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OF A HIGH-FREQUENCY-MODULATED ELECTRON BEAM.

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DYNAMIC STABILIZATION OF PLASMAS BY MEANS OF A
HIGH-FREQUENCY-MODULATED ELECTRON BEAM

by
Reza Aghevli

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Electrical Engineering)
in The University of Michigan
1972

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LIST OF SYMBOLS

A	An arbitrary constant.
A_i ($i = 0,6$)	Quantities defined by Eq. 3.110.
A_i ($i = 0,8$)	Quantities defined by Eqs. 2.87-2.95.
$A_n^{(p)}, B_n^{(p)}$	Quantities defined by Eqs. 4.30a-4.30n.
a	Radius of the perfectly conducting tube containing plasma.
B	An arbitrary constant.
$\underline{B}$	Magnetic flux density.
$\underline{B}_0$	The dc magnetic flux density.
b	The electron beam radius.
c	Velocity of light, 299.7925×10^6 m/s.
$\underline{\underline{D}}$	A differential operator defined by Eqs. 4.13-4.16.
D_1	The ac electric induction.
$D_1^{(0)}(\xi)$	Slowly varying part of D_1 .
$D_1^{(1)}(\xi)$	Fast varying part of D_1 .
D_x	A differential operator defined by Eq. 4.15.
$D(\omega, k)$	Dispersion function.
$D_{ }$	A differential operator defined by Eq. 4.16.
$D_{\perp}$	A differential operator defined by Eq. 4.14.
d	A normalized parameter defined by Eq. 3.71.
$d_n(\omega)$	Fourier component in space of $D_1^{(1)}(\xi)$.
E_1	First-order perturbation term for electric field intensity.
e	Electronic charge, 1.602×10^{-19} C.
e	Napier's constant, 2.718281828.

$F_m(kb, b/a)$	A function defined by Eq. 3.44.
$f(\underline{r}, \underline{v}, t)$	Distribution function.
$f_i(\underline{r}, \underline{v}, t)$	Distribution function of a particle of type i.
$f_n(\omega)$	Fourier components of the beam dielectric constant, $\epsilon_b(\omega, \xi)$.
$f_{n,eff}(\omega)$	A quantity defined by Eq. 3.34.
G_m	Geometric reduction factor for mth-harmonic mode.
g	The gravitational acceleration.
h	A parameter of smallness.
$\underline{\underline{I}}$	Unit matrix.
$I_{\underline{k}, \omega}$	Fourier spectrum of the potential correlation function.
I_m	The mth-order modified Bessel functions of the first kind.
i	An integer, dummy index.
J_m	The mth-order Bessel function of the first kind.
j	$\sqrt{-1}$.
K	The constant of separation.
K_m	The mth-order modified Bessel function of the second kind.
k	Propagation constant.
$\underline{k}$	Wave vector.
k_B	Boltzmann's constant, $1.380 \times 10^{-23} \text{ J/}^\circ\text{K}$.
$\underline{k}_m$	The eigenwave vectors defined by Eq. 2.24.
L	Periodicity in space of beam density profile.
l	A length defined by Eq. 2.31.
m	An integer.
m, m_e	Electronic mass, $9.109 \times 10^{-31} \text{ kg}$.
m_i	Ionic mass.

N_m	The mth-order Bessel function of the second kind.
n_o, n_{op}	Unperturbed plasma density.
n_{ob}	Unperturbed beam density.
$n_{ob}(\xi)$	Zeroth-order perturbation term of the total beam density.
n_{1b}, n_{1be}	First-order perturbation term for the beam electron density.
$n_{1b}(\xi)$	First-order perturbation term of the total beam density.
n_{1p}, n_{1pe}	First-order perturbation term for the plasma electron density.
n_{1pi}	First-order perturbation term for the plasma ion density.
$n_b(\xi, t)$	Total beam density.
p	An integer appearing in Eq. 4.29.
p_{1e}	First-order perturbation term for electron gas pressure.
p_{ms}	The sth root of the mth-order Bessel function of the first kind.
Q_j	The total normalized charge on the jth ring.
q_i	Charge of a particle of type i.
R	The Reynolds number.
R_{oj}	The outer radius of the jth ring.
R_{cr}	The critical Reynolds number.
R_{ij}	The inner radius of the jth ring.
R_j	The average position of the jth ring.
r	Radial coordinate in a cylindrical geometry.
r	Ratio of the thermal energy of plasma electrons to the beam dc energy.
$\underline{r}$	General spatial coordinate vector.
S	A parameter of smallness.

T_e	Electronic temperature.
t	Time.
$\underline{u}_0$	The dc velocity vector of the electron beam.
$\underline{v}$	General velocity vector.
$\underline{v}_{1b}, \underline{v}_{1be}$	First-order perturbation term for the beam electron velocity vector.
$\underline{v}_{1p}, \underline{v}_{1pe}$	First-order perturbation term for the plasma electron velocity vector.
$\underline{v}_{1pi}$	First-order perturbation term for the plasma ion velocity vector.
v_g	Group velocity.
v_{ph}	Phase velocity.
v_s	Ion-acoustic velocity.
v_{th}	Thermal velocity.
X	Forcing function in Eq. 2.13.
X_n	Fourier component in time of X .
x	A function of time in Eq. 2.13.
x	A normalized propagation constant, ku_0/ω_p .
x	Different normalized variables defined by Eqs. 3.69 and 3.108.
$Z(\zeta)$	A function of ζ .
z	A complex variable.
z	Axial coordinate in a cylindrical geometry.
α	Depth of modulation.
Γ	The rate of suppression.
$\Gamma(\rho)$	The gamma function of argument ρ .
γ	A quantity defined by Eq. 2.35.
γ	Compression ratio.

Δ	Duty cycle defined by Eq. 2.75.
δ	A quantity defined by Eq. 4.52.
δ_{ij}	Kronecker delta.
$\delta(x)$	The Dirac delta function of argument x.
δ_α	A quantity defined by Eq. 2.46.
δ_γ	A quantity defined by Eq. 2.37.
ϵ	Dielectric constant.
$\underline{\underline{\epsilon}}$	Dielectric tensor.
ϵ_0	Permittivity of free space, 8.854×10^{-12} F/m.
ϵ_I	Dielectric constant of Region I.
ϵ_{II}	Dielectric constant of Region II.
$\epsilon_b(\omega, \xi)$	Beam dielectric constant defined by Eq. 2.9.
$\underline{\underline{\epsilon}}_b(\omega, \xi)$	Dielectric tensor of the beam defined by Eq. 4.18.
ϵ_x	A component of $\underline{\underline{\epsilon}}/\epsilon_0$ defined by Eq. 1.8.
$\epsilon_{ }$	A component of $\underline{\underline{\epsilon}}/\epsilon_0$ defined by Eq. 1.9.
$\epsilon_\perp$	A component of $\underline{\underline{\epsilon}}/\epsilon_0$ defined by Eq. 1.7.
ξ	Longitudinal coordinate in the inertial frame of reference attached to the beam.
η	Different quantities defined by Eqs. 3.108, 3.113 and 4.51.
θ	Azimuthal coordinate in a cylindrical geometry.
$\underline{\underline{\kappa}}$	A tensor of rank two defined by Eqs. 4.7-4.10.
κ_x	A component of $\underline{\underline{\kappa}}$ defined by Eq. 4.9.
$\kappa_{ }$	A component of $\underline{\underline{\kappa}}$ defined by Eq. 4.10.
$\kappa_\perp$	A component of $\underline{\underline{\kappa}}$ defined by Eq. 4.8.
μ	A quantity defined by Eq. 2.47.

μ_v	Fourier component in space of $n_{ob}(\xi)/n_{ob}$.
$\xi_0(t)$	Solution of the differential equation as shown in Eq. 2.16.
ξ_n	Fourier component in time of the fast varying part of $\xi(t)$.
$\xi(t)$	An approximate solution defined by Eq. 2.15.
$\Phi_I(\underline{r}, t)$	The total electric potential of Region I.
$\Phi_{II}(\underline{r}, t)$	The total electric potential of Region II.
Φ_m	The magnitude of the eigenpotentials defined by Eq. 2.24.
$\Phi(\underline{r})$	Scalar potential function.
$\Phi(\underline{r}, t)$	The total scalar potential function.
$\varphi_n(\omega)$	Fourier component in space of $1/\epsilon_b(\omega, \xi)$.
Ω	Normalized frequency.
ω	General radian frequency.
ω_0	The frequency of modulation imposed on the beam.
ω_1	A quantity defined by Eq. 4.27.
ω_{1D}	Doppler-shifted frequency corresponding to ω_1 .
ω_D	Doppler-shifted frequency.
ω_M	A quantity defined by Eq. 4.33.
ω_c	Electron cyclotron frequency.
ω_m	The eigenfrequencies defined by Eq. 2.24.
ω_p	Total plasma frequency defined by Eq. 2.65.
ω_p, ω_{pe}	Plasma electron frequency.
$\omega_{p, eff}$	Effective plasma frequency defined by Eq. 2.81.
ω_r	Real part of ω .
ω_s	A quantity defined by Eq. 4.31.
ω_{th}	Threshold frequency defined by Eq. 2.49.

CHAPTER I. INTRODUCTION

1.1 Historical Background

1.1.1 Early Experimental and Theoretical Studies. Before plasma physics found its own niche in the realm of physics, the subject had been under study indirectly in conjunction with low-pressure arc-type discharges. Indeed the word "plasma" was first used by Langmuir¹ to designate "that portion of an arc-type discharge in which the density of ions and electrons are high but substantially equal."

Although the above definition lacks rigor and fails to establish a set of criteria for ascertaining the existence of a plasma, the attempt to explain the experimental findings gave rise to a simple theory of electronic and ionic oscillations in an ionized gas. This theory was developed by Tonks and Langmuir² who found that an electron gas of zero temperature is capable of maintaining local oscillations with a frequency called the "electron plasma frequency," or in brief the "plasma frequency," defined as follows:

$$\omega_p = \left(\frac{e^2 n_0}{\epsilon_0 m} \right)^{1/2}, \quad (1.1)$$

where n_0 is the macroscopic average of plasma electron density, ϵ_0 is the permittivity of free space, and e and m are the electronic charge and mass respectively.

They studied also the plasma ion oscillations in a plasma composed of cold ions and temperate electrons. They found that at short wavelengths,

i.e., at wavelengths much shorter than the Debye length, the ion oscillations are of the same type as described above for electrons. The frequency of these oscillations is given by the ion plasma frequency. However, at wavelengths much longer than the Debye length, the ion oscillations change their local nature, become of a longitudinally compressional type, and propagate at a velocity given by the following:

$$v_g = (k_B T_e / m_i)^{1/2} , \quad (1.2)$$

where v_g is the group velocity of the wave, k_B is the Boltzmann constant, T_e is the electronic temperature, and m_i is the ionic mass. The latter wave is called the ion-acoustic wave. The theory of Tonks and Langmuir was not very satisfactory; it was incomplete in its description of the physical phenomena occurring in ionized gases and it left many of the experimental findings unexplained.

The first serious and successful attempt at developing a theory of plasma oscillations was made in three consecutive papers by Bohm and Gross.³⁻⁵ In the first paper³ a theory of electronic oscillations of an unbounded plasma was developed, taking into account the effects of random thermal motions, but neglecting collisions. For organized steady-state oscillations of small enough amplitude such that the linear theory applies, they derived a dispersion relation for long wavelength oscillations of plasma. Furthermore they showed that organized oscillations at wavelengths shorter than the Debye length for the electron gas are not possible, a result that Landau⁶ had previously shown in a monumental work in 1946. The former authors outlined also their treatment of large-amplitude oscillations for which the equations are nonlinear.

In their second paper⁴ the theory of electron oscillations in an unbounded plasma was extended to take into account the effects of collisions and beams of particles having well defined ranges of velocities (principally drift velocity) which were introduced into the plasma at an electrode held at a fixed potential. They predicted coherent oscillations of plasma particles as well as instabilities in the system as a result of injecting an electron beam into the plasma. Based on the collective properties of the plasma oscillations, they suggested an explanation for the anomalous scattering of electron beams traversing a plasma, previously observed by Langmuir.⁷

In their third paper⁵ the theory was extended to take into account the effects of boundaries. They showed that electrons taking part in traveling plasma oscillations are reflected at an infinitesimally thin sheath in such a manner that they can participate in traveling oscillations in the reverse direction. This means that sheaths are capable of sustaining standing waves without loss. They discussed also the excitation of the plasma by electron beams and derived expressions for the energy transferred by a beam to oscillations of growing amplitude.

Bohm and Gross laid the foundation for a theory of plasma oscillations, although in their discussion of growing waves there seems to be an inherent ambiguity concerning the waves that grow in time and those that grow in space. The terms are introduced and used as if their meanings were intuitively obvious, whereas they had not been defined. Indeed in many branches of physics, where one deals with the propagation of waves through a complex medium, there existed a lack of clarity as to what exactly the terms "amplifying" and "evanescent" meant.

Among those who sought a criterion to distinguish between the amplifying and evanescent waves, two schools of thought are recognized. The two schools based their arguments on either energy or boundary condition considerations. The former school believes that the amplifying and evanescent waves are distinguished simply by determining whether or not the system in question is feeding energy into the wave.⁸⁻¹¹ The other school argued that the waveform which will be set up in the system by prescribed initial and/or boundary conditions should be used as a criterion to distinguish between the amplifying and evanescent waves.¹²⁻¹⁷

Sturrock¹⁸ criticized both schools and offered a general criterion as a basis for distinguishing between amplifying and evanescent waves, as well as between convective and nonconvective instabilities. According to Sturrock's criterion, "If a quasi-monochromatic spatially growing wave which is, by construction, a time-like packet is also a space-like packet, then the wave under consideration is amplifying; if, on the other hand, the time-like packet is not a space-like packet, then the wave is evanescent." By time- or space-like packets is meant wave packets that have finite extent in the time or space scales respectively.

Interchanging the roles of space and time leads to a statement of the criterion on the basis of which convective and nonconvective instabilities can be distinguished. The statement goes as follows: "If a wave function growing in time, which is composed of a narrow spectrum of wave numbers, and which is therefore a space-like packet, is also a time-like packet, then the instability represented by this wave is convective; if, on the other hand, the space-like packet is not a time-like packet, then the instability is nonconvective." A propagating wave is

said to be convectively unstable if a disturbance grows as it propagates away from the source point. On the other hand, the instability is said to be nonconvective if the disturbance grows both in amplitude and in extent while it embraces the point of origin. It should be clear that the types of instabilities discussed above are not inherent properties of the wave packets in question and depend upon the inertial frame of reference attached to the observer.

Briggs¹⁹ offered a different criterion for determining the type of instability involved in the system, the behavior of which is determined by a steady-state dispersion relation $D(\omega, k) = 0$. Essentially his method consists of a mapping of the lines of $\omega_r = \text{constant}$ into the complex k -plane with a mapping function defined as $D(\omega, k) = 0$. Based on the physical notion of the principle of causality, he constructs a criterion for distinguishing between amplifying and evanescent waves as well as convective and nonconvective (or absolute) instabilities. It should be apparent that such notions as energy and momentum conservation principles are implicitly involved in the derivation of the steady-state dispersion relation for the system under consideration.

Sudan²⁰ offered an instability criterion similar to that of Briggs in the sense that he used the mapping technique for distinguishing between the convective and absolute instabilities. He has, however, chosen the complex group velocity as the mapping function.

The necessity for a rigorous criterion for distinguishing between the convective and nonconvective instabilities had been felt long before. Although the kinetic power theorem due to Chu²¹ and the concepts of positive- and negative-energy-carrying waves have been quite useful in

giving a physical picture of the beam and circuit modes in traveling-wave tubes, they suffered from a lack of mathematical rigor which made them unfit for use as a general criterion applicable to the general problem of waves propagating through a complex medium. The rising interest in the properties of wave propagation through plasmas, and especially the device potential that plasma waveguides offered, when used in combination with electron beams, was a principal motive in the development of the Sturrock and Briggs criteria. This will be discussed further in the sections that follow.

1.1.2 Wave Propagation Through Plasmas. Plasma can be regarded as a medium capable of supporting electromagnetic as well as electrokinetic waves. The additional modes of propagation that are introduced in the system due to the presence of plasma will be designated as electrokinetic waves. (In plasma waveguides such modes are also referred to as plasma-guide modes.) The nomenclature has its origin in the fact that electrokinetic waves are composed of longitudinal and transverse electric waves whose associated energy is the kinetic energy of the particles comprising the plasma. For the electrokinetic waves that are slow, i.e., their phase velocity is much smaller than the velocity of light, an adequate treatment can be obtained using the quasi-static approximations.

The dispersion relation for the electrokinetic waves is derived from the set of Maxwell's equations along with the equations describing the motion of plasma particles. A usual formulation is the adoption of a hydrodynamic model for plasma, in which case the equations describing the motion of plasma particles are the first two moments of the Boltzmann equation. This formalism ignores the effect of micromechanisms such as

Landau or cyclotron damping. The above-mentioned equations, when linearized and solved for field quantities, lead directly to either a dispersion relation for the case of an infinite system or a differential equation for the electric field or the scalar potential subject to the appropriate boundary requirements for the case of a finite system. This analysis is quite similar to the derivation of Hahn²² and Ramo²³ of the ordinary space-charge waves associated with the drifting motion of an electron beam.

The electrokinetic waves and their propagation through plasmas have been the subject of extensive experimental and theoretical studies.²⁴⁻²⁸ A more systematic and comprehensive classification and study of the properties of waves propagating through plasmas are possible by means of the CMA (after P. C. Clemmow, R. F. Mullanly and W. P. Allis) diagram. This diagram is based on the fact that the basic shapes or topological genera of wave-normal surfaces do not change for wide variations of the plasma parameters. A wave-normal surface, or a normal surface in brief, is the locus of the tip of a vector having the direction of the propagation vector relative to the magnetic field and an amplitude equal to that of the phase velocity. The parameter space is defined as one in which the coordinates represent the parameters of the plasma. This space can be divided into subspaces within which the topological genera of the wave-normal surfaces do not change; that is, only by crossing the boundaries of such subspaces does the genus of the normal surfaces change. Thus the plasma waves can be labeled in accordance with the bounding volumes in which they fall.

For a cold, two-component plasma, the CMA diagram is two-dimensional, and the bounded volumes become bounded plane areas, while the bounding

surfaces become bounding lines. In this case the parameter space can be divided into thirteen regions and the properties of waves in each region can be examined. A detailed discussion and labeling of the waves, along with a discussion of cutoffs and resonances and the transitions of topological genera, are given by Allis et al.²⁹ and Stix,³⁰ and will not be repeated. Instead, a brief account will be given of the longitudinal space-charge waves and the cyclotron waves propagating through the plasma, due to their bearing on the present study.

In an infinite, cold, one-dimensional plasma, the small-amplitude disturbances oscillate at the plasma natural frequency and are localized in nature. This is because the nature of the restoring electric field is such that the displacement current due to separation of charges is counterbalanced by the convection current due to particle motion. However, if the electron gas is assumed to have a pressure, corresponding to an electronic mean thermal velocity, the disturbance will no longer remain localized but will propagate with a velocity near the thermal velocity for frequencies near the plasma frequency. These oscillations have been studied in considerable detail by Bohm and Gross³ and by Van Kampen.³¹

If the system has finite transverse dimension, even a cold, one-dimensional model of the plasma will be capable of supporting propagating waves. The properties of the waves supported by a waveguide filled with plasma, in the presence of a magnetic field, have been studied by Agdur³² and by Smullin and Chorney.³³ Agdur showed that the anisotropy, which is introduced into the plasma due to the presence of an axially directed dc magnetic field, has the effect of providing a coupling between TE and TM

modes. The latter authors studied the propagation properties of a uniform, cylindrical waveguide, either partially or completely filled with an ion plasma. They found passbands in the region of the ion cyclotron and plasma frequencies which are much smaller than the empty waveguide cutoff frequency.

Gould and Trivelpiece³⁴ considered an electron plasma and derived dispersion relations for the propagating waves in a cylindrical waveguide filled with plasma, both in the presence of an infinitely large, axially directed dc magnetic field and in the presence of an axially directed dc magnetic field of finite value. For the former case the dispersion relation derived for the system is as follows:

$$k^2 = \frac{\omega^2}{c^2} - \frac{(p_{ms}/a)^2}{1 - (\omega_p/\omega)^2}, \quad (1.3)$$

where k is the propagation constant, p_{ms} is the s th root of the m th-order Bessel function of the first kind, a is the radius of the perfectly conducting tube containing the plasma, and c is the velocity of light. The electric field is in the direction of propagation and its time and space variations are given as follows:

$$E_1(r, \theta, z, t) = (E_1)_{r=0} J_m \left(p_{ms} \frac{r}{a} \right) \exp[j(\omega t - m\theta - kz)] \quad (1.4)$$

Figure 1.1 shows the dispersion diagram for the system described above for the lowest-order plasma mode corresponding to $m = 0$ and $s = 1$. The diagram consists of two branches. The upper branch is the lowest-order transverse magnetic wave whose low-frequency cutoff is increased

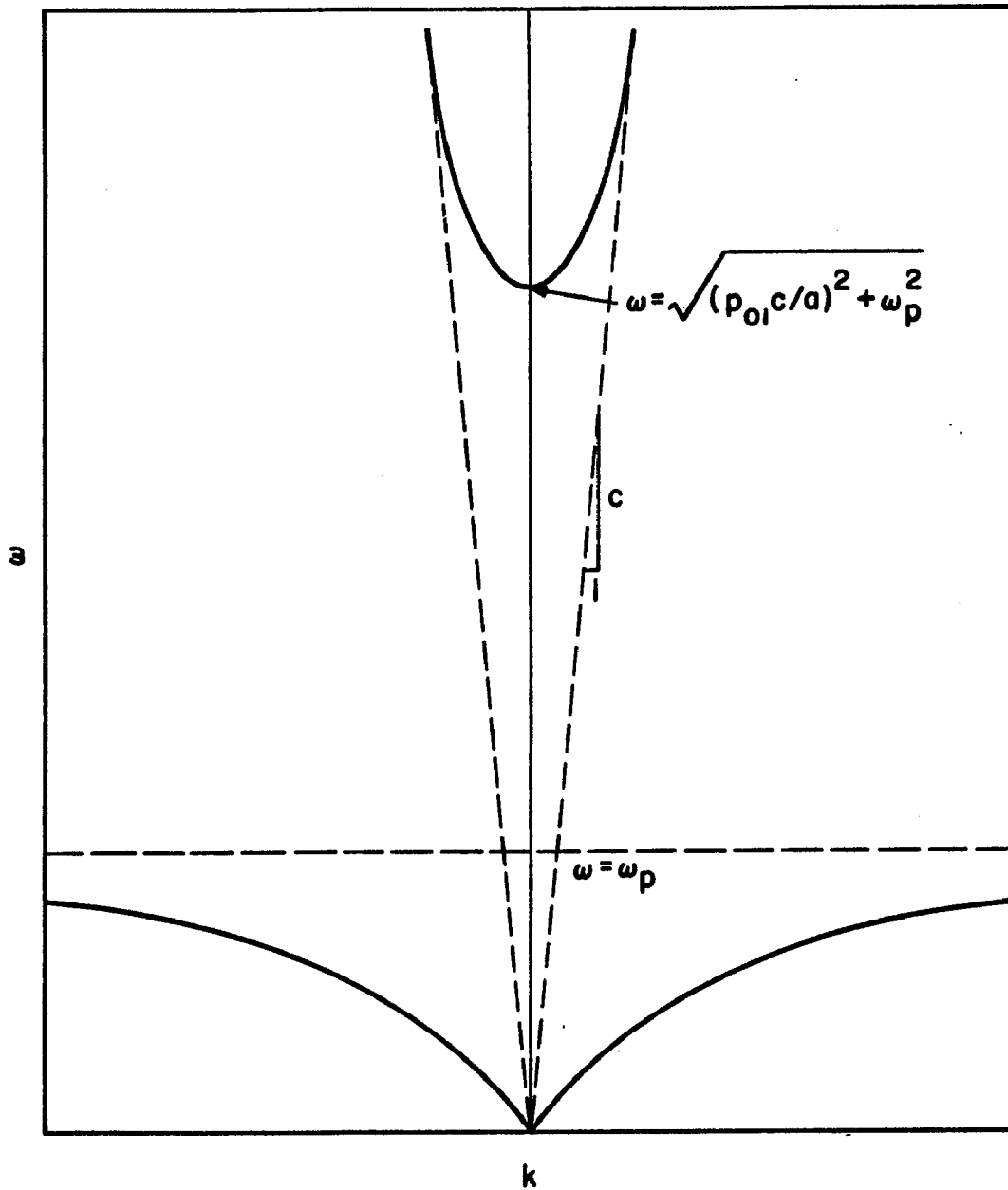


FIG. 1.1 DISPERSION DIAGRAM OF WAVES IN PLASMA-FILLED WAVEGUIDE WITH INFINITE AXIAL MAGNETIC FIELD.

by a factor of $[1 + (a\omega_p/cp_{o1})^2]^{1/2}$ due to the presence of plasma. Higher-order electromagnetic waves can, of course, be supported by the plasma waveguide. Their low-frequency cutoff is increased by a factor of $[1 + (a\omega_p/cp_{ms})^2]^{1/2}$. The lower branch represents the electrokinetic or the plasmaguide mode and has a high-frequency cutoff at the plasma frequency with no low-frequency cutoff. An interesting feature of this mode is that its high-frequency cutoff is independent of the configuration and depends only on the plasma density, whereas for electromagnetic waves the cutoff frequency is extremely sensitive to the geometry. The phase velocity of the electrokinetic wave is given by

$$v_{ph} = \frac{a\omega_p}{p_{ms}} \left(1 - \frac{\omega^2}{\omega_p^2}\right)^{1/2} . \quad (1.5)$$

The phase velocity, as seen above, can in principle be made arbitrarily small by reducing either the radius of the waveguide or the density of the plasma, or by exciting the higher-order plasmaguide modes. [All the higher-order modes ($m > 0$, $s > 1$) will propagate simultaneously with the lowest-order mode ($m = 0$, $s = 1$), if they are excited.] Therefore plasma-filled waveguides can be used as slow-wave structures in the language of electron-beam tubes. Hence, if plasma waveguides are used as slow-wave structures in electron-beam devices, the mechanical problem of making a very fine slow-wave structure is reduced to the maintenance of a plasma column of low density. Of course the not-so-trivial problem of coupling electromagnetic energy into and out of the plasma remains unresolved. This led many researchers to a rather extensive experimental and theoretical study of beam-plasma interactions which in turn led many

to believe that such a system may have device potential for generation and/or amplification of millimeter waves. The problem of beam-plasma interactions will be discussed in Section 1.1.3.

An applied static magnetic field introduces a fascinating complexity into the problem, due to its effect on the dynamics of the individual charged particles. When a charged particle is subject to an applied dc magnetic field, it assumes a preference for motion along the direction of the magnetic field lines. The $\underline{v} \times \underline{B}$ force has the effect of slowing down the particles moving across the magnetic field lines in addition to the bending of their trajectory in the plane normal to the $\underline{B}$ field, whereas the motion of particles moving along the magnetic field lines remains unchanged. The end result of such a preferred individual particle behavior, on the collective scale, is to make the medium anisotropic. The electric properties of such a medium can then be represented in terms of a dielectric tensor.

Assuming a hydrodynamic model for the plasma, the following dielectric tensor can be derived easily from the equation of motion for cold plasma electrons:

$$\underline{\underline{\epsilon}} = \epsilon_0 \begin{bmatrix} \epsilon_{\perp} & j\epsilon_x & 0 \\ -j\epsilon_x & \epsilon_{\perp} & 0 \\ 0 & 0 & \epsilon_{\parallel} \end{bmatrix}, \quad (1.6)$$

where ϵ_0 is the permittivity of free space and

$$\epsilon_{\perp} \equiv 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2}, \quad (1.7)$$

$$\epsilon_x \equiv -\frac{\omega_c}{\omega} \frac{\omega_p^2}{\omega^2 - \omega_c^2}, \quad (1.8)$$

$$\epsilon_{||} \equiv 1 - \frac{\omega_p^2}{\omega^2}, \quad (1.9)$$

where ω_c is the electron cyclotron frequency defined as

$$\omega_c \equiv \frac{eB_0}{m}. \quad (1.10)$$

For a completely filled waveguide, the following dispersion relation results from the solution of the differential equation for the scalar potential function subject to the appropriate boundary conditions at the walls of the perfectly conducting cylindrical tube:

$$k^2 = -\frac{\omega^2(\omega^2 - \omega_p^2 - \omega_c^2)}{(\omega^2 - \omega_p^2)(\omega^2 - \omega_c^2)} \left(\frac{p_{ms}}{a} \right)^2. \quad (1.11)$$

Figure 1.2 shows the dispersion diagram of the system described above for the lowest-order plasma mode corresponding to $m = 0$ and $s = 1$. Note that, because of the quasi-static assumption involved in the derivation of the dispersion equation (1.11), the electromagnetic branch is absent from the diagram. Examination of Fig. 1.2 reveals that, when the value of the axially directed dc magnetic field is made finite, a backward wave is excited in addition to the forward space-charge wave discussed previously. The backward wave corresponds to the electron cyclotron waves. These are transverse electric modes of propagation with circular polarization, whereas the forward wave corresponds to the electron space-charge modes of

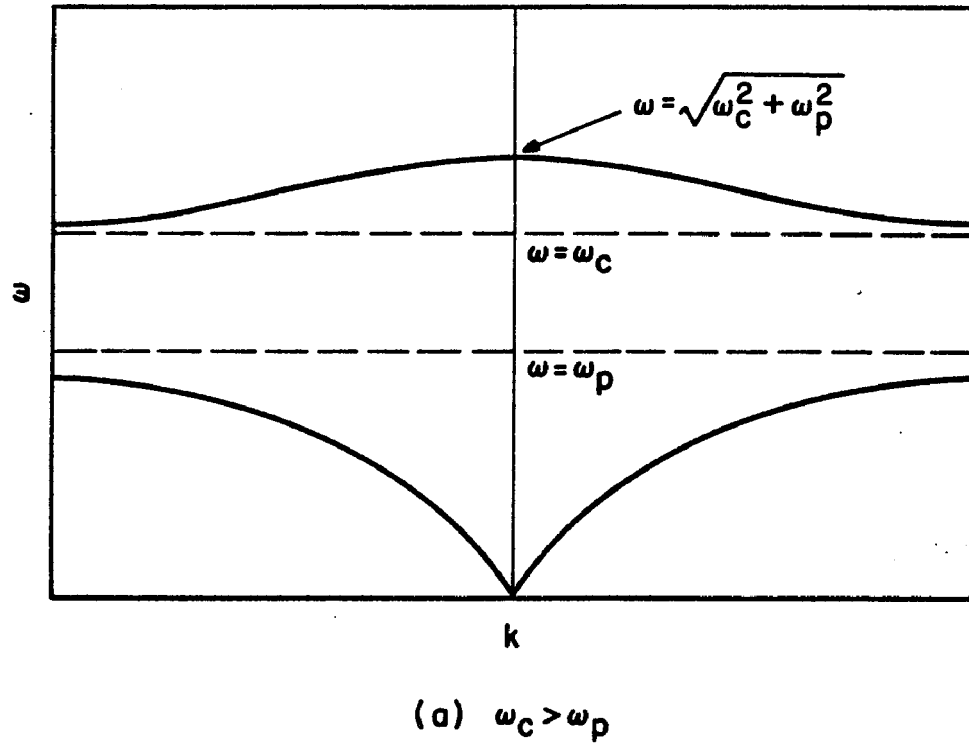
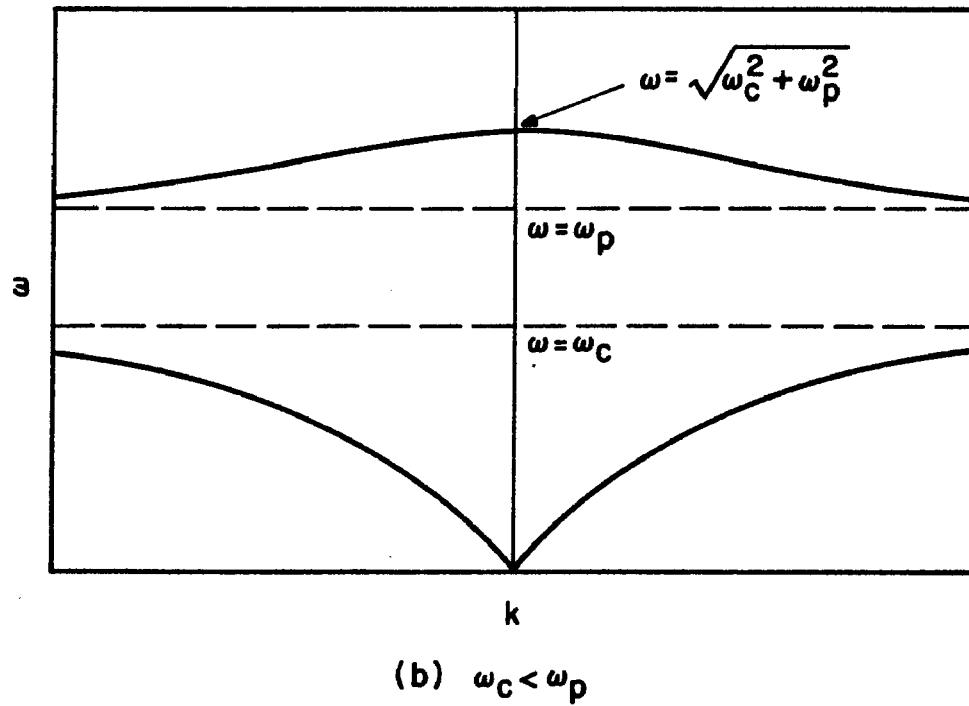


FIG. 1.2 DISPERSION DIAGRAM OF WAVES IN PLASMA-FILLED WAVEGUIDES WITH AN AXIAL H_c MAGNETIC FIELD OF FINITE VALUE.

propagation that are longitudinal electric waves. The high-frequency cutoff for the forward wave is independent of the dimensions of the tube and is given by ω_p or ω_c , whichever is smaller. The higher cutoff frequency for the backward wave is called the upper hybrid resonant frequency. The lower cutoff frequency for the backward wave is given by ω_p or ω_c , whichever has the higher value. Trivelpiece³⁵ has offered a physical explanation for the existence of the backward wave in terms of an equivalent-circuit representation for the system. The low-frequency phase velocity can be easily obtained from the dispersion relation as follows:

$$v_{ph} = \frac{a}{p_{ms}} \frac{\omega_p \omega_c}{(\omega_p^2 + \omega_c^2)^{1/2}}, \quad (1.12)$$

where $\omega^2 \ll \omega_p^2, \omega_c^2$.

In the absence of an axially directed dc magnetic field, propagation of waves is not possible for the completely filled cylindrical waveguide. However, if the waveguide is only partially filled, propagation becomes possible. This is because, for low values of magnetic field such that $\omega_c < \omega_p$, the high-frequency cutoff for the circularly symmetric forward wave in a partially filled waveguide is no longer given by the cyclotron frequency. The physical principle behind the existence of these modes, known as surface waves, lies in charge accumulation on the surface of the plasma column, which in turn causes a "rippling" of the surface and makes the plasma column suitable for supporting propagating waves. For $\omega_c < \omega_p$, if a plot of the potential is made as a function of the radius for different values of the frequency, a discontinuity appearing in the first derivative of the potential is observed as ω passes through ω_c from

lower values. In other words, the mechanism of propagation passes smoothly from a state, involving no charge accumulation at the surface of the plasma column, ($\omega < \omega_c$), to one which involves no charge accumulation inside the plasma, ($\omega > \omega_c$). A detailed discussion of these modes, along with the diagram for the scalar potential is given by Trivelpiece.³⁵

1.1.3 Characteristics of Beam-Plasma Interactions. As discussed earlier, the properties of plasmas, particularly their potential for use as slow-wave structures, led many researchers to extensive experimental and theoretical investigations of beam-plasma systems, even though the difficulties of energy coupling remained an impeding obstacle to the development of the actual device. The advantage of using a beam-plasma system is that the mechanical difficulties of constructing a very fine slow-wave structure to support millimeter and possibly submillimeter waves will be traded for the problem of maintaining a plasma of low density which is much easier to realize. In an electron beam-plasma system either an amplifier or an oscillator can be realized, depending upon whether there is a convective or nonconvective instability respectively.

Although the beam-plasma studies did not culminate in the realization of a competitive microwave device for the purposes of generation, amplification, or oscillation of microwaves, they revealed many interesting and subtle points concerning the nature of wave coupling and energy transfer for wave-particle and wave-wave interactions. Such notions were further developed and made use of in different mechanisms of plasma generation, wave excitation inside a plasma, plasma heating, and particle acceleration, which have found their application in the more promising field of controlled fusion.

In the following, a review of some of the theoretical and experimental work in the areas of beam-plasma and two-stream interactions is made with reference to both the microwave and the fusion aspects of the problem.

The first theoretical investigation of the excitation of the longitudinal oscillations by a beam of charged particles was carried out by Bohm and Gross.⁴ Their analysis led to the discovery that beam instabilities existed if the drift velocity of the beam exceeded the thermal velocity of plasma electrons. The physical reason for this effect can be understood on the basis of the fact that at high drift velocities the coupling of beam and plasma space-charge waves becomes the dominant process of interaction rather than thermalization of the beam electrons by plasma electrons. Similar phenomena arise in the interaction of two or more electron beams. For example, in the case of two electron beams which have drifted Maxwellian distributions, instabilities arise when the following condition is satisfied:

$$|u_{o1} - u_{o2}| > v_{th1} + v_{th2} , \quad (1.13)$$

where u_{oi} and v_{thi} are the drift and thermal velocities of the i th beam respectively.

The first successful experiment to achieve microscopic amplification in a beam-plasma system was carried out by Boyd et al.³⁶ Their experimental device demonstrated the existence of growing waves in plasmas and their measurement of the gain was in good agreement with the predictions of their analysis. They reported a net amplification of 10 dB/cm between input and output levels. Their apparatus was immersed in an axially

directed dc magnetic field large enough to justify a one-dimensional analysis.

Allen and Kino³⁷ performed a similar experiment using helix coupling to a beam traversing a magnetically confined synthetic cesium plasma. They obtained good agreement with their theoretical analysis of the gain mechanism, taking the magnetic field and the finite dimensions of the plasma and the beam into account. Typical gain figures were approximately 5 to 10 dB/cm at 1000 MHz and 10 to 15 dB/cm at 3000 MHz.

In a series of theoretical studies,³⁸⁻⁴⁰ the problem of beam-plasma interaction was investigated in considerable detail. In the first paper,³⁸ a dispersion relation was derived for a system composed of many beams of charged particles in a cylindrical waveguide with an infinitely large axially directed dc magnetic field. Frequency ranges for which the growing waves occur were found and the dispersion relation was solved numerically for the case of a waveguide filled with a beam and a plasma. Numerical examples were given of the dispersion diagram, both for the cases with and without collisions of the plasma electrons. This fairly complete analysis took into account the effects of the ac magnetic field.

In a second paper,³⁹ the authors considered the same problem but for a magnetic field of finite value. They assumed an arbitrary direction for the wave vector and derived a general dispersion relation for a system of many particles on the basis of which the dispersion relation for a beam-plasma system was then constructed. They showed further that the problem, when considered for a cylindrical waveguide, is mathematically equivalent to that of wave propagation through an anisotropic medium in which case the TE and TM modes of propagation are coupled. The problem

of wave propagation through a generalized anisotropic medium had been studied by Van Trier⁴¹ long before.

In a third paper,⁴⁰ a dispersion relation was found for a beam-plasma system of cylindrical geometry immersed in an axially directed dc magnetic field of finite value. The dispersion relation was then solved through the introduction of approximate boundary conditions. It was shown that the problem was mathematically equivalent to that of the propagation of plane waves at an angle to the beam in an unbounded beam-plasma system. The three papers discussed above are characterized by their generality in terms of the formal treatment of the general problem of beam-plasma interaction; yet, partly for that reason, they fail to give a comprehensive understanding of the physics underlying the analysis.

Singhaus⁴² discussed the electrostatic instability of an infinite electron beam penetrating an infinite plasma. The electron beam is assumed to have a Gaussian velocity distribution and have collisions with plasma particles. He concluded that at the high temperature limit, the plasma collisions quench the electrostatic instability, whereas at lower temperatures the collisions tend to enhance the instability. It should be noted that collisions in general reduce the growth rate and increase the bandwidth. As a result regions can be found in which either there is no instability or the growth rate is lower when collisions are absent. This should not lead to the erroneous conclusion that collisions induce instability. The enhancement of the instability in a certain frequency region is simply a result of having a wider bandwidth due to collisions. However for the temperate case the collisional enhancement

is due to another effect. The results of Singhaus' analysis can be explained as follows. In a collisionless plasma if the beam temperature is lowered Landau damping would be enhanced. In the low-temperature limit the dominant process then is the Landau damping, and collisions would make particle trapping difficult thus suppressing Landau damping and enhancing the growth rate. In the high-temperature limit Landau damping is negligible and collisions have the usual effect of suppressing the growth rate.

Trivelpiece³⁵ discussed the beam-plasma interaction and solved the obtained dispersion relation using Pierce's method⁴³ which gives the rate of growth in terms of the interaction impedance. Maxum and Trivelpiece⁴⁴ studied two-stream cyclotron- and plasma-wave interactions and showed that in the case of equal and opposite drift velocities there exists an instability at approximately one half the cyclotron frequency due to a coupling between the slow space-charge wave of one stream and the backward cyclotron wave of the other stream. They also carried out experiments and obtained results in agreement with their theory.

Crawford⁴⁵ discussed the effect of plasma electron temperature by adding a pressure gradient term to the hydrodynamic equation of motion for the plasma electrons. He also used Hahn's approximation²² to investigate the case when the beam boundary is perturbed.

Simpson and Dunn⁴⁶ made a complete and comprehensive study of beam-plasma interaction for finite transverse dimensions and finite magnetic field values. They developed an iteration technique to solve the derived dispersion relation numerically in its exact form. Later Self^{47,48} in two consecutive papers reported his theoretical and experimental studies of beam-plasma interactions. He offered a general

method of treating the finite-length system without the necessity of treating specific boundary conditions. His experimental results agreed well with the theory except for the growth rate being somewhat smaller and spread over a wider frequency band than theory indicates. This was attributed to radial inhomogeneity of plasma.

The above analyses dealt with the small-signal case and ignored the nonlinear effects. Rowe⁴⁹ followed a nonlinear analysis to develop a general theory of the interaction of electromagnetic waves with electron beams. His theory can be applied to electron-beam devices, such as traveling-wave tubes, backward-wave oscillators and klystrons, as well as to multistream or beam-plasma interactions. Apel⁵⁰ carried out experiments in order to observe the effects of nonlinearity in a beam-plasma system during a well developed instability. His experimental findings included the vanishing of the spatial growth rate, generation of harmonics of the fundamental interaction frequency, and a broadening of the emission peaks into a continuous spectrum which is in agreement with the theories of plasma turbulence.⁵¹

1.2 Statement of the Problem and Previous Studies in the Area

The present study deals with the dynamic stabilization of plasma driven by an RF electric field. The RF field is excited in the system through injection of an electron beam of longitudinally nonuniform density. By a beam of longitudinally nonuniform density is meant an electron beam for which the density is a function (here periodic) of the spatial coordinate (along which the beam particles are drifting) measured in the inertial frame of reference attached to the beam.

Therefore the space-time variation of the density of such a beam, written in the inertial frame of reference attached to the beam, is given in the following form:

$$n_b(\xi, t) = n_{ob}(\xi) + n_{1b}(\xi) e^{j\omega t}, \quad (1.14)$$

where ξ is the longitudinal coordinate measured in the moving frame and $n_{ob}(\xi)$ and $n_{1b}(\xi)$ are the zeroth and first-order perturbation terms of the total beam density, $n_b(\xi, t)$, respectively.

Experimental studies^{52,53} have shown that the oscillations excited in the plasma, due to the interaction of such a beam as described above with the plasma, will have considerably smaller growth rates than those due to the interaction of a longitudinally uniform beam with the plasma, so long as the spatial wavelength of the beam inhomogeneity, measured in the moving frame of reference, is considerably smaller than the wavelength of the excited oscillations.

From a wider viewpoint the RF excitation can also be used to suppress the so-called drift-wave instabilities, which are micro-instabilities associated with particle drifts in an inhomogeneous plasma. Recently there has been a growing interest in devising means to suppress the above-mentioned instabilities because of the enhanced diffusion that they cause of plasma across the magnetic field. Experiments⁵⁴⁻⁵⁶ have been carried out in this area to confirm the results of the well established theories.⁵⁷⁻⁵⁹

However, a detailed theoretical examination of the RF stabilization in a beam-plasma system has been lacking. Theoretical papers on the

subject have been few in number and limited in scope.⁶⁰⁻⁶³ The present study is an attempt to study the effects of a beam-produced RF excitation in suppressing the convective instabilities that generally arise in a beam-plasma system, and to realize under what conditions and in what frequency ranges the convective instability is best brought under control.

1.3 Applications

Control of convective instabilities is of interest in at least two areas. The following briefly discusses these two areas.

1.3.1 Heating of the Plasma by Passing an Electron Beam Through It. It has been shown theoretically^{3,4} and experimentally^{53,64} that an electron beam passing through a plasma can excite oscillations and cause instabilities. Instabilities arise as a result of the energy lost by the beam to the plasma. This energy loss is of the order of 10^3 to 10^4 eV/cm for a beam in the density range of 10^7 to 10^8 cm⁻³. Such instabilities can be used for many applications, with plasma heating being one of the most important, from the controlled thermonuclear reaction viewpoint. There exists sufficient experimental evidence⁶⁵ to demonstrate that efficient plasma-electron heating can be achieved in a beam-plasma-type device. However, it is self-evident that the process of energy transfer from electrons to ions, for which the binary collisions are mainly responsible, is very inefficient due to the very large proton to electron mass ratio; therefore, the plasma will be lost to the walls, due to diffusion processes, before the heat transfer from electrons to ions takes place, hence the necessity of direct plasma-ion heating.

One possible and effective way of heating the plasma ions is to employ the ion-cyclotron resonance for which the anomalous doppler effect

is responsible.^{66,67} For efficient heating of the plasma attempts must be made to minimize the other ways by which the beam loses energy to the plasma by suppressing instabilities of all kinds other than the ion-cyclotron resonance. Convective instabilities, which arise as a result of the interaction of a beam-excited, longitudinal wave with the plasma, are among the most dominant instabilities in the entire frequency spectrum. Suppression of these instabilities will increase the efficiency of ion heating by minimizing the rate of energy transfer from the beam to the plasma electrons.

The above argument can be understood on the basis of the following physical picture. Kruer et al.⁶⁸ have shown that heating of the plasma as a result of beam-type instabilities is highly efficient as the result of ion density fluctuations. Such fluctuations in ion density arise after the nonlinear regime is established. How soon the nonlinear regime is established in plasma has to do with the growth rates of different kinds of absolute instabilities present in the system. Before the plasma goes into the nonlinear regime, convective instabilities enhance the coherent oscillations of plasma particles. The higher the growth rates of convective instabilities, the higher the coherency of the collective motion of plasma particles. Coherent oscillations of plasma particles lead to strong radiation of electromagnetic energy from the plasma, which is a deterrent to plasma heating. With this picture in mind it can be asserted that smaller growth rates of convective instabilities improve the efficiency of heating when the process of heating is taken to include both linear and nonlinear modes of operation.

1.3.2 Particle Acceleration in Plasma Waveguides. Fainberg⁶⁹ has shown that plasma waveguides and compensated electron and ion beams can be used as accelerating systems in linear accelerators. The advantage of using plasma waveguides rather than the metallic ones lies in the fact that for metallic waveguides the RF energy losses are much greater. For low plasma densities (in the range of 10^9 to 10^{13} cm^{-3}) the RF energy lost by a charged particle when it passes through the plasma is proportional to the plasma density and is of the order of 10^{-5} to 10^{-6} eV/cm. Such losses are insignificant for beams whose energies lie in the range of 10^2 to 10^3 eV.

There are, however, a number of serious difficulties which have to be overcome. One is the appearance of nonlinear effects in the plasma which results in making the parameters of the plasma dependent upon the accelerating electromagnetic wave propagating through it. Thus plasma parameters should be chosen such that the nonlinear effects are suppressed. It is then necessary that the energy received by a plasma electron over one wavelength be much less than the electric energy stored in the plasma when regarded as a dielectric. Another difficulty has to do with the disturbance of the plasma stability as a result of the passage of the electron beam. It is this particular problem that will be considered. It should be noted, however, that the maintenance of the stability of the plasma in accelerators is a much less serious problem than it is in controlled thermonuclear reaction devices. This is because much higher density values are being dealt with in CTR devices and thus higher growth rates as well as higher diffusion rates are being confronted. Moreover in linear accelerators the difficult task of heating the plasma ions is removed.

1.4 Outline of the Present Study

This investigation deals with the problem of beam-plasma interaction for the case of a longitudinally inhomogeneous beam. Two types of inhomogeneity are considered for the beam density--rectangular and sinusoidal. For each type of inhomogeneity the dispersion relation is solved with the depth of modulation taken as a parameter. The stabilizing effects, due to the inhomogeneity, are investigated for the entire region in which the unstable waves exist in the case of a uniform beam interacting with a plasma.

In Chapter II, a one-dimensional analysis of the problem is developed for the cases of a cold, as well as a warm, infinite plasma. The method of averaging, which will be used throughout the present investigation, is explained and used to solve the differential equation derived for the electric field. The results obtained are discussed at the end of the chapter together with a physical explanation for the stabilizing effect.

Chapter III considers the effects of finite transverse geometry for the case of no externally applied dc magnetic field. As in the previous cases, the fluid model is adopted, both for the beam and the plasma. The dispersion relation derived is solved explicitly using the weak-beam approximation. The results, which are in contrast with those of the previous case, are discussed at the end of the chapter and a physical explanation is offered based on the notions of individual particle dynamics.

Chapter IV is devoted to the two-dimensional analysis of the problem of a cold, inhomogeneous beam interacting with a cold plasma in

the presence of an axially directed dc magnetic field of finite value. An implicit dispersion relation is derived through the solution of the differential equation for the scalar potential and the appropriate boundary requirements. No attempt is made to solve the dispersion relation numerically. The numerical results, however, for moderate values of the applied static magnetic field, must show suppression rates which are lower than those computed for the one-dimensional case. Based on the above fact, arguments are presented to show that the error bound, due to the method of averaging, exceeds the expected rate of suppression except for the case of a very large applied dc magnetic field ($\omega_c \gg \omega_p$), in which case a one-dimensional analysis is applicable.

Chapter V presents a summary of the results as well as the major conclusions. Suggestions are also made for further study.

CHAPTER II. ONE-DIMENSIONAL FORMULATION OF THE INHOMOGENEOUS BEAM-PLASMA INTERACTION

2.1 Introduction

When an electron beam passes through a plasma, amplification of an RF signal impressed on the beam occurs providing that certain conditions are satisfied. In this chapter the effects of a longitudinal inhomogeneity in the beam density on the interaction, in general, and on the spatial growth rate, in particular, are studied. This study is restricted to a case in which the beam and the plasma particles can move only along the longitudinal direction. Physically this corresponds to the case where an external dc magnetic field is applied in the longitudinal direction. The value of the magnetic field is large enough to make the Larmor radius for particles negligibly small compared with their longitudinal displacement due to space-charge restoring forces. Furthermore the system is assumed to have no finite transverse boundaries. In a real physical situation, especially in laboratory systems, there are always finite boundaries where some of the field lines terminate. The assumption of an infinite system in effect requires that the density of the field lines terminating on the boundary be negligibly smaller than the longitudinal field due to the space charge.

Plasmas, as well as electron beams, may often be described mathematically in terms of distribution functions, $f(\underline{r}, \underline{v}, t)$, for each type of particle comprising the system. In principle, these distribution functions can be found by solving a system of Boltzmann equations, taking into account the binary collisions and the actions of electromagnetic

fields on the particles. The fields must satisfy Maxwell's equations and be self-consistent in terms of relating to the macroscopic charge and current densities represented by $\sum_i q_i \int f_i(\underline{r}, \underline{v}, t) d\underline{v}$ and $\sum_i q_i \int \underline{v} f_i(\underline{r}, \underline{v}, t) d\underline{v}$, respectively, where i denotes the type of particle which has an electric charge, q_i , and a distribution function, $f_i(\underline{r}, \underline{v}, t)$. Such a description is mathematically cumbersome and in certain cases even unnecessary. Simplified models can often be employed if there is an interest in the study of the dynamics of the system in the laminar and weakly turbulent regimes. The nature of long-range Coulomb forces in ionized gases allows the adoption of a fluid model for plasmas if the characteristic length of the system is considerably greater than the mean free path. Formally, the equations of a magnetic fluid consist of the first few moments of the distribution functions for the species and Maxwell's equations for the self-consistent fields. In the present analysis the fluid model of the plasma is adopted in order to simplify the analysis.

2.2 Mathematical Description of the System

2.2.1 Assumptions. The following assumptions are made in the course of formulating the problem:

1. The plasma is cold, stationary and uniform with infinitely heavy ions which serve as a neutralizing background.
2. The electron beam is cold and drifting with a velocity, $\underline{u}_0$, parallel to the applied dc magnetic field, $\underline{B}_0 = \infty$.
3. The perturbations are assumed to be of small magnitude so as to make a linear analysis possible.

4. Nonrelativistic mechanics applies.
5. The quasi-static assumption is appropriate.

2.2.2 Field and Moment Equations. On the basis of the preceding assumptions, hydrodynamic equations of motion are written for beam and plasma particles. The beam and plasma space-charge waves are coupled through an ac electric field. This coupling is expressed mathematically by Gauss' law.

The linearized equations of motion, the continuity equation and Gauss' law, when written in an inertial frame of reference attached to the beam, have the following form:

$$\frac{\partial v_{1b}}{\partial t} = - \frac{e}{m} E_1 , \quad (2.1)$$

$$\frac{\partial n_{1b}}{\partial t} + \frac{\partial}{\partial \xi} (n_{0b} v_{1b}) = 0 , \quad (2.2)$$

$$\frac{\partial v_{1p}}{\partial t} - u_0 \frac{\partial}{\partial \xi} v_{1p} = - \frac{e}{m} E_1 , \quad (2.3)$$

$$\frac{\partial n_{1p}}{\partial t} + \frac{\partial}{\partial \xi} (n_{0p} v_{1p} - u_0 n_{1p}) = 0 \quad (2.4)$$

and

$$\frac{\partial}{\partial \xi} E_1 = - \frac{e}{\epsilon_0} (n_{1b} + n_{1p}) , \quad (2.5)$$

where the subscript 1 denotes the first-order perturbation quantities, subscripts b and p refer to the beam and the plasma parameters, respectively, and ξ is the longitudinal coordinate in the moving frame

of reference. A steady-state solution to these equations can be sought by making all the first-order quantities assume an $\exp(j\omega t)$ variation in time. If this consideration is made, the combination of Eqs. 2.1 and 2.2 yields

$$n_{1b} = -\frac{d}{d\xi} \left(\frac{en_{ob}(\xi)}{m\omega^2} E_1 \right) . \quad (2.6)$$

Similarly, a combination of Eqs. 2.3 and 2.4 results in

$$\left(j\omega - u_o \frac{d}{d\xi} \right)^2 n_{1p} = \frac{en_{op}}{m} \frac{d}{d\xi} E_1 . \quad (2.7)$$

Now, the differential operator $[j\omega - u_o(d/d\xi)]^2$ can be operated on both sides of Eqs. 2.5 and 2.6 and the results obtained can be combined with Eq. 2.7 to yield the following differential equation for the longitudinal ac electric field:

$$\left(j\omega - u_o \frac{d}{d\xi} \right)^2 \frac{d}{d\xi} \left(\epsilon_b(\omega, \xi) E_1 \right) + \omega_p^2 \frac{d}{d\xi} E_1 = 0 , \quad (2.8)$$

where $\epsilon_b(\omega, \xi)$ is the beam dielectric constant defined as

$$\epsilon_b(\omega, \xi) \equiv 1 - \frac{e^2 n_{ob}(\xi)}{m\epsilon_o \omega^2} . \quad (2.9)$$

In the absence of an externally applied driving force, Eq. 2.8 can be integrated with respect to ξ with the constant of integration chosen to be zero as follows:

$$\left(j\omega - u_o \frac{d}{d\xi} \right)^2 \left(\epsilon_b(\omega, \xi) E_1 \right) + \frac{\omega_p^2}{u_o^2} E_1 = 0 . \quad (2.10)$$

A transformation of the dependent variable can be made to standardize the above differential equation. Hence if $Z(\xi)$ is defined as

$$Z(\xi) \equiv \epsilon_b(\omega, \xi) E_1(\xi) \exp \left(-j \frac{\omega}{u_o} \xi \right) , \quad (2.11)$$

a direct substitution into Eq. 2.10 yields the following differential equation for $Z(\xi)$:

$$\frac{d^2}{d\xi^2} Z(\xi) + \frac{\omega_p^2}{u_o^2 \epsilon_b(\omega, \xi)} Z(\xi) = 0 . \quad (2.12)$$

A differential equation of this form has been discussed in detail by Kaplan⁷⁰ and Cesari⁷¹ who gave the exact solution of the equation for an arbitrary form of $\epsilon_b(\omega, \xi)$. A special case of particular interest is when $\epsilon_b(\omega, \xi)$ is periodic, in which case Eq. 2.12 is called Hill's equation. For this case, Kaplan and Cesari offered stability criteria to explore the ranges in which the solution is unstable. These methods have the advantage of being exact, but suffer from a lack of applicability to equations of orders higher than two. The next section thus presents an approximate method of solving such equations under certain restrictions imposed upon the form of $\epsilon_b(\omega, \xi)$, aside from its being periodic.

2.3 The Method of Averaging

2.3.1 Introductory Remarks. In oscillatory systems when either the amplitude of the oscillations becomes large, or when there exists a parametric excitation, the differential equation describing the motion becomes either nonlinear or parametric, respectively. In such cases the more conventional methods of solving such equations generally fail to give a clear physical picture of the solution, if the solution can be obtained at all. The most effective tool for analyzing nonlinear oscillations, with or without parametric excitation, is the method of asymptotic expansion in powers of a small parameter. This method can offer an explanation for the detailed features of the oscillatory processes. The method of asymptotic expansion was used by Krylov and Bogoliubov⁷² in a study of nonlinear mechanics. Later Bogoliubov and Mitropolsky,⁷³ as well as Feshchenko et al.,⁷⁴ illustrated the use of this method in a number of problems. Krylov and Bogoliubov showed that a nonlinear differential equation containing a small parameter, which corresponds to a parametric excitation, can be reduced, for most physical situations, to a standard form expressed as

$$\frac{d^2x}{dt^2} + \omega^2 x = hX\left(x, \frac{dx}{dt}, t\right), \quad (2.13)$$

where h is a parameter of smallness. The method of averaging applies if the force function, $hX(x, dx/dt, t)$, is oscillatory with a frequency of oscillation much greater than that of the original system. In other words if

$$X\left(x, \frac{dx}{dt}, t\right) = \sum_n X_n\left(x, \frac{dx}{dt}\right) e^{jnvt}, \quad (2.14)$$

the condition $\nu \gg \omega$ must be satisfied in order to warrant the employment of the method of averaging. The method consists of superimposing on the large-amplitude, slowly varying solution of the time-averaged equation a small-amplitude, fast-varying term whose coefficients are to be determined by a direct substitution into the original equation. Hence the solution of Eq. 2.13 can be expressed as

$$x(t) \cong \xi(t) = \xi_0(t) + \sum_{n \neq 0} \xi_n e^{jnvt}, \quad (2.15)$$

where $\xi_0(t)$ is the solution to the following equation:

$$\frac{d^2 \xi_0}{dt^2} + \omega^2 \xi_0 = hX_0\left(\xi_0, \frac{d\xi_0}{dt}\right), \quad (2.16)$$

and the ξ_n 's are to be determined by substitution of Eq. 2.15 into Eq. 2.13. It can be shown⁷³ that under general conditions the value of $|x(t) - \xi(t)|$ can be made arbitrarily small, for a sufficiently small value of h , for as large an interval $0 < t < T$ as is chosen. No attempt will be made here to present the details of the proof of the theorem stated above; that can be found elsewhere. Instead, the method is illustrated further by solving Eq. 2.10 in the following section using the method of averaging.

2.3.2 Construction of the Solution. In the following the method described previously is directly applied to Eq. 2.10. To simplify the analysis, Eq. 2.10 is rewritten in terms of the ac electric induction, D_1 , as follows:

$$\left(j\omega - u_0 \frac{d}{d\xi} \right)^2 D_1 + \omega_p^2 \frac{D_1}{\epsilon_b(\omega, \xi)} = 0, \quad (2.17)$$

where

$$D_1 = \epsilon_b(\omega, \xi) E_1. \quad (2.18)$$

Now, for an electron beam whose density profile is periodic with a spatial wavelength of $2\pi/k_0$, the beam density and dielectric constant can be expanded in a Fourier series. In particular,

$$\frac{1}{\epsilon_b(\omega, \xi)} = \sum_n \varphi_n(\omega) e^{-jnk_0\xi}. \quad (2.19)$$

For reasons which shall become clear later it is necessary to require that the electron beam support weak pulses; this means that the density variations along the beam are such that the Fourier components of $1/\epsilon_b(\omega, \xi)$ decay rapidly for higher harmonics. In accordance with the method of averaging, $D_1(\xi)$ can be decomposed according to the following:

$$D_1(\xi) = D_1^{(0)}(\xi) + D_1^{(1)}(\xi) = D_1^{(0)}(\xi) + \sum_{n \neq 0} d_n(\omega) e^{-jnk_0\xi}, \quad (2.20)$$

where $D_1^{(1)}(\xi)$ is the fast oscillatory part of the solution, $D_1(\xi)$, whose amplitude is small compared to that of $D_1^{(0)}(\xi)$. Since the periodic

inhomogeneity in beam density is what gives rise to the additional term, $D_1^{(1)}(\xi)$, in the solution, it is intuitively believed, by the notions of perturbation theory, that $D_1^{(1)}(\xi)$ will have a small amplitude in comparison with $D_1^{(0)}(\xi)$ if the pulses are weak. Direct substitution of Eqs. 2.19 and 2.20 into Eq. 2.17, separating the zeroth- and nth-order terms and setting each equal to zero, yields the following results :

$$\left(j\omega - u_o \frac{d}{d\xi}\right)^2 D_1^{(0)} + \omega_p^2 \varphi_o(\omega) D_1^{(0)} + \omega_p^2 \sum_{n \neq 0}' \varphi_n(\omega) d_{-n}(\omega) = 0 \quad (2.21)$$

and

$$-(\omega + nk_o u_o)^2 d_n(\omega) + \omega_p^2 \varphi_n(\omega) D_1^{(0)} + \omega_p^2 \sum_{m \neq 0}' \varphi_{n-m}(\omega) d_m(\omega) = 0 ;$$

$$(n = \pm 1, \pm 2, \dots) \quad (2.22)$$

The next step is to neglect the term $\omega_p^2 \sum_{m \neq 0, n}' \varphi_{n-m}(\omega) d_m(\omega)$ in Eq. 2.22. Physically this corresponds to the weak turbulence approximation. In the following an attempt is made to elucidate the physical meaning underlying the above-mentioned approximation.

2.3.3 Weak Turbulence Approximation. The motion of a viscous fluid in the vicinity of a rigid body can be qualitatively described in two distinct categories, depending upon the value of the Reynolds number, R , associated with the motion. It is certain that for sufficiently small values of the Reynolds number the steady flow is stable and laminar. In the laminar regime, the fluid moves as though it were in separate layers with different velocities with no interaction between the neighboring

layers. If a disturbance, periodic in time, is excited in such a system, the development in time of the spectrum of the disturbance indicates no excitation of modes with frequencies other than the principal frequencies of the original mode, and a decay in time of the original disturbance. When R increases it eventually reaches a value, R_{cr} (the critical Reynolds number), beyond which the flow is unstable with respect to infinitesimal disturbances. Such a flow is said to be turbulent. In contradistinction to laminar flow, the motion of the fluid in the turbulent state is confused in the sense that there exists no unique correspondence between the excitation frequencies and the excited wave numbers. This represents an increase in the number of degrees of freedom of the system. As R increases, both the number of excited waves and the number of degrees of freedom increase. In the limit as R tends to infinity, the number of degrees of freedom also increases indefinitely.⁷⁵

In a similar manner a turbulent plasma is defined as one in which many degrees of freedom are excited to levels considerably above the thermal level.⁷⁶ The presence of turbulent motion in plasma is verified experimentally through measurement of the space-time fluctuations of some parameters in plasma. For example, if the correlation function of the electric potential is measured by means of two electric probes placed inside the plasma at $\underline{r}$ and $\underline{r}'$, the space-time Fourier spectrum can be found in accordance with the following:

$$I_{\underline{k},\omega} = \frac{1}{(2\pi)^4} \int \left\langle \Phi(\underline{r},t) \Phi(0,0) \right\rangle \exp(-j\omega t + j\underline{k} \cdot \underline{r}) d\underline{r} dt \quad . \quad (2.23)$$

The absence or the existence of a turbulence as well as its strength can be determined by observing the form that $I_{\underline{k},\omega}$ assumes in the four-dimensional $(\omega, \underline{k})$ space.

The laminar state represents a group of waves with no mutual interaction between them. Such a state can be mathematically represented by the following:

$$\langle \Phi(\underline{r}, t) \Phi(0, 0) \rangle = \sum_m \Phi_m \exp \left(j(\omega_m t - \underline{k}_m \cdot \underline{r}) \right), \quad (2.24)$$

where the ω_m 's are the eigenfrequencies of the system which are uniquely related to the corresponding wave vectors, $\underline{k}_m$, through a dispersion relation. Direct substitution into Eq. 2.23 yields

$$I_{\underline{k},\omega} = \sum_m \Phi_m \delta(\omega - \omega_m) \delta(\underline{k} - \underline{k}_m), \quad (2.25)$$

where δ represents the Dirac delta function. Hence the spectrum consists of impulse functions with no spread in the $(\omega, \underline{k})$ space. A weak turbulent state represents a group of weakly interacting waves. Such a state is represented by a spectrum of narrow spread in the $(\omega, \underline{k})$ space with the peaks at the eigenwave vectors and the eigenfrequencies as depicted in Fig. 2.1a. Strong turbulence, on the other hand, represents a group of strongly interacting waves and is thus represented by a widened spectrum in the $(\omega, \underline{k})$ space; this is shown in Fig. 2.1b.

In light of the above physical notions, the following statement can be made concerning the problem. Neglecting the term $\omega_p^2 \sum_{m \neq 0, n} \phi_{n-m}(\omega) d_m(\omega)$ in Eq. 2.22 is equivalent to stating that the spectrum of the field has nonzero components only at those frequencies for which the spectrum

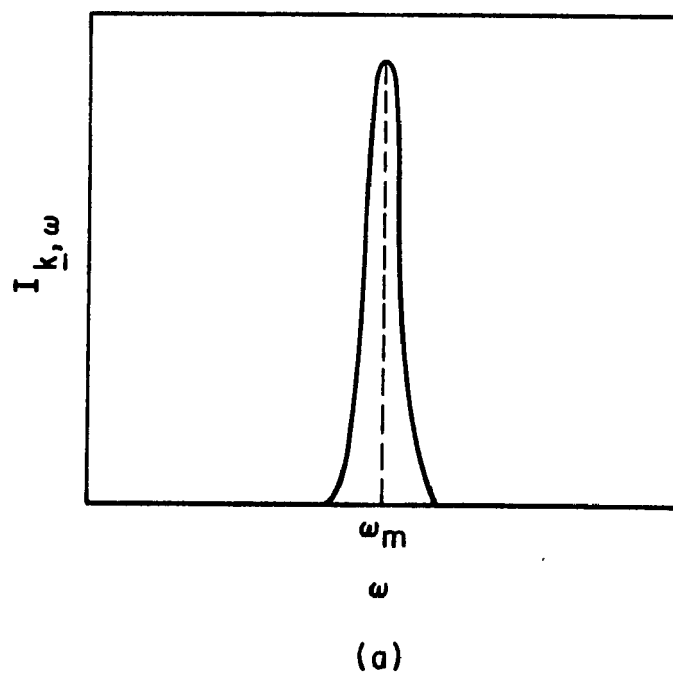
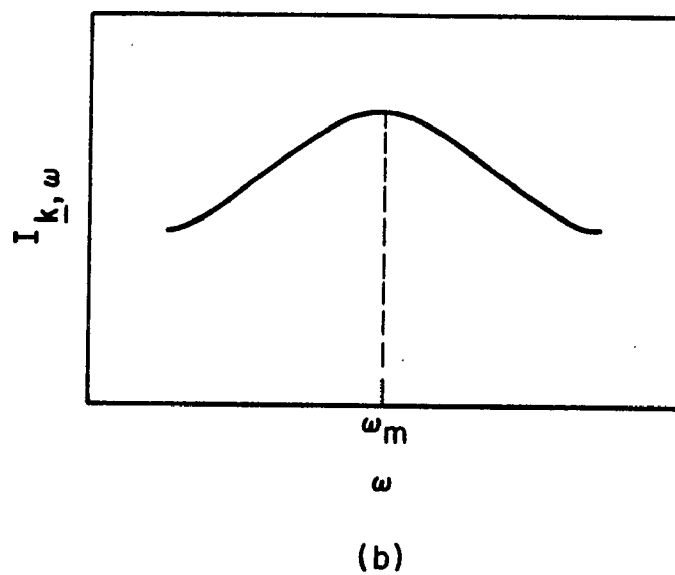


FIG. 2.1 SPECTRUM OF THE ELECTRIC POTENTIAL CORRELATION FUNCTION AS A FUNCTION OF FREQUENCY FOR THE CASE OF (a) A WEAK OR (b) A STRONG TURBULENCE PRESENT IN THE SYSTEM.

of the beam pulse has nonzero components. It should be clear that the above statement is in accordance with the features of the weak turbulence as described above, hence the term "weak turbulence approximation."

The remaining is straightforward and the final form of the solution to Eq. 2.17 can be stated as follows:

$$D_1(\xi) = \left(1 + \sum_{n \neq 0} ' \frac{\omega_p^2 \varphi_n(\omega)}{(\omega + n\omega_0)^2 - \omega_p^2 \varphi_0(\omega)} e^{-jnk_0 \xi} \right) D_1^{(0)}(\xi) , \quad (2.26)$$

where ω_0 is the frequency of the HF modulation on the beam defined as

$$\omega_0 \equiv k_0 u_0 \quad (2.27)$$

and $D_1^{(0)}$ is the solution of the following differential equation:

$$\left(j\omega - u_0 \frac{d}{d\xi} \right)^2 D_1^{(0)} + \omega_p^2 \left(\varphi_0(\omega) + \sum_{n \neq 0} ' \frac{\omega_p^2 |\varphi_n(\omega)|^2}{(\omega - n\omega_0)^2 - \omega_p^2 \varphi_0(\omega)} \right) D_1^{(0)} = 0 . \quad (2.28)$$

In the above derivation use was made of the fact that $\varphi_{-n}(\omega) = \varphi_n^*(\omega)$, an equality which stems from the reality of the beam-dielectric constant, $\epsilon_b(\omega, \xi)$.

2.4 Dispersion Relation

The dispersion relation is obtained by a simple substitution of $D_1^{(0)} \propto e^{-jk\xi}$ into Eq. 2.28 as follows:

$$\omega_p^2 \varphi_0(\omega) - (\omega + ku_0)^2 + \omega_p^4 \sum_{n \neq 0} ' \frac{|\varphi_n(\omega)|^2}{(\omega - n\omega_0)^2 - \omega_p^2 \varphi_0(\omega)} = 0 . \quad (2.29)$$

Now it is noted that the relations $\omega \ll \omega_0$ and $k \ll k_0$, the latter being an assumption without which the method of averaging could not be employed, are equivalent statements throughout the interaction region, which is the region of interest here. To realize this fact it is sufficient to note that from a coupled-mode viewpoint the phase velocity of the wave has to be of the order of beam dc velocity, u_0 ; otherwise the coherent motion of plasma particles, represented by a wave of phase velocity ω/k , will not be sensed by a static wave moving with a uniform velocity u_0 . Therefore, the relation $u_0 \sim \omega/k$ must hold. Hence it is concluded that the relations $k \ll k_0$ and $\omega \ll \omega_0$ are equivalent statements and terms of order ω/ω_0 can be ignored without making further assumptions. Furthermore, Eq. 2.29 is the dispersion relation expressed in terms of parameters measured in the frame of reference attached to the beam. The dispersion relation in the laboratory frame of reference is obtained by a doppler transformation $\omega \rightarrow \omega_D \equiv \omega - ku_0$ of Eq. 2.29.

Having taken these considerations into account, the final form of the dispersion relation is expressed as follows:

$$\varphi_0(\omega_D) - \frac{\omega^2}{\omega_p^2} + 2\omega_p^2 \sum_{n=1}^{\infty} \frac{|\varphi_n(\omega_D)|^2}{n^2\omega_0^2 - \omega_p^2\varphi_0(\omega_D)} = 0 \quad (2.30)$$

It is noted that in the limit when $\varphi_n \rightarrow 0$, i.e., when the beam is not initially pulsed, Eq. 2.30 is reduced to the Bohm-Gross dispersion relation derived for the one-dimensional beam-plasma interaction.⁴ In the following sections the obtained results are applied to two specified forms of beam-density profile: rectangular and sinusoidal.

2.4.1 Rectangular Inhomogeneity. The beam-density profile in this case is given as

$$n_{ob}(\xi) = \begin{cases} n_{ob} & mL \leq \xi < mL + \ell \\ 0 & mL + \ell \leq \xi < (m+1)L \end{cases} \quad (m = 0, \pm 1, \pm 2, \dots) \quad (2.31)$$

Appendix A shows that the dispersion relation, as expressed in Eq. 2.30, assumes the following form for the case of the above density profile:

$$1 = \frac{\omega_D^2}{\omega_b^2} \left(1 - \frac{\omega_p^2}{\omega^2} \right) + \frac{\omega_p^2}{\omega^2} \left(1 - \frac{\ell}{L} \right) - \frac{2}{\pi^2} \frac{\omega_p^4}{\omega^2 \omega_0^2} \frac{\omega_b^2 / \omega_p^2}{1 - \omega_b^2 / \omega_D^2} \sum_{n=1}^{\infty} \frac{1}{n^4} \sin^2 \frac{n\pi\ell}{L}, \quad (2.32)$$

where ω_b is the beam plasma frequency defined as

$$\omega_b \equiv \left(\frac{e^2 n_{ob}}{m \epsilon_0} \right)^{1/2}. \quad (2.33)$$

Throughout the analysis ω_p^2 / ω_0^2 was assumed to be the parameter of smallness and the terms of order ω_p^2 / ω_0^2 or higher were neglected. In the same fashion the third term on the right-hand side of Eq. 2.32 can be ignored and the remaining terms solved for the propagation constant, k . The results can be expressed as

$$k = \frac{\omega}{u_0} \pm \frac{\omega_b}{u_0} \left(\frac{1 - \gamma^2 \omega_p^2 / \omega^2}{1 - \omega_p^2 / \omega^2} \right)^{1/2}, \quad (2.34)$$

where γ is a measure of the fraction of time that the beam remains turned off and is defined as follows:

$$\gamma \equiv \left(1 - \frac{\ell}{L}\right)^{1/2} . \quad (2.35)$$

From Eq. 2.34 the region of instability can easily be found as

$$\gamma < \frac{\omega}{\omega_p} < 1 . \quad (2.36)$$

It is noted that rectangular pulsation puts a lower limit on the extent of the instability region measured on the ω -scale. Throughout the region of instability the growth rate is diminished by a factor of $[1 - \gamma^2(\omega_p^2/\omega^2)]^{1/2}$. The effectiveness of the suppression of convective instabilities by means of a high-frequency, rectangular pulsation is shown in Figs. 2.2 and 2.3. In Fig. 2.2 variations of the normalized growth rate, δ_γ/δ_0 , are shown as a function of γ , with the normalized frequency, ω/ω_p , taken as a constant, where δ_γ is defined as follows:

$$\delta_\gamma \equiv \frac{\omega_b}{u_0} \left(\frac{1 - \gamma^2 \omega_p^2/\omega^2}{\omega_p^2/\omega^2 - 1} \right)^{1/2} , \quad \gamma < \frac{\omega}{\omega_p} < 1 , \quad 0 \leq \gamma < 1 . \quad (2.37)$$

Figure 2.3 shows the variation of the normalized growth rate as a function of normalized frequency, with γ taken as a parameter. The above-mentioned plots give a clear picture of the narrowed spectrum for the region of instability as well as higher suppression rates for the lower frequencies. These results are further discussed in Section 2.4.3 where a physical explanation is also offered.

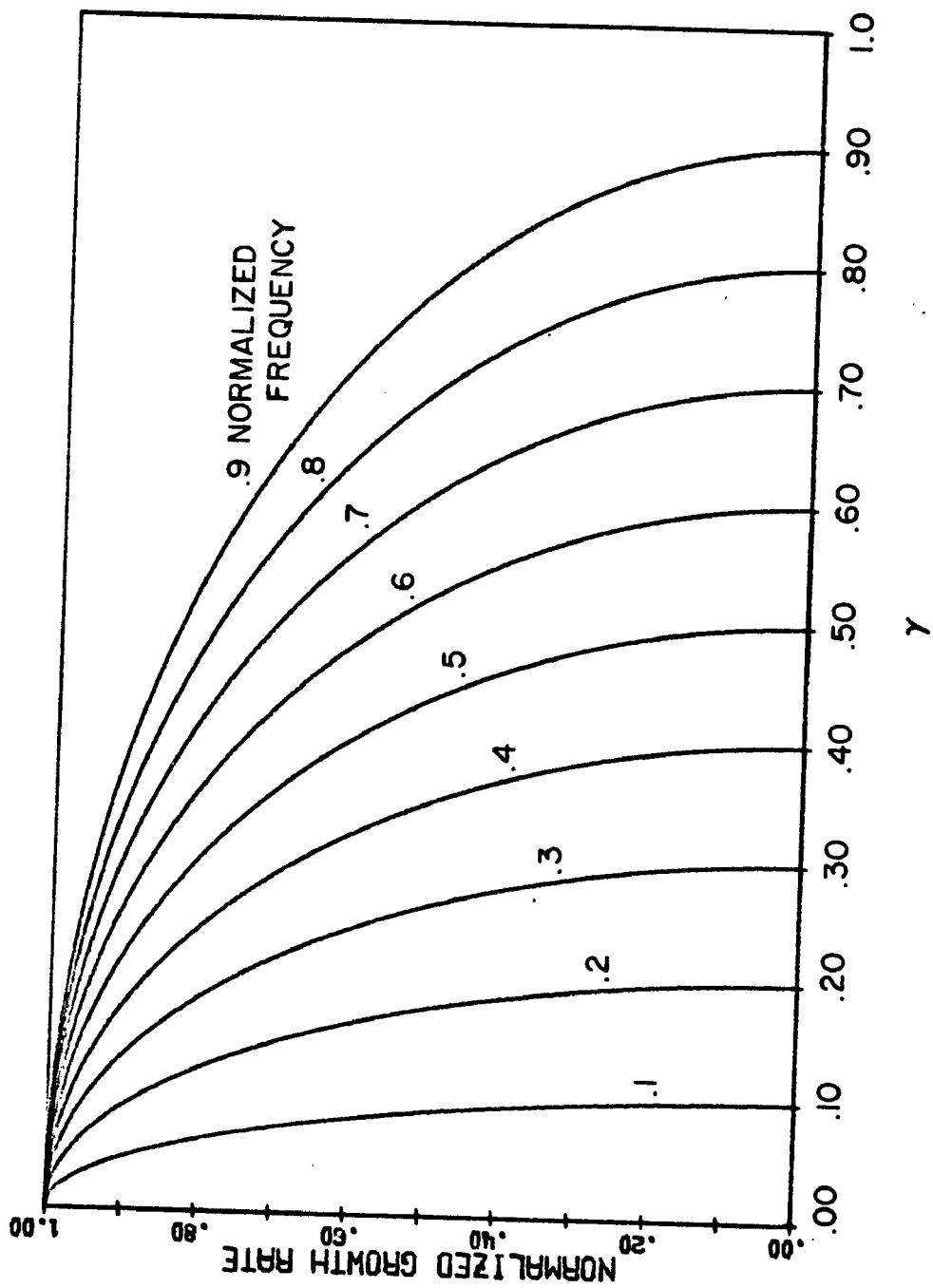


FIG. 2.2 NORMALIZED GROWTH RATE, δ_γ/δ_0 , VS. γ FOR RECTANGULAR INHOMOGENEITY.

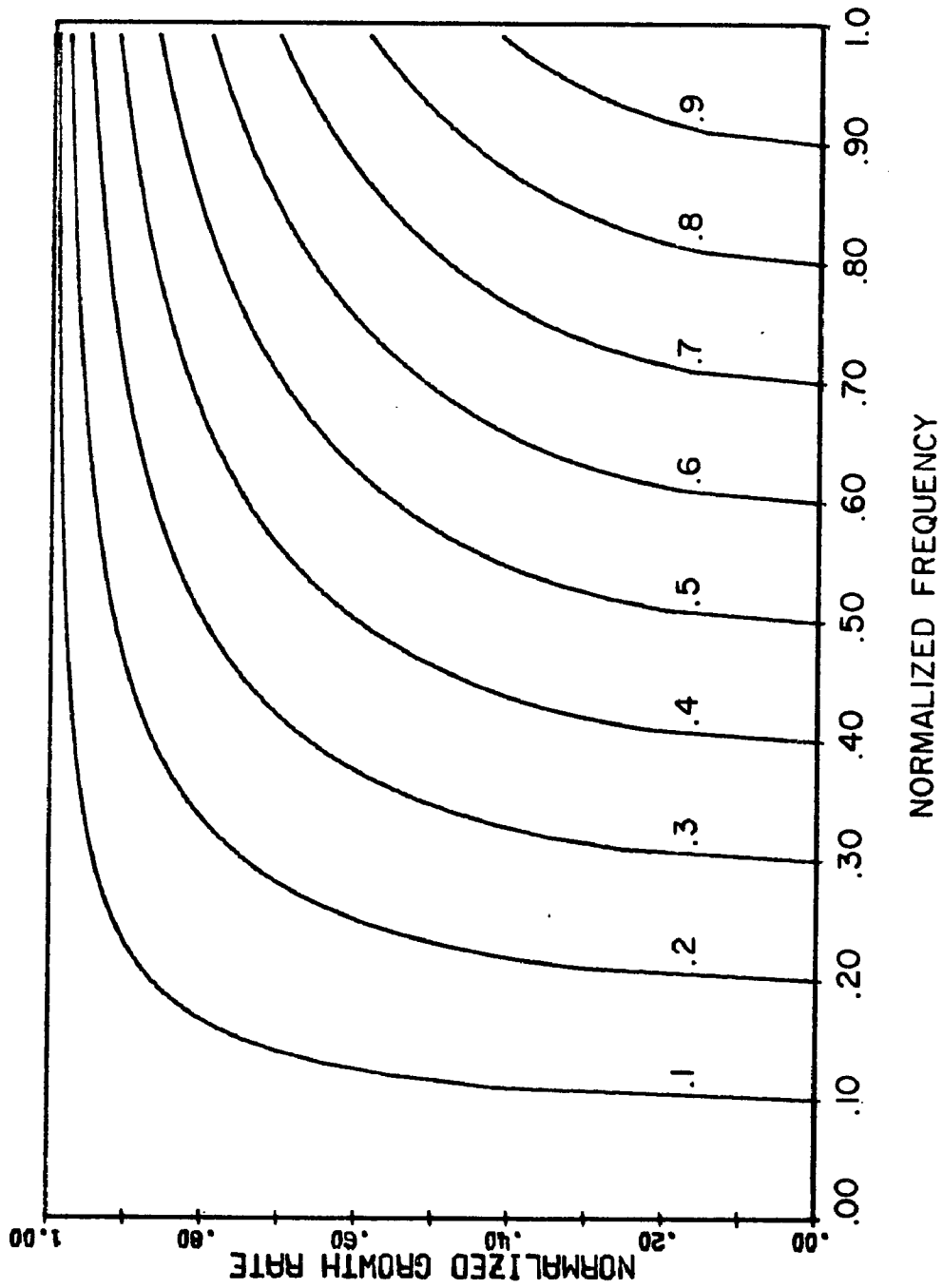


FIG. 2.3 NORMALIZED GROWTH RATE, ϵ/ϵ_0 , VS. NORMALIZED FREQUENCY FOR RECTANGULAR INHOMOGENEITY.

Briggs' stability criterion¹⁹ can be applied to examine the loci of the roots of the dispersion relation, Eq. 2.34, in the complex k-plane. Equation 2.34 can be rewritten as

$$\frac{k u_o}{\omega_p} = \frac{\omega}{\omega_p} \pm \frac{\omega_b}{\omega_p} \left(1 + \frac{l/L}{\omega^2/\omega_p^2 - 1} \right)^{1/2} . \quad (2.38)$$

According to Briggs' criterion the value of ω is set equal to

$$\omega = \omega_r + j\omega_i \quad (2.39)$$

and the loci of the propagation constant k is observed in the complex k-plane as the contour of integration in the complex ω -plane is moved up to the $\omega_i = 0$ axis from the lower half of the ω -plane. In terms of the normalized parameters Eq. 2.38 can be rewritten as

$$\frac{k u_o}{\omega_p} = \Omega_r + j\Omega_i \pm \Omega_b \left(1 + \frac{l/L}{\Omega_r^2 - \Omega_i^2 - 1 + 2j\Omega_r\Omega_i} \right)^{1/2} . \quad (2.40)$$

From Fig. 2.4 it is seen that for $\Omega_r = 1$ as Ω_i approaches zero from infinitely large negative values, one of the roots moves to the upper half of the k-plane corresponding to a convectively unstable mode while the other remains in the lower half-plane. For $\Omega_r = 0.4$ both roots remain in the lower half-plane indicating the absence of unstable modes in agreement with the previous results.

2.4.2 Sinusoidal Inhomogeneity. The beam-density profile in this case is given as

$$n_{ob}(\xi) = n_{ob}(1 + \alpha \cos k_o \xi) , \quad (2.41)$$

where α is the depth of modulation whose magnitude will never exceed unity.

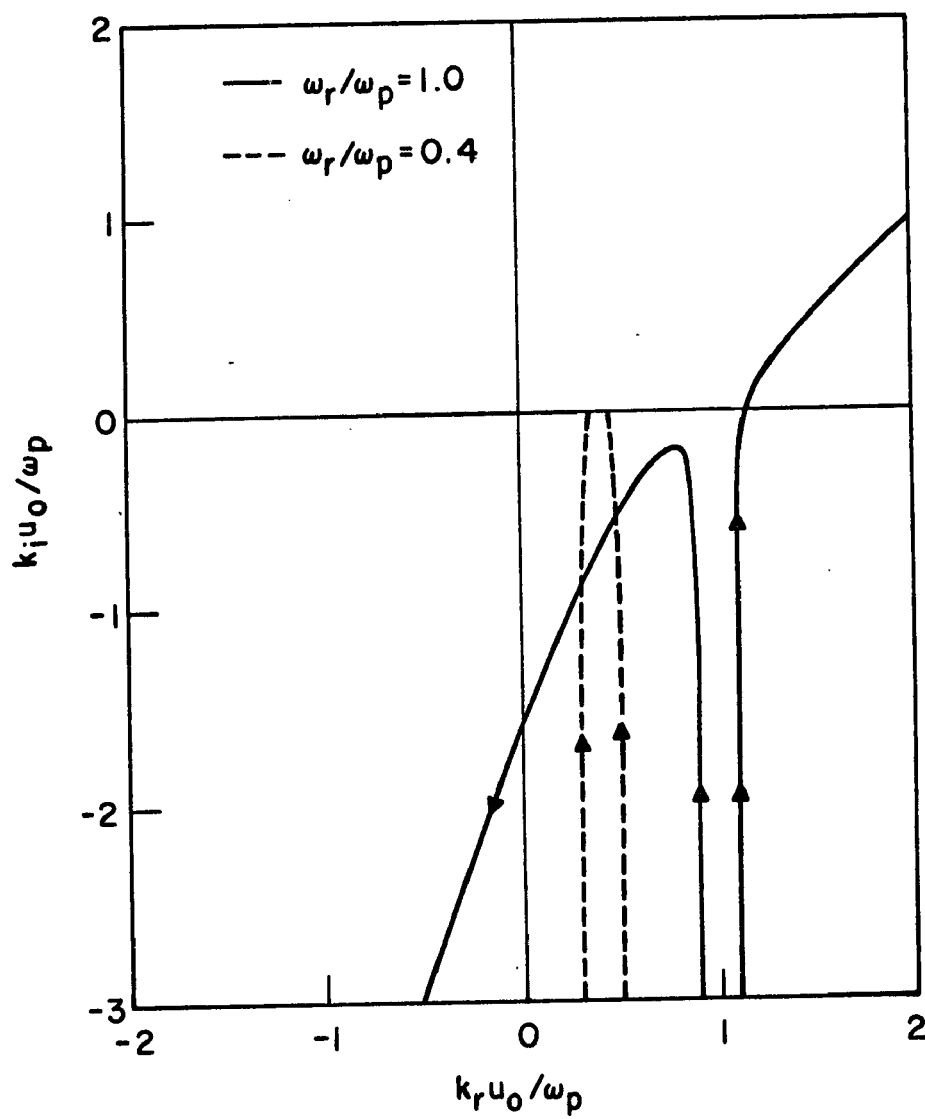


FIG. 2.4 THE LOCI OF THE ROOTS OF THE DISPERSION RELATION, (EQ. 2.34)
FOR $l/L = 0.5$ AND $\omega_b/\omega_p = 0.1$.

It is shown in Appendix B that the dispersion relation, as expressed in Eq. 2.30, assumes the following form for the case of the above density profile:

$$\left[\left(1 - \frac{\omega_b^2}{\omega_D^2} \right)^2 - \alpha^2 \frac{\omega_b^4}{\omega_D^4} \right]^{-1/2} - \frac{\omega^2}{\omega_p^2} + \frac{2}{\pi^2} \frac{\omega_p^2}{\omega_0^2} \sum_{n=1}^{\infty} \frac{I_n^2}{n^2} = 0, \quad (2.42)$$

where I_n is defined as

$$I_n \equiv \int_0^{\pi} \frac{\cos nx \, dx}{1 - \frac{\omega_b^2}{\omega_D^2} - \alpha \frac{\omega_b^2}{\omega_D^2} \cos x}. \quad (2.43)$$

In Appendix C a three-term recurrence formula is developed which can be used to evaluate integrals of the above form. The summation term can again be neglected due to the fact that ω_p^2/ω_0^2 is the natural parameter of smallness used throughout the analysis. The result can be expressed as

$$(1 - \alpha^2) \frac{\omega_b^4}{\omega_D^4} - 2 \frac{\omega_b^2}{\omega_D^2} + 1 - \frac{\omega_p^4}{\omega^4} = 0. \quad (2.44)$$

The above equation is biquadratic in ω_b/ω_D yielding four roots for the propagation constant. Two of them can be discarded due to the fact that they do not reduce to the one-dimensional beam-plasma dispersion relation in the limit of $\alpha \rightarrow 0$. Hence in this sense they do not correspond to the physically real modes of propagation in the system. They are artificially introduced simply through mathematical operations. The other two roots,

however, with appropriate correspondence relations, can be expressed as

$$k = \frac{\omega}{u_o} \pm j \frac{\omega_b}{u_o} \left[\frac{1 - \alpha^2}{\left[\frac{\omega_p^4}{\omega^4} - \alpha^2 \left(\frac{\omega_p^4}{\omega^4} - 1 \right) \right]^{1/2} - 1} \right]^{1/2} . \quad (2.45)$$

The term $\omega_p^4/\omega^4 - \alpha^2[(\omega_p^4/\omega^4) - 1]$ is always positive ($\alpha < 1$) and its value is greater than unity throughout the range $\omega/\omega_p < 1$. Therefore, unlike the case of rectangular inhomogeneity, the range of instability in this case is independent of the depth of modulation. Throughout the region $\omega/\omega_p < 1$ the growth rate of the convective instability is diminished by a factor given as

$$\frac{\delta_\alpha}{\delta_o} = \frac{\mu(1 - \Omega^2)^{1/2}}{[\sqrt{\mu^2(1 - \Omega^4) + \Omega^4} - \Omega^2]^{1/2}} , \quad (2.46)$$

where δ_α is defined as the imaginary part of k as expressed in Eq. 2.45, and μ and Ω are defined as

$$\mu \equiv (1 - \alpha^2)^{1/2} \quad (2.47)$$

and

$$\Omega \equiv \frac{\omega}{\omega_p} . \quad (2.48)$$

Figures 2.5 and 2.6 show variations of the normalized growth rate, δ_α/δ_o , as a function of the depth of modulation, α , and the normalized

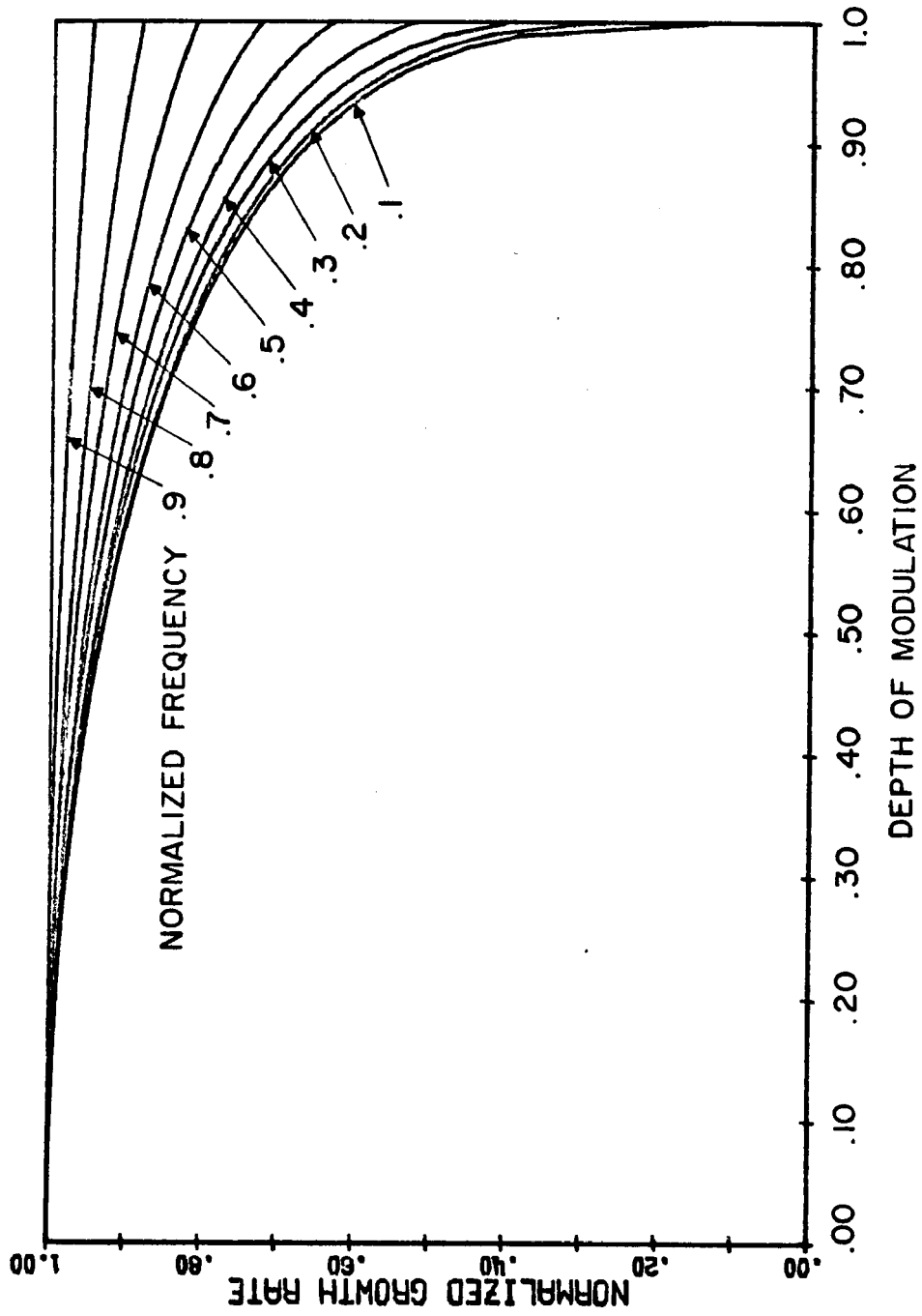


FIG. 2.5 NORMALIZED GROWTH RATE, δ_{α}/δ_0 , VS. DEPTH OF MODULATION, α , FOR SINUSOIDAL INHOMOGENEITY.

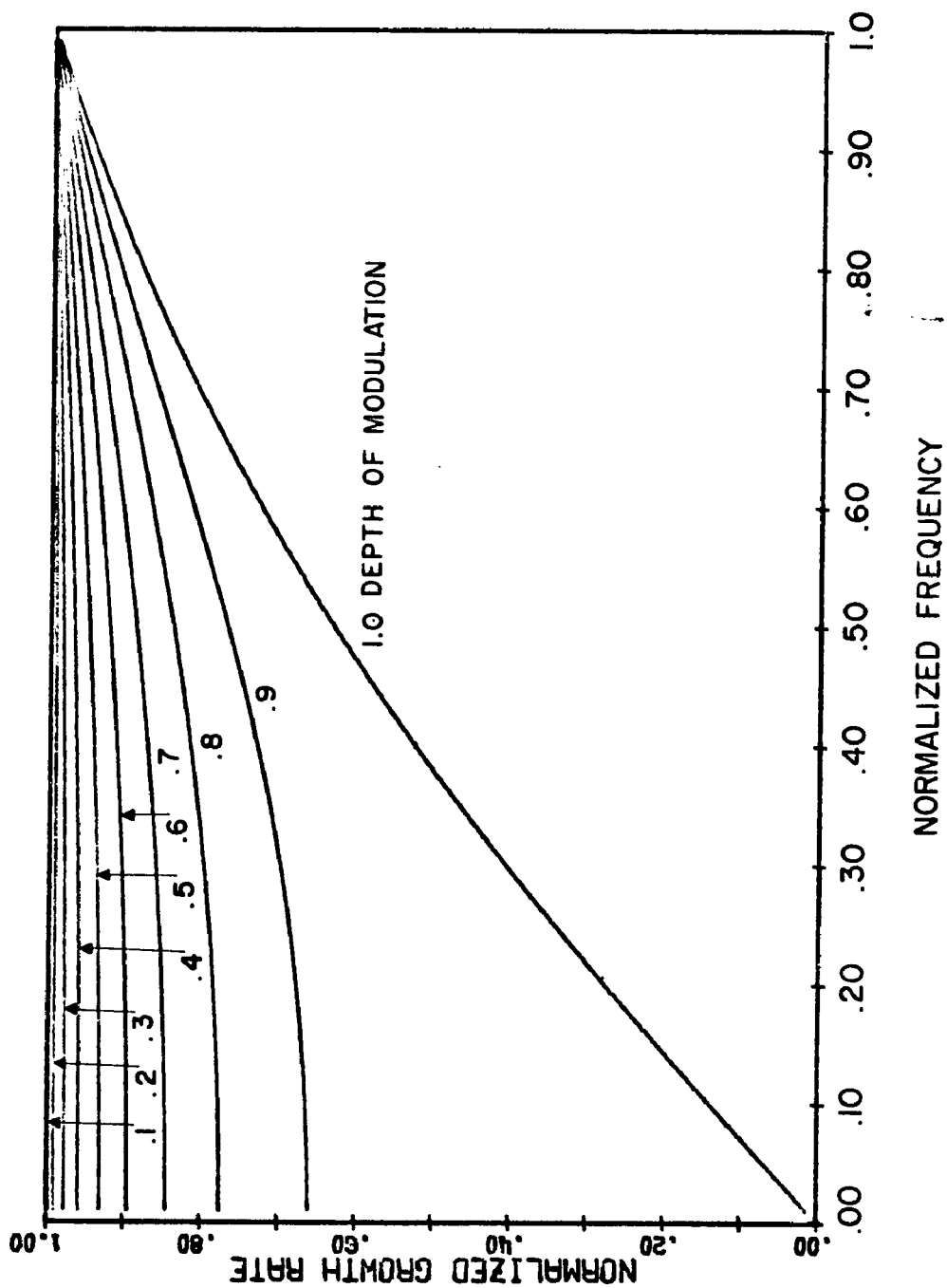


FIG. 2.6 NORMALIZED GROWTH RATE, δ_{α}/δ_o , VS. NORMALIZED FREQUENCY, ω/ω_p , FOR SINUSOIDAL INHOMOGENEITY.

frequency, Ω , respectively. The rate of suppression increases directly with the depth of modulation and inversely with the frequency. The frequency bandwidth over which the instability occurs remains unchanged as the depth of modulation varies, in contradistinction with the case of rectangular inhomogeneity.

2.4.3 Discussion of the Results. Fainberg⁶⁷ has pointed out that if an electron beam is injected into a plasma self-modulation of the beam particles will occur as a result of synchronization between the beam particles and the restoring electric field, maintaining plasma oscillations. This will eventually lead to a coherent oscillation of the particles in the beam, in phase with the restoring field, and thus contributes to the effectiveness of the interaction through enhancement of the field. Hence, on the basis of the above argument, it would seem natural that if the beam particles are somehow prohibited from falling into phase with the restoring field there will be a disruption of the instabilities throughout the spectrum. Previous bunching on the beam density at a frequency other than that of the restoring field is an effective means by which a phase destruction between the beam particles and the restoring field of plasma oscillations is possible.

Consideration should be made of the parametric resonances that a previously bunched beam excites. These are resonances which occur at frequencies near the bunching frequency. Experimental studies,^{52,53} however, have shown that parametric instabilities have a narrow width, and so they may be weakened or even disappear due to the inhomogeneities and collisions that occur in actual systems.

The present theoretical investigation demonstrates this fact. For the case of a high-frequency, rectangular pulsation of the beam density the theory predicts a disappearance of the convective instabilities below a threshold frequency given as

$$\omega_{th} = \omega_p \sqrt{1 - l/L} . \quad (2.49)$$

Furthermore, the growth rate in space of the instability is diminished for the remaining part of the spectrum. A higher degree of suppression is predicted for modes of lower frequencies. The latter is also in agreement with the argument of phase destruction presented earlier. For the case of a high-frequency, sinusoidal pulsation on beam density, the theory predicts similar results with the exception that the width of the unstable region for this case is independent of the depth of modulation. The more effective suppression of the lower-frequency modes can be seen by expanding the right-hand side of Eq. 2.46 in ascending powers of μ as follows:

$$\frac{\delta_\alpha}{\delta_o} = \left(\frac{2\Omega^2}{1 + \Omega^2} \right)^{1/2} \left(1 + \frac{1 - \Omega^4}{8\Omega^4} \mu^2 + O(\mu^4) \right) . \quad (2.50)$$

The highest-order term is a monotonically increasing function of Ω , which results in a behavior for δ_α/δ_o , as presented in Figs. 2.5 and 2.6.

2.5 Finite-Temperature Effects

In this section the analysis is extended to include the plasma-electron temperature effects as well as the plasma-ion motion. The

hydrodynamic model is again adopted for the beam and the plasma under the general conditions as stated in Section 2.1. It should be noted that by adopting a fluid model for plasmas the effects of such micro-mechanisms as Landau and cyclotron damping are inadvertently ignored.

2.5.1 Field and Moment Equations. The equations of motion for plasma electrons and ions, as well as beam electrons, are written in the same manner as before with an additional term in the force equation for plasma electrons which is introduced to take into account the effects of temperate electrons. Beam ions are assumed to be infinitely heavy. In an inertial frame of reference attached to the beam, the equations of motion and Gauss' law are expressed as follows:

$$\frac{\partial v_{1be}}{\partial t} = -\frac{e}{m_e} E_1, \quad (2.51)$$

$$\frac{\partial n_{1be}}{\partial t} + \frac{\partial}{\partial \zeta} (n_{ob} v_{1be}) = 0, \quad (2.52)$$

$$\frac{\partial v_{1pe}}{\partial t} - u_o \frac{\partial v_{1pe}}{\partial \zeta} = -\frac{e}{m_e} E_1 - \frac{\partial}{\partial \zeta} \left(\frac{p_{1e}}{n_{op} m_e} \right), \quad (2.53)$$

$$\frac{\partial v_{1pi}}{\partial t} - u_o \frac{\partial v_{1pi}}{\partial \zeta} = \frac{e}{m_i} E_1, \quad (2.54)$$

$$\frac{\partial n_{1pe,i}}{\partial t} - u_o \frac{\partial n_{1pe,i}}{\partial \zeta} + n_{op} \frac{\partial v_{1pe,i}}{\partial \zeta} = 0 \quad (2.55)$$

and

$$\frac{\partial}{\partial \zeta} E_1 = \frac{e}{\epsilon_o} (n_{1pi} - n_{1pe} - n_{1be}), \quad (2.56)$$

where the additional subscripts e and i denote the electronic and ionic quantities respectively. The first-order perturbation in the plasma electron pressure is denoted by p_{1e} whose definition, in terms of electronic temperature, is as follows:

$$p_{1e} \equiv \gamma k_B T_e n_{1pe} , \quad (2.57)$$

where γ is the compression constant of the electron gas. Under adiabatic conditions γ will tend to a value equal to three at high frequencies (one-dimensional compression), and $5/3$ at somewhat lower frequencies (three-dimensional compression). At very low frequencies the compression will be isothermal and γ tends to unity.

A steady-state solution is sought by assuming an $e^{j\omega t}$ variation for all the first-order quantities. Combining Eqs. 2.51 and 2.52 yields the following result:

$$n_{1be} = - \frac{e}{m_e \omega^2} \frac{d}{d\xi} [n_{ob}(\xi) E_1] . \quad (2.58)$$

Likewise, if Eqs. 2.53 to 2.55 are combined, the following is obtained:

$$\left[\left(j\omega - u_o \frac{d}{d\xi} \right)^2 - v_{th}^2 \frac{d^2}{d\xi^2} \right] n_{1pe} = \frac{en_{op}}{m_e} \frac{dE_1}{d\xi} \quad (2.59)$$

and

$$\left(j\omega - u_o \frac{d}{d\xi} \right)^2 n_{1pi} = - \frac{en_{op}}{m_i} \frac{dE_1}{d\xi} , \quad (2.60)$$

where v_{th} is the mean thermal velocity of plasma electrons defined as

$$v_{th} \equiv (\gamma k_B T_e / m_e)^{1/2} . \quad (2.61)$$

If both sides of Eq. 2.56 are operated on by the differential operator

$$\left[\left(j\omega - u_o \frac{d}{d\zeta} \right)^2 - v_{th}^2 \frac{d^2}{d\zeta^2} \right] \left(j\omega - u_o \frac{d}{d\zeta} \right)^2$$

and the result is combined with Eqs. 2.58 to 2.60, the following differential equation is obtained for the electric induction, D_1 :

$$\begin{aligned} & \left[\left(j\omega - u_o \frac{d}{d\zeta} \right)^2 - v_{th}^2 \frac{d^2}{d\zeta^2} \right] \left(j\omega - u_o \frac{d}{d\zeta} \right)^2 D_1 \\ & + \left[\left(\omega_{pe}^2 + \omega_{pi}^2 \right) \left(j\omega - u_o \frac{d}{d\zeta} \right)^2 - \omega_{pi}^2 v_{th}^2 \frac{d^2}{d\zeta^2} \right] \left(\frac{D_1}{\epsilon_b(\omega, \zeta)} \right) = 0 . \end{aligned} \quad (2.62)$$

2.5.2 Dispersion Relation. An approximate solution to Eq. 2.62 can be found in the same manner as previously demonstrated, i.e., by decomposing D_1 in accordance with Eq. 2.20. The results are

$$\begin{aligned} & \left[\left(j\omega - u_o \frac{d}{d\zeta} \right)^2 - v_{th}^2 \frac{d^2}{d\zeta^2} + \omega_p^2 \phi_o(\omega) \right] \left(j\omega - u_o \frac{d}{d\zeta} \right)^2 D^{(o)} \\ & - \omega_{pi}^2 v_{th}^2 \phi_o(\omega) \frac{d^2}{d\zeta^2} D^{(o)} - \omega_p^2 \sum_{n \neq 0} \phi_{-n}(\omega) d_n(\omega) = 0 \end{aligned} \quad (2.63)$$

and

$$\begin{aligned} & \left[(\omega + n\omega_o)^2 - n^2 k_o^2 v_{th}^2 \right] (\omega + n\omega_o)^2 d_n(\omega) - \omega_p^2 \varphi_n(\omega) \left[\left(n\omega_o + \omega + ju_o \frac{d}{d\xi} \right)^2 \right. \\ & \left. + \frac{\omega_{pi}^2}{\omega_p^2} \frac{v_{th}^2}{u_o^2} \left(n\omega_o + ju_o \frac{d}{d\xi} \right)^2 \right] D^{(o)} - \omega^2 \omega_p^2 \left(2n \frac{\omega_o}{\omega} + 1 \right) \sum_{m \neq 0} \varphi_{n-m}(\omega) d_m(\omega) \\ & = 0, \quad (2.64) \end{aligned}$$

where

$$\omega_p^2 \equiv \omega_{pe}^2 + \omega_{pi}^2. \quad (2.65)$$

Application of the weak turbulence approximation together with the low-frequency approximation, $\omega \ll \omega_o$, results in a dispersion relation in the moving frame of reference attached to the beam. A doppler-shift transformation results in the final form of the dispersion relation

$$\begin{aligned} & \frac{\omega^2 - k^2 v_{th}^2}{\omega_p^2} - \left(1 - \frac{k^2 v_{th}^2}{\omega^2} \frac{\omega_{pi}^2}{\omega_p^2} \right) \varphi_o(\omega_D) - \frac{\omega_p^2}{\omega_o^2} \frac{1 + (\omega_{pi}^2/\omega_p^2)(v_{th}^2/u_o^2)}{1 - v_{th}^2/u_o^2} \\ & \cdot \sum_{n \neq 0} \frac{1}{n^2} |\varphi_n(\omega_D)|^2 = 0. \quad (2.66) \end{aligned}$$

It is noted from the above equation that for a realistic electron-to-proton mass ratio the effect of plasma ion motion is negligible in the context of the approximations employed throughout. Caution in interpreting the physical implications of this approximation is suggested, however. The coefficient of $\varphi_o(\omega_D)$ in Eq. 2.66 can be alternatively expressed as

$$1 - \frac{k^2 v_{th}^2}{\omega^2} \frac{\omega_{pi}^2}{\omega_p^2} = 1 - \gamma \left(\frac{v_s}{\omega/k} \right)^2, \quad (2.67)$$

where the terms of order $(\omega_{pi}/\omega_{pe})^4$ and higher were neglected and v_s is the velocity of ion-acoustic oscillations of plasma defined as

$$v_s \equiv (k_B T_e / m_i)^{1/2}. \quad (2.68)$$

For oscillations of wavelengths long enough such that $\omega/k \sim u_o$, which is in accordance with the previous assumptions of $k \ll k_o$ and $\omega \ll \omega_o$, the right-hand side of Eq. 2.67 tends to unity. This is equivalent to neglecting the motion of plasma ions. For oscillations at wavelengths short enough such that $\omega/k \sim v_s$, the right-hand side of Eq. 2.67 tends to zero. But the basic assumption underlying the method of averaging, namely that $k \ll k_o$, does not allow consideration of short-wavelength oscillations. Other mathematical techniques of solving Eq. 2.62 must therefore be developed if the coupling to the ion-acoustic oscillations is of interest.

Neglecting the motion of plasma ions gives the following dispersion relation:

$$\frac{\omega^2 - k^2 v_{th}^2}{\omega_p^2} - \phi_o(\omega_D) - \frac{\omega_p^2}{\omega_o^2} \left(1 - \frac{v_{th}^2}{u_o^2} \right)^{-1} \sum_{n \neq 0} \frac{1}{n^2} |\phi_n(\omega_D)|^2 = 0. \quad (2.69)$$

It is noted that in the limit when $\phi_n \rightarrow 0$, i.e., when the beam is homogeneous, Eq. 2.69 is reduced to

$$\frac{\omega_p^2}{\omega^2 - (\gamma k_B T_e / m) k^2} + \frac{\omega_b^2}{\omega_D^2} = 1, \quad (2.70)$$

which is the Bohm-Gross dispersion relation for the one-dimensional interaction of an electron beam with a temperate plasma.

2.5.3 Rectangular Inhomogeneity. In this section a density variation for the electron beam in accordance with Eq. 2.31 is considered. For this case, in reference to Appendix A, the derived dispersion relation, Eq. 2.69, can be expressed as

$$1 = \frac{\omega^2 - k^2 v_{th}^2}{\omega_p^2} - \frac{\frac{l}{L} \omega_b^2}{\omega_D^2 - \omega_b^2} + \frac{2}{\pi^2} \frac{\omega_p^2}{\omega_o^2} \frac{(1 - v_{th}^2 / u_o^2)^{-1}}{1 - \omega_D^2 / \omega_b^2} \sum_{n=1}^{\infty} \frac{1}{n^4} \sin^2 \frac{n\pi l}{L}. \quad (2.71)$$

For computational purposes the last term on the right-hand side of the above equation can be ignored. Note that terms of order $(\omega_p / \omega_o)^2$ and higher have been neglected throughout the analysis. Equation 2.71 reduces to

$$1 = \frac{\omega^2 - k^2 v_{th}^2}{\omega_p^2} - \frac{\frac{l}{L} \omega_b^2}{\omega_D^2 - \omega_b^2}. \quad (2.72)$$

The above equation is of fourth order in k (or ω). Now, define the following set of normalized parameters:

$$\Omega \equiv \frac{\omega}{\omega_p}, \quad (2.73)$$

$$\Omega_b \equiv \frac{\omega_b}{\omega_p}, \quad (2.74)$$

$$\Delta \equiv \frac{l}{L}, \quad (2.75)$$

$$r \equiv \frac{\gamma}{2} \frac{k_B T_e}{e V_0} \quad (2.76)$$

and

$$x \equiv \frac{k u_0}{\omega_p}, \quad (2.77)$$

where V_0 is the dc voltage of the beam. In terms of the above normalized parameters, Eq. 2.72 can be written as

$$rx^4 - 2r\Omega x^3 + \left(1 - \Omega^2 + r(\Omega^2 - \Omega_b^2)\right) x^2 - 2\Omega(1 - \Omega^2)x + (1 - \Omega^2)(\Omega^2 - \Omega_b^2) + \Delta\Omega_b^2 = 0 \quad (2.78)$$

For a given value of Ω , Ω_b , Δ and r the above equation yields four values for the normalized propagation constant, $k u_0 / \omega_p$. For the uncoupled situation the four roots correspond to the temperate plasma modes

$$k = \pm (\omega^2 - \omega_p^2)^{1/2} / v_{th}, \quad (2.79)$$

and the slow and fast space-charge modes of the beam

$$k = (\omega \pm \omega_b) / u_0. \quad (2.80)$$

This is shown in Fig. 2.7 where the normalized frequency, ω/ω_p , is plotted vs. the normalized propagation constant, $k u_0 / \omega_p$, for $\Omega_b = 0.1$ and $r = 0.2$. As seen, the frequency of maximum interaction, which occurs between the slow space-charge wave of the beam and the propagating mode of the temperate plasma, is slightly higher than its corresponding value

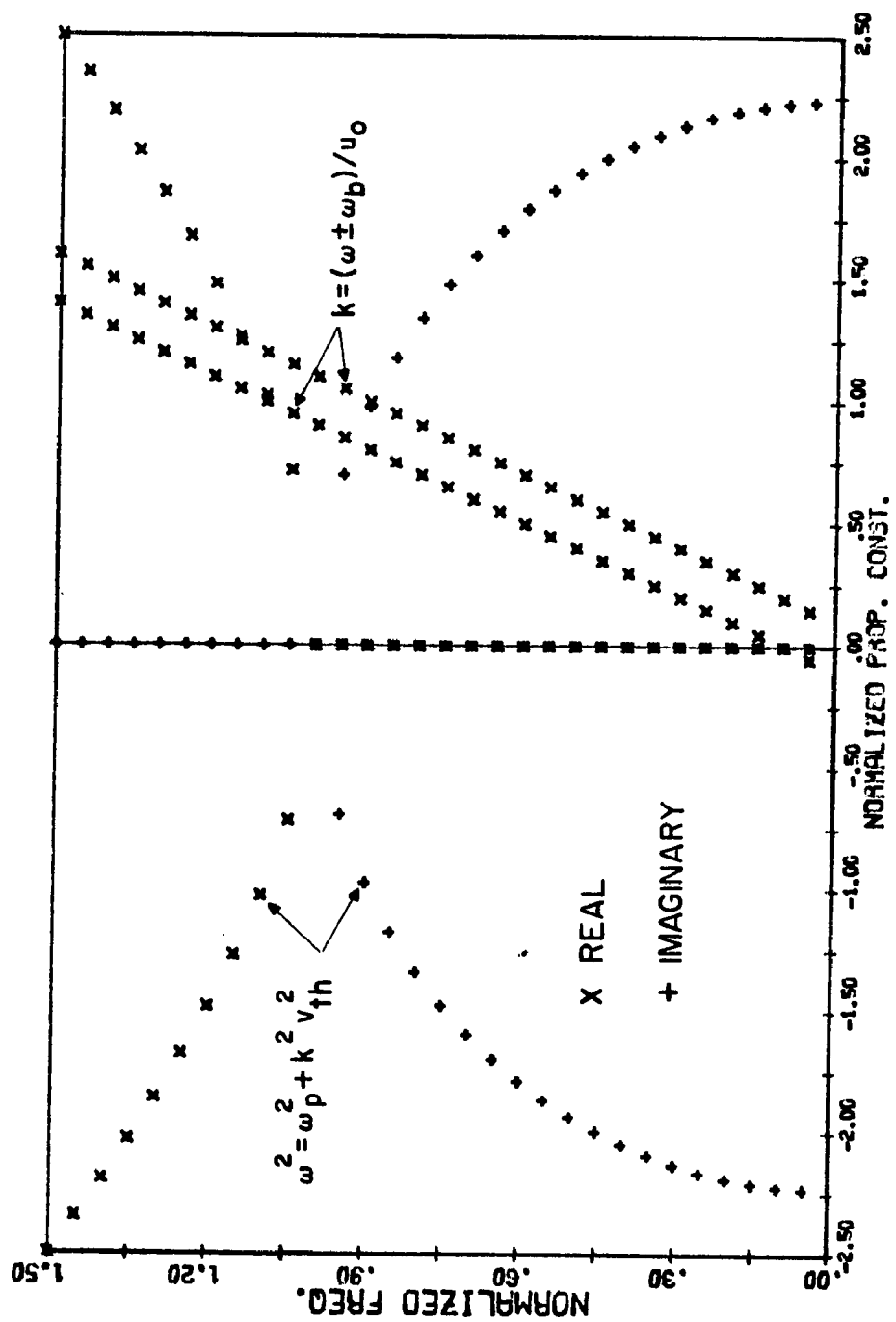


FIG. 2.7 UNCOUPLED MODES OF BEAM SPACE-CHARGE WAVES AND TEMPERATE PLASMA WAVES.

for the cold plasma case. The reason is that for the case of a temperate plasma the diffusion of electrons due to their thermal motion helps restore the plasma neutrality, hence acting in the same manner as the restoring space-charge electric field. This can be expressed in terms of an increase in the value of the plasma frequency

$$\omega_{p,eff} = (\omega_p^2 + v_{th}^2 k^2)^{1/2}, \quad (2.81)$$

which in turn results in a higher value for the frequency at which the maximum coupling is possible.

To study the effect of rectangular inhomogeneity the normalized frequency is plotted vs. the normalized propagation constant with the duty cycle, Δ , taking values equal to 0.1, 0.2, ..., 0.9 and 1.0 in Figs. 2.8 through 2.17, respectively. The numerical values of the other parameters are given as

$$\Omega_b = 0.1 \quad (2.82)$$

and

$$r = 0.2. \quad (2.83)$$

For an electron beam of moderate dc energy, typically 300 eV, a value of $r = 0.2$ corresponds to an electronic temperature of 40 eV. As seen from Figs. 2.8 through 2.17 the spectrum of convective instabilities on the frequency scale can be quenched quite effectively by decreasing the duty cycle. Like the cold plasma case the suppression of lower-frequency modes is much more efficient. Figures 2.18 and 2.19 depict, respectively, the normalized gain, $\text{Im}(ku_o/\omega_p)$, and the normalized growth rate, δ/δ_o , vs. normalized

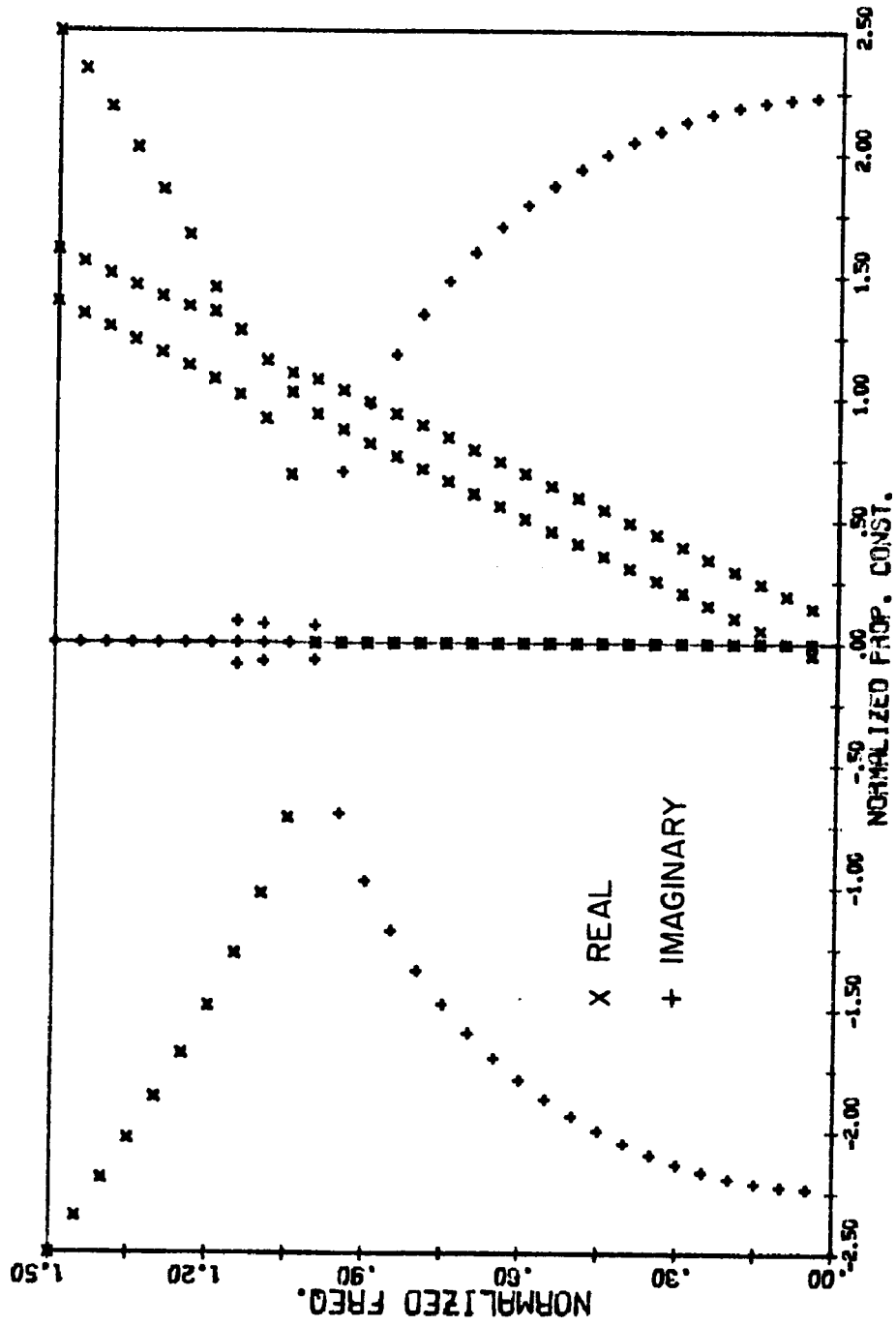


FIG. 2.8 DISPERSION DIAGRAM FOR $\Delta = 0.1$ (RECTANGULAR INHOMOGENEITY).

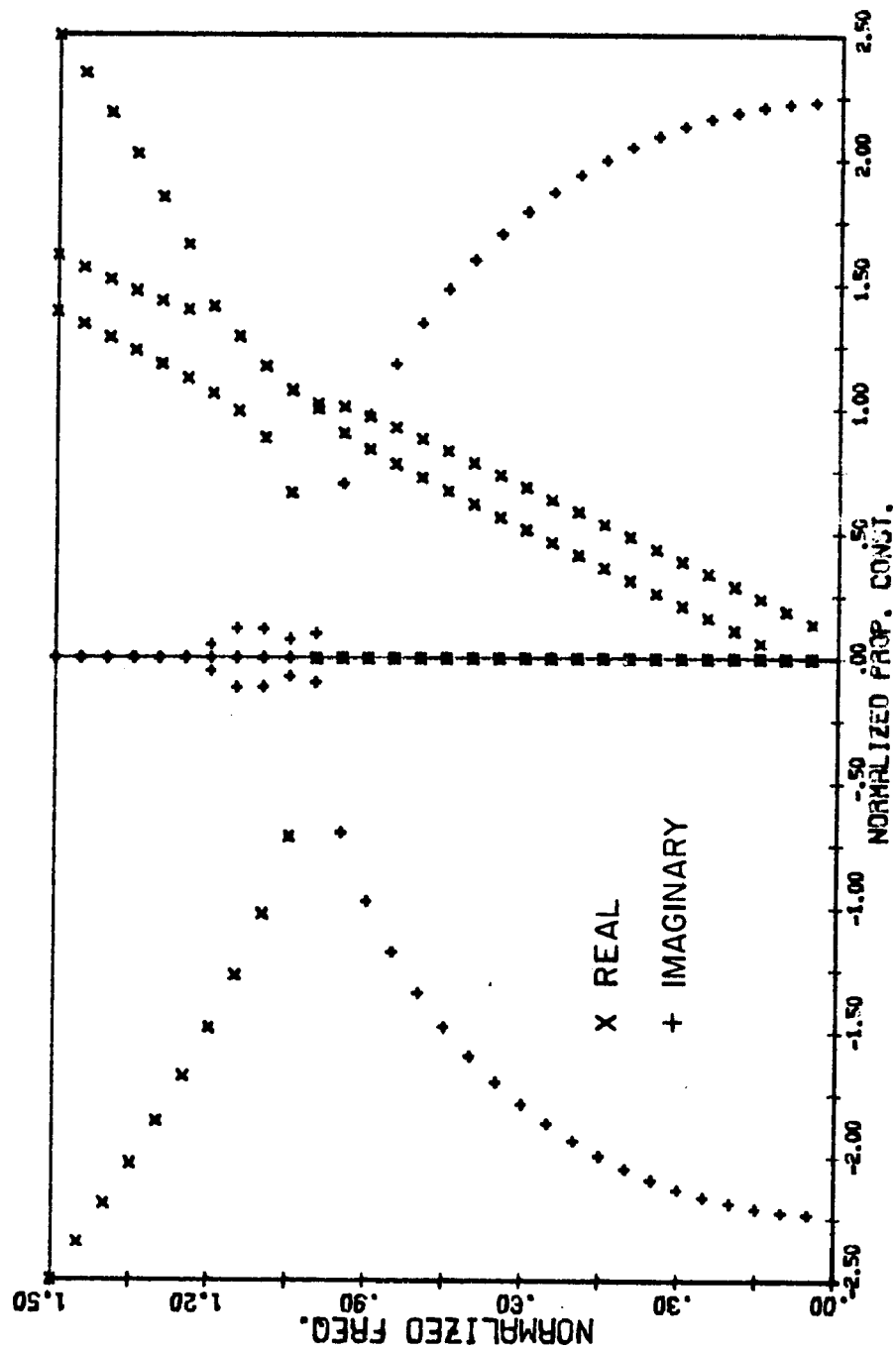


FIG. 2.9 DISPERSION DIAGRAM FOR $\Delta = 0.2$ (RECTANGULAR INHOMOGENEITY).

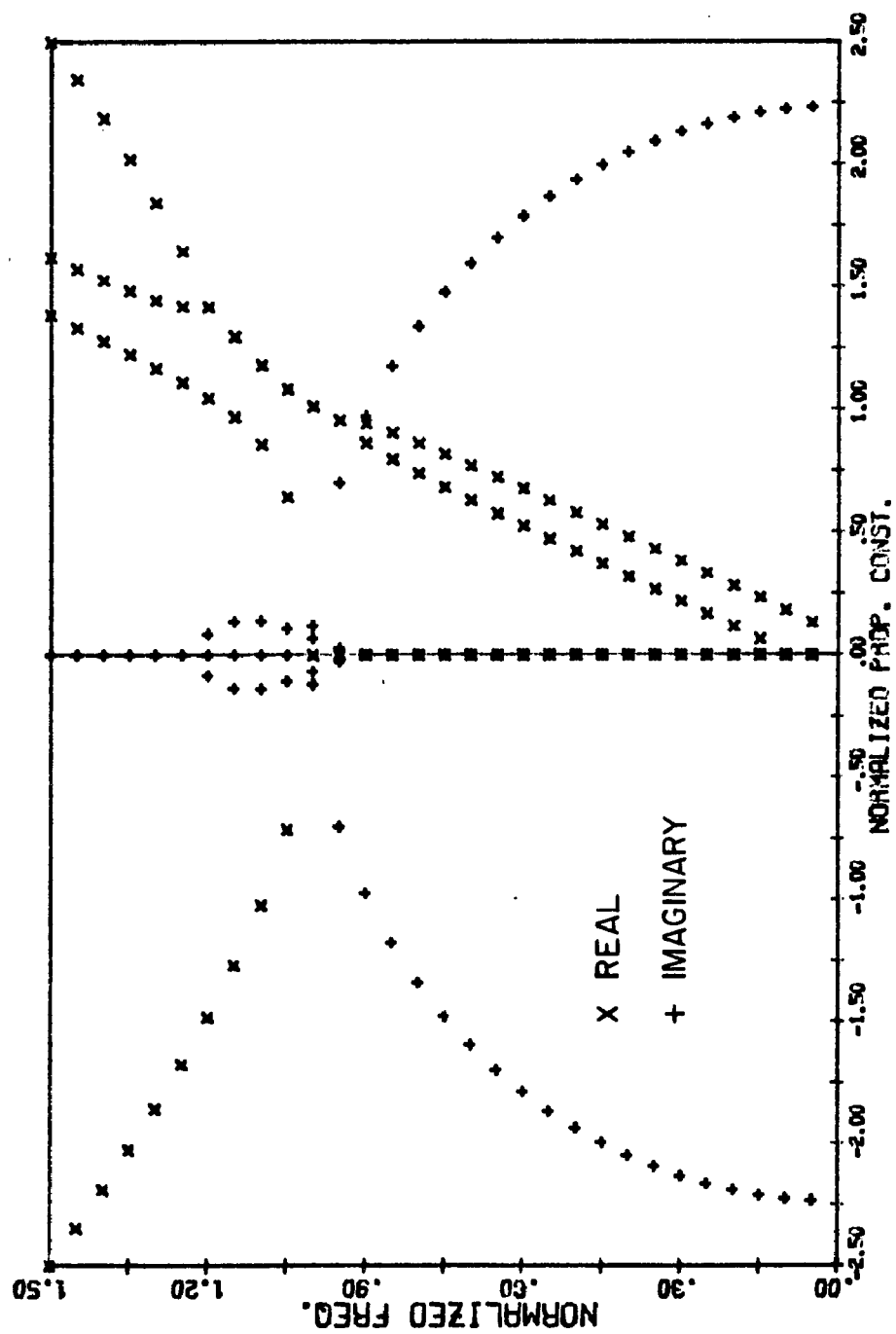


FIG. 2.10 DISPERSION DIAGRAM FOR $\Delta = 0.3$ (RECTANGULAR INHOMOGENEITY).

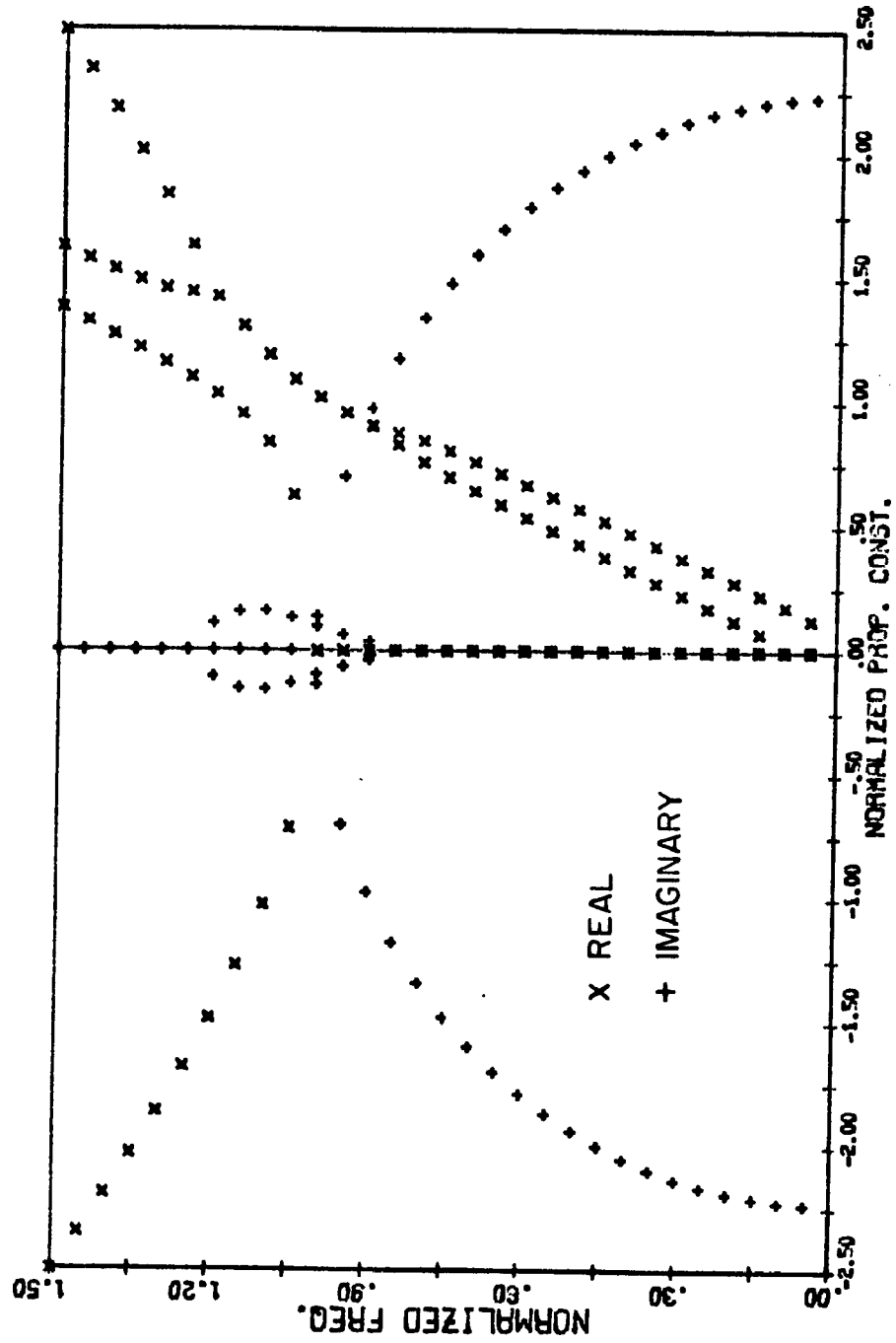
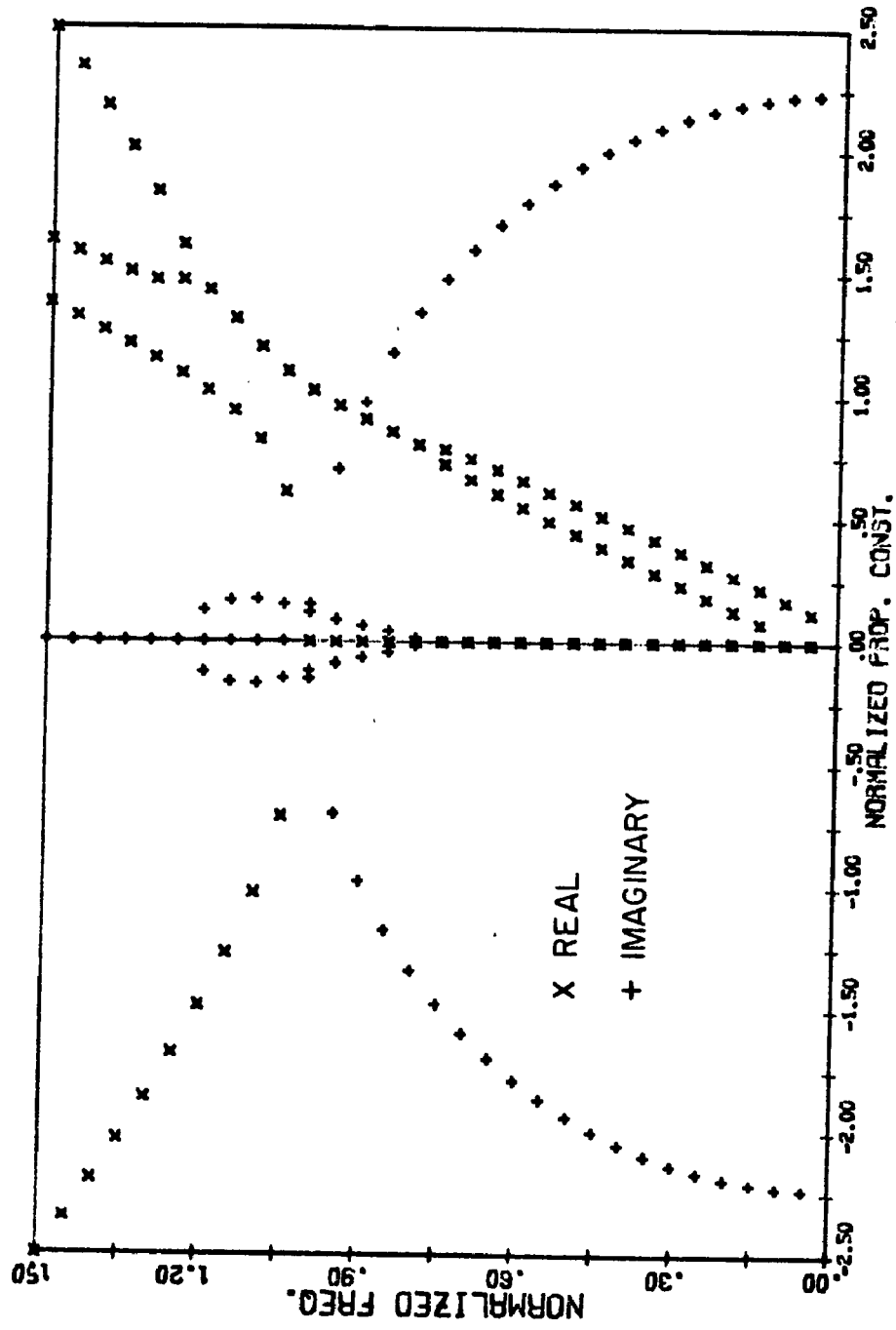
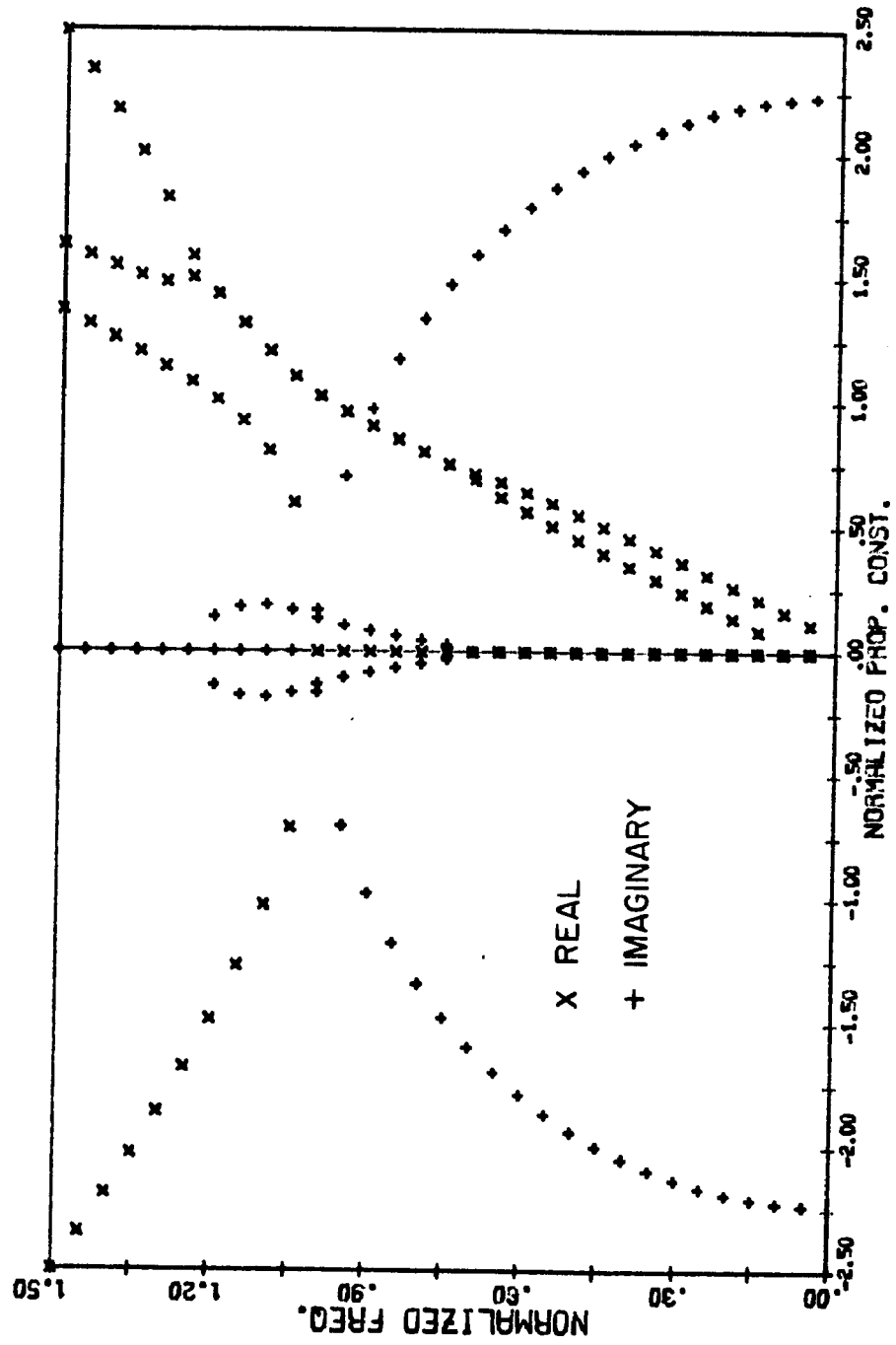


FIG. 2.11 DISPERSION DIAGRAM FOR $\Delta = 0.4$ (RECTANGULAR INHOMOGENEITY).



-FIG. 2.12 DISPERSION DIAGRAM FOR $\Delta = 0.5$ (RECTANGULAR INHOMOGENEITY).



.FIG. 2.13 DISPERSION DIAGRAM FOR $\Delta = 0.6$ (RECTANGULAR INHOMOGENEITY).

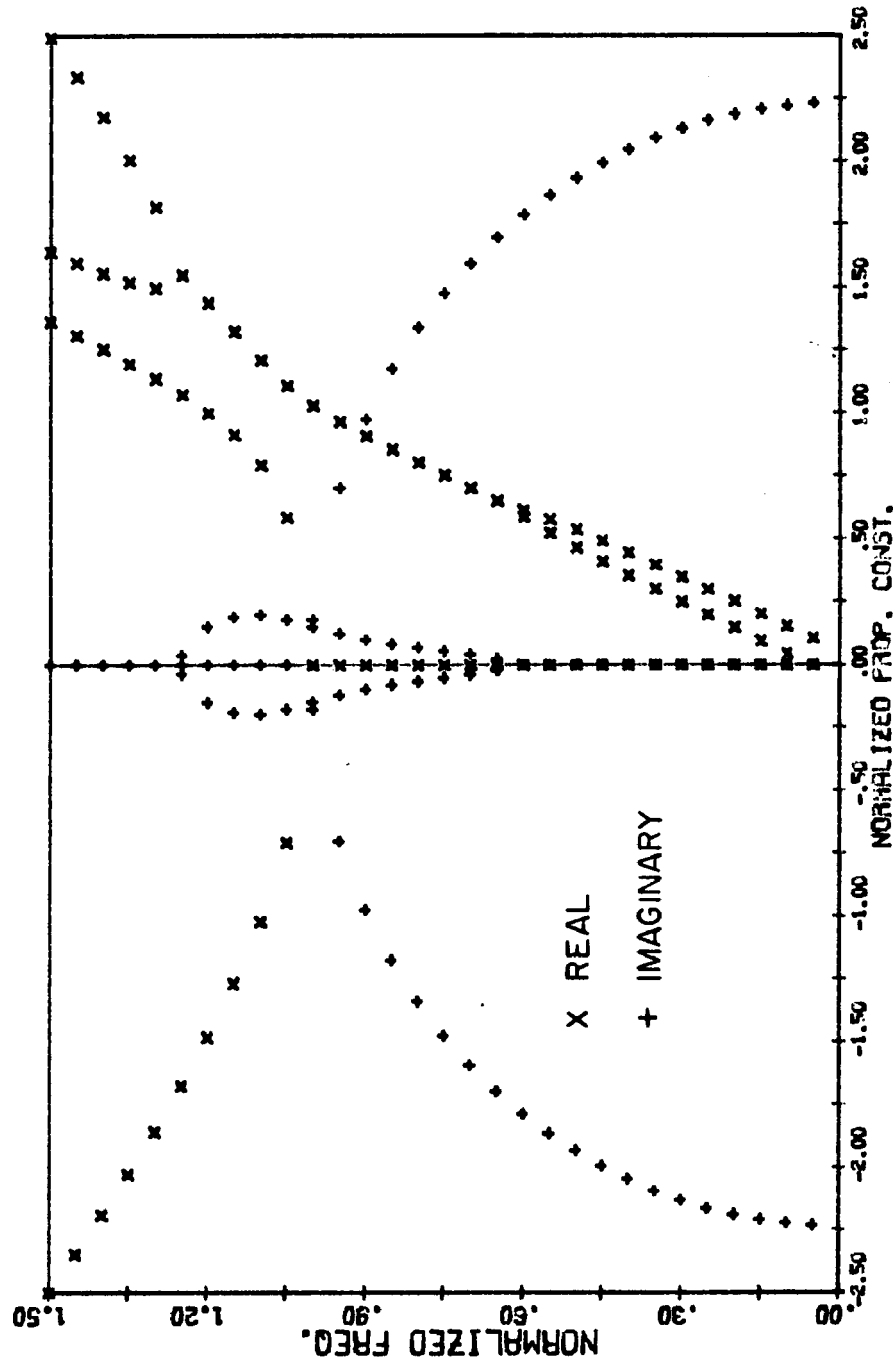


FIG. 2.14 DISPERSION DIAGRAM FOR $\Delta = 0.7$ (RECTANGULAR INHOMOGENEITY).

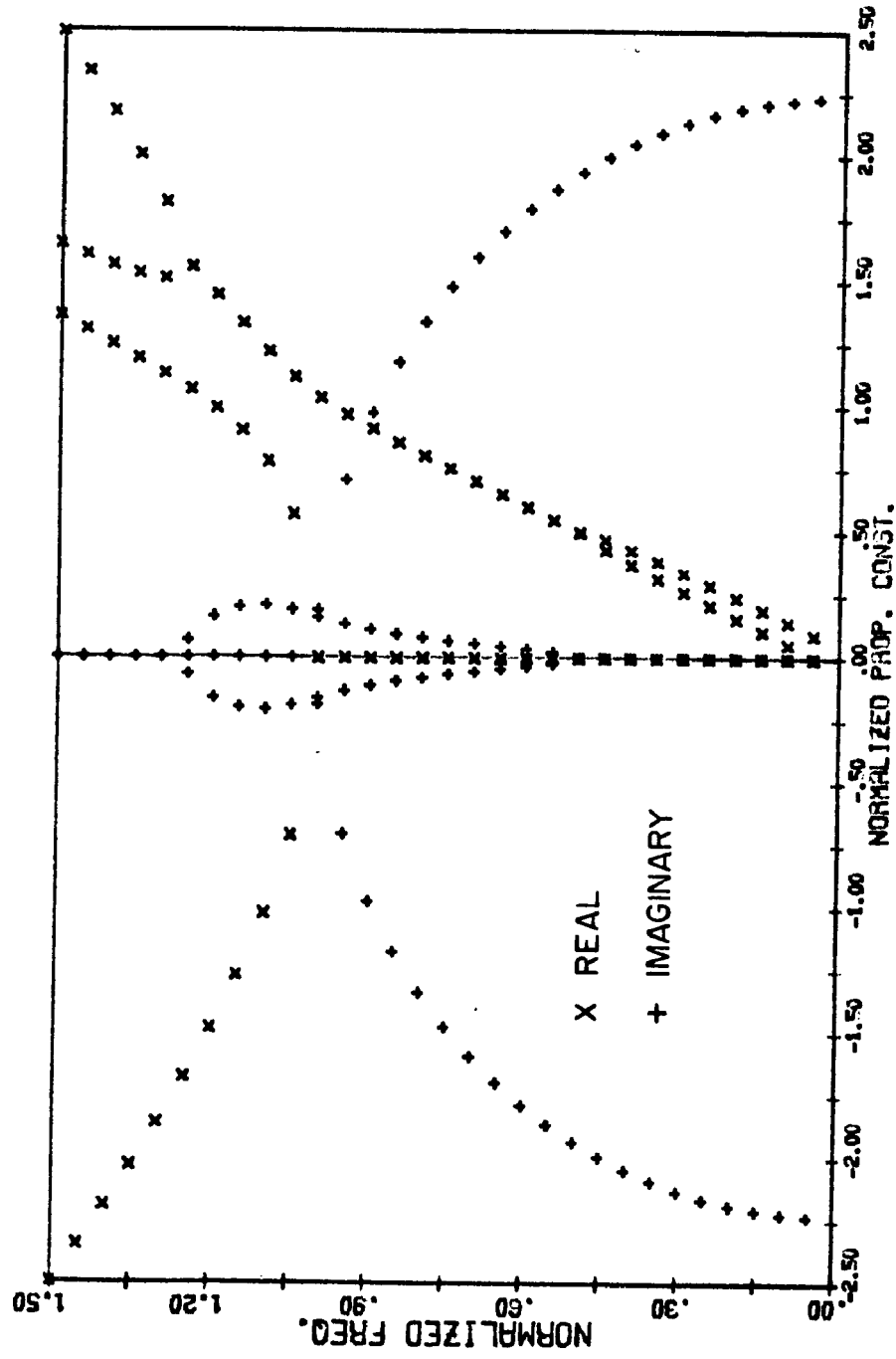


FIG. 2.15 DISPERSION DIAGRAM FOR $\Delta = 0.8$ (RECTANGULAR INHOMOGENEITY).

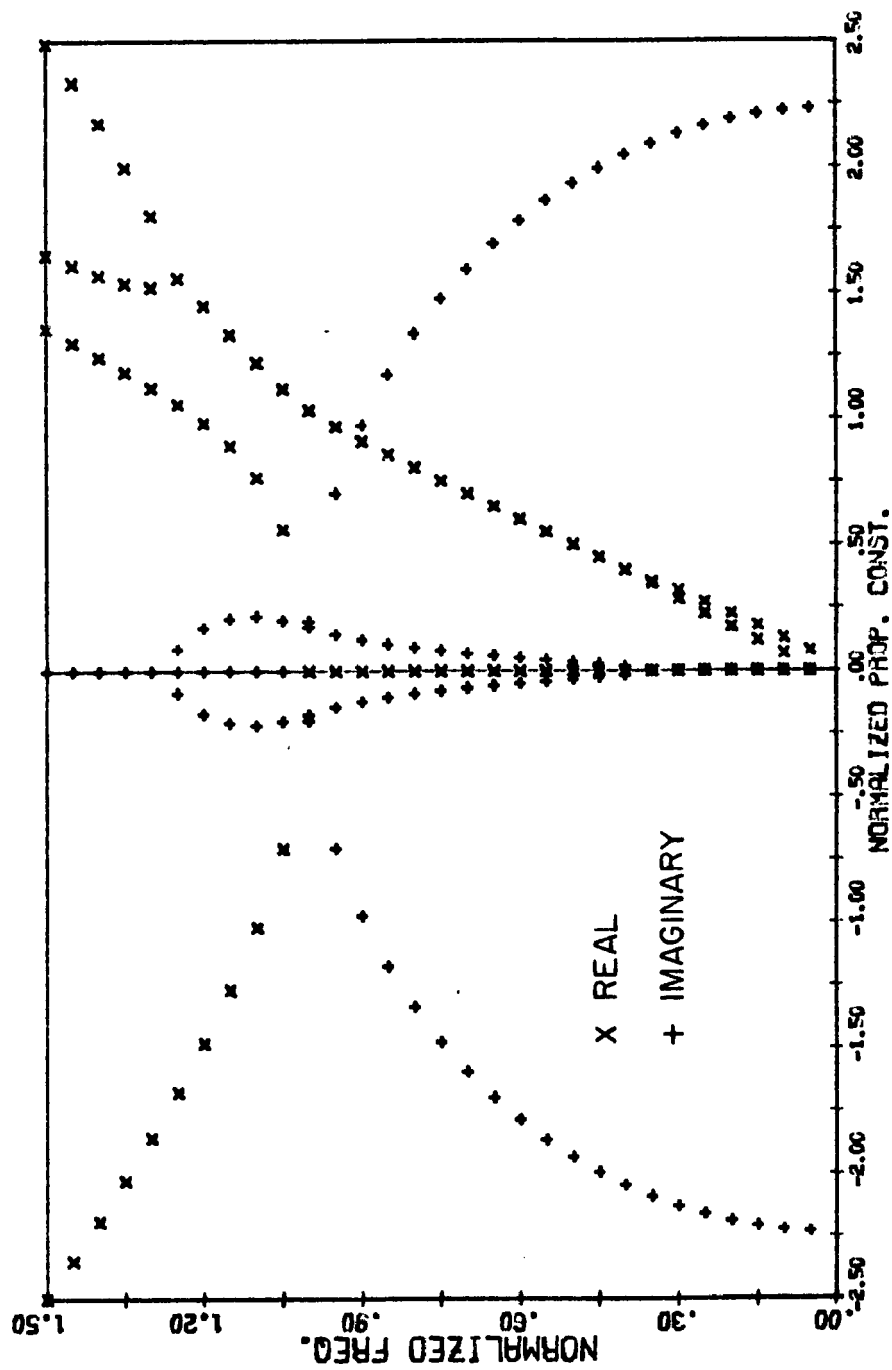


FIG. 2.16 DISPERSION DIAGRAM FOR $\Delta = 0.9$ (RECTANGULAR INHOMOGENEITY).

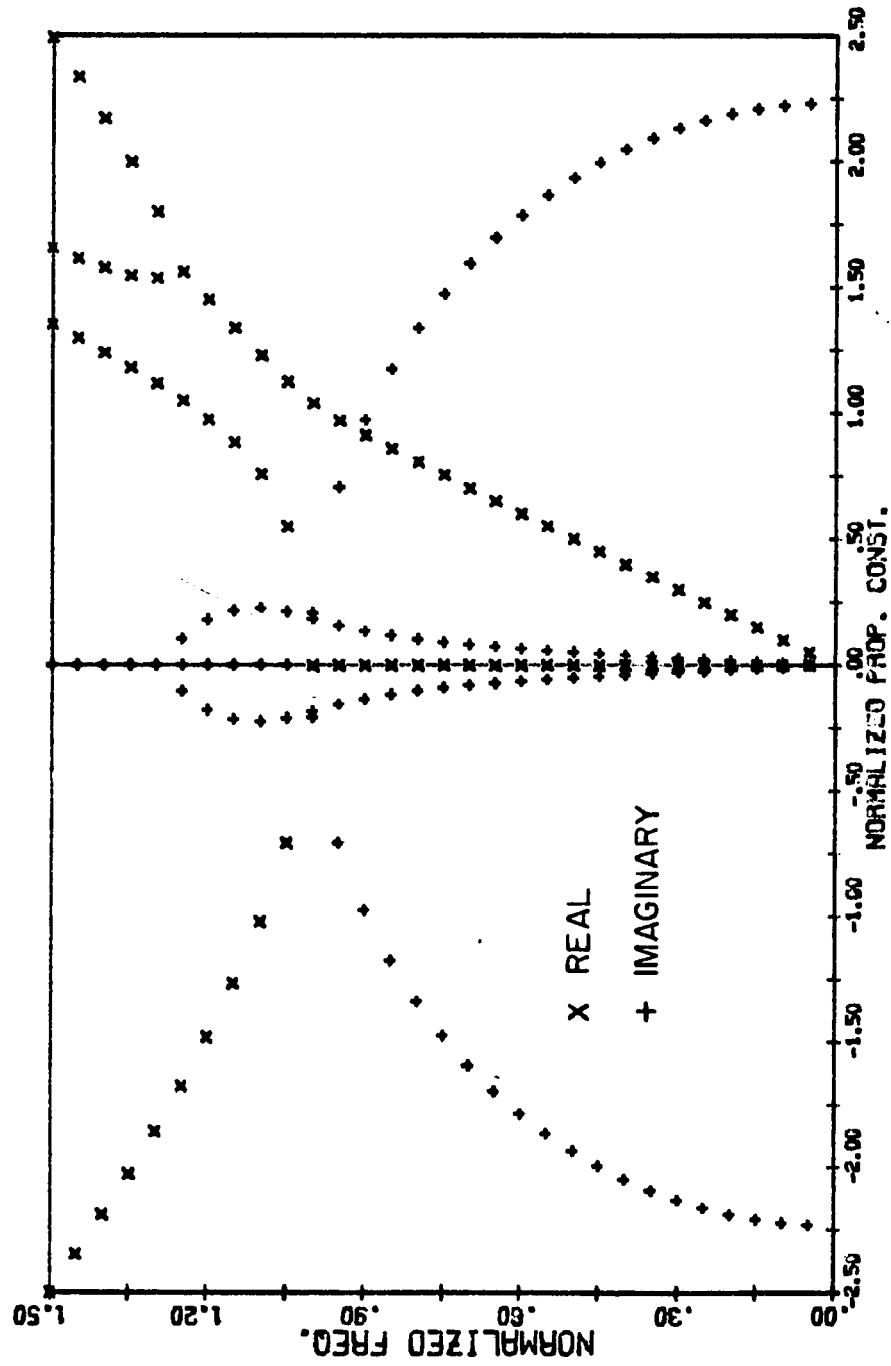


FIG. 2.17 DISPERSION DIAGRAM FOR $\Delta = 1$ (HOMOGENEOUS BEAM).

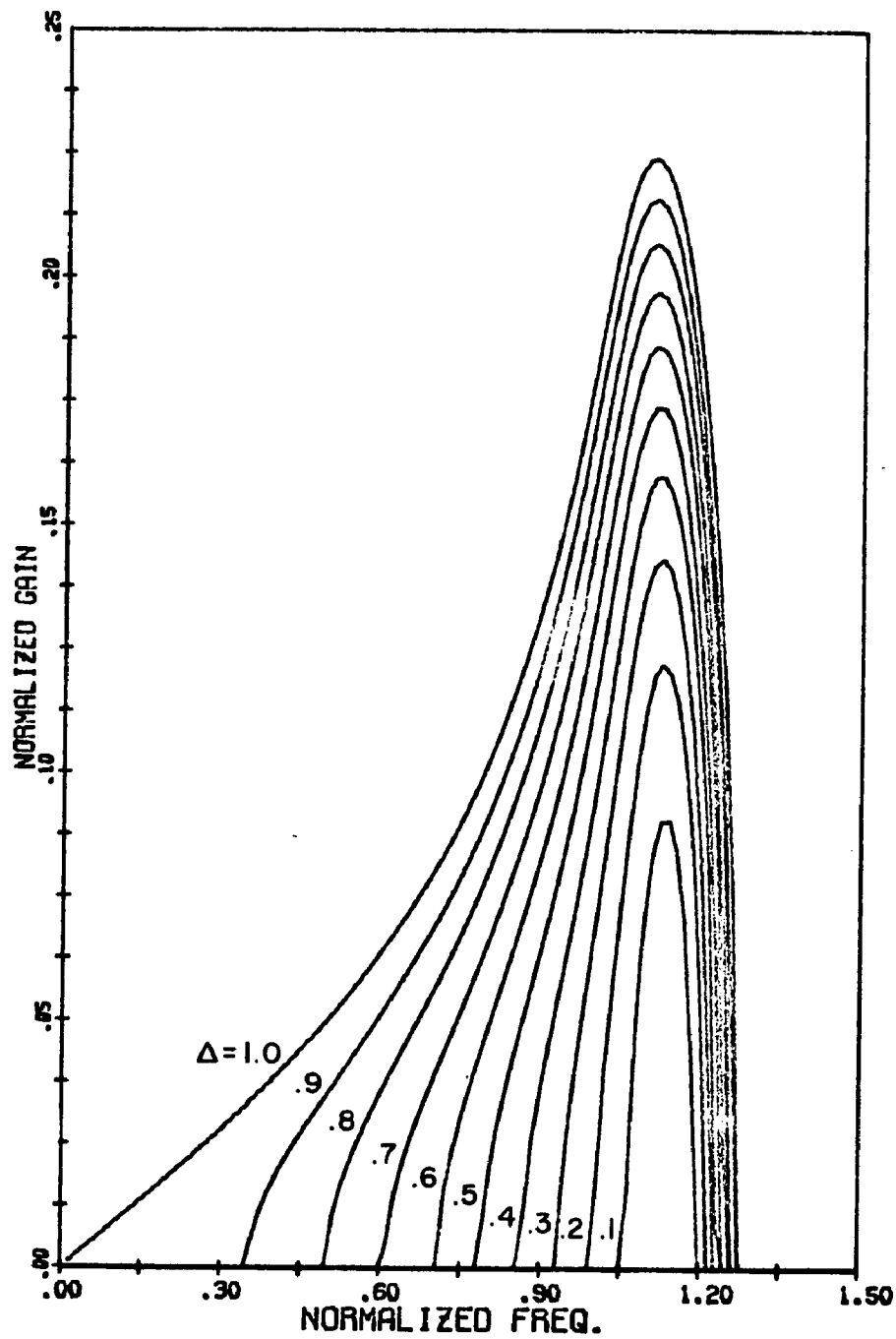


FIG. 2.18 NORMALIZED GAIN, $k_1 u_0 / \omega_p$, VS. NORMALIZED FREQUENCY, ω / ω_p , WITH THE DUTY CYCLE, Δ , TAKEN AS A PARAMETER (RECTANGULAR INHOMOGENEITY). $[\gamma / 2(k_B T_e / eV_0) = 0.2]$

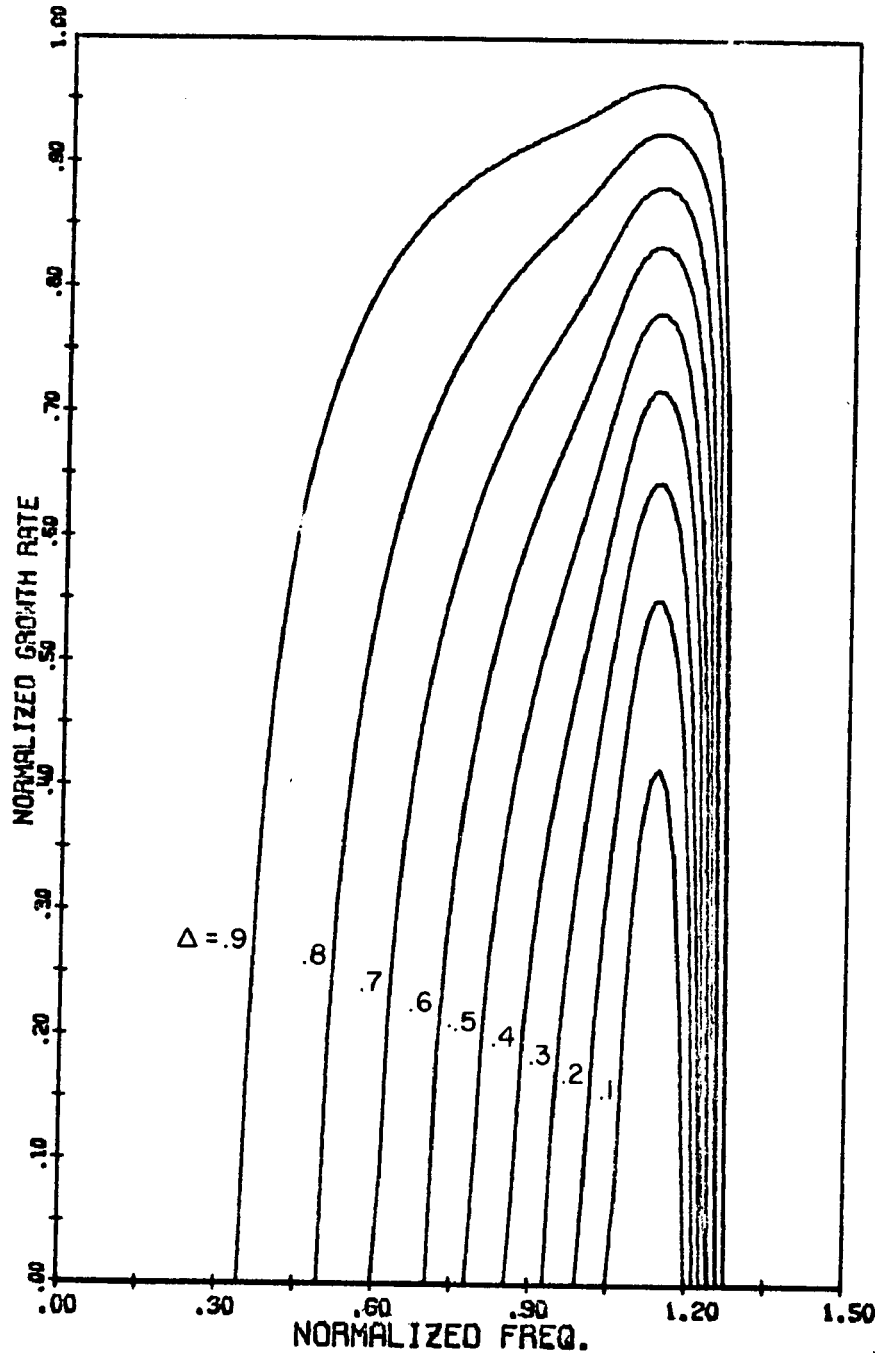


FIG. 2.19 NORMALIZED GROWTH RATE, δ/δ_0 , VS. NORMALIZED FREQUENCY, ω/ω_p , WITH THE DUTY CYCLE, Δ , TAKEN AS A PARAMETER (RECTANGULAR INHOMOGENEITY). $[\gamma/2(k_B T_e/eV_0) = 0.2]$

frequency, ω/ω_p , with the duty cycle, Δ , taken as a parameter. A narrowing of the k-spectrum is observed (Fig. 2.18) which corresponds to smaller growth rates for lower-duty cycles (Fig. 2.19). Similar results are obtained if the electronic temperature is increased. Figures 2.20 and 2.21 repeat the last two figures with the exception that the value of r is assumed to be equal to 0.4, corresponding to an equivalent electron temperature of 80 eV for an electron beam having a dc energy of 300 eV.

2.5.4 Sinusoidal Inhomogeneity. A density variation in accordance with Eq. 2.41 is considered for the beam electrons. In reference to Appendix B and the dispersion relation (Eq. 2.69), the following dispersion relation is obtained for this case:

$$\frac{\omega^2 - k^2 v_{th}^2}{\omega_p^2} - \left[\left(1 - \frac{\omega_b^2}{\omega_D^2} \right)^2 - \alpha^2 \frac{\omega_b^2}{\omega_D^2} \right]^{-1/2} - \frac{2}{\pi^2} \frac{\omega_p^2}{\omega_0^2} \left(1 - \frac{v_{th}^2}{u_0^2} \right)^{-1} \sum_{n=1}^{\infty} \frac{I_n^2}{n^2} = 0, \quad (2.84)$$

where α is the depth of modulation and I_n is a quantity defined by Eq. 2.43. For evaluation of I_n , a method which uses a three-term recurrence relation is offered in Appendix B. In the same manner as before, if terms of order $(\omega_p/\omega_0)^2$ are ignored, the final form of the dispersion relation can be expressed as

$$\left(1 - \frac{\omega_b^2}{\omega_D^2} \right)^2 - \alpha^2 \frac{\omega_b^4}{\omega_D^4} - \left(\frac{\omega_p^2}{\omega^2 - k^2 v_{th}^2} \right)^2 = 0. \quad (2.85)$$

The above equation is of eighth degree in k (or ω). Care must be employed in the interpretation of the physical implications of the degree of the

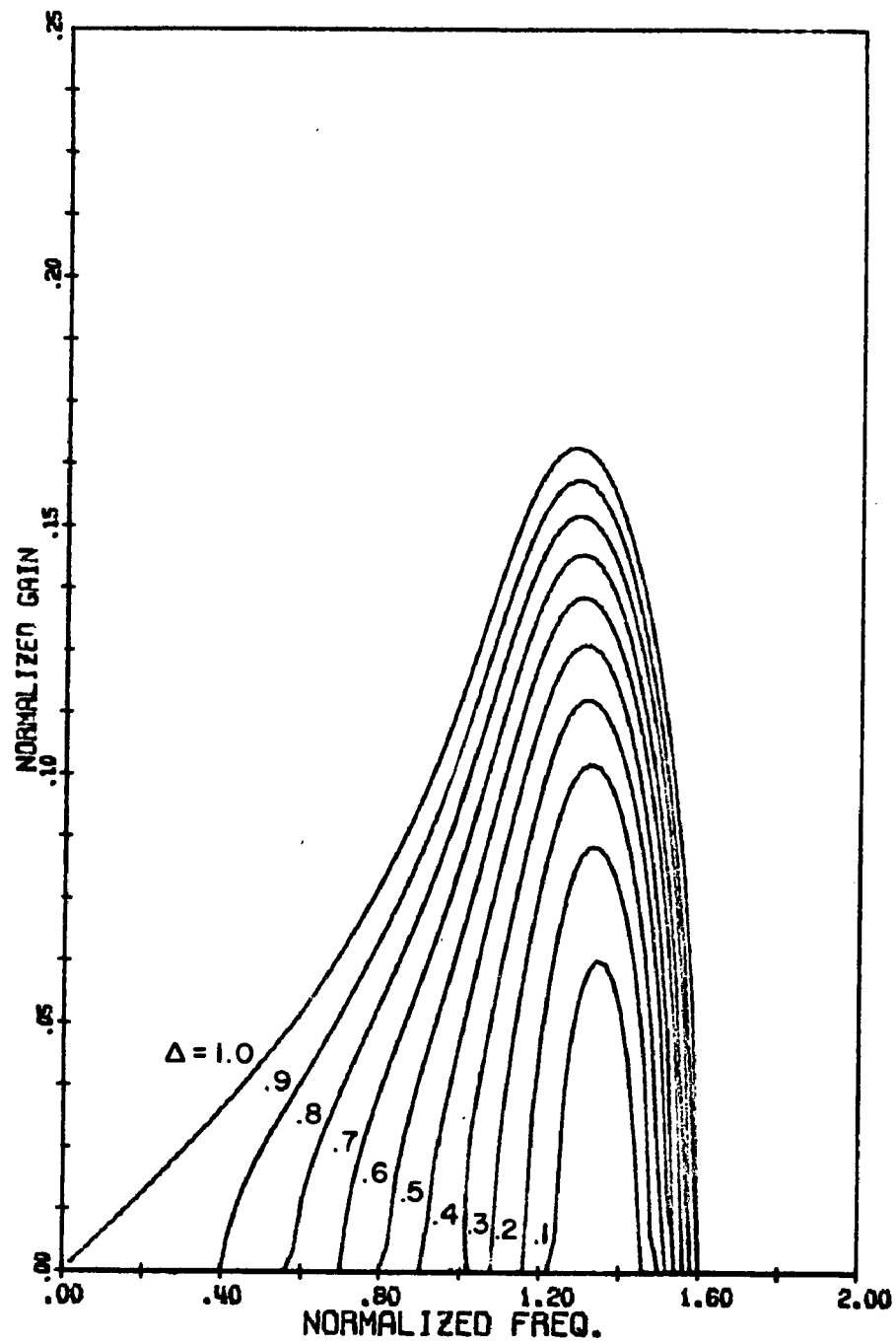


FIG. 2.20 NORMALIZED GAIN, $k_i u_o / \omega_p$, VS. NORMALIZED FREQUENCY, ω / ω_p , WITH THE DUTY CYCLE, Δ , TAKEN AS A PARAMETER (RECTANGULAR INHOMOGENEITY). $[\gamma/2(k_B T_e / eV_o) = 0.4]$

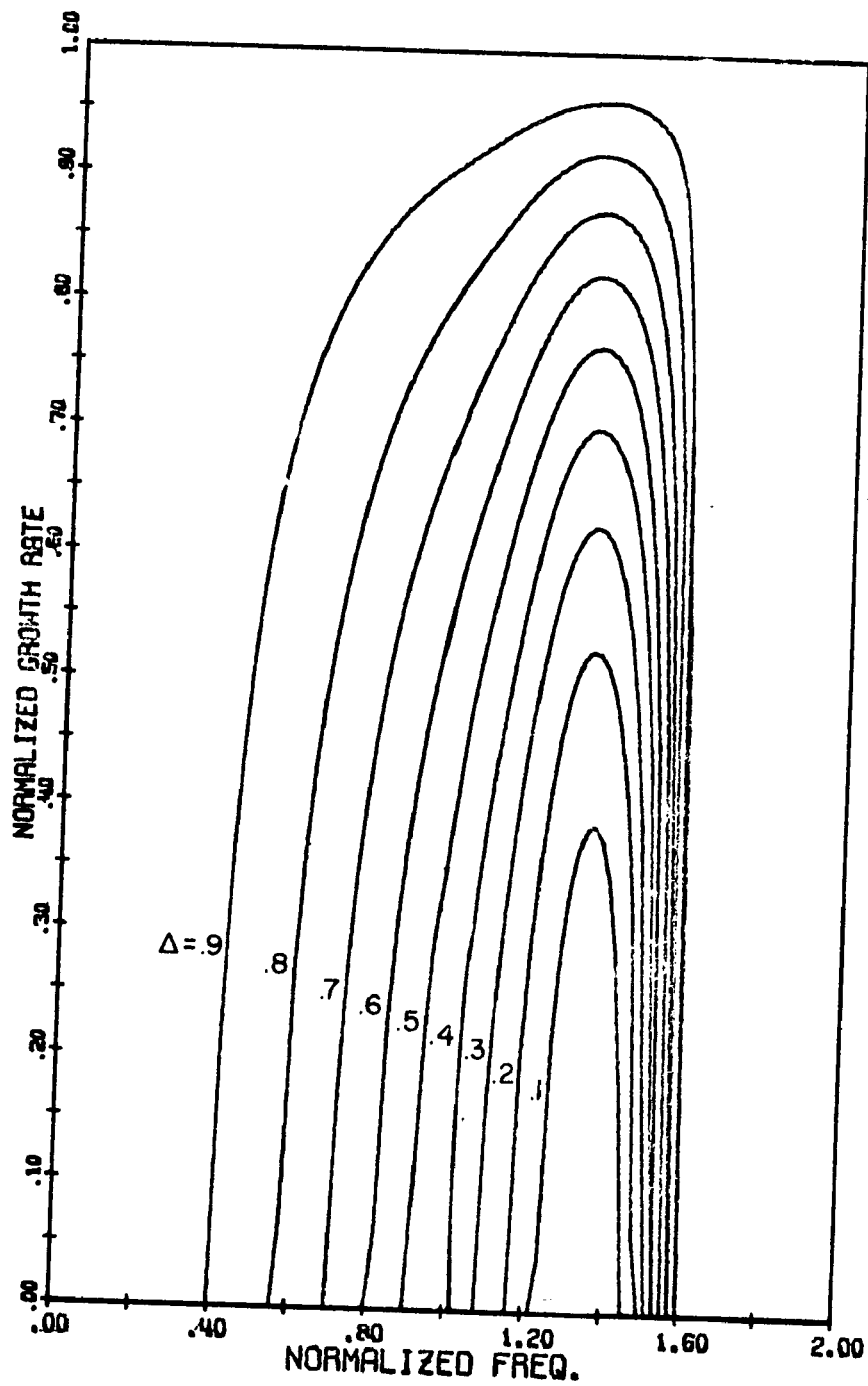


FIG. 2.21 NORMALIZED GROWTH RATE, δ/δ_0 , VS. NORMALIZED FREQUENCY, ω/ω_p , WITH THE DUTY CYCLE, Δ , TAKEN AS A PARAMETER (RECTANGULAR INHOMOGENEITY). $[\gamma/2(k_B T_e / eV_0) = 0.4]$

above dispersion relation. There are only four distinct modes in the system corresponding to the slow and fast space-charge waves of the beam and symmetrical temperate plasma modes. Four extra modes are artificially introduced in the system through mathematical operations involved in the derivation of Eq. 2.85. In tracing the roots of the above equation by means of computers care must therefore be employed in choosing the roots which correspond to the physically real modes.

In terms of the normalized parameters Ω , Ω_b and r , as defined previously, a polynomial of eighth degree in $x \equiv k u_0 / \omega_p$ is found in the form

$$\sum_{i=0}^8 A_i x^i = 0 , \quad (2.86)$$

where the A_i coefficients are given as follows:

$$A_0 \equiv (r - 1)^2 (1 - \alpha^2) \Omega^4 \Omega_b^4 , \quad (2.87)$$

$$A_1 \equiv -4r(r - 1)(1 - \alpha^2) \Omega^3 \Omega_b^4 , \quad (2.88)$$

$$A_2 \equiv -2\Omega^2 \Omega_b^2 [(r - 1)^2 \Omega^2 - r(3r - 1)(1 - \alpha^2) \Omega_b^2] , \quad (2.89)$$

$$A_3 \equiv 4r\Omega \Omega_b^2 [2(r - 1)\Omega^2 - r(1 - \alpha^2) \Omega_b^2] , \quad (2.90)$$

$$A_4 \equiv (r - 1)^2 \Omega^4 - 4r(3r - 1) \Omega^2 \Omega_b^2 + r^2 (1 - \alpha^2) \Omega_b^4 - 1 , \quad (2.91)$$

$$A_5 \equiv -4r\Omega [(r - 1)\Omega^2 - 2r\Omega_b^2] , \quad (2.92)$$

$$A_6 \equiv 2r[(3r - 1)\Omega^2 - r\Omega_b^2] , \quad (2.93)$$

$$A_7 \equiv -4r^2\Omega \quad (2.94)$$

and

$$A_8 \equiv r^2 . \quad (2.95)$$

The effect of a sinusoidal inhomogeneity, on the k - and ω -spectrum is shown in Figs. 2.22 and 2.23 where the normalized gain, $\text{Im}(ku_o/\omega_p)$, and the normalized growth rate, δ_α/δ_o , are plotted vs. normalized frequency, ω/ω_p , respectively, for $\Omega_b = 0.1$ and $r = 0.2$. As seen, unlike the case of a rectangular inhomogeneity and like the zero-temperature, sinusoidal case, there is no narrowing of the ω -spectrum. The narrowing of the k -spectrum is considerably smaller than that which was obtained for the rectangular case. Higher suppression rates are also obtained at lower frequencies. The effect of a temperature increase is shown in Figs. 2.24 and 2.25 for which r is taken to have a value equal to 0.4. The results are qualitatively the same as the previous case. An increase in temperature simply results in an extension of the frequency bandwidth over which the convective instability occurs toward slightly higher frequencies due to an increase in the value of the effective plasma frequency.

2.5.5 Discussion of the Results. The results of the temperate plasma case have indicated suppression of convective instabilities as a result of periodic beam inhomogeneity in a similar manner as obtained

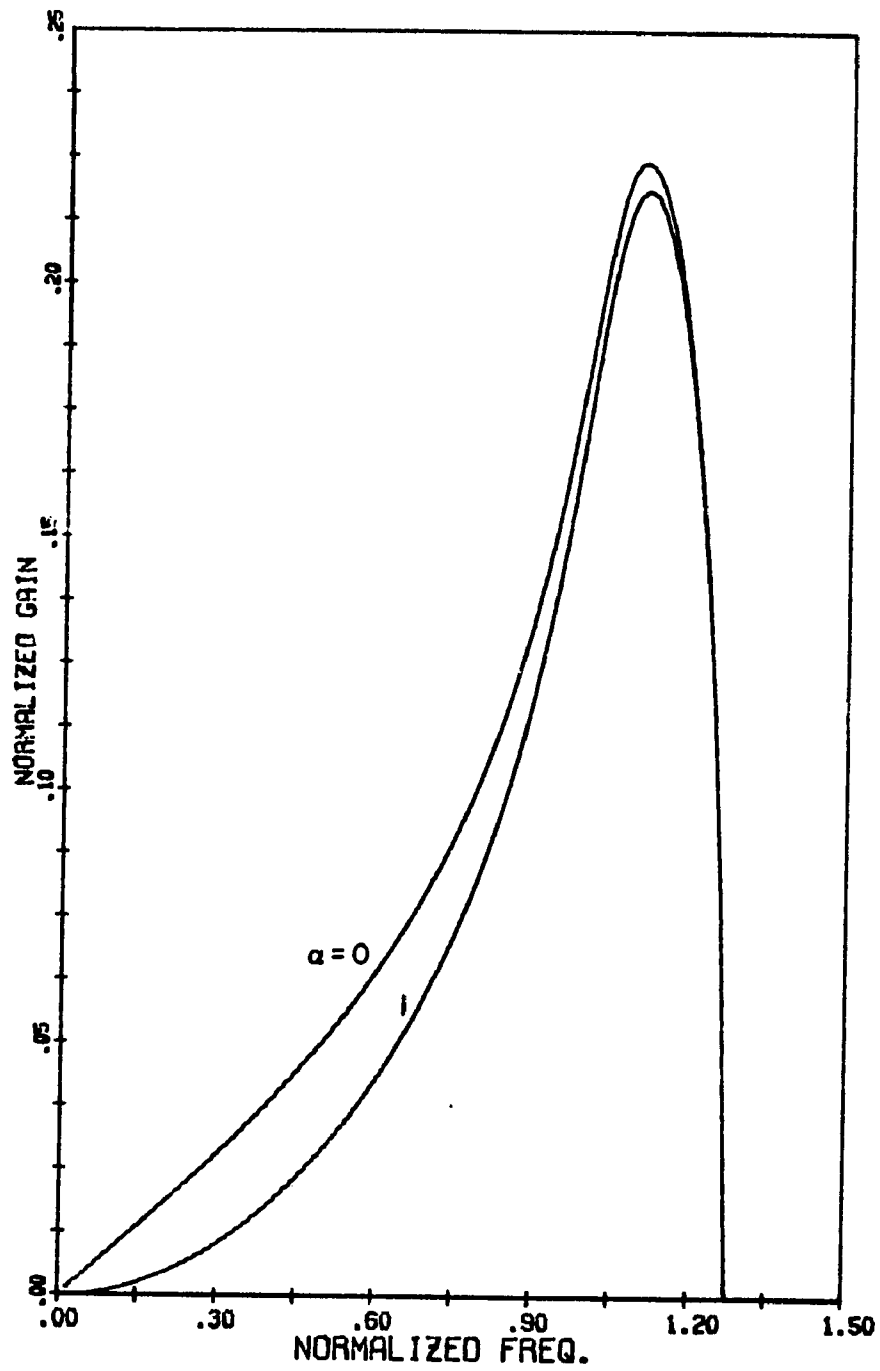


FIG. 2.22 NORMALIZED GAIN, $k_1 u_o / \omega_p$, VS. NORMALIZED FREQUENCY, ω / ω_p , WITH DEPTH OF MODULATION, α , TAKEN AS A PARAMETER (SINUSOIDAL INHOMOGENEITY). $[\gamma / 2(k_B T_e / eV_o) = 0.2]$

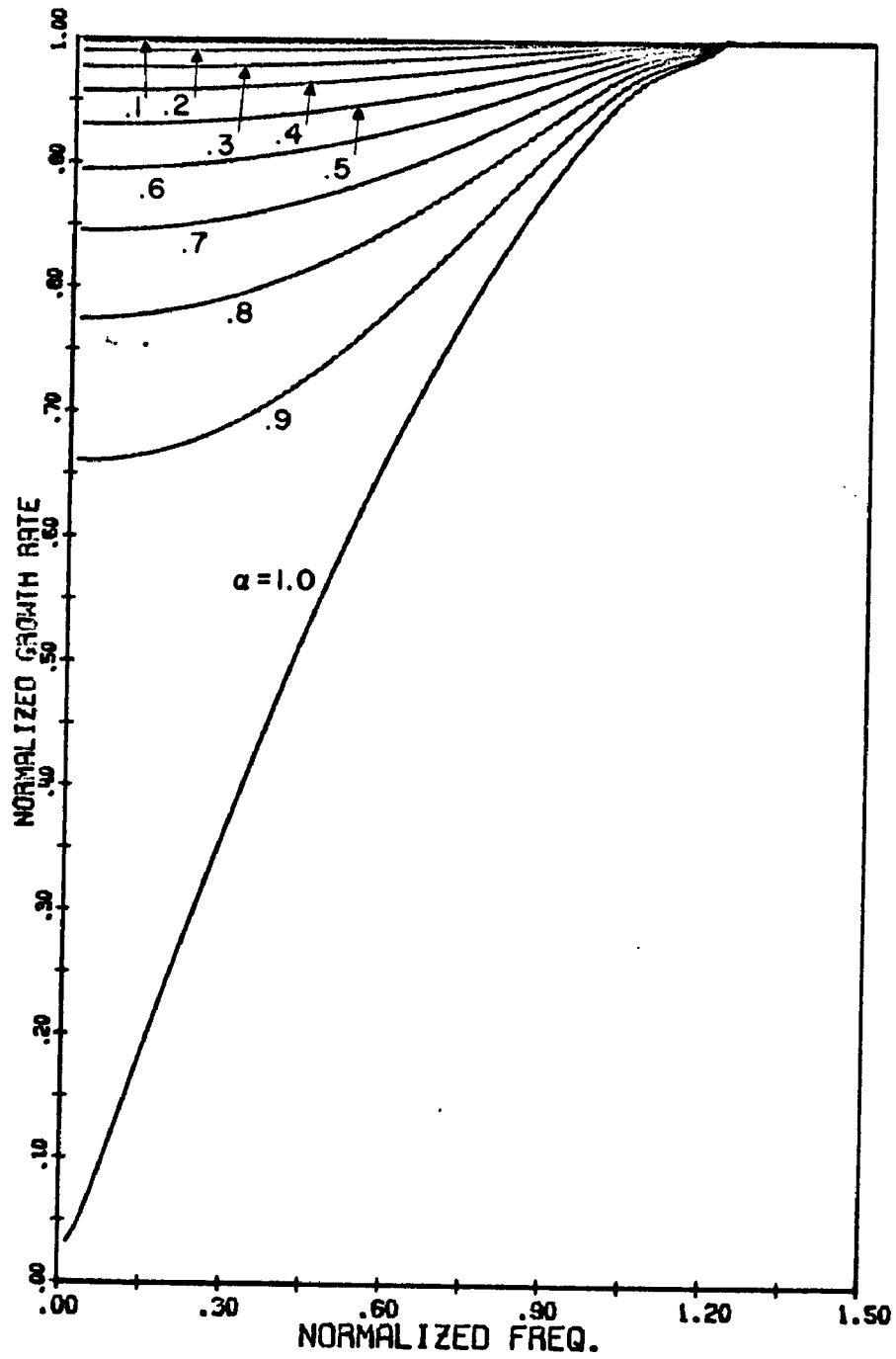


FIG. 2.23 NORMALIZED GROWTH RATE, δ/δ_0 , VS. NORMALIZED FREQUENCY, ω/ω_p , WITH DEPTH OF MODULATION, α , TAKEN AS A PARAMETER (SINUSOIDAL INHOMOGENEITY). $[\gamma/2(k_B T_e / eV_0) = 0.2]$

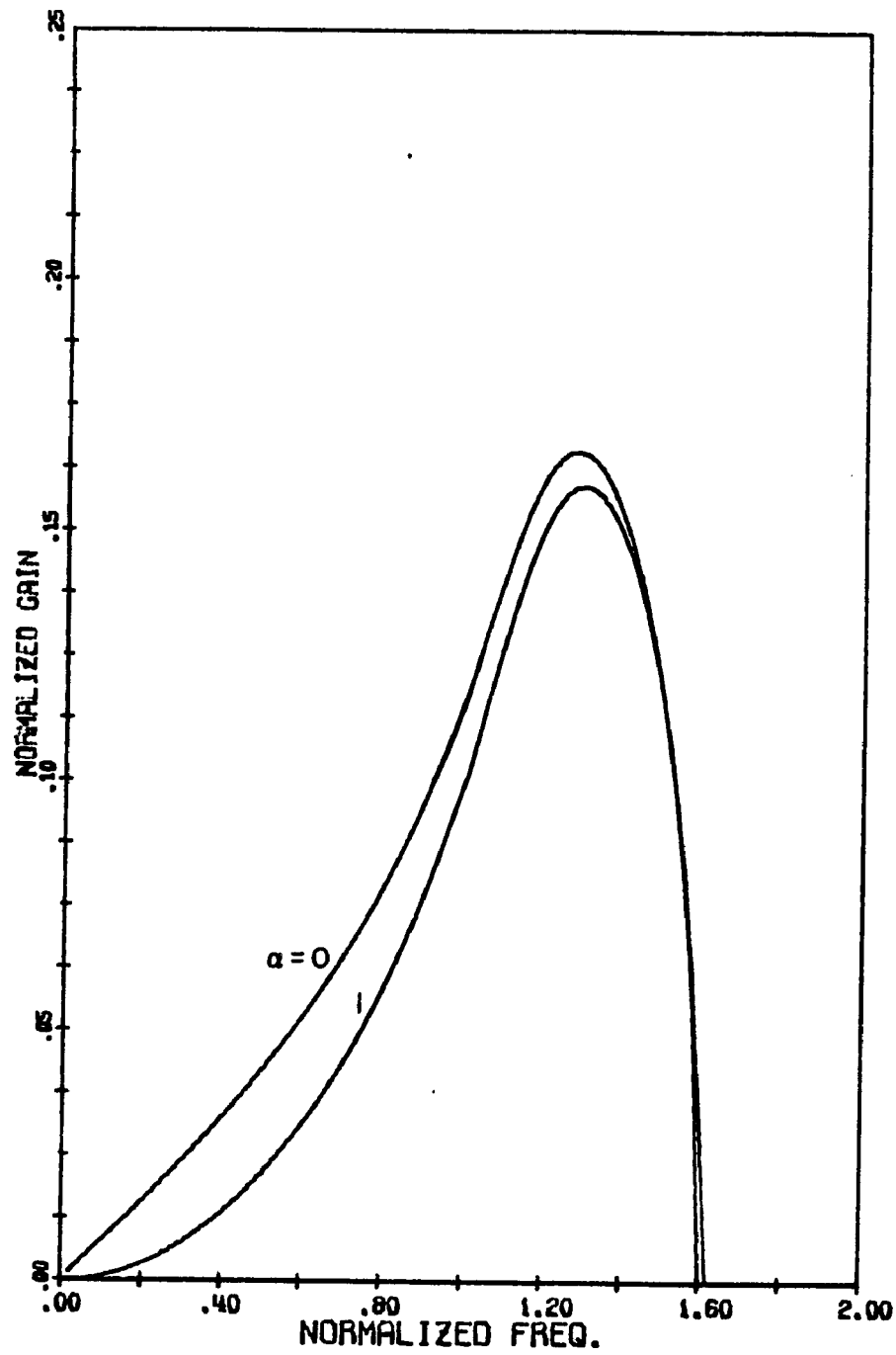


FIG. 2.24 NORMALIZED GAIN, $k_1 u_o / \omega_p$, VS. NORMALIZED FREQUENCY, ω / ω_p , WITH DEPTH OF MODULATION, α , TAKEN AS A PARAMETER (SINUSOIDAL INHOMOGENEITY). $[\gamma / 2(k_B T_e / eV_o) = 0.4]$

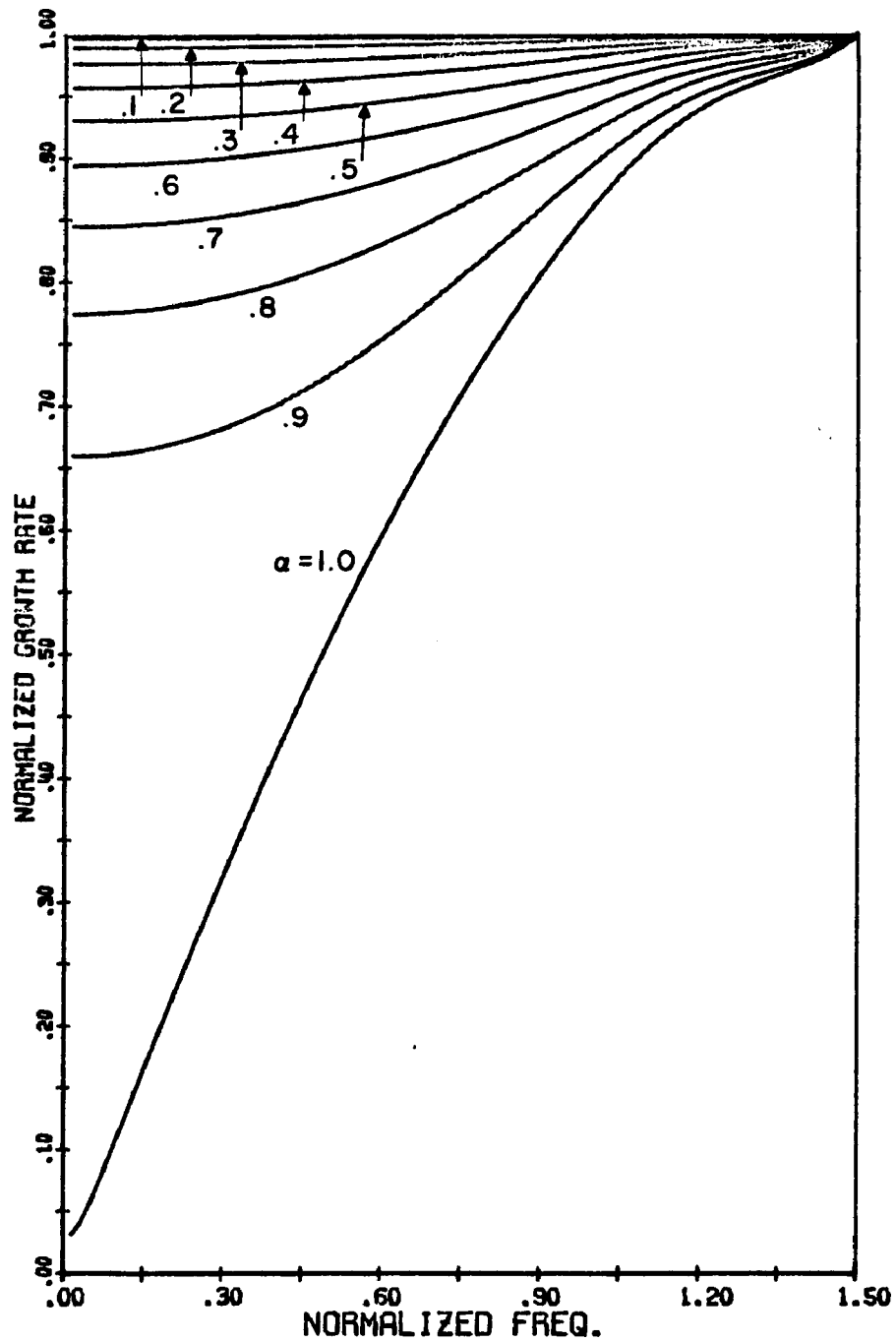


FIG. 2.25 NORMALIZED GROWTH RATE, δ/δ_0 , VS. NORMALIZED FREQUENCY, ω/ω_p , WITH DEPTH OF MODULATION, α , TAKEN AS A PARAMETER (SINUSOIDAL INHOMOGENEITY). $[\gamma/2(k_B T_e / eV_0) = 0.4]$

previously for the cold plasma. The effect of temperature is simply to remove the singularity of the propagation constant at the plasma frequency. A narrowing of both the ω - and k -spectra is observed for the case of a rectangular inhomogeneity, whereas the results of sinusoidal inhomogeneity have indicated a narrowing of the k -spectrum only. Narrowing of both spectra is more effectively achieved by means of rectangular modulation. Insufficiency of the mathematical technique, employed in the present study, to investigate the effect of beam inhomogeneities on the ion-acoustic oscillations of the plasma was shown to be due to the low values for the wavelength of ion-acoustic oscillations which cast doubt on the validity of the long wavelength assumption essential to the use of the method of averaging. The above fact, in addition to the large proton-to-electron mass ratio, allows neglect of the plasma ion motion altogether.

CHAPTER III. TWO-DIMENSIONAL FORMULATION OF THE INHOMOGENEOUS BEAM-PLASMA INTERACTION: ZERO MAGNETIC FIELD CASE

3.1 Introduction

The interaction of a periodically inhomogeneous electron beam of finite transverse extent with its surrounding plasma is considered in this chapter. No applied static magnetic field exists and plasma and beam particles can thus respond to the ac fields with no preference for transverse or longitudinal motion. The surrounding plasma can assume either an infinite or a finite transverse extent. The analysis is extended further to include finite plasma temperature effects.

The hydrodynamic model is adopted for both the beam and the plasma under the general assumptions stated in Chapter II for the validity of the model. The effect of finite temperature of the plasma electrons is taken into account by adding an extra term to the force equation to account for the electron gas pressure.

3.2 Mathematical Description of the System

3.2.1 Assumptions. The following assumptions are made in the course of formulating the problem:

1. The hydrodynamic model applies to both the beam and the plasma.
2. The plasma is cold, stationary and uniform with infinitely heavy ions serving as a neutralizing background.
3. The electron beam is cold, periodically inhomogeneous and drifting with an axial velocity u_0 .
4. There is no applied static magnetic field.

5. Perturbations are assumed to be of small magnitude so as to make a linear analysis possible.

6. The quasi-static approximation is valid.

7. Nonrelativistic mechanics applies.

The physical limitations, as well as the implications, of the above assumptions were discussed in some detail in Chapter II. The quasi-static approximation implies that the ac electric field, due to space-charge effects, is derivable from a scalar potential satisfying an equation of state in the form of an integral or a differential equation subject to proper boundary requirements. The quasi-static approximation loses its validity for waves whose phase velocities are comparable to the speed of light.

3.2.2 Field and Moment Equations. The linearized equation of motion and the continuity equation, along with Gauss' law, which expresses mathematically the coupling between the beam and the plasma space-charge waves, are written for beam and plasma particles on the basis of the above assumptions. The above-mentioned equations, when written in an inertial frame of reference attached to the beam, have the following form:

$$\frac{\partial v_{1b}}{\partial t} = - \frac{e}{m} E_1 , \quad (3.1)$$

$$\frac{\partial n_{1b}}{\partial t} + \nabla \cdot (n_{0b} v_{1b}) = 0 , \quad (3.2)$$

$$\frac{\partial v_{1p}}{\partial t} - \frac{u_0}{c} \cdot \nabla v_{1p} = - \frac{e}{m} E_1 , \quad (3.3)$$

$$\frac{\partial n_{1p}}{\partial t} + n_{op} \nabla \cdot \underline{v}_{1p} - \underline{u}_o \cdot \nabla n_{1p} = 0 \quad (3.4)$$

and

$$\nabla \cdot \underline{E}_1 = - \frac{e}{\epsilon_o} (n_{1b} + n_{1p}) , \quad (3.5)$$

where the differential vector operator ∇ is defined in the moving frame of reference attached to the beam. All other quantities are defined in the same manner as in Chapter II. A steady-state solution to the above equations is sought by assuming an $e^{j\omega t}$ variation in time of all first-order quantities. The combination of Eqs. 3.1 and 3.2 in a steady-state situation yields

$$n_{1b} = -\nabla \cdot \left(\frac{en_{ob}}{m\omega^2} \underline{E}_1 \right) . \quad (3.6)$$

Likewise, the combination of Eqs. 3.3 and 3.4 yields

$$\left(j\omega - u_o \frac{\partial}{\partial \xi} \right)^2 n_{1p} = \frac{en_{op}}{m} \nabla \cdot \underline{E}_1 . \quad (3.7)$$

If the differential operator $[j\omega - u_o (\partial/\partial \xi)]^2$ is applied to both sides of Eq. 3.5, and Eqs. 3.6 and 3.7 are combined with the result, the following equation for the ac electric field vector, $\underline{E}_1$, is obtained:

$$\left(j\omega - u_o \frac{\partial}{\partial \xi} \right)^2 \nabla \cdot [\epsilon_b(\omega, \xi) \underline{E}_1] + \omega_p^2 \nabla \cdot \underline{E}_1 = 0 . \quad (3.8)$$

The above differential equation cannot be solved without assuming further relations among the components of the electric field. The required

relation can be established by exploiting the quasi-static approximation; on this basis the electric field can be approximated by a conservative field derivable from a scalar potential, Φ , as

$$\underline{E}_1 = -\nabla\Phi \quad . \quad (3.9)$$

The total scalar potential function, Φ , is then defined as

$$\Phi(\underline{r}, t) = \Phi(\underline{r}) e^{j\omega t} \quad . \quad (3.10)$$

The substitution of Eq. 3.9 into Eq. 3.8 yields the following differential equation for the scalar potential, Φ :

$$\left(j\omega - u_o \frac{\partial}{\partial \xi} \right)^2 \nabla \cdot [\epsilon_b(\omega, \xi) \nabla \Phi] + \omega_p^2 \nabla^2 \Phi = 0 \quad . \quad (3.11)$$

3.3 Construction of the Solution to the Equation for the Scalar Potential Function

It is assumed that the scalar potential function, Φ , is separable and can thus be expressed in cylindrical geometry as

$$\Phi(r, \theta, \xi) = R(r)Z(\xi) e^{-jm\theta} \quad , \quad (3.12)$$

where R and Z are to be determined and m is a constant which is allowed to take only integer values in compliance with the uniqueness of the solution. The substitution of Eq. 3.12 into Eq. 3.11 yields the following:

$$\left[\frac{1}{Z} \left(j\omega - u_o \frac{d}{d\zeta} \right)^2 (\epsilon_b Z) + \omega_p^2 \right] \left[\frac{1}{rR} \frac{d}{dr} \left(r \frac{dR}{dr} \right) - \frac{m^2}{r^2} \right] + \frac{1}{Z} \left[\left(j\omega - u_o \frac{d}{d\zeta} \right)^2 \frac{d}{d\zeta} \left(\epsilon_b \frac{dZ}{d\zeta} \right) + \omega_p^2 \frac{d^2 Z}{d\zeta^2} \right] = 0 \quad (3.13)$$

The above equation is obviously in separable form and can be resolved into two separate differential equations, for R and Z, as follows:

$$\frac{1}{rR} \frac{d}{dr} \left(r \frac{dR}{dr} \right) + K^2 - \frac{m^2}{r^2} = 0 \quad (3.14)$$

and

$$\left(j\omega - u_o \frac{d}{d\zeta} \right)^2 \left[\frac{d}{d\zeta} \left(\epsilon_b \frac{dZ}{d\zeta} \right) - K^2 \epsilon_b Z \right] + \omega_p^2 \left(\frac{d^2}{d\zeta^2} - K^2 \right) Z = 0, \quad (3.15)$$

where K is an arbitrary constant of separation whose value must be determined by employing the boundary requirements.

Equation 3.14 is recognized as Bessel's differential equation which has two independent solutions denoted by $J_m(Kr)$ and $N_m(Kr)$ of which only the function of the first kind, $J_m(Kr)$, is to be retained. The function of the second kind, $N_m(Kr)$, is excluded from the solution due to its possession of a logarithmic singularity along the axis, $r = 0$; therefore,

$$R(r) = A J_m(Kr), \quad (3.16)$$

where A is an arbitrary constant.

An approximate solution to Eq. 3.15 can be sought by employing the method of averaging as explained and demonstrated in Chapter II. First, rewrite Eq. 3.15 in the form

$$\begin{aligned}
 & K^2 \left[\left(\omega + ju_o \frac{d}{d\xi} \right) \epsilon_b - \omega_p^2 \right] Z + \left\{ \left[2ju_o K^2 \left(\omega + ju_o \frac{d}{d\xi} \right) \right. \right. \\
 & \left. \left. - \frac{d}{d\xi} \left(\omega + ju_o \frac{d}{d\xi} \right)^2 \right] \epsilon_b \right\} \frac{dZ}{d\xi} + \left\{ \omega_p^2 - \left[K^2 u_o^2 + 2ju_o \frac{d}{d\xi} \left(\omega + ju_o \frac{d}{d\xi} \right) \right. \right. \\
 & \left. \left. + \left(\omega + ju_o \frac{d}{d\xi} \right)^2 \right] \epsilon_b \right\} \frac{d^2 Z}{d\xi^2} + \left\{ \left[ju_o - 3ju_o \left(\omega + ju_o \frac{d}{d\xi} \right) \right] \epsilon_b \right\} \frac{d^3 Z}{d\xi^3} \\
 & + u_o^2 \epsilon_b \frac{d^4 Z}{d\xi^4} = 0 \quad . \quad (3.17)
 \end{aligned}$$

If $n_{ob}(\xi)$ is periodic in ξ , the beam dielectric constant $\epsilon_b(\omega, \xi)$ can be expressed in terms of its discrete Fourier components as follows:

$$\epsilon_b(\omega, \xi) = 1 - \frac{e^2 n_{ob}(\xi)}{m \epsilon_o \omega^2} = \sum_n f_n(\omega) e^{-jnk_o \xi} \quad , \quad (3.18)$$

where $f_n(\omega)$ has the property

$$f_n^*(\omega) = f_{-n}(\omega) \quad (3.19)$$

by virtue of the reality of the beam dielectric constant, $\epsilon_b(\omega, \xi)$. In the same manner as before $Z(\xi)$ can be broken into a large-amplitude slowly varying part, $Z_o(\xi)$, and a small-amplitude fast oscillatory part, $Z_1(\xi)$, as follows:

$$Z(\xi) = Z_o(\xi) + \sum_{n \neq 0} z_n(\omega) e^{-jnk_o \xi} \quad . \quad (3.20)$$

A direct substitution of Eqs. 3.18 and 3.20 into Eq. 3.17 results in the following:

Zeroth-Order Equation

$$\left(K^2 - \frac{d^2}{d\xi^2}\right) \left[\left(\omega + ju_o \frac{d}{d\xi}\right)^2 f_o(\omega) - \omega_p^2 \right] Z_o(\xi) + K^2 \omega^2 \sum_{n \neq 0} f_n(\omega) z_{-n}(\omega) = 0. \quad (3.21)$$

Nth-Order Equation

$$\begin{aligned} f_n(\omega) \left(K^2 + jnk_o \frac{d}{d\xi} - \frac{d^2}{d\xi^2}\right) \left(\omega + n\omega_o + ju_o \frac{d}{d\xi}\right)^2 Z_o(\xi) + (K^2 + n^2 k_o^2) \\ \cdot [(\omega + n\omega_o)^2 f_o(\omega) - \omega_p^2] z_n(\omega) + (\omega + n\omega_o)^2 \sum_{m \neq 0, n} (K^2 + nm k_o^2) f_{n-m}(\omega) \\ \cdot z_m(\omega) = 0 \quad (n = \pm 1, \pm 2, \dots), \quad (3.22) \end{aligned}$$

where

$$\omega_o \equiv k_o u_o. \quad (3.23)$$

Application of the weak turbulence approximation to the nth-order equation results in a value for $z_n(\omega)$ in terms of $Z_o(\xi)$ and the known parameters. The expression obtained for $z_n(\omega)$ is then substituted in Eq. 3.20 to yield the final form of the solution to Eq. 3.15 as follows:

$$Z(\xi) = \left[1 - \sum_{n \neq 0} \frac{\left(K^2 + jnk_o \frac{d}{d\xi} - \frac{d^2}{d\xi^2}\right) \left(\omega + n\omega_o + ju_o \frac{d}{d\xi}\right)^2}{(K^2 + n^2 k_o^2) [(\omega + n\omega_o)^2 f_o(\omega) - \omega_p^2]} f_n(\omega) \right] Z_o(\xi). \quad (3.24)$$

The term $Z_o(\xi)$, the solution to the averaged equation, is obtained by combining the zeroth- and nth-order equations and employing the weak turbulence approximation as well as the identity, Eq. 3.19,

$$\left(K^2 - \frac{d^2}{d\xi^2}\right) \left[\left(\omega + ju_0 \frac{d}{d\xi}\right)^2 f_0(\omega) - \omega_p^2 \right] Z_0(\xi) - K^2 \omega^2 \sum_{n \neq 0} |f_n(\omega)|^2$$

$$\cdot \frac{\left(K^2 - jnk_0 \frac{d}{d\xi} - \frac{d^2}{d\xi^2}\right) \left(\omega - n\omega_0 + ju_0 \frac{d}{d\xi}\right)^2}{(K^2 + n^2 k_0^2) [(\omega - n\omega_0)^2 f_0(\omega) - \omega_p^2]} Z_0(\xi) = 0 \quad (3.25)$$

3.4 Dispersion Relation

The dispersion relation is obtained by solving the averaged equation as expressed in Eq. 3.25. For traveling waves propagating in the positive ξ -direction, $Z_0(\xi)$ is given by

$$Z_0(\xi) \sim e^{-jk\xi}, \quad (3.26)$$

where $|k| \ll k_0$ is in compliance with the assumption that the wavelength of the oscillations be long compared with the length characterizing the longitudinal inhomogeneity of the beam. The value of k is found by a direct substitution of Eq. 3.26 into Eq. 3.25, as shown by

$$(k^2 + K^2) [(\omega + ku_0)^2 f_0(\omega) - \omega_p^2] - K^2 \omega^2 \sum_{n \neq 0} \frac{(k^2 - nk_0 k + K^2) (\omega + ku_0 - n\omega_0)^2}{(K^2 + n^2 k_0^2) [(\omega - n\omega_0)^2 f_0(\omega) - \omega_p^2]} \cdot |f_n(\omega)|^2 = 0 \quad (3.27)$$

The requirement that the amplitude of the fast-varying part of the solution, $|Z_1(\xi)|$, be small compared to that of the slowly varying part of the solution, $|Z_0(\xi)|$, is satisfied if $\omega \ll \omega_0$. This is consistent with the long-wavelength assumption, $|k| \ll k_0$, throughout the interaction region. If Eq. 3.28 is doppler-transformed ($\omega \rightarrow \omega_D \equiv \omega - ku_0$) and the above approximations are employed in the result, the following is obtained:

$$(k^2 + K^2) \left(f_o(\omega_D) - \frac{\omega_p^2}{\omega^2} - 2f_o^{-1}(\omega_D) \frac{\omega_D^2}{\omega^2} \sum_{n=1}^{\infty} \frac{K^2}{K^2 + n^2 k_o^2} |f_n(\omega_D)|^2 \right) = 0 . \quad (3.28)$$

The dispersion relation is obtained by combining the above equation with the boundary requirements. Care must be employed, however, in interpretation of the elements of Eq. 3.28 for the case of finite transverse geometry. In the following section an attempt is made to clarify the meanings of the elements of Eq. 3.28 using an argument similar to that of the correspondence principle.

3.4.1 Geometric Reduction Factor. For an electron beam of infinite transverse extent, the space-charge field lines are all contained in the beam, beginning in an electron-deficient region and terminating in an electron-excess region. The situation becomes more complex when the electron beam is of finite radius. In this case some of the lines of force due to the space-charge field extend beyond the transverse extent of the beam; some leave the beam at one point and reenter it at another, while others terminate on the metal tube along which the beam is drifting. As a result the beam plasma frequency, ω_p , will be reduced by a factor depending upon the geometry, signal frequency and the particular mode of propagation considered. The frequency dependence of the reduction factor stems from the fact that at higher frequencies there will be less time for the establishment of the field lines beyond the transverse extent of the beam, corresponding to a smaller leakage of the displacement current from the beam.

The geometric reduction factor is defined here as the factor by which the beam density must be reduced to account for finite transverse

effects. It will be shown in Section 3.4.3 that the exact form of the geometric reduction factor for an electron beam of radius, b , drifting along a perfectly conducting tube of radius, a , filled with plasma is as follows:

$$G_m \left(kb, \frac{b}{a} \right) = -kb \frac{I'_m(kb)}{I_m(ka)} \left(K_m(ka) I_m(kb) - K_m(kb) I_m(ka) \right), \quad (3.29)$$

where m is a positive integer denoting the radial and peripheral modes of propagation, and I_m and K_m denote respectively the m th-order modified Bessel functions of the first and second kind. For the case of an infinitely large tube ($a \rightarrow \infty$) the right-hand side of Eq. 3.29 will appropriately approach a value given as

$$G_m(kb, 0) = kb I'_m(kb) K_m(kb), \quad (3.30)$$

or consequently,

$$G_m \left(kb, \frac{b}{a} \right) = G_m(kb, 0) \left(1 - \frac{K_m(ka) I_m(kb)}{I_m(ka) K_m(kb)} \right). \quad (3.31)$$

In Figs. 3.1 through 3.4 values of $G_m(kb, b/a)$ are plotted vs. kb for values of $b/a = 1/2, 1/4, 1/10$ and 0, respectively, for the first four harmonics. As seen, the geometric reduction factor approaches the value $1/2$ asymptotically for all harmonics provided that $b/a < 1$. The asymptotic behavior of $G_m(kb, b/a)$ can be explored using the large-argument expansions for modified Bessel functions. In Appendix C the limiting values of $G_m(kb, b/a)$ are found for large as well as small values of the argument kb . The results are stated as follows:

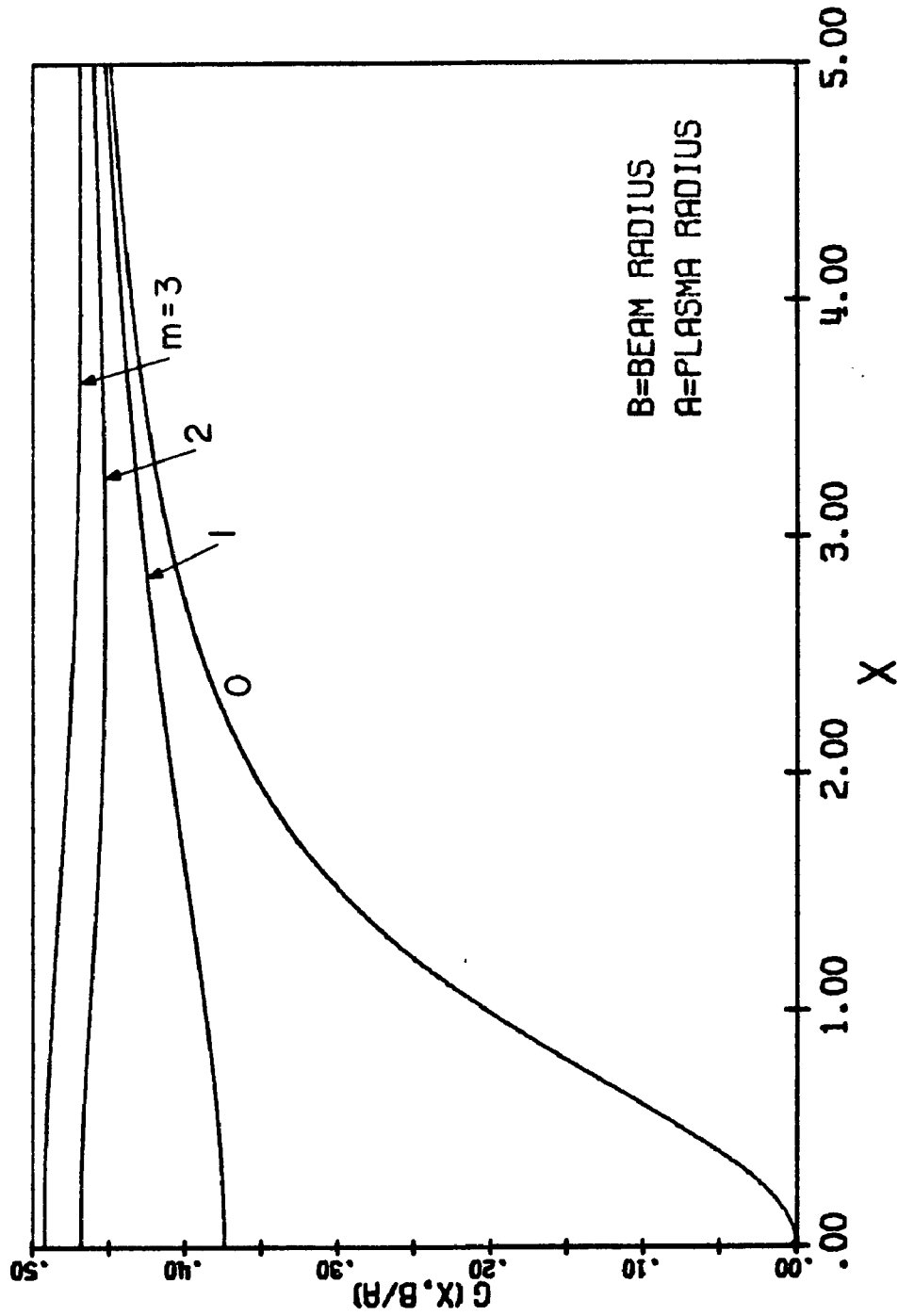


FIG. 3.1 GEOMETRIC REDUCTION FACTOR, $G(x, b/a)$, VS. x FOR THE FIRST FOUR HARMONICS WITH $b/a = 1/2$.

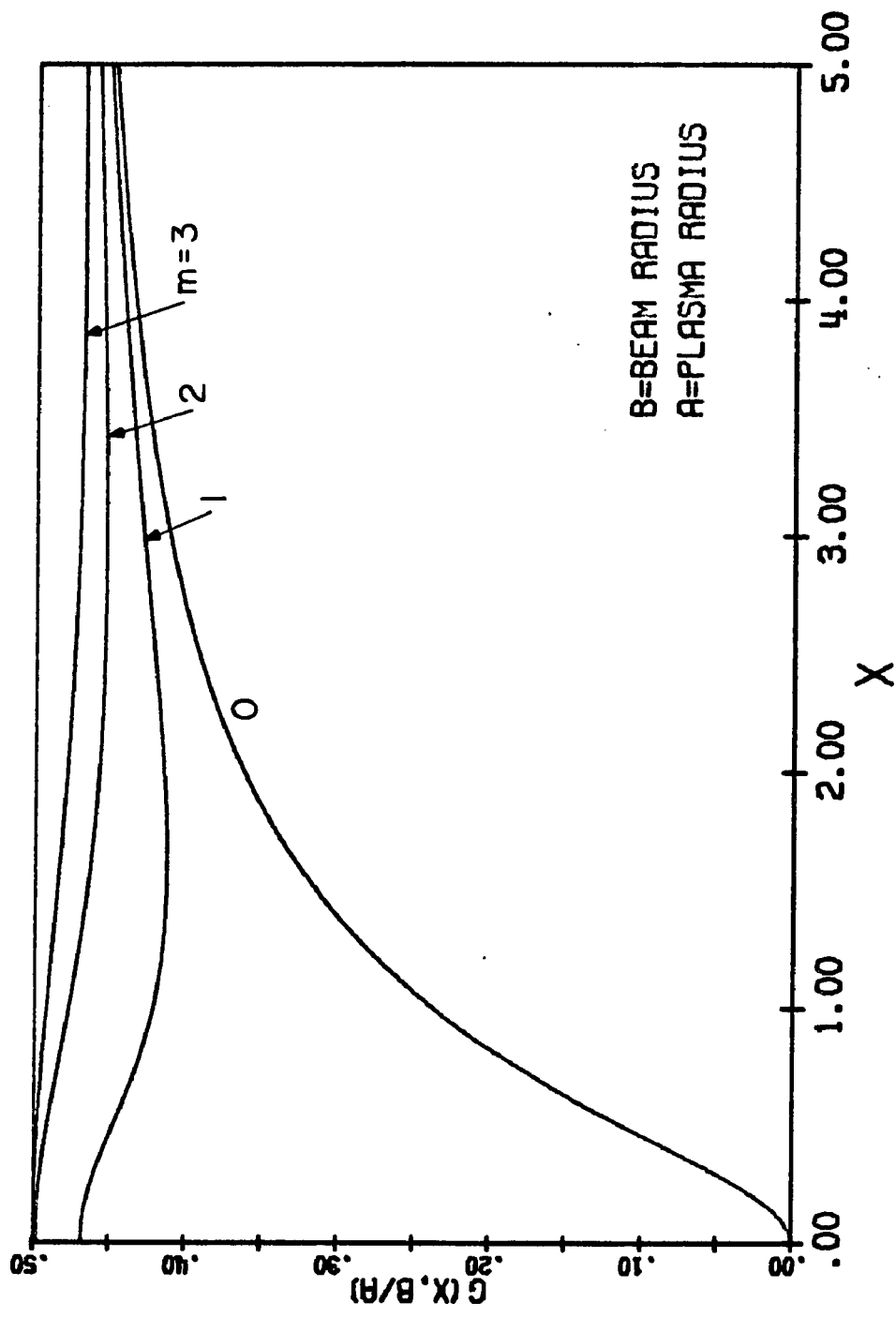


FIG. 3.2 GEOMETRIC REDUCTION FACTOR, $G(x, b/a)$, VS. x FOR THE FIRST FOUR HARMONICS WITH $b/a = 1/4$.

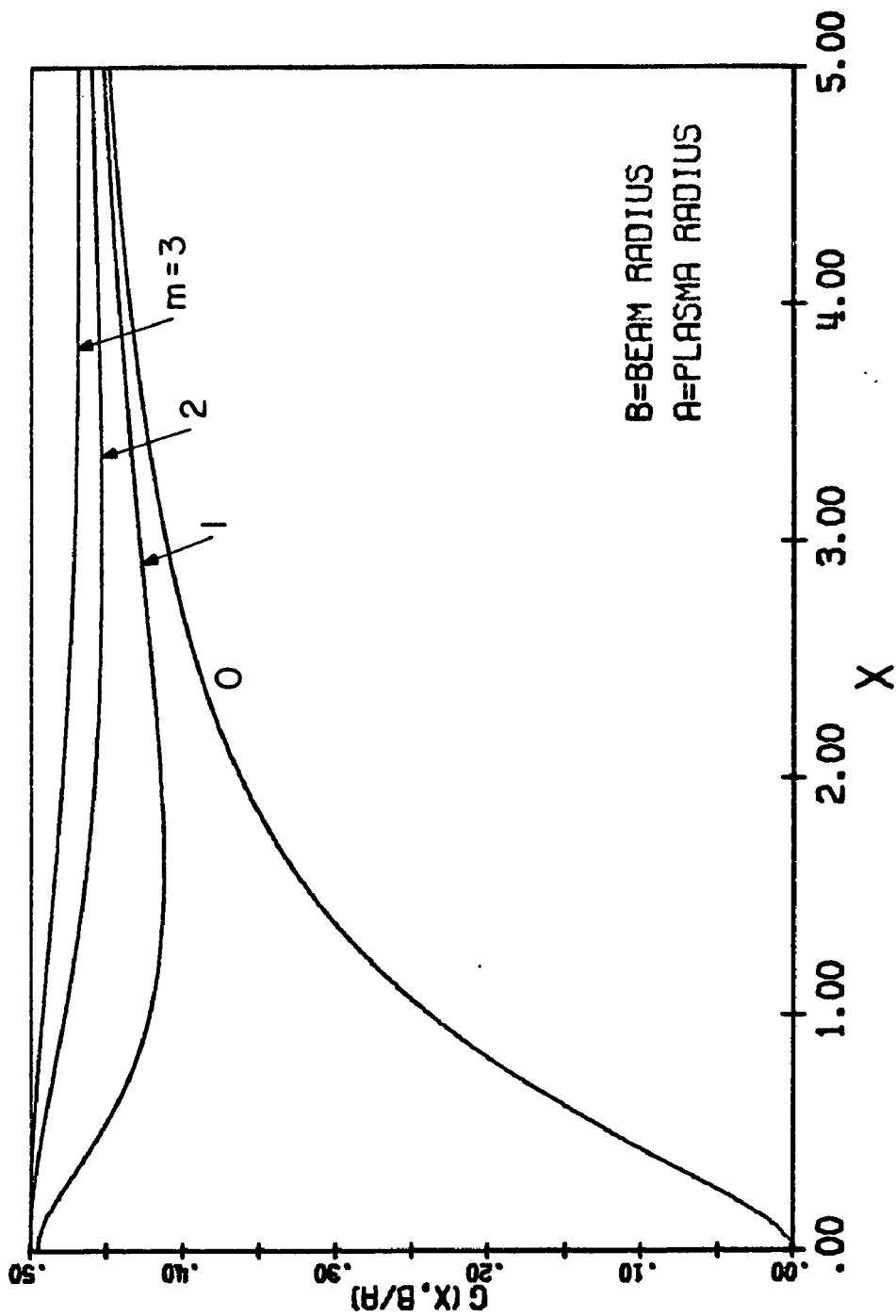


FIG. 3.3 GEOMETRIC REDUCTION FACTOR, $G(x, b/a)$, VS. x FOR THE FIRST FOUR HARMONICS WITH $b/a = 1/10$.

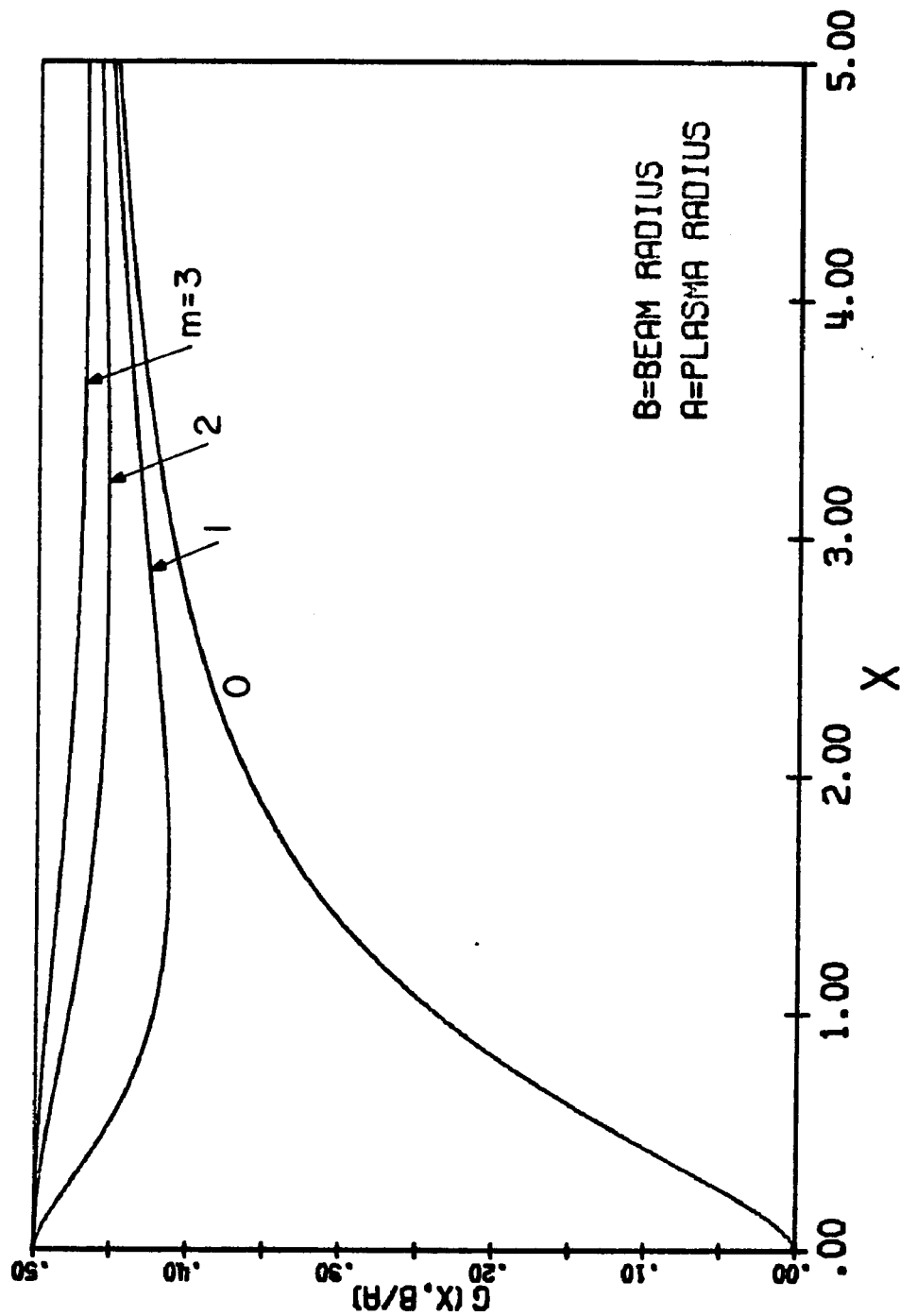


FIG. 3.4 GEOMETRIC REDUCTION FACTOR, $G(x, b/a)$, VS. x FOR THE FIRST FOUR HARMONICS WITH $b/a = 0$.

The Large-Argument Limit

$$\lim_{kb \rightarrow \infty} G_m \left(kb, \frac{b}{a} \right) = \frac{1}{2} ; \quad m = 0, 1, 2, \dots, \quad 0 \leq \frac{b}{a} < 1. \quad (3.32)$$

The Small-Argument Limit

$$\lim_{kb \rightarrow 0} G_m \left(kb, \frac{b}{a} \right) = \begin{cases} 0 & ; \quad m = 0, \quad 0 \leq \frac{b}{a} < 1 \\ \frac{1}{2} \left[1 - \left(\frac{b}{a} \right)^{2m} \right] & ; \quad m = 1, 2, \dots, \quad 0 \leq \frac{b}{a} < 1 \end{cases} \quad (3.33)$$

In reference to Eq. 3.28, the finite transverse effects can be taken into account by introducing the geometric reduction factor. The geometric reduction factor must appropriately appear in conjunction with variables whose values depend upon the beam density. Such variables are readily recognized in Eq. 3.28 as $f_n(\omega_D)$ with $n = 0, 1, 2, \dots$. Therefore, for the case of finite boundary, the values of $f_n(\omega_D)$, $n = 0, 1, 2, \dots$, in Eq. 3.28 must be substituted by their effective values, $f_{n, \text{eff}}(\omega_D)$, $n = 0, 1, 2, \dots$, in accordance with the following definition for $f_{n, \text{eff}}(\omega_D)$:

$$1 - \frac{e^2 n_{ob}(\xi)}{m \epsilon_0 \omega^2} G_m \left(kb, \frac{b}{a} \right) = \sum_n f_{n, \text{eff}}(\omega_D) e^{-jnk_0 \xi}. \quad (3.34)$$

Hence, Eq. 3.28 can be rewritten as

$$(k^2 + K^2) \left(f_{o,eff}(\omega_D) - \frac{\omega_p^2}{\omega^2} - 2f_{o,eff}^{-1}(\omega_D) \frac{\omega_D^2}{\omega^2} \sum_{n=1}^{\infty} \frac{K^2}{K^2 + n^2 k_o^2} |f_{n,eff}(\omega_D)|^2 \right) = 0, \quad (3.35)$$

where the effects of finite transverse boundaries are implicitly taken into account and K is to be eliminated between the above equation and the boundary requirement which will be derived in the next section.

3.4.2 Boundary Requirements. The beam-plasma system that has been considered so far is composed of two distinct regions. The first is the beam-plasma region, designated as Region I, which fills up a cylindrical space of a radius equal to b . The second is the surrounding region, designated as Region II, which contains a plasma of either infinite or finite extent, contained in a perfectly conducting tube of radius a . In the following the boundary requirements for the finite beam-finite plasma case are derived and the results are then extended to include the finite beam-infinite plasma case. It must be remembered that the following derivation of the boundary requirements for the inhomogeneous beam involves an approximation, namely that the polarization effects caused by the short wavelength inhomogeneity in the beam density which is negligibly small are ignored in writing the dielectric constant for Region I.

3.4.2a Finite Beam-Finite Plasma Case. For Region I the dielectric constant and the total scalar potential function are given as

$$\epsilon_I = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_b^2}{\omega_D^2} \quad (3.36)$$

and

$$\Phi_I(\underline{r}, t) = A J_m(Kr) \exp [j(\omega t - m\theta - kz)] \quad (3.37)$$

Likewise, for Region II, it follows that

$$\epsilon_{II} = 1 - \frac{\omega_p^2}{\omega^2} \quad (3.38)$$

and

$$\phi_{II}(\underline{r}, t) = B[K_m(ka)I_m(kr) - I_m(ka)K_m(kr)] \exp [j(\omega t - m\theta - kz)] , \quad (3.39)$$

where A and B are arbitrary constants and the requirement that the electric field be normal at the surface of the wall, $r = a$, has already been accounted for. Two more requirements remain, namely that the potential and the normal component of the electric induction vector be continuous across the beam surface:

$$\phi_I \Big|_{r=b} = \phi_{II} \Big|_{r=b} \quad (3.40)$$

and

$$\epsilon_I \frac{\partial \phi_I}{\partial r} \Big|_{r=b} = \epsilon_{II} \frac{\partial \phi_{II}}{\partial r} \Big|_{r=b} . \quad (3.41)$$

The second requirement, Eq. 3.41, is equivalent to the requirement that there will be no accumulation of surface charge at the boundary, $r = b$, and thus excludes the possibility of the existence of rippling at the beam surface. A combined form of the above two requirements can conveniently be written as

$$\epsilon_I \frac{\partial(\log \phi_I)}{\partial r} \Big|_{r=b} = \epsilon_{II} \frac{\partial(\log \phi_{II})}{\partial r} \Big|_{r=b} . \quad (3.42)$$

A direct substitution of Eqs. 3.36 through 3.39 into the above yields

$$kb \frac{J'_m(Kb)}{J_m(Kb)} = \frac{1 - \omega_p^2/\omega^2}{1 - \omega_p^2/\omega^2 - \omega_b^2/\omega_D^2} F_m \left(kb, \frac{b}{a} \right), \quad (3.43)$$

where $F_m(kb, b/a)$ is defined as

$$F_m(kb, b/a) \equiv kb \frac{K_m(ka)I'_m(kb) - I_m(ka)K'_m(kb)}{K_m(ka)I_m(kb) - I_m(ka)K_m(kb)}. \quad (3.44)$$

Equation 3.43 is the desired boundary requirement which, when combined with Eq. 3.35, yields the dispersion relation.

3.4.2b Finite Beam-Infinite Plasma Case. The boundary requirement for this case can be trivially found either by exploring the limit of $F_m(kb, b/a)$ as a tends to infinity or by initially excluding the modified Bessel function of the first kind from the solution to $\Phi_{II}(\underline{r}, t)$ due to its possession of a singularity for arguments of an infinitely large real part. The result is that

$$kb \frac{J'_m(Kb)}{J_m(Kb)} = \frac{1 - \omega_p^2/\omega^2}{1 - \omega_p^2/\omega^2 - \omega_b^2/\omega_D^2} F_m(kb, 0), \quad (3.45)$$

where

$$F_m(kb, 0) = kb \frac{K'_m(kb)}{K_m(kb)}. \quad (3.46)$$

3.4.3 Homogeneous Beam. It is shown elsewhere⁴⁷ that the dispersion relation for a homogeneous beam-plasma system can be expressed as

$$kb \frac{I'_m(kb)}{I_m(kb)} = \frac{1 - \omega_p^2/\omega^2}{1 - \omega_p^2/\omega^2 - \omega_b^2/\omega_D^2} F_m \left(kb, \frac{b}{a} \right). \quad (3.47)$$

It can be readily seen, using the identity

$$J_m(jz) = j^m I_m(z) , \quad (3.48)$$

that setting the first multiplying factor in Eq. 3.35 equal to zero corresponds to the degenerate case of a homogeneous beam-plasma system. Thus the actual equation of state for the inhomogeneous beam is given by the second multiplying factor in Eq. 3.35 which is written as

$$f_{o,eff}(\omega_D) - \frac{\omega_p^2}{\omega^2} - 2f_{o,eff}^{-1}(\omega_D) \frac{\omega_D^2}{\omega^2} \sum_{n=1}^{\infty} \frac{K^2}{K^2 + n^2 k_o^2} |f_{n,eff}(\omega_D)|^2 = 0 , \quad (3.49)$$

where K is yet to be eliminated using the boundary requirement, Eq. 3.43 or Eq. 3.45. The following shows the mathematical equivalence of Eqs. 3.49 and 3.47 in the limit of vanishing inhomogeneity [$f_{n,eff}(\omega) \rightarrow 0$; $n \neq 0$]. This amounts to a derivation of the geometric reduction factors as expressed in Eqs. 3.29 and 3.30 from the dispersion relation for the homogeneous beam, Eq. 3.47.

Equation 3.47 can be rewritten as follows after having substituted for $F_m(kb, b/a)$ from Eqs. 3.44 and 3.46:

$$1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_b^2}{\omega_D^2} G_m \left(kb, \frac{b}{a} \right) = 0 , \quad (3.50)$$

where

$$G_m \left(kb, \frac{b}{a} \right) = \left(1 - \frac{I_m(kb)}{I'_m(kb)} \frac{K_m(ka)I'_m(kb) - I_m(ka)K'_m(kb)}{K_m(ka)I_m(kb) - I_m(ka)K_m(kb)} \right)^{-1} \quad (3.51)$$

and

$$G_m(kb, 0) = \left(1 - \frac{I_m(kb)}{I'_m(kb)} \frac{K'_m(kb)}{K_m(kb)} \right)^{-1} \quad (3.52)$$

The above equations can be simplified if the recurrence relations for the modified Bessel functions, namely

$$I'_m(z) = I_{m+1}(z) + \frac{m}{z} I_m(z) \quad (3.53)$$

and

$$K'_m(z) = -K_{m+1}(z) + \frac{m}{z} K_m(z) , \quad (3.54)$$

are employed. Equation 3.51 can then be rewritten as

$$G_m\left(kb, \frac{b}{a}\right) = \left[1 - \frac{\frac{m}{kb} + \frac{K_m(ka)I_{m+1}(kb) + I_m(ka)K_{m+1}(kb)}{K_m(ka)I_m(kb) + I_m(ka)K_m(kb)}}{\frac{m}{kb} + \frac{I_{m+1}(kb)}{I_m(kb)}} \right]^{-1} \quad (3.55)$$

Straightforward algebra and the use of the Wronskian

$$W[K_m(z), I_m(z)] = I_m(z)K_{m+1}(z) + I_{m+1}(z)K_m(z) = \frac{1}{z} , \quad (3.56)$$

yield a value for $G_m(kb, b/a)$ as expressed in Eq. 3.29. A similar procedure is involved in rewriting $G_m(kb, 0)$ in a more simple form as expressed in Eq. 3.30.

Now take note of the mathematical equivalence of Eqs. 3.49 and 3.50 in the limit of a vanishing inhomogeneity. In accordance with Eq. 3.34 for a homogeneous beam it is seen that

$$f_{n,\text{eff}}(\omega_D) = \left(1 - \frac{\omega_b^2}{\omega_D^2} G_m(kb, b/a)\right) \delta_{no} , \quad (3.57)$$

where δ_{no} is the Kronecker delta. A direct substitution of Eq. 3.57 into Eq. 3.49 establishes the desired correspondence.

A numerical solution of the dispersion relation for the case of a homogeneous beam interacting with plasmas can be found by solving Eq. 3.50 for complex k (for convective instabilities) in terms of other parameters. This procedure is quite involved due to the transcendental nature of Eq. 3.50. The so-called "weak beam approximation" can, however, be applied in this case. According to this approximation the value of the propagation constant, k , is, as a first approximation, put equal to the real part of k in the limit of vanishing beam density. For our case this value is given by ω/u_o . Therefore, Eq. 3.50 can be rewritten, in the first approximation, as

$$1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_b^2}{\omega_D^2} G_m\left(\frac{b\omega}{u_o}, \frac{b}{a}\right) = 0 . \quad (3.58)$$

The solution to this equation can be found immediately and expressed as

$$k = \frac{\omega}{u_o} \pm j \frac{\omega_b}{u_o} \left(\frac{\omega_p^2}{\omega^2} - 1\right)^{-1/2} G_m^{1/2}\left(\frac{b\omega}{u_o}, \frac{b}{a}\right) . \quad (3.59)$$

In Figs. 3.5 through 3.8, values of the normalized gain, $k_1 u_o / \omega_p$, are plotted vs. normalized frequency, ω / ω_p , for the first three harmonics and for $b/a = 1/2, 1/4, 1/10$ and 0, respectively. The numerical values of the other parameters are

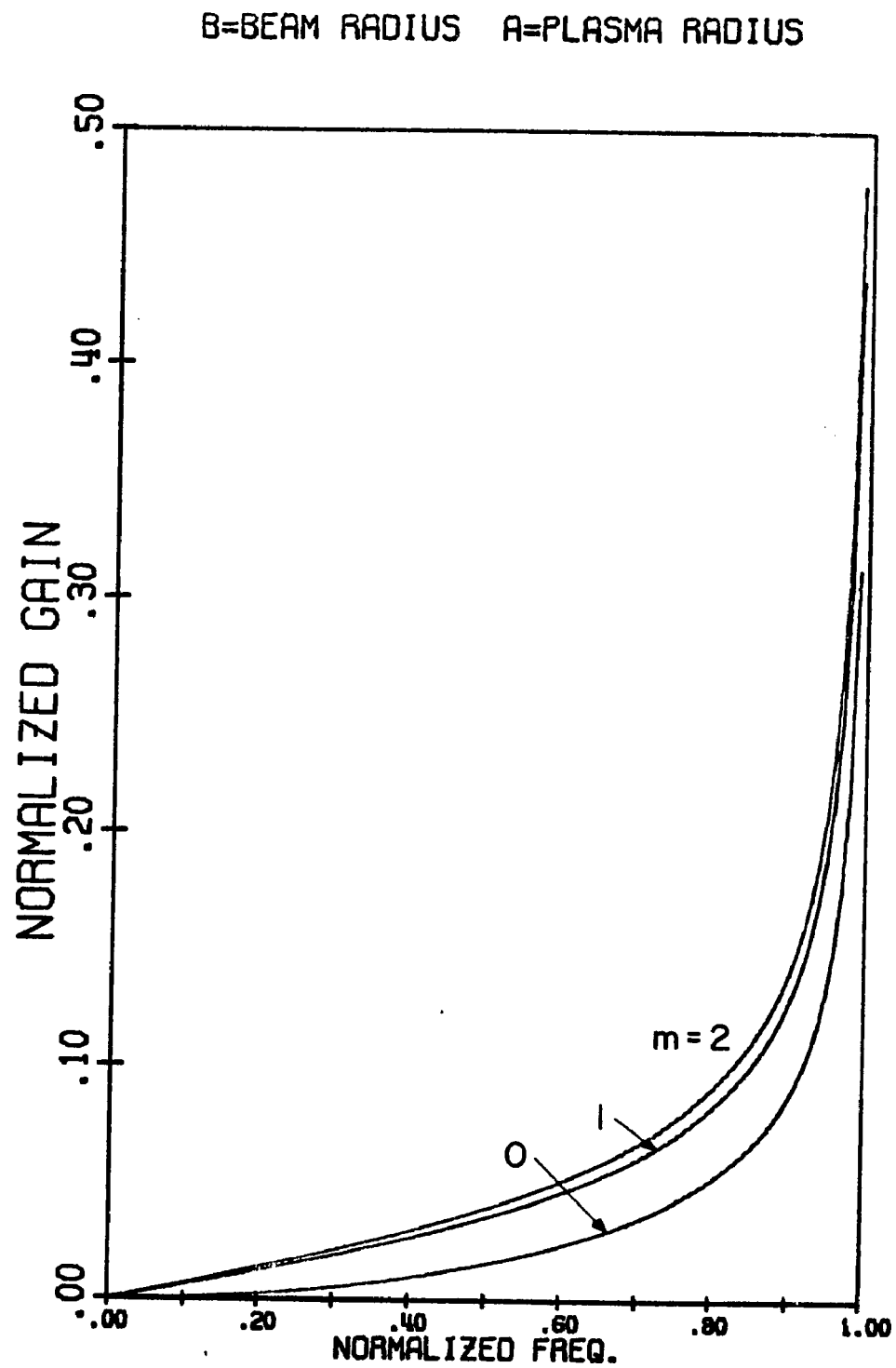


FIG. 3.5 NORMALIZED GAIN VS. NORMALIZED FREQUENCY FOR THE FIRST THREE HARMONICS WITH $b/a = 1/2$.

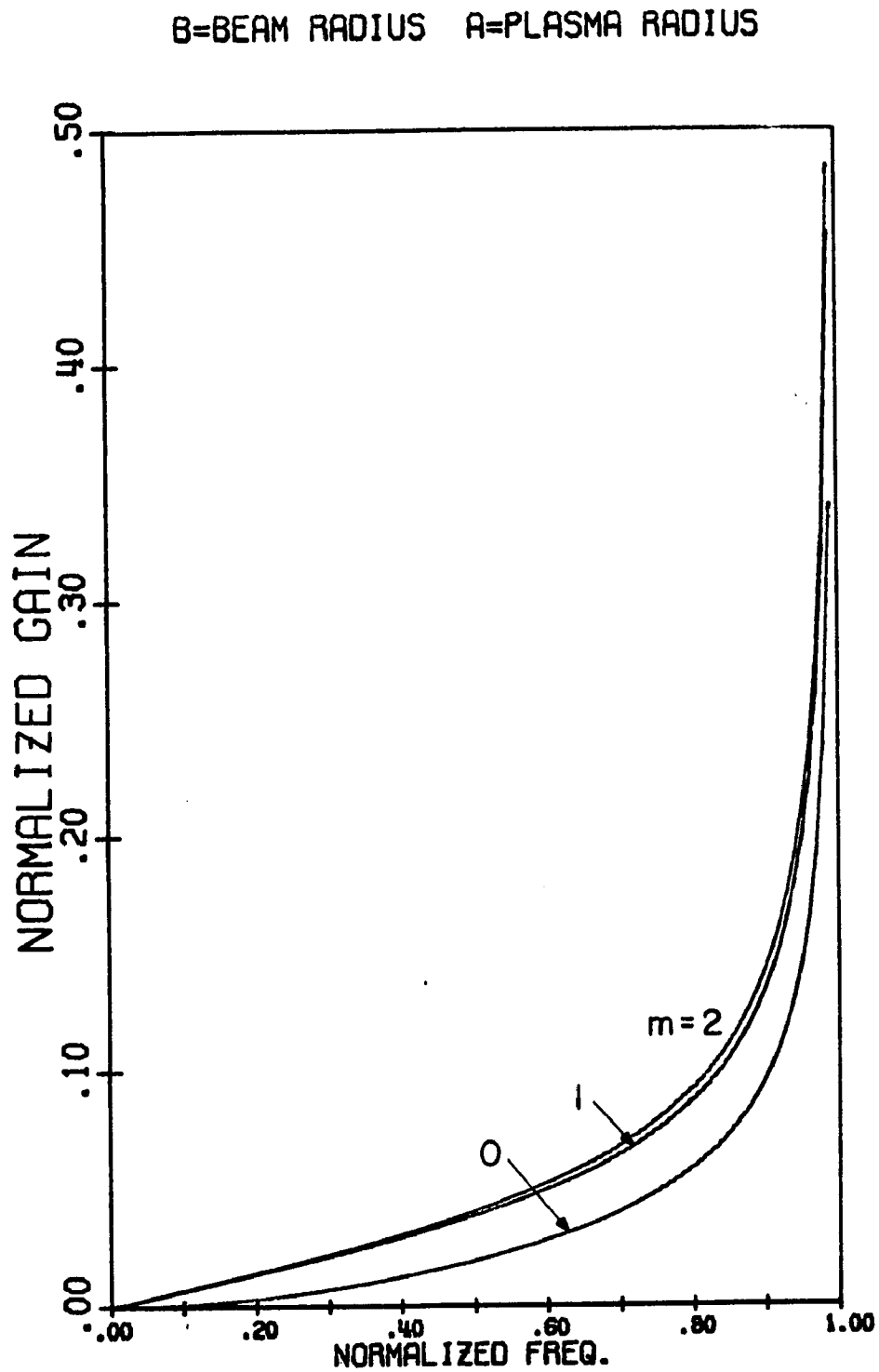


FIG. 3.6 NORMALIZED GAIN VS. NORMALIZED FREQUENCY FOR THE FIRST THREE HARMONICS WITH $b/a = 1/4$.

B=BEAM RADIUS A=PLASMA RADIUS

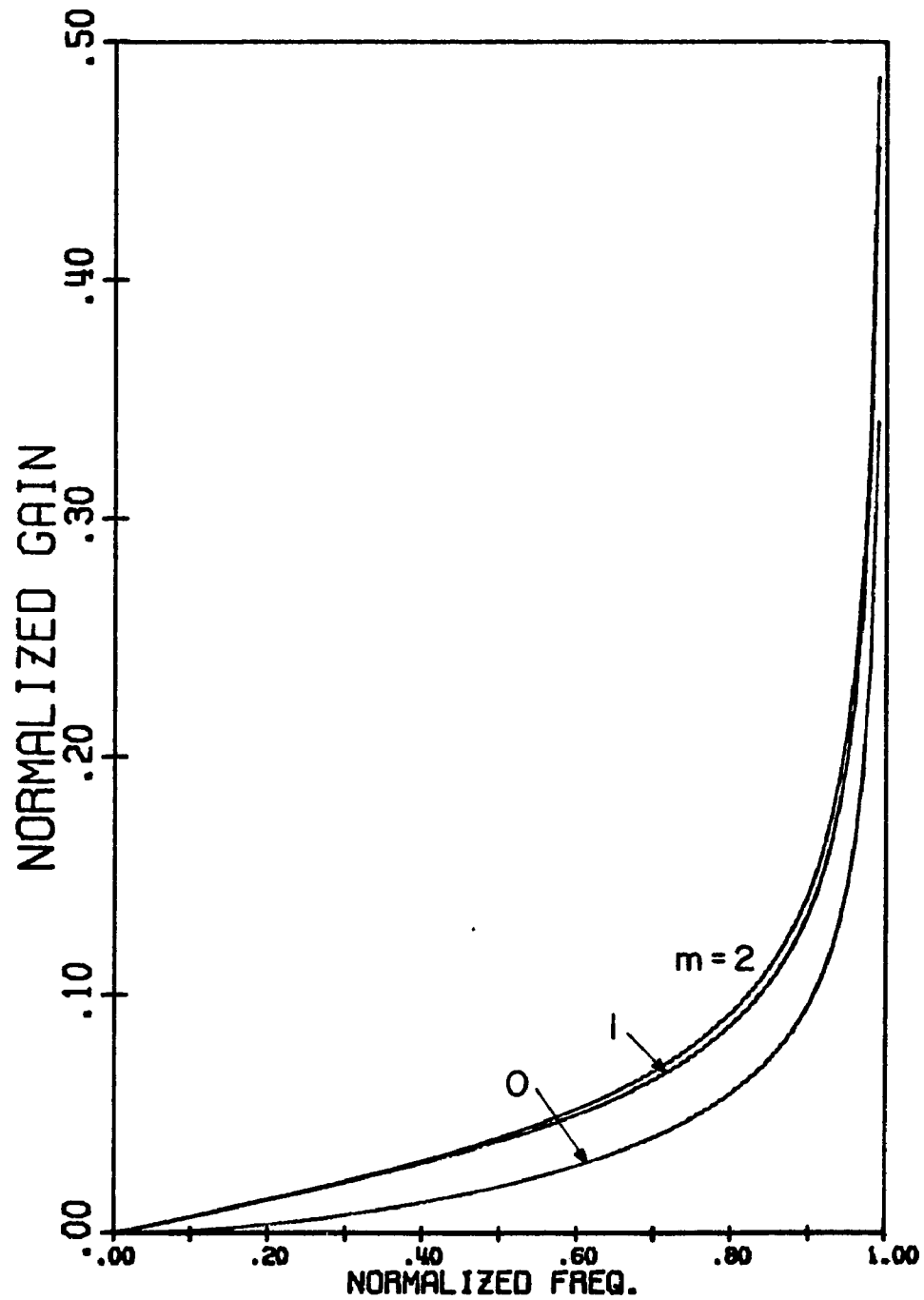


FIG. 3.7 NORMALIZED GAIN VS. NORMALIZED FREQUENCY FOR THE FIRST THREE HARMONICS WITH $b/a = 1/10$.

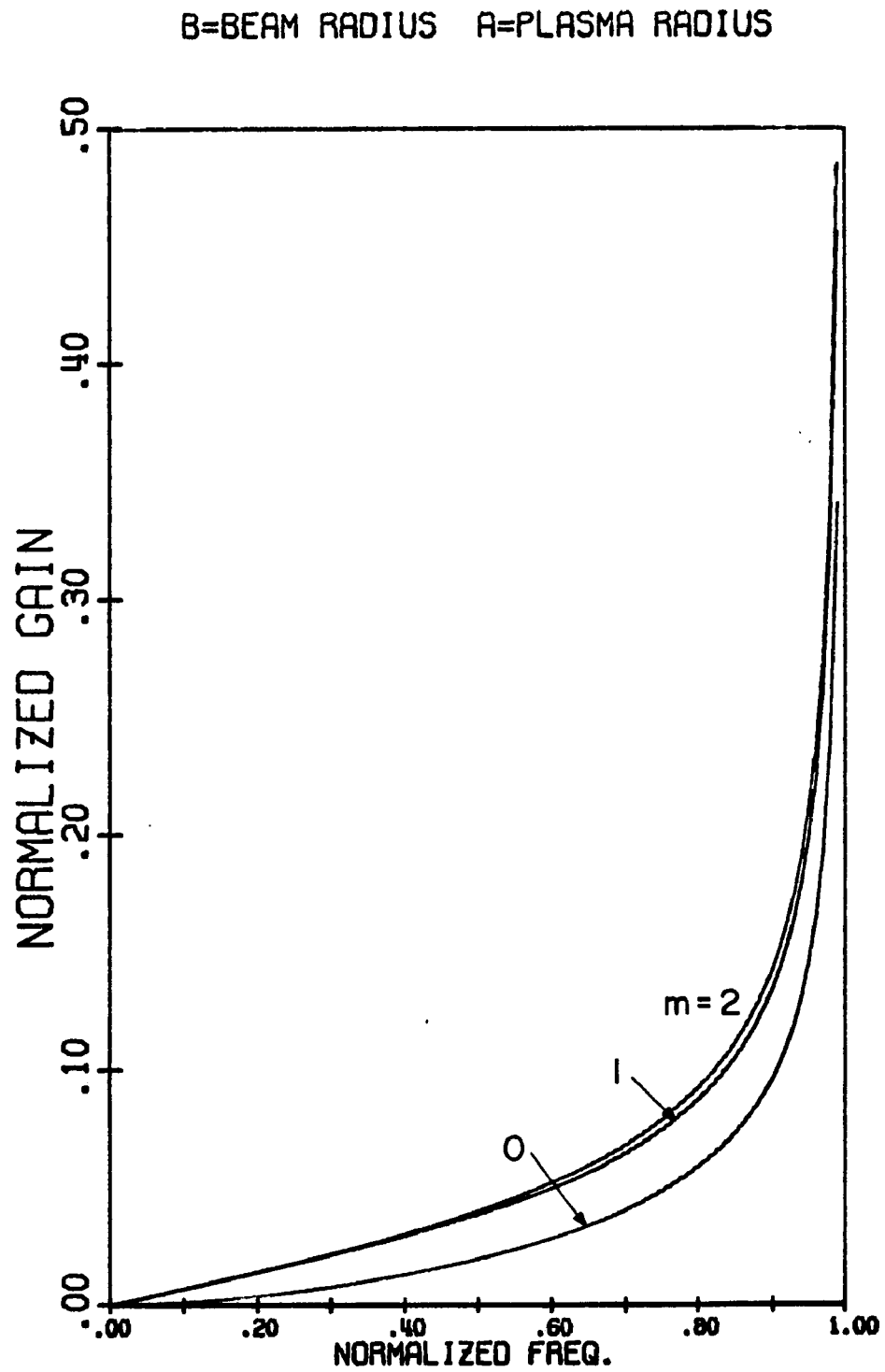


FIG. 3.8 NORMALIZED GAIN VS. NORMALIZED FREQUENCY FOR THE FIRST THREE HARMONICS WITH $b/a = 0$.

$$\beta \equiv b\omega_p/u_o = 1.0 \quad (3.60)$$

and

$$\Omega_b \equiv \omega_b/\omega_p = 0.1 \quad (3.61)$$

Beam perveance is defined in terms of beam dc current, I_o , and voltage, V_o , and is given as

$$P \equiv \frac{I_o}{V_o^{3/2}} \quad (3.62)$$

If Eq. 3.62 is utilized, an expression for beam perveance can be found in terms of the normalized parameters β and Ω_b as follows:

$$P = 2\pi\epsilon_o \left(\frac{2e}{m} \right)^{1/2} (\beta\Omega_b)^2 = 33 \times 10^{-6} (\beta\Omega_b)^2 \quad (3.63)$$

With the numerical values of the normalized parameters chosen as in Eqs. 3.60 and 3.61 a value for the beam perveance results which is equal to 0.33 μ perv.

3.4.4 Inhomogeneous Beam. In this section the effects on the interaction of a previously imposed periodic inhomogeneity of the beam are considered. The section is composed of two parts; the first deals with a rectangular variation of the beam density profile while the second assumes a sinusoidal waveform for the axial inhomogeneity of the beam.

Before each of the above cases is dealt with individually the equation of state for an inhomogeneous beam, Eq. 3.49, is discussed somewhat generally and conditions under which suppression of convective instabilities is possible, if at all, are explored.

Consider a variation of beam density along the axis in accordance with

$$n_{ob}(\xi) = n_{ob} \sum_v \mu_v e^{-jvk_o \xi} . \quad (3.64)$$

Then according to Eq. 3.34 the following results:

$$f_{o,eff}(\omega) = 1 - \mu_o \frac{\omega_b^2}{\omega^2} G_m \quad (3.65)$$

and

$$f_{n,eff}(\omega) = -\mu_n \frac{\omega_b^2}{\omega^2} G_m \quad (n = \pm 1, \pm 2, \dots) . \quad (3.66)$$

Subsequently the equation of state for the inhomogeneous beam can be rewritten as

$$1 - \mu_o \frac{\omega_b^2}{\omega_D^2} G_m - \frac{\omega_p^2}{\omega^2} - 2 \left(1 - \mu_o \frac{\omega_b^2}{\omega_D^2} G_m \right)^{-1} \frac{\omega_b^4}{\omega^2 \omega_D^2} G_m^2 \sum_{n=1}^{\infty} \frac{K^2}{K^2 + n^2 k_o^2} |\mu_n|^2 = 0 , \quad (3.67)$$

where K is to be eliminated using the boundary requirements, Eqs. 3.43 or 3.45, depending upon the plasma having a finite or an infinite transverse extent, respectively. For the sake of argument, it is assumed that for given values of ω_b/ω_p and b/a the value of K can be explicitly determined as a function of ω and k from Eqs. 3.43 or 3.45. Then the weak beam approximation can be written as

$$K(\omega, k) \cong K\left(\omega, \frac{\omega}{u_o}\right) \quad (3.68)$$

throughout the region of instability. In that event Eq. 3.67 assumes the form of a biquadratic equation in ω_D . In terms of the normalized parameters defined as

$$x \equiv \frac{\omega_D^2}{\omega_b^2} \left(\mu_o G_m \right)^{-1}, \quad (3.69)$$

$$\Omega \equiv \frac{\omega}{\omega_p} \quad (3.70)$$

and

$$d \equiv \frac{\omega_b^2}{\omega^2} \frac{G_m}{\mu_o} \sum_{n=1}^{\infty} \frac{K^2}{K^2 + n^2 k_o^2} |\mu_n|^2, \quad (3.71)$$

Eq. 3.67 can be written as

$$(1 - \Omega^{-2})x^2 - 2 \left(1 - \frac{1}{2} \Omega^{-2} + d \right) x + 1 = 0. \quad (3.72)$$

The solution is as follows:

$$x = (1 - \Omega^{-2})^{-1} \left\{ 1 - \left(\frac{1}{2} \Omega^{-2} - d \right) \left[1 - \sqrt{1 + 2d\Omega^2 \left(\frac{1}{2} - d\Omega^2 \right)^{-2}} \right] \right\}, \quad (3.73)$$

where the root was discarded which does not correspond to the appropriate limiting value for x , namely that

$$\lim_{d \rightarrow 0} x = (1 - \Omega^{-2})^{-1}. \quad (3.74)$$

The expression on the right-hand side of Eq. 3.73 can be greatly simplified by noting that

$$|\Omega^2 d| = \left| \frac{\omega_b^2}{\mu_o \omega_p^2} G_m \left(\frac{b\omega}{u_o}, \frac{b}{a} \right) \sum_{n=1}^{\infty} \frac{K^2}{K^2 + n^2 k_o^2} |\mu_n|^2 \right| \ll 1 \quad (3.75)$$

The above condition is always satisfied if the set $\{|\mu_n|\}$ represents a rapidly converging sequence and if $K \neq \pm jnk_o$. The condition where $K = \pm jnk_o$, however, as noted later, contradicts the requirement $\omega \ll \omega_o$ upon which the entire analysis is based. An expansion of the right-hand side of Eq. 3.73 in ascending powers of $\Omega^2 d$ yields the following expression for x :

$$x \cong (1 - \Omega^{-2})^{-1} (1 + 2\Omega^2 d + 8\Omega^4 d^2 + \dots) \quad (3.76)$$

Straightforward algebra and further simplification yield the following final result which expresses the propagation constant, k , in terms of other parameters:

$$k \cong \frac{\omega}{u_o} \pm j \frac{\omega_b}{u_o} \left(\frac{\mu_o G_m}{\omega_p^2 / \omega^2 - 1} \right)^{1/2} \left(1 + 2 \frac{\omega_b^2}{\omega_p^2} \frac{G_m}{\mu_o} \sum_{n=1}^{\infty} \frac{K^2}{K^2 + n^2 k_o^2} |\mu_n|^2 \right) \quad (3.77)$$

The growth rate, when normalized to that of a homogeneous beam, can be expressed as

$$\frac{\delta}{\delta_o} = \mu_o^{1/2} + 2 \frac{\omega_b^2}{\omega_p^2} \mu_o^{-1/2} G_m \left(\frac{b\omega}{u_o}, \frac{b}{a} \right) \sum_{n=1}^{\infty} \frac{K^2}{K^2 + n^2 k_o^2} |\mu_n|^2 \quad (3.78)$$

which exhibits appropriate limiting behavior in the limit of a vanishing inhomogeneity.

Suppression of convective instabilities is possible if (1) K is pure imaginary and (2) $|K| \ll k_o$. The above conditions will be shown to be

generally satisfied unless the beam fills the entire region inside the perfectly conducting cylinder, in which case the value of K is given by p_{ms}/a , where p_{ms} is the s th root of the m th-order Bessel function of the first kind.

To derive the appropriate form of the function $K = K(\omega, k)$ from the boundary requirements, Eqs. 3.43 or 3.45, note the fact that according to Eq. 3.77 the propagation constant for the homogeneous beam, as expressed by Eq. 3.59, differs from that of an inhomogeneous beam by a correction term added to the growth rate. That the value of the correction term, which is designated by λ , is small can be realized by noting that the value of λ does not exceed an upper limit given by the beam-to-plasma density ratio. This stems from the fact that

$$G_m \left(\frac{b\omega}{u_0}, \frac{b}{a} \right) \leq \frac{1}{2}$$

and

$$\left| \sum_{n=1}^{\infty} \frac{K^2}{K^2 + n^2 k_0^2} |\mu_n|^2 \right| < 1 .$$

Therefore the form of $K(\omega, k)$ can be approximated by the form that its limit assumes when the inhomogeneity vanishes. This was shown to have the form

$$K(\omega, k) \rightarrow \pm jk . \quad (3.79)$$

Thus, in accordance with the previous contention, K can be substituted as a first approximation by $\pm j\omega/u_0$ in Eqs. 3.77 and 3.78 yielding the following final results:

$$k \cong \frac{\omega}{u_o} \pm j \frac{\omega_b}{u_o} \left(\frac{\mu_o G_m}{\omega_p^2/\omega^2 - 1} \right)^{1/2} \left(1 - 2 \frac{\omega_b^2}{\omega_p^2} \frac{G_m}{\mu_o} \sum_{n=1}^{\infty} \frac{\omega^2}{n^2 \omega_o^2 - \omega^2} |\mu_n|^2 \right) \quad (3.80)$$

and

$$\frac{\delta}{\delta_o} \cong \mu_o^{1/2} - 2 \frac{\omega_b^2}{\omega_p^2} \mu_o^{-1/2} G_m \left(\frac{b\omega}{u_o}, \frac{b}{a} \right) \sum_{n=1}^{\infty} \frac{\omega^2}{n^2 \omega_o^2 - \omega^2} |\mu_n|^2. \quad (3.81)$$

3.4.4a Rectangular Inhomogeneity. The results obtained previously can be directly applied to a rectangular density profile, where the variation of the density along the axis is given by Eq. 2.31. Straight-forward Fourier decomposition results in the following:

$$\mu_o = \Delta, \quad (3.82)$$

$$\mu_n = (e^{j2n\pi\Delta} - 1)/j2n\pi \quad (n = \pm 1, \pm 2, \dots), \quad (3.83)$$

where Δ is the duty cycle. A direct substitution into Eqs. 3.80 and 3.81 yields

$$k \cong \frac{\omega}{u_o} \pm j \frac{\omega_b}{u_o} \left(\frac{\Delta G_m}{\omega_p^2/\omega^2 - 1} \right)^{1/2} \left[1 - \frac{2}{\pi^2} \frac{\omega_b^2}{\omega_p^2} \frac{G_m}{\Delta} \sum_{n=1}^{\infty} \frac{\omega^2}{n^2 \omega_o^2 - \omega^2} \left(\frac{\sin n\pi\Delta}{n} \right)^2 \right] \quad (3.84)$$

and

$$\frac{\delta}{\delta_o} \cong \Delta^{1/2} - \frac{2}{\pi^2} \frac{\omega_b^2}{\omega_p^2} \Delta^{-1/2} G_m \left(\frac{b\omega}{u_o}, \frac{b}{a} \right) \sum_{n=1}^{\infty} \frac{\omega^2}{n^2 \omega_o^2 - \omega^2} \left(\frac{\sin n\pi\Delta}{n} \right)^2. \quad (3.85)$$

A measure of suppression of the convective instabilities due to the beam inhomogeneity can be found by using Eq. 3.85. Care must be employed, however, in the manner in which the rate of suppression is defined. For a beam of rectangular density profile the dc power carried by the beam is reduced from that of the homogeneous beam by a factor equal to the duty cycle. This, in turn, results in a decrease of the growth rate by a factor of $\Delta^{1/2}$. Therefore the first term in Eq. 3.85 does not truly represent the suppression of convective instabilities due to inhomogeneity and is merely an indication of the fact that the input dc power has been reduced by a factor of Δ . The rate of suppression, Γ , of the convective instabilities due to axial inhomogeneity of beam density can then be represented as

$$\Gamma = 1 - \frac{\delta/\delta_0}{\Delta^{1/2}} = \frac{2}{\pi^2} \frac{\omega_b^2}{\omega_p^2} \frac{G_m}{\Delta} \sum_{n=1}^{\infty} \frac{\omega^2}{n^2 \omega_0^2 - \omega^2} \left(\frac{\sin n\pi\Delta}{n} \right)^2 . \quad (3.86)$$

According to the previous assumption, $\omega_0 \gg \omega$, the appearance of $\omega^2/(n^2 \omega_0^2 - \omega^2)$ in the expression for Γ makes the quantity negligibly small. Figures 3.9 through 3.16 show the variations of the rate of suppression, Γ , once as a function of normalized frequency, ω/ω_p , with duty cycle, Δ , taken as a parameter (Figs. 3.9 through 3.12) and once as a function of the duty cycle with the normalized frequency taken as a parameter (Figs. 3.13 through 3.16) for the first two harmonics and for $b/a = 0$ and $1/2$. The other parameters are assumed to have the following values:

$$\omega_b/\omega_p = 0.1 , \quad b\omega_p/u_0 = 1.0 , \quad \omega_0/\omega_p = 10 . \quad (3.87)$$

$$WB/WP = .1 \quad B \times WP/U_0 = 1. \quad W_0/WP = 10.$$

B=BEAM RADIUS A=PLASMA RADIUS

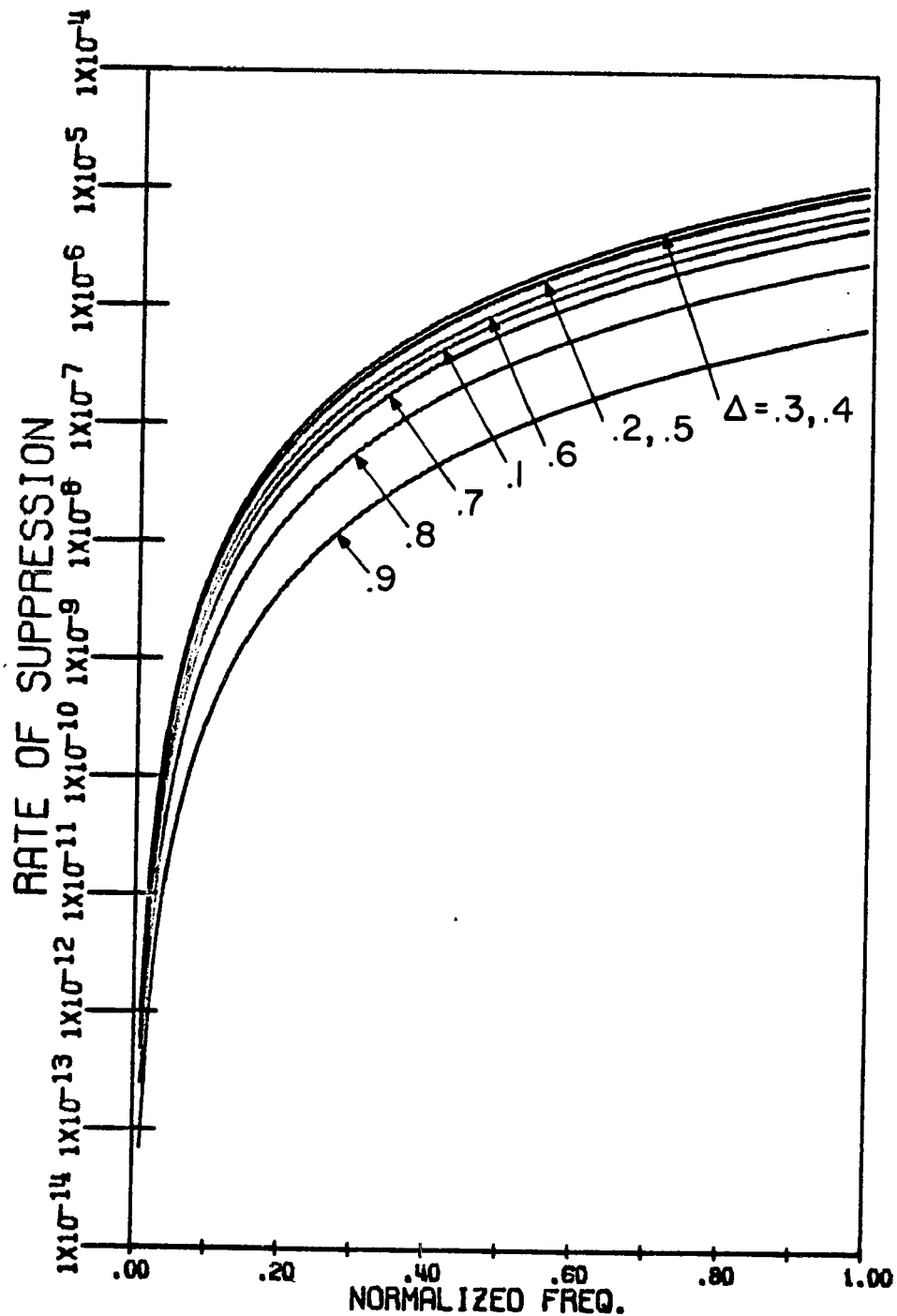


FIG. 3.9 THE RATE OF SUPPRESSION VS. NORMALIZED FREQUENCY WITH THE DUTY CYCLE TAKEN AS A PARAMETER FOR $b/a = 0$ AND $m = 0$ (RECTANGULAR INHOMOGENEITY).

$$WB/WP=.1 \quad B \times WP/U0=1. \quad W0/WP=10.$$

B=BEAM RADIUS A=PLASMA RADIUS

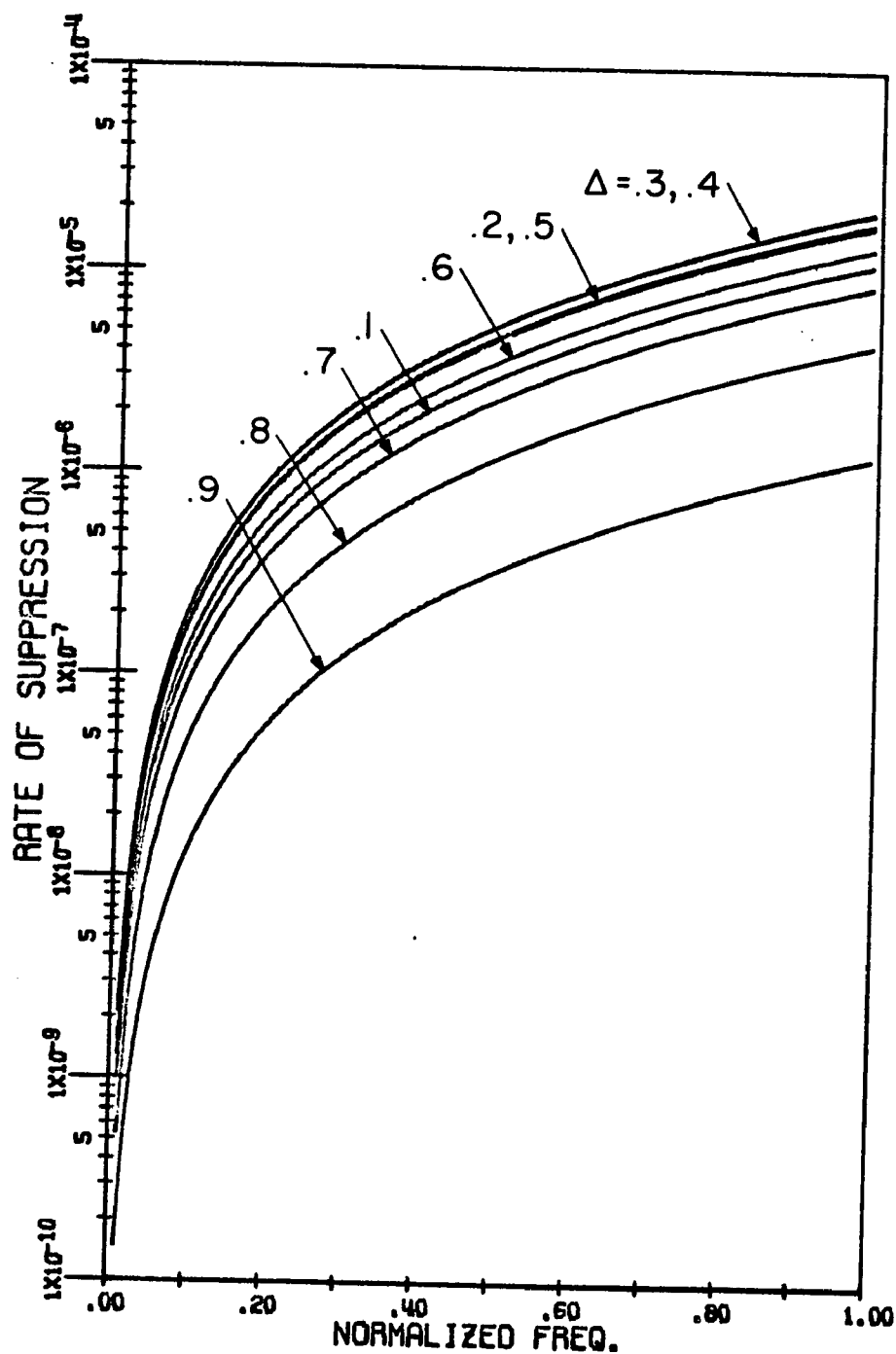


FIG. 3.10 THE RATE OF SUPPRESSION VS. NORMALIZED FREQUENCY WITH THE DUTY CYCLE TAKEN AS A PARAMETER FOR $b/a = 0$ AND $m = 1$ (RECTANGULAR INHOMOGENEITY).

$$WB/WP=.1 \quad B \times WP/U0=1. \quad W0/WP=10.$$

B=BEAM RADIUS A=PLASMA RADIUS

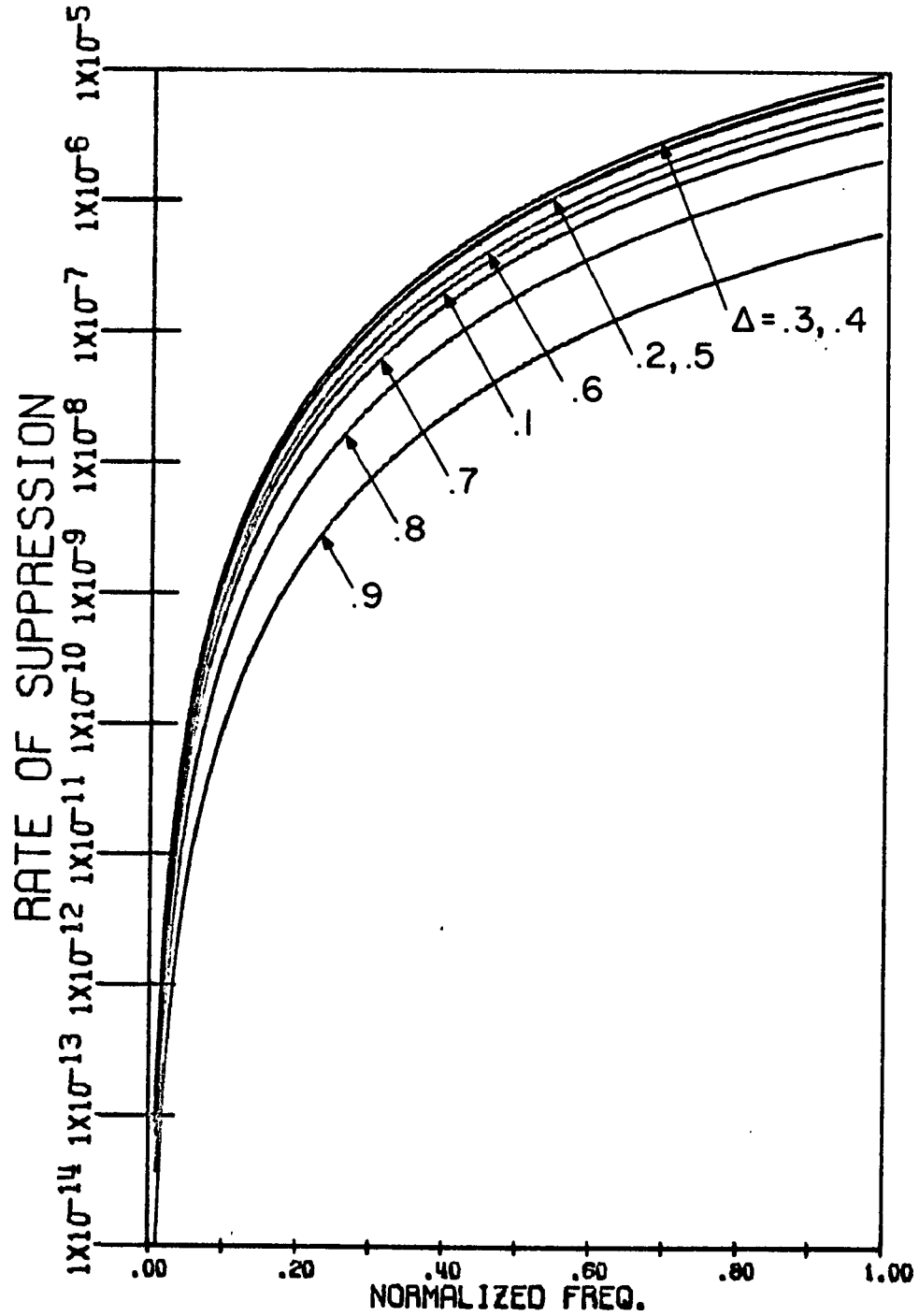


FIG. 3.11 THE RATE OF SUPPRESSION VS. NORMALIZED FREQUENCY WITH THE DUTY CYCLE TAKEN AS A PARAMETER FOR $b/a = 1/2$ AND $m = 0$ (RECTANGULAR INHOMOGENEITY).

$$WB/WP=.1 \quad B \times WP/U0=1. \quad W0/WP=10.$$

B=BEAM RADIUS A=PLASMA RADIUS

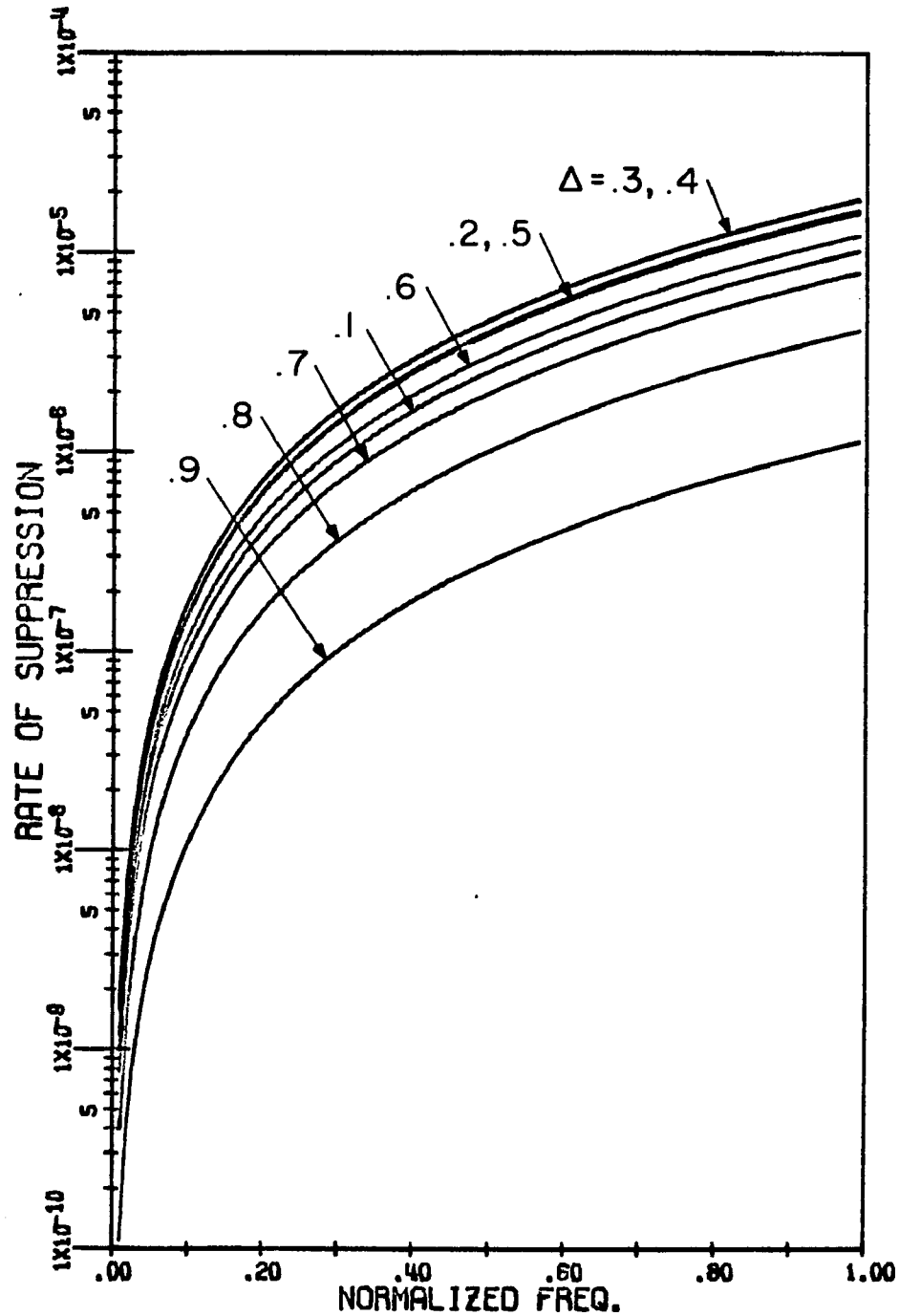


FIG. 3.12 THE RATE OF SUPPRESSION VS. NORMALIZED FREQUENCY WITH THE DUTY CYCLE TAKEN AS A PARAMETER FOR $b/a = 1/2$ AND $m = 1$ (RECTANGULAR INHOMOGENEITY).

$$WB/WP = .1 \quad B \times WP/U0 = 1. \quad W0/WP = 10.$$

B=BEAM RADIUS A=PLASMA RADIUS

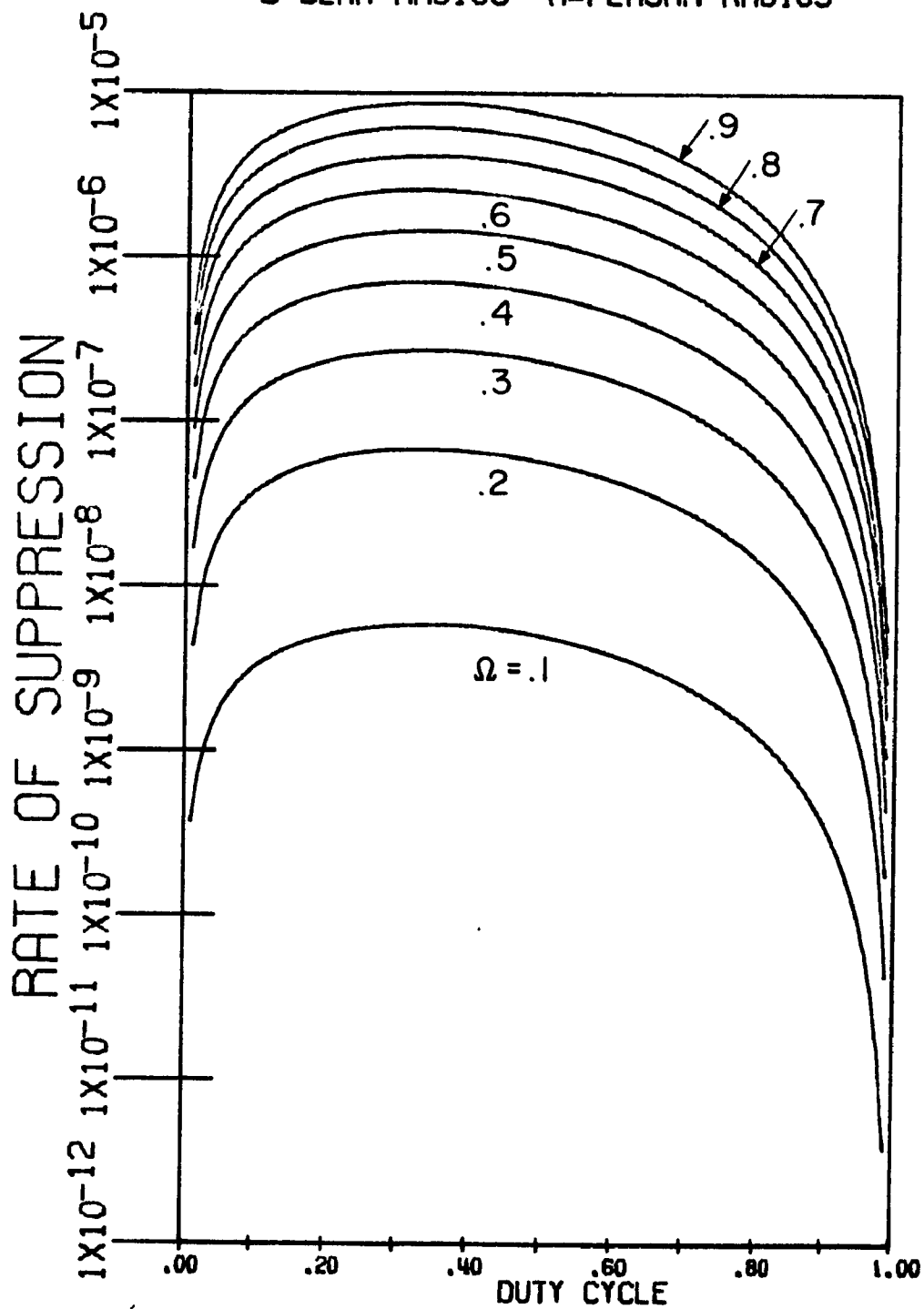


FIG. 3.13 THE RATE OF SUPPRESSION VS. DUTY CYCLE WITH NORMALIZED FREQUENCY TAKEN AS A PARAMETER FOR $b/a = 0$ AND $m = 0$ (RECTANGULAR INHOMOGENEITY).

$$WB/WP=.1 \quad B \times WP/U0=1. \quad W0/WP=10.$$

B=BEAM RADIUS A=PLASMA RADIUS

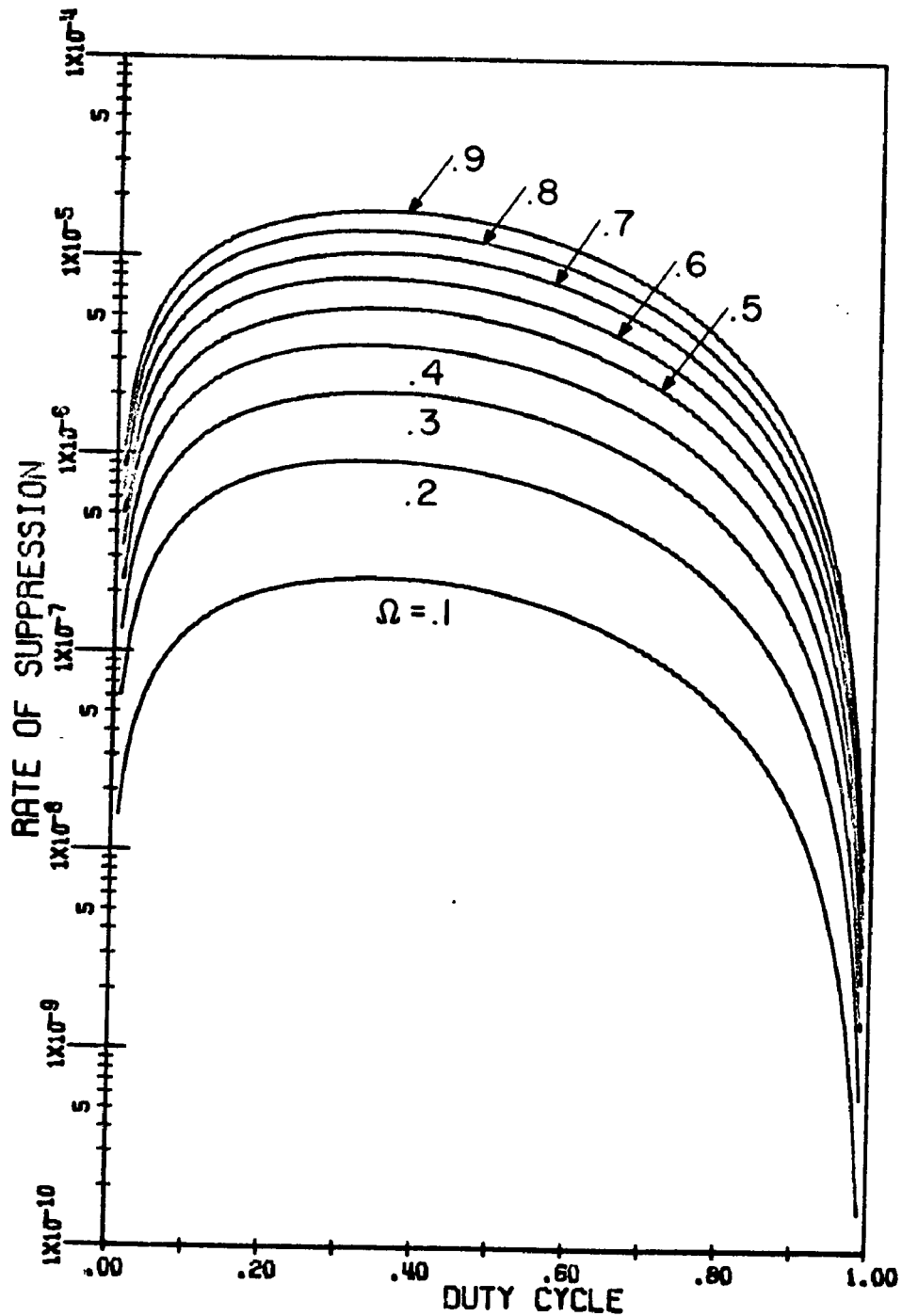


FIG. 3.14 THE RATE OF SUPPRESSION VS. DUTY CYCLE WITH NORMALIZED FREQUENCY TAKEN AS A PARAMETER FOR $b/a = 0$ AND $m = 1$ (RECTANGULAR INHOMOGENEITY).

$$WB/WP = .1 \quad B \times WP/U0 = 1. \quad W0/WP = 10.$$

B=BEAM RADIUS A=PLASMA RADIUS

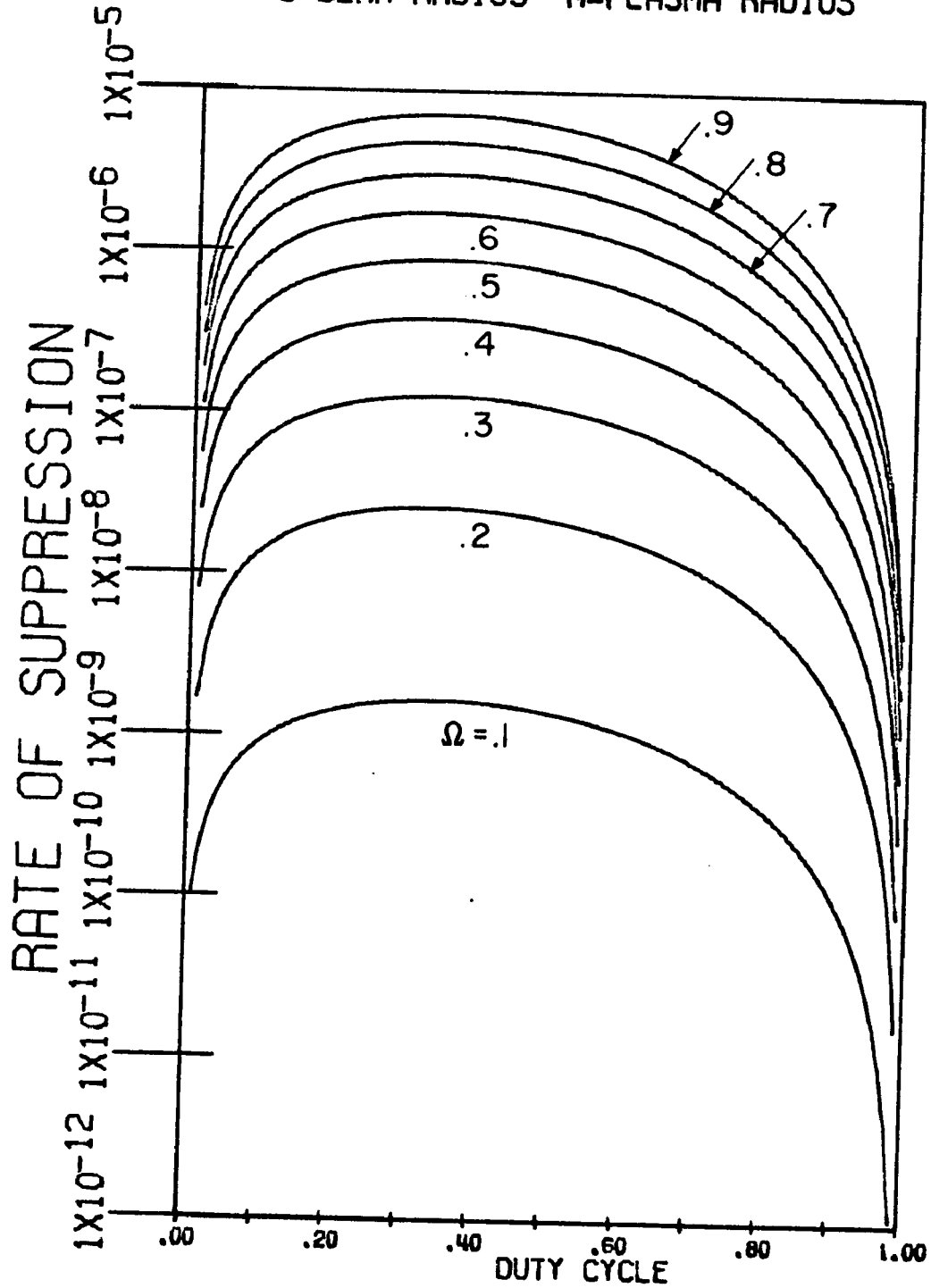


FIG. 3.15 THE RATE OF SUPPRESSION VS. DUTY CYCLE WITH NORMALIZED FREQUENCY TAKEN AS A PARAMETER FOR $b/a = 1/2$ AND $m = 0$ (RECTANGULAR INHOMOGENEITY).

$$WB/WP=.1 \quad B \times WP/U0=1. \quad W0/WP=10.$$

B=BEAM RADIUS A=PLASMA RADIUS

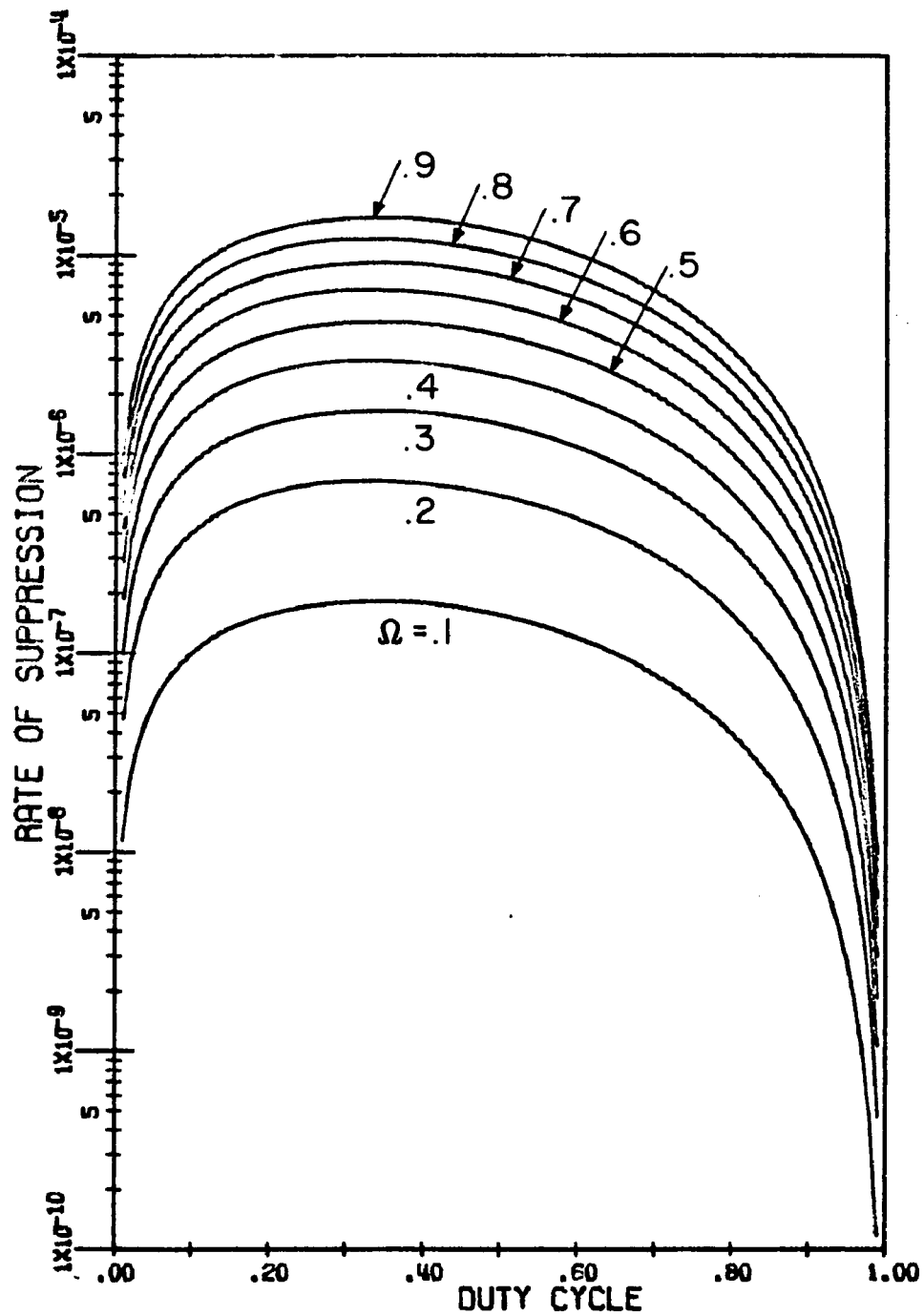


FIG. 3.16 THE RATE OF SUPPRESSION VS. DUTY CYCLE WITH NORMALIZED FREQUENCY TAKEN AS A PARAMETER FOR $b/a = 1/2$ AND $m = 1$ (RECTANGULAR INHOMOGENEITY).

The rate of suppression remains negligibly small throughout the region of instability; thus it is concluded that an inhomogeneity in beam density, which was previously shown to suppress the convectively unstable waves in a one-dimensional model, has no suppressive effects on the convective instabilities excited in the system if the externally imposed dc magnetic field is removed. The physical reason lies in the difference in the way an individually charged particle behaves in the presence or absence of a dc magnetic field. The point is further discussed in Section 3.4.5 in connection with a physical analogy which serves to represent a simple dynamical model for the system.

3.4.4b Sinusoidal Inhomogeneity. For a beam with density variations in accordance with Eq. 2.41, it is immediately concluded that

$$\mu_0 = 1, \quad (3.88)$$

$$\mu_{\pm 1} = \frac{\alpha}{2}, \quad \mu_n = 0 \quad (n = \pm 2, \pm 3, \dots) \quad (3.89)$$

Hence the propagation constant, k , and the normalized growth rate, δ/δ_0 , can be expressed as

$$k \approx \frac{\omega}{u_0} \pm j \frac{\omega_b}{u_0} \left(\frac{G_m}{\omega_p^2/\omega^2 - 1} \right)^{1/2} \left(1 - \frac{\alpha^2}{2} \frac{\omega_b^2}{\omega_p^2} \frac{\omega^2}{n^2\omega_0^2 - \omega^2} G_m \right) \quad (3.90)$$

and

$$\frac{\delta}{\delta_0} \approx 1 - \frac{\alpha^2}{2} \frac{\omega_b^2}{\omega_p^2} \frac{\omega^2}{n^2\omega_0^2 - \omega^2} G_m \quad (3.91)$$

The rate of suppression in this case can appropriately be defined as

$$\Gamma = 1 - \frac{\delta}{\delta_0} = \frac{\alpha^2}{2} \frac{\omega_b^2}{\omega_p^2} \frac{\omega^2}{n^2 \omega_0^2 - \omega^2} G_m . \quad (3.92)$$

The variation of Γ is shown in Figs. 3.17 through 3.24, once as a function of normalized frequency, ω/ω_p , with depth of modulation, α , taken as a parameter (Figs. 3.17 through 3.20) and once as a function of the depth of modulation with the normalized frequency taken as a parameter (Figs. 3.21 through 3.24) for the first two harmonics and for $b/a = 0$ and $1/2$.

A behavior similar to that of the rectangular inhomogeneity is observed, namely that the rate of suppression is negligibly small.

3.4.5 Discussion of the Results. It was shown in Section 3.4.4 that a longitudinally inhomogeneous beam, in the absence of an externally applied dc magnetic field, exhibits no stabilizing effects on the instabilities that grow naturally in a beam-plasma system. This is in contradistinction with the stabilizing effects that electron beams of the same type exhibit in a one-dimensional beam-plasma model which represents, in physical terms, a beam-plasma system with an externally applied dc magnetic field of infinitely large value. It can therefore be stated that the magnetic field plays a decisive role in dynamic stabilization of the beam-plasma-type, convectively unstable modes. Any attempt to explain the phenomenon must therefore be directed toward exploring the consequences of the difference in behavior that charged particles exhibit in the presence or absence of an externally imposed dc magnetic field. In what follows, the results are explained in connection with a physical analog

$$WB/WP=.1 \quad B \times WP/U0=1. \quad W0/WP=10.$$

B=BEAM RADIUS A=PLASMA RADIUS

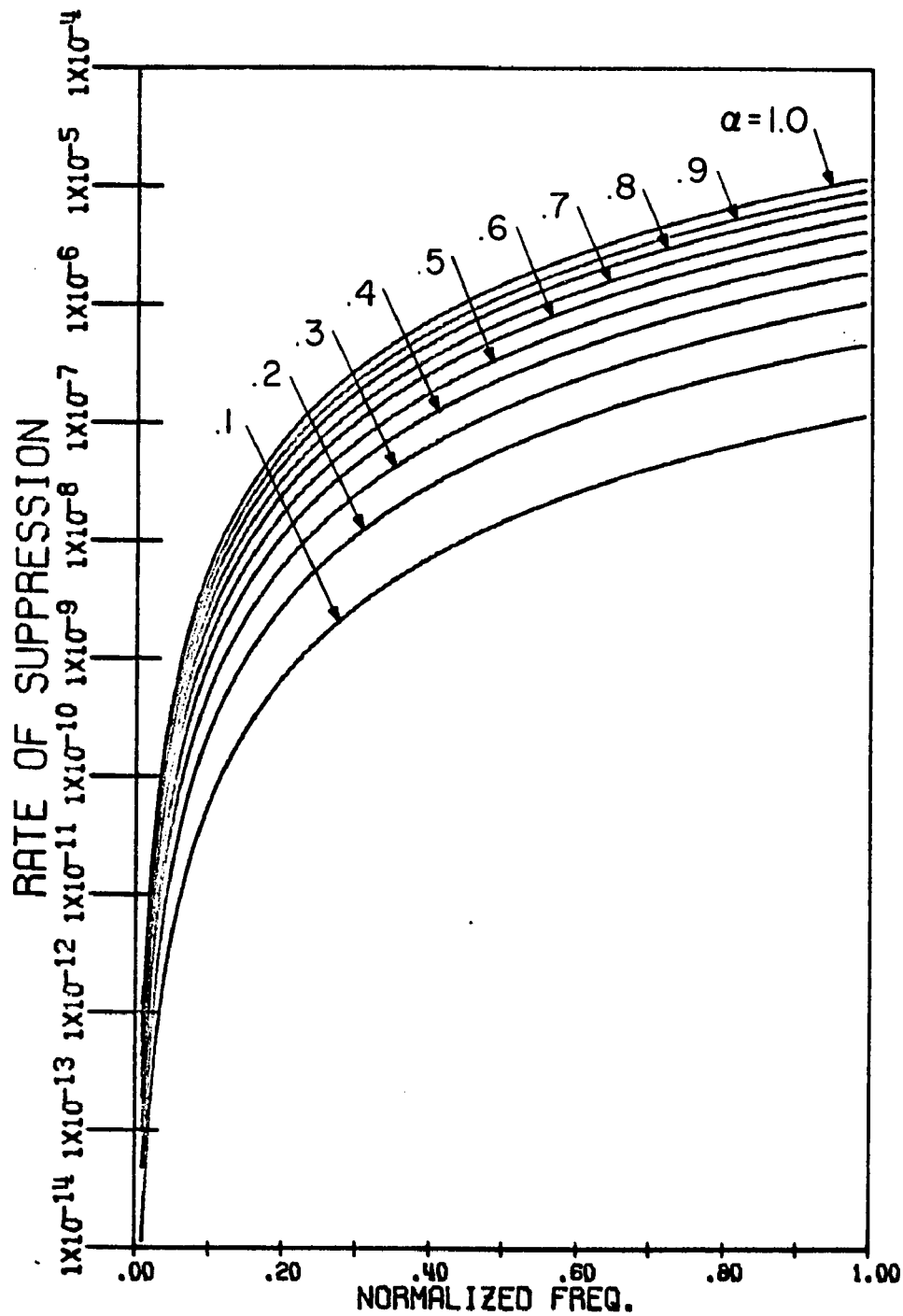


FIG. 3.17 THE RATE OF SUPPRESSION VS. NORMALIZED FREQUENCY WITH DEPTH OF MODULATION TAKEN AS A PARAMETER FOR $b/a = 0$ AND $m = 0$ (SINUSOIDAL INHOMOGENEITY).

$$WB/WP=.1 \quad B \times WP/U0=1. \quad W0/WP=10.$$

B=BEAM RADIUS A=PLASMA RADIUS.

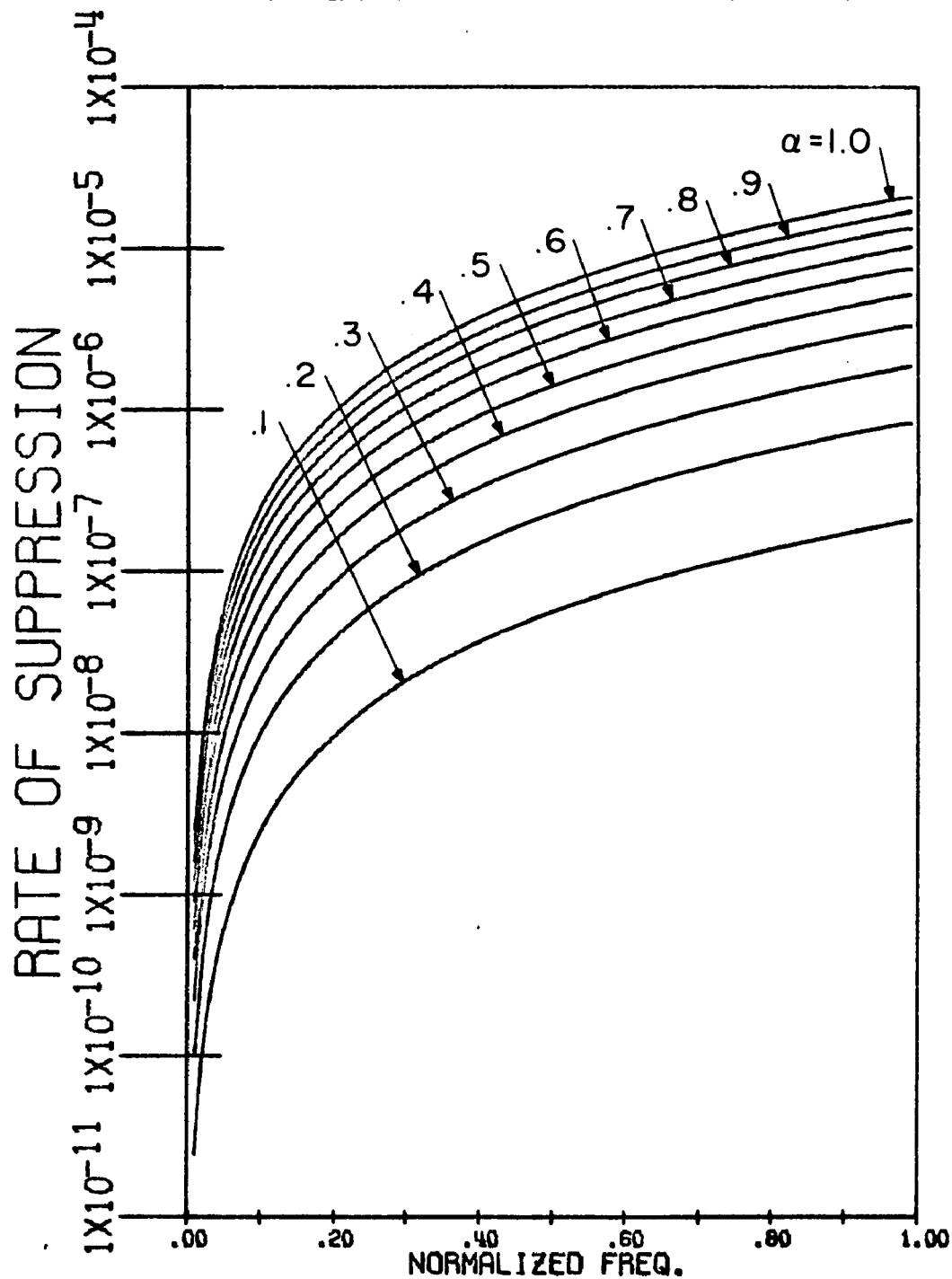


FIG. 3.18 THE RATE OF SUPPRESSION VS. NORMALIZED FREQUENCY WITH DEPTH OF MODULATION TAKEN AS A PARAMETER FOR $b/a = 0$ AND $m = 1$ (SINUSOIDAL INHOMOGENEITY).

$$WB/WP=.1 \quad B \times WP/U0=1. \quad W0/WP=10.$$

B=BEAM RADIUS A=PLASMA RADIUS

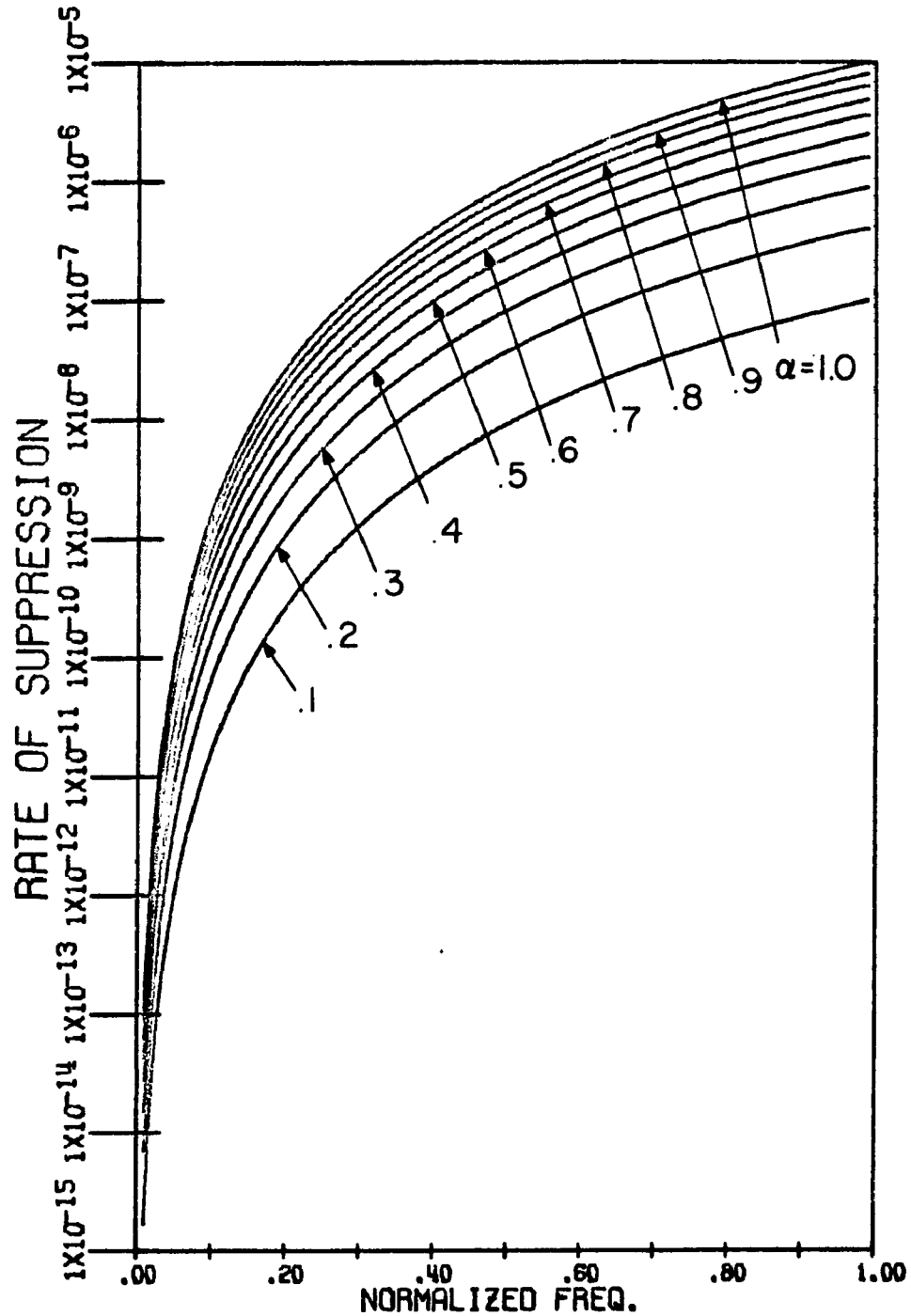


FIG. 3.19 THE RATE OF SUPPRESSION VS. NORMALIZED FREQUENCY WITH DEPTH OF MODULATION TAKEN AS A PARAMETER FOR $b/a = 1/2$ AND $m = 0$ (SINUSOIDAL INHOMOGENEITY).

$$WB/WP=.1 \quad B \times WP/U0=1. \quad W0/WP=10.$$

B=BEAM RADIUS A=PLASMA RADIUS

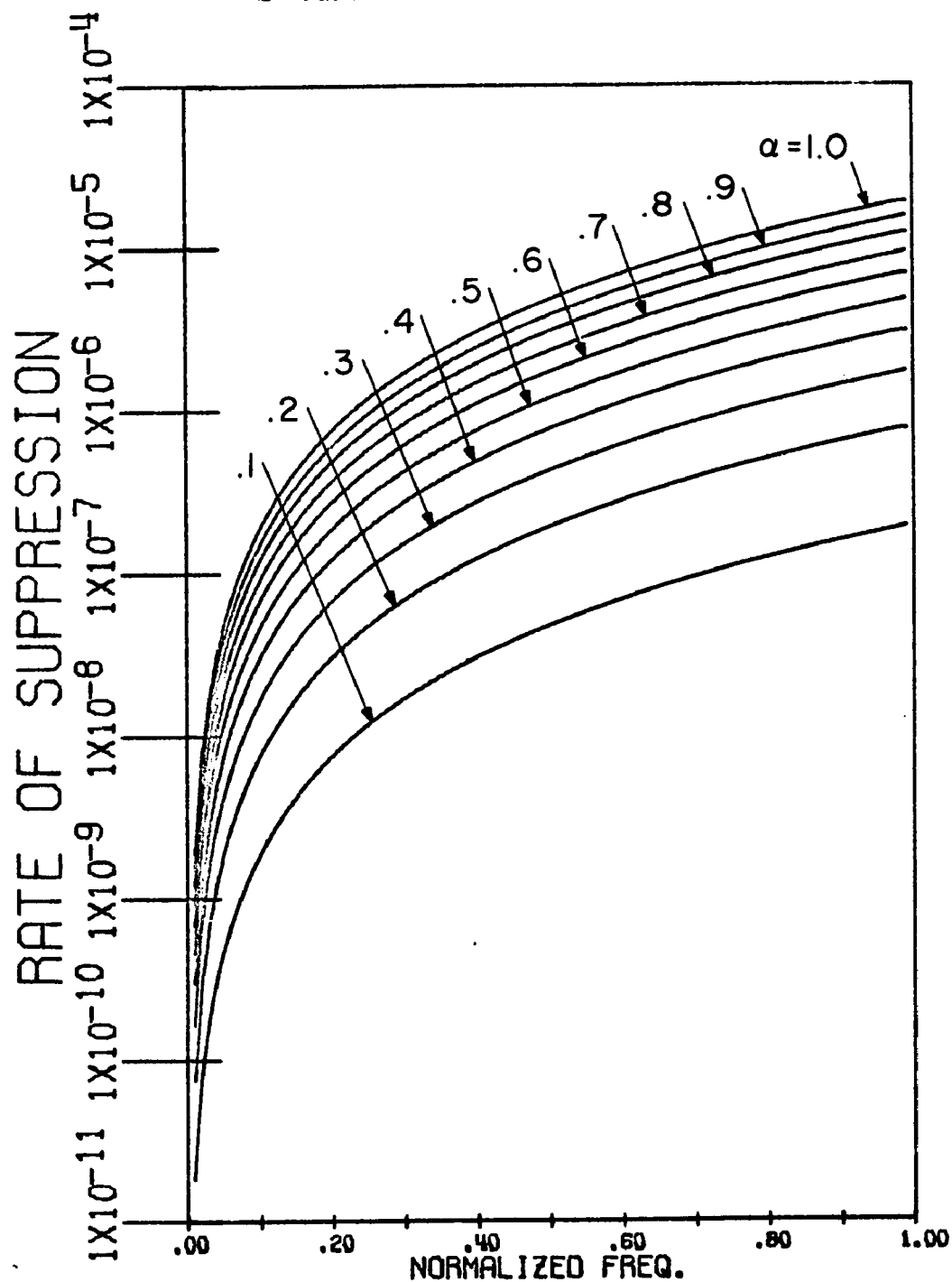


FIG. 3.20 THE RATE OF SUPPRESSION VS. NORMALIZED FREQUENCY WITH DEPTH OF MODULATION TAKEN AS A PARAMETER FOR $b/a = 1/2$ AND $m = 1$ (SINUSOIDAL INHOMOGENEITY).

$$WB/WP=.1 \quad B \times WP/U0=1. \quad W0/WP=10.$$

B=BEAM RADIUS A=PLASMA RADIUS

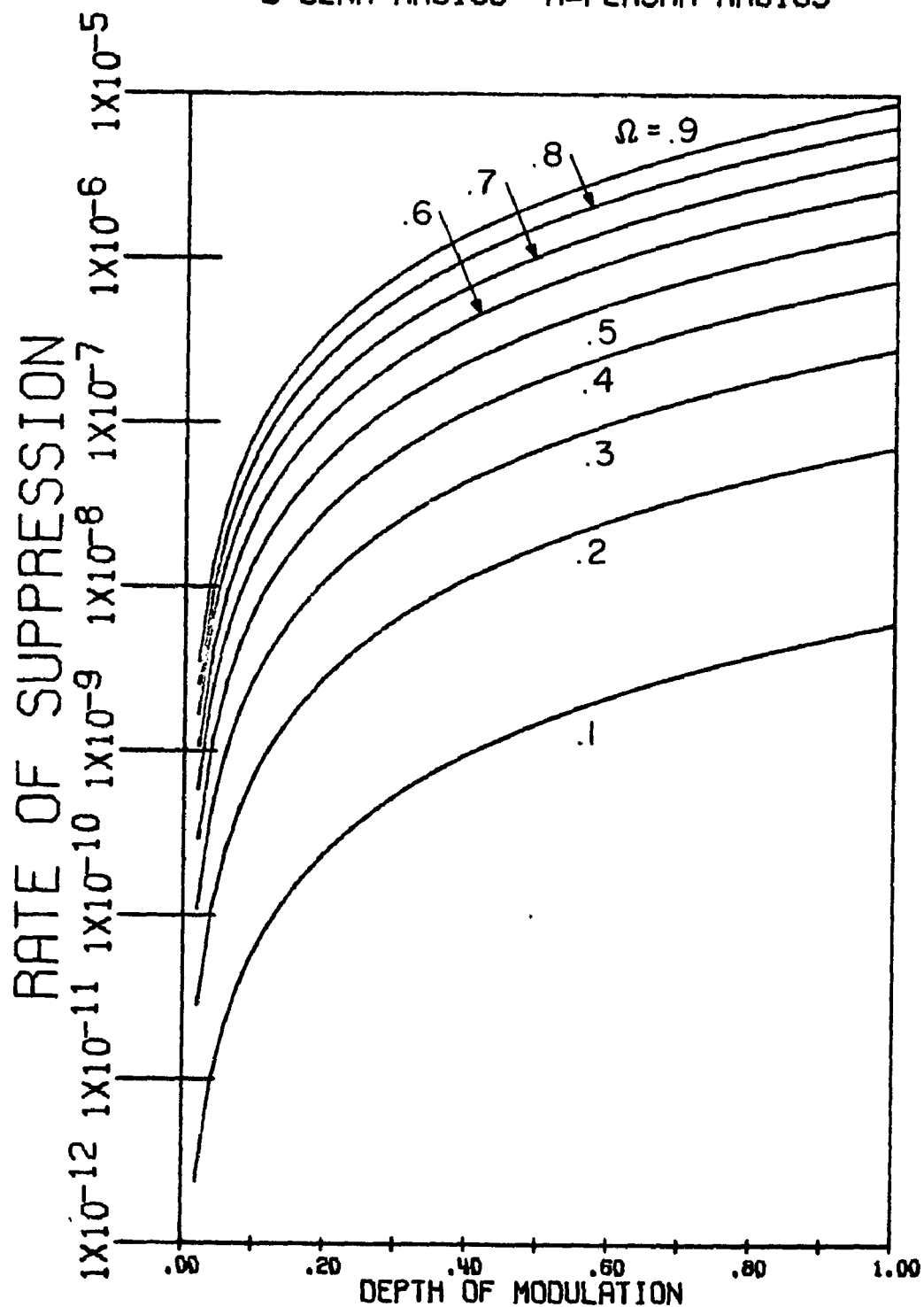


FIG. 3.21 THE RATE OF SUPPRESSION VS. DEPTH OF MODULATION WITH
NORMALIZED FREQUENCY TAKEN AS A PARAMETER FOR $b/a = 0$
AND $m = 0$ (SINUSOIDAL INHOMOGENEITY).

$$WB/WP=.1 \quad B \times WP/U0=1. \quad W0/WP=10.$$

B=BEAM RADIUS A=PLASMA RADIUS

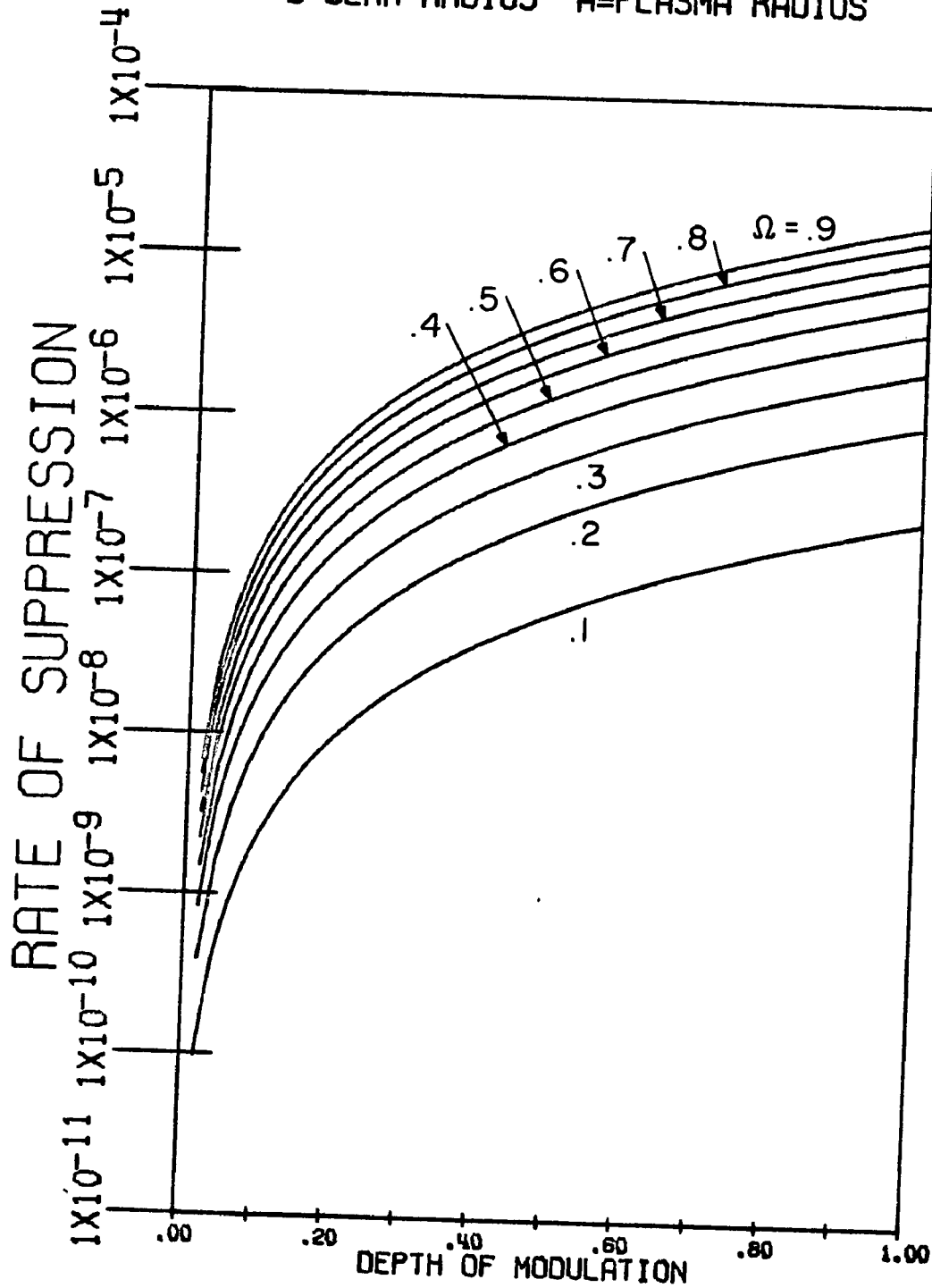


FIG. 3.22 THE RATE OF SUPPRESSION VS. DEPTH OF MODULATION WITH
NORMALIZED FREQUENCY TAKEN AS A PARAMETER FOR $b/a = 0$
AND $m = 1$ (SINUSOIDAL INHOMOGENEITY).

$$WB/WP=.1 \quad B \times WP/U0=1. \quad W0/WP=10.$$

B=BEAM RADIUS A=PLASMA RADIUS

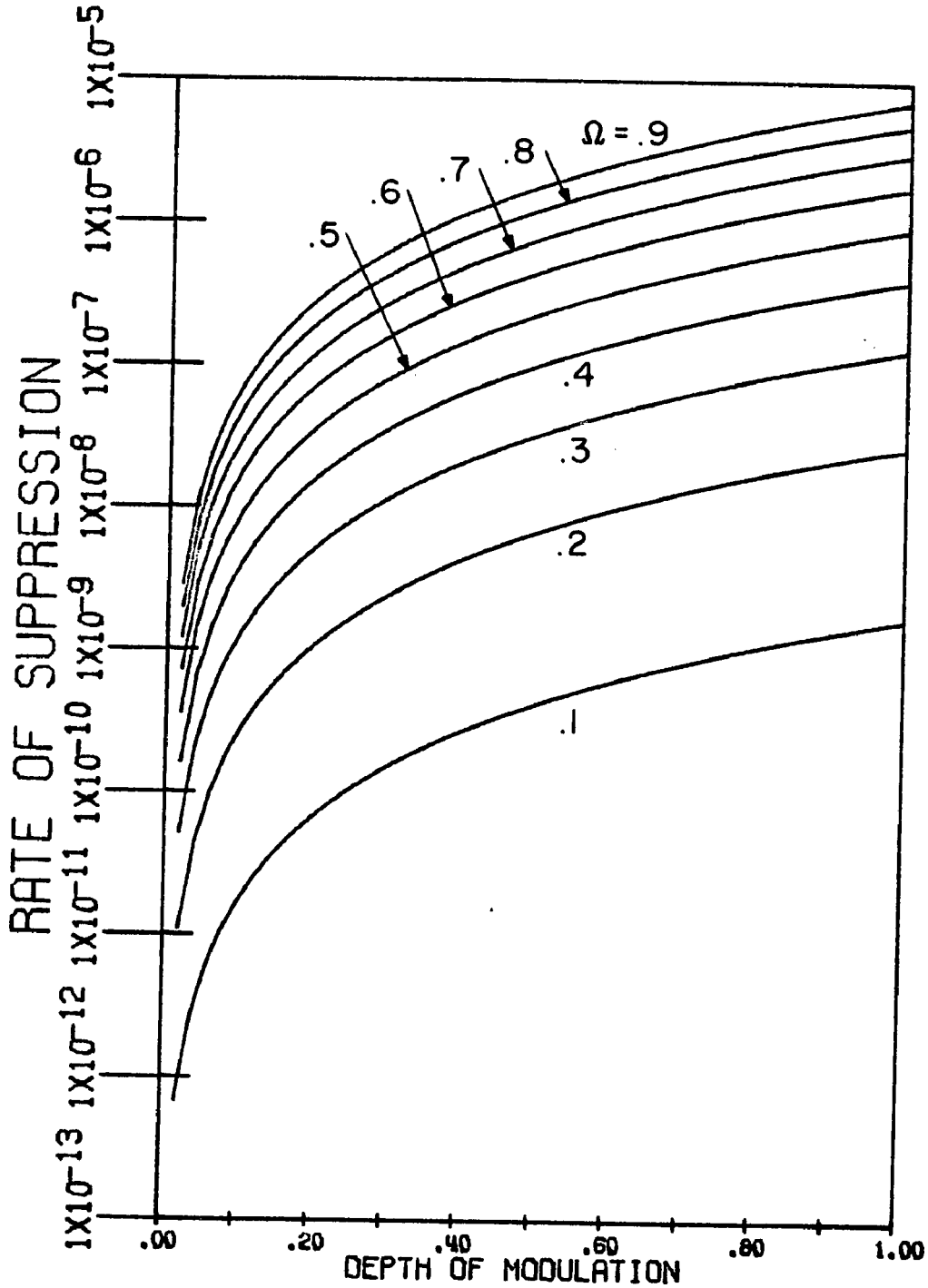


FIG. 3.23 THE RATE OF SUPPRESSION VS. DEPTH OF MODULATION WITH
NORMALIZED FREQUENCY TAKEN AS A PARAMETER FOR $b/a = 1/2$
AND $m = 0$ (SINUSOIDAL INHOMOGENEITY).

$$WB/WP=.1 \quad B \times WP/U0=1. \quad W0/WP=10.$$

B=BEAM RADIUS A=PLASMA RADIUS

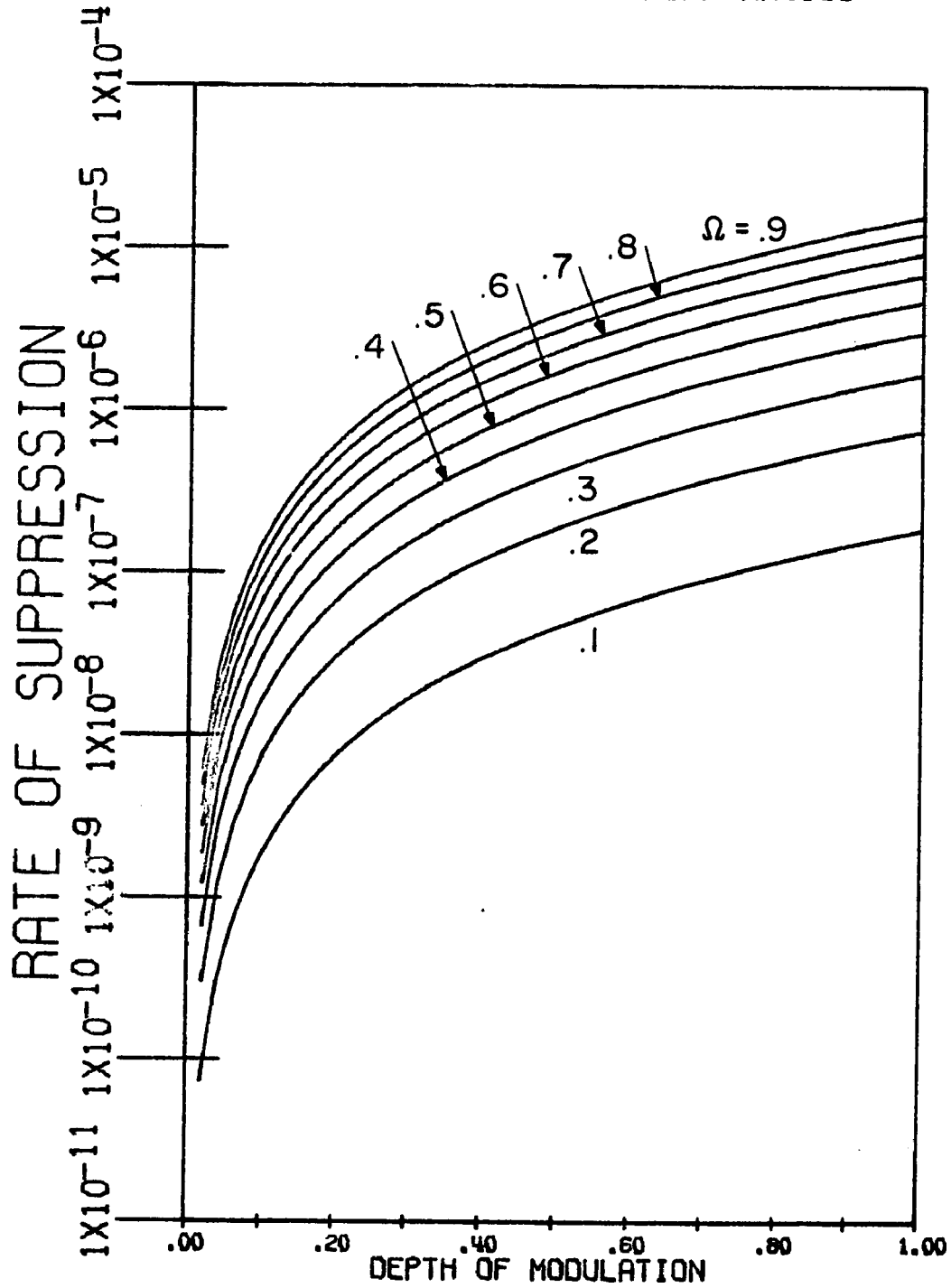


FIG. 3.24 THE RATE OF SUPPRESSION VS. DEPTH OF MODULATION WITH
NORMALIZED FREQUENCY TAKEN AS A PARAMETER FOR $b/a = 1/2$
AND $m = 1$ (SINUSOIDAL INHOMOGENEITY).

representing the original system in a less intricate fashion. The arguments are then translated into the language of the original system to make the analogy complete and to further the understanding of the physics involved.

Consider a rod of length L which is made to stand on a rigid pivot representing its state of unstable equilibrium in a gravitational force field. It is obvious that a small-angle deviation of the rod from its position of unstable equilibrium will upset the equilibrium by setting the rod into a nonlinear pendular motion about its position of stable equilibrium. The nonlinearity of the motion follows from the fact that a deviation from the position of unstable equilibrium by a small angle, $\delta\theta_0$, corresponds to a large-angle deviation ($\theta_0 = \pi - \delta\theta_0$) from the position of stable equilibrium about which the system oscillates. It is shown elsewhere⁷⁷ that, if the pivot is excited externally to perform a small-amplitude oscillation of frequency, ω_0 , along the lines of force of the field, a small-angle deviation from the position of unstable equilibrium will no longer upset the system provided that $\omega_0 \gg \sqrt{g/L}$, where g denotes the gravitational acceleration. The system will then be set into a linear motion composed of a low-frequency, ($\omega = \sqrt{g/L}$), large-amplitude (respectively speaking), regular pendular motion about its position of unstable equilibrium superimposed by a high-frequency, small-amplitude oscillatory motion of frequency ω_0 .

The physical reason for stabilization lies in the particular manner in which the inertia is transferred from the external source of oscillation to the rod. This is analogous to the one-dimensional formulation of the inhomogeneous beam-plasma problem presented in Chapter II. The pendulum is the analogy of the beam-plasma system, and the externally

excited, high-frequency oscillations of the pivot are analogous to the small-amplitude, high-frequency modulation of the beam density profile.

If, on the other hand, the pivot is fed with a noise source, enabling it to make oscillations of random orientations, the high-frequency drive will no longer lead to stabilization. This is analogous to the two-dimensional formulation of the inhomogeneous beam-plasma problem in the absence of an externally imposed dc magnetic field. The argument of phase destruction, advanced by Fainberg⁶⁷ and used in Chapter II to explain the stabilizing effects in the one-dimensional case, no longer applies in this case. This is due to the fact that the absence of an imposed dc magnetic field makes it impossible for the previously formed high-frequency bunches of the beam to couple effectively to the longitudinal component of the space-charge wave. Thus there is no destruction of the phases between the beam and the plasma space-charge waves.

3.5 Finite Temperature Effects

In this section the effects of a finite temperature for plasma electrons are investigated. For analytical reasons, plasma electrons are assumed to be unidimensionally temperate in the sense that the temperature is assumed to be due to one-dimensional vibrations of the plasma electrons. The hydrodynamic model is again adopted for the beam as well as the plasma electrons, with due justifications and limitations as outlined in some detail in Chapter II.

3.5.1 Field and Moment Equations. On the basis of the above assumptions, equations of motion are written for the beam as well as the plasma electrons. Beam and plasma ions are assumed to be infinitely heavy and serve merely as a neutralizing background. In the inertial frame

of reference attached to the beam, equations of motion, continuity and Gauss' law are expressed as

$$\frac{\partial v_{1b}}{\partial t} = -\frac{e}{m} E_1, \quad (3.93)$$

$$\frac{\partial n_{1b}}{\partial t} + \nabla \cdot [n_{ob}(\xi) v_{1b}] = 0, \quad (3.94)$$

$$\frac{\partial v_{1p}}{\partial t} - u_o \cdot \nabla v_{1p} = -\frac{e}{m} E_1 - \frac{\partial}{\partial \xi} \left(\frac{p_{1e}}{n_{op} m} \right) \hat{\xi}, \quad (3.95)$$

$$\frac{\partial n_{1p}}{\partial t} - u_o \cdot \nabla n_{1p} + n_{op} \nabla \cdot v_{1p} = 0 \quad (3.96)$$

and

$$\nabla \cdot E_1 = -\frac{e}{\epsilon_o} (n_{1b} + n_{1p}), \quad (3.97)$$

where the differential operator, ∇ , is defined in the moving frame of reference attached to the beam and p_{1e} is the one-dimensional electronic pressure defined by Eq. 2.57.

A steady-state solution is sought by assuming an $e^{j\omega t}$ variation for all the first-order quantities. A combination of Eqs. 3.93 and 3.94 yields

$$n_{1b} = -\frac{e}{m\omega^2} \nabla \cdot [n_{ob}(\xi) E_1]. \quad (3.98)$$

Likewise, if Eqs. 3.95 and 3.96 are combined, the following is obtained:

$$\left[\left(j\omega - u_o \frac{\partial}{\partial \xi} \right)^2 - v_{th}^2 \frac{\partial^2}{\partial \xi^2} \right] n_{1p} = \frac{en_{op}}{m} \nabla \cdot \underline{E}_1, \quad (3.99)$$

where v_{th} is defined by Eq. 2.61. If both sides of Eq. 3.97 are operated on by the differential operator $[j\omega - u_o(\partial/\partial \xi)]^2 - v_{th}^2(\partial^2/\partial \xi^2)$, and the result is combined with Eqs. 3.98 and 3.99, a differential equation is obtained for the ac electric field, $\underline{E}_1$, which in turn can be written in terms of the scalar potential function as

$$\left[\left(j\omega - u_o \frac{\partial}{\partial \xi} \right)^2 - v_{th}^2 \frac{\partial^2}{\partial \xi^2} \right] \nabla \cdot [\epsilon_b(\omega, \xi) \nabla \Phi] + \omega_p^2 \nabla^2 \Phi = 0. \quad (3.100)$$

If the one-dimensional restriction on the thermal oscillations of the plasma electrons had not been imposed, the resulting differential equation would have appeared in the same form with the exception of $v_{th}^2 \nabla^2$ appearing instead of $v_{th}^2(\partial^2/\partial \xi^2)$ in Eq. 3.100.

3.5.2 Dispersion Relation. It can be shown, in a similar manner to that given in Section 3.3, that the scalar potential function, $\Phi(r, \theta, \xi)$, is separable. Arguments of uniqueness and finiteness of the solution can be used to obtain the following result for $\Phi(r, \theta, \xi)$:

$$\Phi(r, \theta, \xi) = A J_m(Kr) Z(\xi) e^{-jm\theta},$$

where $Z(\xi)$ satisfies the following differential equation:

$$\left[\left(j\omega - u_o \frac{d}{d\zeta} \right)^2 - v_{th}^2 \frac{d^2}{d\zeta^2} \right] \left[\frac{d}{d\zeta} \left(\epsilon_b \frac{dZ}{d\zeta} \right) - K^2 \epsilon_b Z \right] + \omega_p^2 \left(\frac{d^2}{d\zeta^2} - K^2 \right) Z = 0 \quad (3.101)$$

An approximate solution can be sought to the above differential equation by using the method of averaging under the general assumptions and limitations as outlined in Chapter II. The final form of the solution can be stated as

$$Z(\zeta) = \left[1 + \sum_{n \neq 0} \frac{f_n(\omega) e^{-jnk_o \zeta} \left[\left(j(n\omega_o + \omega) - u_o \frac{d}{d\zeta} \right)^2 - v_{th}^2 \left(\frac{d}{d\zeta} - jnk_o \right)^2 \right] \cdot \left(\frac{d^2}{d\zeta^2} - jnk_o \frac{d}{d\zeta} - K^2 \right)}{(n^2 k_o^2 + K^2) (\omega_p^2 - [(n\omega_o + \omega)^2 - n^2 k_o^2 v_{th}^2] f_o(\omega))} \right] Z_o(\zeta), \quad (3.102)$$

where $f_n(\omega)$ is defined by Eq. 3.18 and $Z_o(\zeta)$ is the solution to the averaged equation stated as

$$\left\{ f_o(\omega) \left[\left(j\omega - u_o \frac{d}{d\zeta} \right)^2 - v_{th}^2 \frac{d^2}{d\zeta^2} \right] + \omega_p^2 \right\} \left(\frac{d^2}{d\zeta^2} - K^2 \right) Z_o - K^2 \omega^2 \sum_{n \neq 0} \frac{\left[\left\{ \left(j(n\omega_o + \omega) - u_o \frac{d}{d\zeta} \right)^2 - v_{th}^2 \left(\frac{d}{d\zeta} - jnk_o \right)^2 \right\} \cdot \left(\frac{d^2}{d\zeta^2} - jnk_o \frac{d}{d\zeta} - K^2 \right) Z_o \right]}{[(n\omega_o + \omega)^2 - n^2 k_o^2 v_{th}^2] f_o(\omega) - \omega_p^2} (n^2 k_o^2 + K^2)} |f_n(\omega)|^2 = 0 \quad (3.103)$$

The equation of state in the moving frame of reference is obtained simply by solving the averaged equation, i.e., by setting $Z_0 \sim e^{-jk\xi}$. A doppler transformation, $(\omega \rightarrow \omega_D \equiv \omega - ku_0)$, results in the final form of the equation of state expressed in the laboratory frame of reference:

$$\begin{aligned} & \left(f_0(\omega_D)(\omega^2 - k^2 v_{th}^2) - \omega_p^2 \right) (K^2 + k^2) \\ & - K^2 \omega_D^2 \sum_{n \neq 0} \frac{n^2 \omega_0^2 (1 - v_{th}^2/u_0^2) (K^2 - nk_0 k + k^2)}{[n^2 \omega_0^2 (1 - v_{th}^2/u_0^2) f_0(\omega_D) - \omega_p^2] (n^2 k_0^2 + K^2)} |f_n(\omega_D)|^2 = 0, \end{aligned} \quad (3.104)$$

where terms of order $(\omega/\omega_0)^2$ and $(k/k_0)^2$ and higher were neglected. The above equation can be simplified further by taking note of the fact that

$$\left| \omega_0^2 \left(1 - \frac{v_{th}^2}{u_0^2} \right) f_0(\omega_D) \right| \gg \omega_p^2$$

and that in carrying out the summation over integers n the term $nk_0 k$ vanishes due to symmetry. The former conclusion follows from the fact that $\omega_0 \gg \omega_p$, $v_{th} < u_0$ and that near the interaction region ($\omega \approx ku_0$) the value of $|f_0(\omega_D)|$ is large compared to unity. The effects of finite transverse boundaries of the beam can also be taken into account by considering an effective value for the beam density equal to that of its actual value multiplied by an appropriate reduction factor, $G_m(b\omega/u_0, b/a)$, derived previously. The final result is stated as

$$\begin{aligned} & (k^2 + K^2) \left(f_{0,eff}(\omega_D)(\omega^2 - k^2 v_{th}^2) - \omega_p^2 \right) \\ & - 2 \frac{\omega_D^2}{f_{0,eff}(\omega_D)} \sum_{n=1}^{\infty} \frac{K^2}{n^2 k_0^2 + K^2} |f_{n,eff}(\omega_D)|^2 = 0, \end{aligned} \quad (3.105)$$

where K is to be eliminated between the above equation of state and the boundary requirement

$$\frac{J'_m(Kb)}{J_m(Kb)} = \frac{1 - \omega_p^2/(\omega^2 - k^2 v_{th}^2)}{1 - \omega_p^2/(\omega^2 - k^2 v_{th}^2) - \omega_b^2/\omega_D^2} F_m \left(kb, \frac{b}{a} \right) . \quad (3.106)$$

It is noted that the equation of state exhibits appropriate limiting behavior when the inhomogeneity vanishes. Quite similar to the cold-plasma case, Eq. 3.105 can be divided through by $k^2 + K^2$, since the condition $k^2 + K^2 = 0$ is satisfied only in the limit of vanishing inhomogeneity.

3.5.3 Rectangular Inhomogeneity. The equation of state, Eq. 3.105, for the case of a rectangularly inhomogeneous beam of duty cycle Δ is written as

$$\begin{aligned} & (\omega^2 - k^2 v_{th}^2) \left(1 - \Delta \frac{\omega_b^2}{\omega_D^2} G_m \right) - \omega_p^2 \\ & + \frac{2}{\pi^2} \frac{\omega_b^4/\omega_D^2}{1 - \Delta \omega_b^2 G_m/\omega_D^2} G_m^2 \sum_{n=1}^{\infty} \frac{\omega^2}{n^2 \omega_o^2 - \omega^2} \left(\frac{\sin n\pi\Delta}{n} \right)^2 = 0 \quad , \quad (3.107) \end{aligned}$$

where the weak beam approximation was employed in the same manner as before in eliminating K between Eqs. 3.105 and 3.106. In terms of the normalized variables

$$\begin{aligned} \Omega & \equiv \frac{\omega}{\omega_p} , \quad \Omega_{br} \equiv (\Delta G_m)^{1/2} \frac{\omega_b}{\omega_p} , \quad \Omega_o \equiv \frac{\omega_o}{\omega_p} , \quad r \equiv \frac{v_{th}^2}{u_o^2} , \\ x & \equiv \frac{\omega_D}{\omega_p} , \quad \eta \equiv \frac{2}{\pi^2} \frac{1}{\Delta^2} \sum_{n=1}^{\infty} \frac{\omega^2}{n^2 \omega_o^2 - \omega^2} \left(\frac{\sin n\pi\Delta}{n} \right)^2 \quad , \quad (3.108) \end{aligned}$$

Eq. 3.107 can be written as a sixth-degree polynomial in x as

$$\sum_{i=0}^6 A_i x^i = 0 , \quad (3.109)$$

where

$$A_0 = -(1 - r)\Omega_{br}^4 \Omega^2 ,$$

$$A_1 = -2r\Omega_{br}^4 \Omega ,$$

$$A_2 = -\Omega_{br}^2 [1 - (r - \eta)\Omega_{br}^2 - 2(1 - r)\Omega^2] ,$$

$$A_3 = 4r\Omega_{br}^2 \Omega ,$$

$$A_4 = 1 - 2r\Omega_{br}^2 - (1 - r)\Omega^2 ,$$

$$A_5 = -2r\Omega$$

and

$$A_6 = r . \quad (3.110)$$

Solving the above equation for x is a trivial matter when using the Scientific Subroutine Package, *SSP, available in the Fortran IV library. If x_0 is a root of Eq. 3.109, the normalized gain, $k_i u_0 / \omega_p$, is given by

$$k_i u_0 / \omega_p = -x_{0i} , \quad (3.111)$$

where x_{oi} denotes the imaginary part of x_o . The results indicate, however, that the gain variation is proportional to the one-half power of the duty cycle. Indeed the normalized growth rate, δ/δ_o , defined as the ratio of the growth rate of an inhomogeneous beam of duty cycle Δ to the growth rate of a homogeneous beam differs from $\Delta^{1/2}$ by an infinitesimal value not exceeding the error bound of the subroutine involved in computation of the roots of the polynomial. The rate of suppression, Γ , defined as $1 - (\delta/\delta_o)/\Delta^{1/2}$, is then small enough to be neglected altogether. It is thus concluded that a finite temperature for plasma electrons does not affect the interaction insofar as the suppression of the convective instabilities, due to beam inhomogeneity, is concerned.

3.5.4 Sinusoidal Inhomogeneity. The equation of state for the case of a sinusoidally inhomogeneous beam characterized by its depth of modulation, α , is given as

$$(\omega^2 - k^2 v_{th}^2) \left(1 - \frac{\omega_b^2}{\omega_D^2} G_m \right) - \omega_p^2 + \frac{\alpha^2}{2} \frac{\omega_b^4/\omega_D^2}{1 - \omega_b^2 G_m/\omega_D^2} \frac{\omega^2}{\omega_o^2 - \omega^2} G_m^2 = 0 \quad (3.112)$$

A polynomial, identical with Eq. 3.109, can be obtained using the same set of normalized parameters, defined by Eq. 3.108, with the exception of η defined as

$$\eta \equiv \frac{\alpha^2}{2} \frac{\omega^2}{\omega_o^2 - \omega^2} \quad (3.113)$$

A choice of $\Omega_o = 10$ leads to a value of η equal to $\alpha^2/200$ for frequencies near the plasma frequency. A look at the form of the A_i coefficients (Eq. 3.110) reveals that for such low values of η as $\alpha^2/200$ the roots will

differ from that of a homogeneous beam ($\eta = 0$) only by a negligible amount within the error bound of the computer program. Thus it is concluded that an inhomogeneous beam has no suppressive effects on the convectively unstable modes of a beam-plasma system in the absence of a dc magnetic field whether the beam is made to interact with a cold or a temperate plasma.

CHAPTER IV. TWO-DIMENSIONAL FORMULATION OF THE INHOMOGENEOUS BEAM-PLASMA INTERACTION: FINITE MAGNETIC FIELD CASE

4.1 Introduction

In this chapter the interaction of a longitudinally inhomogeneous electron beam with the surrounding plasma is considered in the presence of an axially directed, uniform, dc magnetic field of finite value. The study is encouraged by the results obtained in Chapters II and III indicating that the absence of a static magnetic field is detrimental to the suppression of convectively unstable, low-frequency waves due to a beam axial inhomogeneity in the case of an infinitely large magnetic field value. A study is made of the system in some intermediate region in which the intensity of the imposed dc magnetic field is neither large enough to warrant a one-dimensional analysis as considered in Chapter II, nor does it have a zero value as considered in Chapter III. As much as an infinitely large, static magnetic field simplifies the formulation, a dc magnetic field of finite value introduces complexities into the system by imposing a preferred motion for the charged particles along the magnetic field lines and thus indicates an anisotropic behavior for the system.

4.2 Mathematical Description of the System

4.2.1 Assumptions. The following assumptions are made in the course of formulating the problem:

1. The hydrodynamic model is adopted for both the beam and the plasma.
2. The plasma is cold, stationary and uniform with infinitely heavy ions serving as a neutralizing background.

3. The electron beam is cold, longitudinally inhomogeneous and drifting with an axial velocity, u_0 .

4. A static magnetic field of finite value, B_0 , is externally imposed in the direction along which the beam is drifting.

5. Perturbations are assumed to be of small magnitude so as to warrant a linear analysis.

6. The quasi-static approximation is valid.

7. Nonrelativistic mechanics applies.

The above assumptions, along with their physical limitations, are discussed in some detail in Chapters II and III.

4.2.2 Field and Moment Equations. The linearized equations of motion, along with the continuity equation and Gauss' law, are written for beam and plasma electrons in the inertial frame of reference attached to the beam:

$$\frac{\partial v_{1b}}{\partial t} = -\frac{e}{m} (\underline{E}_1 + \underline{v}_{1b} \times \underline{B}_0) , \quad (4.1)$$

$$\frac{\partial n_{1b}}{\partial t} + \nabla \cdot (n_{0b} \underline{v}_{1b}) = 0 , \quad (4.2)$$

$$\frac{\partial v_{1p}}{\partial t} - u_0 \cdot \nabla \underline{v}_{1p} = -\frac{e}{m} (\underline{E}_1 + \underline{v}_{1p} \times \underline{B}_0) , \quad (4.3)$$

$$\frac{\partial n_{1p}}{\partial t} + n_{0p} \nabla \cdot \underline{v}_{1p} - u_0 \cdot \nabla n_{1p} = 0 \quad (4.4)$$

and

$$\nabla \cdot \underline{E}_1 = -\frac{e}{\epsilon_0} (n_{1b} + n_{1p}) , \quad (4.5)$$

where the differential operator, ∇ , is defined in the moving frame of reference. A steady-state solution to the problem is sought by considering an explicit time dependence of all the first-order quantities in the form of $e^{j\omega t}$. Combination of Eqs. 4.1 and 4.2 then yields

$$n_{1b} = -\nabla \cdot \left(\frac{en_{ob}}{m\omega^2} \underline{\kappa} \cdot \underline{E}_1 \right), \quad (4.6)$$

where $\underline{\kappa}$ is a Hermitian tensor of rank two defined as

$$\underline{\kappa} \equiv \begin{bmatrix} \kappa_{\perp} & j\kappa_x & 0 \\ -j\kappa_x & \kappa_{\perp} & 0 \\ 0 & 0 & \kappa_{\parallel} \end{bmatrix}, \quad (4.7)$$

where

$$\kappa_{\perp} \equiv \frac{1}{1 - \omega_c^2/\omega^2}, \quad (4.8)$$

$$\kappa_x \equiv \frac{\omega_c}{\omega} \frac{1}{1 - \omega_c^2/\omega^2}, \quad (4.9)$$

$$\kappa_{\parallel} \equiv 1 \quad (4.10)$$

and ω_c is the electron cyclotron frequency defined as

$$\omega_c \equiv \frac{eB_0}{m}. \quad (4.11)$$

Likewise, combination of Eqs. 4.3 and 4.4 results in

$$\left(j\omega - u_0 \frac{\partial}{\partial \xi} \right)^2 \left[\left(j\omega - u_0 \frac{\partial}{\partial \xi} \right)^2 + \omega_c^2 \right] n_{1p} = \frac{en_{op}}{m} \nabla \cdot (\underline{D} \cdot \underline{E}_1), \quad (4.12)$$

where $\underline{\underline{D}}$ is a differential operator in the form of a skew-symmetric matrix defined as follows in terms of three differential operators-- $D_{\perp}$, D_x and $D_{\parallel}$:

$$\underline{\underline{D}} \equiv \begin{bmatrix} D_{\perp} & D_x & 0 \\ -D_x & D_{\perp} & 0 \\ 0 & 0 & D_{\parallel} \end{bmatrix}, \quad (4.13)$$

where

$$D_{\perp} \equiv \left(j\omega - u_o \frac{\partial}{\partial \xi} \right)^2, \quad (4.14)$$

$$D_x \equiv \omega_c \left(j\omega - u_o \frac{\partial}{\partial \xi} \right) \quad (4.15)$$

and

$$D_{\parallel} \equiv \left(j\omega - u_o \frac{\partial}{\partial \xi} \right)^2 + \omega_c^2. \quad (4.16)$$

If Eqs. 4.6 and 4.12 are combined with Gauss' law, Eq. 4.5, and use is made of the quasi-static assumption, the following differential equation is obtained for the scalar potential function, $\Phi(r, \theta, \xi)$:

$$D_{\perp} D_{\parallel} \nabla \cdot \left(\underline{\underline{\epsilon}}_b(\omega, \xi) \cdot \nabla \Phi \right) + \omega_p^2 \nabla \cdot \left(\underline{\underline{D}} \cdot \nabla \Phi \right) = 0, \quad (4.17)$$

where $\underline{\underline{\epsilon}}_b(\omega, \xi)$ is the dielectric tensor of the beam defined as

$$\underline{\underline{\epsilon}}_b(\omega, \xi) \equiv \underline{\underline{I}} - \frac{e^2 n_{ob}(\xi)}{m \epsilon_o \omega^2} \underline{\underline{\kappa}}, \quad (4.18)$$

where $\underline{\underline{I}}$ is the 3 x 3 unit matrix.

4.3 Construction of the Solution to the Equation for the Scalar Potential Function

Equation 4.17 is separable, i.e., the scalar potential function, $\Phi(r, \theta, \xi)$, can be expressed as

$$\Phi(r, \theta, \xi) = R(r)Z(\xi) e^{-jm\theta} , \quad (4.19)$$

where a cylindrical geometry is under consideration. Equation 4.17, expressed in terms of $R(r)$ and $Z(\xi)$, is as follows:

$$\begin{aligned} \left(D_{\perp} D_{\parallel} (\epsilon_x Z) + \omega_p^2 D_{\perp} Z \right) \left[\frac{1}{rR} \frac{d}{dr} \left(r \frac{dR}{dr} \right) - \frac{m^2}{r^2} \right] + D_{\perp} D_{\parallel} \frac{d}{d\xi} \left(\epsilon_{\parallel} \frac{dZ}{d\xi} \right) \\ + \omega_p^2 D_{\parallel} \frac{d^2 Z}{d\xi^2} = 0 , \quad (4.20) \end{aligned}$$

where ϵ_x and $\epsilon_{\parallel}$ are components of $\underline{\epsilon}_b(\omega, \xi)$ in the 11 (or 22) and 33 positions of the 3×3 tensor, respectively. Equation 4.20 can therefore be represented as two separate differential equations, coupled through a constant of separation. The results can be expressed as

$$R(r) = A J_m(Kr) \quad (4.21a)$$

and

$$D_{\perp} D_{\parallel} \left[\frac{d}{d\xi} \left(\epsilon_{\parallel}(\omega, \xi) \frac{dZ}{d\xi} \right) - K^2 \epsilon_x(\omega, \xi) Z \right] + \omega_p^2 \left(D_{\parallel} \frac{d^2}{d\xi^2} - K^2 D_{\perp} \right) Z = 0 , \quad (4.21b)$$

where the solution to $R(r)$ was obtained in compliance with the requirement that the potential function remain finite throughout the region, in particular on the axis, $r = 0$.

The values of $\epsilon_x(\omega, \xi)$ and $\epsilon_{||}(\omega, \xi)$ can be found using the definition of $\epsilon_{||}(\omega, \xi)$ as it appears in Eq. 4.18:

$$\epsilon_x(\omega, \xi) = 1 - \frac{e^2 n_{ob}(\xi)}{m \epsilon_0 (\omega^2 - \omega_c^2)} \quad (4.22)$$

and

$$\epsilon_{||}(\omega, \xi) = 1 - \frac{e^2 n_{ob}(\xi)}{m \epsilon_0 \omega^2} \quad (4.23)$$

For a periodic form of $n_{ob}(\xi)$, the right-hand side of Eq. 4.23 can be represented as a discrete summation over the orthogonal Fourier modes:

$$\epsilon_{||}(\omega, \xi) = \sum_n f_n(\omega) e^{-j n k_0 \xi}, \quad (4.24)$$

where

$$f_n(\omega) = \frac{k_0}{2\pi} \int_0^{2\pi/k_0} \epsilon_{||}(\omega, \xi) e^{j n k_0 \xi} d\xi \quad (n = 0, \pm 1, \dots) \quad (4.25)$$

It is evident, from comparison of Eqs. 4.22 and 4.23, that

$$\epsilon_x(\omega, \xi) = \sum_n f_n(\omega_1) e^{-j n k_0 \xi}, \quad (4.26)$$

where

$$\omega_1^2 \equiv \omega^2 - \omega_c^2 \quad (4.27)$$

An approximate solution to Eq. 4.21b is then sought using the method of averaging described in Chapter II. According to this method $Z(\xi)$ can be represented as follows:

$$Z(\xi) = Z_0(\xi) + \sum_{n \neq 0} z_n(\omega) e^{-jnk_0 \xi}, \quad (4.28)$$

where $Z_0(\xi)$ is the large-amplitude (comparatively speaking) slowly varying part of $Z(\xi)$, representing the averaged behavior, and the summation term represents small-amplitude, fast-varying components, indicative of deviations from the average behavior, for which the axial inhomogeneity of the beam density is responsible. Substituting Eqs. 4.24, 4.26 and 4.28 into the differential equation for $Z(\xi)$, Eq. 4.21b, and carrying out the complicated algebra involved result in the following:

$$\begin{aligned} & \sum_{p=0}^6 \left\{ \left[\sum_n j^p \left(A_n^{(p)} f_n(\omega) + B_n^{(p)} f_n(\omega_1) \right) e^{-jnk_0 \xi} \right] \right. \\ & \cdot \left(\frac{d^p Z_0}{d\xi^p} + \sum_{n \neq 0} (-jnk_0)^p z_n(\omega) e^{-jnk_0 \xi} \right) \left. \right\} + \omega_p^2 \omega_c^2 \frac{d^2 Z_0}{d\xi^2} + \omega_p^2 \left(j\omega - u_0 \frac{d}{d\xi} \right)^2 \\ & \cdot \left(\frac{d^2}{d\xi^2} - K^2 \right) Z_0 + \omega_p^2 \sum_{n \neq 0} \left(-n^2 k_0^2 \omega_c^2 + (K^2 - n^2 k_0^2) (\omega + n\omega_0)^2 \right) z_n(\omega) e^{-jnk_0 \xi} = 0, \end{aligned} \quad (4.29)$$

where $A_n^{(p)}$ and $B_n^{(p)}$, $p = 0, 1, \dots, 6$ and $n = 0, \pm 1, \pm 2, \dots$ are defined as

$$A_n^{(0)} = 0, \quad (4.30a)$$

$$B_n^{(0)} = K^2 \omega_s^2 (\omega_c^2 - \omega_s^2), \quad (4.30b)$$

$$A_n^{(1)} = nk_0 \omega_s^2 (\omega_c^2 - \omega_s^2), \quad (4.30c)$$

$$B_n^{(1)} = 2K^2 u_0 \omega_s (\omega_c^2 - 2\omega_s^2), \quad (4.30d)$$

$$A_n^{(2)} = \omega_s \left(\omega_s (\omega_c^2 - \omega_s^2) + 2n\omega_o (\omega_c^2 - 2\omega_s^2) \right) , \quad (4.30e)$$

$$B_n^{(2)} = K^2 u_o^2 (\omega_c^2 - 6\omega_s^2) , \quad (4.30f)$$

$$A_n^{(3)} = u_o \left(2\omega_s (\omega_c^2 - 2\omega_s^2) + n\omega_o (\omega_c^2 - 6\omega_s^2) \right) , \quad (4.30g)$$

$$B_n^{(3)} = -4K^2 u_o^2 \omega_s , \quad (4.30h)$$

$$A_n^{(4)} = u_o^2 (\omega_c^2 - 6\omega_s^2 - 4n\omega_o \omega_s) , \quad (4.30i)$$

$$B_n^{(4)} = -K^2 u_o^2 , \quad (4.30j)$$

$$A_n^{(5)} = -u_o^3 (4\omega_s + n\omega_o) , \quad (4.30k)$$

$$B_n^{(5)} = 0 , \quad (4.30l)$$

$$A_n^{(6)} = u_o^4 , \quad (4.30m)$$

$$B_n^{(6)} = 0 , \quad (4.30n)$$

and

where

$$\omega_s \equiv \omega + n\omega_o . \quad (4.31)$$

Tedious algebra is involved in obtaining the zeroth-order as well as the nth-order equations. However, since the procedures involved are the same as outlined in Chapters II and III, it will merely suffice to state the results.

Zeroth-Order Equation

$$\left[\omega_p^2 \left(K^2 \omega_M^2 + k^2 (\omega_M^2 - \omega_c^2) \right) - \omega_M^2 (\omega_M^2 - \omega_c^2) \left(k^2 f_o(\omega) + K^2 f_o(\omega_1) \right) \right] Z_o(\xi) \\ - K^2 \omega_M^2 (\omega^2 - \omega_M^2) \sum_{n \neq 0} f_n(\omega_1) z_{-n}(\omega) = 0 , \quad (4.32)$$

where

$$\omega_M \equiv \omega + ku_0 \quad . \quad (4.33)$$

Physically ω_M is the measured frequency, in the moving frame of reference, of a wave whose frequency when measured in the laboratory frame of reference is given by ω . Unlike the previous cases the substitution $Z_0(\xi) \propto \exp(-jk\xi)$ was made at an earlier stage in the analysis. This makes the equations algebraic at an earlier stage of the problem than in the previous cases.

Nth-Order Equation

$$\begin{aligned} & (\omega_M + n\omega_0)^2 \left((\omega_M + n\omega_0)^2 - \omega_c^2 \right) \left(k(k + nk_0)f_n(\omega) + K^2 f_n(\omega_1) \right) Z_0(\xi) \\ & + \omega_s^2 (\omega_s^2 - \omega_c^2) \sum_{m \neq 0} \left(mnk_0^2 f_{n-m}(\omega) + K^2 f_{n-m}(\omega_1) \right) z_m(\omega) \\ & - \omega_p^2 \left(n^2 k_0^2 (\omega_s^2 - \omega_c^2) + K^2 \omega_s^2 \right) z_n(\omega) = 0 \quad . \quad (4.34) \end{aligned}$$

Application of the weak turbulence approximation permits neglect of the term

$$\omega_s^2 (\omega_s^2 - \omega_c^2) \sum_{m \neq 0, n} \left(mnk_0^2 f_{n-m}(\omega) + K^2 f_{n-m}(\omega_1) \right) z_m(\omega)$$

in Eq. 4.34 and thus makes it possible for $z_n(\omega)$ to be directly derived from the above equation. Consequently an approximate solution, $Z(\xi)$, to Eq. 4.21b can be expressed as

$$Z(\xi) \cong C \left(1 - \sum_{n \neq 0} \tilde{z}_n(\omega) e^{-jnk_o \xi} \right) e^{-jk\xi}, \quad (4.35)$$

where C is a constant whose value is to be determined by the initial conditions and $\tilde{z}_n(\omega)$ is defined as

$$\tilde{z}_n(\omega) \equiv \frac{(\omega_M + n\omega_o)^2 \left((\omega_M + n\omega_o)^2 - \omega_c^2 \right) \left(k(k + nk_o) f_n(\omega) + K^2 f_n(\omega_1) \right)}{\omega_s^2 (\omega_s^2 - \omega_c^2) \left[n^2 k_o^2 \left(f_o(\omega) - \omega_p^2 / \omega_s^2 \right) + K^2 \left(f_o(\omega_1) - \omega_p^2 / (\omega_s^2 - \omega_c^2) \right) \right]}. \quad (4.36)$$

4.4 Dispersion Relation

The equation of state for the system is obtained simply through an elimination of $z_n(\omega)$ between Eqs. 4.32 and 4.34 with the weak turbulence approximation applied to Eq. 4.34. If terms of order ω/ω_o and higher are neglected, in compliance with seeking a solution of the form of Eq. 4.28, the following equation of state is obtained for the system:

$$k^2 \left(f_o(\omega_D) - \frac{\omega_p^2}{\omega^2} \right) + K^2 \left(f_o(\omega_{1D}) - \frac{\omega_p^2}{\omega^2 - \omega_c^2} \right) - 2 \frac{\omega_D^2 - \omega_c^2}{\omega^2 - \omega_c^2} \\ \cdot \sum_{n=1}^{\infty} \frac{K^2 \left(k^2 f_n^*(\omega_D) + K^2 f_n^*(\omega_{1D}) \right) f_n(\omega_{1D})}{n^2 k_o^2 \left(f_o(\omega_D) - \omega_p^2 / n^2 \omega_o^2 \right) + K^2 \left(f_o(\omega_{1D}) - \omega_p^2 / (n^2 \omega_o^2 - \omega_c^2) \right)} = 0, \quad (4.37)$$

where

$$\omega_{1D}^2 \equiv \omega_D^2 - \omega_c^2 ,$$

and a doppler transformation ($\omega \rightarrow \omega_D \equiv \omega - ku_0$) was made in going from the moving frame of reference to the laboratory frame of reference. Equation 4.37 exhibits the appropriate limiting behavior by reducing to Eq. 3.28 in the limit of $\omega_c \rightarrow 0$. It also exhibits appropriate limiting behavior in the limit of a vanishing inhomogeneity as shown in the discussion of the boundary requirements. In Eq. 4.37 the constant of separation, K, is found through the requirements at the boundary, as discussed in the following section.

4.4.1 Boundary Requirements. The beam-plasma system under consideration is composed of two distinct regions. The beam-plasma region, designated as Region I, has a radius b equal to that of the drifting electron beam. This region is characterized by a dielectric tensor $\underline{\epsilon}_I$ and a scalar potential function Φ_I specified as

$$\underline{\epsilon}_I = \begin{bmatrix} \epsilon_{1,I} & j\epsilon_{x,I} & 0 \\ -j\epsilon_{x,I} & \epsilon_{1,I} & 0 \\ 0 & 0 & \epsilon_{\parallel,I} \end{bmatrix} , \quad (4.38)$$

where

$$\begin{aligned} \epsilon_{1,I} &= 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} - \frac{\omega_b^2}{\omega_D^2 - \omega_c^2} , \\ \epsilon_{x,I} &= -\frac{\omega_c}{\omega} \frac{\omega_p^2}{\omega^2 - \omega_c^2} - \frac{\omega_c}{\omega_D} \frac{\omega_b^2}{\omega_D^2 - \omega_c^2} , \\ \epsilon_{\parallel,I} &= 1 - \frac{\omega_p^2}{\omega^2} - \frac{\omega_b^2}{\omega_D^2} \end{aligned} \quad (4.39)$$

and

$$\Phi_I(\underline{r}, t) = A J_m(Kr) \exp[j(\omega t - m\theta - kz)] ; \quad (4.40)$$

the effect of the beam axial inhomogeneity was ignored in writing the expressions for $\underline{\epsilon}_I$ and Φ_I . The effects of such inhomogeneities have already been expressed through k in connection with the equation of state, Eq. 4.37. The second region, referred to as Region II, is the plasma region. Geometrically, this region forms a hollow cylinder of outer and inner radii a and b , respectively. It is characterized by a dielectric tensor $\underline{\epsilon}_{II}$ and a scalar potential function Φ_{II} specified as

$$\underline{\epsilon}_{II} = \begin{bmatrix} \epsilon_{l,II} & j\epsilon_{x,II} & 0 \\ -j\epsilon_{x,II} & \epsilon_{l,II} & 0 \\ 0 & 0 & \epsilon_{||,II} \end{bmatrix} , \quad (4.41)$$

where

$$\begin{aligned} \epsilon_{l,II} &= 1 - \frac{\omega_p^2}{\omega^2 - \omega_c^2} , \\ \epsilon_{x,II} &= -\frac{\omega_c}{\omega} \frac{\omega_p^2}{\omega^2 - \omega_c^2} , \\ \epsilon_{||,II} &= 1 - \frac{\omega_p^2}{\omega^2} \end{aligned} \quad (4.42)$$

and

$$\Phi_{II}(\underline{r}, t) = B \left(N_m(T_2 a) J_m(T_2 r) - J_m(T_2 a) N_m(T_2 r) \right) \exp[j(\omega t - m\theta - kz)] , \quad (4.43)$$

Here it is seen that

$$T_2^2 \equiv -k^2 \epsilon_{\parallel, II} / \epsilon_{\perp, II} \quad (4.44)$$

and the requirement that the electric field be normal to the surface of the drift tube has already been accounted for. Continuity of the tangential component of the electric field and the normal component of the electric induction vector at the beam-plasma interface can be combined and expressed as

$$\left(\underline{\epsilon}_I \cdot \nabla \log \Phi_I \right) \cdot \hat{r} \Big|_{r=b} = \left(\underline{\epsilon}_{II} \cdot \nabla \log \Phi_{II} \right) \cdot \hat{r} \Big|_{r=b}, \quad (4.45)$$

where $\hat{r}$ denotes the unit vector in the radial direction. If substitution is made into Eq. 4.45 from Eqs. 4.38, 4.40, 4.41 and 4.43, the following is obtained:

$$\epsilon_{\perp, I}^{Ka} \frac{J'_m(Ka)}{J_m(Ka)} + m\epsilon_{\parallel, I} = \epsilon_{\perp, II} T_2^a \frac{N_m(T_2^a) J'_m(T_2^b) - J_m(T_2^a) N'_m(T_2^b)}{N_m(T_2^a) J_m(T_2^b) - J_m(T_2^a) N_m(T_2^b)} + m\epsilon_{\parallel, II} \cdot \quad (4.46)$$

Equation 4.46 is the desired boundary condition. The dispersion relation is obtained by elimination of K between the above equation and the equation of state, Eq. 4.37.

It remains to show that the combination of Eqs. 4.37 and 4.46 exhibits appropriate limiting behavior in the limit of a vanishing inhomogeneity. For a homogeneous beam the following is determined simply in reference to Eq. 4.24:

$$f_o(\omega) = 1 - \frac{\omega_p^2}{\omega^2} \quad (4.47)$$

and

$$f_n(\omega) = 0 \quad (n = \pm 1, \pm 2, \dots) \quad . \quad (4.48)$$

Substitution of the above two equations into the equation of state, Eq. 4.37, yields

$$K^2 = -k^2 \epsilon_{\parallel, I} / \epsilon_{\perp, I} \quad . \quad (4.49)$$

It is shown elsewhere,³⁵ however, that the combination of Eqs. 4.44, 4.46 and 4.49 represents the dispersion relation for a homogeneous beam-plasma system in the presence of an axially directed dc magnetic field of finite value, thus completing the proof.

4.4.2 Determination of the Constant of Separation. It was shown in Chapter III that in the absence of an externally applied magnetic field the boundary requirements could be incorporated in the equation of state. This is done by introducing the notion of a geometric reduction factor, thus making it possible for the dispersion relation of the inhomogeneous beam-plasma system to be represented as a single equation relating ω and k implicitly. For the case of an applied dc magnetic field of finite value, incorporation of the boundary requirements in the equation of state is not possible. However, the set of equations, Eqs. 4.37, 4.44 and 4.46, which represent the dispersion relation for the latter case can be reduced in the sense that the dispersion relation can be made to involve only two implicit variables instead of three. In this section a procedure is suggested to determine in the first approximation the constant of separation, K , explicitly in terms of other parameters, from the equation of state, Eq. 4.37. The resulting dispersion relation which

would involve the remaining two variables, ω and k , implicitly, is in a form to be solved numerically using the numerical techniques^{46,78} developed to solve implicit dispersion relations with the aid of digital computers.

The equation of state, Eq. 4.37, is composed of three terms. The last term was shown to vanish in the limit of a vanishing inhomogeneity. In the presence of inhomogeneity in the beam, the last term, although nonzero, is small compared to the other two terms. Indeed, the entire argument underlying the applicability of the method of averaging, which made it possible to express the solution, $Z(\xi)$, in the form of Eq. 4.28 will no longer hold if the last term in Eq. 4.37 assumes an amplitude comparable to those of the first two terms. In light of the above observation K can be determined to an arbitrary degree of accuracy using the method of successive approximation. It should be noted, however, that no virtue is gained by attempting to determine K from Eq. 4.37 with an error bound exceedingly smaller than ω/ω_0 . This is due to the fact that the very derivation of Eq. 4.37 involved ignoring the terms of order ω/ω_0 and higher.

From the above argument the zeroth term for K , in the language of the perturbation theory, is found simply by ignoring the last term in Eq. 4.37:

$$K_0^2 = -\eta k^2, \quad (4.50)$$

where

$$\eta \equiv \frac{f_0(\omega_D) - \omega_p^2/\omega^2}{f_0(\omega_{1D}) - \omega_p^2/(\omega^2 - \omega_c^2)}. \quad (4.51)$$

The first-order correction term is found by substituting K_0 , as expressed in Eq. 4.50, for K into the third term of Eq. 4.37 and then solving the

resulting equation for K. If only the first-order correction term for K is included by terminating the series at this point, the following expression is obtained for K:

$$K^2 \cong -\eta k^2(1 + \delta) , \quad (4.52)$$

where

$$\delta \equiv 2k^2 \frac{\omega_D^2 - \omega_c^2}{\omega^2 - \omega_c^2} \left(f_o(\omega_{1D}) - \frac{\omega_p^2}{\omega^2 - \omega_c^2} \right)^{-1} \\ \cdot \sum_{n=1}^{\infty} \frac{[f_n^*(\omega_D) - \eta f_n^*(\omega_{1D})] f_n(\omega_{1D})}{n^2 k_o^2 f_o(\omega_D) - \eta k^2 [f_o(\omega_{1D}) - \omega_p^2 / (\omega^2 - \omega_c^2)]} \quad (4.53)$$

The set of Eqs. 4.44, 4.46 and 4.52 represents the desired dispersion relation that relates ω and k implicitly. The roots of such dispersion relations can be traced on the complex- ω , complex- k hyperplane using various numerical techniques, one of which is described briefly in the next section.

4.4.3 Description of a Numerical Technique for Tracing the Roots of Dispersion Relations. Gillanders⁷⁸ has developed a numerical technique, along with a FORTRAN IV computer program, that allows tracing the root loci of dispersion relations in a four-dimensional complex- ω , complex- k space. This provides the user with a technique for exploring different kinds of instabilities that arise in a system and for determining their relative strengths, to any desired accuracy, provided that the dispersion relation is known for the system under consideration. The following is a brief description of the numerical method.

In the beginning, the numerical values of a set of control variables are initialized. These are the values of (a) those normalized parameters that remain constant throughout the program, (b) a parameter of smallness indicating the desired accuracy, (c) a step size and (d) the coordinates, (ω, k) , of an initial point corresponding to a root of the dispersion relation. In beam-plasma systems a pair, (ω, k) , corresponding to a root of the dispersion relation can be found simply by determining the asymptotic behavior of a mode of the beam or the plasma far from the interaction region. Three more control variables are initialized through a subroutine which determines the values of the dispersion function and its partial derivatives at the initial point. It is then determined along which axis the function is varying most rapidly, and the corresponding variable is then identified as the independent variable. Newton's method is used to correct the two dependent variables until the desired accuracy is obtained. The independent variable is then increased by the step size and the dependent variables are increased by a value equal to the product of the step size and their derivatives with respect to the independent variable. This yields the predicted coordinates of the second point and the procedure is reiterated. There are provisions in the program to halve the step size if, after an initially prescribed number of iterations of Newton's method, the desired accuracy is not obtained.

4.4.4 Expected Results and Conclusions. Although the above method can be used to trace the roots of the dispersion relation, Eq. 4.46, the approximations involved in the derivation of the dispersion relation would make such an effort fruitless in the light of the expected results as explained in the following.

It was shown in Chapter II that in a beam-plasma system a periodic inhomogeneity in beam density has the effect of stabilizing convective instabilities if the system is immersed in an infinitely large, static magnetic field. On the other hand, the results of Chapter III indicated that the presence of a magnetic field is essential to stabilization, in the sense that there will be no reduction in the value of the growth rate of convectively unstable waves due to beam inhomogeneity if the applied magnetic field is removed from the system. In numerical terms the rate of suppression is of the order of 10^{-1} for the one-dimensional case and zero for the two-dimensional, zero magnetic field case. The error bound in both cases is $(\omega_p/\omega_o)^2$. Having realized that the electron inertia sets an upper limit on the value of ω_o , the error bound cannot be made arbitrarily small while maintaining a reasonably dense plasma. Values as high as 3000 MHz for ω_o have been reported in some experiments.⁵³ For such values of ω_o and moderately dense plasmas (densities in the range of 10^8 to 10^9 cm⁻³) the numerical value of the error bound, $(\omega_p/\omega_o)^2$, equals 10^{-2} .

On the other hand, it is plausible to expect some measure of suppression for the finite magnetic field case. The value of static magnetic field is expected to have a minimum threshold below which suppressive effects are either negligible or nonexistent. Above the threshold, however, suppression of convective instabilities is expected with suppression rates smaller than 10^{-1} . If it is noted that the dispersion relation, Eq. 4.46, is valid within an error of ω/ω_o , it can easily be concluded that the expected suppression is of the same order of magnitude as the error bound in the more important parts of the

spectrum, $\omega \sim \omega_c$ and $\omega \sim \omega_p$. Consequently, the task of solving the obtained dispersion relation numerically under the said approximations would not lead to any conclusive results and therefore is not undertaken here. Instead a computer simulation of the problem is suggested to study this particular case.

CHAPTER V. CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER STUDY

5.1 Summary and Conclusions

This study is concerned with a theoretical investigation of the suppressive effects, due to the presence of a longitudinal inhomogeneity of the beam, on convectively unstable modes of a beam-plasma system. The longitudinal inhomogeneity of the beam is assumed to have a periodic nature whose spatial wavelength, when measured in the frame of reference attached to the beam, is much shorter than the wavelength of the oscillations under consideration. The formalism is based on a collisionless, hydrodynamic model for the beam as well as the plasma. Therefore micro-mechanisms, such as Landau and cyclotron damping, are not included. Further assumptions made in the course of formulating the problem have, in essence, involved a restriction on the total number of degrees of freedom of the system. This implies that the system is in its early stages of development moving toward a fully developed turbulent state.

For the one-dimensional model the theoretical developments have indicated an effective suppression of the convectively unstable modes. Two types of inhomogeneity, rectangular and sinusoidal, are particularly considered. For the case of a high-frequency, rectangular pulsation of the beam density the theory predicts a cancellation of convective instabilities below a threshold frequency. Furthermore, the spatial growth rate of the instability is diminished for the remaining part of the spectrum. A higher rate of suppression is obtained for lower-frequency modes. Similar results are obtained from a one-dimensional analysis of the

sinusoidal-type inhomogeneity with the exception that the spectral width of the unstable region for this case is independent of the depth of modulation.

The one-dimensional analysis is extended to include the effects of temperate plasma electrons. The results obtained are similar to those of the cold plasma case with the expected exception that the unstable region is extended, on the frequency scale, beyond the plasma frequency. For the case of a rectangular inhomogeneity, a narrowing of both the ω - and k -spectra is obtained. The narrowing of the ω -spectrum is distinct from that of the cold plasma case by its two sharp cutoffs, as opposed to one, at both ends of the spectrum. For the case of a sinusoidal inhomogeneity, the narrowing occurs only in the k -spectrum. These results are explained physically using the arguments of phase destruction advanced by Fainberg.⁶⁷

The results of the two-dimensional analysis have indicated that a longitudinally inhomogeneous beam, in the absence of an externally applied magnetic field, exhibits no stabilizing effects on the instabilities that grow naturally in a beam-plasma system. This is in contradistinction to the stabilizing effects that electron beams of the same type exhibit in a one-dimensional beam-plasma model which represents, in physical terms, a beam-plasma system with an externally applied static magnetic field of infinitely large value. A physical explanation is offered in connection with a physical analog representing the original system in a less intricate fashion.

Finally, for the case of an externally applied static magnetic field of finite value, the dispersion relation is obtained for the inhomogeneous beam-plasma system. No attempt is made to solve the dispersion relation numerically in light of the fact that the expected rate of suppression is of the same order of magnitude as the error bound involved in the derivation of the dispersion relation.

5.2 Recommendations for Further Study

In the present study one of the approximations used throughout the analysis, namely that the spatial wavelength of the inhomogeneity is much smaller than wavelengths of the oscillations considered, makes the scope of the study limited by setting an upper limit for both the ω - and the k -spectra of the oscillations under consideration. This is unfortunate, particularly if there is interest in considering the instabilities in connection with the parametric resonance, even though it is argued⁶² that such resonances are of extremely narrow width and that the inhomogeneities and collisions which occur in actual systems may cause these instabilities to be weakened or even disappear.

An examination of the parametric resonance, for which the wavelength is of the same order as that of inhomogeneity, is readily possible when the depth of modulation of the beam is small. If the depth of modulation is denoted by α , Eq. 2.12 can be written as

$$\frac{d^2 Z(\xi)}{d\xi^2} + \left(h_0(\omega) + \alpha h_1(\omega) \cos k_0 \xi + \alpha^2 h_2(\omega) \cos^2 k_0 \xi + \dots \right) Z(\xi) = 0 \quad (5.1)$$

This equation is correct to terms of the n th order in α , where n is the highest exponent of α which is included in Eq. 5.1. The values of $h_0(\omega)$, $h_1(\omega)$, ... are to be determined using Eq. 2.12. A solution, to a desired degree of accuracy, can be sought to Eq. 5.1 using the method of characteristic exponents of Floquet and Poincaré as shown in Kaplan.⁷⁰

Another area of interest is the stabilization of drift instabilities in an inhomogeneous plasma by means of an applied uniform high-frequency electric field. A broad class of drift instabilities along with the hydromagnetic instabilities has hindered a successful plasma confinement scheme.⁷⁹ Recently there has been a growing interest in theoretical and experimental studies of collisionless drift instabilities as well as the drift dissipative instabilities.⁵⁵⁻⁵⁹

However, an analytical study of the time evolution of an inhomogeneous plasma, driven by a high-frequency electric field, involves integration of differential equations which are more complicated than those encountered in the present work, even with the simplest models chosen to represent the plasma. More complexity is added to the problem if there is an interest in including inhomogeneities in the temperature, as well as the density, of the particles comprising the plasma. Indeed, any serious attempt to study the drift instabilities, collisionless or dissipative, in actual devices would defeat its purpose if it did not take into account the two types of inhomogeneities mentioned above.

In such complicated situations as described it is encouraging to study the system under consideration using a many-particle simulation technique, which traces out the simultaneous motion of many particles or

their representatives using a high-speed computer with a large memory. The dynamical formulation of the system involves no approximation in that from a set of initial conditions for the velocities and positions of the particles the fields are computed using the fundamental laws of classical electromagnetic theory. The force equation can then be written using the derived fields. Two successive integrations over an infinitesimally small time increment result in new velocities and positions. The time evolution of the system is then obtained by repeating the cycle as many times as desired. The approximations come in when an attempt is made to make the study economically feasible. The limitations on the total computer time spent for each run and the difficulties involving large memory allocation over an appreciable amount of time force the exploitation of artificial models to represent the actual systems.

In principle as many models can be thought of as there are experimental situations. Particle models as well as sheet or ring models can be employed to represent the system. All models have one characteristic in common, namely that each sample particle, each sheet or each ring represents many particles of the actual system. The validity of these models hinges upon the fact that the collective motion of the plasma particles is of interest rather than having extremely intricate and detailed information on how each particle behaves;⁸⁰ this notion is inherent in methods of statistical physics and is essential for its development. In the following, a computer experiment for a study of drift instabilities is proposed, along with a brief description of the model chosen to represent the plasma.

For a system of cylindrical geometry the choice of a ring model is a natural one. This choice allows a two-dimensional representation

involving only the radial and the axial variables of the actual three-dimensional system. The rings are initially spaced in a manner so as to yield a constant ring density equal to three rings per Debye circle throughout the cross section. In the axial direction the initial state is such that each Debye length contains three sets of rings. Therefore the density, n , in terms of rings/cm³, is given by

$$n = \frac{9}{\pi} \lambda_D^{-3} . \quad (5.2)$$

Formally there are $N_D = 12$ rings in the Debye sphere; this assures that the model can appropriately represent a plasma. The system is assumed to be finite in size with a total length, $L = 16 \lambda_D$, and a diameter, $D = 8 \lambda_D$. The total number of rings in the entire system is given as

$$N = \frac{\pi}{4} D^2 L n = 2304 . \quad (5.3)$$

Systems which are longer in the axial extent can be included by application of periodic boundary conditions at $z = 0$ and $z = 16 \lambda_D$ to the proposed system.

An initially prescribed particle inhomogeneity in the transverse direction can be accounted for in either of the following ways. A constant number of rings can be maintained in the Debye sphere throughout the system as described earlier, but the total charge of each ring can be made to depend on the average radius of the ring in a monotonically decreasing manner. In other words the total normalized charge on the j th ring, Q_j , is given by

$$Q_j = \pi (R_{oj}^2 - R_{ij}^2) , \quad (5.4)$$

where R_{oj} and R_{ij} are the normalized outer and inner radii of the j th ring, respectively, and Q_j assumes a functional dependence on R_j , where

$$R_j = \frac{1}{2} (R_{oj} + R_{ij}) . \quad (5.5)$$

This prescription has the disadvantage of causing a congestion of outer rings by leaving a very small radial separation between them and thus increasing the numerical error involved in the computation of the potential.

The particle inhomogeneity can alternatively be represented by maintaining a constant radial distance between the two neighboring rings, as well as a constant total charge on each ring. This automatically assures that the density is a decreasing function of radius. The above-mentioned model appears much simpler than the former with the apparent disadvantage that it is apt to be doubted as a valid model to represent the plasma before the sheath region formed at the boundaries. This is due to the fact that in this case the average number of rings in the Debye sphere decreases as shown by the following:

$$N'_D = \frac{N_D}{2n + 1} , \quad (5.6)$$

where N_D is the number of rings in the Debye sphere based on the value of the density between $R_o = 0$ and $R_1 = \lambda_D$, and N'_D is the number of rings in the Debye sphere based on the value of the density between $R_n = n\lambda_D$ and $R_{n+1} = (n + 1)\lambda_D$. If the latter model is chosen, provisions must be made to avoid unacceptable values for N'_D near the sheath.

In the temperate case the particles are represented by rings moving randomly about their average position, R_j . A temperature gradient can be represented by making the randomness of the ring motion dependent upon the average position of the ring.

With the choice of a two-dimensional ring model, composed of N rings, the number of memory locations on the magnetic storage core required to store the instantaneous positions and velocities of each and every ring is given by $4N$. For such typical values of N as discussed previously the memory requirement is well within the capacity of the storage devices.

The total computer time spent for each run depends significantly on a host of factors, the most important of which is the total number of rings. The usual methods of solving Poisson's equation numerically⁸¹ involve a check on the values of the potential by calculating the total energy of the system at each cycle. If this procedure is adopted and a first-order correction term is sought to the value of the potential at every time step, the total computer time will have an N^2 dependence. Care must be employed in the numerical differentiation of the potential function. This step usually introduces gross numerical errors into the problem if a suitable differentiation algorithm is not chosen.

The total time, T , over which the evolution of the system is observed is usually expressed in units of the plasma period, ω_p^{-1} . It is evident that the total computer time spent is in direct relationship with $T/\Delta t$, where Δt is the time increment of each cycle. However, to decrease the total computer time spent for a given T , Δt cannot be increased at will due to other limiting factors involved. In addition to the increased

numerical error introduced due to the choice of a larger time increment, all those processes of the system whose frequency exceeds $(\Delta t)^{-1}$ are being ignored, or in effect averaged over. For the system considered above, if interest is in observing the phenomenon over a spectral band which is to include the second-harmonic effects, a choice of $\Delta t = 0.4 \omega_0^{-1}$ is sufficient where ω_0 is the frequency of the driving RF electric field.

APPENDIX A. THE DERIVATION OF THE EQUATION OF STATE

In accordance with the definition of $\varphi_n(\omega)$, expressed by Eq. 2.19, it is seen that

$$\varphi_n(\omega) = \frac{k_o}{2\pi} \int_0^{2\pi/k_o} \frac{e^{jnk_o\xi}}{\epsilon_b(\omega, \xi)} d\xi \quad . \quad (A.1)$$

For a rectangular beam density profile, with density variations along the beam specified as in Eq. 2.31, the following is simply arrived at:

$$\varphi_n(\omega) = \frac{1}{L} \left[\int_0^L \left(1 - \frac{\omega_b^2}{\omega^2}\right)^{-1} e^{jnk_o\xi} d\xi + \int_L^L e^{jnk_o\xi} d\xi \right] \quad (A.2)$$

or

$$\varphi_n(\omega) = j \frac{\omega_b^2/\omega^2}{1 - \omega_b^2/\omega^2} \frac{1 - e^{jnk_o L}}{2\pi n} \quad (n = \pm 1, \pm 2, \dots) \quad , \quad (A.3)$$

$$\varphi_o(\omega) = 1 + \frac{L}{1 - \omega_b^2/\omega^2} \quad . \quad (A.4)$$

Substitution of Eqs. A.3 and A.4 into Eq. 2.30 and neglect of terms of order $(\omega_p/\omega_o)^4$ and higher yield Eq. 2.32.

APPENDIX B. THE DEVELOPMENT OF A THREE-TERM RECURRENCE FORMULA

In the derivation of the dispersion relation, Eq. 2.42, the Fourier components of $1/\epsilon_b(\omega, \zeta)$ are to be evaluated for a sinusoidal beam density profile given by Eq. 2.41. The Fourier components for this case are given by

$$\varphi_n(\omega) = \frac{k_o}{2\pi} \int_{-\pi/k_o}^{\pi/k_o} \left(1 - \frac{\omega_b^2}{\omega^2} - \alpha \frac{\omega_b^2}{\omega^2} \cos k_o \zeta \right)^{-1} e^{jnk_o \zeta} d\zeta$$

(n = 0, ±1, ±2, ...) . (B.1)

The right-hand side of the above equation is composed of two parts: a real part which involves an integral whose integrand is an even function of ζ , and an imaginary part which involves an integral whose integrand is an odd function of ζ . The imaginary part vanishes due to the symmetry of the interval of integration. Therefore $\varphi_n(\omega)$ can be expressed as

$$\varphi_n(\omega) = \frac{1}{\pi} \int_0^\pi \frac{\cos nx \, dx}{A + B \cos x} \quad (n = 0, \pm 1, \pm 2, \dots) , \quad (B.2)$$

where a change of the dummy variable is made and A and B are defined as

$$A \equiv 1 - \frac{\omega_b^2}{\omega^2}$$

and

$$B \equiv -\alpha \frac{\omega_b^2}{\omega^2} .$$

Now a three-term recurrence formula for I_n is developed where

$$I_n = \int_0^\pi \frac{\cos nx \, dx}{A + B \cos x} \quad (B.3)$$

First, rewrite $n = n - 1 + 1$ and expand the numerator

$$I_n = \int_0^\pi \frac{\cos(n-1)x \cdot \cos x}{A + B \cos x} dx - \int_0^\pi \frac{\sin(n-1)x \cdot \sin x}{A + B \cos x} dx \quad (B.4)$$

The first term in Eq. B.4 can be rewritten as

$$\frac{1}{B} \int_0^\pi \left(1 - \frac{A}{A + B \cos x} \right) \cos(n-1)x \, dx = -\frac{A}{B} I_{n-1} \quad (B.5)$$

The integrand of the second term on the right-hand side of Eq. B.4 can be decomposed to yield the following result:

$$\int_0^\pi \frac{\sin(n-1)x \cdot \sin x}{A + B \cos x} dx = -\frac{1}{2} I_n + \frac{1}{2} I_{n-2} \quad (B.6)$$

Combination of Eqs. B.4 through B.6 results in

$$I_{n+2} = -\frac{2A}{B} I_{n+1} - I_n \quad (n = 0, 1, 2, \dots) \quad (B.7)$$

The above equation is the desired recurrence relation that allows evaluation of the I_n 's for $n \geq 2$ once I_0 and I_1 are found. Note that by definition of I_n the following is valid:

$$I_{-n} = I_n \quad (B.8)$$

The evaluation of I_1 in terms of I_0 is trivial; in fact, it can be seen that

$$I_1 = \int_0^\pi \frac{\cos x}{A + B \cos x} dx = \frac{\pi - AI_0}{B}, \quad (\text{B.9})$$

where

$$I_0 = \int_0^\pi \frac{dx}{A + B \cos x}. \quad (\text{B.10})$$

The above integration can be carried out by expressing $\cos x$ in terms of $\tan (x/2)$ and changing the variable to $u = \tan (x/2)$ as shown by

$$\begin{aligned} \int \frac{dx}{A + B \cos x} &= \int \frac{\left(1 + \tan^2 \frac{x}{2}\right) dx}{A + B + (A - B) \tan^2 \frac{x}{2}} \\ &= 2 \int \frac{du}{A + B + (A - B)u^2}. \end{aligned}$$

Now set $u = [(A + B)/(A - B)]^{1/2} \tan \alpha$ to show that

$$\begin{aligned} \int \frac{dx}{A + B \cos x} &= \frac{2}{A - B} \int \frac{\left(\frac{A + B}{A - B}\right)^{1/2} (1 + \tan^2 \alpha)}{\left(\frac{A + B}{A - B}\right) (1 + \tan^2 \alpha)} d\alpha \\ &= 2(A^2 - B^2)^{-1/2} \alpha \end{aligned}$$

or

$$\int \frac{dx}{A + B \cos x} = \frac{2}{\sqrt{A^2 - B^2}} \tan^{-1} \left[\left(\frac{A - B}{A + B}\right)^{1/2} \tan \frac{x}{2} \right];$$

therefore, for $|A| > |B|$

$$I_0 = \pi(A^2 - B^2)^{-1/2} \operatorname{sgn}(A). \quad (\text{B.11})$$

APPENDIX C. LIMITING VALUES OF THE GEOMETRIC REDUCTION FACTOR

The asymptotic behavior of the geometric reduction factor, $G_m(x, b/a)$, can be explored using the following expansions for the modified Bessel functions of the argument:

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}} \left[1 - \frac{4\nu^2 - 1}{8z} + o\left(\frac{1}{z^2}\right) \right], \quad (C.1)$$

$$I'_\nu(z) = \frac{e^z}{\sqrt{2\pi z}} \left[1 - \frac{4\nu^2 + 3}{8z} + o\left(\frac{1}{z^2}\right) \right] \quad (C.2)$$

and

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left[1 + \frac{4\nu^2 - 1}{8z} + o\left(\frac{1}{z^2}\right) \right]. \quad (C.3)$$

Substitution of the above formulas in Eq. 3.30 yields

$$\lim_{x \rightarrow \infty} G_m(x, 0) = \lim_{x \rightarrow \infty} x I'_m(x) K_m(x) = \frac{1}{2}. \quad (C.4)$$

Likewise, it can be seen that

$$\lim_{k \rightarrow \infty} \frac{K_m(ka) I_m(kb)}{I_m(ka) K_m(kb)} = e^{2k(b-a)} \rightarrow 0 \text{ if } b < a \text{ and } k_r > 0. \quad (C.5)$$

Therefore, according to Eq. 3.31, it can be stated that

$$\lim_{kb \rightarrow \infty} G_m(kb, b/a) = \frac{1}{2}; \quad m = 0, 1, \dots, 0 \leq b/a < 1. \quad (C.6)$$

For the small values of the argument, however, $G_m(kb, b/a)$ approaches a value which depends upon the ratio b/a unless $m = 0$. To explore the small-argument behavior, the ascending power series is used for the modified Bessel functions:

$$I_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu + 1)} [1 + o(z^2)] , \quad (C.7)$$

$$K_\nu(z) = \frac{1}{2} \Gamma(\nu) \left(\frac{z}{2} \right)^{-\nu} [1 + o(z^2)] \quad (C.8)$$

and

$$K_0(z) = -\log_e \left(\frac{z}{2} \right) [1 + o(z^2)] , \quad (C.9)$$

where $\Gamma(\rho)$ denotes the gamma function of argument ρ . Substitution of the above formulas in Eq. 3.30 yields

$$\lim_{x \rightarrow 0} G_m(x, 0) = \lim_{x \rightarrow 0} x I'_m(x) K_m(x) = \begin{cases} 0 & ; \quad m = 0 \\ \frac{1}{2} & ; \quad m = 1, 2, \dots \end{cases} \quad (C.10)$$

Likewise, it can be seen that

$$\lim_{k \rightarrow 0} \frac{K_m(ka) I_m(kb)}{I_m(ka) K_m(kb)} = \left(\frac{b}{a} \right)^{2m} . \quad (C.11)$$

Therefore, in accordance with Eq. 3.31, it can be stated that

$$\lim_{kb \rightarrow 0} G_m \left(kb, \frac{b}{a} \right) = \begin{cases} 0 & ; \quad m = 0 , \\ \frac{1}{2} \left[1 - \left(\frac{b}{a} \right)^{2m} \right] & ; \quad m = 1, 2, \dots, \end{cases} \quad 0 \leq \frac{b}{a} < 1 . \quad (C.12)$$

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