}\, B/\mathbf{G}_m) \to \mathcal{D}_{qc}(X^{\mathrm{dR}})$ given by $\mathcal{M} \mapsto \mathrm{pr}_{X^{\mathrm{dR}}*}(\mathcal{M} \otimes \mathcal{O}(i))$, where $\mathrm{pr}_{X^{\mathrm{dR}}} : X^{\mathrm{dR}} \times \mathrm{Spf}\, B/\mathbf{G}_m \to X^{\mathrm{dR}}$.

III.4: The equivalence $\mathcal{D}_{qc,[0,p-2]}(W(k)^{syn}) \to \mathscr{DMF}^{\mathrm{big}}_{[0,p-2]}(W(k))$

III.4.1: The main result

Theorem III.39. *The functor* (III.3.4) *induces an equivalence of categories*

$$\Phi_{\mathrm{Maz}} \colon \mathcal{D}_{qc,[0,p-2]}(W(k)^{syn}) \xrightarrow{\sim} \mathscr{DMF}^{\mathrm{big}}_{[0,p-2]}(W(k)). \tag{III.4.1}$$

Moreover, it restricts to an equivalence on perfect complexes

$$\mathrm{Perf}_{[0,p-2]}(W(k)^{syn}) \xrightarrow{\sim} \mathcal{D}^b\left(\mathscr{MF}_{[0,p-2]}(W(k))\right) \tag{III.4.2}$$

that identifies the subcategories[21] $\mathrm{Coh}_{[0,p-2]}(W(k)^{syn}) \xrightarrow{\sim} \mathscr{MF}_{[0,p-2]}(W(k))$.

We now give the outline for the proof of Theorem III.39.

Step 1: we reduce (III.4.1) to proving the equivalence

$$\Phi_{\mathrm{Maz}} \otimes \mathbf{F}_p : \mathcal{D}_{qc,[0,p-2]}(W(k)^{syn}) \otimes \mathbf{F}_p \xrightarrow{\sim} \mathscr{DMF}^{\mathrm{big}}_{[0,p-2]}(W(k)) \otimes \mathbf{F}_p. \tag{III.4.3}$$

We say that $\mathcal{F} \in \mathcal{D}_{qc}(W(k)^{syn} \otimes \mathbf{F}_p)$ has weights in $[a,b]$ if and only if the pullbacks of $\mathcal{F}$ along both composites $\mathrm{Spec} k[v_\pm]/\mathbf{G}_{m,k} \to k^{\mathcal{N}} \otimes \mathbf{F}_p \to W(k)^{syn} \otimes \mathbf{F}_p$ lie in

[21] By definition, $\mathrm{Coh}_{[0,p-2]}(W(k)^{syn})$ is a full subcategory of $\mathrm{Coh}(W(k)^{syn})$ that consists of objects lying in $\mathcal{D}_{qc,[0,p-2]}(W(k)^{syn})$.

$\mathcal{D}_{qc,[a,b]}((\mathbf{A}^1_{\pm}/\mathbf{G}_m) \otimes \mathbf{F}_p)$ in the sense of Definition III.1.

We show that the restriction along $W(k)^{syn} \otimes \mathbf{F}_p \to W(k)^{syn}$ induces an equivalence

$$\mathcal{D}_{qc,[0,p-2]}(W(k)^{syn}) \otimes \mathbf{F}_p \simeq \mathcal{D}_{qc,[0,p-2]}\left(W(k)^{syn} \otimes \mathbf{F}_p\right). \tag{III.4.4}$$

Step 2: Let $W(k)^{syn}_{red}$ be the reduced locus of $W(k)^{syn} \otimes \mathbf{F}_p$. Let $\mathcal{D}_{qc,[a,b]}(W(k)^{syn}_{red})$ be the full subcategory of $\mathcal{D}_{qc}(W(k)^{syn}_{red})$ consising of objects with Hodge-Tate weights in $[a,b]$ defined via the restriction along $(\mathbf{A}^1_{\pm}/\mathbf{G}_m) \otimes \mathbf{F}_p \to W(k)^{syn}_{red}$ as in Step 1. We show in Proposition III.4.2 that the restriction functor

$$r_{[0,p-2]} : \mathcal{D}_{qc,[0,p-2]}(W(k)^{syn} \otimes \mathbf{F}_p) \to \mathcal{D}_{qc,[0,p-2]}(W(k)^{syn}_{red}) \tag{III.4.5}$$

is an equivalence of categories.

Step 3: Use description of $W(k)^{syn}_{red}$ from [Dri24, §7] to complete the proof.

Remark III.40. We show in Step 3 that the functor $\Phi_{\mathrm{Maz}} \otimes \mathbf{F}_p$ factors through the reduced locus

$$\begin{array}{ccc} \mathcal{D}_{qc,[0,p-1]}(W(k)^{syn} \otimes \mathbf{F}_p) & \xrightarrow{\Phi_{\mathrm{Maz}} \otimes \mathbf{F}_p} & \mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(W(k)) \otimes \mathbf{F}_p \\ & {\scriptstyle r_{[0,p-1]}} \searrow \quad \nearrow {\scriptstyle \Phi^{\mathrm{Maz}}_{red}} & \\ & \mathcal{D}_{qc,[0,p-1]}(W(k)^{syn}_{red}) & \end{array}$$

Moreover, we construct an equivalence

$$\Phi_{\mathrm{DI}} : \mathcal{D}_{qc,[0,p-1]}(W(k)^{syn}_{red}) \simeq \mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(W(k)) \otimes \mathbf{F}_p. \tag{III.4.6}$$

Thus, via (III.4.6), $\Phi^{\mathrm{Maz}}_{red}$ can be viewed as an endo-functor on $\mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(W(k)) \otimes \mathbf{F}_p$. We shall see that $\Phi^{\mathrm{Maz}}_{red}$ is not an equivalence of categories, but becomes an equivalence when restricted to $\mathscr{DMF}^{\mathrm{big}}_{[0,p-2]}(W(k)) \otimes \mathbf{F}_p$.

III.4.2: Proof of the Main Theorem

We now prove Theorem III.39.

Step 1:

For a $\mathbf{Z}$-linear presentable stable ∞-category $\mathcal{C}$ and a commutative ring A we will denote by $\mathcal{C} \otimes A$ the category $\mathcal{C} \otimes_{\mathcal{D}(\mathbf{Z})} \mathcal{D}(A)$ defined via Lurie's tensor product in Pr^L_{st}; see [Lur17,

§4.8]. Informally, an object of $\mathcal{C} \otimes_{\mathcal{D}(\mathbf{Z})} \mathcal{D}(A)$ is merely an A-module in $\mathcal{C}$. We say that such $\mathcal{C}$ is p-complete if the natural functor $\mathcal{C} \to \lim \mathcal{C} \otimes \mathbf{Z}/p^n$ is an equivalence.

Lemma III.41. *Let $f : \mathcal{C}_1 \to \mathcal{C}_2$ be a map between stable $\mathbf{Z}$-linear presentable p-complete ∞-categories. If the induced functor $f \otimes \mathbf{F}_p : \mathcal{C}_1 \otimes \mathbf{F}_p \to \mathcal{C}_2 \otimes \mathbf{F}_p$ is an equivalence, then f is an equivalence.*

Proof. Observe that, for any $X, Y \in \mathcal{C}_1$, we have

(III.4.7)

$$\mathrm{Hom}_{\mathcal{C}_1 \otimes \mathbf{Z}/p^n}(X \otimes \mathbf{Z}/p^n, Y \otimes \mathbf{Z}/p^n) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}_1}(X, Y \otimes \mathbf{Z}/p^n) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}_1}(X, Y) \otimes \mathbf{Z}/p^n.$$

Indeed, each mapping spectrum in (III.4.7) is naturally identified with the cone of $\mathrm{Hom}_{\mathcal{C}_1}(X, Y) \xrightarrow{p^n} \mathrm{Hom}_{\mathcal{C}_1}(X, Y)$. In particular, $\mathrm{Hom}_{\mathcal{C}_1}(X, Y) \xrightarrow{\sim} \lim \mathrm{Hom}_{\mathcal{C}_1}(X, Y) \otimes \mathbf{Z}/p^n$ *i.e.*, $\mathrm{Hom}_{\mathcal{C}_1}(X, Y)$ is p-complete. The same applies to $\mathrm{Hom}_{\mathcal{C}_2}(X', Y')$, for $X', Y' \in \mathcal{C}_2$.

Let us check that f is fully faithful *i.e.*, the map $\mathrm{Hom}_{\mathcal{C}_1}(X, Y) \to \mathrm{Hom}_{\mathcal{C}_2}(f(X), f(Y))$ is an equivalence. By p-completeness, it suffices to check that $\mathrm{Hom}_{\mathcal{C}_1}(X, Y) \otimes \mathbf{F}_p \to \mathrm{Hom}_{\mathcal{C}_2}(f(X), f(Y)) \otimes \mathbf{F}_p$ is an equivalence. Using (III.4.7), this is equivalent to showing that $\mathrm{Hom}_{\mathcal{C}_1 \otimes \mathbf{F}_p}(X \otimes \mathbf{F}_p, Y \otimes \mathbf{F}_p) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}_2 \otimes \mathbf{F}_p}(f(X) \otimes \mathbf{F}_p, f(Y) \otimes \mathbf{F}_p)$. Consequently, it suffices to show that, for any X in $\mathcal{C}_1$ the natural map $f(X) \otimes \mathbf{F}_p \to f_1(X \otimes \mathbf{F}_p)$, where $f_n := f \otimes \mathbf{Z}/p^n : \mathcal{C}_1 \otimes \mathbf{Z}/p^n \to \mathcal{C}_2 \otimes \mathbf{Z}/p^n$, is an equivalence in $\mathcal{C}_2 \otimes \mathbf{F}_p$. To this end, observe that the functor $\mathcal{C}_2 \otimes \mathbf{F}_p \to \mathcal{C}_2$ is conservative. Indeed, $\mathcal{C}_2 \otimes \mathbf{F}_p$ is identified with the category $\mathrm{Fun}_{\mathrm{Mod}_{\mathbf{Z}}(\mathrm{Pr}^L_{st})}(\mathcal{D}(\mathbf{F}_p), \mathcal{C}_2) \in \mathrm{Pr}^L_{st}$ of colimit-preserving $\mathbf{Z}$-linear functors. This equivalence carries the functor $\mathcal{C}_2 \otimes \mathbf{F}_p \to \mathcal{C}_2$ to the evaluation-at-$\mathbf{F}_p$ functor $\mathrm{Fun}_{\mathrm{Mod}_{\mathbf{Z}}(\mathrm{Pr}^L_{st})}(\mathcal{D}(\mathbf{F}_p), \mathcal{C}_2) \to \mathcal{C}_2$. It remains to observe that $\mathcal{D}(\mathbf{F}_p)$ is generated by $\mathbf{F}_p$ under colimits and finite limits. By conservativity, it is enough to show that $f(X) \otimes \mathbf{F}_p \to f_1(X \otimes \mathbf{F}_p)$ is an isomorphism in $\mathcal{C}_2$. But this is clear: each side is naturally identified with the cone of $f(X) \xrightarrow{p} f(X)$.

We claim that f_n is fully faithful, for every n: representing $\mathcal{C}_i \otimes \mathbf{Z}/p^n$ as the inner Hom from $\mathcal{D}(\mathbf{Z}/p^n)$ to $\mathcal{C}_i$ the claim follows from a general fact that a fully faithful f induces a fully faithful functor on inner Hom's, see [Lur18a, TAG 04Q6].

It remains to check that f_n is essentially surjective. We check it by induction on n. For any $Z \in \mathcal{C}_2 \otimes \mathbf{Z}/p^n$, we have a fiber sequence $Z \otimes_{\mathbf{Z}/p^n} \mathbf{Z}/p^{n-1} \to Z \to Z \otimes_{\mathbf{Z}/p^n} \mathbf{F}_p$. By induction assumption the boundary terms are in the essential image. Since f_n is fully faithful it follows that Z is in the essential image. □

In particular, the lemma reduces proving equivalence (III.4.1) to (III.4.3).

pr The restriction along $W(k)^{syn} \otimes \mathbf{F}_p \to W(k)^{syn}$ induces an equivalence $\mathcal{D}_{qc}(W(k)^{syn} \otimes \mathbf{F}_p) \simeq \mathcal{D}_{qc}(W(k)^{syn}) \otimes \mathbf{F}_p$, where the right-hand side stands for the category of $\mathbf{F}_p$-linear

objects in $\mathcal{D}_{qc}(W(k)^{syn})$. Moreover, this equivalence respects Hodge-Tate weights. Thus, it induces $\mathcal{D}_{qc,[0,p-2]}(W(k)^{syn} \otimes \mathbf{F}_p) \simeq \mathcal{D}_{qc,[0,p-2]}(W(k)^{syn}) \otimes \mathbf{F}_p$. The proof of this proposition is given at the end of *Step 1*. We start with some preliminary results.

Lemma III.42. *Let A be a ring, $(p, f_1, \cdots, f_n)$ is a regular sequence in A, $I \subset A$ the ideal generated by this sequence, and let $\bar{I} \subset A/p$ be its image. Assume that A is I-complete. Denote by* $\mathrm{Spf}\, A$ *(resp.* $\mathrm{Spf}\, A/p$*) the I-adic (resp. $\bar{I}$-adic) formal scheme. Then the restriction along $i : \mathrm{Spf}(A) \otimes \mathbf{F}_p \to \mathrm{Spf}(A)$ induces $\mathcal{D}_{qc}(\mathrm{Spf}(A)) \otimes \mathbf{F}_p \simeq \mathcal{D}_{qc}(\mathrm{Spf}(A/p))$.*

Proof. Identify $\mathcal{D}_{qc}(\mathrm{Spf}(A))$ with $\mathcal{D}_{I-comp}(A) \subset \mathcal{D}(A)$ and $\mathcal{D}_{qc}(\mathrm{Spf}(A/p))$ with $\mathcal{D}_{\bar{I}-comp}(A/p) \subset \mathcal{D}(A/p)$. Under this equivalence, the functor $\Phi : \mathcal{D}_{qc}(\mathrm{Spf}(A)) \otimes \mathbf{F}_p \to \mathcal{D}_{qc}(\mathrm{Spf}(A/p))$ is compatible with the functor $\mathcal{D}(A) \otimes \mathbf{F}_p \to \mathcal{D}(A/p)$ induced by $\mathrm{Spec}(A/p) \to \mathrm{Spec}(A)$, which is an equivalence by [Lur17, Theorems 4.8.5.11 and 4.8.5.16]. Since $- \otimes \mathcal{D}(\mathbf{F}_p) \simeq \mathrm{Fun}_{\mathrm{Mod}_{\mathbf{Z}}(\mathrm{Pr}^L_{st})}(\mathcal{D}(\mathbf{F}_p), -)$ as functors, and the latter preserves fully faithful embeddings, we see that Φ is fully faithful. Let us show that it is essentially surjective. Observe that A/p is in the image since $\Phi(A) = A/p$. Thus, it suffices to check that A/p generates $\mathcal{D}_{\bar{I}-comp}(A/p)$. Note that the $\bar{I}$-completion functor $\mathcal{D}(A/p) \to \mathcal{D}_{\bar{I}-comp}(A/p)$ is essentially surjective and it commutes with colimits since it is left adjoint to the embedding. Since A/p generates $\mathcal{D}(A/p)$ the claim follows. □

Lemma III.43. *Let $\mathfrak{X}$ be a stack which admits a faithfully flat cover by an affine formal scheme* $\mathrm{Spf}(A)$*. Denote by* $\mathrm{Spf}(A)^{\bullet/\mathfrak{X}}$ *the Cech nerve of* $\mathrm{Spf}(A) \to \mathfrak{X}$ *and assume that* $\mathrm{Spf}(A)^{\bullet/\mathfrak{X}}$ *is represented by an $I^\bullet$-adic affine simplicial formal scheme* $\mathrm{Spf}(A^\bullet)$ *with each $I^\bullet \subset A^\bullet$ satisfying the assumption of Lemma III.42. Then the natural map $\mathcal{D}_{qc}(\mathfrak{X}) \otimes \mathbf{F}_p \to \mathcal{D}_{qc}(\mathfrak{X} \otimes \mathbf{F}_p)$ is an equivalence.*

Proof. The assumption implies that the functor $\mathcal{D}_{qc}(\mathfrak{X}) \to \lim \mathcal{D}_{qc}(\mathrm{Spf}(A^\bullet))$ is an equivalence. The natural map $\mathcal{X} \otimes \mathbf{F}_p \to \mathcal{X} \otimes^L \mathbf{F}_p$ is an isomorphism since A is flat over $\mathbf{Z}_p$ and flatness is fpqc-local. Thus, by the base change stability of Cech nerves we get the following diagram

$$\begin{array}{ccc} \mathcal{D}_{qc}(\mathcal{X}) \otimes \mathbf{F}_p & \longrightarrow & (\lim \mathcal{D}_{qc}(\mathrm{Spf}\, A^\bullet)) \otimes \mathbf{F}_p \\ \downarrow & & \downarrow \\ \mathcal{D}_{qc}(\mathcal{X} \otimes \mathbf{F}_p) & \longrightarrow & \lim \mathcal{D}_{qc}(\mathrm{Spf}(A^\bullet \otimes_{\mathbf{Z}_p} \mathbf{F}_p)). \end{array}$$

Note that horizontal maps are equivalences by descent. The right vertical map is an equivalence by combining Lemma III.42 with (self)dualizability of $\mathcal{D}(\mathbf{F}_p)$ in Pr^L_{st} (the latter

implies that, for any diagram $\mathcal{C}_i$ in Pr^L_{st}, the natural map $(\lim \mathcal{C}_i) \otimes \mathbf{F}_p \to \lim \mathcal{C}_i \otimes \mathbf{F}_p$ is an equivalence since, for every $\mathcal{C} \in \mathrm{Pr}^L_{st}$, we have $\mathcal{C} \otimes \mathbf{F}_p \xrightarrow{\sim} \mathrm{Fun}(\mathcal{D}(\mathbf{F}_p), \mathcal{C}))$. Thus, all maps in the diagram are equivalences. □

Proof of Proposition III.4.2. Denote by C the p-adic completion of an algebraic closure of $\mathrm{Frac}\, W(k)$ and by $\mathcal{O}_C \subset C$ the subring of integral elements. Then $W(k) \to \mathcal{O}_C$ induces a faithfully flat cover $\mathcal{O}_C^{\mathcal{N}} \to W(k)^{\mathcal{N}}$ by [Bha23, Remark 6.6.12]. If $\mathcal{O}_C^{\mathcal{N}} \to \mathbf{A}^1_-/\mathbf{G}_m$ is the Rees map, then $\mathcal{O}_C^{\mathcal{N}} \times_{\mathbf{A}^1_-/\mathbf{G}_m} \mathbf{A}^1_-$ is a formal affine scheme $\mathrm{Spf}\, A$ such that the Cech nerve $\mathrm{Spf}(A)^{\bullet/W(k)^{\mathcal{N}}}$ satisfies the assumption of Lemma III.43; see [Bha23, Corollary 5.5.11]. Thus, Lemma III.43 gives $\mathcal{D}_{qc}((W(k)^{\mathcal{N}}) \otimes \mathbf{F}_p \simeq \mathcal{D}_{qc}(W(k)^{\mathcal{N}} \otimes \mathbf{F}_p)$. Similarly, $\mathcal{D}_{qc}((W(k)^{\mathbf{\Delta}}) \otimes \mathbf{F}_p \simeq \mathcal{D}_{qc}(W(k)^{\mathbf{\Delta}} \otimes \mathbf{F}_p)$. This implies the first statement since $\mathcal{D}(W(k)^{syn}) \simeq \mathrm{Eq}(\mathcal{D}(W(k)^{\mathcal{N}}) \rightrightarrows \mathcal{D}(W(k)^{\mathbf{\Delta}})$. To show that it preserves Hodge-Tate weights it is enough to observe that the equivalence $\mathcal{D}(\mathbf{A}^1_-/\mathbf{G}_m) \otimes \mathbf{F}_p \to \mathcal{D}(\mathbf{A}^1_-/\mathbf{G}_m \otimes \mathbf{F}_p)$ restricts to an equivalence $\mathcal{D}_{qc,[a,b]}(\mathbf{A}^1_-/\mathbf{G}_m) \otimes \mathbf{F}_p \to \mathcal{D}_{qc,[a,b]}(\mathbf{A}^1_-/\mathbf{G}_m \otimes \mathbf{F}_p)$ which is clear.

Step 2:

Recall from [Dri24, §5.10.8 and Prop. 8.11.2] that $W(k)^{syn}_{red}$ is a Cartier divisor of $W(k)^{syn} \otimes \mathbf{F}_p$, given by a nonzero section $v_1 \in H^0(W(k)^{syn} \otimes \mathbf{F}_p, \mathcal{O}\{p-1\})$. Let $(W(k)^{syn} \otimes \mathbf{F}_p)_{v_1^i=0}$ denote the zero locus of v_1^i inside $W(k)^{syn} \otimes \mathbf{F}_p$. pr For each $i > 1$, the restriction map

$$\mathcal{D}_{qc,[0,p-2]}((W(k)^{syn} \otimes \mathbf{F}_p)_{v_1^i=0}) \to \mathcal{D}_{qc,[0,p-2]}((W(k)^{syn} \otimes \mathbf{F}_p)_{v_1^{i-1}=0}) \tag{III.4.8}$$

is an equivalence. In particular, the functor $r_{[0,p-2]}$ from (III.4.5) is an equivalence.

Proof. Observe that the closed embedding

$$\alpha_i : (W(k)^{syn} \otimes \mathbf{F}_p)_{v_1^{i-1}=0} \hookrightarrow (W(k)^{syn} \otimes \mathbf{F}_p)_{v_1^i=0} \tag{III.4.9}$$

is a square-zero extension with sheaf of ideals given by

$$\mathcal{J}_i = \mathcal{O}_{W(k)^{syn}_{red}}\{-(i-1)(p-1)\}. \tag{III.4.10}$$

We shall use the following generalization of a fact from [Bha23, Example 6.5.13].

Lemma III.44. *Let n be an arbitrary integer. For any $\mathcal{E} \in \mathcal{D}_{qc,[n,\infty]}(W(k)^{syn}_{red})$ and any $\mathcal{E}' \in \mathcal{D}_{qc,(-\infty,n-1]}(W(k)^{syn}_{red})$, we have* $\mathrm{RHom}(\mathcal{E}', \mathcal{E}) = 0$.

Proof. Recall from [Dri24, §7] and [Bha23, §6.2] the geometry of the reduced locus $W(k)^{\mathcal{N}}_{red}$, which has two irreducible components given by the Hodge-Tate component D_{HT} and the de

Rham component D_{dR} meeting transversely along the closed substack D_{Hod}. The Hodge-Tate component D_{HT} is identified with $\mathbf{A}^{1,\mathrm{dR}}_{+}/\mathbf{G}_{m,k} \simeq \mathbf{G}_{a,k}/(\mathbf{G}^{\sharp}_{a,k} \rtimes \mathbf{G}_{m,k})$. The de Rham component D_{dR} is identified with the classifying stack of a commutative group scheme $\mathcal{G}$ over $(\mathbf{A}^1_{-}/\mathbf{G}_m) \otimes \mathbf{F}_p$. The closed substack $D_{Hod} = B(F_*\mathbf{G}^{\sharp}_{a,k} \rtimes \mathbf{G}_{m,k})$ is embedded into D_{dR} as the fibre over $B\mathbf{G}_{m,k} \subset (\mathbf{A}^1_{-}/\mathbf{G}_m) \otimes \mathbf{F}_p$, and it is embedded into D_{HT} as $\alpha_p/(\mathbf{G}^{\sharp}_{a,k} \rtimes \mathbf{G}_{m,k})$. Consequently, $\mathcal{D}_{qc}(W(k)^{\mathcal{N}}_{red})$ is a full subcategory of the fibre product $\mathcal{D}_{qc}(D_{\mathrm{HT}}) \times_{\mathcal{D}_{qc}(D_{Hod})} \mathcal{D}_{qc}(D_{\mathrm{dR}})$. We refer the reader to [Lur18b, Theorem 16.2.0.2] for a much more general result the above assertion follows from.

Set $D_{und} := B\mathbf{G}^{\sharp}_{m,k}$. The stack $W(k)^{\mathcal{N}}_{red}$ contains two disjoint open substacks isomorphic to D_{und}. The embedding of $D_{und} \subset D_{\mathrm{HT}} = \mathbf{G}_{a,k}/(\mathbf{G}^{\sharp}_{a,k} \rtimes \mathbf{G}_{m,k})$ identifies D_{und} with $(\mathbf{G}_{a,k} \setminus \{0\})/(\mathbf{G}^{\sharp}_{a,k} \rtimes \mathbf{G}_{m,k})$. The embedding of $D_{und} \subset D_{\mathrm{dR}}$ identifies D_{und} with the complement to D_{Hod} inside D_{dR}. The stack $W(k)^{syn}_{red}$ is obtained from $W(k)^{\mathcal{N}}_{red}$ by identifying the two copies of D_{und} inside D_{HT} and D_{dR} via the arithmetic Frobenius automorphism $\mathrm{Id} \times \mathrm{Frob}$ of $B\mathbf{G}^{\sharp}_{m,k} = B\mathbf{G}^{\sharp}_{m,\mathbf{F}_p} \times \mathrm{Spec}k$. In particular, by this pushout presentation of $W(k)^{syn}_{red}$, we have:

$$\begin{aligned}\mathrm{RHom}(\mathcal{E}', \mathcal{E}) = \mathrm{Fib}\Big(&\mathrm{RHom}(\mathcal{E}'|_{D_{\mathrm{HT}}}, \mathcal{E}|_{D_{\mathrm{HT}}}) \oplus \mathrm{RHom}(\mathcal{E}'|_{D_{\mathrm{dR}}}, \mathcal{E}|_{D_{\mathrm{dR}}}) \\ &\to \mathrm{RHom}(\mathcal{E}'|_{D_{und}}, \mathcal{E}|_{D_{und}}) \oplus \mathrm{RHom}(\mathcal{E}'|_{D_{Hod}}, \mathcal{E}|_{D_{Hod}})\Big).\end{aligned} \tag{III.4.11}$$

Now we claim that under our assumption on Hodge-Tate weights, the following hold:

(i) $\mathrm{RHom}(\mathcal{E}'|_{D_{\mathrm{HT}}}, \mathcal{E}|_{D_{\mathrm{HT}}}) = 0$.

(ii) The restriction map induces an isomorphism $\mathrm{RHom}(\mathcal{E}'|_{D_{\mathrm{dR}}}, \mathcal{E}|_{D_{\mathrm{dR}}}) \xrightarrow{\sim} \mathrm{RHom}(\mathcal{E}'|_{D_{und}}, \mathcal{E}|_{D_{und}})$.[22]

(iii) $\mathrm{RHom}(\mathcal{E}'|_{D_{Hod}}, \mathcal{E}|_{D_{Hod}}) = 0$.

To prove (i), note $\mathcal{D}_{qc}(D_{\mathrm{HT}}) \simeq \mathcal{D}_{qc}(\mathbf{A}^{1,\mathrm{dR}}_{+}/\mathbf{G}_{m,k})$ is a full subcategory of the derived category $\mathcal{D}_{gr}(\mathcal{A}_1)$ of graded $\mathcal{A}_1$-modules, where $\mathcal{A}_1 := k\{v_+, D\}/(Dv_+ - v_+D - 1)$ is the algebra of differential operators on the affine line; see [Bha23, §6.5.4]. Let $G' = \bigoplus G'_i$ (resp. $G = \bigoplus G_i$) be the graded module corresponding to $\mathcal{E}'|_{D_{\mathrm{HT}}}$ (resp. $\mathcal{E}|_{D_{\mathrm{HT}}}$) under the above inclusion. Using the assumption on weights and Remark III.11, we conclude that G_i is acyclic for $i \leq n-1$ and $G'_i \xrightarrow{v_+} G'_{i+1}$ is an isomorphism for $i+1 \geq n$.

We claim that $\mathrm{RHom}_{\mathcal{D}_{gr}(k[v_+])}(G', G(j)) = 0$, $j \leq 0$, where $G(j)_i := G_{i+j}$. Indeed, let G'' be the free graded $k[v_+]$-module on G'_{n-1}, *i.e.*, $G''_i = G'_i$ for $i \geq n-1$ and 0

[22] The argument below will prove this under weaker assumptions on the weights of $\mathcal{E}$, namely, it has weights $\geq n-1$ instead of $\geq n$.

otherwise. Then $\mathrm{RHom}_{\mathcal{D}_{gr}(k[v_+])}(G'', G(j)) \xrightarrow{\sim} \mathrm{RHom}_{\mathcal{D}_{gr}(k)}(G'_{n-1}, G_{n-1+j})$, which is 0 for $j \leq 0$ since $G_{n-1+j} = 0$. It remains to check that for $G''' := \mathrm{cone}(G'' \to G')$, we have $\mathrm{RHom}_{\mathcal{D}_{gr}(k[v_+])}(G''', G(j)) = 0$ for $j \leq 0$. Indeed, G'''_i is acyclic for $i > n-2$. Then we have

$$\mathrm{RHom}_{\mathcal{D}_{gr}(k[v_+])}(G''', G(j)) = \mathrm{Fib}\Big(\mathrm{RHom}_{\mathcal{D}_{gr}(k)}(G''', G(j)) \xrightarrow{\mathrm{ad}_{v_+}} \mathrm{RHom}_{\mathcal{D}_{gr}(k)}(G''', G(j+1))\Big),$$

$$\mathrm{RHom}_{\mathcal{D}_{gr}(k)}(G''', G(j)) = \prod_i \mathrm{RHom}_k(G'''_i, G_{i+j}) = 0, \quad \text{for } j \leq 1, \tag{III.4.12}$$

since for every i, either G'''_i or G_{i+j} is acyclic (for $j \leq 1$). Next, we have

$$\mathrm{RHom}_{\mathcal{D}_{gr}(\mathcal{A}_1)}(G', G) = \mathrm{Fib}(\mathrm{RHom}_{\mathcal{D}_{gr}(k[v_+])}(G', G) \xrightarrow{\mathrm{ad}_D} \mathrm{RHom}_{\mathcal{D}_{gr}(k[v_+])}(G', G(-1))). \tag{III.4.13}$$

By the vanishing of $RHom_{\mathcal{D}_{gr}(k[v_+])}(G', G(j)) = 0$ for $j = 0, -1$, we deduce vanishing of the left-hand side in (III.4.13). Thus $RHom(\mathcal{E}'|_{D_{\mathrm{HT}}}, \mathcal{E}|_{D_{\mathrm{HT}}}) = 0$ and (i) holds.

To prove (ii), recall from [Bha23, §6.5.3] that $\mathcal{D}_{qc}(D_{\mathrm{dR}})$ can be viewed as a full subcategory of $\mathcal{D}_{gr}(k[v_-, \Theta])$, where $\deg(v_-) = -1$ and $\deg(\Theta) = -p$. Denote by $F' = \bigoplus F'^i$ (resp. $F = \bigoplus F^i$) the module corresponding to $\mathcal{E}'$ (resp. $\mathcal{E}$). Using the assumption on weights and Remark III.11, we have that F'^i is acyclic for $i \geq n$, and $F^i \xrightarrow{v_-} F^{i-1}$ is an isomorphism for $i \leq n$.

Likewise, $\mathcal{D}_{qc}(D_{und})$ is a full subcategory of $\mathcal{D}_{gr}(k[v_-, v_-^{-1}, \Theta]) = \mathcal{D}(k[\Theta])$. Consider the embedding $j : D_{und} \hookrightarrow D_{\mathrm{dR}}$ lying over $\bar{j} : (\mathbf{A}^1_- \setminus \{0\})/\mathbf{G}_m \otimes \mathbf{F}_p \hookrightarrow \mathbf{A}^1_-/\mathbf{G}_m \otimes \mathbf{F}_p$. Under the identifications above, the pullback along j corresponds to the functor $j^* : \mathcal{D}_{gr}(k[v_-, \Theta]) \to \mathcal{D}_{gr}(k[v_-, v_-^{-1}, \Theta])$ given by the tensor product with $k[v_-, v_-^{-1}, \Theta]$ over $k[v_-, \Theta]$. We shall prove that under the above assumption on F and F', the map

$$j^* : \mathrm{RHom}_{\mathcal{D}_{gr}(k[v_-, \Theta])}(F', F) \to \mathrm{RHom}_{\mathcal{D}_{gr}(k[v_-, v_-^{-1}, \Theta])}(j^*F', j^*F) \tag{III.4.14}$$

is an isomorphism. The left-hand side of (III.4.14) can be computed as

$$\mathrm{RHom}_{\mathcal{D}_{gr}(k[v_-, \Theta])}(F', F) = \mathrm{Fib}\Big(\mathrm{RHom}_{\mathcal{D}_{gr}(k[v_-])}(F', F) \xrightarrow{\mathrm{ad}_\Theta} \mathrm{RHom}_{\mathcal{D}_{gr}(k[v_-])}(F', F(-p))\Big),$$

and the right-hand side of (III.4.14) can be computed as

$$\begin{aligned} &\mathrm{RHom}_{\mathcal{D}_{gr}(k[v_-, v_-^{-1}, \Theta])}(j^*F', j^*F) \\ =&\mathrm{Fib}\Big(\mathrm{RHom}_{\mathcal{D}_{gr}(k[v_-, v_-^{-1}])}(j^*F', j^*F) \xrightarrow{\mathrm{ad}_\Theta} \mathrm{RHom}_{\mathcal{D}_{gr}(k[v_-, v_-^{-1}])}(j^*F', j^*F(-p))\Big). \end{aligned}$$

We will show the stronger statement that, for any $F' \in \mathcal{D}_{gr}(k[v_-])$ with F'^i being acyclic for $i \geq n$, and any $F \in \mathcal{D}_{gr}(k[v_-])$ with $F^i \xrightarrow{v_-} F^{i-1}$ being an isomorphism for $i \leq n-1$, we have an isomorphism

$$\overline{j}^* : RHom_{\mathcal{D}_{gr}(k[v_-])}(F', F) \xrightarrow{\sim} RHom_{\mathcal{D}_{gr}(k[v_-, v_-^{-1}])}(\overline{j}^* F', \overline{j}^* F). \tag{III.4.15}$$

This will imply the desired isomorphism in (III.4.14).

Now, the functor $\overline{j}^*$ admits a right adjoint $\overline{j}_*$ given by the restriction along $k[v_-] \hookrightarrow k[v_-, v_-^{-1}]$. By adjunction, $\mathrm{RHom}_{\mathcal{D}_{gr}(k[v_-, v_-^{-1}])}(\overline{j}^* F', \overline{j}^* F) \simeq \mathrm{RHom}_{\mathcal{D}_{gr}(k[v_-])}(F', \overline{j}_* \overline{j}^* F)$. Let $F'' := \mathrm{cone}(F \to \overline{j}_* \overline{j}^* F)$. To prove (III.4.15), it suffices to show $\mathrm{RHom}(F', F'') = 0$. We have $F''^i = 0$ for $i < n$. To show the desired vanishing, we use the equivalence $\mathcal{D}_{gr}(k[v_-]) \xrightarrow{\sim} \mathcal{D}_{gr}(k[v_+])$ given by $F \mapsto G$ with $G_i = F^{-i}$. Now the vanishing of $\mathrm{RHom}(F', F'') \simeq \mathrm{RHom}(G', G'')$ follows from (III.4.12).

To prove (iii), recall from [Bha23, §6.5.2] that $\mathcal{D}_{qc}(D_{Hod})$ can be embedded as a full subcategory of $\mathcal{D}_{gr}(k[\Theta])$. Denote by $V' = \bigoplus V'_i$ (resp. $V = \bigoplus V_i$) the corresponding graded modules for $\mathcal{E}'|_{D_{Hod}}$ (resp. $\mathcal{E}|_{D_{Hod}}$). Observe that

$$\mathrm{RHom}_{\mathcal{D}_{gr}(k[\Theta])}(V', V) = \mathrm{Fib}\Big(\mathrm{RHom}_{\mathcal{D}_{gr}(k)}(V', V) \xrightarrow{\mathrm{ad}_\Theta} \mathrm{RHom}(V', V(-p))\Big). \tag{III.4.16}$$

Using the assumption on weights, V_i is acyclic for $i < n$ and V'_i is acyclic for $i \geq n$. Thus both the source and the target of ad_Θ on the right-hand side of (III.4.16) are zero. Therefore, $\mathrm{RHom}_{\mathcal{D}_{qc}(D_{Hod})}(\mathcal{E}'|_{D_{Hod}}, \mathcal{E}|_{D_{Hod}}) = \mathrm{RHom}_{\mathcal{D}_{gr}(k[\Theta])}(V', V) = 0$. □

We now check that (III.4.8) is fully faithful. Let $\mathcal{F}, \mathcal{F}' \in \mathcal{D}_{qc,[0,p-2]}((W(k)^{syn} \otimes \mathbf{F}_p)_{v_1^i=0})$. The morphism $\mathrm{RHom}(\mathcal{F}, \mathcal{F}') \to \mathrm{RHom}(\alpha_i^* \mathcal{F}, \alpha_i^* \mathcal{F}') \simeq \mathrm{RHom}(\mathcal{F}, \alpha_{i*} \alpha_i^* \mathcal{F}')$ is induced by the map $\mathcal{F}' \to \alpha_{i*} \alpha_i^* \mathcal{F}'$. Let $\mathcal{G}$ be its fibre. By (III.4.10), we have $\mathcal{G} := \mathcal{F}' \otimes \mathcal{J}_i = \mathcal{F}'\{-(i-1)(p-1)\}|_{W(k)^{syn}_{red}}$. We need to show that

$$0 = \mathrm{RHom}(\mathcal{F}, \mathcal{G}) = \mathrm{RHom}_{\mathcal{D}_{qc}(W(k)^{syn}_{red})}\left(\mathcal{F}|_{W(k)^{syn}_{red}}\ ,\ \mathcal{F}'\{-(i-1)(p-1)\}|_{W(k)^{syn}_{red}}\right).$$

Note that $\mathcal{F}|_{W(k)^{syn}_{red}}$ has weights $\leq p-2$, and $\mathcal{F}'\{-(i-1)(p-1)\}|_{W(k)^{syn}_{red}}$ has weights $\geq (i-1)(p-1)$. Thus the desired vanishing follows by Lemma III.44. This finishes the proof of fully faithfulness of functor (III.4.8).

To show that (III.4.8) is essentially surjective we shall use the following result.

Lemma III.45. *Let $i : \mathcal{X} \hookrightarrow \mathcal{Y}$ be a square-zero extension of algebraic stacks over a field k. Let $\mathcal{I}$ be the corresponding sheaf of ideals. Let $\mathcal{F} \in \mathcal{D}_{qc}(\mathcal{X})$. Assume that $Ext^2_{\mathcal{D}_{qc}(\mathcal{X})}(\mathcal{F}, \mathcal{F} \otimes \mathcal{I}) = 0$. Then there exists an $\widetilde{\mathcal{F}} \in \mathcal{D}_{qc}(\mathcal{Y})$ such that $\widetilde{\mathcal{F}}|_{\mathcal{X}} = \mathcal{F}$.*

Proof. We first consider the case where $\mathcal{Y}$ is an affine scheme. Let R be any associative algebra over k, and $I \subset R$ a two-sided ideal with $I^2 = 0$. Set $\overline{R} := R/I$. Define an object $M \in \mathcal{D}(\overline{R} \otimes \overline{R}^{\mathrm{op}})$ as follows: $M := \tau_{\leq 1}(\overline{R} \overset{\mathbb{L}}{\otimes}_R \overline{R})$. We have $H_0(M) = \bar{R}$, $H_1(M) = I$ and $H_i(M) = 0$ for all $i \neq 0, 1$. For the duration of this proof, we denote by Ψ the functor $\mathcal{D}(\bar{R}) \to \mathcal{D}(\bar{R})$ given by $\Psi(N) := M \otimes_{\bar{R}} N$. Then we have a fibre sequence of functors

$$I \otimes_{\bar{R}} N[1] \to \Psi(N) \to N \xrightarrow{\mathfrak{o}(N)} I \otimes_{\bar{R}} N[2]. \tag{III.4.17}$$

We refer to $\mathfrak{o}(N)$ as the obstruction class. A choice of a null-homotopy for $\mathfrak{o}(N)$ determines a map $\delta : \Psi(N) \to I \otimes_{\bar{R}} N[1]$ splitting the fibre sequence (III.4.17). Composing δ with the map $\bar{R} \otimes_R N \simeq (\overline{R} \overset{\mathbb{L}}{\otimes}_R \overline{R}) \otimes_{\bar{R}} N \to \Psi(N)$, we obtain a map $\bar{R} \otimes_R N \to I \otimes_{\bar{R}} N[1]$, which yields a map $\bar{\delta} : N \to I \otimes_{\bar{R}} N[1]$ in $\mathcal{D}(R)$. Let

$$\widetilde{N} := \mathrm{Fib}(N \xrightarrow{\bar{\delta}} I \otimes_{\bar{R}} N[1]) \in \mathcal{D}(R). \tag{III.4.18}$$

The morphism $\widetilde{N} \to N$ in $\mathcal{D}(R)$ induces a map in $\mathcal{D}(\bar{R})$:

$$\bar{R} \otimes_R \widetilde{N} \to N. \tag{III.4.19}$$

We claim that the map (III.4.19) is an isomorphism. To see this, we compute the obstruction class $\mathfrak{o}(N)$ of N as follows. By [Dri04, B.3 Lemma] we can assume that N is represented by a semi-free complex of $(N^{\cdot}, d)$ of $\bar{R}$-modules, *i.e.*, there exists an increasing exhaustive filtration $0 = F_0 N^{\cdot} \subset F_1 N^{\cdot} \subset \cdots \subset (N^{\cdot}, d)$ such that each quotient $F_{i+1}N^{\cdot}/F_i N^{\cdot}$ is isomorphic to a direct sum of DG-modules of the form $\bar{R}[n]$. For each i, pick a lift of the filtered $\bar{R}$-module $F_{\bullet}N^i$ to a filtered R-module $F_{\bullet}\widetilde{N}^i$ such that $F_{j+1}\widetilde{N}^i/F_j\widetilde{N}^i$ is a free R-module, for every j. We can lift d to a homomorphism of filtered R-modules $\tilde{d} : \widetilde{N}^i \to \widetilde{N}^{i+1}$ such that $\tilde{d}$ takes $\widetilde{F}_{\bullet}\widetilde{N}^{\cdot}$ to $\widetilde{F}_{\bullet-1}\widetilde{N}^{\cdot+1}$. In general, $\tilde{d}^2 \neq 0$. Note that $\tilde{d}^2$ factors through the projection $\widetilde{N}^i \to N^i$ and its image lies in $I \otimes_{\bar{R}} N^{i+2}$. Moreover, since $\tilde{d}^2$ commutes with $\tilde{d}$, we get a morphism of complexes $\tilde{d}^2 : (N^{\cdot}, d) \to I \otimes_{\bar{R}} (N^{\cdot}, d)[2]$.

Claim III.46. *The above morphism $\tilde{d}^2 : N \to I \otimes_{\bar{R}} N[2]$ is equal to $\mathfrak{o}(N)$.*

Proof. Consider the DG-algebra $\bar{R}^{dg} := (I \to R)$ coming from the resolution $0 \to I \to R \to \bar{R} \to 0$. Let us define a DG-$\bar{R}^{dg}$-module N' whose underlying graded R-module is $\widetilde{N} \oplus I \otimes_{\bar{R}} N[1]$ and the differential in degree i is given by $D := \begin{pmatrix} \tilde{d} & (-1)^i\beta \\ (-1)^i\tilde{d}^2 & d \end{pmatrix}$, where $\beta : I \otimes_{\bar{R}} N^i \hookrightarrow \widetilde{N}^i$ is the kernel of $\widetilde{N}^i \to N^i$. The action $I[1]$ on N' is given by the lower triangular matrix whose only non-zero entry is given by $I[1] \otimes_R \widetilde{N} \simeq I \otimes_{\bar{R}} N[1]$. Note

that by construction N' is semi-free over $\bar{R}^{dg}$. Consider the map $N' \to N$ which takes $(n, x) \in \widetilde{N}^i \oplus I \otimes_{\bar{R}} N^{i+1}$ to the image of n under the map $\widetilde{N}^i \to N^i$. It is a map of DG-$\bar{R}^{dg}$-modules and, moreover, it is a quasi-isomorphism. Under the equivalence $\mathcal{D}(\bar{R}) \simeq \mathcal{D}(\bar{R}^{dg})$, the functor $\Psi : \mathcal{D}(\bar{R}) \to \mathcal{D}(\bar{R})$ corresponds to the functor $\Psi' : \mathcal{D}(\bar{R}^{dg}) \to \mathcal{D}(\bar{R})$ given by $\Psi'(N) = \tau_{\leq 1}(\bar{R} \otimes_R \bar{R}^{dg}) \otimes^{\mathbb{L}}_{\bar{R}^{dg}} N$. Note that the natural map $\tau_{\leq 1}(\bar{R} \otimes^{\mathbb{L}}_{\bar{R}} \bar{R}^{dg}) \to \tau_{\leq 1}(\bar{R} \otimes_R \bar{R}^{dg})$ in $\mathcal{D}(\bar{R} \otimes (\bar{R}^{dg})^{op})$ is an isomorphism, and $\tau_{\leq 1}(\bar{R} \otimes_R \bar{R}^{dg})$ is represented by $\bar{R} \oplus I[1]$ with the zero differential. Observe that the right action of $\bar{R}^{dg}$ has the property that $\bar{R} \otimes_R (\bar{R}^{dg})^{-1} \to I$ is an isomorphism. Thus, $\Psi'(N)$ is identified with $C := (N \oplus I \otimes_{\bar{R}} N[1], \bar{D})$, where $\bar{D}$ in degree i is given by $\begin{pmatrix} d & 0 \\ (-1)^i \widetilde{d^2} & d \end{pmatrix}$. Denote $u : I \otimes_{\bar{R}} N[1] \to C$ the inclusion and by $\alpha : C \to N$ the projection. To complete the proof, it is enough to construct a quasi-isomorphism $f : I \otimes_{\bar{R}} N[2] \to \mathrm{Cone}(\alpha)$ such that the two morphisms $\mathfrak{o}(N)$ and $\widetilde{d^2} \circ f : N \to \mathrm{Cone}(\alpha)$ are equal. Consider the following diagram

$$\begin{array}{ccccccc} N & \xrightarrow{\mathfrak{o}(N)} & \mathrm{Cone}(\alpha) & & \longrightarrow & C[1] & \xrightarrow{\alpha[1]} N[1] \\ \| & & \uparrow f & & \nearrow_{u[1]} & & \nearrow_{0} \\ N & \xrightarrow{\widetilde{d^2}} & I \otimes_{\bar{R}} N[2]. & & & & \end{array}$$

The composition $\alpha[1] \circ u[1] : I \otimes_{\bar{R}} N[2] \to N[1]$ is zero. Thus, there exists a map $f : I \otimes_{\bar{R}} N[2] \to \mathrm{Cone}(\alpha)$ satisfying $f \circ \widetilde{d^2} = \mathfrak{o}(N)$. Moreover, f is a quasi-isomorphism. □

Let us now return to the proof of the isomorphism property of (III.4.19). Since $(N^{\cdot}, d)$ is semi-free, the null-homotopy for $\mathfrak{o}(N)$ determines a collection $h : N^i \to I \otimes_{\bar{R}} N^{i+1}$ with $hd + dh = \widetilde{d^2}$. Setting $\widetilde{d'} = \widetilde{d} - h : \widetilde{N}^i \to \widetilde{N}^{i+1}$ we have that $(\widetilde{d'})^2 = 0$. Then $\widetilde{N}$ is represented by the complex $(\widetilde{N}^{\cdot}, \widetilde{d'})$. The isomorphism property of (III.4.19) follows.

This implies the lemma in the affine case. Indeed, if $\mathrm{Ext}^2_{\bar{R}}(N, N \otimes I) = 0$, then $\mathfrak{o}(N)$ is homotopic to zero. Therefore, there exists $\widetilde{N} \in \mathcal{D}(R)$ together with a map (III.4.19) which is an isomorphism.

For the general case, observe that $\mathcal{D}_{qc}(\mathcal{Y}) = \lim \mathcal{D}_{qc}(R)$ where the limit is taken over all flat maps $\mathrm{Spec} R \to \mathcal{Y}$. Thus applying the previous construction to each R, we obtain an obstruction class $\mathfrak{o}(\mathcal{F}) : \mathcal{F} \to \mathcal{F} \otimes \mathcal{I}[2]$. Moreover, a choice of null-homotopy for $\mathfrak{o}(\mathcal{F})$ determines an object $\widetilde{\mathcal{F}} \in \mathcal{D}_{qc}(\mathcal{Y})$ together with a map $\widetilde{\mathcal{F}}|_{\mathcal{X}} \to \mathcal{F}$. By the local computations in the affine case, this map is an isomorphism. This completes the proof of the lemma. □

Let us check that (III.4.8) is essentially surjective. Applying Lemma III.45 to α_i

in (III.4.9), it suffices to verify that for every $\mathcal{F} \in \mathcal{D}_{qc,[0,p-2]}\left((W(k)^{syn} \otimes \mathbf{F}_p)_{v_1^{i-1}=0}\right)$, the group $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F} \otimes \mathcal{I}_i) = 0$. Using (III.4.10), we have that $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F} \otimes \mathcal{I}_i) = \mathrm{Ext}^2(\mathcal{F}|_{W(k)^{syn}_{red}}, \mathcal{F}|_{W(k)^{syn}_{red}}\{-(i-1)(p-1)\})$. Note that $\mathcal{F}|_{W(k)^{syn}_{red}}$ has weights $\leq p-2$, and $\mathcal{F}|_{W(k)^{syn}_{red}}\{-(i-1)(p-1)\}$ has weights $\geq (i-1)(p-1) \geq p-1$ (since $i-1 \geq 1$). Thus the vanishing for Ext^2 follows from Lemma III.44.

For the last assertion of the proposition, it suffices to observe that the functor

$$\mathcal{D}_{qc,[0,p-2]}(W(k)^{syn} \otimes \mathbf{F}_p) \to \varprojlim \mathcal{D}_{qc,[0,p-2]}((W(k)^{syn} \otimes \mathbf{F}_p)_{v_1^i=0}) \tag{III.4.20}$$

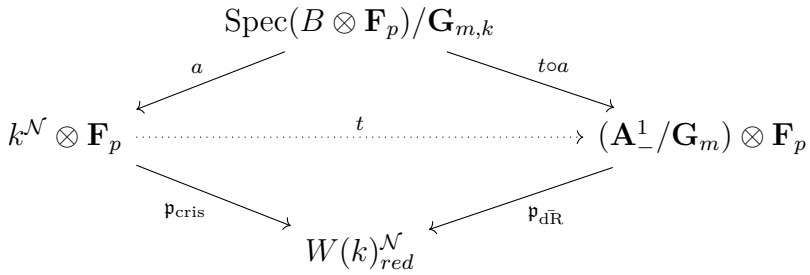
is an equivalence by the definition of reduced locus. □

Step 3:

We shall prove that the functor $(\Phi_{\mathrm{Maz}} \otimes \mathbf{F}_p) \circ r^{-1}_{[0,p-2]} : \mathcal{D}_{qc,[0,p-2]}(W(k)^{syn}_{red}) \to \mathscr{DMF}^{\mathrm{big}}_{[0,p-2]}(W(k)) \otimes \mathbf{F}_p$ is an equivalence. Although this result does not hold with $r_{[0,p-2]}$ replaced by $r_{[0,p-1]}$, some intermediate results hold in the larger range. Aimed at some geometric applications (see Remark III.59), we explain the results in full generality.

We start by showing that the functor $\Phi_{\mathrm{Maz}} \otimes \mathbf{F}_p : \mathcal{D}_{qc,[0,p-1]}(W(k)^{syn} \otimes \mathbf{F}_p) \to \mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(W(k)) \otimes \mathbf{F}_p$ factors through the restriction $r_{[0,p-1]}$ to the reduced locus $W(k)^{syn}_{red}$.[23] Since $k^{\mathcal{N}} \otimes \mathbf{F}_p$ (resp. $\mathbf{A}^1_-/\mathbf{G}_m \otimes \mathbf{F}_p$) is reduced, the morphism $\mathfrak{p}_{\mathrm{cris}} : k^{\mathcal{N}} \otimes \mathbf{F}_p \to W(k)^{\mathcal{N}} \otimes \mathbf{F}_p$ (resp. $\mathfrak{p}_{\mathrm{d\bar{R}}} \otimes \mathbf{F}_p : \mathbf{A}^1_-/\mathbf{G}_m \otimes \mathbf{F}_p \to W(k)^{\mathcal{N}} \otimes \mathbf{F}_p$) factors through $W(k)^{\mathcal{N}}_{red}$. Thus we have the following diagram obtained from (III.2.1) by tensoring with $\mathbf{F}_p$:

(III.4.21)



By Theorem III.21, we have an isomorphism $\Psi_{\mathrm{Maz}} \otimes \mathbf{F}_p : \mathfrak{p}_{\mathrm{cris}} \circ a \xrightarrow{\sim} \mathfrak{p}_{\mathrm{d\bar{R}}} \circ t \circ a$ in diagram (III.4.21) above. Consequently, this gives a functor

$$\mathcal{D}_{qc}(W(k)^{syn}_{red}) \to \mathcal{D}_{qc}(k^{syn} \otimes \mathbf{F}_p) \times_{\mathcal{D}_{qc}(\mathrm{Spec}(B \otimes \mathbf{F}_p)/\mathbf{G}_{m,k})} \mathcal{D}_{qc}((\mathbf{A}^1_-/\mathbf{G}_m) \otimes \mathbf{F}_p)$$

sending $\mathcal{D}_{qc,[0,p-1]}(W(k)^{syn}_{red})$ to the corresponding subcategory of the right-hand side (as in

[23] This assertion would be obvious if $[0, p-1]$ is replaced by $[0, p-2]$ because $r_{[0,p-2]}$ is an equivalence by *Step 2*.

Remark III.34). Combining this with (III.3.8), we obtain

$$\Phi^{\mathrm{Maz}}_{red} : \mathcal{D}_{qc,[0,p-1]}(W(k)^{syn}_{red}) \to \mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(W(k)) \otimes \mathbf{F}_p \tag{III.4.22}$$

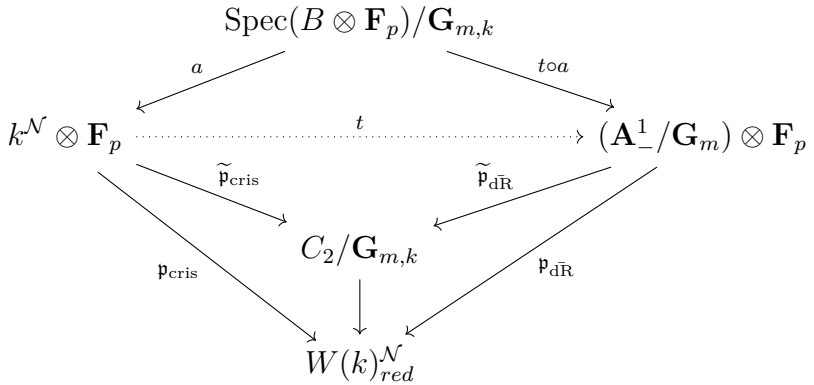
together with an isomorphism $\Phi_{\mathrm{Maz}} \otimes \mathbf{F}_p \simeq \Phi^{\mathrm{Maz}}_{red} \circ r_{[0,p-1]}$. Eventually in this step, we shall prove that $\Phi^{\mathrm{Maz}}_{red}$ is an equivalence in the range $[0, p-2]$.

Note that the construction of $\Phi^{\mathrm{Maz}}_{red}$ makes use of a particular choice of an isomorphism $\mathfrak{p}_{\mathrm{cris}} \circ a \simeq \mathfrak{p}_{\mathrm{d\bar{R}}} \circ t \circ a$ provided by $\Psi_{\mathrm{Maz}} \otimes \mathbf{F}_p$. We will make use of another isomorphism $\Psi_{\mathrm{DI}} : \mathfrak{p}_{\mathrm{cris}} \circ a \xrightarrow{\sim} \mathfrak{p}_{\mathrm{d\bar{R}}} \circ t \circ a$ constructed as follows.[24] Let C_2 be the spectrum of the subalgebra of $k[v_+, v_-]/(v_+v_-)$ generated by v_+^p and v_-, *i.e.*, $C_2 := \mathrm{Spec} k[v_+^p, v_-]/(v_+^p v_-)$. We endow the affine curve C_2 with a natural action of $\mathbf{G}_{m,k}$ given by $\deg v_+^p = p$ and $\deg v_- = -1$. Recall from [Dri24, §5.16.10] the morphism $C_2/\mathbf{G}_{m,k} \to W(k)^{\mathcal{N}}_{red}$, whose precomposition with $k^{\mathcal{N}} \otimes \mathbf{F}_p \to C_2/\mathbf{G}_{m,k}$ is $\mathfrak{p}_{\mathrm{cris}} \otimes \mathbf{F}_p$ (see also (III.1.17)). Let us recall how the map $C_2/\mathbf{G}_{m,k} \to W(k)^{\mathcal{N}}_{red}$ is constructed. Note $C_2/\mathbf{G}_{m,k}(R)$ is the groupoid of diagrams

$$L \xrightarrow{v_-} R \xrightarrow{v_+^p} L^{\otimes p} \tag{III.4.23}$$

where L is an invertible R-module and $v_+^p v_- = 0$. To any such diagram we associate an admissible Cartier-Witt divisor $M \to W$ over R. Here M is defined to be the fiber product of $F_* W_R \xrightarrow{F_*[v_+^p]} F_*[L^{\otimes p}]$, where $[L] \in \mathrm{Pic}(W(R))$ is the Teichmüller representative of L as in [Dri24, §3.11], and the Frobenius $[L] \to F_*[L^{\otimes p}]$. The map $M \to W$ is the restriction of $[L] \times_R F_* W_R \xrightarrow{([v_-], V)} W_R$. We have a commutative diagram (the commutativity data are explained below)

(III.4.24)



The map $\widetilde{\mathfrak{p}}_{\mathrm{cris}}$ is given by the embedding $k[v_+^p, v_-]/(v_+^p v_-) \hookrightarrow k[v_+, v_-]/(v_+ v_-)$, and the map $\widetilde{\mathfrak{p}}_{\mathrm{d\bar{R}}}$ is given by the map $k[v_+^p, v_-]/(v_+^p v_-) \to k[v_-]$. Since $v_+^p = p \cdot \frac{v_+^p}{p}$ is zero in

[24] Here "DI" refers to Deligne–Illusie.

$B \otimes \mathbf{F}_p$, we obtain an isomorphism $\widetilde{\mathfrak{p}}_{\mathrm{cris}} \circ a \simeq \widetilde{\mathfrak{p}}_{\mathrm{d\bar{R}}} \circ t \circ a$. To see the commutativity of the bottom right triangle, note $\tilde{\mathfrak{p}}_{\mathrm{d\bar{R}}} : ((\mathbf{A}^1_-/\mathbf{G}_m) \otimes \mathbf{F}_p)(R) \to (C_2/\mathbf{G}_{m,k})(R)$ is given by taking $(L, v_- : L \to R) \in ((\mathbf{A}^1_-/\mathbf{G}_m) \otimes \mathbf{F}_p)(R)$ to a diagram $L \xrightarrow{v_-} R \xrightarrow{0} L^{\otimes p}$. Using the commutativity datum for the lower left triangle in (III.4.24) from [Dri24, §7.8] (which gives rise to a commutativity datum of the lower right triangle), we obtain an isomorphism $\Psi_{\mathrm{DI}} : \mathfrak{p}_{\mathrm{cris}} \circ a \xrightarrow{\sim} \mathfrak{p}_{\mathrm{d\bar{R}}} \circ t \circ a$.

The composition $\psi := \Psi_{\mathrm{DI}} \circ (\Psi_{\mathrm{Maz}} \otimes \mathbf{F}_p)^{-1}$ is an automorphism of the point $\mathfrak{p}_{\mathrm{d\bar{R}}} \circ t \circ a$. We now describe it explicitly. Recall that the composition of $\mathfrak{p}_{\mathrm{d\bar{R}}} \circ t \circ a$ with the projection $\mathrm{Spec}(B \otimes \mathbf{F}_p) \to \mathrm{Spec}(B \otimes \mathbf{F}_p)/\mathbf{G}_{m,k}$ is given by the morphism $W^{(F)}_{B\otimes\mathbf{F}_p} \oplus F_* W_{B\otimes\mathbf{F}_p} \xrightarrow{(v_-, V)} W_{B\otimes\mathbf{F}_p}$ of admissible W_B-modules. Recall from [Bha23, Proposition 5.2.1 (2)] an isomorphism

$$\mathrm{Hom}_{W_{B\otimes\mathbf{F}_p}}(F_* W_{B\otimes\mathbf{F}_p}, W^{(F)}_{B\otimes\mathbf{F}_p}) \xrightarrow{\sim} \ker\left(W^{(F)}(B \otimes \mathbf{F}_p) \xrightarrow{w_1} B \otimes \mathbf{F}_p\right), \tag{III.4.25}$$

where the map w_1 is the first Witt coordinate $w_1 : W \to \mathbf{G}_a$. The map in (III.4.25) is the precomposition with Frobenius $F : W \to F_* W$, *i.e.*,

$$\mathrm{Hom}_{W_{B\otimes\mathbf{F}_p}}(F_* W_{B\otimes\mathbf{F}_p}, W^{(F)}_{B\otimes\mathbf{F}_p}) \xrightarrow{-\circ F} \mathrm{Hom}_{W_{B\otimes\mathbf{F}_p}}(W_{B\otimes\mathbf{F}_p}, W^{(F)}_{B\otimes\mathbf{F}_p}) = W^{(F)}(B \otimes \mathbf{F}_p).$$

Let $\{\frac{[v_+^p]}{p}\}$ be the image of $\frac{[v_+^p]}{p} \in W(B)$ inside $W(B \otimes \mathbf{F}_p)$. Recall that by Lemma III.22, the element $[v_+^p]$ is indeed divisible by p inside $W(B)$.

Lemma III.47. *The map ψ, viewed as an automorphism of $W^{(F)}_{B\otimes\mathbf{F}_p} \oplus F_* W_{B\otimes\mathbf{F}_p}$, is given by the matrix $\begin{pmatrix} \mathrm{Id} & V(\{\frac{[v_+^p]}{p}\}) \\ 0 & \mathrm{Id} \end{pmatrix}$, where the entry $V(\{\frac{[v_+^p]}{p}\})$ in the matrix is viewed as an element of $\mathrm{Hom}_{W_{B\otimes\mathbf{F}_p}}(F_* W_{B\otimes\mathbf{F}_p}, W^{(F)}_{B\otimes\mathbf{F}_p})$ via the isomorphism (III.4.25).*

Proof. Recall from (III.1.19) that the isomorphism Ψ_{DI}:

$$\mathrm{coker}\left(W^{(F)}_{B\otimes\mathbf{F}_p} \to W^{(F)}_{B\otimes\mathbf{F}_p} \oplus W_{B\otimes\mathbf{F}_p}\right) \xrightarrow{\sim} W_{B\otimes\mathbf{F}_p} \underset{F_* W_{B\otimes\mathbf{F}_p}}{\times} F_* W_{B\otimes\mathbf{F}_p} = W^{(F)}_{B\otimes\mathbf{F}_p} \oplus F_* W_{B\otimes\mathbf{F}_p}$$

is given by the matrix $X := \begin{pmatrix} \mathrm{Id} & [v_+] \\ 0 & F \end{pmatrix}$. Recall from the proof of Theorem III.21 that $\Psi_{\mathrm{Maz}} \otimes \mathbf{F}_p$ is given by $Y := \begin{pmatrix} 1 & f \\ 0 & F \end{pmatrix}$, where $f := [v_+] - V(\{\frac{[v_+^p]}{p}\})$. Our desired lemma amounts to the statement that $\psi Y = X$. This is immediate from our construction of the isomorphism (III.4.25). □

Corollary III.48. *The restrictions of* Ψ_{DI} *and* $\Psi_{\mathrm{Maz}} \otimes \mathbf{F}_p$ *to the closed subscheme* $\left(\mathrm{Spec} k[v_+, v_-]/(v_+^{p-1}, v_+v_-)\right)/\mathbf{G}_{m,k} \hookrightarrow (\mathrm{Spec} B \otimes \mathbf{F}_p)/\mathbf{G}_{m,k}$ *are equal.*

Proof. This follows from the fact that the image of $\frac{[v_+^p]}{p} \in W(B)$ inside $W\left(k[v_+, v_-]/(v_+^{p-1}, v_+v_-)\right)$ is zero. □

Using the construction from Remark III.34 once again, we obtain a functor

$$\Phi_{\mathrm{DI}} : \mathcal{D}_{qc,[0,p-1]}(W(k)^{syn}_{red}) \to \mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(W(k)) \otimes \mathbf{F}_p. \tag{III.4.26}$$

Corollary III.49. *The restrictions of* $\Phi^{\mathrm{Maz}}_{red}$ *and* Φ_{DI} *to the subcategory* $\mathcal{D}_{qc,[0,p-2]}(W(k)^{syn}_{red})$ *are isomorphic.*

The following proposition will complete the proof of the main Theorem III.39. pr Φ_{DI} is an equivalence of categories.

Proof. Let us first outline the argument. Recall from the proof of Lemma III.44 a fully faithful embedding

$$\mathcal{D}_{qc}(W(k)^{syn}_{red}) \hookrightarrow \mathcal{D}_{qc}(D_{\mathrm{HT}}) \underset{\mathcal{D}_{qc}(D_{und}) \times \mathcal{D}_{qc}(D_{Hod})}{\times} \mathcal{D}_{qc}(D_{\mathrm{dR}}).$$

The first part of the proof consists of showing that the restriction $\mathcal{D}_{qc,[0,p-1]}(D_{\mathrm{HT}}) \to \mathcal{D}_{qc,[0,p-1]}(D_{Hod})$ is an equivalence (see Corollary III.53). This yields a fully faithful embedding

$$\mathcal{D}_{qc,[0,p-1]}(W(k)^{syn}_{red})) \hookrightarrow \mathrm{Eq}\left(\mathcal{D}_{qc,[0,p-1]}(D_{\mathrm{dR}}) \underset{T^* \circ \mathfrak{p}^*_{\bar{\mathrm{dR}}}}{\overset{\widetilde{i}_1^*}{\rightrightarrows}} \mathcal{D}_{qc}(D_{und})\right), \tag{III.4.27}$$

where $\widetilde{i}_1^*$ is the restriction and T is the composition $B\mathbf{G}^{\sharp}_{m,\mathbf{F}_p} \otimes k \xrightarrow{\mathrm{id} \times \mathrm{Frob}} B\mathbf{G}^{\sharp}_{m,\mathbf{F}_p} \otimes k \to B\mathbf{G}_{m,k} \to (\mathbf{A}^1_-/\mathbf{G}_m) \otimes \mathbf{F}_p$. Next, we show that pullbacks along $\mathfrak{p}_{\bar{\mathrm{dR}}}$ and the inclusion $D_{und} \hookrightarrow D_{\mathrm{dR}}$ induce an equivalence $\mathcal{D}_{qc,[0,p-1]}(D_{\mathrm{dR}}) \xrightarrow{\sim} \mathcal{D}_{qc,[0,p-1]}((\mathbf{A}^1_-/\mathbf{G}_m) \otimes \mathbf{F}_p) \times_{\mathcal{D}(k)} \mathcal{D}_{qc}(D_{und})$, see Lemma III.54. This, together with a category theory result, Lemma III.55, identifies (III.4.27) with

$$\mathcal{D}_{qc,[0,p-1]}(W(k)^{syn}_{red})) \hookrightarrow \mathrm{Eq}\left(\mathcal{D}_{qc,[0,p-1]}((\mathbf{A}^1_-/\mathbf{G}_m) \otimes \mathbf{F}_p) \underset{F^* \circ i_0^*}{\overset{i_1^*}{\rightrightarrows}} \mathcal{D}(k)\right). \tag{III.4.28}$$

The functor Φ_{DI} is isomorphic to (III.4.28). This proves fully faithfulness of Φ_{DI}. For the essential surjectivity we use Lemma III.9.

Let us explain the details. Consider the maps $(C_2)_{v_-=0}/\mathbf{G}_{m,k} \overset{f'}{\twoheadrightarrow} D_{\mathrm{HT}} = (\mathbf{A}^{1,\mathrm{dR}}_+/\mathbf{G}_m) \otimes \mathbf{F}_p \overset{f}{\to} B\mathbf{G}_{m,k}$ and a section $B\mathbf{G}_{m,k} = (C_2)_{v_+^p=v_-=0}/\mathbf{G}_{m,k} \overset{g}{\hookrightarrow} (C_2)_{v_-=0}/\mathbf{G}_{m,k}$. We first show the following.

Lemma III.50. *The pullback functors along f and f' induce equivalences*[25]

$$\mathcal{D}_{qc,[0,p-1]}(B\mathbf{G}_{m,k}) \xrightarrow{\sim} \mathcal{D}_{qc,[0,p-1]}(D_{\mathrm{HT}}) \xrightarrow{\sim} \mathcal{D}_{qc,[0,p-1]}\left((C_2)_{v_-=0}/\mathbf{G}_{m,k}\right).$$

The inverse to the composite is given by g^*.

Proof. Since $f \circ f' \circ g = \mathrm{Id}$, we have that $g^* \circ f'^* \circ f^* \simeq \mathrm{Id}$. Thus, it suffices to check that f^* and $f'^* \circ f^*$ are equivalences.

Let us show that f^* is fully faithful. Let us recall from [BL22b, §6.5.4] that the category $\mathcal{D}_{qc}(D_{\mathrm{HT}}) = \mathcal{D}_{qc}(\mathbf{G}_a^{dR}/\mathbf{G}_{m,k})$ is identified with the full subcategory $\mathcal{D}_{gr,D^p\text{-nilp}}(k[D^p, v_+^p])$ of the derived category $\mathcal{D}_{gr}(k[D^p, v_+^p])$ of graded modules over the polynomial algebra $k[D^p, v_+^p]$ with $\deg v_+^p = -\deg D^p = p$. Objects of $\mathcal{D}_{gr,D^p\text{-nilp}}(k[D^p, v_+^p])$ consist of complexes $\mathcal{M}$ such that the action of D^p on $\bigoplus_i H^i(\mathcal{M})$ is locally nilpotent. The natural map $\mathbf{G}_a^{dR} := \mathbf{G}_{a,k}/\mathbf{G}_{a,k}^\sharp \to \mathbf{G}_{a,k}/\alpha_p = F_*\mathbf{G}_{a,k}$ coming from the factorization $\mathbf{G}_{a,k}^\sharp \to \alpha_p \to \mathbf{G}_{a,k}$ exhibits the source as the $BF_*\mathbf{G}_{a,k}^\sharp$-torsor over the target. A flat lift of $\mathbf{G}_{a,k}$ to $W(k)$ together with a lift of the Frobenius splits the torsor; [BL22b, Remark 5.13]. The standard choice of the Frobenius lift gives a $\mathbf{G}_{m,k}$-equivariant equivalence $\mathbf{G}_a^{dR} \simeq F_*\mathbf{G}_{a,k} \oplus BF_*\mathbf{G}_{a,k}^\sharp$ and $\mathcal{D}_{qc}((F_*\mathbf{G}_{a,k} \oplus BF_*\mathbf{G}_{a,k}^\sharp)/\mathbf{G}_{m,k}) \simeq \mathcal{D}_{gr,D^p\text{-nilp}}(k[D^p, v_+^p])$. Under the equivalence above, $\mathcal{D}_{qc,[0,p-1]}(D_{\mathrm{HT}})$ is identified with the subcategory $\mathcal{D}_{gr,[0,p-1]}(k[v_+^p, D^p])$ of $\mathcal{D}_{gr,D^p\text{-nilp}}(k[D^p, v_+^p])$ that consists of objects $\mathcal{M} = \bigoplus_i \mathcal{M}_i$ with $\mathcal{M}_i$ acyclic for $i \le 0$ and $\mathrm{cone}(\mathcal{M}_{i-p} \xrightarrow{v_+^p} \mathcal{M}_i)$ acyclic for $i \ge p$.

The category $\mathcal{D}_{qc,[0,p-1]}(B\mathbf{G}_{m,k})$ is identified with the full subcategory of the derived category $\mathcal{D}_{gr}(k)$ of graded complexes $C_\bullet = \bigoplus_i C_i$ such that for any $i \notin [0, p-1]$, the grading degree i component C_i is acyclic. Under the above identifications, the functor f^* sends C_i to $C_i[v_+^p] := C_i \otimes_k k[v_+^p]$ with the trivial action of D^p. The complex $\mathrm{RHom}_{\mathcal{D}_{gr}(k[D^p, v_+^p])}(C_i[v_+^p], C'_j[v_+^p])$ is computed by

$$\text{(III.4.29)}\quad \mathrm{Fib}\left(\mathrm{RHom}_{\mathcal{D}_{gr}(k[v_+^p])}(C_i[v_+^p], C'_j[v_+^p]) \xrightarrow{\mathrm{ad}_{D^p}} \mathrm{RHom}_{\mathcal{D}_{gr}(k[v_+^p])}(C_i[v_+^p], C'_j[v_+^p](-p))\right),$$

where $(-p)$ stands for the degree shift, *i.e.*, $\mathcal{M}(-p)_i = \mathcal{M}_{i-p}$.

[25] By definition, $\mathcal{D}_{qc,[0,p-1]}\left((C_2)_{v_-=0}/\mathbf{G}_m\right)$ is a full subcategory of $\mathcal{D}_{qc}\left((C_2)_{v_-=0}/\mathbf{G}_m\right)$ formed of objects whose pullbacks to $\mathbf{A}^1_+/\mathbf{G}_m \otimes \mathbf{F}_p$ have weights in $[0, p-1]$.

If $i, j \in [0, p-1]$, then $\mathrm{RHom}_{\mathcal{D}_{gr}(k[v_+^p])}(C_i[v_+^p], C'_j[v_+^p](-p)) = 0$, and therefore by (III.4.29) we get

$$\mathrm{RHom}_{\mathcal{D}_{gr}(k)}(C_i, C'_j) \xrightarrow{\sim} \mathrm{RHom}_{\mathcal{D}_{gr}(k[v_+^p])}(C_i[v_+^p], C'_j[v_+^p]). \tag{III.4.30}$$

This proves fully faithfulness of f^*. Formula (III.4.30) shows fully faithfulness of $f'^* \circ f^*$.

Now we show essential surjectivity. It is enough to show that, for every $i \in [0, p-1]$ and $\mathcal{M} \in \mathcal{D}_{gr,[0,p-1]}(k[D^p, v_+^p])$ such that the grading degree i' part $\mathcal{M}_{i'} \in \mathcal{D}(k)$ of $\mathcal{M}$ is acyclic for $i' \not\equiv i \mod p$, $\mathcal{M}$ is quasi-isomorphic to $\mathcal{M}_i[v_+^p]$. Let $\tilde{\mathcal{M}}$ be a graded complex of $k[D^p, v_+^p]$-modules, representing $\mathcal{M}$, such that each term $\tilde{\mathcal{M}}^j$ of $\tilde{\mathcal{M}}$ (we use cohomological notation) is a free graded module over $k[D^p, v_+^p]$ and the grading degree i' part $\tilde{\mathcal{M}}^j_{i'} \in \mathcal{D}(k)$ of $\tilde{\mathcal{M}}_j$ vanishes for $i' \not\equiv i \mod p$.[26] In particular, the map $v_+^p : \tilde{\mathcal{M}}_j \to \tilde{\mathcal{M}}_{j+p}$ is a term-wise injective morphism of complexes. Let $\tilde{\mathcal{M}}'$ be a graded complex of $k[D^p, v_+^p]$-modules defined as follows: in each cohomological degree, the graded $k[D^p, v_+^p]$-module $\tilde{\mathcal{M}}'^n$ is a quotient of $\tilde{\mathcal{M}}^n$ by the submodule generated by $\tilde{\mathcal{M}}^n_j$ for $j \leq 0$. Then the canonical map $\tilde{\mathcal{M}} \to \tilde{\mathcal{M}}'$ is a quasi-isomorphism. Since $\tilde{\mathcal{M}}'_j$ is a term-wise zero complex for $i \geq j$, the operator D^p acts trivially on $\tilde{\mathcal{M}}'_i$. Hence, there exists a unique morphism of graded complexes of $k[D^p, v_+^p]$-modules $\tilde{\mathcal{M}}_i[v_+^p] \to \tilde{\mathcal{M}}'$, which is the canonical projection $\tilde{\mathcal{M}}_i \to \tilde{\mathcal{M}}_i/v_+^p\tilde{\mathcal{M}}_{i-p} = \tilde{\mathcal{M}}'_i$ in degree i. By the weight assumption on $\mathcal{M}$, this morphism is a quasi-isomorphism. Thus we have essential surjectivity of f^*.

Since the map $f' : (C_2)_{v_-=0}/\mathbf{G}_{m,k} \to D_{\mathrm{HT}} = BH|_{(C_2)_{v_-=0}/\mathbf{G}_{m,k}}$ has a left inverse[27] f'', *i.e.*, $f'' \circ f' = \mathrm{Id}$, we have the essential surjectivity of f'^* as desired. Since f^* is an equivalence and $f'^* \circ f^*$ is fully faithful, it implies that $f'^* \circ f^*$ is an equivalence. □

Corollary III.51. *The pullback along the projection $t : C_2/\mathbf{G}_{m,k} \to \mathbf{A}^1_-/\mathbf{G}_m \otimes \mathbf{F}_p$ induces an equivalence $t^* : \mathcal{D}_{qc,[0,p-1]}(\mathbf{A}^1_-/\mathbf{G}_m \otimes \mathbf{F}_p) \xrightarrow{\sim} \mathcal{D}_{qc,[0,p-1]}(C_2/\mathbf{G}_{m,k})$.*

Proof. Using [Lur18b, Theorem 16.2.0.2], the category $\mathcal{D}_{qc,[0,p-1]}(C_2/\mathbf{G}_{m,k})$ is a full subcategory of the fibre product

$$\mathcal{D}_{qc,[0,p-1]}\left((C_2)_{v_-=0}/\mathbf{G}_{m,k}\right) \times_{\mathcal{D}_{qc,[0,p-1]}\left((C_2)_{v_+^p=v_-=0}/\mathbf{G}_{m,k}\right)} \mathcal{D}_{qc,[0,p-1]}\left((C_2)_{v_+^p=0}/\mathbf{G}_{m,k}\right).$$

By Lemma III.50, composition of t^* with the above embedding is an equivalence. □

[26] Here we make use of the following general fact: for a graded ring R, every complex of graded R-modules is quasi-isomorphic to a complex of free graded R-modules. See, for instance, [Dri04, Lemma B.3.] for a proof of the analogous statement for ordinary rings and [Dri04, §C.8.] for a very general assertion that applies to the graded setting.

[27] Recall that H is a group stack over $C_2/\mathbf{G}_m$. The left inverse f'' is defined to be the restriction of the structure map $BH \to C_2/\mathbf{G}_m$ to the fiber over $(C_2)_{v_-=0}/\mathbf{G}_{m,k}$.

Lemma III.52. *The pullback along the composition $D_{Hod} \hookrightarrow D_{\mathrm{HT}} \to B\mathbf{G}_{m,k}$ induces an equivalence of categories $\mathcal{D}_{qc,[0,p-1]}(B\mathbf{G}_{m,k}) \xrightarrow{\sim} \mathcal{D}_{qc,[0,p-1]}(D_{Hod})$.*

Proof. Recall from [Bha23, §6.5] that $\mathcal{D}_{qc}(D_{Hod})$ is identified with the full subcategory $\mathcal{D}_{gr,\Theta\text{-nilp}}(k[\Theta])$ of $\mathcal{D}_{gr}(k[\Theta])$ that consists of objects $\mathcal{M}$ such that the action of Θ on $\bigoplus_i H^i(\mathcal{M})$ is locally nilpotent. Under this identification, the pullback functor $\mathcal{D}_{qc}(B\mathbf{G}_{m,k}) \to \mathcal{D}_{qc}(D_{Hod})$ sends the graded complex $C_\cdot = \oplus C_i$ to $C_\cdot$ with a trivial action of Θ. The rest is straightforward.

To see fully faithfulness: we have

$$\mathrm{RHom}_{\mathcal{D}_{gr}(k[\Theta])}(C_i, C_j) = \mathrm{Fib}\left(\mathrm{RHom}_{\mathcal{D}_{gr}(k)}(C_i, C_j) \xrightarrow{\mathrm{ad}_\Theta = 0} \mathrm{RHom}_{\mathcal{D}_{gr}(k)}(C_i, C_j(-p))\right).$$

For $i, j \in [0, p-1]$, we have $\mathrm{RHom}_{\mathcal{D}_{gr}(k)}(C_i, C_j(-p)) = 0$. Fully faithfulness follows.

To see essential surjectivity: observe that every $\mathcal{M} \in \mathcal{D}_{qc,[0,p-1]}(D_{Hod})$ can be represented by a complex of graded $k[\Theta]$-modules with the trivial action of Θ. □

Combining the previous two lemmas, we obtain the following:

Corollary III.53. *The pullback along the closed embedding $D_{Hod} \hookrightarrow D_{\mathrm{HT}}$ induces an equivalence $\mathcal{D}_{qc,[0,p-1]}(D_{\mathrm{HT}}) \xrightarrow{\sim} \mathcal{D}_{qc,[0,p-1]}(D_{Hod})$. The restriction to the open substack $\mathcal{D}_{qc,[0,p-1]}(D_{\mathrm{HT}}) \to \mathcal{D}_{qc}(D_{und})$ is isomorphic to the composition*

$$\mathcal{D}_{qc,[0,p-1]}(D_{\mathrm{HT}}) \to \mathcal{D}_{qc}(D_{Hod}) \to \mathcal{D}_{qc}(B\mathbf{G}_{m,k}) \to \mathcal{D}_{qc}(D_{und}),$$

which is induced by pullback along $D_{und} = B\mathbf{G}^\sharp_{m,k} \to B\mathbf{G}_{m,k} \to D_{Hod} \hookrightarrow D_{\mathrm{HT}}$.[28]

Proof. We just need to explain the second assertion. Note that the post-composition of $D_{und} \to B\mathbf{G}_{m,k} \to D_{Hod} \hookrightarrow D_{\mathrm{HT}}$ with $D_{\mathrm{HT}} \to B\mathbf{G}_{m,k}$ is isomorphic to the standard map $B\mathbf{G}^\sharp_{m,k} \to B\mathbf{G}_{m,k}$. The same is true for the open immersion $D_{und} \to D_{\mathrm{HT}}$. This is enough since $f^* : \mathcal{D}_{qc,[0,p-1]}(B\mathbf{G}_{m,k}) \to \mathcal{D}_{qc,[0,p-1]}(D_{\mathrm{HT}})$ is an equivalence by Lemma III.50. □

Lemma III.54. *The commutative diagram of stacks*

$$\begin{array}{ccc} ((\mathbf{A}^1_- \setminus \{0\})/\mathbf{G}_m) \otimes \mathbf{F}_p & \longrightarrow & (\mathbf{A}^1_-/\mathbf{G}_m) \otimes \mathbf{F}_p \\ \downarrow & & \downarrow \\ D_{und} & \hookrightarrow & D_{\mathrm{dR}} \end{array} \tag{III.4.31}$$

[28] For the reader's convenience, let us recall the definitions of the displayed morphisms of stacks. The identification $D_{und} = B\mathbf{G}^\sharp_{m,k}$ is that from [Bha23, Remark 6.2.5]; the map $B\mathbf{G}^\sharp_{m,k} \to B\mathbf{G}_{m,k}$ is induced by the natural homomorphism of group schemes; $B\mathbf{G}_{m,k} \to D_{Hod} = B(F_*\mathbf{G}^\sharp_{a,k} \rtimes \mathbf{G}_{m,k})$ is given by $\mathbf{G}_{m,k} \xrightarrow{(0,\mathrm{Id})} F_*\mathbf{G}^\sharp_{a,k} \rtimes \mathbf{G}_{m,k}$, and the last map realizes D_{Hod} as a closed substack of D_{HT}.

induces an equivalence of categories

$$\mathcal{L} : \mathcal{D}_{qc,[0,p-1]}(D_{\mathrm{dR}}) \xrightarrow{\sim} \mathcal{D}_{qc,[0,p-1]}((\mathbf{A}^1_-/\mathbf{G}_m) \otimes \mathbf{F}_p) \times_{\mathcal{D}(k)} \mathcal{D}_{qc}(D_{und}), \tag{III.4.32}$$

where $\mathcal{D}(k)$ *is identified with* $\mathcal{D}_{qc}(((\mathbf{A}^1_- \setminus \{0\})/\mathbf{G}_m) \otimes \mathbf{F}_p)$.

Proof. To show fully faithfulness of the functor (III.4.32): we shall use the embedding $\mathcal{D}_{qc}(D_{\mathrm{dR}}) \hookrightarrow \mathcal{D}_{gr}(k[v_-, \Theta])$ as in the proof of Lemma III.44 and the references therein. For $\mathcal{F}, \mathcal{F}' \in \mathcal{D}_{qc,[0,p-1]}(D_{\mathrm{dR}})$, let $F, F' \in \mathcal{D}_{gr,[0,p-1]}(k[v_-, \Theta])$ be the corresponding graded modules. We have

$$\mathrm{RHom}(F', F) = \mathrm{Fib}\left(\mathrm{RHom}_{\mathcal{D}_{gr}(k[v_-])}(F', F) \xrightarrow{\mathrm{ad}_\Theta} \mathrm{RHom}_{\mathcal{D}_{gr}(k[v_-])}(F', F(-p))\right).$$

On the other hand,

$$\mathrm{RHom}(\mathcal{L}F', \mathcal{L}F) = \mathrm{Fib}\left(\mathrm{RHom}_{\mathcal{D}_{gr}(k[v_-])}(F', F) \xrightarrow{\mathrm{ad}_\Theta} \mathrm{RHom}_{\mathcal{D}_{gr}(k[v_-])}(\overline{j}^* F', \overline{j}^* F(-p))\right).$$

Using (III.4.15) and the assumption on weights, the morphism

$$\overline{j}^* : \mathrm{RHom}_{\mathcal{D}_{gr}(k[v_-])}(F', F(-p)) \to \mathrm{RHom}_{\mathcal{D}_{gr}(k[v_-])}(\overline{j}^* F', \overline{j}^* F(-p))$$

is an isomorphism. This proves fully faithfulness of $\mathcal{L}$.

To show essential surjectivity, observe that any object of the right-hand side of (III.4.32) can be represented by a complex $F = \bigoplus F^i$ of graded $k[v_-]$-modules, where F^i is term-wise zero for $i \geq p$, and the map $v_- : F^i \to F^{i-1}$ is a term-wise isomorphism for $i \leq 0$, together with a $\theta : F^0 \to F^0$. Then there exists a unique $\Theta : F^\bullet \to F^{\bullet - p}$ that commutes with v_- and such that $\Theta : F^0 \to F^{-p}$ is precisely $v_-^p \theta$. This proves essential surjectivity as desired. □

Let us return to the proof of the proposition. Consider the maps

$$\begin{array}{ccc}
\mathrm{Eq}\Big(\mathcal{D}_{qc,[0,p-1]}(\mathbf{A}^1_-/\mathbf{G}_m \otimes \mathbf{F}_p) \underset{F^* \circ i_p^*}{\overset{i_1^*}{\rightrightarrows}} \mathcal{D}_{qc}(\mathrm{Spec} k)\Big) & & \\
\simeq \big\downarrow t^* & & \simeq \big\downarrow \mathrm{Id} \\
\mathrm{Eq}\Big(\mathcal{D}_{qc,[0,p-1]}(C_2/\mathbf{G}_{m,k}) \rightrightarrows \mathcal{D}_{qc}(\mathrm{Spec} k)\Big) & & \\
\uparrow & & \uparrow \\
\mathrm{Eq}\Big(\mathcal{D}_{qc,[0,p-1]}(W(k)^{\mathcal{N}}_{red}) \rightrightarrows \mathcal{D}_{qc}(D_{und})\Big) & &
\end{array} \tag{III.4.33}$$

The equalizer in the first row is precisely the category $\mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(W(k)) \otimes \mathbf{F}_p$ from Definition III.6. The two horizontal arrows in the second row are given by the restriction to the point $(v_+^p = 0, v_- = 1)$, and the restriction to the point $(v_+^p = 1, v_- = 0)$ post-composed with the Frobenius. The downward arrows induce an equivalence of equalizers by Corollary III.51. The equalizer at the bottom arrow is $\mathcal{D}_{qc,[0,p-1]}(W(k)^{syn}_{red})$. The functor Φ_{DI} can be interpreted as the composition of the upward map on equalizers composed with the inverse of the downward equivalence on equalizers. Thus it remains to check that the upward map on equalizers is an equivalence.

Recall (*cf.* Lemma III.8) that the category $\mathcal{D}_{qc,[0,p-1]}(W(k)^{syn}_{red})$ is a full subcategory of the fiber product

$$\mathcal{C} := \mathcal{D}_{qc,[0,p-1]}(D_{\mathrm{HT}}) \underset{\mathcal{D}_{qc}(D_{und})\times\mathcal{D}_{qc}(D_{Hod})}{\times} \mathcal{D}_{qc,[0,p-1]}(D_{\mathrm{dR}}). \tag{III.4.34}$$

where the functors $\mathcal{D}_{qc}(D_{\mathrm{HT}}),\ \mathcal{D}_{qc}(D_{\mathrm{dR}}) \to \mathcal{D}_{qc}(D_{Hod})$ and $\mathcal{D}_{qc}(D_{\mathrm{dR}}) \to \mathcal{D}_{qc}(D_{und})$ are given by the $W(k)$-*linear* embeddings of underlying stacks; the functor $\mathcal{D}_{qc}(D_{\mathrm{HT}}) \to \mathcal{D}_{qc}(D_{und})$ is the pullback along the morphism (*cf.* the end of §III.1.3)

$$D_{und} = B\mathbf{G}^{\sharp}_{m,\mathbf{F}_p} \otimes k \xrightarrow{\mathrm{id}\times\mathrm{Frob}} B\mathbf{G}^{\sharp}_{m,\mathbf{F}_p} \otimes k = D_{und} \hookrightarrow D_{\mathrm{HT}}.$$

In Corollary III.53, we showed that the restriction functor $\mathcal{D}_{qc,[0,p-1]}(D_{\mathrm{HT}}) \to \mathcal{D}_{qc,[0,p-1]}(D_{Hod})$ is an equivalence and described explicitly its inverse composed with the other restriction $\mathcal{D}_{qc,[0,p-1]}(D_{\mathrm{HT}}) \to \mathcal{D}_{qc,[0,p-1]}(D_{und})$. Using this result, we obtain the following description of $\mathcal{C}$

$$\mathcal{D}_{qc,[0,p-1]}(W(k)^{syn}_{red}) \hookrightarrow \mathcal{C} \xrightarrow{\sim} \mathrm{Eq}\left(\mathcal{D}_{qc,[0,p-1]}(D_{\mathrm{dR}}) \underset{T^*\circ\mathfrak{p}^*_{\mathrm{d\bar{R}}}}{\overset{\widetilde{i}_1^*}{\rightrightarrows}} \mathcal{D}_{qc}(D_{und})\right). \tag{III.4.35}$$

Here $\widetilde{i}_1^*$ is the restriction along the open embedding $\widetilde{i}_1 : D_{und} \hookrightarrow D_{\mathrm{dR}}$, and $\mathfrak{p}^*_{\mathrm{d\bar{R}}}$ is the pullback along $\mathfrak{p}_{\mathrm{d\bar{R}}} : (\mathbf{A}^1_-/\mathbf{G}_m) \otimes \mathbf{F}_p \hookrightarrow D_{\mathrm{dR}}$, and T^* is the pullback along the composition $T : B\mathbf{G}^{\sharp}_{m,\mathbf{F}_p} \otimes k \xrightarrow{\mathrm{id}\times\mathrm{Frob}} B\mathbf{G}^{\sharp}_{m,\mathbf{F}_p} \otimes k \to B\mathbf{G}_{m,k} \to (\mathbf{A}^1_-/\mathbf{G}_m) \otimes \mathbf{F}_p$.

The functor Φ_{DI} factors through (III.4.35) and the pullback functor

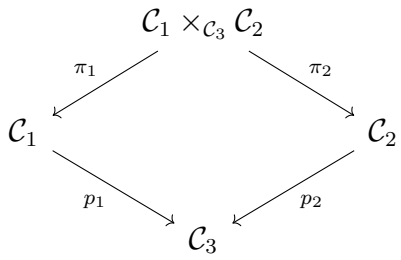
$$\mathrm{Eq}\left(\mathcal{D}_{qc,[0,p-1]}(D_{\mathrm{dR}}) \underset{T^*\circ\mathfrak{p}^*_{\mathrm{d\bar{R}}}}{\overset{\widetilde{i}_1^*}{\rightrightarrows}} \mathcal{D}_{qc}(D_{und})\right) \to \mathrm{Eq}\left(\mathcal{D}_{qc,[0,p-1]}((\mathbf{A}^1_-/\mathbf{G}_m) \otimes \mathbf{F}_p) \underset{q^*T^*}{\overset{i_1^*}{\rightrightarrows}} \mathcal{D}(k)\right) \tag{III.4.36}$$

induced by $\mathfrak{p}_{\mathrm{d\bar{R}}}$ and $q : \mathrm{Spec}k \to B\mathbf{G}^{\sharp}_{m,k}$ is the map classifying the trivial $\mathbf{G}^{\sharp}_{m,k}$-torsor. Note that $T \circ q = i_0 \circ F$, and thus the equalizer on the right-hand side of (III.4.36) is precisely

the category $\mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(W(k)) \otimes \mathbf{F}_p$. Lemma III.54, together with the following easy Lemma III.55, shows that the functor (III.4.36) is an equivalence.

Lemma III.55. *Let*

(III.4.37)



be a pullback square of ∞-categories, and let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a functor (not necessarily commuting with p_1 and p_2). Then the equalizers are equivalent:

$$\mathrm{Eq}\ (\mathcal{C}_1 \times_{\mathcal{C}_3} \mathcal{C}_2 \underset{F\circ\pi_1}{\overset{\pi_2}{\rightrightarrows}} \mathcal{C}_2) \ \simeq \mathrm{Eq}\ (\mathcal{C}_1 \underset{p_2\circ F}{\overset{p_1}{\rightrightarrows}} \mathcal{C}_3)\ .$$

The functor from left to right is given by sending $(X \in \mathcal{C}_1, Y \in \mathcal{C}_2, \alpha : p_1X \simeq p_2Y, \beta : Y \simeq FX) \in \mathrm{Eq}\ (\mathcal{C}_1 \times_{\mathcal{C}_3} \mathcal{C}_2 \underset{F\circ\pi_1}{\overset{\pi_2}{\rightrightarrows}} \mathcal{C}_2)$ to $(X, \gamma := (p_2\beta)\circ\alpha : p_1X \simeq p_2FX) \in \mathrm{Eq}\ (\mathcal{C}_1 \underset{p_2\circ F}{\overset{p_1}{\rightrightarrows}} \mathcal{C}_3)$. The functor from right to left is given by sending $(X, \gamma : p_1X \simeq p_2FX) \in \mathrm{Eq}\ (\mathcal{C}_1 \underset{p_2\circ F}{\overset{p_1}{\rightrightarrows}} \mathcal{C}_3)$ to $(X, FX, \gamma, \mathrm{Id}) \in \mathrm{Eq}\ (\mathcal{C}_1 \times_{\mathcal{C}_3} \mathcal{C}_2 \underset{F\circ\pi_1}{\overset{\pi_2}{\rightrightarrows}} \mathcal{C}_2)$.

Proof. This is immediate. □

Combining (III.4.35) and equivalence (III.4.36), we conclude that Φ_{DI} is the composition of a fully faithful embedding into the fiber product category and an equivalence:

$$\mathcal{D}_{qc,[0,p-1]}(W(k)^{syn}_{red}) \hookrightarrow \mathcal{C} \xrightarrow{\sim} \mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(W(k)) \otimes \mathbf{F}_p. \tag{III.4.38}$$

It remains to prove that the composition is essentially surjective. Given any $(\mathcal{M}, \varphi) \in \mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(W(k)) \otimes \mathbf{F}_p$, let $\mathcal{F}$ be the corresponding object of $\mathcal{C}$. By Lemma III.9, it is enough to check that the pullback of $\mathcal{F}$ to $\mathcal{D}_{qc}\left((C_2)_{v_-=0}/\mathbf{G}_m\right) \times_{\mathcal{D}_{qc}\left((C_2)_{v_+^p=v_-=0}/\mathbf{G}_m\right)} \mathcal{D}_{qc}\left((C_2)_{v_+^p=0}/\mathbf{G}_m\right)$ lies in $\mathcal{D}_{qc}(C_2/\mathbf{G}_m)$. But by diagram (III.4.33), the pullback of $\mathcal{F}$ is isomorphic to $t^*\mathcal{M} \in \mathcal{D}_{qc}(C_2/\mathbf{G}_m)$. This completes the proof of Proposition III.4.2 as well as the proof of equivalence (III.4.1). □

Remark III.56. Note that in *Step 3*, the only result that uses the fact that weights are in $[0, p-2]$ and not merely in $[0, p-1]$, is Corollary III.49. Moreover, the corollary

is not true without this assumption on weights; see Remark III.59 for an explanation. We also note that this assumption is essential for (III.4.8), *i.e.*, to show that the restriction $\mathcal{D}_{qc,[0,p-2]}(W(k)^{syn} \otimes \mathbf{F}_p) \to \mathcal{D}_{qc,[0,p-2]}(W(k)^{syn}_{red})$ is an equivalence. However, it is only needed in the first step of the deformation argument, *i.e.*, to deform from $W(k)^{syn}_{red}$ to $(W(k)^{syn} \otimes \mathbf{F}_p)_{v_1^2=0}$. In particular, the restriction $\mathcal{D}_{qc,[0,p-1]}((W(k)^{syn} \otimes \mathbf{F}_p)_{v_1^3=0}) \to \mathcal{D}_{qc,[0,p-1]}((W(k)^{syn} \otimes \mathbf{F}_p)_{v_1^2=0})$ is an equivalence.

End of proof of Theorem 3. Since perfectness is preserved under pullbacks the functor Φ_{Maz} restricts to

$$\mathrm{Perf}_{[0,p-1]}(W(k)^{syn}) \to \mathcal{D}^b\left(\mathscr{MF}_{[0,p-1]}(W(k))\right).$$

To prove (III.4.2) we have to show that, for every $\mathcal{F} \in \mathrm{Perf}_{[0,p-2]}(W(k)^{syn})$, the object $\mathcal{M} := \Phi_{\mathrm{Maz}}(\mathcal{F}) \in \mathcal{D}^b\left(\mathscr{MF}_{[0,p-2]}(W(k))\right)$ is perfect. It suffices to check that the pullback of $\Phi_{\mathrm{Maz}}(\mathcal{F})$ to $W(k)^{\mathcal{N}}_{red}$ is perfect. Moreover, since the map $C_2/\mathbf{G}_{m,k} \to W(k)^{\mathcal{N}}_{red}$ is faithfully flat (see [Dri24, §7.7.1]), it suffices to check that the pullback of $\Phi_{\mathrm{Maz}}(\mathcal{F})$ to $C_2/\mathbf{G}_{m,k}$ is perfect. But the latter is isomorphic to the pullback of $\mathcal{M}$ along the composition $C_2/\mathbf{G}_{m,k} \to \mathbf{A}^1_-/\mathbf{G}_m \otimes \mathbf{F}_p \hookrightarrow \mathbf{A}^1_-/\mathbf{G}_m$. Indeed, both these maps $C_2/\mathbf{G}_{m,k} \to \mathbf{A}^1_-/\mathbf{G}_m$ send a triple $(L, v_- : L \to R, u : R \to L^{\otimes p}) \in C_2/\mathbf{G}_{m,k}(R)$ as in (III.4.23) to $(v_- : L \to R) \in \mathbf{A}^1_-/\mathbf{G}_m(R)$. This proves (III.4.2).

Let us prove the last assertion of Theorem III.39. Recall from [Bha23] that every coherent sheaf on $W(k)^{syn}$ is perfect. In particular, Φ_{Maz} carries $\mathrm{Coh}_{[0,p-1]}(W(k)^{syn})$ into $\mathcal{D}^b\left(\mathscr{MF}_{[0,p-1]}(W(k))\right)$. Let $\mathcal{M} \in \mathscr{MF}_{[0,p-2]}(W(k))$ with $p\mathcal{M} = 0$. We claim that $E := \Phi^{-1}_{\mathrm{Maz}}(\mathcal{M})$ is a vector bundle over $W(k)^{syn} \otimes \mathbf{F}_p$. Since $W(k)^{\mathcal{N}} \otimes \mathbf{F}_p \to W(k)^{syn} \otimes \mathbf{F}_p$ is an étale cover it suffices to verify that the pullback of E to $W(k)^{\mathcal{N}} \otimes \mathbf{F}_p$ is a vector bundle. An object of $\mathcal{D}_{qc}(W(k)^{\mathcal{N}} \otimes \mathbf{F}_p)$ is a vector bundle if and only if its pullback to $W(k)^{\mathcal{N}}_{red}$ is a vector bundle. Since the morphism $C_2/\mathbf{G}_m \to W(k)^{\mathcal{N}}_{red}$ is faithfully flat, it is enough to show the assertion for the pullback of E to $C_2/\mathbf{G}_m$. But the latter is the pullback $\mathcal{M}$ under $C_2/\mathbf{G}_m \to \mathbf{A}^1_-/\mathbf{G}_m \otimes \mathbf{F}_p$. Now the claim follows from Propiosition III.1.2 (2).

By dévissage, we infer that $\Phi^{-1}_{\mathrm{Maz}}(\mathcal{M}) \in \mathrm{Coh}_{[0,p-2]}(W(k)^{syn})$ for every $\mathcal{M} \in \mathscr{MF}_{[0,p-2]}(W(k))$. Thus the functor $\Phi^{-1}_{\mathrm{Maz}} : \mathcal{D}^b(\mathscr{MF}_{[0,p-2]}(W(k))) \to \mathcal{D}_{qc}(W(k)^{syn})$ is t-exact. Moreover, as we observed above, its essential image contains $\mathrm{Coh}_{[0,p-2]}(W(k)^{syn})$. This completes the proof. □

Remark III.57. The functor Φ_{Maz} is not an equivalence. Indeed, there exists a non-zero morphism $v_1 : \mathcal{O}_{W(k)^{syn}} \otimes \mathbf{F}_p \to \mathcal{O}_{W(k)^{syn}}\{p-1\} \otimes \mathbf{F}_p$ whose restriction to $W(k)^{syn}_{red}$ is equal to 0. Since the functor $\Phi_{\mathrm{Maz}} \otimes \mathbf{F}_p$ factors through the restriction to the reduced locus (see (III.4.22)) we conclude that $\Phi_{\mathrm{Maz}}(v_1) = 0$.

Remark III.58. The last part of the proof of Theorem III.39 shows that every $\mathcal{F} \in \mathrm{Coh}_{[0,p-2]}(W(k)^{syn} \otimes \mathbf{F}_p)$ is a vector bundle over $W(k)^{syn} \otimes \mathbf{F}_p$. Consequently, every p-torsionfree $\mathcal{G} \in \mathrm{Coh}_{[0,p-2]}(W(k)^{syn})$ is a vector bundle over[29] $W(k)^{syn}$. The assumption on Hodge-Tate weights cannot be weakened: for example, the sheaf $\mathcal{O}_{W(k)^{syn}_{red}}\{p-1\} = \mathrm{Coker}(\mathcal{O}_{W(k)^{syn}} \otimes \mathbf{F}_p \xrightarrow{v_1} \mathcal{O}_{W(k)^{syn}}\{p-1\} \otimes \mathbf{F}_p)$ has Hodge-Tate weights 0 and $p-1$, and it is not a vector bundle over $W(k)^{syn} \otimes \mathbf{F}_p$. Also, we have

$$\begin{aligned}\Phi_{\mathrm{Maz}}(\mathcal{O}_{W(k)^{syn}_{red}}\{p-1\}) &\xrightarrow{\sim} \Phi_{\mathrm{Maz}}(\mathcal{O}_{W(k)^{syn}} \otimes \mathbf{F}_p)[1] \oplus \Phi_{\mathrm{Maz}}(\mathcal{O}_{W(k)^{syn}}\{p-1\} \otimes \mathbf{F}_p)\\ &\xrightarrow{\sim} k[1] \oplus k(p-1).\end{aligned}$$

In particular, $\Phi_{\mathrm{Maz}}(\mathcal{O}_{W(k)^{syn}_{red}}\{p-1\})$ is not concentrated in cohomological degree zero. Thus, the Hodge-Tate weight assumption in the last assertion of Theorem III.39 cannot be weakened.

Remark III.59 (The Mazur *versus* Deligne-Illusie decompositions). Recall from the proof of Theorem III.39 the functors

$$\Phi_{\mathrm{DI}}, \Phi^{\mathrm{Maz}}_{red} : \mathcal{D}_{qc,[0,p-1]}(W(k)^{syn}_{red}) \to \mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(W(k)) \otimes \mathbf{F}_p.$$

By Proposition III.4.2 Φ_{DI} is an equivalence of categories. Consequently, we can consider

$$\Phi^{\mathrm{Maz}}_{red} \circ \Phi^{-1}_{\mathrm{DI}} : \mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(W(k)) \otimes \mathbf{F}_p \to \mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(W(k)) \otimes \mathbf{F}_p. \tag{III.4.39}$$

By Corollary III.49 the restriction of this endofunctor to the subcategory $\mathscr{DMF}^{\mathrm{big}}_{[0,p-2]}(W(k)) \otimes \mathbf{F}_p$ is isomorphic to Id. However, the functor (III.4.39) is not even an auto-equivalence. In fact, using Lemma III.47 one can describe $\Phi^{\mathrm{Maz}}_{red} \circ \Phi^{-1}_{\mathrm{DI}}$ explicitly. We shall just state the answer. Recall that an object of $\mathscr{DMF}^{\mathrm{big}}_{[0,p-1]}(W(k)) \otimes \mathbf{F}_p$ consists of an effective object $F^\bullet \in \mathcal{D}_{gr}(k[v_-])$ with $F^p = 0$ together with a homotopy equivalence

$$\varphi : \bigoplus_{i=0}^{p-1} F^* \,\mathrm{Gr}^i_F F^{-\infty} \xrightarrow{\sim} F^{-\infty}.$$

[29] Indeed, recall from §III.1.5 that $W(k)^{syn}$ admits a faithfully flat cover $\mathrm{Spf}\, R \to W(k)^{syn}$, where R is a *Noetherian* ring complete with respect to an ideal $p \in J \subset R$, and $\mathrm{Spf}\, R$ is the corresponding J-adic formal scheme. A quasi-coherent sheaf $\mathcal{G}$ on $W(k)^{syn}$ is a vector bundle (resp. a coherent sheaf) if and only if its pullback to $\mathrm{Spf}\, R$ is a finite projective R-module (resp. a finite R-module). Thus, our claim follows from the following elementary general statement: a p-torsion-free finite module M over a p-complete Noetherian ring, $p \in J$, is projective if and only if M/p is a projective R/p-module. Indeed, it is enough to check that, for every maximal ideal $\mathfrak{m} \in R$, one has $\mathrm{Tor}^R_1(M, R/\mathfrak{m}) = 0$. Since R is p-complete, $p \in \mathfrak{m}$. Thus $\mathrm{Tor}^R_1(M, R/\mathfrak{m}) = \mathrm{Tor}^{R/p}_1(M/p, R/\mathfrak{m}) = 0$.

Consider the endomorphism

$$\Theta' = \sum_{i=0}^{p-1} (-i)\,\mathrm{Id}_{F^*\,\mathrm{Gr}_F^i\,F^{-\infty}}$$

of the left-hand side and let $\Theta = \varphi \circ \Theta' \circ \varphi^{-1}$ be the corresponding endomorphism of $F^{-\infty}$. The endomorphism Θ is called the Sen operator (see [Bha23, §6.5]). Consider the composition $\bar{\Theta}$

$$F^{p-1} \xrightarrow{v_-^{p-1}} F^0 \xrightarrow{\sim} F^{-\infty} \xrightarrow{\Theta} F^{-\infty} \xrightarrow{\sim} F^0 \to F^0/F^1. \tag{III.4.40}$$

Here we use that the canonical map $F^0 \to \varinjlim_{v_-} F^i = F^{-\infty}$ is an equivalence, which follows from the effectivity of $F^\bullet$. Define a nilpotent endomorphism α of $\bigoplus_{i=0}^{p-1} F^*\,\mathrm{Gr}_F^i\,F^{-\infty}$ to be zero on all $F^*\,\mathrm{Gr}_F^i\,F^{-\infty}$, with $i \neq p-1$, and $\alpha|_{F^*\,\mathrm{Gr}_F^{p-1}\,F^{-\infty}}$ being the composition

$$F^*\,\mathrm{Gr}_F^{p-1}\,F^{-\infty} \xrightarrow{\sim} F^*F^{p-1} \xrightarrow{F^*\bar{\Theta}} F^*\,\mathrm{Gr}_F^0\,F^{-\infty} \hookrightarrow \bigoplus_{i=0}^{p-1} F^*\,\mathrm{Gr}_F^i\,F^{-\infty}.$$

We claim that the functor (III.4.39) carries each $(F^\bullet, \varphi)$ to $(F^\bullet, \varphi \circ (\mathrm{Id} - \alpha))$.

For example, if $(F^\bullet, \varphi) \in \mathscr{MF}(\mathbf{Z}_p)$ is a p-torsion Fontaine-Laffaille module, which is an extension of $\mathbf{F}_p(1-p)$ by $\mathbf{F}_p$ then the functor (III.4.39) carries $(F^\bullet, \varphi)$ to $\mathbf{F}_p(1-p) \oplus \mathbf{F}_p$, that is, the endomorphism of $\mathrm{Ext}^1_{\mathscr{MF}(\mathbf{Z}_p)\otimes\mathbf{F}_p}(\mathbf{F}_p(1-p), \mathbf{F}_p) \xrightarrow{\sim} \mathbf{F}_p$ induced by (III.4.39) is equal to 0.

Let X be a smooth p-adic formal scheme over $W(k)$ of dimension $\leq p-1$. By Remark III.36, we have $\widetilde{\Phi}_{\mathrm{Maz}}(X) \xrightarrow{\sim} \Phi_{\mathrm{Maz}} \circ \mathcal{H}_{syn}(X)$. On the other hand, by [LM24, Theorem 5.10] the p-torsion Fontaine-Laffaille module $\Phi_{\mathrm{DI}}(\mathcal{H}_{syn}(X)|_{W(k)^{syn}_{red}})$ is given by the Deligne-Illusie decomposition ([DI87]). Thus, in the geometric context, the above formula for $\Phi^{\mathrm{Maz}}_{red} \circ \Phi^{-1}_{\mathrm{DI}}$ describes the relation between the modulo p reduction of (I.0.1) and the Deligne-Illusie decomposition. For $\dim X < p-1$ the two decompositions coincide, but in dimension $p-1$ they are different. For example, if X is an elliptic curve over $\mathbf{Z}_2$ with ordinary reduction then Fontaine-Laffaille module $H^1_{\mathrm{dR}}(X) \otimes \mathbf{F}_2$ constructed using (I.0.1) (*i.e.*, $H^1(\Phi_{\mathrm{Maz}} \circ \mathcal{H}_{syn}(X)) \otimes \mathbf{F}_2$) is isomorphic to $\mathbf{F}_2 \oplus \mathbf{F}_2(-1)$ whereas the Fontaine-Laffaille structure on $H^1_{\mathrm{dR}}(X) \otimes \mathbf{F}_2$ induced by the Deligne-Illusie decomposition is a nonsplit extension of $\mathbf{F}_2(-1)$ by $\mathbf{F}_2$ unless $X \times \mathrm{Spec}\mathbf{Z}/4$ is the *canonical* lift of the ordinary elliptic curve $X \times \mathrm{Spec}\mathbf{F}_2$.

Remark III.60 (Syntomic cohomology in small weights). Given an F-gauge $\mathcal{F} \in \mathcal{D}_{qc,[0,p-2]}(W(k)^{syn})$ one can use Theorem III.39 to compute the syntomic cohomology

$R\Gamma(W(k)^{syn}, \mathcal{F}\{i\})$, for $i \leq p-2$. Let us briefly explain how, using the functor

$$\mathbf{\Phi}_{\mathrm{Maz}} : \mathcal{D}_{qc,[0,+\infty]}(W(k)^{syn}) \to \mathrm{Mod}_{\mathrm{Maz}}(W(k)), \quad \mathcal{F} \mapsto (\mathfrak{p}^*_{\mathrm{d\bar{R}}}(\mathcal{F}), \varphi_\bullet) \tag{III.4.41}$$

constructed in Remark III.35, the computation can be generalized to all effective F-gauges. Let $\mathcal{F} \in \mathcal{D}_{qc,[0,\infty]}(W(k)^{syn})$ be an effective F-gauge, $\mathbf{\Phi}_{\mathrm{Maz}}(\mathcal{F}) = (F^\bullet, \varphi_\bullet)$. We claim that, for $i \leq p-2$, the functor $\mathbf{\Phi}_{\mathrm{Maz}}$ induces equivalences

$$\begin{aligned} R\Gamma(W(k)^{syn}, \mathcal{F}\{i\}) &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Mod}_{\mathrm{Maz}}(W(k))} \\ &\qquad \Big(\mathbf{\Phi}_{\mathrm{Maz}}(\mathcal{O}_{W(k)^{syn}}\{-i\}), \mathbf{\Phi}_{\mathrm{Maz}}(\mathcal{F})\Big) \\ &\simeq \mathrm{Eq}\left(F^i \underset{v^i_-}{\overset{\varphi_i}{\rightrightarrows}} F^0\right). \end{aligned} \tag{III.4.42}$$

To see the last isomorphism, note that by [NS18, Proposition II.1.5], the mapping spectrum $\mathrm{Hom}_{\mathrm{Mod}_{\mathrm{Maz}}(W(k))}(W(k)(-i), F^\bullet)$ is identified with the fiber of $\mathrm{Hom}(W(k)(-i), F^\bullet) \xrightarrow{t} \mathrm{Hom}(W(k)(-i), F_* F^0 \otimes B)$, where mapping spectra are computed in $\widehat{\mathcal{D}}^{\mathrm{eff}}_{gr}(W(k)[v_-])$ and t takes $f : W(k)(-i) \to F^\bullet$ to the difference $\varphi_\bullet \circ f - f \circ \varphi_\bullet$. Note that the source of t can be identified with F^i and the target with F^0. Under this identification one always has $\varphi_\bullet \circ f = \varphi_i(f)$ and, provided $i < p$, one has $f \circ \varphi_\bullet = v^i_-(f)$. For a proof, it suffices to check that the map is an equivalence after reducing modulo p. The proof of the mod p statement is explained[30] in [Bha23, Remark 6.5.15].

For $\mathcal{F}$ of the form $\mathcal{H}_{syn}(X)$, the formula (III.4.42) recovers [AMMN22, Theorem F (2)].

Remark III.61 (Lattices in crystalline representations). Let $\mathscr{MF}^f_{[0,p-2]}(W(k)) \subset \mathscr{MF}_{[0,p-2]}(W(k))$ be the full subcategory, whose objects are p-torsionfree Fontaine–Laffaille modules. In ([FL82, §7.14]), Fontaine and Laffaille constructed an exact fully faithful functor

$$\mathcal{T}^{FL}_{\acute{e}t} : \mathscr{MF}^f_{[0,p-2]}(W(k)) \hookrightarrow \mathrm{Rep}_{G_K}(\mathbf{Z}_p) \tag{III.4.43}$$

to the category of continuous $\mathbf{Z}_p$-linear representations of the absolute Galois group G_K of $K := \mathrm{Frac}(W(k))$. Moreover, Breuil identified, in [Bre99, Proposition 3], the essential image of (III.4.43) with the full subcategory $\mathrm{Rep}^{crys,f}_{G_K,[0,p-2]}(\mathbf{Z}_p) \subset \mathrm{Rep}_{G_K}(\mathbf{Z}_p)$, formed by all $\Lambda \in \mathrm{Rep}_{G_K}(\mathbf{Z}_p)$, such that Λ is a finite free as a $\mathbf{Z}_p$-module and $\Lambda \otimes \mathbb{Q}_p$ is a crystalline representation with Hodge–Tate weights lying in $[0, p-2]$.

On the other hand, using results from [BS23], Bhatt and Lurie established in [Bha23,

[30] Bhatt and Lurie do not use the functor $\mathbf{\Phi}_{\mathrm{Maz}}$; instead they construct φ_i, $i < p-1$, with $\mathbf{Z}/p$-coefficients, using the Sen operator.

Theorem 6.6.13] an equivalence of categories

$$\mathcal{T}^{BL}_{\acute{e}t} : \mathrm{Coh}^{\mathrm{refl}}(W(k)^{syn}) \xrightarrow{\sim} \mathrm{Rep}^{crys,f}_{G_K}(\mathbf{Z}_p)$$

between the category *reflexive* coherent sheaves on $W(k)^{syn}$ and the category of lattices in crystalline representations of G_K with arbitrary Hodge–Tate weights.

By Remark III.58 the functor (III.4.2) restricts to an equivalence of categories

$$\Phi_{\mathrm{Maz}} : \mathrm{Vect}_{[0,p-2]}(W(k)^{syn}) \xrightarrow{\sim} \mathscr{MF}^{f}_{[0,p-2]}(W(k)) \tag{III.4.44}$$

between the subcategory $\mathrm{Vect}_{[0,p-2]}(W(k)^{syn}) \subset \mathrm{Coh}_{[0,p-2]}(W(k)^{syn})$ of vector bundles and the subcategory $\mathscr{MF}^{f}_{[0,p-2]}(W(k)) \subset \mathscr{MF}_{[0,p-2]}(W(k))$ of p-torsionfree Fontaine-Laffaille modules.

We claim that

$$\mathcal{T}^{FL}_{\acute{e}t} \xrightarrow{\sim} \mathcal{T}^{BL}_{\acute{e}t} \circ \Phi^{-1}_{\mathrm{Maz}} : \mathscr{MF}^{f}_{[0,p-2]}(W(k)) \to \mathrm{Rep}^{crys,f}_{G_K,[0,p-2]}(\mathbf{Z}_p). \tag{III.4.45}$$

The argument below was communicated to us by Akhil Mathew. Recall from [Bha23, Construction 6.3.1 and Corollary 6.6.5] and [BS23, Theorem 5.2] that, for any vector bundle $E \in \mathrm{Vect}_{(-\infty,0]}(W(k)^{syn})$ with non-positive weights, the corresponding Galois representation is computed by the formula

$$\mathcal{T}^{BL}_{\acute{e}t}(E) = H^0(\mathcal{O}_C^{syn}, f^*_{syn}E) \simeq H^0(W(k)^{syn}, f_{syn*}f^*_{syn}E) \simeq \tag{III.4.46}$$

$$\simeq H^0(W(k)^{syn}, E \otimes f_{syn*}\mathcal{O}_{\mathcal{O}_C^{syn}}), \tag{III.4.47}$$

where $\mathcal{O}_C$ stands for the ring of integers of completed algebraic closure of K and $f_{syn} : \mathcal{O}_C^{syn} \to W(k)^{syn}$ is induced by $\operatorname{Spf} \mathcal{O}_C \to \operatorname{Spf} W(k)$. Note that $f_{syn*}\mathcal{O}_{\mathcal{O}_C^{syn}}$ is an effective F-gauge, $f_{syn*}\mathcal{O}_{\mathcal{O}_C^{syn}} \in \mathcal{D}_{qc,[0,\infty]}(W(k)^{syn})$. Thus, if $E \in \mathrm{Vect}_{[2-p,0]}(W(k)^{syn})$, we have $E \otimes f_{syn*}\mathcal{O}_{\mathcal{O}_C^{syn}} \in \mathcal{D}_{qc,[2-p,\infty]}(W(k)^{syn})$. Consequently, (III.4.47) can be computed using formula (III.4.42). Note that by [Bha12, Proposition 9.9] the p-completed derived de Rham cohomology of $\operatorname{Spf} \mathcal{O}_C$ is identified with Fontaine's period ring A_{crys} matching the Hodge filtration on the derived de Rham cohomology and the filtration by divided powers of $I := \ker(A_{crys} \twoheadrightarrow \mathcal{O}_C)$. Now, let $(\mathcal{M}, \varphi) \in \mathscr{MF}^{f}_{[0,p-2]}(W(k))$ and let $M := \mathcal{M}_{v_-=1}$ be the associated $W(k)$-module equipped with the Hodge filtration (see (III.1.2)). We apply the above computation to $E = \left(\Phi^{-1}_{\mathrm{Maz}}(\mathcal{M}, \varphi)\right)^*$. By the construction of Φ_{Maz}, the de Rham realization $\mathfrak{p}_{\mathrm{d\bar{R}}}(E)$, regarded as a filtered $W(k)$-module, is M^* with its Hodge filtration.

Using (III.4.42), we have

$$\mathcal{T}^{BL}_{\acute{e}t}(\Phi^{-1}_{\mathrm{Maz}}(\mathcal{M},\varphi)) \xrightarrow{\sim} \left(\mathcal{T}^{BL}_{\acute{e}t}(E)\right)^* \xrightarrow{\sim} \left(\ker\left(F^0(M^*\otimes A_{crys}) \xrightarrow{\varphi-\mathrm{Id}} M^*\otimes A_{crys}\right)\right)^*.$$

Here $M^*\otimes A_{crys}$ is equipped with the tensor product filtration, $F^0(M^*\otimes A_{crys}) = \sum_{i\geq 0} F^{-i}M^*\otimes I^{[i]}$, and $\varphi = \varphi_{M^*}\otimes\varphi_{A_{crys}}$. The right-hand side of the above formula is $\mathcal{T}^{FL}_{\acute{e}t}(\mathcal{M},\varphi)$. This proves (III.4.45).

As a consequence of (III.4.45), we conclude that the Bhatt–Lurie equivalence carries every object of $\mathrm{Rep}^{crys,f}_{G_K,[0,p-2]}(\mathbf{Z}_p)$ to a *vector bundle* over $W(k)^{syn}$.

Bibliography

[AMMN22] Benjamin Antieau, Akhil Mathew, Matthew Morrow, and Thomas Nikolaus. On the Beilinson fiber square. *Duke Math. J.*, 171(18):3707–3806, 2022.

[Aut19] The Stacks Project Authors. Stacks project, 2019.

[Bha12] Bhargav Bhatt. p-adic derived de rham cohomology, 2012.

[Bha23] Bhargav Bhatt. Prismatic f-gauges, 2023.

[Bha24] Bhargav Bhatt. Private communications, 2024.

[BHL17] Bhargav Bhatt and Daniel Halpern-Leistner. Tannaka duality revisited. *Adv. Math.*, 316:576–612, 2017.

[BK90] Spencer Bloch and Kazuya Kato. L-functions and Tamagawa numbers of motives. In *The Grothendieck Festschrift, Vol. I*, volume 86 of *Progr. Math.*, pages 333–400. Birkhäuser Boston, Boston, MA, 1990.

[BL22a] Bhargav Bhatt and Jacob Lurie. Absolute prismatic cohomology. *arXiv:2201.06120*, 2022.

[BL22b] Bhargav Bhatt and Jacob Lurie. The prismatization of p-adic formal schemes, 2022.

[BMS18] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. Integral p-adic Hodge theory. *Publications mathématiques de l'IHÉS*, 128(1):219–397, 2018.

[BMS19] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. Topological Hochschild homology and integral p-adic Hodge theory. *Publ. Math. Inst. Hautes Études Sci.*, 129:199–310, 2019.

[BO78a] Pierre Berthelot and Arthur Ogus. *Notes on crystalline cohomology*. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1978.

[BO78b] Pierre Berthelot and Arthur Ogus. Notes on crystalline cohomology. (mn-21), 1978.

[Bre99] Christophe Breuil. Une remarque sur les représentations locales p-adiques et les congruences entre formes modulaires de Hilbert. *Bull. Soc. Math. France*, 127(3):459–472, 1999.

[BS22] Bhargav Bhatt and Peter Scholze. Prisms and prismatic cohomology. *Ann. of Math. (2)*, 196(3):1135–1275, 2022.

[BS23] Bhargav Bhatt and Peter Scholze. Prismatic F-crystals and crystalline Galois representations. *Camb. J. Math.*, 11(2):507–562, 2023.

[CF00] Pierre Colmez and Jean-Marc Fontaine. Construction des représentations p-adiques semi-stables. *Invent. Math.*, 140(1):1–43, 2000.

[CL19] Bryden Cais and Tong Liu. Breuil-Kisin modules via crystalline cohomology. *Trans. Amer. Math. Soc.*, 371(2):1199–1230, 2019.

[ČS24] Kęstutis Česnavičius and Peter Scholze. Purity for flat cohomology. *Ann. of Math. (2)*, 199(1):51–180, 2024.

[DI87] Pierre Deligne and Luc Illusie. Relèvements modulo p^2 et décomposition du complexe de de Rham. *Invent. Math.*, 89(2):247–270, 1987.

[Dri04] Vladimir Drinfeld. DG quotients of DG categories. *J. Algebra*, 272(2):643–691, 2004.

[Dri21] Vladimir Drinfeld. On a notion of ring groupoid, 2021.

[Dri24] Vladimir Drinfeld. Prismatization. *Selecta Math. (N.S.)*, 30(3):Paper No. 49, 2024.

[Eke86] Torsten Ekedahl. *Diagonal complexes and F-gauge structures*. Travaux en Cours. Hermann, Paris, 1986.

[Fal89] Gerd Faltings. Crystalline cohomology and p-adic Galois representations. In *Algebraic Analysis, Geometry, and Number Theory*, pages 25–80. Johns Hopkins University Press, Baltimore, MD, 1989.

[FJ21] Jean-Marc Fontaine and Uwe Jannsen. Frobenius gauges and a new theory of p-torsion sheaves in characteristic p. *Doc. Math.*, 26:65–101, 2021.

[FL82] Jean-Marc Fontaine and Guy Laffaille. Construction de représentations p-adiques. *Ann. Sci. École Norm. Sup. (4)*, 15(4):547–608, 1982.

[Kat87] Kazuya Kato. On p-adic vanishing cycles (application of ideas of fontaine–messing). In *Algebraic Geometry, Sendai, 1985*, volume 10 of *Advanced Studies in Pure Mathematics*, pages 207–251. North-Holland, Amsterdam, 1987.

[LM24] Shizhang Li and Shubhodip Mondal. On endomorphisms of the de Rham cohomology functor. *Geom. Topol.*, 28(2):759–802, 2024.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[Lur17] Jacob Lurie. Higher algebra. *https://www.math.ias.edu/∼lurie/papers/HA.pdf*, 2017.

[Lur18a] Jacob Lurie. Kerodon. https://kerodon.net, 2018.

[Lur18b] Jacob Lurie. Spectral algebraic geometry. *https://www.math.ias.edu/∼lurie/papers/SAG-rootfile.pdf*, 2018.

[Maz73] Barry Mazur. Frobenius and the hodge filtration (estimates). *Annals of Mathematics*, 98(1):58–95, 1973.

[MM25] Akhil Matthew and Shubhodip Mondal. Affine stacks and derived rings, 2025.

[NS18] Thomas Nikolaus and Peter Scholze. On topological cyclic homology. *Acta Math.*, 221(2):203–409, 2018.

[Ogu21] Arthur Ogus. The saturated de Rham–Witt complex for schemes with toroidal singularities, 2021. Preprint.

[OV07] A. Ogus and V. Vologodsky. Nonabelian Hodge theory in characteristic p. *Publ. Math. Inst. Hautes Études Sci.*, 106:1–138, 2007.

[Pet23] Alexander Petrov. Non-decomposability of the de rham complex and non-semisimplicity of the sen operator, 2023.

[Pet25] Alexander Petrov. Decomposition of de rham complex for quasi-f-split varieties, 2025.

[Rak20] Arpon Raksit. Hochschild homology and the derived de rham complex revisited, 2020.

[Sch05] Daniel Kenneth Schepler. *Logarithmic nonabelian Hodge theory in characteristic p*. ProQuest LLC, Ann Arbor, MI, 2005. Thesis (Ph.D.)–University of California, Berkeley.

[Shi15] Atsushi Shiho. Notes on generalizations of local Ogus-Vologodsky correspondence. *J. Math. Sci. Univ. Tokyo*, 22(3):793–875, 2015.

[Sim97] Carlos Simpson. The Hodge filtration on nonabelian cohomology. In *Algebraic geometry—Santa Cruz 1995*, volume 62, Part 2 of *Proc. Sympos. Pure Math.*, pages 217–281. Amer. Math. Soc., Providence, RI, 1997.

[Sta20] The Stacks Project Authors. *Stacks Project*. https://stacks.math.columbia.edu, 2020.

[Win84] Jean-Pierre Wintenberger. Un scindage de la filtration de Hodge pour certaines variétés algébriques sur les corps locaux. *Ann. of Math. (2)*, 119(3):511–548, 1984.

[Xu19] Daxin Xu. Lifting the Cartier transform of Ogus-Vologodsky modulo p^n. *Mém. Soc. Math. Fr. (N.S.)*, 163:133, 2019.