Modular Degree of Elliptic Curves Over Function Fields in Relation to Jacquet-Langlands

by

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Dedication

To everyone who was there for me

For nothing was simply one thing

Virginia Woolf, *To the Lighthouse*

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TABLE OF CONTENTS

Dedication . . . . . **ii**

Acknowledgments . . . . . **iii**

List of Figures . . . . . **vii**

Abstract . . . . . **viii**

Chapter

I Introduction **1**

1.1 Background from the number field case 1
1.2 Analogies between number fields and function fields 2
1.3 Definitions 4
1.4 Bad reduction of Moduli of Drinfeld-Stuhler modules 8
1.5 Level Lowering 10
1.6 Open Questions 10
1.7 Structure of the thesis 11

II Preliminaries **13**

2.1 Table of Notations 13
2.2 Analogies between number fields and function fields 14
2.3 Setting and Function Field Background 15
2.4 Inner Forms of GL_2 15
2.5 Quaternion algebras 16

III Automorphic Forms and Drinfeld Modular Forms **21**

3.1 Drinfeld Upper Half-Plane and its relation to the Bruhat-Tits building 22
3.2 Harmonic cochains 27
3.3 Relating harmonic cochains to automorphic representations 32
3.4 Making harmonic cochains explicit for computations 33
3.5 Drinfeld modular forms 35
3.6 Theta series 40

IV Geometric Side **41**

4.1 A-motives 41

4.2 Drinfeld modules . . . 45
4.3 Drinfeld-Stuhler modules . . . 48
4.4 The motivation behind shtukas . . . 50
4.5 $\mathcal{D}$-elliptic sheaves . . . 51
4.6 Taguchi duality . . . 54
4.7 Level structures and Moduli space . . . 56
4.8 Various modularity theorems . . . 58

V Uniformization and Bad Reduction . . . 62

5.1 Setting for Chapter V . . . 63
5.2 Divisible local Anderson modules . . . 63
5.3 $A_{\mathfrak{p}}$-Dieudonné modules . . . 64
5.4 $\mathbb{F}_{\mathfrak{p}}$-Dieudonné modules . . . 66
5.5 $\mathcal{D}$-Dieudonné modules . . . 67
5.6 Exceptional and superspecial objects . . . 70
5.7 Drinfeld level structures . . . 74
5.7.1 Drinfeld level structures of Drinfeld modules . . . 75
5.7.2 Drinfeld-level structure of Dieudonné module . . . 76
5.7.3 Drinfeld-level structure of $\mathcal{D}$-elliptic sheaf . . . 77
5.8 Integral model . . . 78
5.9 Analogue of Deligne-Rapoport . . . 81
5.10 Analogue of Cerednik-Drinfeld . . . 85
5.11 Uniformization at ∞ . . . 86

VI Ribet Short Exact Sequence and Level Lowering . . . 90

6.1 Component groups and Character groups . . . 90
6.2 Atkin-Lehner involution on character group . . . 96
6.3 Analogue of Ribet's Short Exact sequence . . . 97
6.4 Eisenstein Property of Component group . . . 108
6.5 Background on Hecke algebra . . . 111
6.5.1 Hecke algebra . . . 111
6.5.2 Galois representation associated to maximal ideals . . . 113
6.6 Level Lowering . . . 115
6.6.1 Mazur's level lowering at a good prime . . . 116
6.6.2 Prime Swapping in the level . . . 118
6.6.3 Proof of Level Lowering . . . 123
6.6.4 Level Lowering and Component groups . . . 124
6.6.5 Level Lowering and Fermat's Last Theorem . . . 124
6.7 Analogue of Ribet-Takahashi . . . 126
6.8 Ratio of Petersson norms in weight 2 . . . 132
6.8.1 Analytic description of the modular degree . . . 133
6.8.2 Ratio of Peterson Norms . . . 136
6.9 An extended example . . . 137

VII List of Open Questions 142
7.1 Ihara's Lemma 142
7.1.1 Theta groups Background 142
7.1.2 Ihara's Lemma and Level-Raising 144
7.2 Oort-splitting of Drinfeld-Stuhler modules 150
7.3 Multiple Legs and Higher Modularity 155
7.4 Partitions may know irreducibles 159
7.5 Lifting Drinfeld modular forms to mixed characteristic 161
Appendices 163
Bibliography 166

LIST OF FIGURES

FIGURE

III.1 Bruhat-Tits tree for $q = 2$ (image taken [KP23]) 23
III.2 Comparison of the Drinfeld upper half plane and Bruhat-Tits tree (image taken from [Mar]) 26

IV.1 Table of analogies between Drinfeld modules of various ranks and elliptic curves (image taken from [Poo]) 47

V.1 Drawing of the reduction of $X_0(\mathfrak{p})$ [Gek86] 81

VI.1 Dual Graph of $J^{\mathfrak{pq}}$ [PW15] 141

VII.1 Sage-code to test partition over function fields 159

ABSTRACT

In this thesis, we study the geometry of automorphic forms in the function field setting. The primary goal of this thesis prove a formula relating degrees of modular parametrization of an elliptic curve by different Drinfeld modular curves. This is analogous to the similar result of Ribet-Takahashi [RT97] in the number field setting, which relates degrees of modular parameterizations of an elliptic curve over $\mathbf{Q}$ by varying Shimura curves. To prove this result, I prove a result analogous to Ribet's short exact sequence [Rib90a] which relates the special fibers of Shimura varieties at different primes. Using this technical result I deduce (in the function field case) level-lowering results akin to those of Ribet [Rib90a] and relations between Petersson inner products of modular forms that are related by the Jacquet-Langlands correspondence similar to the work of K. Prasanna [Pra03].

CHAPTER I

Introduction

1.1 Background from the number field case

The modularity theorem of Wiles [Wil95] and Taylor-Wiles [TW95] asserts that for any elliptic curve E over $\mathbb{Q}$ with conductor N, there exists a cuspidal holomorphic modular form f of weight 2 and level $\Gamma_0(N)$ such that

$$L(E, s) = L(f, s).$$

As a consequence of Faltings' isogeny theorem [Fal83], there is an equivalent geometric formulation of modularity, namely that there exists a (non-constant) morphism

$$X_0(N) \to E.$$

The modular form f (or rather the differential form attached to it) can be recovered as the pullback of a non-zero holomorphic differential on E. Moreover, the curve $X_0(N)$ is a (coarse) moduli space for elliptic curves with $\Gamma_0(N)$-level structure.

Let D be an indefinite quaternion algebra over $\mathbb{Q}$ with discriminant $\mathrm{disc}(D)$ dividing N. The Jacquet-Langlands correspondence implies that the form f transfers to an automorphic form on the group $D^\times$. To geometrize the study of $D^\times$-automorphic forms, one generalizes the notion of $X_0(N)$: the analogous objects are called Shimura curves, and they can also be defined as suitable moduli spaces, as we now recall.

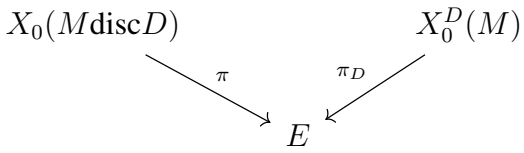
Fix a maximal order $\mathcal{O}$ in D. A D-abelian surface is a pair (A, ι) consisting of an abelian surface A with a ring morphism

$$\iota : \mathcal{O} \to \mathrm{End}(A).$$

The choice of a maximal order is turns out to be auxiliary and is therefore suppressed from the notation. These D-abelian surfaces behave very similarly to elliptic curves and are often called

"*fake elliptic curves*" (see for instance [Buz97]). For an integer M such that $M \cdot \mathrm{disc}(D) = N$, one then gets a curve $X_0^D(M)$, which is a (coarse) moduli space of D-abelian surfaces with $\Gamma_0(M)$-level structure. Note that the category of Mat_2-abelian surfaces is equivalent to the category of elliptic curves, so in the special case $D = \mathrm{Mat}_2(\mathbb{Q})$, this construction (essentially) recovers the usual modular curves $X_0(N)$.

Now, the $D^\times$-automorphic form associated with f may be viewed as a differential form on the curve $X_0^D(M)$. Moreover, the elliptic curve E admits a parameterization by $X_0^D(M)$, so we get a diagram as below:



The relation between the degrees of π and π_D was studied by Ribet-Takahashi in [RT97], where they prove that an explicit formula relating the degrees. If $\sim^S$ denotes equality up to Eisenstein primes of E (i.e., primes ℓ such that the ℓ-torsion $E[\ell]$ is not irreducible as a Galois module) and 2, then in particular the particular case that $\mathrm{disc}D = pq$, we have

$$\deg(\pi_D) \sim^S \frac{\deg(\pi)}{c_p c_q}$$

where and c_p and c_q denote the order of the component groups of E at p and q respectively. One of the main goals of this thesis is to generalize this result to the *function field* case.

1.2 Analogies between number fields and function fields

The ring of integers $\mathbb{Z}$ possess several similarities with $\mathbb{F}_p[t]$. For instance both of these rings have infinitely many primes, have unique prime factorization and the residue fields of their primes are finite fields. These somewhat superficial similarities lead to the hope that they share more subtle similarities. Some of the earliest work in this directions arose in the work of Dedekind, who proved quadratic reciprocity for $\mathbb{F}_p[t]$. This parallel was further pursued in the work of Hasse, Artin and others, extending the analogy to finite extensions, i.e., showing that finite extensions of $\mathbb{Q}$ behave analogously to finite extensions of $\mathbb{F}_p(t)$. Structurally there are many similarities between the behavior of various invariants of these fields, for instance in the context of Artin reciprocity, Class number formulas etc. A very important parallel is *Ostrowski's theorem*: for $\mathbb{Q}$, it says that any non-trivial absolute value arises either from the valuation associated with a prime ideal of the integers or the usual absolute value. For $\mathbb{F}_p(t)$ Ostrowski's theorem implies

that all non-trivial absolute values arise from prime ideals of its ring of integers or the valuation associated to $\frac{1}{t}$. In particular we can think of $\infty \in \mathbb{P}^1_{\mathbb{F}_p}$ as the function field analogue of the euclidean norm (despite **not** being archimedean). Maybe most famously, Andre Weil gave a proof the Riemann hypothesis for function fields [Wei41], which sparked the interest in what we now call the Weil conjectures.
Most relevant to this thesis is the function field analogue of the Langlands program. This was started by the work of Drinfeld. At the time the Modularity conjecture had been formulated, however was not yet proven. The moduli space of elliptic curves is **not** the right space to consider, that is we should not expect there to exist a non-trivial morphism

$$\{\text{moduli of elliptic curves over } \mathbb{F}_p[t]\} \to E.$$

(we save the notation $X_0(\mathfrak{n})$ for the "right" object). This already manifests itself in the role that elliptic curves play in class field theory. Consider, for example, the Kronecker-Weber theorem:

Theorem 1.2.1 *The maximal abelian extension of $\mathbb{Q}$ is generated by the roots of unity.*

A geometric interpretation of this is that the roots of unity are the torsion points of $\mathbb{G}_m$, and thus in particular

$$\mathbb{Q}^{\mathrm{ab}} \cong \bigcup_n \mathbb{Q}(\mathbb{G}_m[n]).$$

However this fails for function fields: adjoining the roots of unity only yields constant, and thus unramified, extensions. Therefore we would like a "geometric" object generalizing $\mathbb{G}_m$[1]. Another reason to expect that elliptic curves are not enough is to recall Kronecker's Judgendtraum for number fields:

Theorem 1.2.2 *Let $K/\mathbb{Q}$ be an imaginary quadratic extension. If E is an elliptic curve with complex multiplication by $\mathcal{O}_K$, then the Hilbert class field of K can be obtained by adjoining the j-invariant of E, that is we have*

$$K^{Hilb} \cong K(j(E)).$$

We have a natural analogue of an "imaginary" extension over function fields determined by its splitting behaviour at infinity (see Def 2.3.4). However no elliptic curve over $\mathbb{F}_p(t)$ can have complex multiplication by the integers of a function field! Hence we are looking for a different object in the function-field case which mimics the behaviour of elliptic curves.
Based on the work of Carlitz [Car38], Hayes [Hay74] and Drinfeld [Dri74] defined objects which

[1] we note that we implicitly think of $\mathbb{G}_m$ as a "degenerate" elliptic curve

we now call *Drinfeld modules*. These behave philosophically like elliptic curves in many ways and their endomorphism algebras are of positive characteristic. Later Drinfeld introduced objects called *shtukas*, which behave like the associated motives. Drinfeld used this notion to prove Langlands in the GL_2-setting by studying the moduli space of GL_2-shtukas, which was extended by L. Lafforgue [Laf98] to prove Langlands for GL_n. This was then generalized by V.Lafforgue to prove the "automorphic to Galois" direction for a reductive group G using the moduli of G-shtukas in [Laf18a]. In particular the moduli space of $D^\times$-shtukas, which we denote by $X_0^{\mathfrak{D}}(\mathfrak{M})$, is a "good" analogue of the moduli space of D-abelian surfaces (for a list of parallels see Section 2.2). Thanks to these results, for a non-isotrivial elliptic curve E over $\mathbb{F}_p(t)$ with conductor $\mathfrak{n} = \mathfrak{D}\mathfrak{M}$, we now have again two different coverings

$$\begin{array}{ccc} X_0(\mathfrak{M}\mathfrak{D}) & & X_0^{\mathfrak{D}}(\mathfrak{M}) \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi_{\mathfrak{D}}} & \\ & E & \end{array}$$

and we can now, analogously to the number field case, ask what the relation between the covering degree of π and $\pi_{\mathfrak{D}}$ is. This is the question inspiring this thesis.
In the remainder of the introduction, we shall now outline the content of this thesis in more detail.

1.3 Definitions

As mentioned above, a good analogue of D-abelian surfaces over function fields are $D^\times$-shtukas. These are equivalent to the category of Drinfeld-Stuhler modules, whose definition we recall:

Definition 1.3.1 *For any $\mathbb{F}_p$-algebra R, we denote by $R[\tau]$ the skew-polynomial ring generated by τ such that for all $r \in R$*

$$\tau r = r^p \tau.$$

This corresponds to the algebra of additive endomorphisms of $\mathbb{A}^1_R$.

Definition 1.3.2 (Drinfeld-Stuhler modules) *Let D be a central simple algebra over $\mathbb{F}_p(t)$ of dimension 4 and fix a maximal $\mathbb{F}_p[t]$-order $\mathcal{O}_D$ in D. A* Drinfeld-Stuhler $\mathcal{O}_D$-module over an $\mathbb{F}_p[t]$-algebra R *is an $\mathbb{F}_p$-algebra morphism*

$$\varphi : \mathcal{O}_D \to \mathrm{End}_{\mathbb{F}_p}(\mathbb{A}^2_R) \cong M_2(R[\tau]), \quad b \mapsto \varphi_b$$

such that

1. *For all $b \in \mathcal{O}_D \cap D^\times$, the kernel $\ker \varphi_b \subset \mathbb{A}^2_R$ is a finite group scheme over R of order $|\mathcal{O}_D/b\mathcal{O}_D|$.*

2. *The morphism φ respects $\mathbb{F}_p[t]$, that is under the morphism*

$$\mathbb{F}_p[t] \to \mathcal{O}_D \to M_2(R[\tau]) \to M_2(R),$$

(where the last morphism is induced by $\sum r_i\tau^i \mapsto r_0$), any element $a \in \mathbb{F}_p[t]$ maps to $\mathrm{diag}(a, a)$.

In analogy with abelian surfaces, Drinfeld-Stuhler modules admit a notion of level structure (see Def 4.7.3). The corresponding moduli functor admits a coarse moduli space.

Proposition-Definition 1.3.3 (Moduli of Drinfeld-Stuhler modules [LRS93]) *Let D be a quaternionic algebra over $\mathbb{F}_p(t)$ with discriminant $\mathfrak{D}$ and let $\mathfrak{m}$ be an ideal of $\mathbb{F}_p[t]$. Then the coarse moduli space of Drinfeld-Stuhler modules with $\Gamma_0(\mathfrak{m})$-level structure exists. It is a smooth proper scheme of dimension one over $\mathrm{Spec}\,\mathbb{F}_p[t][\frac{1}{\mathfrak{D}\mathfrak{m}}]$ and is denoted by $X_0^{\mathfrak{D}}(\mathfrak{m})$.*

These moduli spaces are function field analogues of Shimura curves, and have similar properties. In particular they maybe used to formulate geometric modularity similar to the number field case.

Theorem 1.3.4 (Geometric Modularity, [DK73]) *Let $\mathfrak{n}$ be an ideal of $\mathbb{F}_p[t]$ and let E be an elliptic curve over $\mathbb{F}_p(t)$ with split multiplicative reduction at ∞ and of conductor $\mathfrak{n}\infty$. Then for any decomposition $\mathfrak{n} = \mathfrak{D}\mathfrak{m}$ where $\mathfrak{D}$ is the product of an even number of primes and $(\mathfrak{D}, \mathfrak{m}) = (1)$, the elliptic curve E is a quotient of the Jacobian $\mathrm{Jac}(X_0^{\mathfrak{D}}(\mathfrak{m}))$.*

This form of geometric modularity gives rise to coverings of elliptic curves, of which we will study a distinguished subclass.

Definition 1.3.5 *We say a morphism $X_0^{\mathfrak{D}}(\mathfrak{m}) \to E$ is an* optimal covering *if the induced morphism on the Jacobians $E \to \mathrm{Jac}(X_0^{\mathfrak{D}}(\mathfrak{m}))$ is injective.*

Proposition-Definition 1.3.6 (Modular degree) *Let $\mathfrak{n} = \mathfrak{D}\mathfrak{m}$ as above. In any isogeny class of elliptic curves of conductor $\mathfrak{n}\infty$, there exists a unique representative E which admits an optimal covering $X_0^{\mathfrak{D}}(\mathfrak{m}) \to E$. The degree of this covering of curves is called the* modular degree of the isogeny class *and denoted by $\delta^{\mathfrak{D}}(\mathfrak{m})$.*

One of the main results of my thesis is the following analogue of Ribet-Takahashi [RT97].

Theorem 1.3.7 (Thm 6.7.4) *Let $\mathfrak{p}, \mathfrak{q}$ be primes which are coprime to $\mathfrak{m}$, E, E' be two isogenous elliptic curves over $\mathbb{F}_p(t)$ which admit optimal coverings by $X_0^{\mathfrak{D}}(\mathfrak{mpq})$ and $X_0^{\mathfrak{D}\mathfrak{pq}}(\mathfrak{m})$ respectively. Denote by $c'_{\mathfrak{p}}, c_{\mathfrak{q}}$ denote the order of the component groups of E, E' at $\mathfrak{p}$ and $\mathfrak{q}$ respectively. If $\delta^{\mathfrak{D}}(\mathfrak{pqm}), \delta^{\mathfrak{pq}\mathfrak{D}}(\mathfrak{m})$ denote the corresponding modular degrees, then*

$$\delta^{\mathfrak{pq}\mathfrak{D}}(\mathfrak{m}) \sim^S \frac{\delta^{\mathfrak{D}}(\mathfrak{pqm})}{c'_{\mathfrak{p}} c_{\mathfrak{q}}}$$

where $\sim^S$ denotes equality up to

- *Eisenstein primes of E, E', i.e. those primes ℓ such that $E[\ell]$ is reducible as a representation of $\mathrm{Gal}(\mathbb{F}_p(t)^{sep}/\mathbb{F}_p(t))$,*
- *primes dividing $p-1$,*
- *p.*

Remark 1.3.8 This is similar to the number field case in that $p-1$ is analogous to 2: indeed if $\ell \mid p-1$ then $\mathbb{F}_p(t)$ has ℓ-th roots of unity. However the possibility of p-factors is different.

Using this result, we can deduce a theorem that relates the Petersson inner products of (the function field analogues of) modular forms that are Jacquet-Langlands lifts of each other. To explain this, we recall the notion of modular forms in this context. The usual upper-half plane is replaced by the *Bruhat-Tits tree $\mathcal{T}$ of* $\mathrm{PGL}_2(\mathbb{F}_p((1/t)))$, i.e., the directed graph whose vertices classify lattices in $\mathbb{F}_p((1/t))^2$ up to scaling with an edge from $[\mathcal{L}]$ to $[\mathcal{L}']$ if there exist representatives $\mathcal{L} \subset \mathcal{L}'$ such that $\mathcal{L}'/\mathcal{L} \cong \mathbb{F}_p$.

Definition 1.3.9 (Harmonic cochain) *For any commutative ring Λ, an* Λ-valued harmonic cochain on $\mathcal{T}$ *is a function*

$$f : \mathrm{Ed}(\mathcal{T}) \to \Lambda$$

such that for all $e \in \mathrm{Ed}(\mathcal{T}), v \in \mathrm{Ver}(\mathcal{T})$ we have

$$f(e) + f(\bar{e}) = 0 \text{ and } \sum_{e \in \mathrm{Ed}(\mathcal{T}), t(e)=v} f(e) = 0,$$

where for any edge $e \in \mathrm{Ed}(\mathcal{T})$ the inversely oriented edge is denoted by $\bar{e}$ and $t(e)$ denotes the terminus of e. We denote the Λ-module of Λ-valued harmonic cochains by $\mathcal{H}(\mathcal{T}, \Lambda)$. If a group Γ acts on $\mathcal{T}$, then we denote the Λ-module of Γ-equivariant cochains by $\mathcal{H}(\mathcal{T}, \Lambda)^{\Gamma}$. A

Γ-*equivariant harmonic cochain with compact support as function on* $\mathcal{T}/\Gamma$ *is said to be* cuspidal *and the* Λ*-module generated by these is denoted by* $\mathcal{H}_!(\mathcal{T},\Lambda)^\Gamma$.

Remark 1.3.10 *As explained in [GR96], the* $\mathbb{Q}$*-module of cuspidal harmonic cochains with coefficients in* $\mathbb{Q}$ *is isomorphic to the rational first homology group of the the quotient graph* $\mathcal{T}/\Gamma$*, i.e.* $\mathcal{H}_!(\mathcal{T},\mathbb{Q})^\Gamma \cong H_1(\mathcal{T}/\Gamma,\mathbb{Q})$.

The Γ's which are relevant for this thesis are the following: For any ideals $\mathfrak{D},\mathfrak{m}$, let $\mathcal{O}$ be an Eichler order of level $\mathfrak{m}$ of in the indefinite quaternionic algebra D over $\mathbb{F}_p(t)$ with discriminant $\mathfrak{D}$. Denote by $\Gamma_0(\mathfrak{D},\mathfrak{m})$ the image of

$$\mathcal{O}^\times \to (D\otimes\mathbb{F}_p((t)))^\times \cong \mathrm{GL}_2(\mathbb{F}_p((t))) \to \mathrm{PGL}_2(\mathbb{F}_p((t))).$$

The corresponding cuspidal harmonic cochains are the function field analogue of modular forms for $\mathfrak{D}=1$ and quaternionic modular forms for $\mathfrak{D}\neq 1$ respectively. By work of V. Drinfeld, E.U.-Gekeler and Gekeler-Reversat, we have the following modularity theorem.

Theorem 1.3.11 (eg. [GR96, §8]) *For any decomposition* $\mathfrak{n}=\mathfrak{D}\mathfrak{m}$ *as above, there exists a canonical bijection between the set of isogeny classes of elliptic curves* $E/\mathbb{F}_p(t)$ *with conductor* $\mathfrak{n}\infty$ *and split multiplicative reduction at* ∞ *and the set of Hecke-eigenforms in* $\mathcal{H}_!(\mathcal{T},\mathbb{Q})^{\Gamma_0(\mathfrak{D},\mathfrak{m})}$ *which are* $\mathfrak{m}$*-new.*

Furthermore there exists an analogue of the Peterson inner product.

Definition 1.3.12 (Petersson inner product) *The vector space* $\mathcal{H}_!(\mathcal{T},\mathbb{Q})^{\Gamma_0(\mathfrak{D},\mathfrak{m})}$ *has a natural inner product*

$$\langle\cdot,\cdot\rangle : \mathcal{H}_!(\mathcal{T},\mathbb{Q})^{\Gamma_0(\mathfrak{D},\mathfrak{m})}\times\mathcal{H}_!(\mathcal{T},\mathbb{Q})^{\Gamma_0(\mathfrak{D},\mathfrak{m})}\to\mathbb{Q},\quad (f,g)\mapsto \sum_{e\in \mathrm{Ed}(\Gamma_0(\mathfrak{D},\mathfrak{m})\backslash\mathcal{T})}\left|\frac{\Gamma_0(\mathfrak{D},\mathfrak{m})_e}{\mathbb{F}_q^\times}\right| f(e)g(e).$$

Just as in the number field case, the Petersson inner product of a harmonic cochain is closely related to the modular degree of the associated isogeny class of elliptic curves (see for instance [Pap13, Theorem 8.19]). This allows us to study the behaviour of the Petersson inner product under Jacquet-Langlands by relating it to the modular degree of the associated elliptic curves and then leverage Thm 1.3.7, yielding the following result:

Corollary 1.3.13 (Thm 6.8.14) *Let* E *be an elliptic curve* $E/\mathbb{F}_p(t)$ *with split multiplicative reduction at* ∞*. Let* $f\in\mathcal{H}_!(\mathcal{T},\mathbb{Q})^{\Gamma_0(1,\mathfrak{n})}$*,* $g\in\mathcal{H}_!(\mathcal{T},\mathbb{Q})^{\Gamma_0(\mathfrak{D},\mathfrak{m})}$ *cuspidal harmonic cochain*

associated to it, which are Jacquet-Langlands lifts of each other and that are integrally normalized. We have the following equality (up to Eisenstein primes, primes dividing $p-1$ and p)

$$\frac{\langle f,f\rangle}{\langle g,g\rangle} \sim^S \frac{\delta^{\mathfrak{D}}(\mathfrak{m})}{\delta^{(1)}(\mathfrak{D}\mathfrak{m})} \sim^S \prod_{\mathfrak{q}|\mathfrak{D}} c_{\mathfrak{q}}.$$

To prove Theorem 1.3.7, we follow the strategy of the proof in the number field case. However, this required establishing the function field analogs of two major results:

- The study of the bad reduction of integral models at primes dividing the level, i.e. the analogue of Buzzard's extension [Buz97] of Deligne-Rapoport.
- Level lowering results analogous to those of Ribet in [Rib90a].

1.4 Bad reduction of Moduli of Drinfeld-Stuhler modules

As mentioned in Prop 1.3.3, the moduli spaces $X_0^{\mathfrak{D}}(\mathfrak{m})$ are defined over $\operatorname{Spec}\mathbb{F}_p[t][\frac{1}{\mathfrak{D}\mathfrak{m}}]$, in analogy with Shimura curves $X_0^D(m)$ which are defined over $\operatorname{Spec}\mathbb{Z}[\frac{1}{Dm}]$. In [DR73], Deligne-Rapoport constructed integral models for modular curves at primes dividing the level. This was later extended to Shimura curves (at primes not dividing the discriminant) by Buzzard in his PhD thesis [Buz97]. While the analogue of Deligne-Rapoport in the function field setting is known (by Gekeler [Gek86] in the prime level case and Papikian-Wei [PW15] in the non-prime level case), the analogue of Buzzard's thesis over function fields was not worked out previously. This turns out to be an important technical ingredient of this thesis:

Theorem 1.4.1 (see Thm 5.9.3) *Let $\mathfrak{p}\nmid\mathfrak{m}\mathfrak{D}$. There exists a proper flat $\mathbb{F}_p[t]_{\mathfrak{p}}$-scheme $\mathcal{X}_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{m})$ which is an integral model of $X_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{m})$ such that the special fiber is geometrically reduced, consisting of two irreducible components isomorphic to $\mathcal{X}_0^{\mathfrak{D}}(\mathfrak{m})_{\mathbb{F}_{\mathfrak{p}}}$ which intersect transversely at the supersingular points, that is at those Drinfeld-Stuhler modules whose $\mathfrak{p}$-torsion scheme is connected.*

To prove Thm 5.9.3 we adapt the proof of [Buz97]. If $\mathcal{O}_D$ is a maximal order of a quaternionic algebra over $\mathbb{Q}$ of discriminant D, the associated Shimura curve is the moduli space of abelian surfaces with an action of $\mathcal{O}_D$. To extend the integral models of Shimura curves to $\mathbb{Z}_\ell$, Buzzard (following Deligne-Rapoport) introduces a "new" level structure at ℓ. However, this is the adaptation of the notion of a *Drinfeld $\Gamma_0(\ell^n)$-level structure*, which was introduced by Drinfeld to extend the notion of level structures of Drinfeld modules to primes dividing the level. A Drinfeld

$\Gamma_0(\ell^n)$-level structure on a $\mathcal{O}_D$-abelian surface A is an $\mathcal{O}_D$-invariant? finite locally free closed subscheme $\mathbb{H} \subset A[\ell^n]$ of rank ℓ^n which admits a generator, that is an an A-linear morphism

$$\iota : (\mathbb{Z}/\ell^n)^2 \to \mathbb{H}$$

such that we have an equality

$$\mathbb{H} = \sum_{x \in (\mathbb{Z}/\ell^n)^2} \iota(x) \subset A[\ell^n]$$

of Cartier divisors of A. This is often referred to as a "cyclicity" condition. However, if $n = 1$ the "cyclicity" assumption is automatic, and thus a Drinfeld $\Gamma_0(\ell)$-level structure is a closed subgroup of $A[\ell]$, which is finite and locally free of rank l^2. As we assume that ℓ is not ramified, we have an isomorphism $\mathcal{O}_D/\ell \cong \mathrm{Mat}_2(\mathbb{F}_\ell)$. One can use this to reduce the study of ℓ-level structures on $\mathcal{O}_D$-abelian surfaces to the study of ℓ-level structures on elliptic curves. In particular if A is ordinary, then $A[\ell]$ has two such subgroups, which are furthermore Cartier-dual, and if A is supersingular there exists exactly one such subgroup. To adapt this strategy, we need to use the function field analogue of p-divisible groups, which are $A_\mathfrak{p}$-Dieudonné modules. We adapt Drinfeld level structure to Drinfeld-Stuhler modules. Now the category of $\mathcal{O}_{D,\mathfrak{p}} \cong \mathrm{Mat}_2(\mathbb{F}_p[t]_\mathfrak{p})$-Dieudonné modules is Morita equivalent to the category of $A_\mathfrak{p}$-Dieudonné modules. In particular one can reduce the study of Drinfeld $\Gamma_0(\mathfrak{p})$-level structures on Drinfeld-Stuhler modules to the study of Drinfeld $\Gamma_0(\mathfrak{p})$-level structures on Drinfeld modules. If $\mathcal{E}$ is a Drinfeld module over $(\mathbb{F}_p[t]/\mathfrak{p})^{sep}$, then the scheme of $\mathfrak{p}$-torsion points is

$$\mathcal{E}[\mathfrak{p}] = \begin{cases} \alpha_{|\mathfrak{p}|} \times \left(\underline{\mathbb{F}_p[t]/\mathfrak{p}}\right), & \text{if } \mathcal{E} \text{ is ordinary;} \\ \alpha_{|\mathfrak{p}|^2}, & \text{if } \mathcal{E} \text{ is supersingular.} \end{cases}$$

At first sight this differs from what we would expect from the torsion of an abelian surface in positive characteristic: a key difference is that while μ_ℓ and $\underline{\mathbb{Z}/\ell}$ are Cartier dual, $\alpha_{|\mathfrak{p}|}$ and $\underline{\mathbb{F}_p[t]/\mathfrak{p}}$ are not! The solution consists of using a duality theory specific to function fields, namely ***Taguchi duality***, first introduced by Y. Taguchi in [Tag95]. Indeed $\alpha_{|\mathfrak{p}|}$ and $\underline{\mathbb{F}_p[t]/\mathfrak{p}}$ are Taguchi-dual. The final ingredient to adapt Buzzards proof is the fact that Dieudonné modules and Drinfeld-Stuhler modules satisfy the analogue of the Serre-Tate theorem. These formal ingredient allow us to reduce the proof of regularity of the integral model we construct to the trivial discriminant case, which is known by work of Gekeler and Papikian-Wei.

1.5 Level Lowering

Another ingredient that we need is a function field analogue of Ribet's famous level lowering results [Rib90a]. The first step towards proving this is a study of integral models of $X_0^{\mathfrak{D}}(\mathfrak{m})$ at primes dividing the discriminant, namely the analogue of Cerednik-Drinfeld for Shimura curves. This had been done by T.Hausberger in his Ph.D. thesis [Hau05], where he in particular showed that the singular points of the special fiber correspond to *exceptional Drinfeld-Stuhler modules*. Using classification results of Papikian [Pap10], we show that there exists a bijection

$$\mathcal{X}_0^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{m})_{\mathfrak{p}}^{\text{sing}} \cong \mathcal{X}_0^{\mathfrak{D}}(\mathfrak{m}\mathfrak{p}\mathfrak{q})_{\mathfrak{q}}^{\text{sing}}$$

between singular points of the special fibers (at different primes!) of integral models. In fact even more subtle information can be transferred between the special fibers, namely how these singularities arise as the intersection of irreducible components. A convenient way to encode this information is via character groups.

We prove that there exists an analogue of Ribet's short exact sequence, which was one of the main technical ingredients of [Rib90a]:

$$0 \to \mathcal{X}(J_0^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{m}), \mathfrak{p}) \xrightarrow{\iota} \mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{m}\mathfrak{p}\mathfrak{q}), \mathfrak{q}) \to \mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{q}\mathfrak{m}), \mathfrak{q})^2 \to 0.$$

From this we are able to deduce the following analogue of Ribet's level lowering:

Theorem 1.5.1 (see Thm 6.6.1) *Let $\mathbb{T}$ be the ring generated by the Hecke operators of level $\mathfrak{n} = \mathfrak{D}\mathfrak{p}\mathfrak{q}\mathfrak{m}$. Fix a maximal ideal $\mathfrak{M}$ in $\mathbb{T}$ such that char $(\mathbb{T}/\mathfrak{M}) =: \ell \neq p$ and assume that $\ell \nmid p-1$. Let $\rho_{\mathfrak{M}}$ be the associated two-dimensional Galois representation with coefficients in $\mathbb{T}/\mathfrak{M}$. If $\rho_{\mathfrak{M}}$ is irreducible and $\rho_{\mathfrak{M}}$ is finite at $\mathfrak{p}$, then $\rho_{\mathfrak{M}}$ is modular of level $\mathfrak{D}\mathfrak{q}\mathfrak{m}$.*

This is analogous to [Rib90a, Thm 1.1], where the one has to assume that $\ell \neq 2$. As in Ribet's original paper [Rib90a] we first prove this under the additional assumption that $|\mathfrak{p}| \neq 1 \mod \ell$. However, the strategy of Ribet [Rib90a] to reduce to this case (relying on Ihara's Lemma) does not seem to immediately translate to the function field setting. Instead we use a modified strategy analogous to Ribet's subsequent paper [Rib94] which allows us to bypass the need for Ihara's Lemma.

1.6 Open Questions

Finally we mention some open questions in this thesis which we cover in Chapter VII.

- This thesis addresses questions of level-lowering. In the number field setting, Ribet [Rib90b] show that any modular representation of level n with coefficients in $\mathbb{F}_\ell$, and $p \nmid \ell n$, if

$$T_p = \pm(p+1)$$

then ρ is a p-newform of level pn. In Prop 7.1.26, we prove a function field analogue of this result, assuming a function field analogue of Ihara's Lemma holds (see Conj 7.1.11).

- The proof of the function field analogue is based Deuring-type classification results of supersingular and mixed exceptional Drinfeld-Stuhler modules over finite fields. In particular these are dependent on the choice of a "base-point". In [Rib89] Ribet constructed a base-point free classification which classifies abelian surfaces via their endomorphism algebra with the additional datum of an orientation. As a first step, we prove such a classification for supersingular Drinfeld-modules (see Prop 7.2.5). It would thus be interesting to adapt the results of [Rib89] to our setting. For this it would be useful to have a function field analogue of the *Oort invariant*. The Oort invariant has been used in [Hel10] to construct a stratification of Shimura varieties associated to unitary groups. The study of this stratification allows for the geometric realization of Jacquet-Langlands and this could be an interesting line of inquiry once a good analogue of the Oort-invariant has been established.

- The moduli spaces of Drinfeld-Stuhler admit a generalization which has no analogue in the number field setting, namely the moduli space of r-legged shtukas. These moduli spaces are key to proving the Langlands program in the function field setting. More recently they've also proven useful to give "higher" analogues of results known over number fields. For instance [YZ17] computes the r-th derivative of L-functions generalizing the resulsts of Gross-Zagier (see [YZ17, Rem 1.3]). In a similar vein, conjecturally, elliptic curves are r-modular (see for instance [LW22]). It could be interesting to establish a higher analogue of the modular degree and understand what arithmetic information this encodes.

1.7 Structure of the thesis

We briefly explain the structure of this thesis.

- In Chapter II We give an overview of the definitions and results concerning quaternion algebras which we will use in this thesis.

- In Chapter III we give an overview of the theory of automorphic forms over function fields and their explicit representation in terms of harmonic cochains.

- In Chapter IV we introduce the categories of Drinfeld-Stuhler modules/$\mathcal{D}$-elliptic sheaves, their associated moduli spaces and the corresponding Modularity theorems.
- In Chapter V we study the uniformization results of moduli spaces $X_0^{\mathfrak{D}}(\mathfrak{N})$ at bad primes, in particular their special fibers.
- In Chapter VI we first introduce component and character groups as a convenient way to package information about the reduction of $X_0^{\mathfrak{D}}(\mathfrak{N})$. Using this language we are give the analogue of Ribet's short exact sequence. We then leverage this to give level lowering results and prove the desired relation between modular degrees of elliptic curves.
- In Chapter VII we list some questions that arose during work on this thesis that the author thinks merit further thought.

CHAPTER II

Preliminaries

2.1 Table of Notations

τ	($\mathbb{F}_q$-linear) Frobenius
σ	partial Frobenius
$\mathbb{C}_\infty$	completion of Q_∞
$\mathcal{H}(\mathcal{T}, \Lambda)$	Harmonic cochains on $\mathcal{T}$ with values in Λ (see Def 3.2.8)
$\mathcal{H}_!(\mathcal{T}, \Lambda)$	cuspidal harmonic cochains (see Def 3.2.14)
$\langle \cdot, \cdot \rangle$	Petersson inner product (see Def 3.2.18)
$\mathcal{T}$	Bruhat-Tits building (see Def 3.1.3)
Ω^d	Drinfeld symmetric space (see Def 3.1.1)
Γ	congruence subgroup
$\overline{\Gamma} := \Gamma^{\mathrm{ab}}/\mathrm{tor}(\Gamma^{\mathrm{ab}})$	abelianization modulo torsion (see Def 3.4.1)
$\Gamma_0^{\mathfrak{D}}(\mathfrak{N})$	specific congruence subgroup
$X_0^{\mathfrak{D}}(\mathfrak{N})$	moduli space of Drinfeld-Stuhler modules with level $\mathfrak{N}$
$J_0^{\mathfrak{D}}(\mathfrak{N}) := \mathrm{Jac}(X_0^{\mathfrak{D}}(\mathfrak{N}))$	Jacobian of $X_0^{\mathfrak{D}}(\mathfrak{N})$
D	quaternion algebra with ramification $\mathfrak{D}$
$\mathcal{D}$	sheaf of quaternionic algebra
$\Sigma^{\mathfrak{D}}(\mathfrak{N})$	set of Drinfeld-Stuhler modules with extra structure
$\mathcal{X}$	character group (see Def 6.1.4)
Φ	component group (see Def 6.1.2)
$c_{\mathfrak{p}}$	Tamagawa number of elliptic curve E at $\mathfrak{p}$
$\delta^{\mathfrak{D}}(\mathfrak{N})$	degree of $X_0^{\mathfrak{D}}(\mathfrak{N}) \to E$

2.2 Analogies between number fields and function fields

Analogies between global fields	
Number fields	Function fields
$\mathbb{Z}$	$\mathbb{F}_p[t]$
$\lvert\cdot\rvert : \mathbb{Q} \to \mathbb{R}$	$\lvert\cdot\rvert_\infty : \mathbb{F}_p(t) \to \mathbb{R}$
$\mathbb{R}$	completion of $\mathbb{F}_p(t)$ along $\lvert\cdot\rvert_\infty$
$\overline{\mathbb{R}} = \mathbb{C}$	$\mathbb{C}_\infty$
Elliptic curves over $\mathbb{Q}$	Elliptic curves over $\mathbb{F}_p(t)$
Quaternion algebras over $\mathbb{Q}$ determined by their discriminant, an ideal in $\mathbb{Z}$	Quaternion algebras over $\mathbb{F}_p(t)$ determined by their discriminant, an ideal in $\mathbb{F}_p[t]$
Abelian surfaces with an action of $\mathcal{O}_D$	Drinfeld-Stuhler sheaves
Shimura curves	Moduli spaces of Drinfeld-Stuhler sheaves
Upper half-plane $\mathbb{H}$	Drinfeld half-plane or Bruhat-Tits building $\mathcal{T}$
Modular forms	Harmonic cochains
p-divisible groups	$A_{\mathfrak{p}}$-Dieudonne modules
Serre-Tate theorem	Serre-Tate theorem
Buzzard's extension of Deligne-Rapoport	Thm 5.9.3
Singular fiber of Deligne-Rapoport correspond to supersingular $\mathcal{O}_D$-abelian surfaces	Singular fiber of Deligne-Rapoport correspond to supersingular Drinfeld-Stuhler modules
Cerednik-Drinfeld	Hausberger's analogue of Cerednik-Drinfeld (see Thm 5.10.6)
Singular fiber of Cerednik-Drinfeld model correspond to mixed-exceptional $\mathcal{O}_D$-abelian surfaces	Singular fiber of Cerednik-Drinfeld model correspond to mixed-exceptional Drinfeld-Stuhler modules

2.3 Setting and Function Field Background

Let X be a smooth proper curve over $\mathbb{F}_q$ and let ∞ be a closed point thereof. Then we set

$$A = \mathcal{O}_X(X \backslash \{\infty\}) \quad F = \mathrm{Quot}(A).$$

We denote by τ the $\mathbb{F}_q$-linear Frobenius. For any A-algebra R, we denote by $R[\tau]$ the skew-commutative polynomial ring with the commutativity relation

$$\tau r = r^q \tau.$$

Definition 2.3.1 *For any $\mathbb{F}_q$-algebra R, the* partial Frobenius σ *on $A \otimes R$ is the endomorphism defined by*

$$\sigma : A \otimes R \to A \otimes R, \quad a \otimes r \mapsto a \otimes r^q.$$

Remark 2.3.2 The literature is not consistent in its choice of convention for τ, σ.

Definition 2.3.3 *Let R be an A-algebra with structure morphism γ. We say that R has* A-characteristic $\mathrm{char}(R) := \ker \gamma$. *If* $\mathrm{char}(R) = 0$, *we say it has* generic characteristic.

Definition 2.3.4 *A field extension $K/\mathbb{F}_q(t)$ is said to be* imaginary quadratic *if it is a degree two extension in which ∞ is ramified.*

number fields	$\mathbb{Z}$	$\mathbb{Q}$	$\mathbb{R}$	$\mathbb{C}$
function fields	$\mathbb{F}_q[t]$	$\mathbb{F}_q(t)$	$(\mathbb{F}_q(t))_\infty$	$\mathbb{C}_\infty$

2.4 Inner Forms of GL_2

Definition 2.4.1 *Let G be an algebraic group over a field F. A* form *is an algebraic group G' such that there exits an isomorphism $\varphi : G_{F^{\mathrm{sep}}} \xrightarrow{\sim} G'_{F^{\mathrm{sep}}}$. If the action*

$$G_F \to \mathrm{Aut}(G'), \quad \sigma \mapsto \varphi^{-1} \circ \sigma \circ \varphi$$

is inner, that is acts by inner automorphisms, then we say that G' is an inner form.

If we have an inner form G' of GL_2, then we can use the corresponding cocycle to form a twist of $\mathrm{Mat}_2(F)$. Since by Hilbert 90 the first cohomology group $H^1(F, \mathrm{GL}_2)$ is trivial, this twist remains a four-dimensional k-vector space with a (central) algebra structure. In particular G' arises as the unit group of a four-dimensional central simple algebra. This motivates the following definition:

Definition 2.4.2 *A* quaternion algebra B *over a field* F *is a central simple* F*-algebra of dimension four as a* F*-vector space.*

Proposition 2.4.3 *Let* B *be a quaternion algebra over* F*. Then the functor*

$$B^\times : \mathrm{CAlg}_F \to \mathit{Set}, \quad R \mapsto (B \otimes R)^\times$$

is an algebraic group. Furthermore it is an inner from of GL_2 *and all inner forms arise in this way.*

Proposition 2.4.4 *Let* G' *be an inner form of* G*. Then there exists an isomorphism between the associated* L*-groups* ${}^LG \xrightarrow{\sim} {}^LG'$

2.5 Quaternion algebras

In this section, we recap some relevant results about quaternion algebras. Throughout this section, let F be a global function field. For this exposition, the primary source shall be [Voi21].

Definition 2.5.1 *An* involution *of an* F*-algebra* B *is a* F*-linear morphism* $\overline{-} : B \to B$ *such that*

$$\overline{1} = 1, \quad \overline{\overline{b}} = b$$

for all $b \in B$ *and* $\overline{\alpha}\overline{\beta} = \overline{\beta}\overline{\alpha}$ *for all* $\alpha, \beta \in B$*.*

Definition 2.5.2 *An involution is said to be* standard *if* $b\overline{b} \in F$ *for all* $b \in B$*.*

Proposition 2.5.3 ([Voi21, Cor 3.4.4]) *There exists at most one involution on any* F*-algebra.*

Remark 2.5.4 This allows us to talk henceforth of the *the* involution of an algebra.

Definition 2.5.5 *Let* B *be an* F*-algebra with a standard involution. Then the* reduced norm associated with the involution *is the morphism*

$$\mathrm{nrd} : B \to F, \quad \alpha \mapsto \alpha\bar{\alpha}.$$

Example 2.5.6 If $B = \mathrm{Mat}_2(F)$, then the involution determined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

is a standard involution. The associated reduced norm is the determinant.

Proposition 2.5.7 *If* char$F \neq 2$*, then an F-algebra B is a quaternion algebra if and only if there exists $i, j \in B$ such that $1, i, j, ij$ form a F-linear basis of B, $i^2, j^2 \in F^\times$ and*

$$ij = -ji.$$

If char$F = 2$, then an F-algebra B is said to be a quaternion algebra if and only if there exists $i, j \in B$ such that $1, i, j, k$ form a F-linear basis of B, $i^2 + i \in F, j^2 \in F^\times$ and

$$k = ij = j(i + 1).$$

Example 2.5.8 The simplest example of a quaternion algebra is $\mathrm{Mat}_2(F)$. The corresponding generators are

$$i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Historically the first quaternion algebra was Hamilton's quaternion algebra, which is the $\mathbb{R}$-algebra such

$$i^2 = -1, \quad j^2 = -1.$$

Definition 2.5.9 *Let $\mathfrak{p}$ be a place of F. We say that a quaternionic algebra B is* ramified at $\mathfrak{p}$ *if $B \otimes_F F_\mathfrak{p}$ is a division algebra. If $B \otimes_F F_\mathfrak{p}$ is not a division algebra, then we say that B* is unramified/split at $\mathfrak{p}$. *We denote the set of places at which B is ramified by* Ram$B$.

Definition 2.5.10 *A quaternion algebra which is ramified at the places at infinity is said to be* indefinite.

Proposition-Definition 2.5.11 *For any quaternionic algebra B, the set of ramified places is finite and therefore the ideal*

$$\mathrm{disc}B := \prod_{\mathfrak{p} \in \mathrm{Ram}B} \mathfrak{p}$$

is well-defined, which we call the discriminant of B.

Proposition 2.5.12 ([Voi21, Thm 14.6.1]) *The set of ramified places is of even cardinality and determines B up to isomorphism. Furthermore every such set occurs as the set of ramification places.*

Hence we will often implicitly describe a quaternion algebra by its discriminant throughout this thesis. As a matter of convention, we shall often denote

$$\mathfrak{D} := \mathrm{disc}B.$$

This is justified, as the choice of a representative in a given isomorphism class will not matter.

Proposition 2.5.13 ([Voi21, Prop 14.6.7]) *Let K/F be a separable quadratic extension and B a quaternionic algebra over F. Then the following are equivalent:*

1. *there exists an embedding $K \to B$ of F-algebras*
2. *every place $\nu \in \mathrm{Ram}(B)$ does not split in K.*

Theorem 2.5.14 ([Voi21, Thm 14.7.4]) *Let F be a function field and B a quaternion F-algebra. Then the reduced norm is surjective, that is we have* $\mathrm{nrd}(B^\times) = F^\times$.

Definition 2.5.15 (Lattice) *Let V be a finite dimensional F-vector space, then a finitely generated A-module $L \subset V$ such that $L \otimes F \cong V$ is said to be a* lattice.

Definition 2.5.16 (Order) *An A-lattice $\mathcal{O} \subset B$ which is also a subring is said to be an A-*order. *If an A-order is not properly contained in any other A-order, then we say it is a* maximal order.

Example 2.5.17 The simplest example of an order is $\mathrm{Mat}_2(A) \subset \mathrm{Mat}_2(F)$ as this is indeed an algebra, finitely generated over A and we have $\mathrm{Mat}_2(A) \otimes_A F \cong \mathrm{Mat}_2(F)$. Notice that $\mathrm{Mat}_2(A)$ is isomorphic to the endomorphism algebra of the lattice $A^2 \subset \mathrm{Mat}_2(F)$. In fact every maximal order arises in this way by the following result.

Proposition 2.5.18 ([Voi21, Lemma 10.5.4]) *For any maximal order $\mathcal{O} \subset B$, there exists a lattice M such that $\mathcal{O} = \mathrm{End}(M)$.*

Definition 2.5.19 (Eichler order) *Let B be a quaternion algebra. An* Eichler order *$\mathcal{O} \subset B$ is the intersection of two maximal orders.*

Note that in the previous definition we have explicitly **not** excluded the case where the two maximal orders are the same. Locally, the structure of Eichler orders is described by the following theorem.

Proposition 2.5.20 ([Voi21, Prop 23.4.3]) *Let R be a local ring and consider an A-order $\mathcal{O} \subset \mathrm{Mat}_2(\mathrm{Quot}(R))$. Then the following are equivalent:*

1. *$\mathcal{O}$ is an Eichler order*
2. *$\mathcal{O}$ is the intersection of a (uniquely determined) pair of maximal orders*
3. *$\mathcal{O} \cong \begin{pmatrix} R & R \\ \mathfrak{p}^e & R \end{pmatrix}$*
4. *$\mathcal{O}$ contains an R-subalgebra that is $\mathrm{Mat}_2(\mathrm{Quot}(R))$-conjugate to $\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$.*

Proposition 2.5.21 *Every central simple F-algebra is cyclic, that is there exists a cyclic Galois extension K/F, say of order d, such that*

$$B \cong \bigoplus_{i=0}^{d-1} K z^i$$

where $f \in A \backslash \{0\}$, we have $z^d = f$, $zr = \sigma(r)z$ for $r \in K$ and a generator σ of the Galois group.

Proposition 2.5.22 *After extension of scalars, every quaternion algebra B is isomorphic to the matrix algebra, that is we have*

$$B \otimes F^{\mathrm{sep}} \cong \mathrm{Mat}_2(F^{\mathrm{sep}}).$$

Proposition 2.5.23 (Tsen Theorem) *For any algebraically closed field k, every quaternion algebra over $k(t)$ is split. In particular for any quaternion algebra B over $\mathbb{F}_q(t)$ there exists an n such that $B \otimes \mathbb{F}_{q^n}(t) \cong \mathrm{Mat}_2(\mathbb{F}_{q^n}(t))$.*

Proposition 2.5.24 *Let R be a complete DVR. Let $\mathcal{O}_1, \mathcal{O}_2 \subset \mathrm{Mat}_2(\mathrm{Quot}(R))$ be maximal orders and $S := \mathcal{O}_1 \cap \mathcal{O}_2$ the Eichler order given by their intersection. If M_1 is a free rank one $\mathcal{O}_1$-module and M_2 is a free rank one $\mathcal{O}_2$-module which are lattices in a rank one $\mathrm{Mat}_2(\mathrm{Quot}(R))$-module, then the intersection of the modules*

$$M = M_1 \cap M_2$$

is a free rank one S-module.

Proof. Since all maximal orders are conjugated, we may assume without loss of generality that

$$\mathcal{O}_1 = \mathrm{Mat}_2(R), \mathcal{O}_2 = \begin{pmatrix} \varpi^r & 0 \\ 0 & 1 \end{pmatrix}^{-1} \mathrm{Mat}_2(R) \begin{pmatrix} \varpi^r & 0 \\ 0 & 1 \end{pmatrix}$$

and thus $S = \begin{pmatrix} R & R \\ \varpi^r R & R \end{pmatrix}$. Furthermore we may assume that $M_1 = \mathrm{Mat}_2(R)$ and $M_2 = \begin{pmatrix} \varpi^t & 0 \\ 0 & 1 \end{pmatrix} \mathcal{O}_2$. Thus we have

$$\begin{aligned} M &= M_1 \cap M_2 \\ &= \mathrm{Mat}_2(R) \cap \begin{pmatrix} \varpi^t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R & \varpi^{-r} R \\ \varpi^r R & R \end{pmatrix} \\ &= \begin{pmatrix} \varpi^{\max(t,0)} R & \varpi^{\max(t-r,0)} R \\ \varpi^r R & R \end{pmatrix} \\ &= \begin{pmatrix} \varpi^{\max(t,0)} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R & R \\ \varpi^r R & R \end{pmatrix} \end{aligned}$$

Thus M is a free right S-module. □

CHAPTER III

Automorphic Forms and Drinfeld Modular Forms

In this chapter we present the necessary background on automorphic forms and their associated objects in the function field setting. In the number field setting, an automorphic form on a reductive group G is a function

$$G(\mathbb{A}_{\mathbb{Q}}) \to \mathbb{C}$$

which is $G(\mathbb{Q})$-equivariant and satisfies certain finiteness conditions. This definition can be adapted to the function field setting by replacing the adeles of $\mathbb{Q}$ with the adeles over a global function field, see Def 3.3. On the other hand, in the number field setting, GL_2-automorphic forms are closely related to modular forms, of which we will need an analogue. If one thinks of modular forms as functions on the (complex) upper half-plane, then one needs a function field-version of the upper half-plane. The correct analogue was introduced by Drinfeld, and is thus called the *Drinfeld upper half-plane* Ω. Associated to the Drinfeld upper half-plane is a graph, namely the *Bruhat-Tits building* $\mathcal{T}$. These two spaces are related by the so called "building-map"

$$\lambda : \Omega \to \mathcal{T}(\mathbb{R})$$

where $\mathcal{T}(\mathbb{R})$ denotes the topological realization of the graph $\mathcal{T}$. This can be (philosophically) seen as a tropicalization (see Rem 3.1.15). Using the Bruhat-Tits building as an analogue of the upper-half plane, we arrive at the notion of a *harmonic cochain*, which is a function on the Bruhat-Tits building subject to certain conditions (see Def 3.2.14). Thus by analogy

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \to \mathbb{C} \leftarrow \mathrm{SL}_2(\mathbb{F}_p[t])\backslash\mathcal{T}$$

harmonic cochains on quotients of the Bruhat-Tits tree are analogous to modular forms.
Another way of thinking of modular forms is to think of them as differential forms on the moduli space of elliptic curves. In particular this allows us to give a notion of modular forms where the

target is also function field valued, that is we have the analogy

$$\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \to \mathbb{C}, \quad \mathbb{C}_\infty \leftarrow \mathrm{SL}_2(\mathbb{F}_p[t])\backslash\Omega$$

The adaption of this to the function field setting leads to the notion of *Drinfeld modular forms*. These are closely related to the modulo p reductions of harmonic cochains, as we will see in Prop 3.5.13.

A key property of "classical" modular forms is that one can associate Galois representations to them. This is also true for both harmonic cochains and Drinfeld modular forms. The Galois representations associated to Drinfeld modular forms are *one-dimensional*, and thus in particular abelian. Essentially, this is due to the "usual" issue that cohomology in "bad" characteristic is too small. This will be explained in Rem 3.5.14.

number fields	$\mathbb{H}$	Automorphic forms	$C_c^\infty(\mathbb{H}/\Gamma, \mathbb{C})$	$H^0(\mathbb{H}/\Gamma, \Omega)$
function fields	$\mathcal{T}$	Automorphic forms	$\mathcal{H}_!(\mathcal{T}, \mathbb{Q})^\Gamma$	$M_{k,l}(\Gamma)$

3.1 Drinfeld Upper Half-Plane and its relation to the Bruhat-Tits building

Throughout this section let K be a global function field, ∞ a place of K, K_∞ the corresponding completion and $\pi_\infty \in K_\infty$ a uniformizer.

Definition 3.1.1 (Drinfeld symmetric space) *For any d, the* Drinfeld symmetric space *is*

$$\Omega^d := \mathbb{P}^{d-1}(\mathbb{C}_\infty) - \bigcup_H H(\mathbb{C}_\infty)$$

where H runs through the set of all K_∞-rational hyperplanes. In the special case $d = 2$, we shall simplify the notation and write $\Omega := \Omega^2$ and call it the Drinfeld upper half-plane.

Remark 3.1.2 *This is rigid analytic variety. To prove this, one constructs a map to the Bruhat-Tits tree (at a level of points), which is fibered in discs (see Rem 3.1.15). For a modular interpretation of the Drinfeld symmetric space, see Section 5.10.*

Definition 3.1.3 *The* Bruhat-Tits simplex $\mathcal{T}$ of $\mathrm{GL}_2(K_\infty)$ *is the graph with vertices*

$$\mathrm{Ver}(\mathcal{T}) := \mathrm{GL}_2(K_\infty)/\,\mathrm{GL}_2(\mathcal{O}_\infty)Z(K_\infty)$$

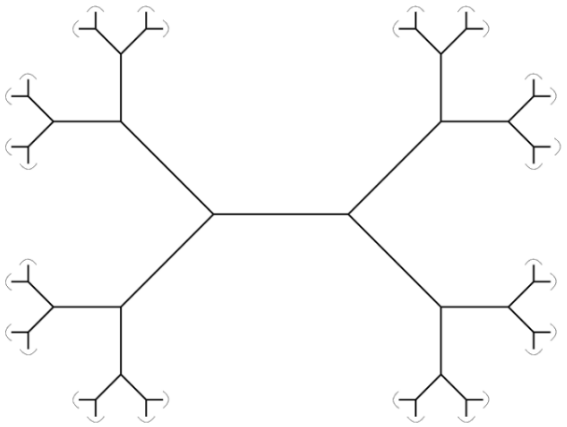


Figure III.1: Bruhat-Tits tree for $q = 2$ (image taken [KP23])

where $Z(K_\infty)$ denotes the center of $\mathrm{GL}_2(K_\infty)$. Denote by

$$\mathcal{I} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_\infty) \mid c = 0 \textit{ mod } \pi_\infty \right\}$$

the Iwahori group*. The edges of the Bruhat-Tits simplex are*

$$\mathrm{Ed}(\mathcal{T}) := \mathrm{GL}_2(K_\infty)/Z(K_\infty)\mathcal{I}.$$

Remark 3.1.4 This simplex comes with an action of $\mathrm{GL}_2(K_\infty)$ which factors through $\mathrm{PGL}_2(K_\infty)$. For more general G, the G-Bruhat-Tits complex satisfies similar properties. In particular, one can think of the Bruhat-Tits complex as being analogous to the symmetric space associated to a reductive group G over $\mathbb{R}$ (see [KP23]).

Proposition 3.1.5 *The Bruhat-Tits simplex of $\mathrm{GL}_2(K_\infty)$ is a $q+1$-regular tree.*

Figure III.1 represents the Bruhat-Tits tree for $q = 2$.

Definition 3.1.6 *An $\mathcal{O}_\infty$-lattice in K_∞^2 is a rank two $\mathcal{O}_\infty$-submodule of K_∞^2. We say two lattices $\mathcal{L}, \mathcal{L}'$ are* equivalent *if there exists a $x \in K_\infty^\times$ such that*

$$\mathcal{L}' = x\mathcal{L}.$$

We shall denote the equivalence class of a lattice by $[\mathcal{L}]$. *Two equivalence classes* $[\mathcal{L}], [\mathcal{L}']$ *are said to be* adjacent *if there exists a representative* $\mathcal{L}'' \in [\mathcal{L}']$ *such that* $\mathcal{L}'' \subset \mathcal{L}$ *and the quotient* $\mathcal{L}/\mathcal{L}''$ *is a length one* $\mathcal{O}_\infty$*-module.*

Proposition 3.1.7 *The vertices of the Bruhat-Tits tree correspond to equivalence classes of lattices and the edges correspond to adjacent lattices.*

We spend the rest of the section explaining how to relate the Drinfeld-upper half plane and Bruhat-Tits tree.

Definition 3.1.8 ((Geometric) Realization of graph) *For any graph* $\mathcal{G}$*, the* realization of $\mathcal{G}$ *is the topological space obtained by glueing unit-intervals for every edge of* $\mathcal{G}$ *according to the incidence relations of* $\mathcal{G}$*. We denote this space by* $\mathcal{G}(\mathbb{R})$.

Remark 3.1.9 This is nothing but the restriction of the geometric realization functor

$$\text{sSet} \to \text{Top}$$

(which realizes simplical sets to topological spaces) restricted to graphs. In particular, this construction is functorial, which implies, for instance, that the geometric realization $\mathcal{T}(\mathbb{R})$ of the Bruhat-Tits building inherits an action of $\mathrm{GL}_2(K_\infty)$.

Definition 3.1.10 *A* non-archimedean norm *on a* K_∞*-vector space* V *is a map*

$$|-| : V \to \mathbb{R}$$

such that for all $v, w \in V$*,* $a \in K_\infty$ *the map* $|-|$ *satisfies*

- $|v| \geqslant 0$
- $|v| = 0 \iff v = 0$
- $|av| = |a||v|$
- $|v + w| \leqslant \max\{|v|, |w|\}$.

Definition 3.1.11 (Similarity classes of norms) *Let* V *be a* K_∞*-vector space. We say that two norms* $|-|, |-|'$ *are* equivalent *or* similar *if they differ by a constant* $r \in \mathbb{R}^\times$.

Remark 3.1.12 The group $\mathrm{GL}(V)$ acts on the the set of norms on V, which respects equivalence classes, in particular $\mathrm{GL}(V)$ acts on the set of equivalence classes of norms on V.

Theorem 3.1.13 (Goldman-Iwahori, see [GR96, Thm 1.4.3]) *There exists a bijection between $\mathcal{T}(\mathbb{R})$ and the set of similarity classes of non-archimedean norms on K_∞^2. This bijection is canonical and $\mathrm{GL}_2(K_\infty)$-equivariant*[1].

We sketch the construction here: A vertex in $\mathcal{T}$ corresponds to a lattice $L \subset V$. To it we can associate the following norm

$$|-|_L : V \to \mathbb{R}, \quad v \mapsto |v|_L := \inf\{|r| \mid r \in K_\infty, v \in rL\}.$$

One checks that, up to equivalence of norms, this does not depend on the equivalence class of lattices, and thus we shall denote it by $|-|_{[L]}$. To extend this to all of $\mathcal{T}(\mathbb{R})$, note that any point on $\mathcal{T}(\mathbb{R})$ lies on an edge, which corresponds to a pair of (equivalence classes of) lattices $([L], [L'])$, and thus can be written as

$$(1-t)[L] + t[L']$$

for some $t \in [0, 1]$. Then we can associate the norm

$$|-|_P : V \to \mathbb{R}, \quad v \mapsto |v|_P := \sup\{|v|_{[L]}, |\pi|_\infty^t |v|_{[L']}\}.$$

Definition 3.1.14 (Building Map) *The* building map $\lambda : \Omega \to \mathcal{T}(\mathbb{R})$ *is defined by*

$$\lambda(z)((a, b)) := |az + b|.$$

Remark 3.1.15 The pre-image of λ any point of $\mathcal{T}(\mathbb{R})$ is a disc which allows us to see the Bruhat-Tits tree as the "skeleton" of the Drinfeld upper half-plane. While we do not do this here, it allows one to endow the Drinfeld upper half plane with the structure of an adic space (be it a rigid analytic variety or Berkovich space) by glueing these discs together (see for instance [GR96, Section 1.2]).

Proposition 3.1.16 ([GR96, Prop 1.5.3]) *The building map is $\mathrm{GL}_2(K_\infty)$-equivariant.*

Remark 3.1.17 The building map allows us to connect (rigid-)analytic geometry (in the form of the Drinfeld symmetric space) and combinatorial geometry (in the form of the Bruhat-Tits tree).

[1] Goldman-Iwahori in fact showed that for $\mathrm{GL}_n(k)$, the points on $\mathcal{B}(\mathrm{GL}_n)(\mathbb{R})$ correspond to non-archimedean norms on k^n.

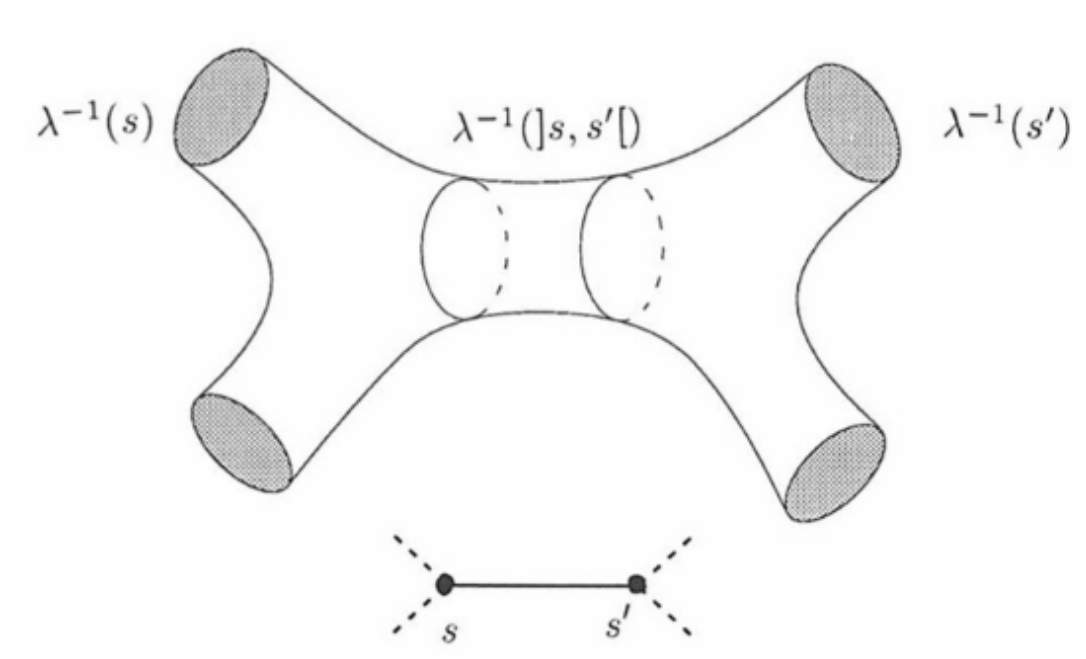


Figure III.2: Comparison of the Drinfeld upper half plane and Bruhat-Tits tree (image taken from [Mar])

3.2 Harmonic cochains

Definition 3.2.1 *Let G be an oriented graph. For any $e \in \mathrm{Ed}(G)$, we denote by $\bar{e}$ the edge with the reverse orientation.*

Definition 3.2.2 *Let G be an oriented graph acted upon by the group Γ. We say that Γ* acts with inversion *if there exists an edge $e \in \mathrm{Ed}(G)$ and $g \in \Gamma$ such that $ge = \bar{e}$.*

Proposition 3.2.3 *Let G be an oriented graph acted upon by Γ. If Γ acts without inversion, then the quotient graph exists and*

$$\mathrm{Ed}(G/\Gamma) = \mathrm{Ed}(G)/\Gamma \quad \mathrm{Ver}(G/\Gamma) = \mathrm{Ver}(G)/\Gamma.$$

Definition 3.2.4 *For any oriented edge $e \in \mathrm{Ed}(\mathcal{G})$, we denote by $t(e)$ the* terminus of the edge e.

Example 3.2.5 The vertices $\Lambda_n := \begin{pmatrix} t^n & 0 \\ 0 & 1 \end{pmatrix}$ of the Bruhat-Tits building form a fundamental domain for the action of $\mathrm{GL}_2(\mathbb{F}_q[t])$. For all n there is an edge $\Lambda_n\Lambda_{n+1}$ and the stabilizer of Λ_n acts transitively on the edges of Λ_n.

Definition 3.2.6 *For any ideal $\mathfrak{D}, \mathfrak{m}$, let $\mathcal{O}$ be an Eichler order of level $\mathfrak{m}$ of in "the" indefinite quaternionic algebra D over $\mathbb{F}_p(t)$ with discriminant $\mathfrak{D}$. Denote by $\Gamma_0(\mathfrak{D}, \mathfrak{m})$ the image of*

$$\mathcal{O}^\times \to (D \otimes \mathbb{F}_p((1/t)))^\times \cong \mathrm{GL}_2(\mathbb{F}_p((1/t))) \to \mathrm{PGL}_2(\mathbb{F}_p((1/t))).$$

Example 3.2.7 The groups $\Gamma_0(\mathfrak{D}, \mathfrak{m})$ act on the Bruhat-Tits tree $\mathcal{T}$ without inversion. (see for instance [Pap12])

Definition 3.2.8 (Harmonic cochain) *Let G be an oriented graph. For any commutative ring Λ, a Λ-*valued harmonic cochain on *G is a function*

$$f : \mathrm{Ed}(G) \to \Lambda$$

such that for all $e \in \mathrm{Ed}(G)$ we have

$$f(e) + f(\bar{e}) = 0$$

and for all $v \in \mathrm{Ver}(G)$

$$\sum_{e \in \mathrm{Ed}(G), t(e)=v} f(e) = 0.$$

We denote the Λ*-module of* Λ*-valued harmonic cochains by* $\mathcal{H}(G, \Lambda)$*. If a group* Γ *is acting on* G*, then we denote the* Γ*-equivariant cochains by* $\mathcal{H}(G, \Lambda)^{\Gamma}$*.*

Proposition 3.2.9 (Functoriality in coefficients) *Let* G *be an oriented graph. Then*

$$\mathcal{H}(G, -) : \mathrm{CAlg} \to \mathrm{Ab}, \quad \Lambda \mapsto \mathcal{H}(G, \Lambda)$$

is a functor.

One difference between the Drinfeld upper-half plane and the complex upper-half plane is that it has non-trivial covers, and thus has non-trivial (étale) cohomology.

Proposition 3.2.10 (Drinfeld [Dri74]) *For* $\ell \neq \mathrm{char}(K_\infty)$*, there exists a* $\mathrm{GL}_2(K_\infty)$*-equivariant isomorphism*

$$H^1_{\text{ét}}(\Omega_{\mathbb{C}_\infty}, \mathbb{Z}/\ell^n) \cong \mathcal{H}(\mathcal{T}, \mathbb{Z}/\ell^n).$$

Remark 3.2.11 This was later generalized by work of Schneider-Stuhler [SS91], de Shalit [DS01] and Amrane [AA06], who showed that the correct generalization of harmonic cochains gives a combinatorial description of the cohomology of Drinfeld symmetric spaces

$$\mathcal{H}^\bullet(\mathcal{T}, \mathbb{Z}/\ell^n) \cong H^\bullet(\Omega^d_{\mathbb{C}_\infty}, \mathbb{Z}/\ell^n)$$

$$\mathcal{H}^\bullet(\mathcal{T}, \mathbb{C}_p) \cong H^\bullet_{dR}(\Omega^d_{\mathbb{C}_p})$$

if $\ell \neq p$. Furthermore, one can identify these cohomologies with the duals of Steinberg representations. This was extended to the $\ell = p$ case by Colmez-Dospinescu-Nizioł first rationally and then integrally in [CDN20] and [CDN21] respectively.

Proposition-Definition 3.2.12 (Weight of edge) *For any edge* e*, the* weight of e *is defined to be*

$$w(e) := [\Gamma_{t(e)} : \Gamma_e]$$

This only depends on its image in $\Gamma \backslash \mathcal{T}$*, hence we can define* $w(\tilde{e}) := w(e)$*.*

Proposition 3.2.13 *For any subgroup $\Gamma \subset \mathrm{GL}_2(F_\infty)$ acting on $\mathcal{T}$ without inversion, a function f on the quotient graph $\Gamma\backslash\mathcal{T}$ comes from a Γ-invariant harmonic cochain if*

$$f(e) + f(\overline{e}) = 0$$

for all $e \in \mathrm{Ed}(\Gamma\backslash\mathcal{T})$ and for all $v \in \mathrm{Ver}(\Gamma\backslash\mathcal{T})$

$$\sum_{e \in \mathrm{Ed}(\Gamma\backslash\mathcal{T}), t(e)=v} w(e)f(e) = 0.$$

Definition 3.2.14 (Cuspidal harmonic cochain) *Let G be a graph with a group Γ acting on it. An Λ-valued Γ-equivariant harmonic cochain f is said to be* cuspidal *if it has compact support as a function on the quotient graph G/Γ. We denote the set of all cuspidal harmonic cochains by $\mathcal{H}_!(G, \Lambda)^\Gamma$.*

Remark 3.2.15 One can show that for any arithmetic subgroup Γ, the quotient graph $\Gamma\backslash\mathcal{T}$ is the union of a finite graph and a finite number of half-lines (see [Ser03, Thm 9 a)]). We refer to the half-lines as the *cusps*.

Example 3.2.16 (No level 1 forms [PW15, Ex 2.4]) The quotient $\mathrm{GL}_2(A)\backslash\mathcal{T}$ is given by vertices corresponding to $\begin{pmatrix} T^n & 0 \\ 0 & 1 \end{pmatrix}$ for all $n \geqslant 0$, which we denote by v_n. The edge with origin v_{i-1} and terminus v_i we denote by e_i. Hence the quotient is

$$v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \dots$$

The weights are

$$w(e_i) = q \quad w(\overline{e_1}) = q + 1 \quad w(\overline{e_{i>1}}) = 1.$$

The harmonicity condition at the vertex v_0 implies that

$$0 = w(e_1)f(\overline{e_1}) = -(q+1)f(e_1).$$

Then at any other vertex v_n we have

$$w(e_n)f(e_n) = w(\overline{e_{n+1}})f(e_{n+1}) = f(e_{n+1}).$$

Hence f is uniquely determined by $f(e_1)$ and thus for any Λ, we have

$$\mathcal{H}(1,\Lambda) = \Lambda[q+1] \quad \mathcal{H}_!(1,\Lambda) = 0$$

where $\Lambda[q+1]$ denotes the $(q+1)$-torsion subgroup of Λ.

Proposition-Definition 3.2.17 (Hecke operator) *For any ideal $\mathfrak{m}$ the $\mathfrak{m}$-th Hecke operator $T_\mathfrak{m}$ on $\mathcal{H}(\mathcal{T},\Lambda)^{\Gamma_0(\mathfrak{D},\mathfrak{n})}$ is*

$$T_\mathfrak{m} : \mathcal{H}(\mathcal{T},\Lambda)^{\Gamma_0(\mathfrak{D},\mathfrak{n})} \to \mathcal{H}(\mathcal{T},\Lambda)^{\Gamma_0(\mathfrak{D},\mathfrak{n})}, \quad f \mapsto \sum f| \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

where a, d are monic such that $ad = \mathfrak{m}$, a is coprime to $\mathfrak{n}$ and $\deg(b) < \deg(d)$. Furthermore these operators preserve cuspidal harmonic cochains. Furthermore, they satisfy the following relations

$$T_{\mathfrak{m}\mathfrak{m}'} = T_\mathfrak{m} T_{\mathfrak{m}'} \text{ for coprime } \mathfrak{m}, \mathfrak{m}'$$

$$T_{\mathfrak{p}^n} = (T_\mathfrak{p})^n \text{ for any } \mathfrak{p}|\mathfrak{n}$$

$$T_{\mathfrak{p}^n} = T_{\mathfrak{p}^{n-1}} T_\mathfrak{p} - |\mathfrak{p}| T_{\mathfrak{p}^{i-2}} \text{ for any } \mathfrak{p} \not| \mathfrak{n}.$$

Definition 3.2.18 (Petersson inner product) *The vector space $\mathcal{H}_!(\mathcal{T},\mathbb{Q})^{\Gamma_0(\mathfrak{D},\mathfrak{n})}$ has a natural inner product*

$$\langle \cdot, \cdot \rangle : \mathcal{H}_!(\mathcal{T},\mathbb{Q})^{\Gamma_0(\mathfrak{D},\mathfrak{n})} \times \mathcal{H}_!(\mathcal{T},\mathbb{Q})^{\Gamma_0(\mathfrak{D},\mathfrak{n})} \to \mathbb{Q}$$

$$(f,g) \mapsto \sum_{e \in \mathrm{Ed}(\Gamma_0(\mathfrak{D},\mathfrak{n})\backslash\mathcal{T})} \frac{1}{|\Gamma_0(\mathfrak{D},\mathfrak{n})_e/\mathbb{F}_q^\times|} f(e)g(e).$$

In the literature, one often finds the following notation

$$n(e) := |\Gamma_0(\mathfrak{D},\mathfrak{n})_e/\mathbb{F}_q^\times|.$$

Proposition 3.2.19 *For all $\mathfrak{m}$ coprime to $\mathfrak{n}$, the Hecke operators $T_\mathfrak{m}$ are self-adjoint.*

Remark 3.2.20 For any $\mathfrak{m} \mid \mathfrak{n}$, the inclusion $\Gamma_0(\mathfrak{D},\mathfrak{m}) \supset \Gamma_0(\mathfrak{D},\mathfrak{n})$ induces a natural inclusion

$$\mathcal{H}_!(\mathcal{T},\mathbb{Z})^{\Gamma_0(\mathfrak{D},\mathfrak{m})} \to \mathcal{H}_!(\mathcal{T},\mathbb{Z})^{\Gamma_0(\mathfrak{D},\mathfrak{n})}.$$

Definition 3.2.21 *The* $\mathfrak{p}$-old subspace $\mathcal{H}(\mathcal{T},\mathbb{Q})^{\Gamma_0(\mathfrak{D},\mathfrak{p}\mathfrak{m}),\mathfrak{p}\text{-}old}$ *of $\mathcal{H}(\mathcal{T},\mathbb{Q})^{\Gamma_0(\mathfrak{D},\mathfrak{p}\mathfrak{m})}$ is the subspace generated by the image of the natural injections $\mathcal{H}(\mathcal{T},\mathbb{Q})^{\Gamma_0(\mathfrak{D},\mathfrak{m})} \to \mathcal{H}(\mathcal{T},\mathbb{Q})^{\Gamma_0(\mathfrak{D},\mathfrak{p}\mathfrak{m})}$. The integral*

$\mathfrak{p}$-*old subspace is*

$$\mathcal{H}(\mathcal{T},\mathbb{Z})^{\Gamma_0(\mathfrak{D},\mathfrak{pm}),\mathfrak{p}\text{-}old} := \mathcal{H}(\mathcal{T},\mathbb{Z})^{\Gamma_0(\mathfrak{D},\mathfrak{pm})} \cap \mathcal{H}(\mathcal{T},\mathbb{Q})^{\Gamma_0(\mathfrak{D},\mathfrak{pm}),\mathfrak{p}\text{-}old}.$$

The $\mathfrak{p}$-newspace *is the orthogonal complement of* $\mathcal{H}(\mathcal{T},\mathbb{Z})^{\Gamma_0(\mathfrak{D},\mathfrak{pm}),\mathfrak{p}\text{-}old}$ *in* $\mathcal{H}(\mathcal{T},\mathbb{Z})^{\Gamma_0(\mathfrak{D},\mathfrak{pm})}$.

Theorem 3.2.22 ([vdP83, Prop 1.1]) *Let* $\mathcal{O}(\Omega)$ *denote the ring of rigid holomorphic functions on* Ω*. There exists a* $\mathrm{GL}_2(K_\infty)$*-equivariant short exact sequence*

$$0 \to K^\times \to \mathcal{O}(\Omega)^\times \xrightarrow{r} \mathcal{H}(\mathcal{T},\mathbb{Z}) \to 0.$$

The morphism $r : \mathcal{O}(\Omega)^\times \to \mathcal{H}(\mathcal{T},\mathbb{Z})$ is often referred to as the *van der Put map*. We can now give some first examples of harmonic cochains, using the van der Put map. For this we need to give non-vanishing functions on the Drinfeld upper half-plane. These will eventually give rise to the function field analogues of Eisenstein series.

Definition 3.2.23 ([GR96, (2.9.2)]) *For any* $k > 0$*, the* Eisenstein series *is*

$$E^k(-) : \Omega \to \mathbb{C}_\infty, \quad z \mapsto \Sigma_{a,b\in\mathbb{F}_q[t]} \frac{1}{(az+b)^k}.$$

Definition 3.2.24 (Drinfeld discriminant function) *The* Drinfeld discriminant function *is the function*

$$\Delta : \Omega \to \mathbb{C}_\infty, \quad z \mapsto (T^{q^2}-T)E^{q^2-1}(z) + (T^{q^2}-T^q)E^{(q-1)^{q+1}}(z).$$

This is a holomorphic and non-vanishing function on Ω*. For any ideal* $\mathfrak{n} = (n)$*, the function* $\Delta_\mathfrak{n}(z) := \Delta(nz)$ *is independent of the choice of generator.*

Remark 3.2.25 Note that the functions E^k, Δ behave similarly to their number field analogues, which are in particular "classical" modular forms. These function field valued functions are in fact examples of *Drinfeld modular forms*, see Section 3.5.

Remark 3.2.26 As the name indicates, this function mimics the behaviour of the discriminant function in the theory of modular forms. In particular, using the van der Put map, which behaves similarly to the logarithmic derivative, we can construct Eisenstein harmonic cochains.

Definition 3.2.27 (Eisenstein harmonic cochains) *For any ideal* $\mathfrak{n}$ *the function*

$$\frac{\Delta}{\Delta_\mathfrak{n}} : \Omega \to \mathbb{C}_\infty, \quad \Lambda \mapsto \frac{\Delta(z)}{\Delta_\mathfrak{n}(z)}$$

is non-vanishing. Denote its image under the van-der Put map (see Thm 3.2.22) by $E_{\mathfrak{n}} := r\left(\frac{\Delta}{\Delta_{\mathfrak{n}}}\right)$.

Then similarly to the number field case, harmonic cochains of prime level are the direct sums of cuspidal harmonic cochains and Eisenstein cochains.

Proposition 3.2.28 ([Pál07, Theorem 3.12]) *We have a natural decomposition*

$$\mathcal{H}(\mathcal{T},\mathbb{C})^{\Gamma_0(\mathfrak{p})} = \mathcal{H}_!(\mathbb{T},\mathbb{C})^{\Gamma_0(\mathfrak{p})} \oplus \mathbb{C}E_{\mathfrak{p}}.$$

3.3 Relating harmonic cochains to automorphic representations

Let D be a division algebra over K which is split at infinity and consider an order $\mathcal{O} \subset D$. The splitting at infinity gives an isomorphism $D_\infty \cong \mathrm{Mat}_2(K_\infty)$. Via this isomorphism we have $D_\infty^\times \cong \mathrm{GL}_2(K_\infty)$, which allows us to identify $\mathcal{O}^\times$ with a subgroup of $\mathrm{GL}_2(K_\infty)$, whose image in $\mathrm{PGL}_2(K_\infty)$ we denote by Γ.

Definition 3.3.1 *Let $\mathcal{K}$ be an open compact subgroup of $D^\times(\mathbb{A})$. A* automorphic form on $D^\times$ of level $\mathcal{K}$ *is a $\mathbb{C}$-valued function*

$$D^\times(F)\backslash D^\times(\mathbb{A})/(Z(K_\infty)\mathcal{K}) \longrightarrow \mathbb{C}.$$

We denote the $\mathbb{C}$-vector space of these by $S(\mathcal{K})$.

Example 3.3.2 A way to generate open compact subgroups of $D^\times(\mathbb{A})$ is to take an open compact subgroup of $\mathcal{K}_\infty \subset \mathrm{GL}_2(\mathcal{O}_\infty)$ and to consider

$$\mathcal{K} := \prod_{\mathfrak{p}\in\mathrm{Spec}\,A} \mathcal{D}_{\mathfrak{p}}^\times \times \mathcal{K}_\infty.$$

Definition 3.3.3 *The space*

$$S(\mathcal{D}^\times) := \bigcup_{\mathcal{K}_\infty} S\left(\prod_{\mathfrak{p}\in\mathrm{Spec}\,A} \mathcal{D}_{\mathfrak{p}}^\times \times \mathcal{K}_\infty\right)$$

(where $\mathcal{K}_\infty$ ranges over all open compact subgroups of $\mathrm{GL}_2(K_\infty)$) carries a natural action of $\mathrm{GL}_2(K_\infty)$: for any $f \in S(\prod_{\mathfrak{p}\in\mathrm{Spec}\,A} \mathcal{D}_{\mathfrak{p}}^\times \times \mathcal{K}_\infty)$ any $g \in \mathrm{GL}_2(K_\infty)$ the function

$$x \mapsto f(xg)$$

lies in $S\left(g\mathcal{K}_\infty g^{-1} \cap \mathrm{GL}_2(\mathcal{O}_\infty)\right)$*, and as* $g\mathcal{K}_\infty g^{-1} \cap \mathrm{GL}_2(\mathcal{O}_\infty)$ *is open compact, this next function lies in* $S(\mathcal{D}^\times)$*.*

Definition 3.3.4 (Special representation) *Denote by* V_{sp} *the space of locally constant functions modulo constants on* $\mathbb{P}^1(K_\infty)$*. Note that this carries a natural action by* $\mathrm{GL}_2(K_\infty)$ *by acting on the space.*

Proposition 3.3.5 (Drinfeld, [Dri74], [GR96]) *We have an isomorphism*

$$\mathcal{H}(\mathcal{T}, \mathbb{C})^\Gamma \cong \mathrm{Hom}_{\mathrm{GL}_2(K_\infty)}(V_{\mathrm{sp}}, S(\mathcal{D}^\times)).$$

This isomorphism comes from the bijection (provided by the Strong Approximation theorem)

$$D^\times(K)\backslash D^\times(\mathbb{A})/\prod_{\mathfrak{p}\in\mathrm{Spec}\,A} \mathcal{D}_\mathfrak{p}^\times \times \mathcal{K}_\infty \mathcal{I}_\infty Z(K_\infty) \cong \mathcal{D}^\times \backslash \mathrm{GL}_2(K_\infty)/\mathcal{I}_\infty Z(K_\infty) \cong \mathrm{Ed}(\Gamma\backslash\mathcal{T}).$$

UPSHOT: harmonic cochains correspond to subspaces of $S(\mathcal{D}^\times)$ which are "special" at infinity.

3.4 Making harmonic cochains explicit for computations

A feature of harmonic cochains is that they have very explicit descriptions, allowing for "hands-on" computations. In this section, we review these descriptions, which will be useful both to prove results and compute examples (for instance in Section 6.9).

Definition 3.4.1 *For any group* Γ*, denote by* $\overline{\Gamma} := \Gamma^{\mathrm{ab}}/\mathrm{tor}(\Gamma^{\mathrm{ab}})$.

Definition 3.4.2 *A* geodesic path *in a tree is a non-back tracking path.*

Proposition 3.4.3 *Since the Bruhat-Tits building* $\mathcal{T}$ *is a tree, there exists a unique geodesic between any two vertices.*

Definition 3.4.4 ([GR96, (3.3)]) *For any vertices* $v, w \in \mathrm{Ver}(\mathcal{T})$*, let* $c(v,w) \in \mathrm{Ed}(\mathcal{T})$ *be the unique geodesic connecting them. Fix a base vertex* $v_0 \in \mathrm{Ver}(\mathcal{T})$*. For every edge* $e \in \mathrm{Ed}(\mathcal{T})$ *and any* $\alpha, \beta \in \Gamma$*, define*

$$i(e,\alpha,\beta,v_0) = \begin{cases} 1 & \text{, if } \beta(e) \text{belongs to } c(v_0, \alpha v_0) \\ -1 & \text{, if } \beta(e) \text{belongs to } c(\alpha v_0, v_0) \\ 0 & \text{, otherwise} \end{cases}.$$

Then set

$$\varphi_{\alpha,v_0}(e) := \frac{1}{|\Gamma \cap Z(K)|} \sum_{\beta \in \Gamma} i(e, \alpha, \beta, v_0).$$

Proposition 3.4.5 *The integer φ_{α,v_0} is independent of the choice of v_0 and defines a group homomorphism*

$$j : \overline{\Gamma} \to \mathcal{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma, \quad \alpha \mapsto [e \mapsto \varphi_\alpha(e)].$$

Proof. Note that since $i(\gamma e, \alpha, \beta, v_0) = i(e, \alpha, \beta\gamma, v_0)$ we have $\varphi_{\alpha,\gamma e} = \varphi_{\alpha,e}$ and therefore $\varphi_{\alpha,v_0}(-)$ lies in $\mathcal{H}(\mathcal{T}, \mathbb{Z})^\Gamma$. Furthermore, by the definition of $i(-)$, the support of φ_{α,v_0} lies in the geodesic $c(v_0, \alpha v_0)$ and thus φ_{α,v_0} lies in $\mathcal{H}_!(\mathcal{T}, \mathbb{Z})$. □

Proposition 3.4.6 ([GR96, Section 6.4.5]) *The homomorphism $j : \overline{\Gamma} \to \mathcal{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma$ is injective and has finite cokernel whose order is not divisible by p. It is an isomorphism in at least two cases:*

1. *if $A = \mathbb{F}_p[t]$*
2. *if for all primes $\ell \neq p$ the quotient $\Gamma/(\Gamma \cap Z(K))$ is ℓ-torsion free*[2].

The group $\bar{\Gamma}$ can also be seen as the (first) homology group of the quotient $\Gamma\backslash\mathcal{T}$. This utilizes the following description of fundamental groups.

Proposition 3.4.7 ([Ser03, I, Thm 13, Cor 1]) *Let G be a group acting without inversion on a connected, nonempty tree T. Denote by N the smallest normal subgroup generated by the stabilizers of the vertices. Then the fundamental group of the quotient $G\backslash T$ is isomorphic to G/N.*

Proposition 3.4.8 (see for instance [GR96, p.49]) *The fundamental group of $\Gamma\backslash\mathcal{T}$ is isomorphic to Γ/Γ_f, where Γ_f is the (normal) subgroup generated by elements of finite order. Hence the fundamental groups of $\Gamma\backslash\mathcal{T}$ are free. This induces (via the Hurewvitz map) an isomorphism*

$$(\Gamma/\Gamma_f)^{\mathrm{ab}} \xrightarrow{\sim} H_1(\Gamma\backslash\mathcal{T}, \mathbb{Z}).$$

Furthermore the projection morphism induces an isomorphism

$$\bar{\Gamma} \xrightarrow{\sim} (\Gamma/\Gamma_f)^{\mathrm{ab}}.$$

[2] in [GR96] this condition is called *being p'-torsion free*

Proof. This is now a direct consequence of Prop 3.4.7. □

UPSHOT: For any arithmetic subgroup Γ, we have isomorphisms

$$H_1(\mathcal{T}/\Gamma, \mathbb{Z}) \xleftarrow{\sim} \overline{\Gamma} \xrightarrow{\sim} \mathcal{H}_!(\mathcal{T}, \mathbb{Z})^{\Gamma}.$$

In particular this shows a connection between between an arithmetic object $\mathcal{H}_!(\mathcal{T}, \mathbb{Z})^{\Gamma}$ which is linked with the geometry of modular curves to a group theoretic object in the form of $\overline{\Gamma}$.

Remark 3.4.9 In [BB12, Algorithm 6.5.], an algorithm is given which computes an explicit presentation of the quotient $\Gamma\backslash\mathcal{T}$. In particular this algorithm terminates if the quotient graph is finite (see [BB12, Thm 6.8]). As a consequence of the Strong Approximation Theorem, this happens if Γ is the unit group of a maximal order of a split quaternion algebra over $\mathbb{F}_q[t]$[3]. In that case, the complexity of the algorithm is $O\left(q^{2\deg(\mathrm{disc}D)-6}\deg(\mathrm{disc}D)^5\right)$ (see [BB12, Prop 9.7]).

Proposition 3.4.10 ([Pap12, Lemma 5.1]) *Let D be a quaternion algebra split at ∞ and $\mathcal{O}$ a maximal A-order. Then the quotient graph $\mathcal{O}^\times\backslash\mathcal{T}$ is finite.*

3.5 Drinfeld modular forms

In this section, we study an analogue of Modular forms, which are internal to function fields, in that these are functions on the Drinfeld Upper half plane valued in $\mathbb{C}_\infty$. These are arise as differential forms on the corresponding quotient of the Drinfeld Upper half plane. As we shall see in Rem 3.5.14 the associated Galois representations are characters, not GL_2-representations. Nevertheless, they may be viewed as the "reduction modulo p" of harmonic cochains and are useful in studying the geometry of moduli spaces of Drinfeld modules.

Definition 3.5.1 *For any lattice $\Lambda \subset K^2$ and any ideal $\mathfrak{n} \subset A$, denote by*

$$\Gamma(\Lambda) := \{g \in \mathrm{GL}_2(K) \mid g\Lambda = \Lambda\}$$

$$\Gamma(\Lambda, \mathfrak{n}) := \ker\left(\mathrm{GL}(\Lambda) \to \mathrm{GL}(\Lambda/\mathfrak{n})\right)$$

Definition 3.5.2 (Arithmetic subgroup) *A subgroup Γ of $\mathrm{GL}_2(K)$ is said to be* arithmetic *if it is an intermediate group*

$$\Gamma(\Lambda, \mathfrak{n}) \subset \Gamma \subset \Gamma(\Lambda)$$

for some $\mathfrak{n}$ and A-lattice Λ (i.e. a projective rank two submodule of K^2).

[3] this is reminiscent to the fact that in the number field case, the quotient $\mathbb{H}/\Gamma$ is compact, and equivalently that the corresponding Shimura curve is proper.

Definition 3.5.3 (Drinfeld modular forms) *Let Γ be an arithmetic subgroup of $\mathrm{GL}_2(K)$. A* Drinfeld modular form for Γ of weight k and type m *is a function*

$$f : \Omega \to \mathbb{C}_\infty$$

such that

1. $f(\gamma z) = (\det \gamma)^{-m}(cz + d)^k f(z)$
2. f *is rigid holomorphic*
3. f *is holomorphic at the cusps*

We denote by $M_{k,m}(\Gamma)$ the space of Drinfeld modular forms of weight k and type m.

Remark 3.5.4 What does rigid holomorphic mean? A function $f : \Omega \to \mathbb{C}_\infty$ is rigid holomorphic if for each point in $\mathbb{C}_\infty$ we can write it as a formal power series in some sufficiently small rigid analytic disc around it (such that it is absolutely and uniformly convergent).
What does holomorphic at the cusps mean? The analogue of the exponential function is

$$e_A(z) = z \prod_{0 \neq a \in A} \left(1 - \frac{z}{a}\right)$$

and its logarithmic derivative is

$$u(z) := \frac{e_A(z)'}{e_A(z)} = \sum_{a \in A} \frac{1}{z + a}.$$

Then for any $z \in \Omega$ with sufficiently large "imaginary" part (i.e. $|z|_i := \min_{x \in F_\infty} |z - a|$) we may write $f(z) = \sum_{n \in \mathbb{Z}} a_n u(z)^n$.

Definition 3.5.5 *We may say that $f = \sum_{n \in \mathbb{Z}} a_n u(z)^n$ is* holomorphic at the cusps *if $a_n = 0$ for $n < 0$. We say that a modular form is n-*th cuspidal *if $a_k = 0$ for all $k \leqslant n$. A 0-cuspidal modular form is said to be* cuspidal. *We denote the space of n-th cuspidal Drinfeld modular forms of weight k, type m and level Γ by $M^n_{k,m}$.*

Remark 3.5.6 Apriori the type of a Drinfeld modular form is an integer. However it only matters modulo the order of the image of the determinant $|\det(\Gamma)| \mid q - 1$, i.e. we have

$$M_{k,m}(\Gamma) = M_{k,m+|\det(\Gamma)|}(\Gamma).$$

Remark 3.5.7 While we have used a "$\mathbb{Z}$-expansion" of a rigid holomorphic functions to define cuspidality, there exists the notion of an "A_+-expansion", that is a decomposition of a function indexed by irreducible polynomials. We briefly review this in Subsection 7.4.

Example 3.5.8 ([GR96, (2.9)]) As mentioned in Rem 3.2.25, a family of examples of Drinfeld modular forms are the *Eisenstein series*: Consider a lattice $\Lambda \subset K^2$. For any $z \in \Omega = \mathbb{C}_\infty \backslash K$, consider the map

$$\iota_z : K^2 \to \mathbb{C}_\infty, \quad (1,0) \mapsto z, (0,1) \mapsto 1.$$

Then the formal sum

$$E^k(z) := \sum_{0 \neq x \in \Lambda} \iota_z(x)^{-k}$$

is called the *Eisenstein series* for $k \notin (q-1)$ and is an element of $M_{k,0}(\Gamma)$.

We now want to understand the relation between Drinfeld modular forms of level Γ and differentials on a curve associated to Γ. For classical modular forms, one can prove that modular forms of level Γ correspond to differential forms on a compactification of $\mathbb{H}/\Gamma$ and that this quotient is a smooth projective curve defined over $\mathbb{C}$. We can now study the quotient Ω/Γ (or rather a compactification $\overline{\Omega/\Gamma}$ thereof) and we shall later see that this is a smooth proper curve $X(\Gamma)$ defined over $\mathbb{C}_\infty$ whose analytification is isomorphic to $\overline{\Omega/\Gamma}$ (see for instance [Dri74, Section 6 (2.2.1)]) We postpone this to chapter IV.

Proposition 3.5.9 ([GR96, Prop 2.10.2]) *We have natural isomorphisms*

$$M^2_{2,1}(\Gamma) \to H^0(X(\Gamma), \Omega^1)$$

$$M^1_{2,1}(\Gamma) \to H^0(X(\Gamma), \Omega^1(\textit{cusp}))$$

which are given by $f \mapsto f(z)dz$.

Remark 3.5.10 ([GR96, Section 2.10]) This might seem a bit surprising at first, as (globally holomorphic) differential forms now only see double cuspidal modular forms. This comes from the fact that $e_A(z)$ has constant derivative 1, and thus $dz = -e_A(z)^2 d\left(\frac{1}{e_A(z)}\right)$.

Definition 3.5.11 *Denote by $\mathcal{H}_{!!}(\mathcal{T}, \mathbb{F}_p)^\Gamma$ the image of the reduction modulo p, i.e.*

$$\mathcal{H}_{!!}(\mathcal{T}, \mathbb{Z})^\Gamma := \text{im}\left[\mathcal{H}_!(\mathcal{T}, \mathbb{Z})^\Gamma \to \mathcal{H}_!(\mathcal{T}, \mathbb{F}_p)^\Gamma\right].$$

Next we shall define the *residue of a differential form* at the *edge* of the Bruhat-Tits tree.

Proposition-Definition 3.5.12 ([GR96, (1.8)]) *Let $e \in \mathrm{Ed}(\mathcal{T})$ and $\omega \in H^0(X(\Gamma), \Omega)$. Choose $x \in \Omega$ which reduces to the origin of the edge e and a sufficiently integer n such that the inner circle of the annulus*

$$D^0_{x,n} := \left\{ z \in \Omega \mid \frac{1}{n+1} <\mid z - x \mid< \frac{1}{n} \right\}$$

reduces to the origin of e. Then the restriction of ω to $D^0_{x,n}$ admits an expansion

$$\omega = \sum_{k \in \mathbb{Z}} a_k \left((z-x)\pi_\infty^n \right)^k$$

and we define the residue of ω at e *to be*

$$\mathrm{res}(\omega, e) := a_{-1}.$$

Proposition-Definition 3.5.13 ([Tei91],[GR96, Thm 6.5.3]) *For any $v \in \mathrm{Ver}(\mathcal{T})$ and $\omega \in H^0(X(\Gamma), \Omega)$, the residue theorem implies that*

$$\sum_{e \in \mathrm{Ed}(\mathcal{T}), o(e)=v} \mathrm{res}(\omega, e) = 0.$$

In particular the residue of a differential form on $X(\Gamma)$ yields a morphism

$$H^0(X(\Gamma), \Omega) \to \mathcal{H}(\mathcal{T}, \mathbb{C}_\infty), \quad \omega \mapsto [e \to \mathrm{res}(\omega, e)].$$

For any arithmetic subgroup Γ this morphism yields the following isomorphisms

$$M^1_{2,1}(\Gamma) \cong \mathcal{H}_!(\mathcal{T}, \mathbb{C}_\infty)^\Gamma.$$

Furthermore, if we denote by $M^2_{2,1}(\Gamma, \mathbb{F}_p)$ those Drinfeld modular forms which at all edges haves residues contained in $\mathbb{F}_p$, then we have

$$M^2_{2,1}(\Gamma, \mathbb{F}_p) \cong \mathcal{H}_{!!}(\mathcal{T}, \mathbb{F}_p)^\Gamma.$$

UPSHOT: (double cuspidal, weight two and type one, with $\mathbb{F}_p$-residues) Drinfeld modular forms are the mod p reductions of automorphic forms.

For any $f \in \mathcal{O}(\Omega)^\times$ and $e \in \mathrm{Ed}(\mathcal{T})$ we have the following equality

$$\mathrm{res}\left(\frac{df}{f}, e\right) = r(f)(e) \in \mathbb{F}_p$$

(see for instance [GR96, 1.8.6]). This equivalent to the following diagram commuting (where the right vertical arrows are induced by functoriality)

$$\begin{array}{ccc} \mathcal{O}(\Omega)^\times & \xrightarrow{\qquad r \qquad} & \mathcal{H}(\mathcal{T}, \mathbb{Z}) \\ \Big\downarrow{\scriptstyle \mathrm{dlog}} & & \Big\downarrow \\ & & \mathcal{H}(\mathcal{T}, \mathbb{F}_p) \\ & & \Big\downarrow \\ \{\text{holomorphic differentials on } \Omega\} & \xrightarrow{\mathrm{res}} & \mathcal{H}(\mathcal{T}, \mathbb{C}_\infty). \end{array}$$

Remark 3.5.14 Analogously to classical modular forms and indeed harmonic cochains (see Prop 6.5.11), given a cuspidal eigenform f (with $T_\mathfrak{p} f = \lambda_\mathfrak{p} f$), there exists a Galois representation

$$\rho_f : \mathrm{Gal}(\bar{F}/F) \to \mathbb{C}_\infty^\times$$

such that for all but finitely many $\mathfrak{p}$ we have $\rho_f(\mathrm{Frob}_\mathfrak{p}) = \lambda_\mathfrak{p}$ (see for instance Prop 8.5 [Böc15]). Notice the discrepancy to what we might expect: these are characters, that is one-dimensional representations, rather then GL_2-representations! While the general construction is more involved, the special case when f is of weight two and type one explains why this happens: In this case, there exits an analogue of the Eichler-Shimura isomorphism

$$J_0(\mathfrak{n})[p](F^{\mathrm{sep}}) \otimes_{\mathbb{F}_p} \mathbb{C}_\infty \cong \mathcal{H}_!(\Gamma_0(\mathfrak{n}), \mathbb{Z}) \otimes \mathbb{C}_\infty.$$

The Hecke eigenform f now spans a subspace of $J_0(\mathfrak{n})[p](F^{\mathrm{sep}}) \otimes_{\mathbb{F}_p} \mathbb{C}_\infty$, which defines the Galois representation ρ_f. However if $p \neq \ell$, we have

$$\dim_{\mathbb{F}_p} J_0(\mathfrak{n})[p] = \frac{1}{2} \dim_{\mathbb{F}_\ell} J_0(\mathfrak{n})[\ell]$$

as the p-torsion of schemes in characteristic p is typically poorly behaved.

3.6 Theta series

Definition 3.6.1 (Theta function [GR96, (5.1)]) *Let Γ be an arithmetic subgroup. A* holomorphic/meromorphic theta function for Γ *is a non-vanishing holomorphic/meromorphic function $\Omega \to \mathbb{C}_\infty$ having neither zeros nor poles at the cusps such that for all $\alpha \in \Gamma$, there exists a function $c_u : \Gamma \to \mathbb{C}_\infty^\times$ we have*

$$u(az) = c_u(a)u(z).$$

Denote by $\Theta_h(\Gamma) \subset \Theta_m(\Gamma)$ the respective groups of holomorphic and meromorphic theta functions.

Example 3.6.2 ([GR96, (5.2)]) Let $a, b \in \Omega$. Then the product

$$\Theta_\Gamma(a, b, z) := \prod_{\gamma \in \Gamma/(\Gamma \cap Z(K))} \frac{z - \gamma a}{z - \gamma b}$$

is a meromorphic theta function for Γ.

Proposition 3.6.3 (Logarithmic derivative is modular form) *The logarithmic derivative of a theta function is a Drinfeld modular form, i.e. $\frac{u'(z)}{u(z)} \in M^2_{2,1}(\Gamma)$.*

Proposition-Definition 3.6.4 ([GR96, Thm 5.4.1]) *The holomorphic function*

$$z \mapsto u_\alpha(z) := \Theta(\omega, \alpha\omega, z)$$

is independent of the choice of $\omega \in \Omega$ and only depends on α in $\overline{\Gamma} := \Gamma^{\mathrm{ab}}/(\text{torsion})$. Denote by

$$c_\alpha : \overline{\Gamma} \to \mathbb{C}_\infty^\times$$

the corresponding multiplier, which induces a group morphism

$$\overline{\Gamma} \to \mathrm{Hom}(\overline{\Gamma}, \mathbb{C}_\infty^\times), \quad \alpha \mapsto c_\alpha.$$

Remark 3.6.5 Over number fields, the theory of theta series allows to study the geometry of modular curves. Over function fields, this line of thinking yields a uniformization of the Jacobian of the Drinfeld modular curve (at ∞). We revisit this in Section V.

CHAPTER IV

Geometric Side

The strategy of [Pra03] is to leverage the Modularity theorem to relate the behaviour of modular forms under Jacquet-Langlands to geometric questions. Of course the reason for the involvement of the modular curve is ultimately its role in proving the Langlands conjecture. It was an insight of Drinfeld [Dri74] that the correct analogue of the modular curve of elliptic curves in the function field setting is not the moduli space of elliptic curves but rather the moduli of what we now call Drinfeld modules.

However to study "modular parametrization" of elliptic curves related to quaternionic modular forms we need Shimura curves, which essentially are moduli of abelian surfaces with a quaternionic action on them. The correct analogue of these abelian surfaces was suggested by Ulrich Stuhler, namely what we now call $\mathcal{D}$-elliptic sheaves. These were used by Laumon, Rapoport and Stuhler to prove the GL_n-local Langlands correspondence in the function field setting.

A philosophical reason why the moduli spaces of these objects are relevant from a Langlands perspective, and therefore from a modularity perspective, is that underlying them is a function field internal analogue of motives, namely *A-motives*. While there are a number of favorable parallels between the theory of A-motives and "real" motives, it should be noted that as of yet there is no way to associate a A-motive to an scheme.

4.1 A-motives

Let C be a smooth proper curve over $\mathbb{F}_q$ and let ∞ be a closed point thereof. Then we set

$$A = \mathcal{O}_C(C\backslash\{\infty\}).$$

Furthermore we denote by τ the $\mathbb{F}_q$-linear Frobenius. Following convention in the field, for any A-algebra R, we denote by τ also the endomorphism

$$\tau := (\mathrm{id}_A \otimes_{\mathbb{F}_q} \tau) : A \otimes_{\mathbb{F}_q} R \to A \otimes_{\mathbb{F}_q} R.$$

In this section we introduce the category of A-motives over R.

Definition 4.1.1 *Let R be an A-algebra and denote*

$$\Delta := \ker \left[A \otimes_{\mathbb{F}_q} R \to R \right].$$

Proposition 4.1.2 ([Gaz24, Lemma 2.1]) *The ideal $\Delta \subset A \otimes_{\mathbb{F}_q} R$ is invertible.*

Remark 4.1.3 Hence geometrically the ideal $\Delta \subset A \otimes_{\mathbb{F}_q} R$ cuts out a divisor, and in the special case that $R = A$ this divisor is the diagonal of $\mathrm{Spec}\, A \otimes_{\mathbb{F}_q} A$.

An A-motive will now be a $A \otimes_{\mathbb{F}_q} R$-module M with a morphism

$$\tau^* M \to M.$$

However, we shall impose the condition that "away from the diagonal", this morphism is an isomorphism. For this we introduce the following notation:

Definition 4.1.4 *Let R be an A-algebra. For any $A \otimes_{\mathbb{F}_q} R$-module M and set*

$$M[\Delta^{-1}] := \mathrm{colim}_n M \otimes_{A \otimes_{\mathbb{F}_q} R} \Delta^{-n}.$$

Remark 4.1.5 This definition is chosen such that for any A-algebra R we have

$$\mathrm{Spec}\left(A \otimes_{\mathbb{F}_q} R[\Delta^{-1}]\right) \cong \mathrm{Spec}\left(A \otimes_{\mathbb{F}_q} R\right) \setminus V(\Delta).$$

Furthermore, for any $A \otimes_{\mathbb{F}_q} R$-module M, the pullback of the quasicoherent sheaf $\tilde{M}$ on $\mathrm{Spec}\left(A \otimes_{\mathbb{F}_q} R\right)$ along $j : \mathrm{Spec}\left(A \otimes_{\mathbb{F}_q} R[\Delta^{-1}]\right) \hookrightarrow \mathrm{Spec}\left(A \otimes_{\mathbb{F}_q} R\right)$ satisfies

$$j^* \tilde{M} \cong \widetilde{M[\Delta^{-1}]}.$$

Definition 4.1.6 (A-motive) *Let R be an A-algebra. An A-*motive over R *is a pair (M, τ_M) consisting of a locally free $A \otimes_{\mathbb{F}_q} R$-module of finite constant rank with an isomorphism*

$$\tau^* M[\Delta^{-1}] \to M[\Delta^{-1}].$$

A morphism of A-motives *is an $A \otimes R$-linear morphism $M \xrightarrow{f} N$ such that the diagram*

$$\begin{array}{ccc} \tau^* M[\Delta^{-1}] & \xrightarrow{\tau^* f} & \tau^* N[\Delta^{-1}] \\ \downarrow{\scriptstyle \tau_M} & & \downarrow{\scriptstyle \tau_N} \\ M[\Delta^{-1}] & \xrightarrow{f} & N[\Delta^{-1}] \end{array}$$

commutes. The resulting category is denoted by $\mathcal{M}_R$.

Example 4.1.7 *Let $A = \mathbb{F}_q[t]$ and $R = \mathbb{F}_q[z]$ whose structure morphism as an A-algebra is given by $t \mapsto z$. In this case we have*

$$A \otimes_{\mathbb{F}_q} R[\Delta^{-1}] \cong \mathbb{F}_q[t, z]\left[\frac{1}{t-z}\right].$$

Hence an A-motive over R consists of a locally free $\mathbb{F}_q[t, z]$-module M with a morphism $\tau^* M \to M$ whose kernel and cokernel are annihilated by some power of $(t - z)$.

Remark 4.1.8 We follow here the naming convention of [Gaz21], but these would sometimes be called *finite* or *abelian* A-motives. Equally sometimes the term A-motive is replaced by *Anderson* motives, *Anderson A-motives* or *abelian T-motives* [Gos96] [BP06].[1]

Definition 4.1.9 (Isogeny) *A morphism $f : M \to N$ is said to be an* isogeny *if one of the following equivalent conditions is satisfied*

1. *M, N have the same rank and f is injective*
2. *there exists an $a \in A\backslash\{0\}$ such that f induces an isomorphism $M[\frac{1}{a}] \cong N[\frac{1}{a}]$*
3. *there exists an $a \in A\backslash\{0\}$ and $g : (N, \tau_N) \to (M, \tau_M)$ such that*

$$f \circ g = a \cdot \mathrm{id}_N \text{ and } g \circ f = a \cdot \mathrm{id}_M.$$

If there exists an isogeny between (M, τ_M) and (N, τ_N), then we say that they are isogenous.

Remark 4.1.10 Notice the difference to the corresponding notion for abelian varieties, where we say that a morphism

$$A \to A'$$

[1] Anderson did **not** call them A-motives, he used the term *t-motives* as he studied the case $A = \mathbb{F}_p[t]$.

is an isogeny if it is surjective with finite kernel. This reflects the fact that A-motives are analogous to $h^1(A)$.

Definition 4.1.11 (Tensor product) *Let $(M, \tau_M), (N, \tau_N)$ be two A-motives over R. Then the tensor product of the underlying $A \otimes R$-modules is an A-motive*

$$(M, \tau_M) \otimes (N, \tau_N) := (M \otimes_{A \otimes_{\mathbb{F}_q} R} N, \tau_M \otimes \tau_N).$$

Example 4.1.12 (Unit Motive) *The* unit motive $A(0)$ *is*

$$A(0) := (A \otimes R, \mathrm{id}).$$

The following is the A-motivic analogue of the Tate motive.

Example 4.1.13 (Carlitz motive) *Let $\kappa : A \to R$ be the structure morphism. The* Carlitz motive over R *is the A-motive with underlying module $A \otimes R$*

$$\tau^*(A \otimes R) \to A \otimes R, \quad m \mapsto (t - \kappa(t))\tau(m).$$

Its m-tensor product is denoted by $A(n)$ with

$$\tau^*(A \otimes R) \to A \otimes R, \quad m \mapsto (t - \kappa(t))^n\tau(m).$$

Remark 4.1.14 This category is not abelian as morphisms may not admit cokernels. However one would expect a category of motives to be abelian. One can remedy this with the right modifications and construct a *Tannakian category.*

Remark 4.1.15 The goal of this chapter is to give analogues of abelian surfaces with a quaternionic action on them. Abelian varieties can be seen as approximations of motives. If we take the analogy between motives and A-motives, this would mean that one possible analogue of an abelian variety with an action by a quaternion algebra would be an A-motive M of rank 4 with a morphism

$$\mathcal{O}_D \to \mathrm{End}(M)$$

which is compatible with the action by A. This action endows the $A \otimes_{\mathbb{F}_q} R$-module M with the structure of a $\mathcal{O}_D^{\mathrm{opp}} \otimes_{\mathbb{F}_q} R$-module and is in fact locally free of rank one.

This leads to the following definition, which allows for a central simple algebra of dimension d^2 (as opposed to restricting to quaternion algebras).

Definition 4.1.16 ([Pap18, Def 3.1]) *Let F be a field equipped with the structure of a A-algebra with structure morphism $\kappa : A \to F$. Furthermore let D be a central simple algebra of dimension d^2 and fix a maximal A-order $\mathcal{O}_D$ in D. An* $\mathcal{O}_D$-motive over F *is a left $\mathcal{O}_D^{\text{opp}} \otimes F[\tau]$-module M such that*

1. *M is a locally free $\mathcal{O}_D^{\text{opp}} \otimes F$-module of rank one*
2. *M is a free $F[\tau]$-module of rank d*

and for all $a \in A$ we have

$$(a \otimes 1 - 1 \otimes \kappa(a))M \otimes F^{\text{sep}} \subset \tau(M \otimes F^{\text{sep}}).$$

Remark 4.1.17 The last condition shrinks down the category of $\mathcal{O}_D$-motives such that the functor defined in Prop-Def 4.3.6 is essentially surjective. Specifically it ensures that condition (2) in Def 4.3.2 is met.

4.2 Drinfeld modules

After the "motivic" nature of the previous definitions, we now give a definition analogous to the "analytic" definition of elliptic curves over $\mathbb{C}$. As elliptic curves over $\mathbb{C}$ are algebraic curves whose analytification is given by $\mathbb{C}/\Lambda$, where Λ is a lattice, it was a natural idea for Drinfeld to study the corresponding quotient over $\mathbb{C}_\infty$.
As before, let C be a smooth proper curve over $\mathbb{F}_q$ and let ∞ be a closed point thereof. Then we set

$$A = \mathcal{O}_C(C \backslash \{\infty\}).$$

Definition 4.2.1 *A finitely generated, projective, discrete A-submodule of $\mathbb{C}_\infty$ is said to be an* A-lattice.

Remark 4.2.2 Unlike $\mathbb{Z}$-lattices in $\mathbb{C}$, there are A-lattices of arbitrary large size since the extension $\mathbb{C}_\infty/\mathbb{F}_p(t)_\infty$ has infinite degree.

Definition 4.2.3 *Let Λ_1, Λ_2 be A-lattices. A* morphism of A-lattices *from Λ_1 to Λ_2 is*

$$\text{Hom}(\Lambda_1, \Lambda_2) := \begin{cases} \{c \in \mathbb{C}_\infty \mid c\Lambda_1 \subset \Lambda_2\}, & \text{if } \Lambda_1, \Lambda_2 \text{are of the same rank} \\ \varnothing, & \text{otherwise} \end{cases}$$

Proposition-Definition 4.2.4 *The set of A-lattices together with morphisms of A-lattices forms a category.*

Contrary to the characteristic zero situation, for any lattice $\Lambda \subset \mathbb{C}_\infty$, the quotient is analytically isomorphic to $\mathbb{C}_\infty$.

Proposition-Definition 4.2.5 *Let Λ be an A-lattice of rank r. Then*

$$\exp_\Lambda(z) := z \prod_{0 \neq \lambda \in \Lambda} \left(1 - \frac{z}{\lambda}\right)$$

converges for every $z \in \mathbb{C}_\infty$. Furthermore this yields a short exact sequence

$$0 \to \Lambda \to \mathbb{C}_\infty \xrightarrow{\exp_\Lambda} \mathbb{C}_\infty \to 0.$$

Via this isomorphism $\mathbb{C}_\infty$ has now a "new" A-action, often referred to as *exotic*. This "exotic" action satisfies the following property

Proposition 4.2.6 *For any $a \in A$, there exists a polynomial φ_a making the diagram*

$$\begin{array}{ccc} \mathbb{C}_\infty/\Lambda & \xrightarrow{\exp_\Lambda} & \mathbb{C}_\infty \\ \downarrow a & & \downarrow \varphi_a \\ \mathbb{C}_\infty/\Lambda & \xrightarrow{\exp_\Lambda} & \mathbb{C}_\infty \end{array}$$

commute. Indeed the following construction does the job

$$\varphi : A \to \mathbb{C}_\infty[\tau], \quad a \mapsto ax \prod_{0 \neq \lambda \in a^{-1}\Lambda/\Lambda} \left(1 - \frac{x}{\exp_\Lambda(\lambda)}\right).$$

Definition 4.2.7 *Let R be an A-algebra. Denote by ∂ the following ring morphism*

$$\partial : R[\tau] \to R, \quad \Sigma_{n \geqslant 0} r_n \tau^n \mapsto r_0.$$

Definition 4.2.8 *Let R be an A-algebra. A* Drinfeld module of rank d over R *is a homomorphism*

$$\varphi : A \to R[\tau]$$

such that $\deg_\tau(\varphi_a) = r \deg(a)$ and $\partial\varphi_a = a$. A morphism $f : \varphi \to \varphi'$ *over R is an element*

$f \in R[\tau]$ such that for all $a \in A$ the following diagram commutes

$$\begin{array}{ccc} R & \xrightarrow{f} & R \\ \downarrow{\scriptstyle \varphi_a} & & \downarrow{\scriptstyle \varphi'_a} \\ R & \xrightarrow{f} & R \end{array}$$

We denote the category of Drinfeld-modules by DrM_R *and by* $\mathrm{DrM}_R^{\mathrm{rk}=r}$ *the full subcategory of Drinfeld-modules of rank r over R.*

Proposition-Definition 4.2.9 *Let $\varphi : A \to \mathbb{C}_\infty[\tau]$ be a Drinfeld A-module over $\mathbb{C}_\infty$. Then there exists a unique $\exp_\varphi(z) = \sum_{i \geqslant 0} e_i z^i \in \mathbb{C}_\infty[[z]]$ where $e_0 = 1$ such that*

$$\varphi_a(\exp_\varphi(z)) = \exp_\varphi(a \cdot z),$$

which we call the exponential function of φ.[2]

Theorem 4.2.10 *The category of Drinfeld modules of rank r over $\mathbb{C}_\infty$ is equivalent to the category of rank r lattices of $\mathbb{C}_\infty$.*

Remark 4.2.11 For elliptic curves, one can reconstruct the lattice associated to it via singular cohomology. Thus the functor associating to a Drinfeld module its corresponding lattice via Thm 4.2.10 can be seen as a function field analogue of singular cohomology. It should be noted that Drinfeld modules of rank two behave the most like elliptic curve. See Figure IV.1 for comparison between Drinfeld modules of various ranks and elliptic curves

rank 0 Drinfeld module $\longleftrightarrow \mathbb{G}_a$
rank 1 Drinfeld module $\longleftrightarrow \mathbb{G}_m$ or CM elliptic curve
(if E has CM by $\mathscr{O}$, view its lattice as rank 1 $\mathscr{O}$-module)
rank 2 Drinfeld module $\longleftrightarrow$ elliptic curve
rank ≥ 3 Drinfeld module $\longleftrightarrow$? (if only such geometric objects existed...)

Figure IV.1: Table of analogies between Drinfeld modules of various ranks and elliptic curves (image taken from [Poo])

Remark 4.2.12 Let R be an A-algebra. The non-commutative ring $R[\tau]$ is isomorphic to the ring of $\mathbb{F}_q$-linear endomorphisms of the additive group scheme $\mathbb{G}_{a,R}$, i.e. we have

$$R[\tau] \cong \mathrm{End}_{\mathbb{F}_q}(\mathbb{G}_{a,R}).$$

[2]We've suppressed the fact that $a \in A$ acts on z via the inclusion $A \to \mathrm{Quot}(A) \to \mathrm{Quot}(A)_\infty \to \mathbb{C}_\infty$.

This description of $\mathbb{F}_q$-linear endomorphisms generalizes to $\mathbb{G}_{a,R}^d$ for $d \geqslant 0$, in which case we have ([vdH04, Prop 1.1])

$$\mathrm{End}_{\mathbb{F}_q}\left(\mathbb{G}_{a,R}^d\right) \cong \mathrm{Mat}_d(R[\tau]).$$

4.3 Drinfeld-Stuhler modules

In this section we introduce a generalization of Drinfeld modules by adding an action by a central simple algebra. For most of this thesis, this central simple algebra will be a quaternion algebra. In particular Drinfeld-Stuhler modules will be analogues of abelian surfaces with a quaternionic multiplication.

Definition 4.3.1 *For any A-algebra R denote by*

$$\partial : \mathrm{Mat}_d(R[\tau]) \to \mathrm{Mat}_d(R), \quad \sum_{i \geqslant 0} M_i \tau^i \mapsto M_0.$$

Definition 4.3.2 *Let D be a central simple algebra over F of dimension d^2 and fix a maximal A-order $\mathcal{O}_D$. Then a* Drinfeld-Stuhler $\mathcal{O}_D$-module over an A-algebra R *is an embedding*

$$\varphi : \mathcal{O}_D \to \mathrm{Mat}_d(R[\tau]), \quad b \mapsto \varphi_b$$

such that

1. *thanks to the isomorphism $\mathrm{End}_{\mathbb{F}_q}\left(\mathbb{G}_{a,R}^d\right) \cong \mathrm{Mat}_d(R[\tau])$, we can consider φ_b as an endomorphism of $\mathbb{G}_{a,R}$. For all $b \in \mathcal{O}_D \cap D^\times$ the kernel $\ker \varphi_b$ is a finite group scheme over R of order $|\mathcal{O}_D/b\mathcal{O}_D|$*

2. *φ respects A, that is under the morphism*

$$\partial_\varphi : A \to \mathcal{O}_D \xrightarrow{\varphi} \mathrm{Mat}_d(R[\tau]) \xrightarrow{\partial} \mathrm{Mat}_d(R)$$

   a gets mapped to $\mathrm{diag}(\kappa(a), \ldots, \kappa(a))$.

A morphism of Drinfeld-Stuhler modules *φ, ψ over R is an element $u \in \mathrm{Mat}_d(R[\tau])$ such that for all $a \in \mathcal{O}_D$ we have*

$$u\varphi_a = \psi_a u.$$

It is an isomorphism *if u has an inverse in $\mathrm{Mat}_d(R[\tau])$. It is an* isogeny *if the kernel $\ker(u)$ is a finite group scheme and we say it is a* separable isogeny *if $\ker(u)$ is étale.*

Remark 4.3.3 Note that if $\mathcal{O}_D = A$ (and therefore $d = 1$), then the above definition is literally the one of a Drinfeld module over R of rank one. Indeed while it is sometimes imposed that a Drinfeld module φ satisfies

$$\deg_\tau(\varphi_a) = -rv_\infty(a)$$

for all $a \in A$, this is automatically satisfied.

Proposition 4.3.4 ([Pap18, Thm 4.1]) *Let φ be a Drinfeld-Stuhler module over a field k. There exists some $s \mid d$ such that*

$$\mathrm{Aut}(\varphi) \cong \mathbb{F}_{q^s}^\times.$$

Remark 4.3.5 Subsequently we will use the fact that $q - 1 \mid |\mathrm{Aut}(\varphi)|$.

Proposition-Definition 4.3.6 *Let φ be a Drinfeld-Stuhler module over R. The action of $\mathcal{O}_D$ via φ on $\mathbb{G}_{a,R}^d$ endows the module*

$$\mathrm{Hom}_{\mathbb{F}_q}(\mathbb{G}_{a,R}^d, \mathbb{G}_{a,R})$$

with the structure of an $\mathcal{O}_D^{\mathrm{opp}} \otimes R[\tau]$-module. This defines a contra-variant functor from the category of Drinfeld-Stuhler modules over R to the category of $\mathcal{O}_D$-motives over R.

Theorem 4.3.7 ([Pap18, Thm 3.2]) *Let K be a field with the structure of an A-algebra. The categories of Drinfeld-Stuhler modules over K and of $\mathcal{O}_D$-motives over K are anti-equivalent.*

Theorem 4.3.8 (Morita equivalence, [Pap18, Thm 2.20] [LRS93]) *Let R be an A-algebra. The category of Drinfeld-Stuhler modules over R is Morita invariant, that is the category of Drinfeld-Stuhler $\mathrm{Mat}_d(A)$-modules over R is equivalent to the category of Drinfeld A-modules of rank d over R.*

Remark 4.3.9 This is similar to the fact that the category of $\mathrm{Mat}_2(\mathbb{Z})$-abelian surfaces is equivalent to the category of elliptic curves.

While we have not yet introduced the moduli space of Drinfeld-Stuhler modules, we can already give a first parallel with Shimura curves. The following description of the isomorphism set of D-abelian surfaces over $\mathbb{C}$ is parallel "at infinity" for Drinfeld-Stuhler modules. This also shows the analogous use of the (complex) upper half-plane and the Drinfeld symmetric space.

Theorem 4.3.10 (Deligne) *Let D be an indefinite quaternion algebra over $\mathbb{Q}$ with maximal order $\mathcal{O}$. There exists a bijection between pairs (A, ι) consisting of an abelian surface A over $\mathbb{C}$ with an embedding $\iota : \mathcal{O} \hookrightarrow \mathrm{End}(A)$ and the double quotient*

$$D^\times \backslash \mathbb{H}^\pm \times D(\mathbb{A}_f)^\times / \widehat{\mathcal{O}}_D^\times.$$

Proposition 4.3.11 ([BS97, Thm 4.4.11] or see Section 5.11) *There is a bijection between Drinfeld-Stuhler $\mathcal{O}_D$-modules over $\mathbb{C}_\infty$ and the double quotient*

$$D^\times \backslash \Omega^d \times D(\mathbb{A}_f)^\times / \widehat{\mathcal{O}}_D^\times.$$

Remark 4.3.12 One can show that assigning to a Drinfeld-Stuhler module φ over $\mathbb{C}_\infty$ the kernel $\ker(\exp_\varphi)$ of the exponential function is actually a functor. This corresponds essentially to the homology of an abelian variety over $\mathbb{C}$[3].

4.4 The motivation behind shtukas

Inspired by the title of [Gai16], we briefly explain how you could have invented shtukas, thus hopefully motivating the next section. As was pointed out in [hs], the historical perspective is maybe not as illuminating as an a-historical perspective: Let G be a reductive group scheme. Then G-automorphic forms are functions on the double quotient

$$G(F)\backslash G(\mathbb{A})/G(\mathcal{O}_F).$$

By *Weil uniformization*, this is in bijection with the set of G-bundles on the curve X, or equivalently $\mathrm{Bun}_G(\mathbb{F}_q)$. The Hecke operators naturally act on Bun_G. Furthermore, so does the Frobenius. However apriori, these two operations do not interact in a "controlled" way. For this we want to consider G-bundles $\mathcal{E}$ such that the Frobenius pullback $\mathrm{Frob}^* \mathcal{E}$ is a "modification" of $\mathcal{E}$, that is we have a map

$$\mathrm{Frob}^* \mathcal{E} \to \mathcal{E}$$

which is an isomorphism almost everywhere. It turns out that this structure is respected by Hecke operators, thus if we study the moduli space of G-bundles with this Frobenius modification, we get a scheme where Hecke operators and Frobenius act on in a "controlled way". This rough outline is what leads to the notion of a *shtuka*, which we study in the next chapter.

[3] it should be mentioned that Drinfeld's inspiration for Drinfeld modules went the other way: elliptic curves over $\mathbb{C}$ are quotient of $\mathbb{C}$ by lattices thereof. Therefor Drinfeld decided to study the quotients $\mathbb{C}_\infty/\Lambda$, but realized that one has in fact $\mathbb{C}_\infty/\Lambda \cong \mathbb{C}_\infty$. However the action of $\mathbb{F}_p[t]$ on the left induces a *new* action on $\mathbb{C}_\infty$. This lead him to the notion of *elliptic modules*, now called Drinfeld modules.

4.5 $\mathcal{D}$-elliptic sheaves

In the study of the moduli space of Drinfeld-Stuhler modules, it will often be convenient to have an equivalent category consisting of sheaves, which we review in this section.
Let C be a smooth proper curve over $\mathbb{F}_q$ and let ∞ be a closed point thereof. Then we set

$$A := \mathcal{O}_C(C\backslash\{\infty\}), \quad F := \mathrm{Quot}(A).$$

Let D be a central simple algebra over F of dimension d^2 which is split at ∞, i.e such that $D \otimes F_\infty \cong \mathrm{Mat}_d(F_\infty)$. Furthermore fix a coherent, locally free sheaf $\mathcal{D}$ of $\mathcal{O}_C$-algebras such that its generic fiber is isomorphic to D and such that for all $x \in |C|$

$$\mathcal{D}_x \subset D \otimes F_x$$

is a maximal order.

Definition 4.5.1 (Partial Frobenius) *Let X be a $\mathbb{F}_q$-scheme. The* partial Frobenius on $C \times X$ *is the morphism*

$$C \times_{\mathbb{F}_q} X \overset{\mathrm{id}_C \times \mathrm{Frob}}{\longrightarrow} C \times_{\mathbb{F}_q} X.$$

We will denote this morphism by τ.

Definition 4.5.2 ([LRS93]) *Let X be a $\mathbb{F}_q$-scheme. A $\mathcal{D}$-elliptic sheaf over X is a sequence $(\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$ where*

1. *$\mathcal{E}_i$ are locally free $\mathcal{O}_{C\times_{\mathbb{F}_q} X}$-modules of rank d^2 with a right action of $\mathcal{D}$*
2. *$j_i : \mathcal{E}_i \to \mathcal{E}_{i+1}$ injective $\mathcal{D}$-linear morphisms*
3. *$t_i : \tau^*\mathcal{E}_i \to \mathcal{E}_{i+1}$ injective $\mathcal{D}$-linear morphisms*

such that

1. *for all $i \in \mathbb{Z}$ the following diagrams commute*

$$\begin{array}{ccc} \mathcal{E}_i & \xrightarrow{j_i} & \mathcal{E}_{i+1} \\ {\scriptstyle t_{i-1}}\uparrow & & \uparrow{\scriptstyle t_i} \\ \tau^*\mathcal{E}_{i-1} & \xrightarrow{\tau^* j_{i-1}} & \tau^*\mathcal{E}_i \end{array}$$

2. *for all i we have $\mathcal{E}_{i+d\deg(\infty)} \cong \mathcal{E}_i \otimes \mathcal{O}_C(\infty)$*

3. *the composition*
$$\mathcal{E}_i \to \mathcal{E}_{i+1} \to \ldots \to \mathcal{E}_{i+d\deg(\infty)} = \mathcal{E}_i \otimes \mathcal{O}_C(\infty)$$
agrees with the one induced by $\mathcal{O}_C \to \mathcal{O}_C(\infty)$,

4. *the direct image along the projection* $\mathrm{pr}_X : C \times_{\mathbb{F}_q} X \to X$ *of* coker j_i *is a locally free* $\mathcal{O}_X$*-module of rank* d

5. *there exists a morphism* $\iota_0 : X \to C$ *whose image is contained in* $C\backslash\{\infty\}$ *such that*

   (a) *for all* $i \in \mathbb{Z}$ *the sheaf* coker t_i *is supported on the graph of* ι_0

   (b) *for all* $i \in \mathbb{Z}$*, there exists a locally free* $\mathcal{O}_X$*-module* $\mathcal{V}_i$ *such that*
$$\text{coker } t_i \cong (\iota_0 \times \mathrm{id})_* \mathcal{V}_i.$$

This morphism is called the zero of $(\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$.

A morphism of $\mathcal{D}$-elliptic sheaves over X *consists of a sequence of* $\mathcal{D}$*-equivariant morphisms*
$$\varphi_i : \mathcal{E}_i \to \mathcal{E}'_{i+n}$$
for some fixed $n \in \mathbb{Z}$ *which commute with* j_i *and* t_i.

Remark 4.5.3 Important convention: Throughout this thesis, we will assume that $\mathcal{D}$-elliptic sheaves are defined for bases X which have a fixed $X \to C$ and that the zero of the $\mathcal{D}$-elliptic sheaves agree with this structure morphism (see for instance [Pap18] for an example of this convention). In the terminology of [AHKP24], we restrict ourselves to *sound* $\mathcal{D}$-elliptic sheaves.

Definition 4.5.4 *An* isogeny between two $\mathcal{D}$-elliptic sheaves *over* S *is a morphism*
$$(\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}} \xrightarrow{f} (\mathcal{E}'_i, j'_i, t'_i)_{i\in\mathbb{Z}}$$
such that for all $i \in \mathbb{Z}$

- $f_i : \mathcal{E}_i \to \mathcal{E}'_i$ *is injective*
- coker(f_i) *is locally free of finite rank as an* $\mathcal{O}_S$*-module*
- *there exists an effective divisor* $Z_i \subset X$ *such that* coker f_i *is supported on* $Z_i \times S$

Definition 4.5.5 *We say that an isogeny* $(\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}} \xrightarrow{f} (\mathcal{E}'_i, j'_i, t'_i)_{i\in\mathbb{Z}}$ *of* $\mathcal{D}$*-elliptic sheaves on* S *is* $\mathfrak{p}$-torsion *if for all* $i \in \mathbb{Z}$*, the cokernel is supported on* $\operatorname{Spec} A/\mathfrak{p}^n \times S$.

The category of $\mathcal{D}$-elliptic sheaves is equivalent to the categories of $\mathcal{O}_D$-motives and Drinfeld-Stuhler modules. The categories are related by the following construction.

Proposition 4.5.6 *Let* $(\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$ *be a* $\mathcal{D}$*-elliptic sheaf over an* A*-algebra* R*. Then the* $R[\tau]$*-module*

$$H^0((C\backslash\{\infty\} \times_{\mathbb{F}_q} \operatorname{Spec} R, \mathcal{E}_0))$$

where τ *acts via the map induced from* $\tau^*\mathcal{E}_0 \to \mathcal{E}_1$ *is an* $\mathcal{O}_D$*-motive.*

Remark 4.5.7 There is an obvious action of $\mathbb{Z}$ on the category of $\mathcal{D}$-elliptic sheaves by shifting $\mathcal{E}_i \to \mathcal{E}_{i+1}$ etc. It is often convenient to quotient out this action, for instance as this then induces the following theorem.

Via the functors defined in Prop 4.3.6 and Prop 4.5.6, one get the following results, whose upshot is that the three categories we have described (in particular the convention explained in Rem 4.5.3 and Rem 4.5.7), namely Drinfeld-Stuhler modules, $\mathcal{O}_D$-motives and $\mathcal{D}$-elliptic sheaves are all equivalent.

Theorem 4.5.8 ([Pap18, Thm 3.2]) *Let* R *be an* A*-algebra. The category of* $\mathcal{D}$*-elliptic sheaves up to* $\mathbb{Z}$ *over* $\operatorname{Spec} R$ *is anti-equivalent to the category of Drinfeld-Stuhler modules over* R*.*

Theorem 4.5.9 ([Pap18, Thm 3.4]) *Let* R *be an* A*-algebra. The category of* $\mathcal{D}$*-elliptic sheaves up to* $\mathbb{Z}$ *over* $\operatorname{Spec} R$ *is equivalent to the category of* $\mathcal{O}_D$*-motives over* R*.*

Remark 4.5.10 A $\mathcal{D}$-elliptic sheaf over X can also be seen as a flag of the sheaf $\mathcal{E}_0$

$$\mathcal{E}_0(-\infty) = \mathcal{E}_0(-d\deg\infty) \subsetneq \mathcal{E}_{-d\deg\infty+1} \to \ldots \subsetneq \mathcal{E}_{-1} \subsetneq \mathcal{E}_0.$$

Note that $\mathcal{E}_0$ is a vector bundle of rank d^2. The sheaf $\mathcal{E}_0$ is equipped with an action of $\mathcal{D}$, or equivalently a map $\mathcal{D}^\times \to \mathrm{GL}(\mathcal{E}_0)$. This is equipped with a morphism

$$\sigma^*\mathcal{E}_0|_{C\times X\backslash(\Gamma_\infty\cup\Gamma_{\iota_0})} \to \mathcal{E}_0|_{C\times X\backslash(\Gamma_\infty\cup\Gamma_{\iota_0})}$$

which maps $\sigma^*\mathcal{E}_i$ to $\mathcal{E}_{i+1}$. This leads to the notion of a $\mathcal{D}^\times$-shtuka, or more generally a $\mathcal{G}$-shutka. In this case the subschemes $\Gamma_\infty, \Gamma_{\iota_0}$ are what one calls the *legs of the shtuka* $\mathcal{E}_0$.

4.6 Taguchi duality

We briefly recall the relevant results about Taguchi duality (following the exposition of [Hat21]). We need the correct notion of duality, as we saw in Rem 5.7.5. In the case of Drinfeld-modules of rank two over a field, Taguchi introduced the right notion and further results were shown by M. Papanikolas and N. Ramachandran [PR03](which were extended by Hattori in [Hat21]). Unfortunately for our exposition, the results of [Hat21] were proven using the language of v-modules, which we recall for the convenience of the reader. However, the upshot of this section Thm 4.6.9, can be expressed purely in the language of Drinfeld modules. It should be possible to rewrite the work of [Hat21] purely in the language of $\mathbb{F}_{\mathfrak{p}}$-Dieudonne modules.

Definition 4.6.1 ([Hat21, Def 2.9]) *A* t-module over S *is an* A*-module scheme* $\mathcal{G}/S$ *such that the induced action of* A *on the canonical bundle* $\omega_{\mathcal{G}}$ *is the same as the one coming from* $A \to \mathcal{O}_S(S)$ *and that* $\mathcal{G}$ *is a* φ*-module when viewed as a* $\mathbb{F}_q$*-module scheme.*

Definition 4.6.2 ([Hat21, Def 2.11]) *A* v-module over S *is a pair* $(\mathcal{G}, v_{\mathcal{G}})$ *which consists of a* t*-module* $\mathcal{G}$ *and a* $\mathcal{O}_S$*-linear morphism*

$$v_{\mathcal{G}} : \mathcal{E}_{\mathcal{G}} \to \mathcal{E}_{\mathcal{G}}^{(q)}$$

such that

$$\psi_t = \rho + \varphi_{\mathcal{G}} \circ v_{\mathcal{G}} \text{ and } (\psi_t \otimes 1) \circ v_{\mathcal{G}} = v_{\mathcal{G}} \circ \psi_t.$$

A morphism of v-modules *is a morphism of* A*-module schemes which commutes with the* v*-structure and we denote the resulting category by* v*-Mod*$_S$*. We say that a* v*-module is* finite *if the underlying* φ*-module is a finite* φ*-module and we denote the resulting full subcategory by* v-$\mathrm{Mod}_S^{\mathrm{fin}}$.

Proposition 4.6.3 ([Hat21]) *Any Drinfeld module over* S *has a unique* v*-structure and morphisms of Drinfeld modules are compatible with the* v*-structure, that is there exists a functor*

$$\mathrm{DrM}_S \to v\text{-}\mathrm{Mod}_S.$$

Example 4.6.4 In particular, the Carlitz module C can be seen as a v-module.

Proposition 4.6.5 ([Hat21]) *The kernel of any isogeny of Drinfeld modules over* S *has a unique structure as a* v*-module over* S *which is compatible with the inclusion.*

The following is the analogue of Cartier duality in our setting.

Theorem 4.6.6 (Taguchi duality [Hat21, Thm 2.16]) *Let C denote the v-module associated to the Carlitz module. For any finite v-module over S, the sheaf*

$$\mathcal{G}^D := \mathrm{Hom}_{v,S}(\mathcal{G}, C)$$

is a finite v-module over S, which we call the Taguchi dual of $\mathcal{G}$. *This functor is exact, preserves the rank of the v-module, commutes with arbitrary base change and there exists a natural isomorphism of v-modules*

$$\mathcal{G} \to \left(\mathcal{G}^D\right)^D.$$

Example 4.6.7 ([Hat21, Lemma 2.17]) For any $a \in A$ we have an isomorphism

$$C[a]^D \cong \underline{A/(a)}.$$

In particular $C[a]$ "plays the role" of μ_n, as being the (Taguchi and respectively Cartier) dual of the constant group scheme which shows up in the torsion of Drinfeld modules and elliptic curves respectively. This is particularly interesting in view of the next proposition.

Remark 4.6.8 For abelian varieties, one can describe their duality theory via extensions, that is if A is an abelian variety over a field k, then the functor of points of $A^\vee$ is (functorially isomorphic to)

$$A^\vee(R) \cong \mathrm{Ext}^1(A, \mathbb{G}_m).$$

This is known as the *Weil-Barsotti forumula*. In [PR03] Papanikolas-Ramachandran show that if one consider subgroup of $\mathrm{Ext}^1(E, C)$ comprised of those extensions which induce split extensions of their tangent spaces, then one can recover Taguchi-duality in this way. The Taguchi duality behaves as one expects if we restrict to rank two Drinfeld modules (see Def 4.2.8):

Theorem 4.6.9 ([PR03, Thm 1.1] [Hat21, 2.19]) *Let S be any scheme over A. There exists a contravariant endo-functor*

$$\left(\mathrm{DrM}_S^{\mathrm{rk}=2}\right)^{\mathrm{opp}} \to \mathrm{DrM}_S^{\mathrm{rk}=2}, \quad E \mapsto E^D$$

such that if $f : E_1 \to E_2$ is an isogeny, then there exits an isogeny $f^D : E_2^D \to E_1^D$ compatible with Taguchi duality, that is there exists a natural isomorphism

$$\ker(f)^D \cong \ker(f^D).$$

For any Drinfeld module over a field K, we have biduality, i.e. $(E^D)^D \cong E$.

4.7 Level structures and Moduli space

Definition 4.7.1 (Torsion of Drinfeld-Stuhler modules) *Let φ be a Drinfeld-Stuhler module over R. Then for any non-zero ideal $\mathfrak{n} \subset A$, we denote*

$$\varphi[\mathfrak{n}] := \bigcap_{a \in \mathfrak{n}} \ker \varphi_a.$$

Proposition 4.7.2 ([Pap18, Lemma 2.10]) *If the A-characteristic of K does not divide $\mathfrak{n}$, then we have*

$$\varphi[\mathfrak{n}](K^{\mathrm{sep}}) \cong \mathcal{O}_D/\mathfrak{n}.$$

Proof. Denote by $M(\varphi)$ the $\mathcal{O}_D$-motive associated to φ. Then the torsion $\varphi[\mathfrak{n}](K^{\mathrm{sep}})$ is dual to $M(\varphi)/\mathfrak{n}$. Since $M(\varphi)$ is a $\mathcal{O}_D$-motive, it is a locally free left $\mathcal{O}_D^{\mathrm{opp}} \otimes K$-module of rank one. In particular, as $\mathcal{O}_D^{\mathrm{opp}}$-modules, we have

$$M(\varphi)/\mathfrak{n} \cong \mathcal{O}_D^{\mathrm{opp}}/\mathfrak{n}.$$

□

Definition 4.7.3 ((Naive) Level structures on Drinfeld-Stuhler modules) *Let $\mathfrak{n} \subset A$ be an ideal. A* $\Gamma(\mathfrak{n})$-level structure on a Drinfeld-Stuhler module *φ is an isomorphism of $\mathcal{O}_D$-module schemes*

$$\underline{\mathcal{O}_D/\mathfrak{n}} \longrightarrow \varphi[\mathfrak{n}].$$

Definition 4.7.4 *Let $\mathfrak{n} \subset A$ be an ideal. A $\Gamma_0(\mathfrak{n})$-structure on a Drinfeld-Stuhler module φ is a $\mathcal{O}_D$-invariant finite, flat subgroup $\mathbb{H}$ of $\varphi[\mathfrak{n}]$ such that locally there exists a morphism $(A/\mathfrak{n})^d \xrightarrow{\iota} \varphi[\mathfrak{n}]$ such that we have an equality of effective Cartier divisors*

$$\sum_{(A/\mathfrak{n})^d} \iota(a) = \mathbb{H}.$$

Proposition 4.7.5 ([LRS93, 2.2]) *Let $\mathfrak{n}$ be a non-empty closed subscheme of $C\backslash\{\infty\}$, S a $\mathbb{F}_q$-scheme and $(\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$ a $\mathcal{D}$-elliptic sheaf on S. Denote by $\mathcal{D}_\mathfrak{n}$ the restriction of the sheaf of $\mathcal{O}_C$-algebras $\mathcal{D}$ to $\mathfrak{n} \subset C$ by $\mathcal{D}|_\mathfrak{n}$. Then the restrictions $\mathcal{E}_i|_{\mathfrak{n}\times S}$ via j_i are independent and*

furthermore there exists an isomorphism

$$\mathcal{D}|_{\mathfrak{n}} \boxtimes \mathcal{O}_S \to \mathcal{E}_i|_{\mathfrak{n}\times S}.$$

Definition 4.7.6 ((Naive) Level structure on $\mathcal{D}$-elliptic sheaves) *Let $\mathfrak{n}$ be a non-empty closed subscheme of $C\backslash\{\infty\}$, S a $\mathbb{F}_q$-scheme and $(\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$ a $\mathcal{D}$-elliptic sheaf on S. Denote by $\mathcal{E}|_{\mathfrak{n}\times S} = \mathcal{E}_i|_{\mathfrak{n}\times S}$, which is independent of i by Prop 4.7.5, and $t = t_i|_{\mathfrak{n}\times S}$. Then an* $\mathfrak{n}$-level structure on *a $\mathcal{D}$-elliptic sheaf on S is the data of an isomorphism $\iota : \mathcal{D}|_{\mathfrak{n}} \boxtimes \mathcal{O}_S \to \mathcal{E}|_{\mathfrak{n}\times S}$ which is compatible with the right action of $\mathcal{D}|_{\mathfrak{n}}$ and making the following diagram commute:*

$$\begin{array}{ccc} {}^{\tau}\mathcal{E}|_{\mathfrak{n}\times S} & \xrightarrow{\ t\ } & \mathcal{E}|_{\mathfrak{n}\times S} \\ {\scriptstyle \tau_\iota}\uparrow & \nearrow_{\iota} & \\ \mathcal{D}|_{\mathfrak{n}} \boxtimes \mathcal{O}_S & & \end{array}$$

Theorem 4.7.7 (Existence of Moduli-spaces) *For $\mathfrak{n} \neq 0$, the functors*

$$\mathcal{E}ll_{C,\mathcal{D},\mathfrak{n}} : \text{Schemes} \longrightarrow \text{Sets}, \quad S \mapsto \{\mathcal{D}\text{-elliptic sheaves with } \mathfrak{n}\text{-level structure}/S\}$$

$$Y^{\mathfrak{D}}(\mathfrak{n}) : \text{Schemes} \longrightarrow \text{Sets}, \quad S \mapsto \{\text{Drinfeld-Stuhler modules with } \Gamma(\mathfrak{n})\text{-level structure } /S\}$$

$$Y_0^{\mathfrak{D}}(\mathfrak{n}) : \text{Schemes} \longrightarrow \text{Sets}, \quad S \mapsto \{\text{Drinfeld-Stuhler modules with } \Gamma_0(\mathfrak{n})\text{-level structure}/S\}$$

are represented by smooth, quasi-projective schemes over $C\backslash\{\mathfrak{n}, \mathrm{Ram}(\mathcal{D}), \infty\}$ of relative dimension one which we denote by $\mathcal{E}ll_{C,\mathcal{D},\mathfrak{n}}, Y^{\mathfrak{D}}(\mathfrak{n})$ and $Y_0^{\mathfrak{D}}(\mathfrak{n})$ respectively. Furthermore, if $\mathfrak{D}$ is non-trivial, then $Y_0^{\mathfrak{D}}(\mathfrak{n})$ is proper over $C\backslash\{\mathfrak{n}, \mathrm{Ram}(\mathcal{D}), \infty\}$.

Proposition-Definition 4.7.8 *There exists a canonical regular compactification of $Y^{\mathfrak{D}}(\mathfrak{n})$ and $Y_0^{\mathfrak{D}}(\mathfrak{n})$ which we denote by $X^{\mathfrak{D}}(\mathfrak{n})$ and $X_0^{\mathfrak{D}}(\mathfrak{n})$ respectively.*

Remark 4.7.9 *If $\mathfrak{D} \neq 1$, the moduli spaces in Thm 4.7.7 are already proper. As the canonical regular compactification of a proper curve is itself, in this case the compactification does not affect the construction.*

While this was originally proven in [Dri74], this was explained in more detail by Lehmkuhl [Leh09] in his Habilitationsschrift. For a modular interpretation, which also yields a compactification of the moduli space of Drinfeld modules of higher rank, R. Pink introduced the notion of *generalized Drinfeld modules* (see [Pin13]). Following the terminology of Ribet [Rib90a], we will use the following later on:

Definition 4.7.10 (Enhanced Drinfeld-Stuhler module) *We refer to a pair $(\varphi, C_{\mathfrak{n}})$ consisting of a Drinfeld-Stuhler module and a $\Gamma_0(\mathfrak{n})$-level structure as an* enhanced Drinfeld-Stuhler module.

Remark 4.7.11 This is similar to the situation for Shimura curves $X_0^D(N)$ which are smooth, quasi-projective curves defined over $\operatorname{Spec}\mathbb{Z}\backslash\{N, D\}$ (notice that $\operatorname{Spec}\mathbb{Z}$ is analogous to $C\backslash\{\infty\}$). In particular the "naive" level structure is not sufficient to study the reduction at primes dividing the level. This was first resolved by Katz-Mazur [KM85] for trivial discriminants and extended by Buzzard in [Buz97] to arbitrary discriminants.

4.8 Various modularity theorems

In [Pra03] modular forms are related to geometry via the Modularity theorem (formerly known as the Shimura-Taniyama-Weil conjecture) which gives a bijection between Hecke eigenforms and (isogeny) classes of elliptic curves. There exists a direct analogue in the function field setting which follows from the work of Drinfeld on the Langlands program, which was written down in [GR96]. In fact in [Gek95] E.U. Gekeler gave an analytical construction of the elliptic curve associated to a Hecke eigenform. This analytical construction was adapted by M. Papikian in [Pap13] to give a "quaternionic" analogue of the Modularity conjecture.

Definition 4.8.1 (Optimal quotient) *Let E be an elliptic curve and X a smooth proper curve. We say that a morphism* $\mathrm{Jac}(X) \to E$ *is* optimal *if the kernel is connected, or equivalently if the dual morphism $E \to \mathrm{Jac}(X)$ is injective.*

Remark 4.8.2 Sometimes in the literature, an elliptic curve admitting an optimal covering is referred to as a *strong Weil curve* (see for example [Gek95]).

Proposition 4.8.3 *Every isogeny class of elliptic curves admits a unique representative with an optimal curve.*

Proof. By the Modularity theorem, for any representative E of the isogeny class, there exist a non-trivial morphism $\pi : E \to \mathrm{Jac}(X_0^{\mathfrak{D}}(\mathfrak{n}))$. This factors through an injective morphism $E/\ker(\pi) \to \mathrm{Jac}(X_0^{\mathfrak{D}}(\mathfrak{n}))$. □

Theorem 4.8.4 ([DK73], [GR96, Section 8.3.11]) *Let $\mathfrak{n} \subset \mathbb{F}_q[t]$ be an ideal. Then there exists a bijection between*

1. *the set of isogeny classes of elliptic curves over $\mathbb{F}_q(t)$ with conductor $\mathfrak{n}\cdot\infty$ with multiplicative reduction at ∞*

2. *optimal elliptic curves over $\mathbb{F}_q(t)$ with conductor $\mathfrak{n} \cdot \infty$ with multiplicative reduction at ∞*

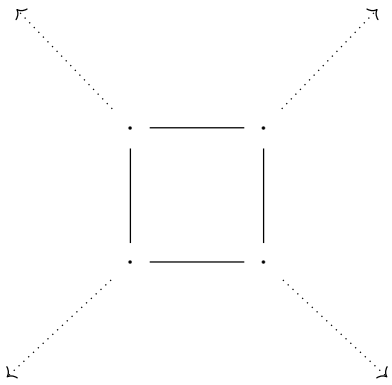
3. *the set of normalized Hecke eigenforms $\mathcal{H}_!^{new}(\mathcal{T}, \mathbb{Q})^{\Gamma_0(\mathfrak{n})}$*

4. *one dimensional isogeny factors of $J_0^{\text{new}}(\mathfrak{n})$.*

Theorem 4.8.5 (Quaternionic Parametrization [Pap13]) *There exists a canonical bijection between quaternionic eigenforms with integrally normalized Hecke-eigenvalues and optimal elliptic curves with multiplicative reduction at $\mathcal{D}$, split multiplicative reduction at ∞ and good reduction at all other places.*

Remark 4.8.6 There exists an "analytical" construction of these optimal elliptic curves due to Gekeler [Gek95] and Flynn [Fly13]. We will explain this at the end of Section 5.11.

Example 4.8.7 ([Gek95, Ex 4.2]) Consider the case of $\mathbb{F}_2(t)$. There exist explicit formulae to compute the genus of the modular curve $X_0(\mathfrak{n})$, see for instance [Gek80]. From them one can see that $X_0(t^3)$ is of genus one. Then the quotient $\Gamma_0(t^3)\backslash\mathcal{T}$ is



where the dotted lines indicate half-lines, which corresponds to the cusps. The Peterson product of a basis φ of $\mathcal{H}_!(\mathcal{T}, \mathbb{Z})^{\Gamma_0(t^3)}$ is $\langle\varphi, \varphi\rangle = 4$. Then Gekeler gives an explicit formula for the strong Weil curve associated to it, namely

$$Y^2 + tXY = X^3 + t^2.$$

Proposition-Definition 4.8.8 (Modular degree) *In any isogeny class of elliptic curves there exists a unique representative E such that the associated morphism $J_0^{\mathfrak{D}}(\mathfrak{n}) \to E$ is optimal. The* modular degree *is the degree of the covering $X_0^{\mathfrak{D}}(\mathfrak{n}) \to E$ which we shall denote by $\delta^{\mathfrak{D}}(\mathfrak{n})$.*

The modular degree is a subtle invariant of an isogeny class of elliptic curves. In particular over number fields, a sufficiently sharp bound on the growth of the modular degree with respect to the conductor of the isogeny class would imply the ABC-conjecture. In the function field case we have such an estimate due to M.Papikian [Pap07b][4]: He showed that the modular degree satisfies

$$\deg \pi_E = q^{2g-2+\mathfrak{n}_E} \frac{\tilde{c}^2 |\text{coker}\pi_{*,\infty}|^2}{|\Phi(E,\infty)|} L(1, Sym^2 \pi_E)$$

where $\tilde{c}$ is the Manin-constant, $\Phi(E,\infty)$ is the component group (see Def 6.1.2) and $\pi_{*,\infty} : \Phi(J_0^{\mathfrak{D}}(\mathfrak{n}),\infty) \to \Phi(E,\infty)$. By [Pál10, Prop 1.2], we know that the Manin-constant is a power p. Via Rankin-Selberg methods one gets

$$\deg \pi = O(q^{\deg \mathfrak{n}_E} (\deg \mathfrak{n}_E)^3)$$

(here one ought to think of $q^{\deg \mathfrak{n}_E}$ as being equivalent to N_E in the number field case).
In the number field case, the modular degree knows certain congruence information. To make this precise, denote by

$$r_E := |\frac{S_2(\mathbb{Z})}{\mathbb{Z}f + (\mathbb{Z}f)^\perp}|.$$

In this case, by the work of Agashe-Ribet-Stein [ARS12], we know that

1. $\deg \pi_E | r_E$
2. if $\text{ord}_p(N) \leqslant 1$, then $\text{ord}_p(\deg \pi_E) = \text{ord}_p(r_E)$
3. if $\text{ord}_p(\deg \pi_E) < \text{ord}_p(r_E)$, then

$$\dim_{\mathbb{T}/\mathfrak{m}} J_0(N)[\mathfrak{m}] > 2$$

that is we have failure of multiplicity one for the maximal ideal $\mathfrak{m} \subset \mathbb{T}$ associated to E.

In the function field case, the results seem, as E.U. Gekeler notes in [Gek95], stronger, as in this case we have

$$\deg \pi_E = r_E$$

(see [Gek95, Corollary 3.20]). Similar interpretations of the modular degree for quaternionic modular forms generalizing the results of Gekeler have been done in the PhD thesis of Ryan Flynn

[4]while this implies the function field version of the ABC-conjecture, this can be proven using more elementary methods, see [Ros13, Thm 7.17]

[Fly13]. For elliptic curves over $\mathbb{Q}$, we also have an "analytical" interpretation of the modular degree as

$$\deg \pi_f \int_{E_f(\mathbb{C})} \omega_E \wedge \overline{\omega_E} = (f, f)_P.$$

This too has a function field analogue, which we shall review in Subsection 6.8.1.

CHAPTER V

Uniformization and Bad Reduction

In this section we will be studying the bad reduction of the moduli space of Drinfeld-Stuhler modules. As with Shimura curves, this arises in two different ways:

1. the bad reduction at $\mathfrak{p}$ if $\mathfrak{p}$ divides the level, which is the situation analogous to the one studied by Deligne-Rapoport and then Buzzard

2. the bad reduction at $\mathfrak{p}$ if $\mathfrak{p}$ divides the discriminant $\mathfrak{D}$, which is the situation analogous to the one studied by Cerednik-Drinfeld.

The analogue of Cerednik-Drinfeld was studied by Hausberger [Hau05] in his PhD thesis and Papikian [Pap10] used this to give the description of the special fiber we need. The analogue of Deligne-Rapoport, or rather more accurately its extension due to Buzzard [Buz97] has not yet been studied in the detail we needed it. Existing work that we should however mention is that of Gekeler [Gek86], who studied the reduction of $X_0(\mathfrak{p})$ at $\mathfrak{p}$, that of Boyer [Boy99] who studied the reduction if one uses $\Gamma(\mathfrak{p}^n)$-level structure, and the work of Biecker [Bie23], who studied $\Gamma_0(\mathfrak{p}^n), \Gamma_1(\mathfrak{p}^n)$-level structures on Drinfeld shtukas.
We adapt the proof of Buzzard to the function field setting, using the work of [Boy99] and [Bie23]. Buzzard's proof relies on several ingredients:

1. If A is an fake elliptic curve over a field of characteristic $p \nmid \mathrm{Ram}(D)$, the p-divisible group satisfies a "Morita"-type property, that is there exists an elliptic curve E such that we have an isomorphism of $\mathcal{O}_D/p \cong \mathrm{Mat}_2(\mathbb{F}_p)$-modules $A[p^\infty] \cong (E[p^\infty])^2$,

2. by the Serre-Tate theorem, the local rings of $X_0^D(pN)$ and $X_0(pN)$ are related, in particular the regularity and flatness of $X_0^D(pN)$ can be deduced from $X_0(pN)$, which was already established in [KM85],

3. the only rank p subgroups $E[p] \cong \mu_p \times \mathbb{Z}/p$ are Cartier-dual to one another and thus on the ordinary loci $X_0^D(N)^h_{\mathbb{F}_p}$ we have an isomorphism

$$X_0^D(N)^h_{\mathbb{F}_p} \coprod X_0^D(N)^h_{\mathbb{F}_p} \xrightarrow{\sim} X_0^D(pN)^h_{\mathbb{F}_p}$$

The correct notion of the level structure at a prime dividing the level (for Drinfeld modules) was first introduced by Drinfeld [Dri74] which inspired the work of Katz-Mazur [KM85]. Furthermore there exists a function field analogue of p-divisible groups, which are called *divisible local Anderson modules*, which we review in Section 5.2. However, it will be more convenient for us to use the function field analogue of Dieudonné modules. These were introduced by Drinfeld [Dri74]. They satisfy the same formal properties as their p-adic counterparts and in particular they control the deformation theory of their global counter-part. The final ingredient that should be mentioned is that Cartier-duality is not the "correct" duality on the torsion of Drinfeld-Stuhler modules $\varphi[\mathfrak{n}]$, that is we have to use Taguchi-duality, which we review in Section 4.6.

5.1 Setting for Chapter V

Throughout this chapter, we shall set

$$A = \mathbb{F}_q[t], \quad X = \mathbb{P}^1_{\mathbb{F}_q}.$$

From Section 5.7 onward, we shall further restrict ourselves from general central simple algebras to quaternion algebras, that is the $d = 2$ case.

5.2 Divisible local Anderson modules

Definition 5.2.1 *A* p-divisible group of height h over $\operatorname{Spec} R$ *is a diagram of finite commutative group schemes*

$$G_0 \to G_1 \to \ldots \to G_n \to \ldots$$

where G_i *is of order* p^{hi} *such that we have a short exact sequence*

$$0 \to G_n \to G_{n+1} \xrightarrow{p^n} G_{n+1}.$$

Example 5.2.2 The standard example comes from the p-torsion points of an abelian variety B: the associated p-divisible group will be of height $2 \dim B$.

We will now give a function field analogue of p-divisible groups. This uses the notion of a *strict module scheme*, for whose definition we refer the reader to Def 1.1.8. The reason for this is that in the words of [Rö18], "this definition is, unfortunately, somewhat unwieldy". Fortunately, one can bypass it entirely by using a category of simpler object, which we study in the next subsection.

Definition 5.2.3 *Let $\mathfrak{p}$ be a maximal ideal of A, and denote by ϖ a uniformizer of $\hat{A}_{\mathfrak{p}}$. Let $S \in \mathrm{Nilp}_{\hat{A}_{\mathfrak{p}}}$ and denote by ζ the image of ϖ in the local sections of S. Then a* ϖ-divisible local Anderson module over S *is a sheaf of $A_{\mathfrak{p}}$-modules G on S_{fppf} such that*

1. *G is ϖ-torsion, i.e. $G = \mathrm{colim} G[\varpi^n]$*
2. *G is ϖ-divisible, i.e. $\varpi : G \to G$ is surjective*
3. *all $G[\varpi^n]$ are representable by strict (see Def 1.1.8) finite $\kappa(\mathfrak{p})$-module schemes over S and,*
4. *locally on S there exists a $d \in \mathbb{N}$ such that $(\varpi - \zeta)^d = 0$ on ω_G, where*

$$\omega_{G[\varpi^n]} := (X \xrightarrow{0} G[\varpi^n])^* \Omega^1_{G[\varpi^n]/X}$$

$$\omega_G := \lim \omega_{G[\varpi^n]}.$$

5.3 $A_{\mathfrak{p}}$-Dieudonné modules

The goal of this section is to introduce a function field object which imitates the Dieudonné module of a divisible group. Hence they are often referred to as *Dieudonné modules*. Our exposition of this will follow [Boy99], [HS19] and [Pap10] (we choose to denote by $A_{\mathfrak{p}}$ what Boyer chooses to call $\mathcal{O}$-Dieudonné modules to avoid notational clash with Papikian).
Let $A_{\mathfrak{p}}$ be the ring of integers of a local field, with residue field κ, $|\kappa| = q$ and denote by ϖ the uniformizer. For any $A_{\mathfrak{p}}$-algebra B, we denote by ζ the image of ϖ.

Definition 5.3.1 ([Boy99, Def 5.2.1]) *Let B be a local $A_{\mathfrak{p}}$-algebra with nilpotent maximal ideal. A* $A_{\mathfrak{p}}$-Dieudonné module (M, F) of rank d on B *is a free $A_{\mathfrak{p}}\hat{\otimes}_{\kappa}B$-module M of rank d with a morphism*

$$(\mathrm{Id} \otimes \mathrm{Frob})^* M \xrightarrow{F} M$$

such that coker$F$ *is a free B-module of finite type.*

Definition 5.3.2 ([HS19, Def 2.1]) *Let B be a local $A_{\mathfrak{p}}$-algebra with nilpotent maximal ideal. An $A_{\mathfrak{p}}$-Dieudonné module (M, F) over B is said to be an* effective local shtuka *if the morphism*

$$(\mathrm{Id} \otimes \mathrm{Frob})^* M \left[\frac{1}{\varpi - \zeta}\right] \xrightarrow{F} M \left[\frac{1}{\varpi - \zeta}\right]$$

is an isomorphism. We call the full subcategory of Dieudonné modules consisting of local shtukas the category of effective local shtukas.

Theorem 5.3.3 ([HS19, Thm 8.3]) *Let B be a local $A_{\mathfrak{p}}$-algebra with nilpotent maximal ideal. The category of effective local shtukas and the category of ϖ-divisible local Anderson modules are anti-equivalent.*

Definition 5.3.4 *We say that an $A_{\mathfrak{p}}$-Dieudonné module (M, F) is* étale *if F is an isomorphism. If there exists some n such that the induced morphism*

$$(\mathrm{Id} \otimes \mathrm{Frob}^n)^* M/\varpi \to M/\varpi$$

is the zero morphism, then we say that (M, F) is topologically nilpotent.

Remark 5.3.5 If G is a p-divisible group over a complete noetherian local ring R, there exists a *connected-étale decomposition of G*, that is we have a short exact sequence

$$0 \to G^0 \to G \to G^{\text{ét}} \to 0$$

where G^0 and $G^{\text{ét}}$ are connected and étale respectively (i.e. their constituent finite group schemes are connected respectively étale). An analogous result exists over function fields (remembering that p-divisible groups and Dieudonné modules are contravariantly related).

Proposition 5.3.6 ([Boy99, Lemma 6.2.3]) *Let (M, F) be an $A_{\mathfrak{p}}$-Dieudonné module over B. Then there exists a canonical short exact sequence*

$$0 \to (M^{\text{ét}}, F^{\text{ét}}) \to (M, F) \to (M^c, F^c) \to 0$$

such that $(M^{\text{ét}}, F^{\text{ét}})$ is an étale Dieudonné-module and (M^c, F^c) is topologically nilpotent. If B is a perfect field, then this extension splits canonically.

Definition 5.3.7 *Let $\breve{A}_{\mathfrak{p}}$ be the completion of the maximal unramified extension of $A_{\mathfrak{p}}$. We shall denote by $\mathcal{C}$ the full subcategory of the category of $\breve{A}_{\mathfrak{p}}$-algebras whose objects consists of those $\breve{A}_{\mathfrak{p}}$-algebras which are complete, noetherian and whose residue fields are isomorphic to $\breve{A}_{\mathfrak{p}}/\mathfrak{p}$.*

Definition 5.3.8 *Let $(\overline{M}, \overline{F})$ be an $A_{\mathfrak{p}}$-Dieudonné module over κ and R an element of $\mathcal{C}$. A* deformation of $(\overline{M}, \overline{F})$ to R *is an $A_{\mathfrak{p}}$-Dieudonné module over R whose reduction modulo $\mathfrak{p}$ is isomorphic to $(\overline{M}, \overline{F})$. We denote by*

$$\mathrm{Def}_{_}(\overline{M}, \overline{F}) : \mathcal{C} \to \mathrm{Set}, \quad R \mapsto \{\text{deformations of } (\overline{M}, \overline{F}) \text{ to } R.\}$$

Proposition 5.3.9 ([Boy99]) *For any $A_{\mathfrak{p}}$-Dieudonné module $(\overline{M}, \overline{F})$ over κ the functor* $\mathrm{Def}_{_}(\overline{M}, \overline{F})$ *is representable.*

5.4 $\mathbb{F}_{\mathfrak{p}}$-Dieudonné modules

Definition 5.4.1 ([HS19, Def 2.6]) *Let R be an arbitrary $\mathbb{F}_q$-algebra. A* $\mathbb{F}_{\mathfrak{p}}$-Dieudonné module over R *is a pair (M, F) consisting of a locally free R-module of finite rank and an R-module morphism*

$$F : \mathrm{Frob}^* M \to M.$$

A morphism of $\mathbb{F}_{\mathfrak{p}}$-Dieudonné modules over R *is a morphism $f : (M, F_M) \to (N, F_N)$ of the underlying R-modules which commutes with F.*

Remark 5.4.2 For any $A_{\mathfrak{p}}$-Dieudonné module (M, F) over R and $n \geqslant 1$, the quotient $(M/\varpi^n, F/\varpi^n)$ is a $\mathbb{F}_{\mathfrak{p}}$-Dieudonné module over R.

Example 5.4.3 *Let $(\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}} \xrightarrow{f} (\mathcal{E}'_i, j'_i, t'_i)_{i\in\mathbb{Z}}$ be an $\mathfrak{p}$-torsion isogeny over $\mathrm{Spec}\, R$. Then the cokernel of f is an $\mathbb{F}_{\mathfrak{p}}$-Dieudonné module over R.*

Remark 5.4.4 It should be mentioned that $\mathbb{F}_{\mathfrak{p}}$-Dieudonné module have an (unfortunately) inconsistent naming convention: in [Dri87] they were introduced as "φ-sheaves", in [Lau96] they are called "$\mathbb{F}_{\mathfrak{p}}$-Dieudonné modules" and in [HS19] they are called "finite $\mathbb{F}_{\mathfrak{p}}$-shtukas".

Definition 5.4.5 *Let R be a $\mathbb{F}_q$-algebra. Then define a contravariant functor*

$$\mathrm{Dr}_q : (\mathbb{F}_{\mathfrak{p}}\text{-Dieudonné modules over } R) \to (\text{group schemes over } R)$$

$$\mathrm{Dr}_q((M, F)) := \underline{\mathrm{Spec}}(\mathrm{Sym}^{\bullet}_R M) / \left(m^{\otimes q} - F(m)\right).$$

Definition 5.4.6 *For any field of characteristic p, denote by Gr_k^+ the full subcategory of abelian group schemes which can be embedded into $\mathbb{A}^S$ for some set S.*

Theorem 5.4.7 ([HS19, Thm 5.2] and [DG0 , IV, 3, 6.7]) *If (M, F) is a $\mathbb{F}_{\mathfrak{p}}$-Dieudonné module of rank r over S, then $\mathrm{Dr}_q((M, F))$ is a finite locally free group scheme of rank $|\mathfrak{p}|^r$ over S. Furthermore (M, F) is étale if and only if $\mathrm{Dr}_q((M, F))$ is. Finally over a field k, we have an anti-equivalence*

$$(\mathbb{F}_p\text{-Dieudonné modules}_k)^{\mathrm{opp}} \xrightarrow{\sim} \mathrm{Gr}_k^+$$

5.5 $\mathcal{D}$-Dieudonné modules

Let $\mathcal{D}$ be an $A_{\mathfrak{p}}$-central simple algebra of dimension d^2. In this section we define $\mathcal{D}$-Dieudonné modules which allow us to classify and study the local behaviour of $\mathcal{D}$-elliptic sheaves. In particular this category encodes the infinitesimal deformation theory of $\mathcal{D}$-elliptic sheaves.

Definition 5.5.1 (Dieudonné $\mathcal{D}$-modules) *An $A_{\mathfrak{p}}$-Dieudonné module (M, F) over R with a right $\mathcal{D}$-action commuting with F of rank d^2 is said to be a* Dieudonné $\mathcal{D}$-module*. A morphism of the underlying Dieudonné modules is said to be a* morphism of Dieudonné $\mathcal{D}$-modules *if it commutes with the $\mathcal{D}$-action.*

Definition 5.5.2 (Dieudonné $\mathcal{D}$-module associated to $\mathbb{E}$) *Let B be a local $A_{\mathfrak{p}}$-algebra and $\mathbb{E} = (\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$ be a $\mathcal{D}$-elliptic sheaf over $\operatorname{Spec} B$ and $\mathfrak{p} \in X$. The* Dieudonné module of $\mathbb{E}$ *is the free $A_{\mathfrak{p}} \otimes B$-module of rank d^2 with right action of $\mathcal{D}_{\mathfrak{p}}$*

$$\mathbb{E}_{\mathfrak{p}} := H^0(\operatorname{Spec} A_{\mathfrak{p}} \otimes B, \mathcal{E}_0).$$

Example 5.5.3 For any scheme S and any $\mathcal{D}$-elliptic sheaf $\mathbb{E} = (\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$ over $\operatorname{Spec} R$, we can consider

$$\mathbb{E}_{\mathfrak{p}}/\mathfrak{p}^n = H^0(\operatorname{Spec} A_{\mathfrak{p}} \otimes B, \mathcal{E}_0) \otimes_{A_{\mathfrak{p}}\otimes R} A_{\mathfrak{p}}/\mathfrak{p}^n \otimes R.$$

This is a $\mathbb{F}_{\mathfrak{p}}$-Dieudoné module and thus we can associate to it a group scheme

$$\mathrm{Dr}_q(\mathbb{E}_{\mathfrak{p}}/\mathfrak{p}^n).$$

Definition 5.5.4 (Scheme of $\mathfrak{p}^n$-torsion points of $\mathcal{E}$) *If $\mathcal{E}$ is a Drinfeld-Stuhler sheaf of rank r over S. Then we call*

$$\mathcal{E}[\mathfrak{p}^n] = \mathrm{Dr}_q(\mathbb{E}_{\mathfrak{p}}/\mathfrak{p}^n)$$

the scheme of $\mathfrak{p}^n$-torsion points of $\mathcal{E}$.

Remark 5.5.5 The chain of anti-equivalences means that isogenies give subgroups of the scheme of torsion points: if $(\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}} \xrightarrow{f} (\mathcal{E}'_i, j'_i, t'_i)_{i\in\mathbb{Z}}$ is an $\mathfrak{p}$-torsion isogeny of $\mathcal{D}$-elliptic sheaves

over S, then we have a quotient $\mathcal{E}' \to \text{coker}(f)$, which at the level of group schemes yields an inclusion

$$\text{Dr}(\text{coker} f) \subset \mathcal{E}'[\mathfrak{p}].$$

Furthermore, if $\mathfrak{p}$ is away from the A-characteristic of S and that $d = 2$, by anti-equivalence Thm 5.4.7, these are étale, finite locally free group schemes over S of order $\mathfrak{p}|$ and $|\mathfrak{p}|^2$ respectively .

The following category bundles all the Dieudoné modules and the generic fiber and forms a category (satisfying the properties that such a collection coming from a $\mathcal{D}$-elliptic sheaf has to satisfy). While this is not required for the work in this thesis, this was used by Papikian to classify Drinfeld-Stuhler modules over finite fields, akin to the work of Deuring.

Definition 5.5.6 *A* φ-space over κ *is a pair* (M, φ) *consisting of a finite rank* $F \otimes \kappa$*-module* M *and a morphism*

$$(\text{id} \otimes \text{Frob})^* M \xrightarrow{\varphi} M.$$

A morphism of φ-pairs *is a morphism of the underlying modules which respects the Frobenius-semi-linear morphisms.*

Definition 5.5.7 *Let* $(\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$ *be a* $\mathcal{D}$*-elliptic sheaf over* κ*. The* generic fiber of $(\mathcal{E}_i, j_i, t_i)_{i\in\mathbb{Z}}$ *is the* φ*-space* $H^0(\text{Spec}\, F \otimes \kappa, \mathcal{E}_0)$ *where* φ *is induced by* t_0*.*

Definition 5.5.8 (DMod [Pap10, Def 4.7]) *The category whose objects are tuples*

$$\big((N, \varphi, \iota), (M_x)_{x\in|X|}\big)$$

consisting of a φ*-space of rank* d^2 *over* $F \otimes k$ *with* $\iota : D^{opp} \to \text{End}(N, \varphi)$ *and a collection of* $\mathcal{D}_x$*-lattices in* (N_x, φ_x) *such that*

1. *if* $x = \infty$*, then*

$$M_\infty \subset \varphi_\infty(M_\infty), \dim_k(\varphi_\infty(M_\infty)/M_\infty) = d, \varphi_\infty^{d\deg\infty}(M_\infty) = \pi_\infty^{-1} M_\infty$$

   where π_∞ *is a uniformizer of* $\mathcal{O}_\infty$

2. *if* $x = \mathfrak{p}$*, then*

$$\pi_\mathfrak{p} M_\mathfrak{p} \subset \varphi_\mathfrak{p}(M_\mathfrak{p}) \subset M_\mathfrak{p}$$

   (where $\pi_\mathfrak{p}$ *is a uniformizer of* $\mathcal{O}_\mathfrak{p}$*) and the quotient* $M_\mathfrak{p}/\varphi_\mathfrak{p}(M_\mathfrak{p})$ *is an* $\kappa(\mathfrak{p}) \otimes k$*-module of length* d *and supported on the connected component of the given embedding* $\kappa(\mathfrak{p}) \to k$

3. *if $x \neq \mathfrak{p}, \infty$, then we have $\varphi_x(M_x) = M_x$*

4. *there exists a basis of the generic fiber N generating N_x for all but finitely many $x \in |X|$.*

A morphism $\big((N,\varphi,\iota),(M_x)_{x\in|X|}\big) \to \big((N,\varphi,\iota),(M_x)_{x\in|X|}\big)$ is a morphism of φ-spaces $f : (N,\varphi) \to (N',\varphi')$ which is compatible with ι and ι' and for all $x \in |X|\backslash\{\infty\}$ we have $f_x(M_x) \subset M'_x$.

The key then is that this collection of local data classifies $\mathcal{D}$-elliptic sheaves uniquely.

Theorem 5.5.9 ([Pap10, Prop 4.8]) *The category of $\mathcal{D}$-elliptic sheaves of characteristic $\mathfrak{p}$ over k is naturally equivalent to the category* DMod *constructed above in Def 5.5.8.*

We will now focus on the local behaviour of $\mathcal{D}$-elliptic sheaves at primes not dividing the discriminant of $\mathcal{D}$. As one would expect from our intuition coming from Morita equivalence, in this case the category of $\mathcal{D}_\mathfrak{p}$-Dieudonné modules is equivalent to the category of $A_\mathfrak{p}$-Dieudonné modules.

Proposition 5.5.10 (Morita equivalence, [Boy99, Prop 6.2.2]) *If $\mathcal{D}_\mathfrak{p} \cong \mathrm{Mat}_d(A_\mathfrak{p})$, then we have an equivalence of categories between $\mathcal{D}$-Dieudonné modules and $A_\mathfrak{p}$-Dieudonné modules.*

Remark 5.5.11 This is of course in perfect analogy with the behaviour abelian surfaces, and this is used by Buzzard [Buz97] to simplify the study of the integral model $\mathcal{X}_0^D(N)$ at primes dividing the level to reduce the study at these primes to the previous work of Katz-Mazur [KM85]. We shall do the same in our proof later on.

For the definition of the deformation functor of a $\mathcal{D}$-elliptic sheaf, we shall use the category $\mathcal{C}$ defined in Def 5.3.7.

Definition 5.5.12 *Let κ be a field of non-zero A-characteristic $\mathfrak{p}$ and let $(\overline{\mathcal{E}},\overline{j}_i,\overline{t}_i)$ be a $\mathcal{D}$-elliptic sheaf over κ. For any $R \in \mathrm{Ob}(\mathcal{C})$, we say that a* deformation of $(\overline{\mathcal{E}},\overline{j}_i,\overline{t}_i)$ to R *is a $\mathcal{D}$-elliptic sheaf over R whose reduction modulo $\mathfrak{p}$ is isomorphic to $(\overline{\mathcal{E}},\overline{j}_i,\overline{t}_i)$. We denote by*

$$\mathrm{Def}_-(\overline{\mathcal{E}}) : \mathcal{C} \to \mathrm{Set}, \quad R \mapsto \{\text{deformations of } (\overline{\mathcal{E}},\overline{j}_i,\overline{t}_i) \text{ to } R.\}$$

Theorem 5.5.13 (Analogue of Serre-Tate [Boy99, Prop 5.3.1]) *Let κ be a field of non-zero A-characteristic $\mathfrak{p}$ and assume that $\mathfrak{p} \nmid \mathrm{disc}(\mathcal{D})$. Let $(\overline{\mathcal{E}},\overline{j}_i,\overline{t}_i)$ be a $\mathcal{D}$-elliptic sheaf over κ and denote by $(\overline{M}_0,\overline{F}_0)$ the associated $A_\mathfrak{p}$-Dieudonné module. Then we have an isomorphism of functors*

$$\mathrm{Def}_-(\overline{\mathcal{E}}) \xrightarrow{\sim} \mathrm{Def}_-(\overline{M}_0,\overline{F}_0).$$

5.6 Exceptional and superspecial objects

In this section we present classification results for the reductions of Drinfeld-Stuhler modules, following the exposition of [Pap10],[Pap18].

Definition 5.6.1 *We say a Drinfeld-Stuhler module over an A-field of characteristic $\mathfrak{q}$ is* supersingular *if $\varphi[\mathfrak{q}]$ is connected.*

Example 5.6.2 Let $\mathfrak{p} = (t)$. Then the Drinfeld module over $\mathbb{F}_\mathfrak{p}$ defined by $t \mapsto \tau^2$ is supersingular. This is for instance checked by computing the t-torsion:

$$\varphi[(t)](\bar{\mathbb{F}}_p) := \{r \in \bar{\mathbb{F}}_p \mid r^{p^2} = 0\} = \{0\}.$$

Remark 5.6.3 (Warning!) Elkies showed that for any elliptic curve over $\mathbb{Q}$ there exists infinitely many primes p such that the reduction at p of E is supersingular (and the same is true for number fields with a real place in general). However it was shown by Bjorn Poonen in [Poo98] that there exist Drinfeld modules with *no primes of supersingular reduction*! A simple example he provides is the following:

$$\varphi(t) = T(1-\tau)^2$$

(assuming q is odd). Another difference in the theory of reductions of elliptic curves over $\mathbb{Q}$ and Drinfeld modules over $\mathbb{F}_p(t)$ is thatthere exist Drinfeld modules with good reduction everywhere.

Proposition 5.6.4 ([Pap18, Lemma 5.6]) *Let k be an A-field of characteristic $\mathfrak{p}$ and assume that $\mathfrak{p} \mid \mathfrak{D} = \mathrm{Ram}(D)$. Then any Drinfeld-Stuhler module is supersingular.*

Proof. (SKetch) It suffices to show that the $\mathfrak{p}$-Tate module is 0, which apriori is an $A_\mathfrak{p}$-module of rank $\leqslant d^2$. If $V_\mathfrak{p}(\varphi) \neq 0$, then $D_\mathfrak{p}$ must act freely, as otherwise say $d \in D_\mathfrak{p}, v \in V_\mathfrak{p}(\varphi)\backslash\{0\}$ the condition $dv = 0$ implies that $d^{-1}dv = v = 0$. In particular then $V_\mathfrak{p}(\varphi)$ must be of dimension d^2. However for any $\varpi \in \mathfrak{p}$, we know that $\partial\varphi(\varpi) = \varpi = 0 \in k$ and thus $\varphi(\varpi) \in \mathrm{Mat}_d(k[\tau])\tau$. Hence $\varphi[\varpi] \subset \mathbb{G}_a^d$ is *not* reduced and thus $V_\varphi(\varpi)$ has to be of dimension less then d^2, showing that $\varphi[\varpi] = 0$. □

For any Drinfeld-Stuhler module over $\bar{\mathbb{F}}_p$ the morphism $\partial_\varphi : \mathcal{O}_D \to \mathrm{Mat}_d(\bar{\mathbb{F}}_p)$ factors through $\mathcal{O}_D/\mathfrak{p}$. Thus there exists $\Pi \in \mathcal{O}_D/\mathfrak{p}$ such that

$$\mathcal{O}_D/\mathfrak{p} \cong \mathbb{F}_{\mathfrak{p}^d} \oplus \ldots \oplus \mathbb{F}_{\mathfrak{p}^d}\Pi^{d-1} \tag{6.4}$$

where $\Pi^d = 0$ and for all α we have $\Pi\alpha = \alpha^{|\mathfrak{p}|}\Pi$.

Proposition-Definition 5.6.5 (Type of Drinfeld-Stuhler module) *Given a Drinfeld-Stuhler module φ, the ring $\mathbb{F}_{\mathfrak{p}^d}$ acts on $\mathbb{F}_{\mathfrak{p}^d}^d$ via*

$$\mathbb{F}_{\mathfrak{p}^d} \to \mathcal{O}_D/\mathfrak{p} \xrightarrow{\partial_\varphi} \mathrm{Mat}_d(\bar{\mathbb{F}}_p).$$

Then we can decompose k^d as the direct sum of

$$V_i := \{v \in V \mid \partial_\varphi(r)v = r^{|\mathfrak{p}|^i} v \text{ for all } r \in \mathbb{F}_{\mathfrak{p}^d}\}.$$

The ordered d-tuple $t(\varphi) := (\dim V_0, \ldots, \dim V_{d-1})$ is the type of φ.

Definition 5.6.6 *For any fixed ordered d-tuple $\mathfrak{t}$, we introduce the notation*

$$\Sigma_{\mathfrak{t}} := \{\text{Drinfeld-Stuhler modules of type } \mathfrak{t}\}_\sim.$$

Remark 5.6.7 For any fixed ordered d-tuple $\mathbb{F}$, M.Papikian [Pap10] denotes by $\mathcal{X}_{\mathfrak{t}}$ the full subcategory of $\mathcal{D}$-elliptic sheaves modules which are of type $\mathfrak{t}$ and by $\mathcal{X}_{\mathfrak{t}}/\mathbb{Z}$ the same category up to shifts (i.e. the category of Drinfeld-Stuhler modules of type $\mathbb{F}$). However this clashes with the notation of the character group of a scheme, and thus to avoid notational overload, we use the above notation. We choose this notation as it is also consistent with the notation in the number field case [Rib90a].

Definition 5.6.8 *We say that φ is*

1. mixed/special[1] *if* $t(\varphi) = (1, \ldots, 1)$
2. pure *if* $t(\varphi) = (0, \ldots, d, 0, \ldots, 0)$
3. exceptional *if* $\partial_\varphi(\Pi) = 0$.
4. superspecial *if φ is both special and exceptional.*

Remark 5.6.9 We shall try to refrain from referring to Drinfeld-Stuhler modules as "superspecial", as this is phonetically close to "super-singular". In analogy with Ribet we will refer to them as mixed-exceptional Drinfeld-Stuhler modules instead.

[1] we include both terminologies here, however "mixed" seems to be dominant for abelian varieties (for example in [Rib89]), whereas "special" seems more common over function fields (see for example [Pap10])

Example 5.6.10 ([Pap18, Ex 5.8]) Consider the quaternion algebra ramified at $t, t-1$. Then $\mathcal{O}_D = \mathbb{F}_{q^2}[t,z]/(z^2 - t(t-1))$ (with $zt = tz$, $t\alpha = \alpha t$ and $z\alpha = \alpha^q z$) is a maximal order. We set $\mathbb{F}_p = (t)$. Then

$$\mathcal{O}_D/\mathfrak{p} = \mathbb{F}_{q^2}[z]/(z^2)$$

and we can set $\Pi = z$. Papikian provides the following examples of Drinfeld-Stuhler modules φ which are

- **special but not exceptional:** the Drinfeld-Stuhler module defined by

$$\varphi(T) = \begin{pmatrix} \tau^2 & 0 \\ 0 & \tau^2 \end{pmatrix}, \varphi(z) = \begin{pmatrix} 0 & 1 \\ \tau^2(\tau^2-1) & 0 \end{pmatrix}, \varphi(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^q \end{pmatrix}$$

  is special but not exceptional.

- **exceptional but not special:** the Drinfeld-Stuhler module defined by

$$\varphi(T) = \begin{pmatrix} \tau^2 & 0 \\ 0 & \tau^2 \end{pmatrix}, \varphi(z) = \begin{pmatrix} 0 & \tau \\ \tau^2(\tau^2-1) & 0 \end{pmatrix}, \varphi(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$$

  is exceptional but not special.

- **superspecial:** the Drinfeld-Stuhler module defined by

$$\varphi(T) = \begin{pmatrix} \tau^2 & 0 \\ 0 & \tau^2 \end{pmatrix}, \varphi(z) = \begin{pmatrix} 0 & \tau^2 \\ \tau^2-1 & 0 \end{pmatrix}, \varphi(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^q \end{pmatrix}$$

  is superspecial.

Definition 5.6.11 *Let (M, F) be a $\mathcal{D}$-Dieudonné module over $\overline{\mathbb{F}}_{\mathfrak{p}}$. Choose a decomposition of $\mathcal{O}_D/\mathfrak{p}$ as in Eq 6.4. Then (M, F) admits a decomposition*

$$M_i := \{m \in M \mid m(r \otimes 1) = m(1 \otimes r^{q^i}), r \in \mathbb{F}_{q^d}\}.$$

Then both Π and F induce morphisms

$$M_i \underset{\Pi_i}{\overset{\varphi_i}{\rightrightarrows}} M_{i+1}.$$

We denote by $f_i := \dim(\text{coker}\varphi_i)$ *and we call the ordered* d*-tuple* $t((M,F)) := (f_0,\dots,f_{d-1})$ *the* type of (M,F). *We say that* (M,F) *is*

1. mixed/special *if* $t((M,F)) = (1,\dots,1)$
2. pure *if* $t((M,F)) = (0,\dots,d,0,\dots,0)$
3. exceptional *if* $\text{im}(\Pi) = \text{im}(F)$
4. superspecial *if it is both special and exceptional*

Remark 5.6.12 This allows us to call $\mathcal{D}$-elliptic sheaves mixed/pure/exceptional/superspecial, and these notions are compatible with the ones for Drinfeld-Stuhler modules.

Theorem 5.6.13 ([Pap10, Thm 5.3]) *For any* $\mathbb{E} \in \Sigma_{\mathfrak{t}}$*, the endomorphism algebra* $\text{End}(\mathbb{E})$ *is a hereditary* A*-order in* $\overline{D}$*, which is maximal at every* $\mathfrak{p} \in |X|\backslash\{\mathfrak{p},\infty\}$ *and*

$$\text{End}(\mathbb{E})_{\mathfrak{p}} \cong M_d(\mathbb{F}, A_{\mathfrak{p}}).$$

We can now give analogues of Deuring's classification.

Theorem 5.6.14 ([Pap10, Thm 5.4]) *For any* $\mathbb{E}_0 \in \Sigma_{\mathfrak{t}}$*, the map*

$$\Sigma_{\mathfrak{t}} \to \text{Pic}(\text{End}(\mathbb{E}_0)), \quad \mathbb{E} \mapsto \text{Hom}(\mathbb{E}, \mathbb{E}_0)$$

is a bijection.

Theorem 5.6.15 ([Pap08, Thm 6.2]) *Let* $\mathfrak{p} \notin Ram(\mathcal{D})$*. For any supersingular* $\mathcal{D}$*-elliptic sheaf* $\mathbb{E}_0$ *of characteristic* $\mathfrak{p}$*, the map*

$$(\textit{sup.sing } \mathcal{D}\textit{-elliptic sheaves}) \to \text{Pic}(\text{End}(\mathbb{E}_0)), \quad \mathbb{E} \mapsto \text{Hom}(\mathbb{E}, \mathbb{E}_0)$$

is a bijection.

We can also upgrade this classification to account for level structures prime to the A-characteristic of the ground field. For ease of notation we introduce the following

Definition 5.6.16 *For any fixed ordered* d*-tuple* $\mathfrak{t}$ *and* $\text{char}_A(k) \nmid \mathfrak{n}$*, we introduce the notation*

$$\Sigma_{\mathfrak{t}}(\mathfrak{n}) := \{\text{Drinfeld-Stuhler modules of type } \mathfrak{t} \text{ with } \Gamma_0(\mathfrak{n})\text{-level structures } \mathfrak{t}\}_{\sim}.$$

Theorem 5.6.17 *Let k be an algebraically closed field of A-characteristic $\mathfrak{p}$. For any $\mathfrak{p} \nmid \mathfrak{n}$, and for any $(\mathbb{E}_0, C_0) \in \Sigma_{\mathfrak{t}}(\mathfrak{n})$, the map*

$$\Sigma_{\mathfrak{t}}(\mathfrak{n}) \to \mathrm{Pic}(\mathrm{End}(\mathbb{E}_0, C_0)), \quad (\mathbb{E}, C) \mapsto \mathrm{Hom}((\mathbb{E}, C), (\mathbb{E}_0, C_0))$$

is a bijection.

Proof. Consider $(E \xrightarrow{\pi} E')$. We first prove that

$$M := \mathrm{Hom}((E \to E'), (E_0 \to E_0/C_0))$$

is an invertible $\mathrm{End}((E_0 \to E_0/C_0))$-module. For this we note that

$$\mathrm{End}((E_0 \to E_0/C_0)) = \mathrm{End}(E_0) \cap \mathrm{End}(E_0/C_0)$$

$$M = \mathrm{Hom}(E, E_0) \cap \mathrm{Hom}(E/C, E_0/C_0).$$

By Thm 5.6.14 we know that $\mathrm{Hom}(E, E_0)$ and $\mathrm{Hom}(E/C, E_0/C_0)$ are invertible $\mathrm{End}(E_0)$ and $\mathrm{End}(E_0/C_0)$-modules respectively. By Prop 2.5.24 the conclusion follows. Conversely, given an invertible $\mathrm{End}((E_0 \to E_0/C_0))$-module M, we can take the tensor products along the maps

$$\mathrm{End}(E_0) \leftarrow \mathrm{End}((E_0 \to E_0/C_0)) \to \mathrm{End}(E_0/C_0)$$

which gives us invertible modules

$$M \otimes \mathrm{End}(E_0) \text{ and } M \otimes \mathrm{End}(E_0/C_0).$$

By Thm 5.6.14, these invertible modules correspond to Drinfeld-Stuhler modules with a corresponding isogeny, which yields the desired inverse. □

5.7 Drinfeld level structures

The goal of this section is to introduce the right notion of level structures. These were originally introduced by Drinfeld and later used by Katz-Mazur to give integral models of moduli spaces of elliptic curves. This in tern later adapted by Buzzard [Buz97] to integral models of Shimura curves. In the context of $\mathcal{D}$-elliptic curves, Drinfeld $\Gamma(\mathfrak{p})$-level structures were used by Boyer in [Boy99] to give integral models, whereas in [Bie23], Biecker constructs integral models

of the moduli space of Drinfeld shtukas. We adapt this body of work to our setting, that is to give Drinfeld $\Gamma_0(\mathfrak{p})$-level structures of $\mathcal{D}$-elliptic sheaves. We will use the fact that $\mathfrak{p} \nmid \mathrm{Ram}(\mathcal{D})$ to use Morita equivalence.

Assumption: From here on out, we shall further restrict D to be a quaternion algebra, that is $d = 2$.

5.7.1 Drinfeld level structures of Drinfeld modules

Definition 5.7.1 ([Wie10, Def 3.1]) *Let $E = (\mathbb{G}_a, e)$ be a Drinfeld module of rank 2 over S, and let $\mathfrak{n} \subset A$ be a non-trivial ideal. A* Drinfeld $\Gamma(\mathfrak{n})$-level structure *is an A-linear morphism*

$$\iota : (A/\mathfrak{n})^2 \to E(S)$$

such that it induces an identity

$$E[\mathfrak{n}] = \sum_{x \in (A/\mathfrak{n})^2} \iota(x).$$

Remark 5.7.2 This identity can be rephrased as a condition of Cartier divisors, by using an embedding $E[\mathfrak{n}] \hookrightarrow \mathbb{A}^1$ provided by Prop 5.4.7.

Remark 5.7.3 This is an updated version of the notion of a Drinfeld level structure compared to the original one from [Dri74]. The original definition also consisted of an A-linear ι, however the condition on the Cartier divisor was that for all $\mathfrak{p} \supset \mathfrak{n}$ one has the identity $E[\mathfrak{p}] = \sum_{x \in (\mathfrak{p}^{-1}/A)^2} \iota(x)$. It was proven in [Wie10, Prop 3.2] that these two conditions are equivalent. The analogous result for abelian surfaces follows from the work of Katz-Mazur [KM85, Rem 1.11.4].

Proposition-Definition 5.7.4 ([Leh09, Cor 2.4] or [Bie23, Prop 3.13]) *If φ is a Drinfeld module of rank r over an algebraically closed field k, there exists an integer h such that*

$$\varphi[\mathfrak{p}^n] \cong \alpha_{|\mathfrak{p}|^{nh}} \times (A_{\mathfrak{p}}/\mathfrak{p}^n)^{r-h}.$$

We call the integer h the height of φ.

Remark 5.7.5 At first glance, this may look alarming, in that it seems to break the analogy with the number field case. For instance the torsion of an ordinary elliptic curve is $\mu_p \times \underline{\mathbb{Z}/p}$. In particular μ_p and $\underline{\mathbb{Z}/p}$ are Cartier dual. However this clearly can not be the case for an ordinary Drinfeld module. The solution will be to replace Cartier duality with *Taguchi duality*, in particular $\alpha_{|\mathfrak{p}|^n}$ and $(A_{\mathfrak{p}}/\mathfrak{p}^n)$ *are* Taguchi-dual.

Definition 5.7.6 *Let $E = (\mathbb{G}_a, e)$ be a Drinfeld module of rank 2 over S, and let $\mathfrak{n} \subset A$ be a non-trivial ideal. A* Drinfeld $\Gamma_0(\mathfrak{p}^n)$-level structure *consists of a finite locally free closed subscheme $\mathbb{H} \subset E[\mathfrak{p}^n]$ of rank $|\mathfrak{p}|^n$ which admits a generator fppf-locally on S, that is an A-linear morphism*

$$\iota : (A/\mathfrak{p}^n) \to \mathbb{H}$$

such that it induces an identity of Cartier divisors

$$\mathbb{H} = \sum_{x \in (A/\mathfrak{p}^n)^2} \iota(x) \subset E[\mathfrak{p}^n].$$

Remark 5.7.7 In [Gek86], an integral model of the moduli space of Drinfeld modules is constructed without reference to "cyclicity". Indeed this is possible as the condition is, as we shall see, tautological for Drinfeld $\Gamma_0(\mathfrak{p})$-level structures. This is analogous to the situation for the moduli of elliptic curves or more generally Shimura curves (see for instance [KM85, Cor 6.8.7] and [Buz97, Rem on page 598])

5.7.2 Drinfeld-level structure of Dieudonné module

Definition 5.7.8 ([Boy99]) *Let (M, F) be a Dieudonné module of rank 2 over R. Then a* Drinfeld $\Gamma(\mathfrak{p}^n)$-level structure on (M, F) *consists of an $A_\mathfrak{p}$-linear morphism*

$$\iota : (A_\mathfrak{p}/\mathfrak{p}^n)^2 \to \mathrm{Dr}_q((M, F))$$

such that we have

$$\sum_{x \in (A_\mathfrak{p}/\mathfrak{p}^n)^2} \iota(x) = \mathrm{Dr}_q((M, F)).$$

Definition 5.7.9 *Let (M, F) be a Dieudonné module of rank 2 over R. Then a* Drinfeld $\Gamma_0(\mathfrak{p}^n)$-level structure on (M, F) *consists of a finite locally free closed subscheme $\mathbb{H} \subset \mathrm{Dr}_q((M, F))$ which admits, fppf-locally, admits an $A_\mathfrak{p}$-linear morphism*

$$\iota : (A_\mathfrak{p}/\mathfrak{p}^n)^2 \to \mathbb{H}$$

such that we have

$$\sum_{x \in (A_\mathfrak{p}/\mathfrak{p}^n)^2} \iota(x) = \mathbb{H} \subset \mathrm{Dr}_q((M, F)).$$

5.7.3 Drinfeld-level structure of $\mathcal{D}$-elliptic sheaf

Definition 5.7.10 (Drinfeld-$\Gamma(\mathfrak{p}^n)$-level structure of $\mathcal{D}$-elliptic sheaf [Boy99, Def 7.1.2]) *A* Drinfeld $\Gamma(\mathfrak{p}^n)$-level structure on $(\mathcal{E}, j_i, t_i)_{i\in\mathbb{Z}}$ *consists of an A-linear morphism*

$$\iota : (A_{\mathfrak{p}}/\mathfrak{p}^n)^2 \to \mathcal{E}[\mathfrak{p}^n]$$

such that we have

$$\sum_{x\in(A_{\mathfrak{p}}/\mathfrak{p}^n)^2} \iota(x) = \mathcal{E}[\mathfrak{p}^n].$$

Definition 5.7.11 (Drinfeld $\Gamma_0(\mathfrak{p}^n)$-level structure of $\mathcal{D}$-elliptic sheaf) *A* Drinfeld $\Gamma_0(\mathfrak{p}^n)$-level structure on $(\mathcal{E}, j_i, t_i)_{i\in\mathbb{Z}}$ *consists of an $\mathcal{O}_D$-invariant finite locally free closed subscheme $\mathbb{H} \subset \mathcal{E}[\mathfrak{p}^n]$ of rank $|\mathfrak{p}|^{2n}$ which fppf-locally on S admits a* generator*, that is an an A-linear morphism*

$$\iota : (A_{\mathfrak{p}}/\mathfrak{p}^n)^2 \to \mathbb{H}$$

such that we have

$$\mathbb{H} = \sum_{x\in(A_{\mathfrak{p}}/\mathfrak{p}^n)^2} \iota(x) \subset \mathcal{E}[\mathfrak{p}^n].$$

Remark 5.7.12 At primes ℓ which do not divide the discriminant, a "new" $\Gamma_0(\ell)$-level structure of an abelian surface A corresponds to a $\mathcal{O}_D$-invariant closed subgroup of $A[\ell]$ of order ℓ^2. There exists an elliptic curve E such that

$$A[\ell] \cong (E[\ell])^2$$

which respect the $\mathcal{O}_D/\ell \cong \mathrm{Mat}_2(\mathbb{F}_\ell)$ action. One can now show that "new" $\Gamma_0(\ell)$-level structures on A are in bijection with "new" $\Gamma_0(\ell)$-level structures on E: If we look at the element $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{F}_\ell)$, then we can decompose the torsion points into

$$A[\ell] \cong \ker(e : A[\ell] \to A[\ell]) \times \ker(1 - e : A[\ell] \to A[\ell]).$$

If K is a closed subscheme of $A[\ell]$ of rank ℓ^2 then, using the above decomposition, we can equally decompose K into $K_1 \times K_2$. Furthermore note that $K_1 \cong K_2$ and that K_1 is finite, locally free of rank ℓ. Conversely if we have a finite flat closed subgroup K of $\ker(e : A[\ell] \to A[\ell]) \cong E[\ell]$, then we denote by K_2 the image of K_1 under the isomorphism $\ker(e : A[\ell] \to A[\ell]) \cong \ker(1 - e : A[\ell] \to A[\ell])$. Then $K_1 \times K_2$ is an $M_2(\mathbb{F}_\ell)$-invariant of rank ℓ^2. This fact is used by Buzzard

[Buz97] to reduce his result to the trivial discriminant case. The following is the corresponding function field analogue.

Proposition 5.7.13 ([Boy99, Lemma 7.4.3]) *Let $\mathfrak{p} \nmid \mathrm{Ram}(\mathcal{D})$. Then the set of Drinfeld $\mathfrak{p}^n$-level structures on a $\mathcal{D}$-elliptic sheaf is in bijection with set of Drinfeld $\Gamma(\mathfrak{p}^n)$-level structures on the corresponding $A_\mathfrak{p}$-Dieudonné module.*

Proposition 5.7.14 *Let $\mathfrak{p} \nmid \mathrm{Ram}(\mathcal{D})$. Then the set of Drinfeld $\Gamma_0(\mathfrak{p}^n)$-level structures on a $\mathcal{D}$-elliptic sheaf is in bijection with the set of Drinfeld $\Gamma_0(\mathfrak{p}^n)$-level structures on the corresponding $\mathcal{D}$-Dieudonné module and thus on the Morita-equivalent $A_\mathfrak{p}$-Dieudonné module.*

Proof. Let $\mathcal{E}$ be a $\mathcal{D}$-elliptic sheaf in the Proposition. By definition a Drinfeld $\Gamma_0(\mathfrak{p}^n)$-level structures on $\mathcal{E}$ is a Drinfeld $\Gamma_0(\mathfrak{p}^n)$-level structures on its associated $\mathcal{D}$-Dieudonné module. By Prop 5.5.10, the $\mathcal{D}$-Dieudonné module associated to $\mathcal{E}$ corresponds to an $A_\mathfrak{p}$-Dieudonné module corresponding to a Drinfeld module. This correspondence is based on Morita equivalence, as $\mathcal{D}$-splits at $\mathfrak{p}$. Let $(\mathbb{H}, \iota)$ be a Drinfeld $\Gamma_0(\mathfrak{p}^n)$-level structure of $\mathcal{E}$. Let $e := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{Mat}_2(A_\mathfrak{p}/\mathfrak{p}^n) \cong \mathcal{O}_\mathcal{D}/\mathfrak{p}^n$. By Prop 5.5.10 the decomposition of $\mathcal{E}[\mathfrak{p}^n]$ along e corresponds to the $\mathfrak{p}^n$-torsion of an $A_\mathfrak{p}$-Dieudonné module. By assumption on ι, there exists an fppf-cover $S' \to S$ such that the generator ι is defined and satisfies the Cartier divisor condition. The module $(A_\mathfrak{p}/\mathfrak{p}^n)^2$ and therefor the morphism ι also gets acted upon by $\mathrm{Mat}_2(A_\mathfrak{p}/\mathfrak{p}^n) \cong \mathcal{O}_\mathcal{D}/\mathfrak{p}^n$ and thus we may decompose them along e and thus we have

$$e(A_\mathfrak{p}/\mathfrak{p}^n)^2 \cong A_\mathfrak{p}/\mathfrak{p}^n \to e\mathbb{H}$$

and thus we have $e\mathbb{H} = \sum_{x \in A_\mathfrak{p}/\mathfrak{p}^n} \iota(x) = e\mathbb{H}$. □

5.8 Integral model

To prove that this defines a representable moduli problem, we use the usual strategy of first proving relative representability.

Definition 5.8.1 (Relative representability [Sta25]Definition 0023) *Let $\mathcal{C}$ be a category and $F \to G$ a morphism of sheaves on $\mathcal{C}$. For $U \in Ob(\mathcal{C})$, denote by $h_U : \mathcal{C}^{\mathrm{opp}} \to$ Set the associated Yoneda pre-sheaf. If for every $U \in Ob(\mathcal{C})$, $g \in G(U)$, the fiber functor*

$$h_U \times_G F$$

is representable, then we say that $F \to G$ is relatively representable.

Proposition 5.8.2 ([Sta25] Lemma 03KC) *Let $\mathcal{C}$ be a category and $F \to G$ a morphism of sheaves on $\mathcal{C}$. If the morphism is relatively representable and G is representable, then F is representable.*

Proposition 5.8.3 ([KM85] and [Bie23]) *If $S_0 = X \backslash \{\mathfrak{p}, \infty\}$, then the functor*

$$[\mathfrak{p}\text{-isog}] : \{\mathcal{D}\text{-elliptic sheaf}/_{S_0}\} \to \text{Set}, \quad \mathcal{E}/S \mapsto \left\{ \begin{array}{l} \mathcal{O}_D\text{-invariant finite locally free closed} \\ \text{subgroup } \mathbb{H} \subset \mathcal{E}[\mathfrak{p}] \text{of rank } |\mathfrak{p}|^2 \end{array} \right\}$$

is representable by a projective scheme.

Proof. A closed subgroup of rank $|\mathfrak{p}|^2$ of $\mathcal{E}[\mathfrak{p}]$ corresponds to a quotient of $\mathcal{O}_{\mathcal{E}[\mathfrak{p}]}$, or equivalently to a quotient of the pushforward along of $\mathcal{O}_{\mathcal{E}[\mathfrak{p}]}$ along $p : \mathcal{O}_{\mathcal{E}[\mathfrak{p}]} \to S$. Since this morphism is finite, the pushforward is coherent. Furthermore this pushforward is a bialgebra (where the second algebra structure comes from the fact that $\mathcal{O}_{\mathcal{E}[\mathfrak{p}]}$ is a group scheme) and closed subgroups of rank $|\mathfrak{p}|^2$ of $\mathcal{E}[\mathfrak{p}]$ correspond to quotients of the $p_*\mathcal{O}_{\mathcal{E}[\mathfrak{p}]}$ which are locally free of rank $|\mathfrak{p}|^2$. Thus this functor is a closed subscheme of the Grassmannian representing all rank $|\mathfrak{p}|^2$-quotients of $p_*\mathcal{O}_{\mathcal{E}[\mathfrak{p}]}$. By [GD71, Chapt I, Prop 9.8.4] the Grassmannian is a projective S_0-scheme, and thus so is $[\mathfrak{p}\text{-isog}]$. □

Definition 5.8.4 *If $S_0 = X \backslash \{\mathfrak{p}, \infty\}$, then we define the functor*

$$[\Gamma_0(\mathfrak{p})] : \{\mathcal{D}\text{-elliptic sheaf}/_{S_0}\} \to \text{Set}, \quad \mathcal{E}/S \mapsto \left(\begin{array}{l} \text{isomorphy classes of} \\ \text{Drinfeld } \Gamma_0(\mathfrak{p})\text{-structures on } \mathcal{E}/S \end{array} \right).$$

Proposition 5.8.5 (Analogue of [KM85, Cor 6.8.7]) *For any Drinfeld $\Gamma_0(\mathfrak{p})$-level structure, the "cyclicity" condition in Def 5.7.11 is automatic, that is every finite locally free closed subgroup $\mathbb{H} \subset \mathcal{E}[\mathfrak{p}]$ of rank $|\mathfrak{p}|^2$ admits a generator fppf-locally on S.*

Proof. Choose an S, and a $\mathcal{O}_D$-invariant $\mathbb{H} \subset \mathcal{E}[\mathfrak{p}]$ of rank $|\mathfrak{p}|$. We will first show that "cyclicty of $\mathbb{H}$ is a closed condition" in the sense that there exists a closed subscheme $W \subset S$ satisfying the following property: for any $T \to S$, the subgroup $\mathbb{H}_T$ is cyclic (in the sense of Def 5.7.11) if and only if $T \to S$ factors through W. This is analogous to [KM85, Thm 6.4.1] for elliptic curves and [Bie23, Thm 5.1 (3)] for Drinfeld shtukas. Cyclicity is a Zariski-local property and

the groups $\mathbb{H}$, $\mathcal{E}[\mathfrak{p}]$ are finitely presented. Hence we may assume that S is affine noetherian. As $\mathfrak{p} \nmid \mathrm{Ram}(\mathcal{D})$ the $\mathcal{D}$-Dieudonné module associated to $\mathcal{E}$ is Morita equivalent to an $A_{\mathfrak{p}}$-Dieudonné module of a Drinfeld shtuka $\mathcal{F}$. The subgroup $\mathbb{H} \subset \mathcal{E}[\mathfrak{p}]$ corresponds to a subgroup $\mathbb{H}' \subset \mathcal{F}[\mathfrak{p}]$ such that cyclicity of $\mathbb{H}$ is equivalent to cyclicity of $\mathbb{H}'$. Thus the statement of "cyclicity being closed" follows from [Bie23, Thm 5.1][2]. In particular the functor $[\Gamma_0(\mathfrak{p})]$ is representable and we have a closed embedding

$$[\Gamma_0(\mathfrak{p})] \to [\mathfrak{p}\text{-isog}].$$

To show that this is an isomorphism, it suffices to prove this on the open subscheme $S \times_A A[1/\mathfrak{p}] \subset S$. However, away from the characteristic, the group $\mathbb{H}$ is isomorphic to $A_{\mathfrak{p}}/\mathfrak{p}$ and thus cyclicity is tautological. □

Remark 5.8.6 In particular we will be able to use this simpler description of level structures. However it would be worthwhile to use the work of [Bie23] to give integral models of $X_0^{\mathfrak{D}}(\mathfrak{p}^n\mathfrak{m})$. This is not required in this thesis so we will not do it.

Definition 5.8.7 *Let D be a quaternion algebra, $\mathfrak{n} = \mathfrak{p}\mathfrak{m}$ where $\mathfrak{p} \nmid \mathfrak{m}, Ram(D) = \mathfrak{D}$. The functor*

$$\mathcal{X}_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{m}) : \mathrm{CAlg}_{A_{\mathfrak{p}}} \to \mathrm{Set}, \quad R \mapsto \left(\begin{array}{l} \mathcal{E} \in X_0^{\mathfrak{D}}(\mathfrak{m})(\mathrm{Spec}\, R)\text{and a finite locally free closed} \\ \mathcal{O}_D\text{-invariant subgroup } \mathbb{H} \subset \mathcal{E}[\mathfrak{p}]\text{of rank } |\mathfrak{p}|^2 \end{array} \right)$$

Proposition 5.8.8 (Local model exists) *Let D be a quaternion algebra, $\mathfrak{n} = \mathfrak{p}\mathfrak{m}$ where $\mathfrak{p} \nmid \mathfrak{m}, Ram(D) = \mathfrak{D}$. The functor $\mathcal{X}_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{m})$ has a coarse moduli space whose generic fiber is $X_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{m})$.*

Proof. By Prop 5.8.2 and Prop 5.8.3, the functor $\mathcal{X}_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{m})$ has a coarse moduli space which is a scheme over $A_{\mathfrak{p}}$. Furthermore, analogously to [Gek86], we shall assume that it is a fine moduli space. The usual way is to add $\mathfrak{n}'$-level strucuture (where $\mathfrak{n}'$ is coprime to $\mathfrak{p}, \mathfrak{m}, \mathfrak{D}$ and then descend the statement to the quotient. If $\mathfrak{p}$ is coprime to the characteristic, a Drinfeld level $\Gamma_0(\mathfrak{p})$ structure is equivalent to a naive one, and thus the statement about the generic fiber follows. □

Question 5.8.9 *It should be noted that while we set-up the study of Drinfeld $\Gamma_0(\mathfrak{p}^n)$-level structures of $\mathcal{D}$-elliptic sheaves analogously to the work of [KM85] and [Bie23], we do not develop*

[2]The proof thereof is the same as in [KM85]: It suffices to prove that the loci where the scheme of generators $\mathbb{H}^\times$ is of rank $|\mathfrak{p}| - 1$ is closed. There exists a coherent sheaf of algebras $\mathcal{F}$ on S such that $\mathbb{H}^\times \cong \mathrm{Spec}\, \mathcal{F}$. The scheme $\mathbb{H}^\times$ being of rank $|\mathfrak{p}| - 1$ is equivalent to $\mathcal{F}$ being free of rank $|\mathfrak{p}| - 1$. The loci where this condition is met is closed by "Mumford's flattening stratification" (see [KM85, Prop 6.4.3] or [Mum66])

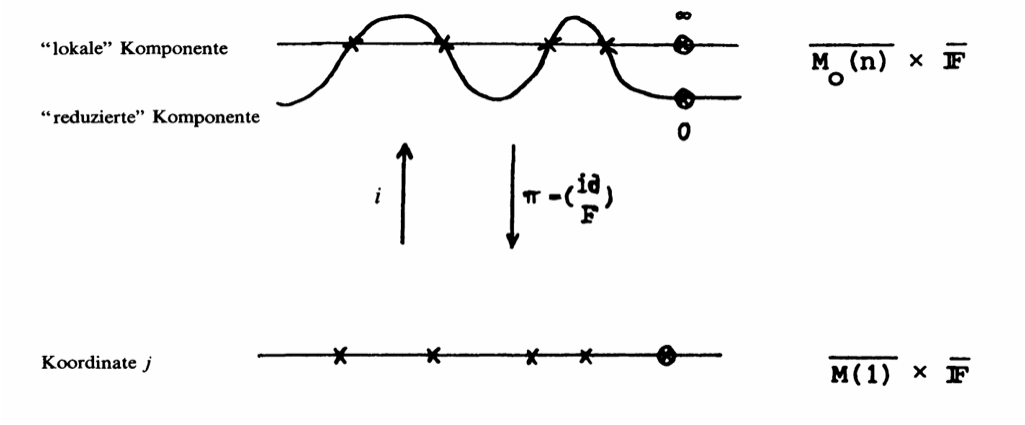


Figure V.1: Drawing of the reduction of $X_0(\mathfrak{p})$ [Gek86]

the theory thereof in this thesis. This is ongoing work of the author. It should be noted that, analogously to [Buz97], this can be done by reducing to the main results of [Bie23].

Definition 5.8.10 (Atkin-Lehner involution) *For any* $\mathfrak{n}$ *which is square-free and any* $\mathfrak{p} \nmid \mathfrak{n}$*, the* Atkin-Lehner involution *is the involution on* $\mathcal{X}_0^{\mathfrak{D}}(\mathfrak{pn})$ *induced by*

$$\omega_{\mathfrak{p}} : (\varphi, C_{\mathfrak{n}}, C_{\mathfrak{p}}) \mapsto (\varphi/C_{\mathfrak{p}}, (C_{\mathfrak{n}} \oplus C_{\mathfrak{p}})/C_{\mathfrak{p}}, \varphi[\mathfrak{p}]/C_{\mathfrak{p}}).$$

By abuse of notation we denote equally all the induced involutions (on the Jacobian, on the Neron model, on the special fiber, on the torus, the character group etc) by $\omega_{\mathfrak{p}}$.

5.9 Analogue of Deligne-Rapoport

In the $\mathfrak{D} = 1$ case, Papikian and Wei mention the following result, which is essentially already proven by Gekeler [Gek86].

Theorem 5.9.1 ([PW15, Thm 5.1]) *Let* $\mathfrak{n} = \mathfrak{pm}$ *where* $\mathfrak{p} \nmid \mathfrak{m}$*. Denote by* ϖ *a uniformizer of* $\mathfrak{p}$*. There exists a proper flat* $A_{\mathfrak{p}}$*-scheme* $\mathcal{X}_0(\mathfrak{n})$ *such that the special fiber is geometrically reduced, consisting of two irreducible components isomorphic to* $\mathcal{X}_0(\mathfrak{m})_{\mathbb{F}_{\mathfrak{p}}}$ *which intersect transversely at the supersingular points. Locally in the étale topology, at any supersingular point* $(\varphi, C_{\mathfrak{m}})$*, the scheme* $X_0(\mathfrak{n})_{A_{\mathfrak{p}}}$ *is given by*

$$XY = \varpi^{\frac{1}{q-1}|\operatorname{Aut}(\varphi, C_{\mathfrak{m}})|}.$$

Recall that we saw in Prop 4.3.4 that $\frac{1}{q-1} \mid |\operatorname{Aut}(\varphi)|$. This yields the usual "braid" from the work of Deligne-Rapoport, see Figure V.1. We will need the extension of this result to

$\mathfrak{D} \neq 1$. In the following, we will follow the strategy of [Buz97], who extended the results of Deligne-Rapoport to the non-trivial discriminant case.

Remark 5.9.2 From here on out, we can in principle proceed as Katz-Mazur [KM85], and we briefly outline this: first one studies the canonical morphism

$$\mathcal{X}_1^{\mathfrak{D}}(\mathfrak{pm}) \to \mathcal{X}_0^{\mathfrak{D}}(\mathfrak{pm}).$$

This morphism is finite flat, and thus the regularity of $\mathcal{X}_0^{\mathfrak{D}}(\mathfrak{pm})$ would follow from the corresponding statement for $\mathcal{X}_1^{\mathfrak{D}}(\mathfrak{pm})$ (using [AK70, VII, Theorem 4.8]). While this strategy should be applicable for non-trivial discriminant, we follow instead the strategy of Buzzard [Buz97], who ultimately reduces the regularity statements of $\mathcal{X}_0^D(pm)$ to the corresponding statement for $\mathcal{X}_0(pm)$.

The following is the analogue of [Buz97, Thm 4.7].

Theorem 5.9.3 (Analogue of Deligne-Rapoport) *Let D be a quaternion algebra, $\mathfrak{n} = \mathfrak{pm}$ where $\mathfrak{p} \nmid \mathfrak{m}, \mathfrak{D} = Ram(D)$. Denote by ϖ a uniformizer of $\mathfrak{p}$. The functor $\mathcal{X}_0^{\mathfrak{D}}(\mathfrak{pm})$ has a coarse moduli space which is a proper flat $A_{\mathfrak{p}}$-scheme whose generic fiber is $X_0^{\mathfrak{D}}(\mathfrak{pm})$, such that the special fiber is geometrically reduced, consisting of two irreducible components isomorphic to $X_0^{\mathfrak{D}}(\mathfrak{m})_{\mathbb{F}_{\mathfrak{p}}}$ which intersect transversely at the supersingular points. Locally in the étale topology, at any supersingular point $(\varphi, C_{\mathfrak{m}})$, the scheme $X_0^{\mathfrak{D}}(\mathfrak{n})_{A_{\mathfrak{p}}}$ is given by*

$$XY = \varpi^{\frac{1}{q-1}|\operatorname{Aut}(\varphi, C_{\mathfrak{m}})|}.$$

Proof. We already saw in Prop 5.8.8 that the integral model $\mathcal{X}_0^{\mathfrak{D}}(\mathfrak{pm})$ exists. Denote by c the forgetful morphism

$$\mathcal{X}_0^{\mathfrak{D}}(\mathfrak{pm}) \xrightarrow{c} \mathcal{X}_0^{\mathfrak{D}}(\mathfrak{m}).$$

As $\mathcal{X}_0^{\mathfrak{D}}(\mathfrak{m})$ is proper, it will suffice to show that c is proper. For this we use the strategy explained by [KM85, (4.5)]. At the level of stacks, by definition, this morphism is proper if for any scheme $T \to \mathcal{X}_0^{\mathfrak{D}}(\mathfrak{m})$, the fiber $\mathcal{X}_0^{\mathfrak{D}}(\mathfrak{pm}) \times_{\mathcal{X}_0^{\mathfrak{D}}(\mathfrak{m})} T$. We saw in Prop 5.8.3 that this fiber is projective. If the moduli problem admit fine moduli spaces, that is if $\mathfrak{m}$ is sufficiently large, then properness at the level of schemes follows. To reduce to this case, one can add auxiliary level structure: let $\mathfrak{m}'$ be such that $\mathcal{X}_0^{\mathfrak{D}}(\mathfrak{mm}')$ is a fine moduli space. Then the morphism $\mathcal{X}_0^{\mathfrak{D}}(\mathfrak{pmm}') \to \mathcal{X}_0^{\mathfrak{D}}(\mathfrak{mm}')$ is proper. Since both morphism

$$\mathcal{X}_0^{\mathfrak{D}}(\mathfrak{pmm}') \to \mathcal{X}_0^{\mathfrak{D}}(\mathfrak{pm}) \quad \mathcal{X}_0^{\mathfrak{D}}(\mathfrak{mm}') \to \mathcal{X}_0^{\mathfrak{D}}(\mathfrak{m})$$

are torsors, properness descends. As c is proper, to show it is finite it suffices to show that the morphism is quasi-finite. For this pick any $x \in \mathcal{X}_0^{\mathfrak{D}}(\mathfrak{m})$ and we proceed dependent of the A-characteristic $\mathfrak{l}$ of x: if $\mathfrak{l} \neq \mathfrak{p}$, then there exists $|\mathfrak{l}| + 1$ level structures and if $\mathfrak{l} = \mathfrak{p}$, then a $\Gamma_0(\mathfrak{p})$-level structure corresponds to a subgroup of rank $\mathfrak{l}$ of $\alpha_{|\mathfrak{p}|} \times (A_{\mathfrak{p}}/\mathfrak{p})$ or $\alpha_{|\mathfrak{p}|^2}$ (depending on whether x lies in the ordinary or supersingular locus). Then there exists either two respectively one such subgroup. In particular there are only finitely many proving finiteness of c.
It remains to prove that $\mathcal{X}_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{m})$ is regular and flat. Denote by

$$\mathcal{U} := \{y \in X_0^{\mathfrak{D}}(\mathfrak{m}) \mid \forall x \in \mathcal{X}_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{m}) \text{ s.t. } c(x) = y, \mathcal{O}_y \to \mathcal{O}_x \text{ regular and flat}\}$$

the regular,flat locus of $X_0^{\mathfrak{D}}(\mathfrak{m})$. Since c is proper, hence closed, the subset $\mathcal{U} \subset X^{\mathfrak{D}}(\mathfrak{m})$ is open. We want to show that $\mathcal{U} = X_0^{\mathfrak{D}}(\mathfrak{m})$, and since $X_0^{\mathfrak{D}}(\mathfrak{m})$ is of finite type, it suffices to show all closed points are contained in $\mathcal{U}$. If a closed point is of characteristic not equal $\mathfrak{p}$, i.e. does not lie above $\mathfrak{p}$, then we saw that a Drinfeld $\Gamma_0(\mathfrak{p})$-structure is equivalent to a "naive" one, in particular this follows from Thm 5.9.1. Thus it remains to study the point in the fiber of $\mathfrak{p}$. For this we use the fact that a noetherian local ring R is regular if and only if $\widehat{R^{s.h.}}$ is. As we saw that c is finite, by Lemma [Sta25, Lemma 00MA], the completion $\widehat{\mathcal{O}}_{\mathcal{X}_0^{\mathfrak{D}}(\mathfrak{m}\mathfrak{p})[[\pi]],x}$ is isomorphic to the tensor product

$$\mathcal{O}_{\mathcal{X}_0^{\mathfrak{D}}(\mathfrak{m}\mathfrak{p})[[\pi]],x} \otimes_{\mathcal{O}_{\mathcal{X}_0^{\mathfrak{D}}(\mathfrak{m})[[\pi]],y}} \widehat{\mathcal{O}}_{\mathcal{X}_0^{\mathfrak{D}}(\mathfrak{m})[[\pi]],y}.$$

This ring represents the Drinfeld $\Gamma_0(\mathfrak{p})$-structures on the universal deformation of the Drinfeld-Stuhler module corresponding to y. As we saw in Prop 5.7.14, at a prime $\mathfrak{p} \nmid \mathrm{Ram}(D) = \mathfrak{D}$, a Drinfeld $\Gamma_0(\mathfrak{p})$-structure on a Drinfeld-Stuhler module corresponds to a Drinfeld $\Gamma_0(\mathfrak{p})$-structure on the "Mortia"-corresponding Drinfeld module. In particular the relative moduli functors $[\Gamma_0(\mathfrak{p})]$ are isomorphic. Thus the regularity of the moduli problem for Drinfeld-Stuhler modules is equivalent to the regularity of the corresponding moduli problem for Drinfeld-modules, which is known by Thm 5.9.1. Furthermore, by Thm 5.9.1, we know that the local ring is étale locally given by

$$XY = \varpi^{\frac{1}{q-1}|\operatorname{Aut}(\varphi, C_{\mathfrak{m}})|}$$

where $(\varphi, C_{\mathfrak{m}})$ is an enhanced supersingular Drinfeld module. As the Morita equivalence is an equivalence of categories, the automorphism groups are isomorphic, and therefor the local statement follows.
To study the special fiber, we construct two natural embeddings

$$X_0^{\mathfrak{D}}(\mathfrak{m})_{\mathbb{F}_{\mathfrak{p}}} \to X_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{m})_{\mathbb{F}_{\mathfrak{p}}}.$$

For any S of characteristic $\mathfrak{p}$ and any Drinfeld-Stuhler module φ over S we have a natural pullback along the absolute Frobenius, which we denote

$$\varphi \xrightarrow{F} \varphi'.$$

This is an isogeny of degree $\mathfrak{p}$, whose dual isogeny we denote by V. Then we have morphisms of schemes

$$e_1 : X_0^{\mathfrak{D}}(\mathfrak{m})_{\mathbb{F}_\mathfrak{p}} \to X_0^{\mathfrak{D}}(\mathfrak{pm})_{\mathbb{F}_\mathfrak{p}}, \quad (\varphi, C_\mathfrak{m}) \mapsto (\varphi, C_\mathfrak{m}, \ker(F))$$

$$e_2 : X_0^{\mathfrak{D}}(\mathfrak{m})_{\mathbb{F}_\mathfrak{p}} \to X_0^{\mathfrak{D}}(\mathfrak{pm})_{\mathbb{F}_\mathfrak{p}}, \quad (\varphi, C_\mathfrak{m}) \mapsto (\varphi', F_*(C_\mathfrak{m}), \ker(V))$$

which gives a morphism

$$X_0^{\mathfrak{D}}(\mathfrak{m})_{\mathbb{F}_\mathfrak{p}} \coprod X_0^{\mathfrak{D}}(\mathfrak{m})_{\mathbb{F}_\mathfrak{p}} \to X_0^{\mathfrak{D}}(\mathfrak{pm})_{\mathbb{F}_\mathfrak{p}}.$$

The composition of e_1 with the forgetful morphism c is the identity, whereas the composition of e_2 with the forgetful morphism c is given by $(\varphi, C_\mathfrak{m}) \mapsto (\varphi', F_* C_\mathfrak{m})$. Hence e_1 is a closed embedding. Now we can apply the Atkin-Lehmer involution, which on S-points is given by

$$w_\mathfrak{p} : X_0^{\mathfrak{D}}(\mathfrak{pm})_{\mathbb{F}_\mathfrak{p}}(S) \to X_0^{\mathfrak{D}}(\mathfrak{pm})_{\mathbb{F}_\mathfrak{p}}(S), \quad (\varphi/C_\mathfrak{p}, (C_\mathfrak{m} \oplus C_\mathfrak{p})/C_\mathfrak{p}, \varphi[\mathfrak{p}]/C_\mathfrak{p}).$$

In particular by construction we have $e_2 = w_\mathfrak{p} \circ e_1$ or more to the point for us $c \circ w_\mathfrak{p} \circ e_2$ is the identity and thus e_2 is also a closed embedding. We now want to show that the induced morphism on the ordinary loci

$$X_0^{\mathfrak{D}}(\mathfrak{m})^h_{\mathbb{F}_\mathfrak{p}} \coprod X_0^{\mathfrak{D}}(\mathfrak{m})^h_{\mathbb{F}_\mathfrak{p}} \to X_0^{\mathfrak{D}}(\mathfrak{pm})^h_{\mathbb{F}_\mathfrak{p}}$$

is an isomorphism. For this recall that to give a Drinfeld $\Gamma_0(\mathfrak{p})$-level structure on a Drinfeld-Stuhler module over $\overline{\mathbb{F}}_\mathfrak{p}$ is to give a Drinfeld $\Gamma_0(\mathfrak{p})$-level structure on the corresponding Drinfeld module of rank 2. If E is an ordinary Drinfeld module over $\overline{\mathbb{F}}_\mathfrak{p}$, by Prop 5.7.4 we have

$$E[\mathfrak{p}] \cong \alpha_{|\mathfrak{p}|} \times A_\mathfrak{p}/\mathfrak{p}.$$

The only rank $|\mathfrak{p}|$ subgroups of $\alpha_{|\mathfrak{p}|} \times A_\mathfrak{p}/\mathfrak{p}$ are $\alpha_{|\mathfrak{p}|}$ and $A_\mathfrak{p}/\mathfrak{p}$. These are Taguchi dual to each other. The two irreducible components intersect in the supersingular loci and we are left to study the behaviour at the intersection. To do this, we reduce to the trivial discriminant case as before: the completed local ring at any supersingular point of $\mathcal{X}^D(\mathfrak{pm})_{\overline{\mathbb{F}}_\mathfrak{p}}$ represents the relative moduli problem $[\Gamma_0(\mathfrak{p})]$, which is isomorphic to the corresponding relative moduli problem for

the corresponding Drinfeld module. Thus étale locally $\mathcal{X}^D(\mathfrak{p}\mathfrak{m})_{\overline{\mathbb{F}}_\mathfrak{p}}$ is given by $XY = 0$ and the transversality follows. □

5.10 Analogue of Cerednik-Drinfeld

We specialize the results of [Hau05] and [Pap10] to the setting we will need them.

Proposition 5.10.1 ([Dri76]) *Denote by $\widehat{\Omega}$ the $A_\mathfrak{p}$-formal scheme associated to the Drinfeld upper half-plane. Then $\widehat{\Omega} \otimes \hat{A}_\mathfrak{p}$ has the following modular interpretation: Fix a special $\mathcal{O}_D$-Dieudonné module $(M_0/\mathfrak{m}, F_0/\mathfrak{m})$ over $\overline{\mathbb{F}}_\mathfrak{p}$[3]. For any $\hat{A}_\mathfrak{p}$-algebra B with nilpotent maximal ideal $\mathfrak{m}$, there exists a bijection between $\widehat{\Omega} \otimes \hat{A}_\mathfrak{p}(B)$ and isomorphy classes of pairs $((M, F), \rho)$ consisting of*

1. *a special $\mathcal{O}_D$-Dieudonné module (M, F) over B*
2. *a quasi-isogeny $(M_0/\mathfrak{m}, F_0/\mathfrak{m}) \xrightarrow{\rho} (M/\mathfrak{m}, F/\mathfrak{m})$.*

Proposition 5.10.2 (Algebraization [Hau05, Lemma 8.5]) *Let $\mathcal{E}_1$ be a $\mathcal{D}$-elliptic sheaf over $A_\mathfrak{p}$ and denote by (M, F) its associated $\mathcal{D}$-Dieudonné module. For any $(\tilde{M}, \tilde{F})$ with a quasi-isogeny $f : (M, F) \to (\tilde{M}, \tilde{F})$ there exists a $\mathcal{D}$-elliptic sheaf $\tilde{\mathcal{E}}$ whose associated $\mathcal{D}$-Dieudone module fits into the following commutative diagram*

$$\begin{array}{ccc} (M, F) & \longrightarrow & \mathcal{E}_1 \\ \downarrow & & \downarrow \\ (\tilde{M}, \tilde{F}) & \longrightarrow & \tilde{\mathcal{E}}. \end{array}$$

Theorem 5.10.3 ([Hau05, Thm 8.1]) *Let D be a quaternionic algebra over F which is split at ∞ and ramified at $\mathfrak{p}$. Then denote by $\bar{D}$ a quaternion algebra which is definite (i.e. not split at ∞), unramified at $\mathfrak{p}$ and such that for all $\mathfrak{q} \neq \mathfrak{p}, \infty$ we have*

$$\bar{D}_\mathfrak{q} \cong D_\mathfrak{q}.$$

For any subscheme $\mathfrak{n} \subset X$, let

$$Z_\mathfrak{n} := K_\mathfrak{n}^{\infty,\mathfrak{p}} \backslash \overline{D}^\times(\mathbb{A}^\infty)/\bar{D}^\times(F).$$

[3] as all these are isogenous, the different moduli problems are all isomorphic to one another

If $\mathfrak{n} \cap \{\infty, \mathfrak{p}\} = \varnothing$, there exists an isomorphism of $A_{\mathfrak{p}}$-formal schemes

$$\Theta : \mathrm{Jac}(\mathcal{E}ll_{X,\mathcal{D},\mathfrak{n}}) \cong \mathrm{GL}_d(F_{\mathfrak{p}})\backslash[\widehat{\Omega}\widehat{\otimes}_{\mathcal{O}_{\mathfrak{p}}}\hat{A}_{\mathfrak{p}}^{unr} \times Z_{\mathfrak{n}}].$$

This isomorphism is compatible the restriction of levels and with the group action by $D^{\times}(\mathbb{A}^{\infty,\mathfrak{p}}) \cong \overline{D}^{\times}(\mathbb{A}^{\infty,\mathfrak{p}})$.

Remark 5.10.4 This result allows us to consider $\widehat{\Omega}\widehat{\otimes}_{A_{\mathfrak{p}}}\hat{A}_{\mathfrak{p}}^{unr}$ as a Rapoport-Zink space. To generalize this to other groups, one thinks of $\mathcal{D}$-Dieudonné modules as $\mathcal{D}$-shtukas and $\mathcal{D}$-elliptic sheaves as $\mathcal{D}^{\times}$-shtukas. In [HJ20] this was used to construct the uniformization of the moduli space of G-shtukas under the assumption that G has connected fibers and reductive generic fiber.

Corollary 5.10.5 *The isomorphism Θ from Thm 5.10.3 induces an isomorphism of $\kappa(\mathfrak{p})$-schemes of the corresponding special fibers*

$$\overline{\Theta} : \mathcal{X}_0^{\mathfrak{D}}(\mathfrak{n}) \otimes \kappa(\mathfrak{p}) \cong \mathrm{GL}_d(F_{\mathfrak{p}})\backslash[\widehat{\Omega} \otimes \kappa(\mathfrak{p}) \times Z_{\mathfrak{n}}].$$

This was used by Papikian to describe the special fiber of $\mathcal{X}_0^{\mathfrak{D}}(\mathfrak{n})$ at a prime dividing $\mathfrak{D}$.

Theorem 5.10.6 ([Pap10, Prop 5.9]) *Let $\mathcal{X}_0^{\mathfrak{D}}(\mathfrak{n})$ be the fine moduli scheme representing special Drinfeld-Stuhler modules with level $\mathfrak{n}$-structure, where $\varnothing \neq \mathfrak{n} \subset X\backslash\{\mathfrak{p}, \infty\}$ and $\mathfrak{p} \mid \mathcal{D}$. Then the fiber at $\mathfrak{p}$ is admissible (i.e a reduced singular curve, whose singular points are ordinary double points), whose normalization is a disjoint union of finitely many rational curves. The singular points of $\mathcal{X}_0^{\mathfrak{D}}(\mathfrak{n}) \times \overline{\mathbb{F}}_{\mathfrak{p}}$ are in bijection with the isomorphism classes of superspecial (i.e. mixed and exceptional) Drinfeld-Stuhler modules over $\overline{\mathbb{F}}_{\mathfrak{p}}$ with level $\mathfrak{n}$-structure, where as the components are in bijection with pure Drinfeld-Stuhler modules over $\overline{\mathbb{F}}_{\mathfrak{p}}$ with level $\mathfrak{n}$-structure.*

Remark 5.10.7 This is in perfect analogy with the situation for Shimura curves, where the singular points of $X_0^D(n)_{\overline{\mathbb{F}}_p}$ correspond to mixed, exceptional $\mathcal{O}_D$-abelian surfaces and connected components correspond to pure $\mathcal{O}_D$-abelian surfaces (see for instance [Rib89, Thm 5.3]).

5.11 Uniformization at ∞

To study the uniformization of the moduli space of $\mathcal{D}$-elliptic sheaves at ∞, we shall follow the work of Spiess [Spi10], whose work shows that this is a consequence of the uniformation of Hauserberger [Hau05]. For this we shall introduce the following notation.

Definition 5.11.1 (Uniformization at the pole [Spi10, p.645]) *We say that* $\mathrm{Ell}_{X,\mathcal{D},I}$ admits uniformization at the pole *if*

$$\widehat{\mathrm{Ell}_{X,\mathcal{D},I}}/\,\mathbf{Spf}(\mathcal{O}_\infty) \cong D^\times \backslash \left(D^\times(\mathbb{A}^\infty) \times \widehat{\Omega}_\infty \right) / K_I^\infty$$

Let $\mathcal{D}, \overline{\mathcal{D}}$ be sheaves of central simple algebras (with generic fibers $D, \overline{D}$) with a prime $\mathfrak{p}$ such that

- for all $\mathfrak{q} \in |X| \backslash \{\infty, \mathfrak{p}\}$, $\mathrm{inv}_\mathfrak{q}(D) = \mathrm{inv}_\mathfrak{q}(\overline{D})$
- $\mathrm{inv}_\infty \overline{D} = 1/2$, $\mathrm{inv}_\infty D = 0$
- $\mathrm{inv}_\mathfrak{p} \overline{D} = 0$ and $\mathrm{inv}_\mathfrak{p} D = 1/2$

Definition 5.11.2 (Cerednik-Drinfeld uniformization [Spi10, p.646]) *We say that* $\mathrm{Ell}_{X,\mathcal{D},I}$ admits Cerednik-Drinfeld uniformization *if*

$$\widehat{\mathrm{Ell}_{X,\mathcal{D},I}}/\,\mathbf{Spf}(\mathcal{O}_\mathfrak{p}) \cong \overline{D}^\times \backslash \left(\overline{D}^\times(\mathbb{A}^\infty) \times \widehat{\Omega}_\infty^d \otimes_{\kappa(\mathfrak{p})} \overline{\kappa(\mathfrak{p})} \right) / K_I^{\infty,\mathfrak{p}}$$

Proposition 5.11.3 ([Spi10, Prop 4.28]) *Let* $\mathfrak{p} \in |X| \backslash \{\infty\}$ *and let* I *be disjoint from* $\mathrm{Ram}(\mathcal{D}) \cup \mathrm{Ram}(\overline{\mathcal{D}}) \cup \{\infty\}$. *Then* $\mathrm{Ell}_{X,\mathcal{D},I}$ *admits uniformization at the pole if and only if* $\mathrm{Ell}_{X,\overline{\mathcal{D}},I}$ *admits Cerednik-Drinfeld uniformization.*

As mentioned by Spiess, the results of Hausberger in [Hau05] (which we covered in Section 5.9) imply that we have uniformization at the pole if $\mathcal{D}$ in indefinite and has a prime at which it ramifies. This also recovers the result claimed in [BS97, Thm 4.4.11]

Proposition 5.11.4 *Let* $\mathcal{D}$ *be a sheaf of indefinite quaternion algebra. Let* I *be a disjoint from the ramification divisor* $\mathrm{Ram}(\mathcal{D})$. *Then* $\mathrm{Ell}_{X,\mathcal{D},I}$ *admits uniformization at the pole, i.e.*

$$\widehat{\mathrm{Ell}_{X,\mathcal{D},I}}/\,\mathbf{Spf}(\mathcal{O}_\infty) \cong D^\times \backslash \left(D^\times(\mathbb{A}^\infty) \times \widehat{\Omega}_\infty^d \right) / K_I^\infty.$$

We will exhibit the analytical construction of uniformization of elliptic curves from haromonic cochains, which has been done by Gekeler [Gek95] and Flynn [Fly13]. Recall that in Section 3.6, we saw that the holomorphic function

$$z \mapsto u_\alpha(z) := \Theta(\omega_0, \alpha\omega_0, z)$$

is independent of the choice of $\omega_0 \in \Omega$ and only depends on α in $\overline{\Gamma} := \Gamma^{\mathrm{ab}}/(\text{torsion})$. We denote the corresponding multiplier by $c_\alpha : \overline{\Gamma} \to \mathbb{C}_\infty^\times$. The induced group morphism

$$\overline{\Gamma} \to \mathrm{Hom}(\overline{\Gamma}, \mathbb{C}_\infty^\times), \quad \alpha \mapsto c_\alpha$$

uniformizes the Jacobian of the Drinfeld modular curve at ∞:

Theorem 5.11.5 ([GR96, Prop 7.3.2 and Thm 7.4.1]) *The quotient* $\mathrm{Hom}(\overline{\Gamma}, \mathbb{C}_\infty^\times)/\overline{\Gamma}$ *is isomorphic to the rigid analytic space over* K_∞ *associated to the Jacobian of the Drinfeld modular curve.*

Proof. (Sketch): We already saw that the Drinfeld modular curve (as a rigid analytic space) is isomorphic to the quotient Ω/Γ. Fix a $\omega_0 \in \Omega$. We denote by

$$\psi : \Omega \to \mathrm{Hom}(\overline{\Gamma}, \mathbb{C}_\infty^\times) \to \mathrm{Hom}(\overline{\Gamma}, \mathbb{C}_\infty^\times)/\overline{\Gamma}, \quad \omega \mapsto c(\omega_0, \omega, -).$$

Then for arbitrary $\alpha \in \Gamma$, $\beta \in \overline{\Gamma}$ we have

$$\frac{\psi(\alpha\omega)(\beta)}{\psi(\omega)(\beta)} = \frac{c(\omega_0, \alpha\omega, \beta)}{c(\omega_0, \omega, \beta)} = c_\beta(\alpha).$$

and thus ψ factors through the quotient Ω/Γ. Since the multiplier is non-zero at the cusps and holomorphic, this extends to a morphism to $X_0(\mathfrak{n}) \to \mathrm{Hom}(\overline{\Gamma}, \mathbb{C}_\infty^\times)/\overline{\Gamma}$. To show that the induced morphism is injective, note any divisor on $X_0(\mathfrak{n})$ may be written $\omega_1 + \ldots + \omega_n - n\omega_0$. If this lies in the kernel, then there exists an $\alpha \in \Gamma$ such that $c_\alpha(-) = \prod_i^n c(\omega_0, \omega, -)$. Thus the divisor of

$$\Omega \to \mathbb{C}_\infty, \quad z \mapsto \frac{1}{u_\alpha(z)} \prod_i^n \Theta(\omega_0, \omega, z)$$

is exactly $\omega_1 + \ldots + \omega_n - n\omega_0$. □

Following the work of Gekeler [Gek95] its quaternionic adaptation due to Papikian [Pap13], we can now give an analytical construction of the optimal elliptic curves associated to eigenforms: since we assumed that $f \in \mathcal{H}_!(\mathbb{T}, \mathbb{Z})^{\Gamma_0^{\mathfrak{D}}(\mathfrak{m})}$, we can think of $f \in \overline{\Gamma_0^{\mathfrak{D}}(\mathfrak{m})}$. In particular evaluation at f yields a morphism

$$\mathrm{ev}_f : \mathrm{Hom}(\overline{\Gamma_0^{\mathfrak{D}}(\mathfrak{m})}, \mathbb{C}_\infty^\times) \to \mathbb{C}_\infty.$$

Our integral normalization now means that $\mathrm{im}\left(\overline{\Gamma_0^{\mathfrak{D}}(\mathfrak{m})}\right) \cong \varpi^{\mathbb{Z}} \subset \mathbb{C}_\infty^\times$. In particular we get the

following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{\Gamma_0^{\mathfrak{P}}(\mathfrak{m})} & \longrightarrow & \mathrm{Hom}(\overline{\Gamma_0^{\mathfrak{P}}(\mathfrak{m})}, \mathbb{C}_\infty^\times) & \longrightarrow & J_0^{\mathfrak{P}}(\mathfrak{m})(\mathbb{C}_\infty) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_f} & & \vdots & & \\ 0 & \longrightarrow & \varpi^{\mathbb{Z}} & \longrightarrow & \mathbb{C}_\infty^\times & \longrightarrow & \mathbb{C}_\infty^\times/\varpi^{\mathbb{Z}} & \longrightarrow & 0 \end{array}$$

Thus there exists an elliptic curve E_f such that $E_f(\mathbb{C}_\infty) \cong \mathbb{C}_\infty^\times/\varpi^{\mathbb{Z}}$ and indeed by our assumptions this is the optimal elliptic curve.

CHAPTER VI

Ribet Short Exact Sequence and Level Lowering

As in Chapter V this chapter, we shall assume that

$$A = \mathbb{F}_q[t], \quad X = \mathbb{P}^1_{\mathbb{F}_q}.$$

Furthermore only consider only the $d = 2$ case, as this yields the closest parallels to Shimura curves.

6.1 Component groups and Character groups

This section is about the reduction of abelian varieties over a local field K with valuation ring R and residue field κ. For any abelian variety A over K, there exists a Neron model $\mathcal{A}$ over R. By Chevalley's theorem, the connected component of the special fiber $\mathcal{A}_\kappa$ is an extension of an abelian variety $\mathbb{A}$ by a connected affine group $T \times U$

$$0 \to T \times U \to \mathcal{A}^0_\kappa \to \mathbb{A} \to 0$$

where T is a torus and U is a unipotent algebraic group.

Definition 6.1.1 *We say that an abelian variety A over K has* toric reduction *if $\mathbb{A}, U$ are trivial. We say that A has* semi-abelian reduction *if U is trivial.*

Definition 6.1.2 (Component group) *Fix a henselian discrete valuation ring R with field of fraction K and an abelian variety A over K. Then $\mathcal{A}_\kappa/\mathcal{A}^0_\kappa$ is a finite constant scheme and the the* group of connected components *of its Neron model $\mathcal{A}$ is*

$$\Phi(A) := \mathcal{A}_\kappa(\overline{\kappa})/\mathcal{A}^0_\kappa(\overline{\kappa}).$$

As we assumed R to be henselian, we have an isomorphism $\mathcal{A}(R)/\mathcal{A}^0(R) \cong \mathcal{A}_\kappa(\kappa)/\mathcal{A}^0_\kappa(\kappa)$. More generally if we have an abelian variety over a base B, then $\Phi(A, b)$ is the group of connected

components of $A \otimes B_b^{\text{hens}}$.

Example 6.1.3 If A has good reduction at b, then $\Phi(A, b) = \{0\}$. If A is a Tate elliptic curve, then Tate computes that $\Phi(A)$ is cyclic of order

$$\operatorname{ord}_K(\Delta_E^{\min}) = -\operatorname{ord}_K(j(E)) = \operatorname{ord}_K(q)$$

(where q is the Tate-period).

Definition 6.1.4 (Character group) *Let A be an abelian variety over K with semi-abelian reduction and denote by T the toric part of the reduction. Then the* character group $\mathcal{X}(A)$ *is defined as*

$$\mathcal{X}(A) := \operatorname{Hom}(T_{\bar{k}}, \mathbb{G}_{m,\bar{k}}).$$

This is a free $\mathbb{Z}$-module with a continuous G_k-action (wrt discrete topology).

Denote by I the inertia subgroup of G_K. For $\ell \neq \operatorname{char} \kappa$, the maximal pro-$\ell$-quotient is $\mathbb{Z}_\ell(1)$, with the quotient morphism given by

$$I \to \mathbb{Z}_\ell(1), g \mapsto \lim_n \frac{g(\varpi^{1/n})}{\varpi^{1/n}}.$$

Definition 6.1.5 (Filtration of Tate module) *Denote by $T_\ell(A)^I$ the* inertia invariant subgroup*, which is isomorphic to $T_\ell(\mathcal{A}_k^0)$. The* toric part $T_\ell(A)^t$ of the Tate module *is defined by the pullback*

$$\begin{array}{ccc} T_\ell(A)^t & \longrightarrow & T_\ell(A)^I \\ \downarrow & & \downarrow \sim \\ T_\ell(T) & \longrightarrow & T_\ell(\mathcal{A}_k^0). \end{array}$$

which yields a filtration $T_\ell(A)^t \subset T_\ell(A)^I \subset T_\ell(A)$.

Theorem 6.1.6 (Grothendieck Orthogonality Theorem [GR72, Thm 2.4]) *The toric part $T_\ell(A)^t$ is orthogonal with regard to the Weil pairing to $T_\ell(A^\vee)^I$.*

Proposition-Definition 6.1.7 *The map*

$$I \to \operatorname{Hom}_{\mathbb{Z}_\ell}(T_\ell(A), T_\ell(A)^t), \quad g \mapsto [x \mapsto (g-1)x]$$

is a continuous group homomorphism.

Proof. For any $y \in T_\ell(A^\vee)^I$, we have

$$\langle (g-1)x, y\rangle = \frac{\langle gx, y\rangle}{\langle x, y\rangle} = \frac{\langle x, g^{-1}y\rangle}{\langle x, y\rangle} = \frac{\langle x, y\rangle}{\langle x, y\rangle} = 1$$

and thus $(g-1)x$ lies in $T_\ell(A)^t$. To prove that this is a group morphism, let $g, h \in I$. Then we have

$$\begin{aligned} 0 = (g-1)(h-1)x &= (gh - h - g + 1)x \\ &= (gh - 1 - (g-1) - (h-1))x \end{aligned}$$

and thus $(gh-1)x = (g-1)x + (h-1)x$. □

Furthermore, for any $x \in T_\ell(A)^I$, we have $(g-1)x = 0$ and thus the above morphism factors through $\mathrm{Hom}(T_\ell(A)/T_\ell(A)^I, T_\ell(A)^t)$. Furthermore, this is a pro-ℓ-group and thus we get the following diagram

$$\begin{array}{ccc} I & \longrightarrow & \mathrm{Hom}(T_\ell(A), T_\ell(A)^t) \\ \downarrow & \nearrow & \uparrow \\ \mathbb{Z}_\ell(1) & \dashrightarrow & \mathrm{Hom}(T_\ell(A)/T_\ell(A)^I, T_\ell(A)^t) \end{array}$$

Furthermore we have

$$T_\ell(A)^t \cong T_\ell(T) \cong \mathcal{X}(A)^*_\ell \otimes \mathbb{Z}_\ell(1)$$

and

$$T_\ell(A)/T_\ell(A)^I \cong \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell(A^\vee)^t, \mathbb{Z}_\ell(1)) \cong \mathrm{Hom}_{\mathbb{Z}_\ell}(T_\ell(T^\vee), \mathbb{Z}_\ell(1)) \cong \mathcal{X}(A^\vee)_\ell.$$

Thus we get a morphism

$$\begin{aligned} \mathbb{Z}_\ell(1) &\to \mathrm{Hom}(T_\ell(A)/T_\ell(A)^I, T_\ell(A)^t) \\ &\cong \mathrm{Hom}(\mathcal{X}(A^\vee)_\ell, \mathcal{X}(A)^*_\ell \otimes \mathbb{Z}_\ell(1)) \\ &\cong [\mathcal{X}(A^\vee)_\ell \otimes \mathcal{X}(A)_\ell]^* \otimes \mathbb{Z}_\ell(1). \end{aligned}$$

This yields a pairing

$$u_{A,\ell} : \mathcal{X}(A)_\ell \times \mathcal{X}(A^\vee)_\ell \to \mathbb{Z}_\ell.$$

A proof of Grothendieck is that this in fact arises from an integral pairing, which we call the *monodromy pairing*.

Proposition-Definition 6.1.8 ([GR72]) *Let A be an abelian variety over a local field K with semi-abelian reduction. Then there exists a unique pairing*

$$u_A : \mathcal{X}(A) \times \mathcal{X}(A^\vee) \to \mathbb{Z}$$

such that for all $\ell \neq char(\kappa)$ the pairing obtained by base-change recovers $u_{A,\ell}$. We call this the monodromy pairing of A.

This pairing is rather difficult to construct. However we shall see that in a nice setting, that is when A is the Jacobian of a curve with mild singularities, there exists an explicit description of this pairing.

Proposition 6.1.9 ([GR72, Thm 10.4 and Thm 11.5]) *The monodromy pairing is non-degenerate and functorial in A. The lack of "perfectness" of this pairing is measured by the component group, that is*

$$0 \to \mathcal{X}(A^\vee) \to \mathrm{Hom}(\mathcal{X}(A), \mathbb{Z}) \to \Phi(A) \to 0.$$

Furthermore for any polarization, the induced pairing

$$\mathcal{X}(A) \times \mathcal{X}(A) \to \mathcal{X}(A) \times \mathcal{X}(A^\vee) \to \mathbb{Z}$$

is symmetric and positive definite.

Remark 6.1.10 The monodromy pairing also induces a pairing on the component group

$$\Phi(A) \times \Phi(A^\vee) \to \mathbb{Q}/\mathbb{Z}.$$

By work of Grothendieck [GR06] and then McCallum [McC86], this is a perfect pairing, which will be useful in Prop 6.6.5.

The goal now is to make both the character group and the monodromy pairing explicit. For this we restrict ourselves to abelian varieties which arise as the Jacobian of curves.

Definition 6.1.11 (Semi-stable model) *A model of a smooth projective geometrically irreducible curve is said to be* semi-stable *if it is flat, proper and if its special fiber is reduced and has only ordinary double points as singularities.*

Example 6.1.12 *The integral models constructed in Thm 5.9.3 and Thm 5.10.6 are semi-stable models of $X_0^{\mathfrak{D}}(\mathfrak{n})$ at primes dividing the level and discriminant respectively.*

Definition 6.1.13 (Dual Graph) *The* dual graph $\mathrm{Gr}(\mathcal{C})$ *of a semi-stable model $\mathcal{C}$ is the graph whose vertices correspond to irreducible components of the model and edges correspond to singular points of the model.*

The dual graph of the moduli spaces of Drinfeld-Stuhler modules is well understood, and in fact this provides the connection to harmonic cochains.

Proposition 6.1.14 ([Mum72], see also [Kur79, 3.2]) *The dual graph associated to the $X_0^{\mathfrak{D}}(\mathfrak{n})$ at ∞ is the quotient $\mathcal{T}/\Gamma_0^{\mathfrak{D}}(\mathfrak{n})$.*

Theorem 6.1.15 (Raynaud) *Let $\mathcal{C}$ be a regular semi-stable model of C. Then we have an isomorphism $\mathcal{X}(\mathrm{Jac}(C)) \cong H_1(\mathrm{Gr}(\mathcal{C}), \mathbb{Z})$, and thus in particular we have a short exact sequence*

$$0 \to \mathcal{X}(\mathrm{Jac}(C)) \to \bigoplus_{x_i \in \mathcal{S}} \mathbb{Z} \cdot x_i \to \bigoplus_{X_j \in \mathcal{I}} \mathbb{Z} \cdot X_j \to \mathbb{Z} \to 0$$

where $\mathcal{S}$ denotes the set of singular points and $\mathcal{I}$ denotes the set of irreducible components of $\mathcal{C}_{\overline{\kappa}}$.

Remark 6.1.16 (Picard-Lefschetz formula) While the construction of the monodromy pairing is somewhat involved, a key feature of regular semi-stable models $\mathcal{C}$ is that the pairing is very explicit: denote by $\mathcal{S} = \{x_i\}$ the set of singular points of $\mathcal{C}$. Then étale locally at x_i the scheme $\mathcal{C}$ is given by

$$XY = \varpi^{n(i)}$$

where $n(i)$ is called the *thickness at* x_i. In that case, the monodromy pairing is given by the restriction of the pairing on $\mathbb{Z}^{\mathcal{S}}$ which is given

$$\langle x_i, x_j \rangle = \begin{cases} n(i), & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases}$$

If all points have thickness one, this was proven by Grothendieck [GR06]. By blow-up at the singular points with $n(i) > 1$, one may reduce to this case, and this was done in [Rib90a], [Pap11]. This allows us to give a description of the monodromy pairing of both the Deligne-Rapoport model and the Cerednik-Drinfeld model. For the Deligne-Rapoport model, if $(\varphi, C_{\mathfrak{m}})$ denotes a supersingular Drinfeld-Stuhler module, i.e. a singular point of $\mathcal{X}_0^{\mathfrak{D}}(\mathfrak{pm})$, by Thm 5.9.3 the

thickness at $(\varphi, C_{\mathfrak{m}})$ is $\frac{1}{q-1}|\operatorname{Aut}(\varphi, C_{\mathfrak{m}})|$.[1] For the Cerednik-Drinfeld model, if $(\mathbb{E}, C_{\mathfrak{m}})$ denotes a mixed-exceptional Drinfeld-Stuhler module, i.e. a singular point of $\mathcal{X}_0^{\mathfrak{p}\mathfrak{D}}(\mathfrak{m})_{A_{\mathfrak{p}}}$, the thickness at $(\mathbb{E}, C_{\mathfrak{m}})$ is equal to $\frac{1}{q-1}|\operatorname{Aut}(\mathbb{E}, C_{\mathfrak{m}})|$.

Definition 6.1.17 *For any finite set S, we shall denote*[2]

$$\left(\mathbb{Z}^S\right)_0 := \ker\left[\mathbb{Z}^S \to \mathbb{Z}, \quad (a_0, \ldots, a_n) \mapsto \sum_{i=0}^{n} a_i\right].$$

Proposition 6.1.18 *Denote by $\Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{N}, \mathfrak{q})$ the set of isomorphism classes of supersingular $\mathcal{D}$-elliptic sheaves with level $\mathfrak{N}$-structure over $\bar{\mathbb{F}}_{\mathfrak{q}}$. Then we have*

$$\mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{N}\mathfrak{q}), \mathfrak{q}) = \left(\mathbb{Z}^{\Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{N}, \mathfrak{q})}\right)_0.$$

Proof. This follows immediately from Thm 5.9.3 and Thm 6.1.15, as the dual graph of $J_0^{\mathfrak{D}}(\mathfrak{q}\mathfrak{M})$ consists of two vertices (corresponding to the two irreducible components) which intersect each other in the superspecial points. □

Proposition 6.1.19 *Denote by $\Sigma^{\mathfrak{D}}_{\text{mixed-exceptional}}(\mathfrak{N}, \mathfrak{p}), \Sigma^{\mathfrak{D}}_{\text{pure}}(\mathfrak{N}, \mathfrak{p})$ the set of isomorphism classes of mixed-exceptional (respectively pure) $\mathcal{D}$-elliptic sheaves with level $\mathfrak{N}$-structure over $\bar{\mathbb{F}}_p$. Then we have*

$$0 \to \mathcal{X}(J_0^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{p}) \to \left(\mathbb{Z}^{\Sigma^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}_{\text{mixed-exceptional}}(\mathfrak{M}, \mathfrak{p})}\right)_0 \to \left(\mathbb{Z}^{\Sigma^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}_{\text{pure}}(\mathfrak{M}, \mathfrak{p})}\right)_0^2 \to 0.$$

Proof. By Prop 5.10.6, and Thm 6.1.15, we have the short exact sequence

$$0 \to \mathcal{X}(J_0^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{p}) \to \mathbb{Z}^{\Sigma^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}_{\text{mixed-exceptional}}(\mathfrak{M}, \mathfrak{p})} \to \mathbb{Z}^{\Sigma^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}_{\text{pure}}(\mathfrak{M}, \mathfrak{p}) \times \{0,1\}} \to \mathbb{Z} \to 0.$$

However, the degree is preserved under $\mathbb{Z}^{\Sigma^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}_{\text{mixed-exceptional}}(\mathfrak{M}, \mathfrak{p})} \to \mathbb{Z}^{\Sigma^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}_{\text{pure}}(\mathfrak{M}, \mathfrak{p}) \times \{0,1\}}$ and so $\mathcal{X}(J_0^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{p})$ must map to elements of degree zero. Thus the claimed short exact sequence follows. □

Proposition 6.1.20 *For $\mathfrak{q} \not| \mathfrak{D}\mathfrak{M}$, we have*

$$\Phi(J^{\mathfrak{D}}(\mathfrak{M}\mathfrak{q}), \mathfrak{q}) := \operatorname{coker}\left[\left(\mathbb{Z}^{\Sigma^{\mathfrak{D}}_{\text{super-singular}}(\mathfrak{M})}\right)_0 \to \left(\mathbb{Z}^{\Sigma^{\mathfrak{D}}_{\text{super-singular}}(\mathfrak{M})}\right)_0^*\right]$$

[1] This is analogous to the corresponding result in [Rib90a], by realizing that $2 = |\mathbb{Z}^\times|$ and $q - 1 = |\mathbb{F}_q[t]^\times|$.
[2] in some of the literature, one may find the same object denoted by $\mathbb{Z}_0^S$ or $(\mathbb{Z}_0)^S$.

Proposition 6.1.21 ([Pap07a, Lemma 3.6]) *For any square-free $\mathfrak{pm}$ we have*

$$|\Phi(J_0(\mathfrak{pm}), \mathfrak{p})| = 1 \bmod p$$

and thus has no non-trivial p-torsion.

Proof. (Sketch): By Raynaud we know that $|\Phi(J_0(\mathfrak{pm}), \mathfrak{p})| = \sum \frac{|\operatorname{Aut}(\varphi, C_{\mathfrak{m}})|}{q-1}$ where the sum ranges over the supersingular points of $X_0(\mathfrak{m})_{\mathfrak{p}}$. The automorphism group is either $\mathbb{F}_q^\times$ or $\mathbb{F}_{q^2}^\times$, both of which are congruent to 1 modulo p. Hence the order of the component group is congruent to the number of supersingular points, which we call s. Furthermore we have $g(X_0(\mathfrak{pm})_{\mathbb{F}_{\mathfrak{p}}} = s - 1 + 2g(X_0(\mathfrak{m}))$. As we assume $\mathfrak{m}$ and $\mathfrak{pm}$ to be square-free, both genii to be congruent to zero, hence we are done. □

Definition 6.1.22 *Let $f : A \to B$ be a surjective morphism of abelian varieties. Then the* saturation of the character group *is*

$$\overline{\mathcal{X}(B)} := (f^*\mathcal{X}(B) \otimes \mathbb{Q}) \cap \mathcal{X}(A).$$

In particular $\mathcal{X}(A)/\overline{\mathcal{X}(B)}$ is torsion free.

Proposition 6.1.23 ([Pap04, Lemma 2.6]) *Let $0 \to C \to B \to A \to 0$ be a short exact sequence of abelian varieties. Then the induced sequence*

$$0 \to \overline{\mathcal{X}(C^\vee)} \to \mathcal{X}(B^\vee) \to \mathcal{X}(A^\vee) \to 0$$

is exact.

6.2 Atkin-Lehner involution on character group

From now on, we will work over $\mathbb{F}_q(t)$ and assume that $\mathfrak{p}, \mathfrak{q}$ do not divide $\mathfrak{D}$ and $\mathfrak{m}$. Furthermore as $\mathbb{F}_q(t)$ will be analogous to $\mathbb{Q}$, we shall denote it by Q.

Proposition 6.2.1 (Analogue of [Rib90a, Prop 3.7]) *Let T denote the toric part of the identity component of Neron of $J_0^{\mathfrak{D}}(\mathfrak{pm})$ at $\mathfrak{p}$. The Atkin-Lehner involution on T is equal to minus the $\mathfrak{p}$-th Hecke operator, i.e.*

$$\omega_{\mathfrak{p}} = -T_{\mathfrak{p}}.$$

Proof. We follow the proof structure of [Rib90a, Prop 3.7]. If we denote as is customary the two degeneracy maps $\alpha, \beta : X_0(\mathfrak{pm}) \to X_0(\mathfrak{m})$, then we see that $w_\mathfrak{p} + T_\mathfrak{p} = \alpha^*\beta_* \in \mathrm{End}\left(J_0^\mathfrak{D}(\mathfrak{pm})\right)$. In particular the restriction to the torus factors through $T \to J_0^\mathfrak{D}(\mathfrak{m})_{\mathbb{F}_\mathfrak{p}}$. As $\mathfrak{p} \nmid \mathfrak{m}$ the Jacobian $J_0^\mathfrak{D}(\mathfrak{m})$ has good reduction at $\mathfrak{p}$. As morphisms from a torus to an abelian variety have to be trivial, it follows that $\alpha^*\beta_*$ is also trivial. □

The following is an analogue of [Rib90a, Prop 3.8] and was already mentioned during the proof of [PW16, Lemma 8.3].

Proposition 6.2.2 *The Atkin-Lehner involution $\omega_\mathfrak{p}$ permutes the two connected components of $X_0^\mathfrak{D}(\mathfrak{pm})_\mathfrak{p}$. The action of $T_\mathfrak{p}$ on $\mathcal{X}(J_0^\mathfrak{D}(\mathfrak{pm}), \mathfrak{p})$ is the one induced by the action of Frobenius on $\Sigma^\mathfrak{D}(\mathfrak{m})$.*

Proof. We saw in Thm 5.9.3 that the connected components of $X_0^\mathfrak{D}(\mathfrak{pm})_\mathfrak{p}$ are two copies of $X_0^\mathfrak{D}(\mathfrak{m})_\mathfrak{p}$. The two embeddings we gave in the proof thereof differ exactly by the Atkin-Lehner involution. In particular the two components get permuted by $\omega_\mathfrak{p}$. By Prop 6.2.1, we know that $T_\mathfrak{p}$ acts via the Atkin-Lehner operator. This action on the supersingular Drinfeld-Stuhler modules corresponds to the action of the Frobenius. □

6.3 Analogue of Ribet's Short Exact sequence

The goal is to prove an analogous result to [Rib90a, Thm 4.1]. The key both in the original and here is that the Cerednik-Drinfeld theorem provides us with an explicit description of the dual graph associated $J_0^{Dpq}(M)$ at p (see [Rib90a, Prop 4.4]).

Theorem 6.3.1 *We have a short exact sequence*

$$0 \to \mathcal{X}(J_0^{\mathfrak{D}\mathfrak{pq}}(\mathfrak{M}), \mathfrak{p}) \xrightarrow{\iota} \mathcal{X}(J_0^\mathfrak{D}(\mathfrak{pq}\mathfrak{M}), \mathfrak{q}) \to \mathcal{X}(J^\mathfrak{D}(\mathfrak{q}\mathfrak{M}), \mathfrak{q})^2 \to 0$$

which is compatible with the Hecke operators $T_\mathfrak{m}$ for all $\mathfrak{m}$ coprime to $\mathfrak{D}\mathfrak{pq}\mathfrak{M}$ and with the monodromy pairings.

We prove this in steps, first proving surjectivity in the short exact sequence claimed in Thm 6.3.1.

Proposition 6.3.2 (Surjectivity in Thm 6.3.1) *The morphism*

$$\mathcal{X}(J_0^\mathfrak{D}(\mathfrak{M}\mathfrak{pq}), \mathfrak{q}) \to \mathcal{X}(J^\mathfrak{D}(\mathfrak{q}\mathfrak{M}), \mathfrak{q}) \times \mathcal{X}(J^\mathfrak{D}(\mathfrak{q}\mathfrak{M}), \mathfrak{q})$$

is surjective.

Before we prove this, we shall need several auxiliary lemmata.

Lemma 6.3.3 ([Pap23, Lem 1.2.6]) *Let N be a free $\mathbb{F}_q[t]$-module and let $f : N \to N$ be an injective $\mathbb{F}_q[t]$-linear endomorphism. Then we have an equality of ideals*

$$\chi(\mathrm{coker}(f)) = (\det f).$$

This is analogous to

Lemma 6.3.4 (for example [Mil08, Lemma 10.8]) *Let α be an injective endomorphism of a free $\mathbb{Z}$-module N of finite rank. Then*

$$|\mathrm{coker}(\alpha)| = (N : \alpha(N)) = |\det(\alpha)|.$$

Example 6.3.5 *The following illuminates the following result in an easier setting: let X be an abelian variety over $\mathbb{C}$. Then we may write $X = \mathbb{C}^{\dim(X)}/H_1(X,\mathbb{Z})$. If α is a self-isogeny of X, then the induced morphism $H_1(X,\mathbb{Z}) \to H_1(X,\mathbb{Z})$ is injective. Furthermore we have $\ker(\alpha) = \alpha^{-1}(H_1(X,\mathbb{Z}))/H_1(X,\mathbb{Z})$. In particular we have a bijection*

$$\ker(\alpha) \cong H_1(X,\mathbb{Z})/\alpha(H_1(X,\mathbb{Z})).$$

Thus by Lemma 6.3.4, we have

$$|\ker(\alpha)| = |H_1(X,\mathbb{Z})/\alpha(H_1(X,\mathbb{Z}))| = |\det(\alpha|H_1(X,\mathbb{Z}))|.$$

This motivates the following definition, if one recalls that the $\mathcal{O}_D$-motive (or the underlying A-motive) of a Drinfeld-Stuhler module corresponds to $H^1(X,\mathbb{Z})$.

Definition 6.3.6 *Let φ be a Drinfeld-Stuhler module over K. Denote by M_φ its associated $\mathcal{O}_D$-motive and by $\overline{M}_\varphi$ its pullback to $\overline{K}$.*

$$\det(u) := \underset{\overline{K}[t]}{\det}(u \mid \overline{M}_\varphi) \in \overline{K}[t].$$

Proposition 6.3.7 (Based on [Pap23, Thm 3.6.13]) *Let φ be a supersingular Drinfeld-Stuhler module over $\mathbb{F}_\mathfrak{q}$. Then for any self-isogeny u and $\mathfrak{l} \neq \mathfrak{q}, \infty$, we have*

$$v_\mathfrak{l}(\det u) = v_\mathfrak{l}(\det T_\mathfrak{l}(u)) = v_\mathfrak{l}\left(\chi(\ker(u))\right).$$

Furthermore, if we denote by $\mathrm{coker}(M(u))^n$ *the nilpotent component of cokernel of the induced morphism of the Dieudonne module associated to* φ*, then*

$$(\det(u)) = (\det T_{\mathfrak{l}}(u)) = \mathfrak{q}^{\mathrm{rkcoker}(M(u))^n}\chi(\ker(u)).$$

Proof. We follow the proof of [Pap23, Thm 3.6.13] and [Pap23, Thm 3.6.6]. By [And86, 1.8.3] or [Har19, Thm 6.6], we know that the Tate module $T_{\mathfrak{l}}(\varphi)$ and $\lim_n(\overline{M}_\varphi/\mathfrak{l}^n)^\tau$ are isomorphic. Thus for all $n \geqslant 0$ we have

$$\begin{aligned}
\det_{\overline{K}[t]}(u \mid \overline{M}_\varphi) \mod \mathfrak{l}^n &= \det_{\overline{K}[t]/\mathfrak{l}^n}(u|\overline{M}_\varphi/\mathfrak{l}^n) \\
&= \det_{\overline{K}[t]/\mathfrak{l}^n}(u|(\overline{M}_\varphi/\mathfrak{l}^n)^\tau \otimes \overline{K}) \\
&\overset{*}{=} \det_{\mathbb{F}_q[t]/\mathfrak{l}^n}(u|(\overline{M}_\varphi/\mathfrak{l}^n)^\tau) \\
&= \det_{\mathbb{F}_q[t]/\mathfrak{l}^n}(u|T_{\mathfrak{l}}(\varphi)) \mod \mathfrak{l}^n
\end{aligned}$$

where we used [And86, Lemma 1.8.2] for $\overset{*}{=}$. Since $\det_{\overline{K}[t]}(u \mid \overline{M}_\varphi)$ is a polynomial which does (by definition) not depend on the choice of $\mathfrak{l}$, we see that it must lie in $\mathbb{F}_q[t]$. Furthermore we see that

$$\det_{\overline{K}[t]}(u \mid \overline{M}_\varphi) = \det_{\mathbb{F}_q[t]/\mathfrak{l}^n}(u|T_{\mathfrak{l}}(\varphi)).$$

In particular the left hand side follows. Furthermore, by Prop 6.3.3, we have

$$v_{\mathfrak{l}}\left(\chi(T_{\mathfrak{l}}(\varphi)/uT_{\mathfrak{l}}(\varphi))\right) = v_{\mathfrak{l}}(\chi(\ker(u))).$$

To study the $\mathfrak{q}$-valuation, we may translate the situation to Dieudonne modules. Denote by $M(u)$ the induced morphism of Dieudonne modules. At the level of strict group schemes, we have

$$\ker(u) \cong \mathbf{Dr}_q(\mathbf{coker}(M(u)))$$

(see for instance [Har19, Thm 5.9]). Both sides admit a canonical decomposition which the "finite shtuka equivalence" (see Thm 5.4.7) respects, i.e.

$$\ker(u)^{\text{ét}} \times \ker(u)^0 \cong \mathbf{Dr}_q(\mathbf{coker}(M(u))^{\text{ét}}) \times \mathbf{Dr}_q(\mathbf{coker}(M(u))^n).$$

The previous comparison between the determinant and the Fitting ideal of the kernel means it

suffices to understand $\mathrm{Dr}_q(\mathrm{coker}(M(u))^n)$. The equivalence Thm 5.4.7 furthermore yields that $\mathrm{Dr}_q(\mathrm{coker}(M(u))^n)$ is of rank $|\mathfrak{q}|^{\mathrm{rk}(\mathrm{coker}(M(u))^n)}$, from which the statement follows. □

Definition 6.3.8 (Eichler condition [Vig80, p.81] or [Rei03, 34.3]) *Let R be a Dedekind domain with quotient field K and B be a central simple K-algebra. Furthermore assume that K is a function field. We say that B* satisfies the Eichler condition relative to *R if there exists a prime of K which does not come from a prime ideal of R which does not ramify in B.*

Theorem 6.3.9 ([Rei03, 34.21]) *Let Λ be a maximal A-order in B and assume that B is Eichler with respect to A. Let I be a right Λ-ideal in B. Then I is principal if and only if the norm* $\mathrm{nrd}(I)$ *is principal.*

Proposition 6.3.10 (Analogue of [Rib90a, Lemma 3.16]) *Let φ be a supersingular Drinfeld-Stuhler module over $\mathbb{F}_\mathfrak{q}$. There exists an endomorphism of φ of odd power $\mathfrak{p}$.*

Proof. Let $\mathfrak{p} = (\varpi)$. By Prop 6.3.7, it suffices to find an $x \in \mathrm{End}(\varphi)$ such that $\det(x) = \varpi$. We want to apply [Vig80, Cor 5.9]. As $\mathrm{End}(\varphi)$ does not ramify at $\mathfrak{p}$, it does satisfy the Eichler condition with respect to $\mathbb{F}_q[t][\frac{1}{\mathfrak{p}}]$. Thus there exists an $x \in \mathrm{End}(\varphi) \otimes \mathbb{F}_q(t)$ which is integral over $\mathbb{F}_q[t][\frac{1}{\mathfrak{p}}]$ whose reduced norm is ϖ.
It remains to show that the determinant of an endomorphism is the reduced norm of the quaternion algebra. To prove this, consider an auxiliary finite prime $\mathfrak{l} \neq \mathfrak{p}, \mathfrak{q}$. Then we have an injection

$$\mathrm{End}(\varphi) \otimes \mathbb{F}_q(t)_\mathfrak{l} \hookrightarrow \mathrm{End}_{\mathbb{F}_q(t)_\mathfrak{l}}(T_\mathfrak{l}(\varphi)).$$

Furthermore, the determinant is compatible with this base change by Prop 6.3.7. Thus both the reduced norm and determinant agree with the determinant after base change to $\mathbb{F}_q(t)_\mathfrak{l}$ and thus they agree. □

Remark 6.3.11 By [Pap18, Thm 5.5], supersingular Drinfeld-Stuhler modules over $\overline{\mathbb{F}}_\mathfrak{q}$ are defined over a finite extension of $\mathbb{F}_\mathfrak{q}$. This might allow to apply the work [BH09]. Indeed Bornhofen-Hartl show that pure semi-simple Anderson motive admits a canonical dual isogeny (see [BH09, Thm 7.3]). Furthermore by [BH09, 6.15], every abelian τ-sheaf over a finite field becomes semi-simple over a finite extension and thus every abelian τ-sheaf is absolutely semi-simple. While this fits the intuition one has from abelian varieties, where by the Poincare reducibility theorem all abelian varieties are isogenous to a product of simple abelian varieties, the reader should be warned that Bornhofen-Hartl prove an example ([BH09, Ex 6.1]) of an abelian τ-sheaf which is *not* semi-simple.

We shall now prove that between any two enhanced supersingular Drinfeld-Stuhler modules, there exists an isogeny whose degree is a power of $\mathfrak{p}$.

Proposition 6.3.12 *Let φ_1 be a supersingular enhanced Drinfeld-Stuhler module over $\mathbb{F}_\mathfrak{q}$. Then $\mathrm{End}(\varphi_1) \otimes \mathbb{F}_p(t)$ is Eichler with respect to $\mathbb{F}_q[t][\frac{1}{\mathfrak{p}}]$.*

Proof. The prime $\mathfrak{p}$ is a prime of $\mathbb{F}_q(t)$ but not of $\mathbb{F}_q[t]\frac{1}{\mathfrak{p}}$. Furthermore $\mathrm{End}(\varphi_1) \otimes \mathbb{F}_q(t)$ ramifies at $\mathfrak{q}, \infty$. □

Proposition 6.3.13 *Let φ_1, φ_2 be supersingular enhanced Drinfeld-Stuhler modules over $\mathbb{F}_\mathfrak{q}$. The right $\mathrm{End}(\varphi_1)$-module $\mathrm{Hom}(\varphi_1, \varphi_2)$ becomes free over $\mathrm{End}(\varphi_1)[\frac{1}{\mathfrak{p}}]$.*

Proof. The maximal $\mathbb{F}_p[t]$-order $\mathrm{End}(\varphi_1)$ is split at $\mathfrak{p}$. We claim that $\mathrm{Hom}(\varphi_1, \varphi_2)[1/\mathfrak{p}]$ is a principal ideal of the maximal order $\mathrm{End}(\varphi_1)[1/\mathfrak{p}]$. By Thm 6.3.9, it suffices to show that the norm of $\mathrm{Hom}(\varphi_1, \varphi_2)[1/\mathfrak{p}]$ is principal. However the norm is an ideal of $\mathrm{nrd}(\mathrm{End}(\varphi_1)[1/\mathfrak{p}])$, which is

$$\begin{aligned}\mathrm{nrd}(\mathrm{End}(\varphi_1)[1/\mathfrak{p}]) &= \mathrm{nrd}(\mathrm{Mat}_2(\mathbb{F}_p[t][1/\mathfrak{p}])) \\ &= \mathbb{F}_p[t][1/\mathfrak{p}].\end{aligned}$$

Since the latter is a principal ideal domain, the condition is met and thus $\mathrm{Hom}(\varphi_1, \varphi_2)[1/\mathfrak{p}]$ is free. □

Remark 6.3.14 By Prop 6.3.13, there exists some $x \in \mathrm{End}(\varphi_1)$ such that $\mathrm{Hom}(\varphi_1, \varphi_2)[\frac{1}{\mathfrak{p}}] \cong x\,\mathrm{End}(\varphi_1)[\frac{1}{\mathfrak{p}}]$. The situation is analogous to the elliptic curve context. Indeed if E_1, E_2 are enhanced elliptic curves over $\mathbb{F}_q$, there exists some $x \in \mathrm{End}(E_1)$ such that $\mathrm{Hom}(E_1, E_2)[\frac{1}{p}] \cong x\,\mathrm{End}(E_1)[\frac{1}{p}]$. In particular the quotient $\mathrm{Hom}(E_1, E_2)/x\,\mathrm{End}(E_1)$ is finite and we have

$$[\mathrm{Hom}(E_1, E_2) : x\,\mathrm{End}(E_1)] = \deg(x)^2.$$

Furthermore the index must be a power of p and thus so must the degree of x. To finish the proof of Prop 6.3.15, we must show a similar interpretation of the degree of isogenies of Drinfeld-Stuhler modules.

Proposition 6.3.15 (Analogue of [Rib90a, Lemma 3.17]) *Let φ_1, φ_2 be supersingular enhanced Drinfeld-Stuhler modules over $\mathbb{F}_\mathfrak{q}$. Let $x \in \mathrm{Hom}(\varphi_1, \varphi_2)$ such that*

$$\mathrm{Hom}(\varphi_1, \varphi_2)[\frac{1}{\mathfrak{p}}] \cong x\,\mathrm{End}(\varphi_1)[\frac{1}{\mathfrak{p}}]$$

(such an element exists by Prop 6.3.13). Then x is a $\mathfrak{p}$-power isogeny.

Proof. For any prime $\mathfrak{l} \neq \mathfrak{p}$, we know that $\mathfrak{p} \in A_{\mathfrak{l}}^{\times}$. In particular, since $\mathrm{Hom}(\varphi_1, \varphi_2)[\frac{1}{\mathfrak{p}}] \cong x \,\mathrm{End}(\varphi_1)[\frac{1}{\mathfrak{p}}]$, we have

$$\mathrm{Hom}(\varphi_1, \varphi_2) \otimes A_{\mathfrak{l}} \cong x \,\mathrm{End}(\varphi_1) \otimes A_{\mathfrak{l}}.$$

If we furthermore assume that $\mathfrak{l} \neq \mathfrak{q}$, thanks to Tate's isogeny theorem for Drinfeld-Stuhler modules, the induced morphism on Tate modules $T_{\mathfrak{l}}(x)$ and the local shtuka/$\mathcal{D}$-Dieudonne module at $\mathfrak{q}$ are isomorphisms. Thus x is an isomorphism on the $\mathfrak{l}$-torsion for all $\mathfrak{l} \neq \mathfrak{p}$. Hence for all $\mathfrak{l} \neq \mathfrak{p}$, we have $v_{\mathfrak{l}}\chi(\ker(x)) = 0$ and thus the support of the kernel is concentrated in $\mathfrak{p}$. □

We can now prove the surjectivity in the analogue of Ribet's short exact sequence:

Proof. [Proof of Prop 6.3.2] We adapt the proof of [Rib90a, Thm 3.15]. It suffices to show that elements of the form $(\varphi_1 - \varphi_2, 0)$ and $(0, \varphi_1 - \varphi_2)$ lie in the image. By Prop 6.3.15 we have an isogeny $\varphi_1 \to \varphi_2$ of degree $\mathfrak{p}^m$ for some m. Furthermore by Prop 6.3.10, there exists an endomorphism of degree $\mathfrak{p}$ of φ. Thus, up to pre-composing with this endomorphism, we have an isogeny $\varphi_1 \to \varphi_2$ of degree $\mathfrak{p}^{2n}$ for some n. We may factor this into compositions of isogenies of degree $\mathfrak{p}$, say $a_0 \circ \ldots \circ a_{2n-1}$. Thus each $a_i, a_i^{\vee}$ is an element of $\Sigma^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}_{\text{mix.exc}}(\mathfrak{m}, \mathfrak{p})$. Then the element

$$a_0 - a_1^{\vee} + a_2 - a_3^{\vee} + \ldots - a_{2n-2}^{\vee} + a_{2n-1} \in \mathbb{Z}^{\Sigma^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}_{\text{mix.exc}}(\mathfrak{m},\mathfrak{p})}$$

actually lies in $\left(\mathbb{Z}^{\Sigma^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}_{\text{mix.exc}}(\mathfrak{m},\mathfrak{p})}\right)_0$ and furthermore maps to $(\varphi_1 - \varphi_2, 0)$. □

Recall that by the function field analogues of Deuring's theorem, (enhanced) supersingular Drinfeld-Stuhler modules are classified as invertible modules of the endomorphism algebra of any fixed supersingular Drinfeld-Stuhler module, often referred to as a *base-point*. Likewise for pure and mixed-exceptional Drinfeld-Stuhler modules. As the latter endomorphism algebra is an Eichler order, the following proposition will allow us to choose compatible base-points.

Proposition 6.3.16 (Analogue of [Rib90a, Prop 3.6]) *Let B, B' be maximal orders of a quaternion algebra of discriminant $\mathfrak{p}\mathfrak{D}$ such that $S := B \cap B'$ is an Eichler order of level $\mathfrak{M}$ in B. Then there exists an enhanced supersingular Drinfeld-Stuhler module $(\varphi, C_{\mathfrak{M}})$ over $\mathbb{F}_{\mathfrak{q}}$ with an isomorphism $\kappa : B \to \mathrm{End}(\varphi)$ which identifies $\mathrm{End}((\varphi, C_{\mathfrak{M}}))$ with S.*

Proof. For this proof, let us fix an enhanced $(\varphi_0, C_{0,\mathfrak{M}})$. Then $\mathrm{End}(\varphi_0) \otimes Q$ has discriminant $\mathfrak{p}\mathfrak{D}$ allowing us to fix an isomorphism $B \otimes Q \cong \mathrm{End}(\varphi_0) \otimes Q$. Henceforth we will assume that B, B' and their intersection $S := B \cap B'$ are orders in $\mathrm{End}(\varphi_0) \otimes Q$.

For any enhanced supersingular Drinfeld-Stuhler module $(\varphi, C_{\mathfrak{M}})$ and non-zero element of $\lambda \in \mathrm{Hom}(\varphi, \varphi_0) \otimes Q$ we get an injection

$$\kappa_\lambda : \mathrm{End}(\varphi, C_{\mathfrak{M}}) \otimes Q \to \mathrm{End}(\varphi_0, C_{0,\mathfrak{M}}), \quad e \mapsto \lambda \circ e \circ \lambda^{-1}.$$

By the Skolem-Noether theorem ([Sta25, Tag 074Q]) all $\kappa : \mathrm{End}(\varphi, C_{\mathfrak{M}}) \otimes Q \to \mathrm{End}(\varphi_0, C_{0,\mathfrak{M}})$ are of the form κ_λ. This yields a bijection between the isomorphism classes of pairs (φ, κ) and

$$(\mathrm{End}(\varphi_0, C_{0,\mathfrak{M}}) \otimes Q)_f / (\mathrm{End}(\varphi_0, C_{0,\mathfrak{M}})_f^\times Q^\times).$$

Let $g \in (\mathrm{End}(\varphi_0, C_{0,\mathfrak{M}}) \otimes Q)_f$ and denote by $(\varphi, C_{\mathfrak{M}})$ its associated pair (φ, κ). By construction of the bijection, the image of $\mathrm{End}(\varphi, C_{\mathfrak{M}})$ under κ is

$$\kappa(\mathrm{End}(\varphi, C_{\mathfrak{M}})) = g \,\mathrm{End}(\varphi_0, C_{0,\mathfrak{M}})_f g^{-1}$$

and

$$\kappa(\mathrm{End}(\varphi)) = g \,\mathrm{End}(\varphi_0)_f g^{-1}.$$

Hence we want to find a $g \in (\mathrm{End}(\varphi_0, C_{0,\mathfrak{M}}) \otimes Q)_f$ such that the two equalities

$$g \,\mathrm{End}(\varphi_0, C_{0,\mathfrak{M}})_f g^{-1} = S_f \text{ and } g \,\mathrm{End}(\varphi_0)_f g^{-1} = B_f$$

are simultaneously satisfied. To construct a $g \in (\mathrm{End}(\varphi_0, C_{0,\mathfrak{M}}) \otimes Q)_f$ such that the two (adelic) equalities are satisfied, it suffices to prove that from any $\mathfrak{l} \in \mathrm{Spec}\, A$, there exists an element of $(\mathrm{End}(\varphi_0, C_{0,\mathfrak{M}}) \otimes Q_\ell$ such that the "$\mathfrak{l}$-adic" equations are satisfied. We will now study these adelic equalities one prime at a time:

- **(Case $\mathfrak{l} \mid \mathfrak{p}\mathfrak{D}$)** The quaternion algebra $\mathrm{End}(\varphi_0, C_{0,\mathfrak{M}}) \otimes Q_\mathfrak{l}$ over $Q_\mathfrak{l}$ has a unique maximal order (see [Voi21, Prop 13.3.4] or [Vig80, Lemme 1.5]), hence the conditions are trivially satisfied.

- **(Case $\mathfrak{l}$ coprime to $\mathfrak{p}\mathfrak{M}$)** Since S is an Eichler order of level $\mathfrak{M}$, at primes $\mathfrak{l}$ away from $\mathfrak{p}\mathfrak{M}$ we have $\mathrm{End}(\varphi_0, C_{0,\mathfrak{M}})_\mathfrak{l} = \mathrm{End}(\varphi_0)_\mathfrak{l}$ and $S_\mathfrak{l} = B_\mathfrak{l}$. Thus the two conditions are equivalent. By [Voi21, Cor 10.5.5] (or [Vig80, Thm 2.3]), all maximal orders in $\mathrm{Mat}_2(Q_\mathfrak{l})$ are conjugate, in particular the maximal orders $\mathrm{End}(\varphi_0)_\mathfrak{l}$ and $B_\mathfrak{l}$ are.

- **(Case $\mathfrak{l}$ divides $\mathfrak{M}$)** Since $\mathrm{End}(\varphi_0, C_{0,\mathfrak{M}})_\mathfrak{l}$ and $S_\mathfrak{l}$ are Eichler orders, they are the intersections of a unique pair of maximal orders (see PropDef 2.5.20). By Prop 2.5.18, these unique orders

correspond to lattices. By section 23 of [Voi21], there exists a bijection between maximal orders in $M_2(Q_\mathfrak{l})$ and vertices in the Bruhat-Tits tree $\mathcal{T}$. The fact that $\mathrm{End}(\varphi_o, C_{0,\mathfrak{M}})$ and S are orders of level $\mathfrak{M}$, we see that the distance between the corresponding vertices is $\mathrm{ord}_\mathfrak{l}(\mathfrak{M})$. We may choose basis e_1, e_2 such that

$$[\mathrm{End}(\varphi_0)_\mathfrak{l}] = [A_\mathfrak{l} e_1 \oplus A_\mathfrak{l} e_2]$$

$$[S] = [A_\mathfrak{l} \oplus \mathfrak{l}^{\mathrm{ord}_\mathfrak{l}(\mathfrak{M})} A_\mathfrak{l} e_2]$$

and a basis f_1, f_2 such that

$$[\mathrm{End}(\varphi_0/C_{0,\mathfrak{M}})_\mathfrak{l}] = [A_\mathfrak{l} f_1 \oplus A_\mathfrak{l} f_2]$$

$$[B'_\mathfrak{l}] = [A_\mathfrak{l} f_1 \oplus \mathfrak{l}^{\mathrm{ord}_\mathfrak{l}(\mathfrak{M})} A_\mathfrak{l} f_2].$$

Thus $g \in \mathrm{GL}_2(Q_\mathfrak{l})$ sending e_i to f_i does the job.

□

Proposition 6.3.17 (Analogue of [Rib90a, Prop 4.7]) *Let $\mathbb{E}$ be a mixed-exceptional Drinfeld-Stuhler module over $\mathbb{F}_\mathfrak{p}$ with $\Gamma_0(\mathfrak{M})$-level structure $C_\mathfrak{M}$ and denote by $\overline{\mathbb{E}}$ a pure quotient thereof. There exists a supersingular $(\varphi_0, C_\mathfrak{M})$ over $\mathbb{F}_\mathfrak{q}$ with cyclic subgroup $C_\mathfrak{p}$ such that there exists an isomorphism*

$$\kappa : \mathrm{End}(\varphi_0) \to \mathrm{End}_\mathcal{D}(\mathbb{E})$$

such that we get compatible isomorphisms

$$\begin{array}{ccc}
\mathrm{End}(\varphi_0) & \xrightarrow{\sim} & \mathrm{End}_\mathcal{D}(\mathbb{E}) \\
\uparrow & & \uparrow \\
\mathrm{End}(\varphi_0, C_\mathfrak{M}) & \xrightarrow{\sim} & \mathrm{End}_\mathcal{D}(\mathbb{E}, C_\mathfrak{M}) \\
\uparrow & & \uparrow \\
\mathrm{End}(\varphi_0, C_\mathfrak{M}, C_\mathfrak{p}) & \xrightarrow{\sim} & \mathrm{End}_\mathcal{D}(\overline{\mathbb{E}}, \overline{C_\mathfrak{M}})
\end{array}$$

Proof. The existence of φ_0 with its level $\mathfrak{m}$-structure follows from Prop 6.3.16. Hence what is left is to associate a $\mathfrak{p}$-level structure compatible with the lower half of the diagram. Via κ, we may identify $\mathrm{End}(\varphi_0, C_\mathfrak{M}, C_\mathfrak{p}) \otimes A_\mathfrak{p}$ with an Eichler order of level $\mathfrak{p}$ in the maximal order

$\mathrm{End}(T_{\mathfrak{p}}(\varphi_0)) \subset \mathrm{End}(V_{\mathfrak{p}}(\varphi_0))$. Then there exists a lattice $U \subset V_{\mathfrak{p}}(\varphi)$ such that

$$S \otimes A_{\mathfrak{p}} = \mathrm{End}(T_{\mathfrak{p}}(\varphi_0)) \cap \mathrm{End}(U)$$

Then we can scale such that $T_{\mathfrak{p}}(\varphi_0) \supset U \supset \mathfrak{p}T_{\mathfrak{p}}(\varphi_0)$. Then the image of

$$T_{\mathfrak{p}}(\varphi)/U \to T_{\mathfrak{p}}(\varphi_0)/\mathfrak{p}T_{\mathfrak{p}}(\varphi_0)$$

gives the desired level structure. □

Proof. [Proof of Thm 6.3.1] Using Prop 6.1.19 to describe $\mathcal{X}(J^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{p})$ and Prop 6.1.18 to describe $\mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}), \mathfrak{q})$ and $\mathcal{X}(J^{\mathfrak{D}}(\mathfrak{q}\mathfrak{M}), q)$, we have the following diagram where we wish to construct the dashed arrows in such a way that we have commuting isomorphisms.

$$\begin{array}{ccccccc}
0 & \longrightarrow & \ker & \longrightarrow & \mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}), \mathfrak{q}) & \longrightarrow & \mathcal{X}(J^{\mathfrak{D}}(\mathfrak{q}\mathfrak{M}), q)^2 \\
 & & & & \downarrow \text{Prop } 6.1.18 & & \downarrow \text{Prop } 6.1.18 \\
 & & & & \left(\mathbb{Z}^{\Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{p}\mathfrak{M}, \mathfrak{q})}\right)_0 & \longrightarrow & \left(\mathbb{Z}^{\Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{M}, \mathfrak{q})}\right)^2_0 \\
 & & & & \vdots & & \vdots \\
0 & \longrightarrow & \mathcal{X}(J_0^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{p}) & \xrightarrow{\text{Prop } 6.1.19} & \left(\mathbb{Z}^{\Sigma^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}_{\text{mixed-exceptional}}(\mathfrak{M}, \mathfrak{p})}\right)_0 & \longrightarrow & \left(\mathbb{Z}^{\Sigma^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}_{\text{pure}}(\mathfrak{M}, \mathfrak{p})}\right)^2_0.
\end{array}$$

For any choice of a supersingular Drinfeld-Stuhler module φ_0 over $\mathbb{F}_{\mathfrak{q}}$ and of a mixed-exceptional Drinfeld-Stuhler module $\mathbb{E}$ over $\mathbb{F}_{\mathfrak{p}}$, we know by the work of Papikian (i.e. Prop 5.6.15 and Prop 5.6.14), that

$$\Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{M}, \mathfrak{q}) \cong \mathrm{Pic}(\mathrm{End}(\varphi_0, C_{\mathfrak{M}})) \text{ and } \Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{p}\mathfrak{M}, \mathfrak{q}) \cong \mathrm{Pic}(\mathrm{End}(\varphi_0, C_{\mathfrak{M}}, C_{\mathfrak{p}}))$$

$$\Sigma^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}_{\text{mixed-exceptional}}(\mathfrak{M}, \mathfrak{p}) \cong \mathrm{Pic}(\mathrm{End}(\mathbb{E}, C_{\mathfrak{M}})) \text{ and } \Sigma^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}_{\text{pure}}(\mathfrak{M}, \mathfrak{p}) \cong \mathrm{Pic}(\mathrm{End}(\overline{\mathbb{E}}, \overline{C}_{\mathfrak{M}})).$$

By Prop 6.3.17, we can choose $(\varphi_0, C_{\mathfrak{M}}, C_{\mathfrak{p}})$ such that the diagram commutes. This yields the desired short exact sequence. To show that this short exact sequence is compatible with the monodromy pairing, we can use the description thereof from the Picard-Lefschetz formula (see Rem 6.1.16): as the pairing is given by the restriction of the pairing on $\mathbb{Z}^{\Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{p}\mathfrak{M}, \mathfrak{q})}$ and $\mathbb{Z}^{\Sigma^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}_{\text{mixed-exceptional}}(\mathfrak{M}, \mathfrak{p})}$, we have to show that

$$|\mathrm{Aut}(\varphi_0, C_{\mathfrak{M}})| = |\mathrm{Aut}(\mathbb{E}, C_{\mathfrak{M}})|.$$

As by construction we have $\mathrm{End}(\varphi_0, C_{\mathfrak{M}}) \cong \mathrm{End}(\mathbb{E}, C_{\mathfrak{M}})$, this is true. To show that the short exact sequence respects the Hecke operators $T_{\mathfrak{r}}$ for $\mathfrak{r}$ coprime to $\mathfrak{pq}\mathfrak{M}$, it suffices to show that "forgetful" and "quotient" morphisms (i.e. the usual degeneracy morphisms)

$$J_0^{\mathfrak{D}}(\mathfrak{rn}) \underset{\beta}{\overset{\alpha}{\rightrightarrows}} J_0^{\mathfrak{D}}(\mathfrak{n})$$

induce the same morphisms on

$$\left(\mathbb{Z}^{\Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{rp}\mathfrak{M},\mathfrak{q})}\right)_0 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \left(\mathbb{Z}^{\Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{p}\mathfrak{M},\mathfrak{q})}\right)_0$$

$$\left(\mathbb{Z}^{\Sigma^{\mathfrak{D}\mathfrak{pq}}_{\text{mixed-exceptional}}(\mathfrak{r}\mathfrak{M},\mathfrak{p})}\right)_0 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \left(\mathbb{Z}^{\Sigma^{\mathfrak{D}\mathfrak{pq}}_{\text{mixed-exceptional}}(\mathfrak{M},\mathfrak{p})}\right)_0.$$

This is true as an $\mathfrak{r}$-isogeny on the supersingular side gets mapped to a $\mathfrak{r}$-isogeny on the mixed side and that forgetting the isogeny gets mapped to forgetting the isogeny. □

In the remainder of this section, we will prove another short exact sequence which will be relevant to prove level-lowering.

Definition 6.3.18 *The degeneracy morphism*

$$J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{q}) \times J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{q}) \xrightarrow{\deg} J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q})$$

induces a morphism of the component groups, whose kernel and cokernel we denote by $\ker(\Phi_{\deg})$ *and* $\mathrm{coker}(\Phi_{\deg})$ *respectively*[3]

$$0 \to \ker(\Phi_{\deg}) \to \Phi(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{q}), \mathfrak{q}) \times \Phi(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{q}), \mathfrak{q}) \to \Phi(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}), \mathfrak{q}) \to \mathrm{coker}(\Phi_{\deg}) \to 0.$$

Proposition 6.3.19 (Analogue of Prop 3.23, Prop 3.25, Thm 4.3 [Rib90a]) *Consider the endomorphism*

$$\mu : \mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{q}), \mathfrak{q}) \times \mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{q}), \mathfrak{q}) \to \mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{q}), \mathfrak{q}) \times \mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{q}), \mathfrak{q})$$

given by

$$\begin{pmatrix} |\mathfrak{p}| + 1 & T_{\mathfrak{p}} \\ T_{\mathfrak{p}} & |\mathfrak{p}| + 1 \end{pmatrix}$$

[3] in [Rib90a] these two groups are called K and C respectively, a notation we change to avoid confusion as K in this thesis often refers to a global function field and C to a smooth proper curve.

This yields an exact sequence

$$0 \to \ker(\Phi_{\deg}) \to \left(\mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{q}), \mathfrak{q})\right)^2 / \mathrm{im}(\mu) \to \Phi(J_0^{\mathfrak{p}\mathfrak{q}\mathfrak{D}}(\mathfrak{M}), \mathfrak{p}) \to \mathrm{coker}(\Phi_{\deg}) \to 0.$$

Furthermore we have an isomorphism

$$\mu(\mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{q}\mathfrak{M}), \mathfrak{q})^2) \cong \left(T_{\mathfrak{p}}^2 - 1\right) \left(\mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{q}\mathfrak{M}), \mathfrak{q})^2\right).$$

Proof. Since we have a short exact sequence

$$0 \to \mathcal{X}(J_0^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{p}) \xrightarrow{\iota} \mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}), \mathfrak{q}) \to \mathcal{X}(J^{\mathfrak{D}}(\mathfrak{q}\mathfrak{M}), \mathfrak{q})^2 \to 0,$$

we get a dual exact sequence

$$0 \to (\mathcal{X}(J^{\mathfrak{D}}(\mathfrak{q}\mathfrak{M}), \mathfrak{q})^*)^2 \to \mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}), \mathfrak{q})^* \to \mathcal{X}(J_0^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{p})^* \to 0.$$

Furthermore from the Monodromy short sequence for $J_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{q}\mathfrak{M})$ at $\mathfrak{q}$, via the embedding

$$\mathcal{X}(J_0^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{p}) \xrightarrow{\iota} \mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}), \mathfrak{q})$$

we get the short exact sequence

$$0 \longrightarrow \mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{q}\mathfrak{M}), \mathfrak{q}) / \mathcal{X}(J_0^{\mathfrak{p}\mathfrak{q}\mathfrak{D}}(\mathfrak{M}), \mathfrak{p}) \longrightarrow \mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{q}\mathfrak{M}), \mathfrak{q})^* / \mathcal{X}(J_0^{\mathfrak{p}\mathfrak{q}\mathfrak{D}}(\mathfrak{M}), \mathfrak{p}) \longrightarrow \Phi(J_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{q}\mathfrak{M}), \mathfrak{q}) \longrightarrow 0.$$

The degeneracy maps

$$J_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{q}\mathfrak{M}) \times J_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{q}\mathfrak{M}) \xrightarrow{\deg} J_0^{\mathfrak{D}}(\mathfrak{q}\mathfrak{M})$$

induces, via the Albanese functoriality, a morphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{q}), \mathfrak{q})^2 & \longrightarrow & (\mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{q}), \mathfrak{q})^*)^2 & \longrightarrow & \Phi(J_0^{\mathfrak{D}}(\mathfrak{q}\mathfrak{M}), \mathfrak{q})^2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \Phi_{\deg}} & & \\
0 & \longrightarrow & \mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}), \mathfrak{q}) / \mathcal{X}(J_0^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{p}) & \longrightarrow & \mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}), \mathfrak{q})^* / \mathcal{X}(J_0^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{p}) & \longrightarrow & \Phi(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}), \mathfrak{q}) & \longrightarrow & 0
\end{array}$$

Furthermore the quotient $\mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}), \mathfrak{q}) / \mathcal{X}(J_0^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{p})$ is isomorphic to $\mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{q}), \mathfrak{q})^2$ and by definition of the Hecke operator, the resulting composition is μ.
The cokernel of the middle vertical map is

$$\mathcal{X}(J_0^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{p})^* / \mathcal{X}(J_0^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{p}) \cong \Phi(J_0^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{p})$$

by the dual of Ribet's short exact sequence. Furthermore both target and source of the middle vertical morphism have the same rank and the cokernel $\Phi(J_0^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M}),\mathfrak{p})$ is finite, the middle vertical morphism is injective. The short exact sequence claimed in the statement thus follows from the snake lemma. It now only remains to prove the claimed automorphism. This follows from the fact that the automorphism

$$\lambda : \mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{q}\mathfrak{M}),\mathfrak{q})^2 \to \mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{q}\mathfrak{M}),\mathfrak{q})^2, \quad (x,y) \mapsto (-x, T_{\mathfrak{p}}x - y)$$

is such that $\mu \circ \lambda = \left(T_{\mathfrak{p}}^2 - 1\right)$. □

6.4 Eisenstein Property of Component group

The goal of this section is to prove a function field analogue of the following theorem:

Theorem 6.4.1 ([Rib88, Thm 2]) *The component group $\Phi(J_0(pN),p)$ is Eisenstein, i.e. for all $q \not| pN$, the operator $T_q - (q+1)$ acts trivially on $\Phi(J_0(pN),p)$.*

Remark 6.4.2 In fact, in [RT97] is mentioned that the same methods extend to the quaternionic case, which is the result whose function field analogue we will prove. We should also mention that this result was anticipated by M. Papikian in [Pap16] in the $D = 1$ case. He also showed that $\Phi(J_0(\mathfrak{p}),\infty)$ is *not* Eisenstein for $|\mathfrak{p}|$ sufficiently large.

Theorem 6.4.3 (Φ is Eisenstein) *Let $\mathfrak{p}$ be a prime which is coprime to $\mathfrak{M}\mathfrak{D}$. The component group $\Phi(J_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{M}),\mathfrak{p})$ is Eisenstein, that is for all $\mathfrak{q}$ coprime to $\mathfrak{p}\mathfrak{M}$, the two natural degeneration maps α,β*

$$J_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{M}) \rightrightarrows J_0^{\mathfrak{D}}(\mathfrak{q}\mathfrak{p}\mathfrak{M})$$

induce the same morphism on the component groups.

Before we prove this, we need two auxiliary Propositions.

Proposition 6.4.4 ([Mad77, Thm 4C]) *Let K be a geometric quadratic extension of $\mathbb{F}_q(x)$ of genus g. Then for any $\alpha \in \mathcal{O}_K \backslash \mathbb{F}_q[x]$, we have*

$$\deg \mathrm{Nm}(\alpha) \geqslant 2g+1.$$

The following is inspired by [Rib88, Prop 2].

Proposition 6.4.5 *Let φ be a supersingular Drinfeld-Stuhler module over $\overline{\mathbb{F}}_{\mathfrak{q}}$ and C a finite cyclic subgroup of order $\mathfrak{p}$. Assume that there exists $\varepsilon \in \left(\mathrm{Aut}(\varphi)\backslash\mathbb{F}_q^{\times}\right)$ which preserves C. Then there exists an isomorphism*

$$\varphi/C \to \varphi.$$

Proof. Denote by K the field generated by the element ε. This is a quadratic field extension of $\mathbb{F}_q(t)$ which is naturally embedded in $\mathrm{End}(\varphi)$. Since ε is a unit in $\mathcal{O}_K$, its norm $\mathrm{Nm}(\varepsilon)$ must also be a unit of $\mathbb{F}_q[t]$. If K were to be geometric, then by Prop 6.4.4, its norm would be of non-zero degree and thus not a unit of $\mathbb{F}_q[t]$. Thus $K/\mathbb{F}_q(t)$ is a constant quadratic extension and thus $\mathbb{F}_{q^2}(t)$. Denote by R the ring of integers of K. Since ε respects C, we thus have a natural morphism

$$\mathrm{End}(\varphi) \supset R \xrightarrow{\rho} \mathrm{End}(C) \cong \mathbb{F}_q[t]/\mathfrak{p}.$$

As the ring of integers of K is a principal ideal domain, the kernel $\ker(\rho)$ is a principal ideal, which is generated by an endomorphism x of φ. In particular we have $C \subset \ker x$. Choose a generator $\mathfrak{p} = (\pi)$. Then locally at $\mathfrak{p}$, we see that $(\det(x)) = \mathfrak{p}$. Thus by Prop 6.3.7, we see that $C = \ker(x)$. □

Proposition 6.4.6 *Let φ be a supersingular Drinfeld-Stuhler module over $\overline{\mathbb{F}}_{\mathfrak{q}}$ and C, C' two finite cyclic subgroups of φ with trivial intersection. Furthermore assume that there exists $\varepsilon \in \left(\mathrm{Aut}(\varphi)/\mathbb{F}_q^{\times}\right)\setminus\{1\}$ which preserves both C and C'. Then there exists an isomorphism*

$$\varphi/C \to \varphi$$

taking $(C \oplus C')/C$ to C'.

Proof. We may apply the construction from the proof of Prop 6.4.5. Thus by Prop 6.4.5, there exists an isomorphism $\varphi/C \to \varphi$. This is, by construction, a polynomial in ε. As ε respects C' the statement follows. □

Proof. [of Thm 6.4.3] We adapt the proof of Thm 2 [Rib88]. Using the combinatorial description of the component group from Prop 6.1.20, to show that α_*, β_* agree on the component group, it suffices to show that the corresponding morphisms $\alpha_*, \beta_* : \left(\mathbb{Z}^{\Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{N},\mathfrak{p})}\right)^*_0 \to \left(\mathbb{Z}^{\Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{q}\mathfrak{N},\mathfrak{p})}\right)^*_0$ agree up to the image of $\left(\mathbb{Z}^{\Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{q}\mathfrak{N},\mathfrak{p})}\right)_0$, or equivalently that the image of

$$\eta := \mathrm{Hom}(\alpha^* - \beta^*, \mathbb{Z}) : \left(\mathbb{Z}^{\Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{N},\mathfrak{p})}\right)^*_0 \to \left(\mathbb{Z}^{\Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{q}\mathfrak{N},\mathfrak{p})}\right)^*_0$$

is contained in the image of $\left(\mathbb{Z}^{\Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{q}\mathfrak{N},\mathfrak{p})}\right)_0$. For notational purposes, let us introduce the notation

$$\sigma : \Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{q}\mathfrak{N},\mathfrak{p}) \to \Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{N},\mathfrak{p}), \quad (\varphi, C_{\mathfrak{N}}, C_{\mathfrak{q}}) \mapsto (\varphi, C_{\mathfrak{N}})$$

$$\tau : \Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{q}\mathfrak{N},\mathfrak{p}) \to \Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{N},\mathfrak{p}), \quad (\varphi, C_{\mathfrak{N}}, C_{\mathfrak{q}}) \mapsto (\varphi/C_{\mathfrak{q}}, (C_{\mathfrak{N}} \oplus C_{\mathfrak{q}})/C_{\mathfrak{q}}).$$

These have the benefit that for any $\varphi \in \left(\mathbb{Z}^{\Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{N},\mathfrak{p})}\right)^*_0$ and $y_1, y_2 \in \Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{q}\mathfrak{N},\mathfrak{p})$, we have

$$\eta(\varphi)(y_1 - y_2) = \varphi(\sigma y_1 - \tau y_1) - \varphi(\sigma y_2 - \tau y_2).$$

Given φ, we now give a candidate

$$l := \sum_{s \in \Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{q}\mathfrak{N},\mathfrak{p})} \varphi(\sigma s - \tau s) s \in \mathbb{Z}^{\Sigma^{\mathfrak{D}}(\mathfrak{q}\mathfrak{N},\mathfrak{p})}$$

such that $\langle l, - \rangle = \eta(\varphi)$. First we have to show that l is of degree zero. Using the fact that we have $\sigma \circ \omega_{\mathfrak{q}} = \tau$, we see that

$$\begin{aligned}
\deg l &= \sum_{s \in \Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{q}\mathfrak{N},\mathfrak{p})} \varphi(\sigma s) - \sum_{s \in \Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{q}\mathfrak{N},\mathfrak{p})} \varphi(\tau s) \\
&= \sum_{s \in \Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{q}\mathfrak{N},\mathfrak{p})} \varphi(\sigma s) - \sum_{s \in \Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{q}\mathfrak{N},\mathfrak{p})} \varphi(\sigma \circ \omega_{\mathfrak{q}} s) \\
&= \sum_{s \in \Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{q}\mathfrak{N},\mathfrak{p})} \varphi(\sigma s) - \sum_{s \in \Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{q}\mathfrak{N},\mathfrak{p})} \varphi(\sigma s) = 0.
\end{aligned}$$

Now for any $y_1, y_2 \in \Sigma^{\mathfrak{D}}_{\text{sup.sing}}(\mathfrak{q}\mathfrak{N},\mathfrak{p})$ we have

$$\langle l, y_1 - y_2 \rangle = \varphi(\sigma y_1 - \tau y_1)\frac{|\operatorname{Aut}(y_1)|}{q-1} - \varphi(\sigma y_2 - \tau y_2)\frac{|\operatorname{Aut}(y_2)|}{q-1}.$$

We want to show that this is equal to

$$\eta(\varphi)(y_1 - y_2) = \varphi(\sigma y_1 - \tau y_1) - \varphi(\sigma y_2 - \tau y_2).$$

If $\frac{|\operatorname{Aut}(y_i)|}{q-1} > 1$, then this automorphism ε will preserve $C_{\mathfrak{q}}$ and $C_{\mathfrak{N}}$. Hence by Prop 6.4.6, there exists an isomorphism r such that $\sigma\varphi = \tau r \varphi$. Hence if $\frac{|\operatorname{Aut}(y_1)|}{q-1} > 1$, the difference $\sigma y_1 - \tau y_1$ vanishes and the equality is clear. If $\frac{|\operatorname{Aut}(y_1)|}{q-1} = \frac{|\operatorname{Aut}(y_2)|}{q-1} = 1$ then we also have equality of $\langle l, - \rangle$ and $\eta(\varphi)$ are thus done. $\square$

Remark 6.4.7 Whilst the Eisenstein-ness of the group of connected components is in perfect parallel with the number field case, there is a difference at the place at infinity. It was shown by Merel in [Mer14] that the group of connected components of $J_0(N)(\mathbb{R})$ is Eisenstein. However by work of Papikian [Pap16], this is is not true of $J_0(\mathfrak{p})$ at ∞ for $\mathfrak{p}$ whose degree is sufficiently large. This follows in a first step from an estimation that

$$\ln|\Phi(J_0(\mathfrak{p}),\infty)| \sim c(q)|\,\mathrm{GL}_2(A):\Gamma_0(\mathfrak{p})|$$

that is the fraction of these two terms goes to one as $|\mathfrak{p}|$ goes to infinity. Then Papikian shows that the submodule of $\Phi(J_0(\mathfrak{p}),\infty)$ which is annihilated by the Eisenstein ideal $\mathfrak{E}(\mathfrak{p})$ is isomorphic to $\mathbb{T}(\mathfrak{p})/\mathfrak{E}(\mathfrak{p}) \cong \mathbb{Z}/N(\mathfrak{p})$ where

$$N(\mathfrak{p}) = \begin{cases} \frac{|\mathfrak{p}|-1}{q-1}, & \deg\mathfrak{p} \text{ is odd} \\ \frac{|\mathfrak{p}|-1}{q^2-1}, & \deg\mathfrak{p} \text{ is even.} \end{cases}$$

6.5 Background on Hecke algebra

6.5.1 Hecke algebra

Definition 6.5.1 *The commutative subalgebra of* $\mathrm{End}(\mathcal{H}_!(\mathcal{T},\mathbb{Z})^{\Gamma_0(\mathfrak{n})})$ *which is generated by the identity and all Hecke operators defined in Def 3.2.17 is called the* Hecke algebra *and denoted by* $\mathbb{T}_\mathfrak{n}$.

For the following result, recall that a harmonic cochain is a function on $\mathrm{GL}_2(\mathbb{F}_q(t)_\infty)$.

Theorem 6.5.2 ([Gek95, Thm 3.17]) *The pairing*

$$\mathbb{T}_\mathfrak{n} \times \mathcal{H}_!(\mathcal{T},\mathbb{Z})^{\Gamma_0(\mathfrak{n})} \to \mathbb{Z}, \quad (T,f) \mapsto -(Tf)\left(\begin{pmatrix} \frac{1}{t^2} & \frac{1}{t} \\ 0 & 1 \end{pmatrix}\right)$$

is non-degenerate and is perfect after inverting p.

Remark 6.5.3 This can be seen the *Fourier coefficient of* Tf *at the trivial divisor* (see [Gek95, p.42] or [Wei71, Chapt III]).

Definition 6.5.4 *The image of* $\mathbb{T}_{\mathfrak{M}_\mathfrak{q}}$ *in the endomorphism ring of the* $\mathfrak{q}$*-old space and its orthogonal complement of* new space *are the rings* $\mathbb{T}_{\mathfrak{M}_\mathfrak{q},old}$ *and* $\mathbb{T}_{\mathfrak{M}_\mathfrak{q},new}$ *respectively.*

Definition 6.5.5 *The image of the degeneracy maps generates an abelian subvariety of* $J_0(\mathfrak{pm})$ *which we call the* $\mathfrak{p}$-old subvariety *and denote by* $J_0(\mathfrak{pm})^{\mathfrak{p}\text{-}old}$. *The quotient is called the* $\mathfrak{p}$-new quotient *and denoted by* $J_0(\mathfrak{pm})_{\mathfrak{p}\text{-}new}$. *The dual of this morphism gives a subvariety of* $J_0(\mathfrak{pm})$ *which we call the* $\mathfrak{p}$-new subvariety *and denote by* $J_0(\mathfrak{pm})^{\mathfrak{p}\text{-}new}$.

Remark 6.5.6 Any subvariety or quotient of an abelian variety with toric reduction has itself toric reduction (see for instance [Pap07a]). In particular since $J_0(\mathfrak{n})$ has toric reduction at infinity, so do the associated new/old sub- and quotient varieties.

Proposition 6.5.7 ([Pap07a, Prop 3.3][4]) *There exists a canonical isomorphism of Hecke modules*

$$\mathcal{H}_!(\mathcal{T},\mathbb{Z})^{\Gamma_0^{\mathfrak{D}}(\mathfrak{n})} \cong \mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{n}),\infty).$$

Proof. Recall that the uniformization at ∞ is given by

$$0 \to \bar{\Gamma} \to \mathrm{Hom}(\bar{\Gamma}, \mathbb{G}_{m,\mathbb{C}_\infty}^{an}) \to J_0^{\mathfrak{D}}(\mathfrak{n})^{an} \to 0.$$

(where the map $\bar{\Gamma} \to \mathrm{Hom}(\bar{\Gamma}, \mathbb{G}_{m,\mathbb{C}_\infty}^{an})$ is given by Theta series, see Section 3.6 and Chapter V). If we denote by $\mathcal{J}^0$ the identity component of the special fiber of the Neron model, then, by work of Raynaud, if we set $\Lambda := \mathrm{Hom}(\hat{\mathcal{J}}^0, \hat{\mathbb{G}}_m)$, we have a short exact sequence

$$0 \to \Lambda \to \mathrm{Hom}(\bar{\Gamma}, \mathbb{G}_{m,\mathbb{C}_\infty}^{an}) \to J_0^{\mathfrak{D}}(\mathfrak{n})^{an} \to 0.$$

In particular we have $\bar{\Gamma} \cong \mathrm{Hom}(\hat{\mathcal{J}}^0, \hat{\mathbb{G}}_m)$. However since the Jacobian has toric reduction, we have

$$\mathrm{Hom}(\hat{\mathcal{J}}^0, \hat{\mathbb{G}}_m) \cong \mathrm{Hom}((\hat{\mathcal{J}}^0)_k, \hat{\mathbb{G}}_{m,k}) \cong \mathrm{Hom}(\mathcal{J}_k^0, \mathbb{G}_{m,k}) = \mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{n}),\infty).$$

□

The following appears in [Pap07a, p.151]).

Proposition 6.5.8 *The Hecke algebra acts faithfully on* $\mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{n}),\infty)$*, i.e. we have a faithful morphism*

$$\mathbb{T} \to \mathrm{End}_{\mathbb{Z}}(\mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{n}),\infty)).$$

Proof. Any endomorphism of $J_0^{\mathfrak{D}}(\mathfrak{n})$ extends to a endomorphism of the Neron model, which we denote by $\mathcal{J}$ (at infinity). Since $J_0^{\mathfrak{D}}(\mathfrak{n})$ has toric reduction at ∞, the induced morphism gives an injection

$$\mathrm{End}_K(J) \hookrightarrow \mathrm{End}_k(\mathcal{J}_k^0).$$

Furthermore endomorphisms of split tori are determined by their action on the character group, and thus we have $\mathrm{End}_k(\mathcal{J}_k^0) \cong \mathrm{End}_{\mathbb{Z}}(\mathcal{X}(J,\infty))$. □

The following was proved by Papikian in the proof of [Pap07a, Prop 3.5]

Proposition 6.5.9 *We have natural isomorphisms*

$$\mathcal{H}_!(\mathcal{T},\mathbb{Q})^{\Gamma_0(\mathfrak{pn}),\mathfrak{p}\text{-}old} \cong \mathcal{X}(J_0(\mathfrak{pn})_{\mathfrak{p}\text{-}old},\infty)\otimes\mathbb{Q}$$

$$\mathcal{H}_!(\mathcal{T},\mathbb{Q})^{\Gamma_0(\mathfrak{pn}),\mathfrak{p}\text{-}new} \cong \mathcal{X}(J_0(\mathfrak{pn})_{\mathfrak{p}\text{-}new},\infty)\otimes\mathbb{Q}.$$

Furthermore the saturations of $\mathcal{X}(J_0(\mathfrak{pn})_{\mathfrak{p}\text{-}old},\infty)$ and $\mathcal{X}(J_0(\mathfrak{pn})_{\mathfrak{p}\text{-}new},\infty)$ are isomorphic to $\mathcal{H}_!(\mathcal{T},\mathbb{Z})^{\Gamma_0(\mathfrak{pn}),\mathfrak{p}\text{-}old}$ and $\mathcal{H}_!(\mathcal{T},\mathbb{Z})^{\Gamma_0(\mathfrak{pn}),\mathfrak{p}\text{-}new}$ respectively.

Proof. Consider the short exact sequence

$$0 \to J_0(\mathfrak{pn})^{\mathfrak{p}\text{-old}} \to J_0(\mathfrak{pn}) \to J_0(\mathfrak{pn})_{\mathfrak{p}\text{-new}} \to 0.$$

Then by Prop 6.1.23, we get a short exact sequence of the component groups at ∞

$$0 \to \overline{\mathcal{X}(J_0(\mathfrak{pn})_{\mathfrak{p}\text{-old}},\infty)} \to \mathcal{X}(J_0(\mathfrak{pn}),\infty) \to \mathcal{X}(J_0(\mathfrak{pn})_{\mathfrak{p}\text{-new}},\infty) \to 0.$$

By Prop 6.5.7, the middle term is the space of automorphic forms of level $\mathfrak{pn}$. As $\mathcal{X}(J_0(\mathfrak{pn})_{\mathfrak{p}\text{-old}},\infty)$ is the image of $\mathcal{X}(J_0(\mathfrak{n}),\infty)$ under the degeneracy maps, so is the component group. □

Proposition 6.5.10 (Analogue of [Rib90a, Prop 3.21]) *The Hecke algebra $\mathbb{T}_{\mathfrak{pqn}}$ acts on $\mathcal{X}(J_0(\mathfrak{nq}),\mathfrak{q})^2$ via the $\mathfrak{q}$-new/$\mathfrak{p}$-old quotient.*

Proof. In a first step, we note that $J_0(\mathfrak{nq})\times J_0(\mathfrak{nq})$ is isogenous to the $\mathfrak{p}$-old subvariety $J_0(\mathfrak{npq})_{\mathfrak{p}\text{-old}} \subset J_0(\mathfrak{npq})$. Hence we have an isogeny

$$(J_0(\mathfrak{nq})\times J_0(\mathfrak{nq}))^{\mathfrak{q}\text{-new quotient}} \sim (J_0(\mathfrak{npq})_{\mathfrak{p}\text{-old}})^{\mathfrak{q}\text{-new}}.$$

The character group at infinity (tensored up $-\otimes_{\mathbb{Z}}\mathbb{Q}$) is isomorphic to the intersection of $\mathcal{H}_!(\Gamma_0(\mathfrak{pqn}),\mathbb{Q})^{\mathfrak{q}-new}$ and $\mathcal{H}_!(\Gamma_0(\mathfrak{pqn}),\mathbb{Q})^{\mathfrak{p}-old}$. □

6.5.2 Galois representation associated to maximal ideals

The following is an analogue of a well known fact, and was already mentioned in [PW15] page 23 (in the proof of Lemma 3.7)

Proposition 6.5.11 (Analogue of [Rib90a, Prop 5.1]) *Let $\mathfrak{m} \subset \mathbb{T}$ be a maximal ideal such that* $\mathrm{char}\ \mathbb{T}/\mathfrak{m} \neq p$. *Then there exists a unique semi-simple representation*

$$\rho_{\mathfrak{m}} : G_{\mathbb{F}_q(t)} \to \mathrm{GL}_2(\mathbb{T}/\mathfrak{m})$$

such that

$$\mathrm{tr}(\rho_{\mathfrak{m}}(\mathrm{Frob}_{\mathfrak{p}})) = T_{\mathfrak{p}} \quad \textit{and}\ \det(\rho_{\mathfrak{m}}(\mathrm{Frob}_{\mathfrak{p}})) = |\mathfrak{p}|.$$

Remark 6.5.12 Note that the condition $\mathrm{char}\ \mathbb{T}/\mathfrak{m} \neq p$ is necessary: as $|\mathfrak{p}|$ is divisible by p, if $\mathrm{char}\ \mathbb{T}/\mathfrak{m} = p$, we would have $\det(\rho_{\mathfrak{m}}(\mathrm{Frob}_{\mathfrak{p}})) = 0$ contradicting $\rho_{\mathfrak{m}}(\mathrm{Frob}_{\mathfrak{p}}) \in \mathrm{GL}_2(\mathbb{T}/\mathfrak{m})$.

Proof. We have a natural isomorphism

$$\mathcal{H}_!(\mathcal{T}, \mathbb{Z})^{\Gamma_0(\mathfrak{n})} \otimes_{\mathbb{Z}} \mathbb{T}/\mathfrak{m} \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{T}, \mathbb{T}/\mathfrak{m}).$$

Then the natural map $\mathbb{T} \to \mathbb{T}/\mathfrak{m}$ corresponds to an element of $\mathcal{H}_!(\mathcal{T}, \mathbb{Z})^{\Gamma_0(\mathfrak{n})} \otimes_{\mathbb{Z}} \mathbb{T}/\mathfrak{m}$, which we denote by f. By its definition, it is an eigenform. We can then lift f to an element of $\mathcal{H}_!(\mathcal{T}, \mathbb{Z})^{\Gamma_0(\mathfrak{n})} \otimes \mathbb{C}$, to which we can assign the prescribed Galois representation. □

Question 6.5.13 *Even though one can not ask for a representation $\rho_{\mathfrak{m}}$ if char $\mathbb{T}/\mathfrak{m} = p$, it still makes sense to ask if there exists an $\mathbb{T}/\mathfrak{m}$-valued two-dimensional* pseudo-representation *T (in the sense of [Tay91]) such that*

$$T(\mathrm{Frob}_{\mathfrak{p}}) = T_{\mathfrak{p}}.$$

Does such a pseudo-representation exist?

Proposition 6.5.14 (Analogue of [Rib90a, Thm 5.2]) *Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}$ of good characteristic ℓ such that $\rho_{\mathfrak{m}}$ is irreducible. Then consider*

$$J_0(\mathfrak{n})[\mathfrak{m}] := \bigcap_{T \in \mathfrak{m}} \ker\left(T : J_0(\mathfrak{n})(\mathbb{F}_q(t)^{\mathrm{sep}}) \to J_0(\mathfrak{n})(\mathbb{F}_q(t)^{\mathrm{sep}})\right) \subset J_0(\mathfrak{n})[\ell].$$

The $\mathbb{T}/\mathfrak{m}[G_{\mathbb{F}_q(t)}]$-module $J_0(\mathfrak{n})[\mathfrak{m}]$ has the following properties:

1. *it is non-zero*
2. $(J_0(\mathfrak{n})[\mathfrak{m}])^{s.s.} \cong \rho_{\mathfrak{m}} \times \ldots \times \rho_{\mathfrak{m}}$
3. *the ideals of the form $(T_{\mathfrak{p}} - 1 - |\mathfrak{p}|$ for all but finitely many $\mathfrak{p})$ and $\mathfrak{m}$ are coprime.*

Proof. The Hecke module $J_0(\mathfrak{n})[\mathfrak{m}]$ is non-zero as the Hecke algebra acts faithfully. Consider the module $J_0(\mathfrak{n})[\mathfrak{m}] \oplus \mathrm{Hom}(J_0(\mathfrak{n})[\mathfrak{m}], \mu_\ell)$. The Eichler-Shimura relation implies that the roots of the characteristic polynomial of $\mathrm{Frob}_\mathfrak{p}$ satisfies

$$x^2 - T_\mathfrak{p} x + |\mathfrak{p}|.$$

Then the characteristic polynomial of $\mathrm{Frob}_\mathfrak{p}$ as an endomorphism of $J_0(\mathfrak{n})[\mathfrak{m}] \oplus \mathrm{Hom}(J_0(\mathfrak{n})[\mathfrak{m}], \mu_\ell)$ satisfies $(x^2 - T_\mathfrak{p} x + |\mathfrak{p}|)^{\dim J_0(\mathfrak{n})[\mathfrak{m}]^{s.s}}$. Hence the semi-simplification must be the same as the product of $\rho_\mathfrak{m}$. If the two ideals in 3 were not coprime, then for almost all $\mathfrak{p}$ we have $T_\mathfrak{p} = 1 + |\mathfrak{p}|$ modulo $\mathfrak{m}$. Hence the corresponding representation $\rho_\mathfrak{m}$ has the same characteristic polynomial as the direct sum of the identity and the ℓ-adic cyclotomic character. However we assumed that $\rho_\mathfrak{m}$ is irreducible, which leads to a contradiction. □

Remark 6.5.15 The cited [Rib90a, Theorem 5.2] includes one more statement, namely K.Ribet shows a weak multiplicity one result, that is he shows that

$$\dim_{\mathbb{T}/\mathfrak{m}} J_0(N)[\mathfrak{m}] = 1.$$

The strategy, which is a generalization of Mazur's work, is to note that $J_0(N)[\mathfrak{m}] \subset J_0(N)[\ell]$. This in turn can be studied by looking at the reduction of $X_0(N)$ at ℓ, using the fact that the reduction then gives a finite flat ℓ-group. The composition of the ℓ-Frobenius with the Verschiebung then equals $\ell \cdot \mathrm{id} = 0$. As the image of $\rho_\mathfrak{m}$ is self-dual, and thus Frob, V are self-dual, we know that the rank of Frob, V must be less or equal to one. Of course this strategy does not work in this situation, as we can not take the reduction at ℓ.
If $\mathfrak{m}$ is Eisenstein, as was pointed out by Papikian in [Pap06], it follows from the work of Pal that multiplicity one holds. As far as the author understands, this does not generalize to the non-Eisenstein setup. However, we would like to point out that F.Diamond has an alternative proof of multiplicity one for number fields in [Dia97] which relies on the Taylor-Wiles construction. We would like to revisit the possibility of proving multiplicity one using this strategy.

6.6 Level Lowering

The goal of this section is to leverage the fact that we have an analogue of Ribet's exact sequence for level-lowering purposes, indeed this was the primary motivation for Ribet in [Rib90a]. We shall prove the following theorem

Theorem 6.6.1 (Level Lowering) *Fix $\mathfrak{p}, \mathfrak{M}$ coprime and a maximal ideal $\mathfrak{m}$ in $\mathbb{T}_{\mathfrak{p}\mathfrak{M}}$ such that* char $(\mathbb{T}_{\mathfrak{p}\mathfrak{M}}/\mathfrak{m}) =: \ell \neq p$*, ensuring that there exists an associated $\rho_{\mathfrak{m}}$ and that $\ell \nmid q-1$. If*

1. *$\rho_{\mathfrak{m}}$ is irreducible*
2. *$\rho_{\mathfrak{m}}$ is unramified at $\mathfrak{p}$*

then $\rho_{\mathfrak{m}}$ is modular of level $\mathfrak{M}$.

The condition that $\ell \nmid q-1$ should be seen as analogous to the assumption in the number field case that $\ell \neq 2$ (see Rem 6.6.13 for more on this). The strategy to prove Thm 6.6.1 is as follows: in a first step, we prove the theorem under the additional condition that $|\mathfrak{p}| \neq 1 \mod \ell$, which is known as *Mazur's level lowering*. In a second step, we shall do a *prime swapping step*, that is find an auxiliary $\mathfrak{q}$ such that $\rho_{\mathfrak{m}}$ is modular of level $\mathfrak{q}\mathfrak{M}$ and which satisfies the additional condition $|\mathfrak{q}| \neq 1 \mod \ell$, allowing us to apply Mazur's level lowering Thm 6.6.2. It should be noted that we can not adapt the exact strategy of [Rib90a] to prove the "prime swapping". Indeed Ribet's strategy is roughly to show that the failure of "prime swapping" would contradict weak multiplicity one. However as pointed out in Rem 6.5.15, it is not apriori clear that weak multiplicity one is true in the function field world. In [Rib94] Ribet provides a modified strategy avoiding the multiplicity assumption. His motivation for this work was in proving that level-lowering results could be applied for number field other than $\mathbb{Q}$. In [Rib94] he studies the multiplicities $\dim J^{pq}(M)[\mathfrak{m}], \dim J(pqM)[\mathfrak{m}]$ by relating them to the corresponding character and component groups, allowing him to derive a different contradiction. We shall follow this strategy in Subsection 6.6.2.

6.6.1 Mazur's level lowering at a good prime

This is the analogue of [Rib90a, Thm 6.1], whose proof strategy we follow (this was originally proven by Mazur in a letter to Mestre).

Theorem 6.6.2 (Mazur at good prime) *Fix $\mathfrak{p}, \mathfrak{M}$ coprime and a maximal ideal in $\mathbb{T}_{\mathfrak{p}\mathfrak{M}}$ such that char $(\mathbb{T}_{\mathfrak{p}\mathfrak{M}}/\mathfrak{m}) =: \ell \neq p$, ensuring that there exists an associated $\rho_{\mathfrak{m}}$. If*

1. *$\rho_{\mathfrak{m}}$ is irreducible*
2. *$\rho_{\mathfrak{m}}$ is unramified at $\mathfrak{p}$*
3. *$|\mathfrak{p}| \neq 1 \mod \ell$*

then $\rho_{\mathfrak{m}}$ is modular of level $\mathfrak{M}$.

Proof. Let V be the $\mathbb{T}/\mathfrak{m}$-vector space corresponding to $\rho_{\mathfrak{m}}$. Since we assume that $\rho_{\mathfrak{m}}$ is unramified at $\mathfrak{p}$, there exists a finite étale $\mathbb{T}/\mathfrak{m}$-vector space scheme $\mathcal{V}$ over $A_{\mathfrak{p}}$. By Prop 6.5.14, there exists a natural map $V \to J_0(\mathfrak{p}\mathfrak{M})[\mathfrak{m}]$. Then since $\mathcal{V}$ is a smooth group scheme, the universal property of Neron models gives a natural extension

$$\begin{array}{ccccc} V & \longrightarrow & J_0(\mathfrak{p}\mathfrak{M})[\mathfrak{m}] & \longrightarrow & J_0(\mathfrak{p}\mathfrak{M}) \\ \downarrow & & & & \downarrow \\ \mathcal{V} & & \cdots\cdots\cdots\

that it must be given by the product of a unramified quadratic character ε with the cyclotomic character, i.e $\varepsilon \cdot \mu_\ell$. Thus the action on the determinant $\det V$ is given by $\varepsilon^2 \mu_\ell^2 = \mu_\ell^2$. However the determinant $\det(V)$ is the cyclotomic character. This means that the cyclotomic character is trivial, is only possible if $|\mathfrak{p}| = 1$ modulo ℓ, which contradicts the extra condition of $\mathfrak{p}$ yielding a contradiction. □

6.6.2 Prime Swapping in the level

The goal of this section is to prove the prime level swapping.

Proposition 6.6.3 (Prime Swapping) *Assume that $\ell \nmid q-1$. Let ρ be of level $\mathfrak{p}\mathfrak{M}$, finite at $\mathfrak{p}$ and let $\mathfrak{q}$ be prime such that $|\mathfrak{q}| \neq 1 \mod \ell$ and ρ is unramified at $\mathfrak{q}$. Then ρ is modular of level $\mathfrak{q}\mathfrak{M}$.*

The following Proposition shows that such an auxiliary $\mathfrak{q}$ always exists. We follow the proof found in [Rib94, p.22] (only replacing the complex conjugation with Frob_∞ in the proof).

Proposition 6.6.4 *Assume that $\ell \nmid q-1$. Let ρ be of level $\mathfrak{p}\mathfrak{M}$, finite at $\mathfrak{p}$. There exists a prime $\mathfrak{q}$ such that $|\mathfrak{q}| \neq 1 \mod \ell$ and ρ is unramified at $\mathfrak{q}$.*

Proof. We have to show that there exists infinitely many primes such that

$$\mathrm{tr}\rho(\mathrm{Frob}_\mathfrak{q}) = |\mathfrak{q}| + 1.$$

First we note that we know that

$$\rho(\mathrm{Frob}_\infty)) = \begin{pmatrix} 1 & 0 \\ 0 & q^{-1} \end{pmatrix}.$$

As the trace is invariant under conjugation it will suffice to find primes such that $\mathrm{Frob}_\mathfrak{q}$ is conjugated to Frob_∞ and

$$|\mathfrak{q}| + 1 = 1 + q^{-1} \mod \ell$$

and thus $|\mathfrak{q}| = q^{-1} \mod \ell$. Then Chebotarev density proves the existence of the $\mathfrak{q}$.

□

Proposition 6.6.5 (Analogue of [Rib94, Prop 6.7]) *The Hecke modules $\Phi(J_0^{\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{q})$ and $\Phi(J_0^{\mathfrak{p}\mathfrak{q}}(\mathfrak{M})^\vee, \mathfrak{q})$ are $\mathbb{Q}/\mathbb{Z}$-dual.*

Proof. In [McC86], it was proven the pairing of component groups $\Phi(A) \times \Phi(A^\vee) \to \mathbb{Q}/\mathbb{Z}$ is perfect for any abelian variety A over an arbitrary DVR, from which the desired result follows. $\square$

Proposition 6.6.6 (Analogue of [Rib94, Cor 6.8]) *The support of* $\Phi(J_0^{\mathfrak{pq}}(\mathfrak{M}), \mathfrak{q})$ *and* $\Phi(J_0^{\mathfrak{pq}}(\mathfrak{M})^\vee, \mathfrak{q})$ *agree. Furthermore for any maximal ideal* $\mathfrak{m}$*, we have*

$$\dim_{\mathbb{T}/\mathfrak{m}} \Phi(J_0^{\mathfrak{pq}}(\mathfrak{M}), \mathfrak{q})/\mathfrak{m} = \dim_{\mathbb{T}/\mathfrak{m}} \Phi(J_0^{\mathfrak{pq}}(\mathfrak{M})^\vee, \mathfrak{q})[\mathfrak{m}]$$

$$\dim_{\mathbb{T}/\mathfrak{m}} \Phi(J_0^{\mathfrak{pq}}(\mathfrak{M})^\vee, \mathfrak{q})/\mathfrak{m} = \dim_{\mathbb{T}/\mathfrak{m}} \Phi(J_0^{\mathfrak{pq}}(\mathfrak{M}), \mathfrak{q})[\mathfrak{m}].$$

Proof. By dualizing the left exact sequence defining $\Phi(J_0^{\mathfrak{pq}}(\mathfrak{M})^\vee, \mathfrak{q})[\mathfrak{m}]$, i.e.

$$0 \to \Phi(J_0^{\mathfrak{pq}}(\mathfrak{M})^\vee, \mathfrak{q})[\mathfrak{m}] \to \Phi(J_0^{\mathfrak{pq}}(\mathfrak{M})^\vee, \mathfrak{q}) \to \prod \Phi(J_0^{\mathfrak{pq}}(\mathfrak{M})^\vee, \mathfrak{q})$$

and using the duality result of Prop 6.6.5 the stated result on dimensions follows, which also shows that the supports must agree. $\square$

Proposition 6.6.7 (Analogue of [Rib94, Thm 6.9]) *Let* $\mathfrak{m} \subset \mathbb{T}$ *be a maximal ideal such that* $\rho_{\mathfrak{m}}$ *is irreducible. Then* $\mathfrak{m}$ *being in the support of* $\Phi(J_0^{\mathfrak{pq}}(\mathfrak{M}), \mathfrak{q})$ *is equivalent to* $\mathfrak{m}$ *lying in the support of* $\mathcal{X}(J_0(\mathfrak{p}\mathfrak{M}) \times J_0(\mathfrak{p}\mathfrak{M}), \mathfrak{p})$ *and containing* $\eta = T_{\mathfrak{p}}^2 - 1$*. If it lies in the support of* $\Phi(J_0^{\mathfrak{pq}}(\mathfrak{M}), \mathfrak{q})$*, then we have*

$$\dim_{\mathbb{T}/\mathfrak{m}} \mathcal{X}(J_0(\mathfrak{p}\mathfrak{M}) \times J_0(\mathfrak{p}\mathfrak{M}), \mathfrak{p}) = \dim_{\mathbb{T}/\mathfrak{m}} \Phi(J_0^{\mathfrak{pq}}(\mathfrak{M})^\vee, \mathfrak{q})/\mathfrak{m}.$$

Proof. As we have assumed $\mathfrak{m}$ to be such that $\rho_{\mathfrak{m}}$ is irreducible, it follows from Prop 6.5.14 that $\mathfrak{m}$ and $(T_{\mathfrak{q}} - |\mathfrak{q}| - 1)$ are coprime. Since the modules $\ker(\Phi_{\text{deg}}), \text{coker}(\Phi_{\text{deg}})$ in Prop 6.3.19 are Eisenstein, it follows that locally at $\mathfrak{m}$ we have an isomorphism

$$(\mathcal{X}(J_0(\mathfrak{p}\mathfrak{M}) \times J_0(\mathfrak{p}\mathfrak{M}), \mathfrak{p})/\eta)_{\mathfrak{m}} \cong \Phi(J_0^{\mathfrak{pq}}(\mathfrak{M})^\vee, \mathfrak{q})_{\mathfrak{m}}.$$

From this the result follows. $\square$

Proposition 6.6.8 ([Rib94, Prop 7.1]) *If* $\mathfrak{m}$ *is not modular of level* $\mathfrak{q}\mathfrak{M}$*, then we have*

$$\mathcal{X}(J_0^{\mathfrak{pq}}(\mathfrak{M}), \mathfrak{p})/\mathfrak{m} \cong \mathcal{X}(J_0(\mathfrak{p}\mathfrak{q}\mathfrak{M}), \mathfrak{q})/\mathfrak{m}$$

and a right exact sequence

$$\mathcal{X}(J_0^{\mathfrak{pq}}(\mathfrak{M}),\mathfrak{q})/\mathfrak{m} \to \mathcal{X}(J_0(\mathfrak{pqM}),\mathfrak{p})/\mathfrak{m} \to \mathcal{X}(J_0(\mathfrak{pM}) \times J_0(\mathfrak{pM}),\mathfrak{p}).$$

In particular we have

$$\dim_{\mathbb{T}/\mathfrak{m}} \mathcal{X}(J_0^{\mathfrak{pq}}(\mathfrak{M}),\mathfrak{p})/\mathfrak{m} = \dim_{\mathbb{T}/\mathfrak{m}} \mathcal{X}(J_0(\mathfrak{pqM}),\mathfrak{q})/\mathfrak{m}$$

and

$$\dim_{\mathbb{T}/\mathfrak{m}} \mathcal{X}(J_0(\mathfrak{pqM}),\mathfrak{p})/\mathfrak{m} \leqslant \dim_{\mathbb{T}/\mathfrak{m}} \mathcal{X}(J_0^{\mathfrak{pq}}(\mathfrak{M}),\mathfrak{q})/\mathfrak{m} + \dim_{\mathbb{T}/\mathfrak{m}} \mathcal{X}(\Phi(J_0(\mathfrak{pM}) \times J_0(\mathfrak{pM}),\mathfrak{p}).$$

Proof. Since we assume $\mathfrak{m}$ not to be modular of level $\mathfrak{qM}$, we know that it generates the unit ideal in the $\mathfrak{p}$-old quotient of $\mathbb{T}$. However by Prop 6.5.10, $\mathbb{T}$ act on $\mathcal{X}(J_0(\mathfrak{qM}) \times J_0(\mathfrak{qM}),\mathfrak{q})$ via its $\mathfrak{p}$-old quotient. In particular $\mathcal{X}(J_0(\mathfrak{qm}) \times J_0(\mathfrak{qM}),\mathfrak{q})_{\mathfrak{m}} = 0$. Thus by localizing the analogue of Ribet's exact sequence (see Prop 6.3.1) and then taking reduction modulo $\mathfrak{m}$ we get the stated isomorphism and right exact sequence respectively, from which the dimension counts are a direct consequence. □

Definition 6.6.9 *Denote by μ, λ the integers such that*

$$J_0(\mathfrak{pqM})[\mathfrak{m}] \cong V^{\lambda} \quad \text{and} \quad J_0^{\mathfrak{pq}}(\mathfrak{M})[\mathfrak{m}] \cong V^{\mu}.$$

Proposition 6.6.10 (Analogue of [Rib94, Prop 7.6]) *If $\mathfrak{m}$ is not modular of level $\mathfrak{qM}$, then*

$$\dim_{\mathbb{T}/\mathfrak{m}} \mathcal{X}(J_0^{\mathfrak{pq}}(\mathfrak{M}),\mathfrak{p})/\mathfrak{m} = 2\mu$$

$$\dim_{\mathbb{T}/\mathfrak{m}} \mathcal{X}(J_0(\mathfrak{pqM}),\mathfrak{p})/\mathfrak{m} = 2\lambda.$$

Proof. By Neron-Ogg-Shaferevich (for abelian varieties), we know that $J_0^{\mathfrak{pq}}(\mathfrak{M})[\ell](\mathbb{F}_{\mathfrak{p}}) \cong J_0^{\mathfrak{pq}}(\mathfrak{M})[\ell](Q_{\mathfrak{p}})^{I_{\mathfrak{p}}}$ and $J_0(\mathfrak{pqM})_{/\mathbb{F}_{\mathfrak{p}}}[\ell](\mathbb{F}_{\mathfrak{p}}) = J_0(\mathfrak{pqM})[\ell](Q_{\mathfrak{p}})^{I_{\mathfrak{p}}}$. Thus

$$J_0^{\mathfrak{pq}}(\mathfrak{M})_{/\mathbb{F}_{\mathfrak{p}}}[\mathfrak{m}] \cong J_0^{\mathfrak{pq}}(\mathfrak{M})[\mathfrak{m}]^{I_{\mathfrak{p}}} = J_0^{\mathfrak{pq}}(\mathfrak{M})[\mathfrak{m}]$$

$$J_0(\mathfrak{pqM})_{/\mathbb{F}_{\mathfrak{p}}}[\mathfrak{m}] = J_0(\mathfrak{pqM})[\mathfrak{m}]^{I_{\mathfrak{p}}} = J_0(\mathfrak{pqM})[\mathfrak{m}].$$

where in the last equality we used the fact that $\mathfrak{p}$ is unramified. Thus

$$\dim_{\mathbb{T}/\mathfrak{m}} J_0^{\mathfrak{pq}}(\mathfrak{M})_{/\mathbb{F}_\mathfrak{p}}[\mathfrak{m}] = \dim_{\mathbb{T}/\mathfrak{m}} J_0^{\mathfrak{pq}}(\mathfrak{M})[\mathfrak{m}]$$

and

$$\dim_{\mathbb{T}/\mathfrak{m}} J_0(\mathfrak{pq}\mathfrak{M})_{/\mathbb{F}_\mathfrak{p}}[\mathfrak{m}] = \dim_{\mathbb{T}/\mathfrak{m}} J_0(\mathfrak{pq}\mathfrak{M})[\mathfrak{m}].$$

The special fiber $J_0^{\mathfrak{pq}}(\mathfrak{M})_{/\mathbb{F}_\mathfrak{p}}$ is the extension of its toric part by its group of components. It will thus suffice for us to show that the toric part of $J^{\mathfrak{pq}}(\mathfrak{M})_{/\mathbb{F}_\mathfrak{p}}[\mathfrak{m}]$ has the same dimension as $J_0^{\mathfrak{pq}}(\mathfrak{M})_{/\mathbb{F}_\mathfrak{p}}$ itself, or equivalently that the group of components has no $\mathfrak{m}$-torsion. By Prop 6.6.7, this is equivalent to the fact that $\mathcal{X}(J_0(\mathfrak{q}\mathfrak{M})^2, \mathfrak{q})/\mathfrak{m}$ is trivial, which follows from the assumption that $\mathfrak{m}$ is not modular of level $\mathfrak{q}\mathfrak{M}$.
Since the group of components of $J_0(\mathfrak{pq}\mathfrak{M})_{/\mathbb{F}_\mathfrak{p}}$ is Eisenstein by Prop 6.4.3, we know in fact that $J_0(\mathfrak{pq}\mathfrak{M})_{/\mathbb{F}_\mathfrak{p}}[\mathfrak{m}]$ is "concentrated" in one connected component, which we denote by $J_0(\mathfrak{pq}\mathfrak{M})^\circ_{/\mathbb{F}_\mathfrak{p}}$. Then consider the short exact sequence

$$0 \to T \to J_0(\mathfrak{pq}\mathfrak{M})^\circ_{/\mathbb{F}_\mathfrak{p}} \to J_0(\mathfrak{q}\mathfrak{M})_{/\mathbb{F}_\mathfrak{p}} \times J_0(\mathfrak{q}\mathfrak{M})_{/\mathbb{F}_\mathfrak{p}} \to 0.$$

Since we are assuming that $\mathfrak{m}$ is not modular of level $\mathfrak{q}\mathfrak{M}$, we know that the first two terms have isomorphic $\mathfrak{m}$-torsion and we are done. □

Proposition 6.6.11 (Analogue of [Rib94, Prop 7.7]) *Assume that $\ell \nmid q-1$. We have*

$$\dim_{\mathbb{T}/\mathfrak{m}} \mathcal{X}(J_0^{\mathfrak{pq}}(\mathfrak{M}), \mathfrak{q})/\mathfrak{m} \leqslant \mu$$

$$\dim_{\mathbb{T}/\mathfrak{m}} \mathcal{X}(J_0(\mathfrak{pq}\mathfrak{M}), \mathfrak{q})/\mathfrak{m} \leqslant \lambda.$$

Proof. We have an injection

$$\mathrm{Hom}(\mathcal{X}(J_0^{\mathfrak{pq}}(\mathfrak{M}), \mathfrak{q})/\mathfrak{m}, \mu_\ell) \to J_0^{\mathfrak{pq}}(\mathfrak{M})[\mathfrak{m}].$$

By Prop 6.6.4, or rather the choice made therein, we know that $\rho_\mathfrak{m}$ is unramified at $\mathfrak{q}$ and that the eigenvalues of $\rho_\mathfrak{m}(\mathrm{Frob}_\mathfrak{q})$ are $1, q^{-1} \mod \ell$ which are distinct. Each of these therefore occurs in $J_0^{\mathfrak{pq}}(\mathfrak{M})[\mathfrak{m}]$ with the multiplicity of $\rho_\mathfrak{m}$. The action of $\mathrm{Frob}_\mathfrak{q}$ on $\mathrm{Hom}(\mathcal{X}(J_0^{\mathfrak{pq}}(\mathfrak{M}), \mathfrak{q})/\mathfrak{m}, \mu_\ell)$ is given by $|\mathfrak{q}|T_\mathfrak{q}$. Since $T_\mathfrak{q}$ acts via an involution and $|\mathfrak{q}| = q^{-1} \mod \ell$, the action of $T_\mathfrak{p}$ is given by $+q^{-1}, -q^{-1}$. Since these appear with multiplicity at most the multiplicity of $\mathfrak{m}$, we get the desired bound. The same argument applies to $J_0(\mathfrak{pq}\mathfrak{M})$. □

Proposition 6.6.12 (Analogue of [Rib94, Prop 7.8]) *Assume that $\ell \nmid q-1$. We have*

$$\dim_{\mathbb{T}/\mathfrak{m}} \mathcal{X}(J_0(\mathfrak{p}\mathfrak{M})^2, \mathfrak{p})/\mathfrak{m} \leqslant \mu.$$

Proof. We have the natural short exact sequence (proven by [GR06] (11.6.6) and (11.6.7) or [Pap06, Proof of Lemma 2.3])

$$0 \to \mathrm{Hom}(\mathcal{X}(J_0^{\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{q})/\ell, \mu_\ell) \to J_0^{\mathfrak{p}\mathfrak{q}}(\mathfrak{M})[\ell] \to \mathcal{X}(J_0^{\mathfrak{p}\mathfrak{q}}(\mathfrak{M})^\vee, \mathfrak{q})/\ell \to 0.$$

We now claim that this sequence localized at $\mathfrak{m}$ is split: indeed it acts on $\mathcal{X}(J_0^{\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{q})/\ell$ via the involution $T_\mathfrak{q}$ and therefore as $|\mathfrak{q}|T_\mathfrak{q}$ whereas it acts on $\mathcal{X}(J_0^{\mathfrak{p}\mathfrak{q}}(\mathfrak{M})^\vee, \mathfrak{q})/\ell$ via $T_\mathfrak{q}$. As $|\mathfrak{q}| = p^{-1} \mod \ell$ these elements are distinct which forces the desired splitting. Thus we may take reduction modulo $\mathfrak{m}$ and the short exact sequence remains exact, which yields

$$0 \to \mathrm{Hom}(\mathcal{X}(J_0^{\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{q})/\ell, \mu_\ell)[\mathfrak{m}] \to J_0^{\mathfrak{p}\mathfrak{q}}(\mathfrak{M})[\mathfrak{m}] \to \mathcal{X}(J_0^{\mathfrak{p}\mathfrak{q}}(\mathfrak{M})^\vee, \mathfrak{q})/\ell \to 0.$$

The characteristic polynomial of $\rho_\mathfrak{m}(\mathrm{Frob}_\mathfrak{q})$ is $(T-1)^\mu(T-q^{-1})^\mu$. If we assume that

$$\mathrm{Hom}(\mathcal{X}(J_0^{\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{q})/\ell, \mu_\ell)[\mathfrak{m}]$$

is of dimension μ_1 and $\mathrm{Frob}_\mathfrak{q}$ acts via the scalar s_1, and that $\mathcal{X}(J_0^{\mathfrak{p}\mathfrak{q}}(\mathfrak{M})^\vee, \mathfrak{q})/\ell$ is of dimension μ_2 and that $\mathrm{Frob}_\mathfrak{q}$ acts via the scalar s_2, then we must have

$$(T-1)^\mu(T-q^{-1})^\mu = (T-s_1)^{\mu_1}(T-s_2)^{\mu_2}.$$

As $s_1 \neq s_2$ and $1, q^{-1}$ are distinct, we have $\mu_1 = \mu_2 = \mu$. From the natural sequence

$$0 \to \mathcal{X}(J_0^{\mathfrak{p}\mathfrak{q}}(\mathfrak{M})^\vee, \mathfrak{q}) \to \mathrm{Hom}(\mathcal{X}(J_0^{\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{q})/\ell, \mu_\ell) \to \Phi(J_0^{\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{q}) \to 0$$

we get that

$$\Phi(J_0^{\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{q})[\ell] \subset \mathcal{X}(J_0^{\mathfrak{p}\mathfrak{q}}(\mathfrak{M})^\vee, \mathfrak{q})/\ell$$

and in particular $\Phi(J_0^{\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{q})[\mathfrak{m}] \subset \mathcal{X}(J_0^{\mathfrak{p}\mathfrak{q}}(\mathfrak{M})^\vee, \mathfrak{q})/\mathfrak{m}$. However we just saw that the latter of these spaces is of $\mathbb{T}/\mathfrak{m}$-dimension μ, whereas the former has the same $\mathbb{T}/\mathfrak{m}$-dimension as $\mathcal{X}(J_0(\mathfrak{p}\mathfrak{M})^2, \mathfrak{p})$ by Prop 6.6.7. □

Proof. [Proof of Prop 6.6.3] Let $\mathfrak{m}$ be as in Prop 6.6.3, that is of level $\mathfrak{p}\mathfrak{M}$, finite at $\mathfrak{p}$ and let $\mathfrak{q}$ be a good prime and assume that $\rho_\mathfrak{m}$ is not modular of level $\mathfrak{q}\mathfrak{M}$. Then by Prop 6.6.10, Prop 6.6.11

and Prop 6.6.12 we know that

$$2\mu \leqslant \lambda \text{ and } 2\mu = 2\lambda.$$

In particular we have that $\mu = \lambda = 0$, which contradicts Prop 6.5.14. Thus $\rho_{\mathfrak{m}}$ must be of level $\mathfrak{q}\mathfrak{M}$. □

6.6.3 Proof of Level Lowering

Proof. [Proof of Prop 6.6.1] By Prop 6.6.4, we may choose a good prime $\mathfrak{q}$. Then we may apply the prime swapping Prop 6.6.3, and thus may assume that ρ is modular of level $\mathfrak{q}\mathfrak{m}$ and then finally we may apply Mazur's level lowering Prop 6.6.2. □

Remark 6.6.13 To prove "prime swapping" (that is in Prop 6.6.12 and Prop 6.6.11) we need to assume that that $\ell \nmid q - 1$ to ensure that the $\rho_{\mathfrak{m}}(\mathrm{Frob}_{\mathfrak{q}})$ has two distinct eigenvalues $1, q$. Equivalently we need to assume that $\mathbb{F}_q(t)$ does not have any non-trivial ℓ-th roots of unity. In the function field/number field analogy, this would thus relate to $\mathbb{Q}$ not having any ℓ-th roots of unity, so in particular ℓ can not be 2. In particular our restriction that $\ell \nmid q - 1$ is consistent with the restriction in [Rib90a] that $\ell \geqslant 3$. A level lowering result for mod 2-representations was proven by K.Buzzard in [Buz00] under the additional assumption that the modular representation satisfy a multiplicity one assumption, that is that

$$\dim_{\mathbb{T}/\mathfrak{m}} J_0(pN)[\mathfrak{m}] = 1.$$

Buzzard adapts Ribet's work in [Rib90a] for mod 2-representations, he does however mention that [Rib94, Prop 7.8] adapts to the $\ell = 2$ case. In this case the multiplicity one with the inequalities stated in the proof of Prop 6.6.3 would suffice to prove Prime Swapping for $\ell = 2$ and thus Level lowering. However it is not clear to the author if this comment is accurate. It does however motivate the following question.

Question 6.6.14 *Fix $\mathfrak{p}, \mathfrak{M}$ coprime and a maximal ideal in $\mathbb{T}_{\mathfrak{p}\mathfrak{M}}$ such that char $(\mathbb{T}/\mathfrak{m}) =: \ell \neq p$, ensuring that there exists an associated $\rho_{\mathfrak{m}}$. Also suppose that that $\ell \mid q - 1$. If*

1. *$\rho_{\mathfrak{m}}$ is irreducible*
2. *$\rho_{\mathfrak{m}}$ is unramified at $\mathfrak{p}$*
3. *$\rho_{\mathfrak{m}}$ appears with multiplicity one in $J_0(\mathfrak{p}\mathfrak{M})[\mathfrak{m}]$*

is $\rho_{\mathfrak{m}}$ modular of level $\mathfrak{M}$?

6.6.4 Level Lowering and Component groups

In [MO89], Mestere and Osterlé showed that $\Phi(J_0(p), p) \to \Phi(E, p)$ is surjective. The proof is essentially an application of the fact that no modular form of level 1 exists. This motivated S.Takahashi [Tak01] to note that

$$\ell \mid \text{coker}\,(\Phi(J_0(pn), p) \to \Phi(E, p))$$

implies that there exists a modular form of level n such that $f = g \mod \ell$ if $E[\ell]$ is irreducible. This is an immediate consequence of Ribet's level lowering theorem. In the number field case, he mentions that he and Ribet are thinking about some consequences of Mazurs principle. The analogous result of Mestere and Oesterlé was shown by Papikian in [Pap05], whereas the extension pointed out by Takahashi in the irreducible case is a consequence of Thm 6.6.1, that is if $E[\ell]$ is irreducible, then if

$$\ell \mid \text{coker}\,(\Phi(J_0(\mathfrak{pn}), \mathfrak{p}) \to \Phi(E, \mathfrak{p}))$$

there exists a modular form g of level $\mathfrak{n}$ which is congruent to f modulo ℓ.

6.6.5 Level Lowering and Fermat's Last Theorem

In characteristic p, the naive analogue of Fermat's last theorem is not true, as for any $f, g \in \mathbb{F}_q[t]$, we have

$$f^p + g^p = (f + g)^p.$$

However one can prove that besides p-powers, it is in fact true:

Theorem 6.6.15 ([Ros13, Prop 7.18]) *Let $\ell \geqslant 3$ be coprime to p. Then there exists no non-constant $a, b, c \in \mathbb{F}_q[t]$ such that*

$$a^\ell + b^\ell = c^\ell.$$

Rosen proved this using the function-field analogue of the ABC-conjecture (which is known to be true). Of course one of the motivations for the work of Ribet [Rib90a] was to prove that Fermat's Last Theorem would follow from the modularity theorem. As this thesis provides an analogue of the Level Lowering results of Ribet [Rib90a] and Modularity is known, the question arises if we can say something about the function field analogue of Fermat's Last theorem. To relate level-lowering questions of representations arising from elliptic curves to the geometry, we will use the following result.

Proposition 6.6.16 (Analogue of [DDT97, Prop 2.12]) *Let K be a global function field and $\mathfrak{p}$ a prime thereof. Let E be an non-isotrivial elliptic curve over K with multiplicative reduction at $\mathfrak{p}$, $\ell \neq \mathrm{char}(K)$ and $\ell \nmid |\kappa(\mathfrak{p})| - 1$. Denote by $\varpi \in K_{\mathfrak{p}}^{\times}$ the Tate parameter of E at $\mathfrak{p}$. If $\ell \mid v_{\mathfrak{q}}(\varpi)$, then the Galois module $E[\ell]$ is unramified at $\mathfrak{p}$.*

I could not find a convenient reference to this fact, hence we include a brief proof of this.

Proof. By considering the ℓ-torsion of the Tate uniformization, we have a short exact sequence of (local) Galois modules

$$0 \to \mu_\ell \to E[\ell] \to \underline{\mathbb{Z}/\ell} \to 0.$$

Since $\ell \neq \mathrm{char}(K_{\mathfrak{p}})$, both μ_ℓ and $\underline{\mathbb{Z}/\ell}$ are unramified. Furthermore such sequences are classified by

$$\mathrm{Ext}^1_{G_{K_{\mathfrak{p}}}}(\mathbb{Z}/\ell, \mu_\ell) \cong H^1(G_{K_{\mathfrak{p}}}, \mu_\ell) \cong K_{\mathfrak{p}}^{\times}/(K_{\mathfrak{p}}^{\times})^{\ell}$$

(for the last isomorphism see for instance [NSW08, Thm 7.1.8]). The extension given above corresponds to the image of the Tate parameter in $K_{\mathfrak{p}}^{\times}/(K_{\mathfrak{p}}^{\times})^{\ell}$. Since $\ell \mid v_{\mathfrak{p}}(\varpi)$ and since $\ell \nmid \mathrm{char}(\kappa(\mathfrak{p})) - 1$, the Tate parameter is an ℓ-th power and thus $E[\ell]$ is unramified. □

Remark 6.6.17 To prove that there exists no non-trivial function field solution to Fermat's Last theorem, we would want to study the Frey-Hellegouarch elliptic curve $E/\mathbb{F}_q(t)$ given by

$$y^2 = x(x - a^\ell)(x + b^\ell).$$

We then want to study the Galois representation associated to $E[\ell]$ and apply our Level Lowering result Thm 6.6.1. This requires that E be modular and that $\ell \nmid q - 1$. However I'm unable to prove that $E/\mathbb{F}_q(t)$ is split multiplicative at ∞. Thus the following result is conditional on this assumption.

Proposition 6.6.18 *Let $\ell \geqslant 3$ be coprime to p and assume that $\ell \nmid q - 1$. Assume that for all non-constant $a, b, c \in \mathbb{F}_q[t]$ such that $a^\ell + b^\ell = c^\ell$ the corresponding Frey-Hellegouarch elliptic curve $y^2 = x(x - a^\ell)(x + b^\ell)$ has split multiplicative reduction at ∞ and multiplicative reduction exactly at the primes dividing abc. Then no such $a, b, c \in \mathbb{F}_q[t]$ can exist.*

Proof. By our assumption, the Frey-Hellegouarch curve has bad reduction at exactly those primes which divide abc. Furthermore, since the Frey-Hellegouarch curve has split multiplicative reduction at ∞, then the Galois representation $E[\ell]$ is modular. Furthermore this representation is unramified at the primes dividing abc by Prop 6.6.16. Thus we may, since $\ell \neq p$, apply the level

lowering theorem Thm 6.6.1. In particular, we see that this representation is modular of trivial level. However as we know that there exist no non-zero cuspidal harmonic cochains of trivial level (see Ex 3.2.16), we have a contradiction. □

Remark 6.6.19 While it is unsatisfactory that this method does not prove the cases where $\ell \mid q-1$ (or at least until the Question Ques 6.6.14 is answered satisfactorily), it should be mentioned that the primes dividing $p-1$ should be seen as analogous to regular primes. Indeed the identity used by Kummer in this proof of Fermat's Last theorem in this case

$$a^\ell + b^\ell = \prod_{i=0}^{\ell-1}(a+\zeta^i b)$$

where ζ is a primitive ℓ-root of unity. Furthermore the ideals $(a+\zeta^i b)$ are mutually coprime if a, b, c are coprime.

6.7 Analogue of Ribet-Takahashi

Throughout this section, let us fix some $\mathfrak{D}, \mathfrak{M}, \mathfrak{p}$ and $\mathfrak{q}$. We assume that $\mathfrak{p}, \mathfrak{q}$ are coprime to $\mathfrak{D}\mathfrak{M}$. We also choose isogenous elliptic curves E, E' over $\mathbb{F}_q(t)$ such that we have the optimal quotients $\xi : J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}) \to E$ and $\xi' : J_0^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M}) \to E'$.

Proposition 6.7.1 ([Pap07b, Lemma 2.8]) *We have* $\xi \circ \xi^\vee = [\deg \xi]$ *and* $\xi' \circ \xi'^\vee = [\deg \xi']$ *respectively.*

Proof. The morphism ξ is a finite morphism of smooth projective, geometrically irreducible curves of degree $\deg \xi$. This induced two morphism on the Picard varieties:

$$\xi^* : \mathrm{Pic}_A \to \mathrm{Pic}_J, \quad \mathcal{L} \mapsto \xi^*\mathcal{L}$$

$$\xi_* : \mathrm{Pic}_J \to \mathrm{Pic}_A, \quad \mathcal{L} \mapsto \det(\xi_*\mathcal{L}) \otimes \det(\xi_*\mathcal{O}_J)^{-1}.$$

By the projection formula we have

$$\begin{aligned}
\det(\xi_*\xi^*\mathcal{L})) \otimes \det(\xi_*\mathcal{O}_J)^{-1} &\cong \det(\mathcal{L} \otimes \xi_*\mathcal{O}_J) \otimes \det(\xi_*\mathcal{O}_J)^{-1} \\
&\cong \det(\mathcal{L}^{\deg \xi}) \otimes \det(\xi_*\mathcal{O}_J)^{-1} \\
&\cong \mathcal{L}^{\deg \xi} \otimes \det \xi_*\mathcal{O}_J \otimes \det(\xi_*\mathcal{O}_J)^{-1} \\
&\cong \mathcal{L}^{\deg \xi}.
\end{aligned}$$

□

Definition 6.7.2 *Let f be the cuspidal harmonic cochain associated to E. We denote the f-*part of the character group *by*

$$\mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}),\mathfrak{q})_f := \{x \in \mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}),\mathfrak{q}) \mid T_{\mathfrak{m}}x = a_{\mathfrak{m}}(f)x \textit{ for all } \mathfrak{m} \nmid \mathfrak{M}\mathfrak{p}\mathfrak{q}\mathfrak{D}\}.$$

Proposition 6.7.3 *We have $\mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}),\mathfrak{q})_f \cong \mathbb{Z}$. Furthermore it is a subgroup of the image of $\mathcal{X}(J_0^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M}),\mathfrak{p})$.*

Proof. The proof is a direct translation of the analogous result of [RT97]. We know that $J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q},\mathfrak{q})$ decomposes up to isogeny into the product of simple abelian varieties. By Langlands correspondence, exactly one of these is isogenous to E, hence

$$\mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}),\mathfrak{q})_f \otimes \mathbb{Q} \cong \mathcal{X}(E,\mathfrak{q}) \otimes \mathbb{Q} \cong \mathbb{Q}$$

since E has multiplicative reduction at $\mathfrak{q}$, which shows that $\mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}),\mathfrak{q})_f$ is of rank one. Using Ribet's short exact sequence, we can see $\mathcal{X}(J_0^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M}),\mathfrak{p})$ as a subspace of $\mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}),\mathfrak{q})$. By the same argument as above, we know that

$$\mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}),\mathfrak{q})_f \cap \mathcal{X}(J_0^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M}),\mathfrak{p})$$

must be of rank one. Therefore the quotient

$$\mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}),\mathfrak{q})_f / \mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}),\mathfrak{q})_f \cap \mathcal{X}(J_0^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M}),\mathfrak{p})$$

must be finite. In particular the image of $\mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}),\mathfrak{q})_f$ in $\mathcal{X}(J^{\mathfrak{D}}(\mathfrak{q}\mathfrak{M}),\mathfrak{q}) \times \mathcal{X}(J^{\mathfrak{D}}(\mathfrak{q}\mathfrak{M}),\mathfrak{q})$ must be finite. However $\mathcal{X}(J^{\mathfrak{D}}(\mathfrak{q}\mathfrak{M}),\mathfrak{q})$ is torsion-free, this image must be trivial. Using Ribet's short exact sequence, we can deduce that $\mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}),\mathfrak{q})_f$ is a subgroup of $\mathcal{X}(J_0^{\mathcal{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M}),\mathfrak{p})$. □

The goal of this section is to prove the following result of Ribet-Takahashi:

Theorem 6.7.4 (Analogue of [RT97, Thm 2]) *We have*

$$\delta^{\mathfrak{p}\mathfrak{q}\mathfrak{D}}(\mathfrak{M}) = \frac{\delta^{\mathfrak{D}}(\mathfrak{p}\mathfrak{q}\mathfrak{M})}{c'_{\mathfrak{p}}c_{\mathfrak{q}}} \left(|\mathrm{im}\xi_*||\mathrm{coker}\xi'_*|\right)^2$$

where

$$\xi_* : \Phi(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}),\mathfrak{q}) \to \Phi(E,\mathfrak{q})$$

$$\xi'_* : \Phi(J_0^{\mathcal{D}^{\mathfrak{pq}}}(\mathfrak{M}), \mathfrak{p}) \to \Phi(E', \mathfrak{p}).$$

Assume that $\mathfrak{M}$ is square-free but not prime. Consider a prime ℓ such that for all $\mathfrak{r} \mid \mathfrak{pq}\mathfrak{D}\mathfrak{M}$ we have $\ell \nmid |\mathfrak{r}| - 1$. Then $\ell \mid (|\mathrm{im}\xi_*||\mathrm{coker}\xi'_*|)$ implies that the Galois representation $E[\ell]$ is reducible. Thus up to Eisenstein primes of E, powers of p and the primes dividing $|\mathfrak{r}| - 1$ (for all $\mathfrak{r} \mid \mathfrak{pq}\mathfrak{D}\mathfrak{M}$), we have

$$\delta^{\mathfrak{pq}\mathcal{D}}(\mathfrak{M}) \sim \frac{\delta^{\mathcal{D}}(\mathfrak{pq}\mathfrak{M})}{c'_{\mathfrak{p}} c_{\mathfrak{q}}}.$$

To understand the behaviour of $(|\mathrm{im}\xi_*||\mathrm{coker}\xi'_*|)$ in the Proof Thm 6.7.4, we shall need the following result.

Proposition 6.7.5 *Let E be a non-isotrivial elliptic curve over $\mathbb{F}_q(t)$ of conductor $\mathfrak{N}$. Let ℓ be such that $E[\ell]$ is irreducible and that for all $\mathfrak{r} \mid \mathfrak{N}$ we have $\ell \nmid |\mathfrak{r}| - 1$. There exists a prime $\mathfrak{r} \mid \mathfrak{N}$ such that $\ell \not| c_{\mathfrak{r}}$.*

Proof. Assume to the contrary that ℓ were to divide $c_{\mathfrak{r}}$ for all $\mathfrak{r}$. Since $c_{\mathfrak{r}}$ is equal to the valuation of the Tate parameter at $\mathfrak{r}$, by Prop 6.6.16, the Galois representation $E[\ell]$ is unramified at all primes $\mathfrak{r}$ dividing $\mathfrak{N}$. Then the modular form (i.e. the harmonic cochain) would satisfy the assumption of Thm 6.6.1 at all primes dividing the level $\mathfrak{N}$. In particular it would be a harmonic cochain of level one, which does not exist (see Prop 3.2.16). □

Proof. [Proof of Thm 6.7.4] Let $x \in \mathcal{X}(E, \mathfrak{q})$ be a generator. Then by Prop 6.7.1 the composition $\xi_* \circ \xi^*$ is multiplication by $\delta^{\mathcal{D}}(\mathfrak{pq}\mathfrak{M})$. Then we have

$$\begin{aligned}
\delta^{\mathcal{D}}(\mathfrak{pq}\mathfrak{M}) c_{\mathfrak{q}} &= \delta^{\mathcal{D}}(\mathfrak{pq}\mathfrak{M}) u_A(x, x) \\
&= u_A(x, \xi_*\xi^* x) \\
&= u_{J_0^{\mathcal{D}}(\mathfrak{M}\mathfrak{pq})}(\xi^* x, \xi^* x).
\end{aligned}$$

Let $y \in \mathcal{X}(J_0^{\mathcal{D}}(\mathfrak{M}\mathfrak{pq}), \mathfrak{q})_f$ be a generator. Since $\xi^* x \in \mathcal{X}(J_0^{\mathcal{D}}(\mathfrak{M}\mathfrak{pq}, \mathfrak{q})_f$, we have

$$\xi^* x = (\mathcal{X}(J_0^{\mathcal{D}}(\mathfrak{M}\mathfrak{pq}), \mathfrak{q})_f : \mathcal{X}(A, \mathfrak{q}))y.$$

In particular we have

$$\delta^{\mathcal{D}}(\mathfrak{pq}\mathfrak{M}) c_{\mathfrak{q}} = (\mathcal{X}(J_0^{\mathcal{D}}(\mathfrak{M}\mathfrak{pq}), \mathfrak{q})_f : \mathcal{X}(E, \mathfrak{q}))^2 u_{J_0^{\mathcal{D}}(\mathfrak{M}\mathfrak{pq})}(y, y).$$

We will now derive an analogous formula for $\delta^{\mathfrak{pq}\mathcal{D}}(\mathfrak{M})$: Again by Prop 6.7.1 the composition

$\xi'_* \circ \xi'^*$ is multiplication by $\delta^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M})$. Let x' be a generator of $\mathcal{X}(E', \mathfrak{p})$. Then we have

$$\delta^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M})c'_{\mathfrak{p}} = u_{J^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M})}(\xi'^* x', \xi'^* x').$$

Then recall that by Prop 6.7.3, we have

$$\begin{array}{ccc} \xi'^* x' \in \mathcal{X}(J_0^{\mathcal{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{p}) & \xrightarrow{\ \iota\ } & \mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}), \mathfrak{q}) \\ & \nwarrow & \uparrow \\ & & \mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}), \mathfrak{q})_f \ni y \end{array}$$

In particular the generator y is in the image of $\mathcal{X}(J_0^{\mathcal{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{p})$, and we thus choose a $y' \in \mathcal{X}(J_0^{\mathcal{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{p})$ such that $\iota y' = y$. Then we have that

$$\xi'^* x' = \left(\mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}), \mathfrak{q})_f : \mathcal{X}(E', \mathfrak{p})\right) y'$$

and thus

$$\begin{aligned} \delta^{\mathfrak{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M})c'_{\mathfrak{p}} &= \left(\mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}), \mathfrak{q})_f : \mathcal{X}(E', \mathfrak{p})\right)^2 u_{J_0^{\mathcal{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M})}(y', y') \\ &= \left(\mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}), \mathfrak{q})_f : \mathcal{X}(E', \mathfrak{p})\right)^2 u_{J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q})}(y, y). \end{aligned}$$

Given these two expression for the modular degrees and the theorem we want to prove, we now want to express the indices in terms of ξ_* and ξ'_*:

Claim:

$$((\mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}), \mathfrak{q})_f : \mathcal{X}(E, \mathfrak{q})) = |\mathrm{coker}(\xi_* : \Phi(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}), \mathfrak{q}) \to \Phi(E, \mathfrak{q}))|$$

and

$$(\mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q})_f : \mathcal{X}(A', \mathfrak{p})) = |\mathrm{coker}(\xi'_* : \Phi(J_0^{\mathcal{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M}), \mathfrak{p}) \to \Phi(E', \mathfrak{p}))|.$$

To study $\mathrm{coker}(\xi_* : \Phi(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}), \mathfrak{q}) \to \Phi(E, \mathfrak{q}))$ consider first the morphism of short exact sequences (given by the monodromy pairing on the character groups)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}), \mathfrak{q}) & \longrightarrow & \mathrm{Hom}(\mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}), \mathfrak{q}), \mathbb{Z}) & \longrightarrow & \Phi(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}), \mathfrak{q}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{X}(E, \mathfrak{q}) & \longrightarrow & \mathrm{Hom}(\mathcal{X}(E, \mathfrak{q}), \mathbb{Z}) & \longrightarrow & \Phi(E, \mathfrak{q}) & \longrightarrow & 0 \end{array}$$

Since $J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}) \to E$ is optimal the dual morphism is injective and thus the induced morphism on character groups $\mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}), \mathfrak{q}) \to \mathcal{X}(E, \mathfrak{q})$ is surjective (see for instance [Pap07b, Cor 2.4]). In particular we get that

$$\text{coker}(\xi_* : \Phi(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}), \mathfrak{q}) \to \Phi(E, \mathfrak{q})) \cong \text{Ext}^1(\mathcal{X}(J, \mathfrak{q})/\mathcal{X}(E, \mathfrak{q}), \mathbb{Z}).$$

Let us **denote** $J := J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q})$. Since we are only interested in the order of

$$\text{coker}(\xi_* : \Phi(J_0^{\mathfrak{D}}(\mathfrak{M}\mathfrak{p}\mathfrak{q}), \mathfrak{q}) \to \Phi(E, \mathfrak{q})),$$

and

$$|\text{Ext}^1(\mathcal{X}(J, \mathfrak{q})/\mathcal{X}(E, \mathfrak{q}), \mathbb{Z})| = |\left(\mathcal{X}(J, \mathfrak{q})/\mathcal{X}(E, \mathfrak{q})\right)_{\text{tor}}|,$$

we are reduced to study the torsion subgroup $\left(\mathcal{X}(J, \mathfrak{q})/\mathcal{X}(E, \mathfrak{q})\right)_{\text{tor}}$. We claim that

$$\left(\mathcal{X}(J, \mathfrak{q})/\mathcal{X}(E, \mathfrak{q})\right)_{tor} \cong \left(\mathcal{X}(J, \mathfrak{q})_f/\mathcal{X}(E, \mathfrak{q})\right)_{tor}.$$

To prove this consider the following diagram (where for ease of notation we denote $\mathcal{X}(-) := \mathcal{X}(-, \mathfrak{q})$):

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathcal{X}(E) & \longrightarrow & \mathcal{X}(E) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{X}(J)_f & \longrightarrow & \mathcal{X}(J) & \longrightarrow & \mathcal{X}(J)/\mathcal{X}(J)_f & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{X}(J)_f/\mathcal{X}(E) & \longrightarrow & \mathcal{X}(J)/\mathcal{X}(E) & \longrightarrow & \mathcal{X}(J)/\mathcal{X}(J)_f & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Then note that $\mathcal{X}(J)/\mathcal{X}(J)_f$ is torsion-free (if $T_n(mx) = a_n(f)mx$ then also $T_n(x) = a_n(f)x$) implies the last claim. The proof of the analogous claim for $J_0^{\mathcal{D}\mathfrak{p}\mathfrak{q}}(\mathfrak{M})$ works just the same.

The upshot is that

$$\begin{aligned}\delta^{\mathfrak{pq}\mathfrak{D}}(\mathfrak{M}) &= \delta^{\mathfrak{D}}(\mathfrak{pq}\mathfrak{M})\frac{c_{\mathfrak{q}}}{c'_{\mathfrak{p}}}\left(\frac{|\mathrm{coker}\xi'_*|}{|\mathrm{coker}\xi_*|}\right)^2 \\ &= \frac{\delta^{\mathfrak{D}}(\mathfrak{pq}\mathfrak{M})}{c'_{\mathfrak{p}}c_{\mathfrak{q}}}\left(|\mathrm{coker}\xi'_*||\mathrm{im}\xi_*|\right)^2.\end{aligned} \tag{1}$$

Thus we are left with showing that $|\mathrm{coker}\xi'_*||\mathrm{im}\xi_*|$ is only divisible by Eisenstein primes and primes dividing $q-1$. Fix henceforth a prime ℓ such that $A[\ell]$ is irreducible and $\ell \nmid q-1$. To simplify notation we introduce

$$e(\mathfrak{D},\mathfrak{p},\mathfrak{q},\mathfrak{M}) := \ell^{\mathrm{ord}_\ell(|\mathrm{coker}\xi'_*||\mathrm{im}\xi_*|)}.$$

What we want to show now is that $e(\mathfrak{D},\mathfrak{p},\mathfrak{q},\mathfrak{M}) = 1$. First note that $\mathrm{ord}_\ell\,|\mathrm{im}\xi_*| = 0$: for any $\mathfrak{r}$ which is prime to $\mathfrak{D}\mathfrak{p}\mathfrak{q}\mathfrak{N}$, by Thm 6.4.3, the Hecke operator $T_{\mathfrak{r}}$ is Eisenstein, i.e. it acts by $1+|\mathfrak{r}|$ on $\Phi(J_0^{\mathfrak{D}}(\mathfrak{pq}\mathfrak{M}),\mathfrak{q})$. By the Eichler-Shimura relation the image is however annihilated by $a_r(f)-|\mathfrak{r}|-1$. Thus if the image is not trivial, then by Thm 6.5.14 this would mean that the representation is reducible. Hence $\mathrm{ord}_\ell\,|\mathrm{im}\xi_*| = 0$ and thus $e(\mathfrak{D},\mathfrak{p},\mathfrak{q},\mathfrak{M})|c_{\mathfrak{p}}$. Since there exists an isogeny of degree coprime to ℓ, the ℓ-primary components of $\Phi(E,\mathfrak{p})$ and $\Phi(E',\mathfrak{p})$ are isomorphic and thus

$$\mathrm{ord}_\ell\, c_{\mathfrak{p}} = \mathrm{ord}_\ell\, c'_{\mathfrak{p}} \text{ and } \mathrm{ord}_\ell\, c_{\mathfrak{q}} = \mathrm{ord}_\ell\, c'_{\mathfrak{q}}.$$

In particular $e(\mathfrak{D},\mathfrak{p},\mathfrak{q},\mathfrak{M})|c'_{\mathfrak{p}}$ and by $e(\mathfrak{D},\mathfrak{p},\mathfrak{q},\mathfrak{M}) = e(\mathfrak{D},\mathfrak{q},\mathfrak{p},\mathfrak{M})$.
To finish the proof, we will need to "permute" prime factors in the error term. To do this, we assume that $\mathfrak{N} = \mathfrak{d}\mathfrak{p}\mathfrak{q}\mathfrak{r}\mathfrak{s}\mathfrak{m}$, where $\mathfrak{p},\mathfrak{q},\mathfrak{r}$ and $\mathfrak{s}$ are prime. In a first instance we have

$$e(\mathfrak{r}\mathfrak{s}\mathfrak{d},\mathfrak{p},\mathfrak{q},\mathfrak{m}) = \ell^{\mathrm{ord}_\ell\,|\mathrm{coker}\xi'_*|} = e(\mathfrak{q}\mathfrak{s}\mathfrak{d},\mathfrak{p},\mathfrak{r},\mathfrak{m}). \tag{2}$$

By applying Eq 1 twice, we see that

$$\ell^{\mathrm{ord}_\ell\left(\frac{\delta^{\mathfrak{d}\mathfrak{p}\mathfrak{q}\mathfrak{r}\mathfrak{s}}(\mathfrak{m})}{\delta^{\mathfrak{d}}(\mathfrak{p}\mathfrak{q}\mathfrak{r}\mathfrak{s}\mathfrak{m})}c_{\mathfrak{p}}c_{\mathfrak{q}}c_{\mathfrak{r}}c_{\mathfrak{s}}\right)} = e(\mathfrak{r}\mathfrak{s}\mathfrak{d},\mathfrak{p},\mathfrak{q},\mathfrak{m})^2 e(\mathfrak{d},\mathfrak{r},\mathfrak{s},\mathfrak{p}\mathfrak{q}\mathfrak{m})^2.$$

By the same procedure in a different order, we have

$$\ell^{\mathrm{ord}_\ell\left(\frac{\delta^{\mathfrak{d}\mathfrak{p}\mathfrak{q}\mathfrak{r}\mathfrak{s}}(\mathfrak{m})}{\delta^{\mathfrak{d}}(\mathfrak{p}\mathfrak{q}\mathfrak{r}\mathfrak{s}\mathfrak{m})}c_{\mathfrak{p}}c_{\mathfrak{q}}c_{\mathfrak{r}}c_{\mathfrak{s}}\right)} = e(\mathfrak{q}\mathfrak{s}\mathfrak{d},\mathfrak{p},\mathfrak{r},\mathfrak{m})^2 e(\mathfrak{d},\mathfrak{q},\mathfrak{s},\mathfrak{p}\mathfrak{r}\mathfrak{m})^2$$

from which it follows that

$$e(\mathfrak{d}, \mathfrak{r}, \mathfrak{s}, \mathfrak{pqm}) = e(\mathfrak{d}, \mathfrak{q}, \mathfrak{s}, \mathfrak{prm}). \tag{3}$$

Furthermore there exists a prime $\mathfrak{r} \mid \mathfrak{N} = \mathfrak{D}\mathfrak{pq}\mathfrak{M}$ such that $\ell \not| c_{\mathfrak{r}}$ by Prop 6.7.5. Hence if we show that $e(\mathfrak{D}, \mathfrak{p}, \mathfrak{q}, \mathfrak{M}) \mid c_{\mathfrak{r}}$ for all $\mathfrak{r} \mid \mathfrak{N}$, then we see that $e(\mathfrak{D}, \mathfrak{p}, \mathfrak{q}, \mathfrak{M}) = 1$. We already showed this for $\mathfrak{r} = \mathfrak{p}, \mathfrak{q}$.
If $\mathfrak{r} \mid \mathfrak{D}$, then may write $\mathfrak{D} = \mathfrak{rsd}$ for some other prime $\mathfrak{s}$. Then by Eq 2 we have $e(\mathfrak{rsd}, \mathfrak{p}, \mathfrak{q}, \mathfrak{M}) = e(\mathfrak{qsd}, \mathfrak{p}, \mathfrak{r}, \mathfrak{M})$ and the latter term divides $c_{\mathfrak{r}}$.
If $\mathfrak{r} \mid \mathfrak{M}$ and as $\mathfrak{M}$ is not prime, we may write $\mathfrak{M} = \mathfrak{rsm}$ where $\mathfrak{s}$ is prime. By Eq 3, we have $e(\mathfrak{D}, \mathfrak{p}, \mathfrak{q}, \mathfrak{rsm}) = e(\mathfrak{D}, \mathfrak{r}, \mathfrak{q}, \mathfrak{psm})$ where the last term divides $c_{\mathfrak{r}}$, which finishes the proof.

□

6.8 Ratio of Petersson norms in weight 2

An interesting phenomenon of L-groups is that different groups may have isomorphic L-groups. For GL_2, as mentioned in Section 2.4, these may be realized at the groups of units of quaternion algebras $\mathcal{D}$. Thus there exists a bijection between global Langlands parameters valued in ${}^L\mathrm{GL}_2$ and ${}^L\mathcal{D}^\times$, and thus via the Langlands correspondence there ought to exist a relation between GL_2-automorphic forms and $\mathcal{D}^\times$-automorphic forms

$$\begin{array}{ccc} \{\mathrm{Gal}(F^{\mathrm{sep}}/F) \to {}^L\mathrm{GL}_2\} & \longrightarrow & \{\mathrm{Gal}(F^{\mathrm{sep}}/F) \to {}^L\mathcal{D}^\times\} \\ \uparrow & & \uparrow \\ \{\mathrm{GL}_2\text{-automorphic reps}\} & \dashrightarrow & \{\mathcal{D}^\times\text{-automorphic reps}\} \end{array}$$

This relation indeed exists and was first proven by Jacquet and Langlands in [JL06]. Using the presentation of automorphic representations in terms of harmonic cochains, one obtains the following result [PW15, Section 9.3]

Theorem 6.8.1 (Jacquet-Langlands) *There exists a Hecke-equivariant isomorphism*

$$\mathcal{H}_!(\mathcal{T}, \mathbb{Q})^{\Gamma_0(\mathfrak{n}), new} \cong \mathcal{H}(\mathcal{T}, \mathbb{Q})^{\Gamma^{\mathfrak{n}}}.$$

Remark 6.8.2 Hartl-Yu studied a version of "mod p" Jacquet-Langlands [HY25], where they proved that assuming a certain conjecture[5] if it is true, there exists a bijection between Drinfeld

[5] [HY25] constructs an arithmetic Satake compactification of the moduli space of Drinfeld modules of rank r with

modular forms valued in $\overline{\mathbb{F}}_p$ and certain locally constant functions

$$D^\times(K)\backslash D^\times(\mathbb{A})/D^\times(K_\infty) \to \overline{\mathbb{F}}_p$$

(see [HY25, Thm 6.13]).

6.8.1 Analytic description of the modular degree

We show in this section the analogue(s) of the formula

$$\deg \pi_f \int_{E(\mathbb{C})} \omega_E \wedge \overline{\omega_E} = (f, f)_P.$$

Given an $\omega \in H^0(\mathcal{E}, \Omega_{\mathcal{E},\mathcal{O}_\infty})$, there exists a natural way to give a measure μ_ω on $E(F_\infty)$ such that

$$\int_{E(F_\infty)} \mu_\omega = |\Phi(E, \infty)|$$

see [Tat75, Thm 5.2]. Hence we shall use $|\Phi(E,\infty)|$ as a substitute for $\int_{E_f(\mathbb{C})} \omega_E \wedge \overline{\omega_E}$.

Proposition 6.8.3 ([Pap13, Thm 8.19]) *Let $f \in \mathcal{H}_!(\mathcal{T}, \mathbb{Q})^{\Gamma_0^{\mathfrak{D}}(\mathfrak{m})}$ be integrally normalized and denote by $\pi_f : J_0^{\mathfrak{D}}(\mathfrak{m}) \to E_f$ be the associated optimal quotient and by f' the image of a generator of $\mathcal{X}(E_f, \infty)$ under $\mathcal{X}(E_f, \infty) \to \mathcal{X}(J_0^{\mathfrak{D}}(\mathfrak{m}), \infty)$. Then we have*

$$\deg(\pi_f)|\Phi(E_f, \infty)| = \langle f', f' \rangle.$$

Since $\mathcal{X}(E_f, \infty) \cong \mathbb{Z}$, the generator is unique up to sign. Furthermore f and f' have the same Hecke-eigenvalues, they are multiple of each others. Therefor once can unambigiously talk of the absolute value of the scalar multiples λ such that $\lambda f = f'$. This absolute value is referred to as the *Manin constant*, which is analogous to the eponymous concept over number fields. Over number fields this is known to be an integer and is conjectured to be one. By work of Pal [Pál10], the Manin constant in the function field setting is known to be a power of p.

Proposition 6.8.4 ([Pál10, Prop 2.4]) *Let E be a non-isotrivial elliptic curve over a function field of characteristic p. Then the Manin constant of E is a power of p.*

level structure. They conjecture that this compactification is Cohen-Macaulay. In the special case where the level is a prime power, this was proven by Pink in forthcoming work, the general case remains open.

Example 6.8.5 This power need not be in general trivial. Indeed, by [Pál10, 8.14], the elliptic curve over $\mathbb{F}_{13}(t)$ defined by

$$y^2 + xy = x^3 - t^{12}$$

has Manin constant 13.

Having established the analogy

$$|\Phi(E, \infty)| \sim \int_{E(\mathbb{C})} \omega_E \wedge \overline{\omega_E}$$

we consider some properties thereof under isogeny. For this we shall need the following definition:

Definition 6.8.6 *Let E be an elliptic curve over a global field K. A prime $\ell \in \operatorname{Spec} \mathcal{O}_K$ is said to be* Eisenstein *if the G_K-representation $E[\ell]$ is reducible.*

Proposition 6.8.7 ([Pra03, p.15]) *Let E_1, E_2 be isogenous elliptic curves over $\mathbb{Q}$. Denote by S the set of Eisenstein primes of the isogeny class containing E_1, E_2. Let ω_1, ω_2 are Neron differentials and let $\sim^S$ denote equality up to Eisenstein primes and 2. Then we have*

$$\int_{E_1(\mathbb{C})} \omega_1 \wedge \overline{\omega_1} \sim \int_{E_2(\mathbb{C})} \omega_2 \wedge \overline{\omega_2}.$$

This result relies on the fact that if we choose an isogeny between E_1 and E_2 of minimal degree, then this degree can only be divisible by Eisenstein primes. In characteristic p, this fails due to inseparable isogenies, as it introduces powerers of p. However we have the following result, which is a special case of [Pap07a, Lemma 2.2].

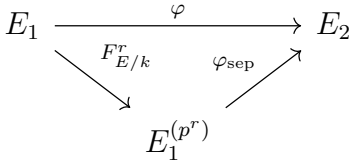
Proposition 6.8.8 ([Pap07a, Lemma 2.2]) *Let E be a Tate elliptic curve over a locally complete field of residue characteristic p. Then we have*

$$|\Phi(E^{(p^r)})| = p^r |\Phi(E)|.$$

To study the behaviour of $|\Phi(-, \infty)\ |$ under isogeny, we use the following structure result.

Proposition 6.8.9 ([Sil09, Chapter II Corollary 2.12]) *Let E_1, E_2 be elliptic curves over a field k of positive characteristic p. For any isogeny $E_1 \to E_2$, there exists an integer r such that there*

exits a factorization



such that φ_{sep} is a separable isogeny.

Thus it suffices to understand the behaviour of $|\Phi(-,\infty)\ |$ under separable isogenies and powers of Frobenius. In Prop 6.8.8 we saw the effect of Frobenius on $|\Phi(-,\infty)|$.

Theorem 6.8.10 ([Pap11, Thm 4.3]) *Let R be a complete DVR whose fraction field we denote by K and assume that the residue field k is algebraically closed. Let A be an abelian variety over K and let H be a finite subgroup of $A(K)$ whose order is coprime to* $\mathrm{char}(k)$*. Denote by*

$$s : H \to A(K) \overset{\textit{reduction}}{\longrightarrow} \mathcal{A}_k(k) \to \Phi(A).$$

Then we have

$$0 \to \ker(s) \to \Phi(A) \to \Phi(A/H) \to \mathrm{im}(s) \to 0.$$

Proposition 6.8.11 *Let $E_1 \to E_2$ be an isogeny of elliptic curves over $\mathbb{F}_q(t)$ and $\ell \nmid q$. If $\ell \nmid \deg_{\mathrm{sep}} \varphi$, then*

$$v_\ell(|\Phi(E_1,\infty)|) = v_\ell(|\Phi(E_2,\infty)|).$$

Proof. After base-change φ along $\operatorname{Spec} \mathbb{F}_p^{\mathrm{alg}}(t)_\infty \to \operatorname{Spec} \mathbb{F}_p(t)_\infty$, we can assume that the residue field is algebraically closed. Futhermore by Prop 6.8.8, we may assume that the induced morphism

$$E_{1,\mathbb{F}_p^{\mathrm{alg}}(t)_\infty} \to E_{2,\mathbb{F}_p^{\mathrm{alg}}(t)_\infty}$$

is separable of degree $\deg \varphi$. We may thus apply Thm 6.8.10. In the notation of Thm 6.8.10, the orders of $\ker(s), \mathrm{im}(s)$ are factors of $|\ker(\varphi)| = \deg \varphi$. Thus if $\ell \nmid \deg \varphi$, we have

$$v_\ell\left(|\Phi(E_{1,\mathbb{F}_p^{\mathrm{alg}}(t)_\infty})|\right) = v_\ell\left(|\Phi(E_{2,\mathbb{F}_p^{\mathrm{alg}}(t)_\infty})|\right).$$

□

Proposition 6.8.12 *Let $E_1 \overset{\varphi}{\to} E_2$ be a separable isogeny of elliptic curves over $\mathbb{F}_q(t)$ and ℓ a prime coprime to q. If $\ell \mid \deg \varphi$, then $E_1[\ell]$ is reducible, i.e. ℓ is an Eisenstein prime for this isogeny class.*

Proof. Consider the ℓ-primary part of the kernel $\ker(\varphi)$. Let $n = \mathrm{ord}_\ell(\deg \varphi)$. Then $\ker(\varphi)[\ell^\infty]$ is a finite subgroup defined over $\mathbb{F}_p(t)$ and since its order is coprime to the characteristic, it is étale. Thus $[\ell^{n-1}]\ker(\varphi)[\ell^n] \subset E_1[\ell]$ is a Galois invariant subspace and thus $E_1[\ell]$ is reducible. □

Proposition 6.8.13 *Let E_1, E_2 be isogenous elliptic curves over $\mathbb{F}_q(t)$ which are split multiplicative over ∞ and $\ell \in \mathrm{Spec}\,\mathbb{Z}$. If*

$$v_\ell(|\Phi(E_1, \infty)|) \neq v_\ell(|\Phi(E_2, \infty)|)$$

then ℓ must be Eisenstein or p.

Proof. This is a consequence of Prop 6.8.11 and Prop 6.8.12. □

6.8.2 Ratio of Peterson Norms

We can now bring all the ingredients together to prove the analogue of [Pra03, Section 1.5.2] :

Theorem 6.8.14 *Let E be an elliptic curve $E/\mathbb{F}_p(t)$ with split multiplicative reduction at ∞. Let $f \in \mathcal{H}_!(\mathcal{T}, \mathbb{Q})^{\Gamma_0(1,\mathfrak{n})}$, $g \in \mathcal{H}_!(\mathcal{T}, \mathbb{Q})^{\Gamma_0(\mathfrak{D},\mathfrak{m})}$ cuspidal harmonic cochains associated to it, which are Jacquet-Langlands lifts of each other and that are integrally normalized. We have the following equality up to Eisenstein primes, primes dividing $q-1$ and p.*

$$\frac{\langle f, f\rangle}{\langle g, g\rangle} \sim \frac{\delta^{\mathfrak{D}}(\mathfrak{m})}{\delta^{(1)}(\mathfrak{D}\mathfrak{m})} \sim \prod_{\mathfrak{q}|\mathfrak{D}} c_{\mathfrak{q}}.$$

Proof. Let f, g be as in the statement of the theorem. By Theorem 4.8.4 and Theorem 4.8.5 we choose elliptic curves E_f, E_g with coverings

$$X_0(\mathfrak{n}\mathfrak{D}) \to E_f$$

$$X_0^{\mathfrak{D}}(\mathfrak{n}) \to E_g$$

corresponding to f, g respectively. These elliptic curves are isogenous. By Prop 6.8.3 respectively, we see that

$$\deg(\pi_f)|\Phi(E_f, \infty)| = \langle f', f'\rangle$$

and

$$\deg(\pi_g)|\Phi(E_g, \infty)| = \langle g', g'\rangle.$$

Since E_f and E_g are isogenous, by Prop 6.8.13 we know that $|\Phi(E_f, \infty)|$ and $|\Phi(E_g, \infty)|$ only differ by Eisenstein primes and powers of p, hence $\frac{\langle f', f' \rangle}{\langle g', g' \rangle} \sim \frac{\deg \pi_f}{\deg \pi_g}$. By Prop 6.7.4 we know that, up to Eisenstein primes and primes dividing $p-1$, we have

$$\frac{\deg \pi_f}{\deg \pi_g} \sim \prod_{\mathfrak{q}|D} c_{\mathfrak{q}}.$$

□

6.9 An extended example

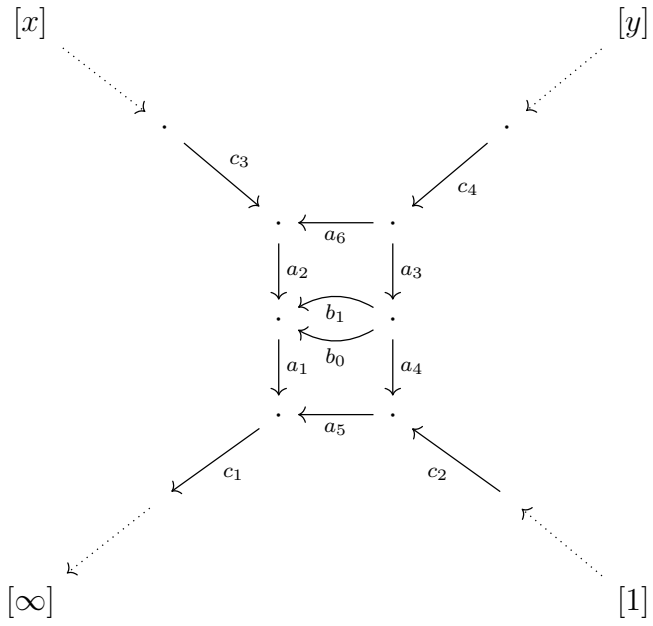
The goal of this section is to work out concretely the relation of the modular degrees in an example. The conductor of an elliptic curve over $\mathbb{F}_q(t)$ is a divisor of degree at least four. We shall consider the minimal case and assume that E has multiplicative reduction at ∞. In [Sch01] the optimal quotients of the modular Jacobian $J_0(\mathfrak{pq})$ were computed if $q = 2^r$, $\deg(\mathfrak{p}) = 1$ and $\deg(\mathfrak{q}) = 2$. Subsequently, under the same assumption, in [Pap11] the component groups of $J_0(\mathfrak{pq})$ and $J^{\mathfrak{pq}}$ were determined for arbitrary q.

Table VI.1: Component groups of the modular Jacobians

	$\Phi(-, \mathfrak{p})$	$\Phi(-, \mathfrak{q})$	$\Phi(-, \infty)$
$J_0(\mathfrak{pq})$	$\mathbb{Z}/5 * 3$	$\mathbb{Z}/3$	$\mathbb{Z}/5 * 3$
$J^{\mathfrak{pq}}$	$\mathbb{Z}/3$	$\mathbb{Z}/5 * 3$	$\mathbb{Z}/3$

In this example, we shall restrict ourselves to $q = 2$. To classify the optimal quotients of $J_0(\mathfrak{pq})$,

we can compute $\mathcal{H}_!(\mathcal{T},\Lambda)^{\Gamma_0(\mathfrak{pq})}$: The quotient $\Gamma_0(\mathfrak{pq})\backslash\mathcal{T}$ is



The weights are such that in fact we have an isomorphism

$$\mathcal{H}_!(\mathcal{T},\mathbb{Z})^{\Gamma_0(\mathfrak{pq})}\to\bigoplus_{u\in\mathbb{F}_2}\mathbb{Z},\quad f\mapsto(f(b_u))_{u\in\mathbb{F}_2}.$$

In particular we have

$$\mathcal{H}_!(\mathcal{T},\mathbb{Q})^{\Gamma_0(\mathfrak{pq}),new}=\mathcal{H}_!(\mathcal{T},\mathbb{Q})^{\Gamma_0(\mathfrak{pq})}\cong\mathbb{Q}^2\neq 0.$$

Thus there are exactly two strong Weil curves which correspond to the sum and difference respectively of the two loops in $\mathcal{T}/\Gamma_0(\mathfrak{pq})$. We shall denote these elliptic curves by E_1, E_2. The information in the table was taken from [Pap11] (with exception of the Peterson inner product, which is also computed in [GR96] section 9.7).

Table VI.2: $J_0(\mathfrak{pq})$-optimal curves and their data

Elliptic curve	Isogeny class	$\lvert c_\mathfrak{p}\rvert$	$\lvert c_\mathfrak{q}\rvert$	$\lvert c_\infty\rvert$	$\langle f,f\rangle$
E_1	I	3	3	3	6
E_2	II	5	1	5	10

As mention in Rem 6.6.4, the image/cokernel of the component group gives information about

Table VI.3: $J^{\mathfrak{pq}}$-optimal curves and their data

Elliptic curve	Isogeny class	$\lvert c'_{\mathfrak{p}}\rvert$	$\lvert c'_{\mathfrak{q}}\rvert$	$\lvert c'_{\infty}\rvert$	$\langle g, g\rangle$
E_1	I	3	3	3	6
E'_2	II	1	5	1	2

the level-lowering. We shall now explicitly show that there exists modular forms of lower level by explicitly show that $\mathcal{H}_!(\mathcal{T},\mathbb{Q})^{\Gamma_0(\mathfrak{p})} = \mathcal{H}_!(\mathcal{T},\mathbb{Q})^{\Gamma_0(\mathfrak{q})} = 0$. The quotient of $\Gamma_0(\mathfrak{p})\backslash\mathcal{T}$ is the graph given with indices indexed by $\begin{pmatrix} T^i & 0 \\ 0 & 1 \end{pmatrix}$ for all i. Then the edges between v_{i-1} and v_i, which we denote by e_i.

$$\ldots \longrightarrow v_{-1} \xrightarrow{e_0} v_0 \xrightarrow{e_1} \ldots$$

Furthermore, the weights of the vertices (i.e. their stabilizers in $\mathrm{GL}_2(A)$) are

$$w(e_i) = \begin{cases} 2 & i \geqslant 0 \\ 1 & i \leqslant -1 \end{cases}$$

$$w(\overline{e_i}) = \begin{cases} 1 & i \geqslant -1 \\ 2 & i \leqslant -2 \end{cases}$$

For any $f \in \mathcal{H}(\mathcal{T},\Lambda)^{\Gamma_0(\mathfrak{p})}$, harmonicity condition now expresses itself as

$$w(e_i)f(e_i) = w(\overline{e_{i+1}})f(e_{i+1}).$$

Hence this is uniquely determined by one value. For this to be cuspidal, i.e. have compact support, the value needs to be annihilated by sufficiently large powers of 2, hence we get

$$\mathcal{H}(\mathcal{T},\Lambda)^{\Gamma_0(\mathfrak{p})} = \Lambda, \quad \mathcal{H}_!(\mathcal{T},\Lambda)^{\Gamma_0(\mathfrak{p})} = \Lambda[2^\infty].$$

In particular, if we choose $\Lambda = \mathbb{Z}, \mathbb{Q}$ there are no cuspidal harmonic forms. We now are going to show that $\mathcal{H}_!(\mathcal{T},\mathbb{Q})^{\Gamma_0(\mathfrak{q})} = 0$: The quotient $\Gamma_0(\mathfrak{p})\backslash\mathcal{T}$ has vertices given by the quotients $\begin{pmatrix} T^i & 0 \\ 0 & 1 \end{pmatrix}$ for all $i \in \mathbb{Z}$, which we denote by v_i and a single class $\begin{pmatrix} T^{-2} & T^{-1} \\ 0 & 1 \end{pmatrix}$, which we denote by u.

The edges between v_{i-1} and v_i are denoted by e_i and the edge with terminus u is denoted by e_u. The corresponding edges are

$$w(e_i) = \begin{cases} q & i \geqslant 0 \\ 1 & i \leqslant -1 \end{cases} \text{ and } w(\overline{e_i}) = \begin{cases} 1 & i \geqslant 0 \\ q & i \leqslant -1 \end{cases}$$

$$w(e_u) = q+1 \quad w(\overline{e_u}) = q-1.$$

The harmonicity condition at the vertex u implies that

$$0 = w(e_u)f(e_u) = (q+1)f(e_u).$$

Hence if we fix $f(e_0) \in \Lambda$ and $f(e_u) \in \Lambda[q+1]$, then the harmonicity condition determines all the other values, in particular we have

$$f(e_i) = \begin{cases} f(e_o)q^i & i \geqslant 0 \\ q^{-i-1}(f(e_o) + (q-1)f(e_u)) & i \leqslant -1 \end{cases}.$$

Hence $\mathcal{H}(\mathcal{T}, \Lambda)^{\Gamma_0(\mathfrak{q})} = \Lambda \oplus \Lambda[q+1]$ and we have

$$\mathcal{H}_!(\mathcal{T}, \Lambda)^{\Gamma_0(\mathfrak{q})} = \begin{cases} \Lambda[2^\infty] & p = 2 \\ \Lambda[p^\infty] \oplus \Lambda[2] & p \neq 2 \end{cases}$$

In particular we have

$$\mathcal{H}_!(\mathcal{T}, \mathbb{Z})^{\Gamma_0(\mathfrak{q})} = \mathcal{H}_!(\mathcal{T}, \mathbb{Q})^{\Gamma_0(\mathfrak{q})} = 0.$$

To compute the Petersson inner product on the quaternionic side, we use the fact that the dual graph of $J^{\mathfrak{pq}}$ is given by Figure VI.1

Since there are no lower-level modular forms, there is no level-lowering that happens, in particular we have $|\text{coker}\,(\Phi(J_0(\mathfrak{pq}), \mathfrak{q}) \to \Phi(E, \mathfrak{q}))\,| = 1$. Thus, applying Thm 6.7.4, we see that

$$\frac{\langle g', g' \rangle}{\langle f', f' \rangle} = \frac{\delta^{\mathfrak{pq}} \cdot c'_\infty}{\delta(\mathfrak{pq}) \cdot c_\infty} |\text{coker}\,(\Phi(J^{\mathfrak{pq}}, \mathfrak{p}) \to \Phi(E, \mathfrak{p}))\,|^2$$

Plugging in the data for the respective isogeny class, we have

$$\frac{6}{6} = \frac{3}{3} \cdot \frac{3}{3} \text{ and } \frac{2}{10} = \frac{1}{5} \cdot \frac{1}{1}.$$

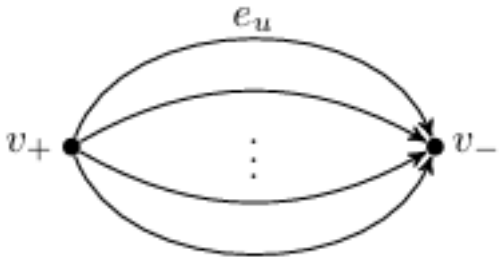


Figure VI.1: Dual Graph of J^{pq} [PW15]

Remark 6.9.1 It should be pointed out that since the results of [Pap11] apply for arbitrary q and the results of [Sch01] apply for $q = 2^r$, one should be able to extend this example to $q = 2^r$.

CHAPTER VII

List of Open Questions

As this thesis contains several open questions, we compile them here for the convenience of the reader interested in pursuing them.

7.1 Ihara's Lemma

While a result of this thesis is with regard to *lowering* the level of modular representations, more is known in the number field setting. In [Rib84] Ribet proved the following result, concerning the *raising* of the level:

Proposition 7.1.1 ([Rib90b, Thm 1]) *Let ρ be a modular $G_{\mathbb{Q}}$-representation of level n with coefficients in $\mathbb{F}_\ell$ and p coprime to ℓn such that we have*

$$T_p = \pm(p+1).$$

Then ρ is a p-newform of level pn.

The proof relies on a result of Ribet's often refered to as "Ihara's lemma", whose proof depends on a result of Ihara, namely Lemma 3.2 of [Iha75]. Ihara's Lemma is a statement about the "simply connectedness" of a system of coverings of fuchsian quotients. Thus there exists the potential for a function field analogue of Ihara's Lemma by replacing the upper half-plane with the Bruhat-Tits building. The question addressed in this section is if therefor results analogous to Prop 7.1.1 hold true. We show that this reduces to the function field analogue of Ihara's Lemma and explain which steps are missing to prove Ihara's Lemma.

7.1.1 Theta groups Background

We briefly recall some basics of Theta groups and the associated Weil pairing, which is relevant for Ribet's work. For this we follow the exposition in [Mum70].

Definition 7.1.2 (Theta group) *A* theta group *is a short exact sequence of group schemes*

$$1 \to \mathbb{G}_m \to G \xrightarrow{\pi} K \to 1$$

where we assume K to be commutative, $\mathbb{G}_m$ to be the center of G and we assume the existence of local sections, that is an open cover $(U_i)_i$ of K such that we have

$$\begin{array}{ccc} & & G \\ & \nearrow & \downarrow \\ U_i & \longrightarrow & K. \end{array}$$

Definition 7.1.3 (Weil pairing) *For any theta group, there exists a skew-symmetric $\mathbb{G}_m$-valued pairing*

$$e : K \times K \to \mathbb{G}_m$$

such that for any $g, h \in G$ we have

$$e(\pi g, \pi h) = [g, h].$$

The primary example we care about arises in the following way:

Proposition-Definition 7.1.4 (Theta group of line bundle [Mum70, Thm 1]) *For an abelian variety X and line bundle $\mathcal{L}$ on it, the functor*

$$S \mapsto \{(x, \varphi) \mid x \in X(S), \textit{isomorphism} \mathcal{L}_S \xrightarrow{\varphi} t_x^* \mathcal{L}_S\}$$

is representable by a group scheme, which we denote by $\mathcal{G}(\mathcal{L})$. The functor

$$S \mapsto \{x : S \to X \quad t_x^*(\mathcal{L}_S) \cong \mathcal{L}_S\}$$

is representable by a commutative group scheme and gives rise to a short exact sequence

$$1 \to \mathbb{G}_m \to \mathcal{G}(\mathcal{L}) \to K(\mathcal{L}) \to 0$$

making this into a theta group.

Proposition 7.1.5 (Morphism associated of line bundle) *Any invertible sheaf $\mathcal{L}$ on an abelian*

variety induces a morphism

$$X \to X^\vee, \quad x \mapsto t_x^*(\mathcal{L}) \otimes \mathcal{L}^{-1}$$

whose kernel is $K(\mathcal{L})$. This is an isogeny if and only if $\mathcal{L}$ is ample and it is an isomorphism if and only if $\mathcal{L}$ is ample and $h^0(\mathcal{L}) = 1$.

Proposition 7.1.6 ([Mum70, p. 228]) *The Weil pairing of a line bundle satisfies the following properties:*

1. *for any $X \xrightarrow{f} Y$, then we have $e^{f^*\mathcal{L}}(x, y) = e^{\mathcal{L}}(f(x), f(y))$.*
2. *$e^{\mathcal{L}_1 \otimes \mathcal{L}_2}(-,-) = e^{\mathcal{L}_1}(-,-) \cdot e^{\mathcal{L}_2}(-,-)$*

Proposition 7.1.7 ([Mum70, Cor p.231]) *For any morphism of abelian varieties $X \xrightarrow{f} Y$ and $\mathcal{L}$ on X, there exists a $\mathcal{L}'$ on Y such that $f^*\mathcal{L}' \cong \mathcal{L}$ if and only if $e^{\mathcal{L}}|_{\ker f \times \ker f} = 1$ and $\ker(f) \subset K(\mathcal{L})$.*

Proposition 7.1.8 ([Mum70, Lemma 2]) *Let $X \xrightarrow{f} Y$ and $\mathcal{L} = f^*\mathcal{L}'$. Then we know that $\ker(f)^\perp \supset \ker(f)$ and we have*

$$K(\mathcal{L}') \cong \ker(f)^\perp / \ker(f).$$

7.1.2 Ihara's Lemma and Level-Raising

Definition 7.1.9 (Unramified covering [Iha75]) *Let $X_0 \leftarrow X_1 \to X_2$ be a system of schemes with surjective morphisms. An* unramified covering of $X_0 \leftarrow X_1 \to X_2$ *consists of unramified morphisms*

$$Y_0 \xrightarrow{f_0} X_0, \quad Y_2 \xrightarrow{f_2} X_2$$

such that

$$X_1 \times_{X_0} Y_0 \cong X_1 \times_{X_2} Y_2$$

satisfying the following condition: denote by F_0, F_2 the smallest Galois coverings of f_0 and f_2 respectively. Then F_0 and φ_0, F_2 and φ_2 are both linearly disjoint.

Definition 7.1.10 (Simply connected) *A system $X_0 \leftarrow X_1 \to X_2$ is said to be* simply connected *if the only unramified covering is the trivial one.*

The following would be a function field analogue of [Iha75, Lemma 3.2]

Conjecture 7.1.11 (Ihara's Lemma, simply connected version) *The system*

$$X(\mathfrak{n}) \leftarrow X(\Gamma(\mathfrak{n}) \cap \Gamma_0(\mathfrak{p})) \rightarrow X(\mathfrak{n})$$

is simply connected.

To simplify comparison with [Iha75], we introduce the following notation

Definition 7.1.12 *Throughout this section, we shall fix a prime ideal* $\mathfrak{p} = (\pi) \subset \mathbb{F}_p[x]$. *For any ideal* $\mathfrak{n} \subset \mathbb{F}_p[x]$, *denote by*

1. $\Gamma_{\mathfrak{n}}$ *the principal congruence group of* $\mathrm{PSL}_2(\mathbb{F}_p[x][\frac{1}{\pi}])$ *of level* $\mathfrak{n}$,
2. $\Delta_{\mathfrak{n}}$ *the principal congruence group of* $\mathrm{PSL}_2(\mathbb{F}_p[x])$ *of level* $\mathfrak{n}$,
3. $\Delta'_{\mathfrak{n}}$ *the conjugation of* $\Delta_{\mathfrak{n}}$ *by* $\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}$,
4. $\Delta^0_{\mathfrak{n}} := \Delta_{\mathfrak{n}} \cap \Delta'_{\mathfrak{n}}$

The proof of [Iha75, Lemma 3.2] is based on the following two properties

- $\Gamma_{\mathfrak{n}}$ is the amalgamated product $\Delta_{\mathfrak{n}} \times_{\Delta^0_{\mathfrak{n}}} \Delta'_{\mathfrak{n}}$
- $\Gamma_{\mathfrak{n}}$ is the normal closure of $\left\{ \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \mid f \in \mathfrak{n} \right\}$.

These are naive analogues of the corresponding number field properties in Ihara. The second property was proven by Serre in [Ser70]. While the proof of the first Property in [Iha08, Lemma 7 p. 91] is probably adaptable to function fields, we shall instead outline a strategy explained in a series of lectures by Serre [Ser77]. It hinges on the following result relating the fundamental group of a quotient of a tree with a particular fundamental domain:

Definition 7.1.13 *A* fundamental domain $T \subset X$ *is a subgraph of* X *such that*

$$X \supset T \cong X/\Gamma.$$

Proposition 7.1.14 ([Ser03, Prop 17]) *If* Γ *acts on a tree* X, *then a fundamental domain exists if and only if* X/Γ *is a tree.*

Proposition 7.1.15 ([Ser03, Thm 6]) *Let Γ be a group acting on a tree X. Let $P, Q \in \text{Vert}(X)$ such that $\overline{PQ} \in \text{Ed}(X)$ be a fundamental domain of Γ acting on X. Then we have*

$$\Gamma \cong \Gamma_P \times_{\Gamma_P \cap \Gamma_Q} \Gamma_Q.$$

In view of this, to prove the first property, it would suffice to prove the following result:

Conjecture 7.1.16 *The quotient $\mathcal{T}/\Gamma_{\mathfrak{n}}$ is isomorphic to a graph with two vertices and one edge connecting them. We define* $\text{Vert}(\mathcal{T}/\Gamma_{\mathfrak{n}}) \cong \{x, y\}$ *and*

$$\text{Stab}_x \cong \Delta_{\mathfrak{n}}, \text{Stab}_y \cong \Delta'_{\mathfrak{n}}.$$

Remark 7.1.17 From here on out, we shall show that the veracity of level raising in the function field setting would be a consequence of Con 7.1.11, adapting the work of Ribet in [Rib90b] [Rib84].

Proposition 7.1.18 (Ihara's Lemma [Rib84, Thm 4.1]) *Assume that Conj 7.1.11 is true. Then if $\mathfrak{p}$ does not divide $\mathfrak{n}$, the morphisms*

$$J(\mathfrak{n}) \times J(\mathfrak{n}) \to J(\Gamma(\mathfrak{n}) \cap \Gamma_0(\mathfrak{p}))$$

$$J_1(\mathfrak{n}) \times J_1(\mathfrak{n}) \to J(\Gamma_1(\mathfrak{n}) \cap \Gamma_0(\mathfrak{p}))$$

are injective.

Proof. We first show that $J(\mathfrak{n}) \times J(\mathfrak{n}) \to J(\Gamma(\mathfrak{n}) \cap \Gamma_0(\mathfrak{p}))$ is injective. It suffices to prove that this morphism is injective on ℓ-torsion points, which means we have to show that

$$H^1(X(\mathfrak{n}), \mathbb{F}_\ell) \oplus H^1(X(\mathfrak{n}), \mathbb{F}_\ell) \to H^1(X(\Gamma(\mathfrak{n}) \cap \Gamma_0(\mathfrak{p})), \mathbb{F}_\ell)$$

is injective. The cohomology groups classify unramified $\mathbb{F}_\ell$-covering groups, and thus injectivity means that

$$X(\mathfrak{n}) \leftarrow X(\Gamma(\mathfrak{n}) \cap \Gamma_0(\mathfrak{p})) \to X(\mathfrak{n})$$

is simply connected in the previous sense, which is the result of Conj 7.1.11. □

Definition 7.1.19 *The kernel of the morphism $J_0(\mathfrak{n}) \to J_1(\mathfrak{n})$ which is induced by the covering $X_1(\mathfrak{n}) \to X_0(\mathfrak{n})$ is a finite group which we call the* Shimura subgroup *and denote by Sh.*

Proposition 7.1.20 ([PW15, Prop 8.5]) *The Shimura subgroup is Eisenstein, that is it is annihilated by all endomorphisms $T_{\mathfrak{q}} - (|\mathfrak{q}| + 1)$ for all $(\mathfrak{q}, \mathfrak{n}) = 1$. The Atkin-Lehner involution acts by -1 on the Shimura subgroup.*

Definition 7.1.21 *The image of the Shimura subgroup under the skew-diagonal embedding is denoted by Σ.*

Proposition 7.1.22 ([Rib84, Thm 4.3]) *The kernel of the morphism*

$$\alpha : J_0(\mathfrak{n}) \times J_0(\mathfrak{n}) \to J_0(\mathfrak{pn})$$

is the group Σ and in particular Eisenstein.

Proof. We will first show that the kernel of α contains Σ. We know that $\omega_{\mathfrak{p}} \circ a = b$, thus it will suffice to show that the Atkin-Lehner involution acts by 1 on $b(Sh)$. The Atkin-Lehner involution $\omega_{\mathfrak{n}}$ acts on the Shimura group by -1 by Prop 7.1.20. The image of the Shimura subgroup is contained in the Shimura subgroup of $J_0(\mathfrak{pn})$, and thus we know that $\omega_{\mathfrak{pn}}$ acts by 1 on the image of the Shimura subgroup. However, we have

$$\omega_{\mathfrak{p}} = \omega_{\mathfrak{pn}} \circ \omega_{\mathfrak{n}}$$

and thus the containment follows. To conclude equality, it suffices to point out that $J_0(\mathfrak{n}) \xrightarrow{b} J_0(\mathfrak{pn})$ is injective, or equivalently that the covering $X_0(\mathfrak{pn}) \to X_0(\mathfrak{n})$ is ramified and admits no non-trivial intermediate covering. □

Proposition 7.1.23 ([Rib90b, Cor p.7]) *The kernel Δ of the endomorphism*

$$\begin{pmatrix} 1 + |\mathfrak{p}| & T_{\mathfrak{p}} \\ T_{\mathfrak{p}} & 1 + |\mathfrak{p}| \end{pmatrix} \in \mathrm{End}(J_0(\mathfrak{n}) \times J_0(\mathfrak{n}))$$

is finite and contains Σ. Thus Σ is Eisenstein.

Proof. The endomorphism in the proposition is given as the composition of

$$J_0(\mathfrak{n}) \times J_0(\mathfrak{n}) \xrightarrow{\alpha} J_0(\mathfrak{pn}) \xrightarrow{\beta} J_0(\mathfrak{n}) \times J_0(\mathfrak{n})$$

from which the containment $\Sigma \subset \Delta$ follows immediately. To show that this is a finite group, we first note that we have a natural morphism

$$\Delta \to \ker(T_{\mathfrak{p}} + (1 + |\mathfrak{p}|)) \times \ker(T_{\mathfrak{p}} + (1 + |\mathfrak{p}|)), \quad (x, y) \to (x + y, x - y).$$

The obvious morphism in the converse direction shows that the kernel of the above morphism consists at most of 2-torsion elements. In partiuclar by the Weil conjectures this kernel has to be finite. □

Proposition-Definition 7.1.24 *The group Δ is equipped with a perfect $\mathbb{G}_m$-valued skew-symmetric pairing. Under this pairing the subgroup Σ is self-orthogonal.*

Proof. Since α and β are dual, we may view Δ as the theta kernel of a polarization of $J_0(\mathfrak{n}) \times J_0(\mathfrak{n})$. From this the existence of the pairing and its properties follows. □

Proposition 7.1.25 ([Rib90b, Thm 2]) *The intersection of the $\mathfrak{p}$-new and $\mathfrak{p}$-old subvariety is equal to $\Sigma^{\perp}/\Sigma$.*

Proof. First consider the line bundle corresponding to the principal polarization of $J_0(\mathfrak{pn})$, which we denote by $\mathcal{L}$. Then we can pullback $\mathcal{L}$ to the $\mathfrak{p}$-old subvariety, which induces an isogeny

$$\varphi_{\mathcal{L}} : J_0(\mathfrak{pn})_{\mathfrak{p}\text{-old}} \to J_0(\mathfrak{pn})^{\vee}_{\mathfrak{p}\text{-old}}.$$

(as pullback of ample along closed embedding is ample). The kernel of this morphism is $K(\mathcal{L})$. By definition, the $\mathfrak{p}$-new subvariety is the kernel of the morphism

$$J_0(\mathfrak{pn}) \to J_0(\mathfrak{pn}) \to J_0(\mathfrak{pn})_{\mathfrak{p}\text{-old}}$$

and thus the intersection $J_0(\mathfrak{pn})_{\mathfrak{p}\text{-old}} \cap J_0(\mathfrak{pn})_{\mathfrak{p}\text{-new}}$ is the kernel of $\varphi_{\mathcal{L}}$, a situation which is summarised by the following diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & J_0(\mathfrak{pn})_{\mathfrak{p}\text{-new}} & \longrightarrow & J_0(\mathfrak{pn}) & \longrightarrow & J_0(\mathfrak{pn}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \cap & \longrightarrow & J_0(\mathfrak{pn})_{\mathfrak{p}\text{-old}} & \xrightarrow{\varphi_{\mathcal{L}}} & J_0(\mathfrak{pn})^{\vee}_{\mathfrak{p}\text{-old}}. \end{array}$$

and thus $\cap = K(\mathcal{L}|_{\mathfrak{p}\text{-old}})$. However we can factor α as

$$J_0(\mathfrak{n}) \times J_0(\mathfrak{n}) \xrightarrow{\beta} J_0(\mathfrak{pn})_{\mathfrak{p}\text{-old}} \longrightarrow J_0(\mathfrak{pn}).$$

Thus by Prop 7.1.8, we have

$$K(\mathcal{L}|_{\mathfrak{p}\text{-old}}) \cong \ker(\alpha)^{\perp}/\ker(\alpha) = \Sigma^{\perp}/\Sigma$$

and we can conclude. □

Proposition 7.1.26 (Level Raising, Analogue of [Rib90b, Thm 1]) *Let $\rho_{\mathfrak{m}}$ be a modular representation of level $\mathfrak{n}$ with coefficients in $\mathbb{F}_\ell$. For any $\mathfrak{p}$ coprime to $\mathfrak{n}$ such that we have*

$$T_{\mathfrak{p}} = \pm(|\mathfrak{p}| + 1).$$

Then ρ is a $\mathfrak{p}$-newform of level $\mathfrak{pn}$.

Proof. Consider the group $J_0(\mathfrak{n})[\mathfrak{m}]$. This is a non-zero finite vector space. If we see $V \times V \subset J_0(\mathfrak{n}) \times J_0(\mathfrak{n})$, then this group must have trivial intersection with Σ, since Σ is Eisenstein and otherwise the irreducibility of $\rho_{\mathfrak{m}}$ would break. In particular we can view (via α) $V \times V$ as a subgroup of the $\mathfrak{p}$-old subvariety.
We first assume that

$$T_{\mathfrak{p}} = -(|\mathfrak{p}| + 1) \text{mod} \ell.$$

We now claim that $V \to V \times V \to J_0(\mathfrak{pn})_{\text{pold}}$ is stable under $\mathbb{T}_{\mathfrak{pn}}$: for $\mathfrak{q}$ coprime to the level, this follows from the Hecke equivariance of α. Then for $T_{\mathfrak{p}}$ this follows from our assumption on $T_{\mathfrak{p}}$ and the fact that we have

$$T_{\mathfrak{p}} \circ \alpha = \alpha \circ \begin{pmatrix} T_{\mathfrak{p}} & |\mathfrak{p}| \\ -1 & 0 \end{pmatrix}.$$

Then in particular we have a homomorphism

$$\mathbb{T}_{\mathfrak{pn}} \to \mathbb{T}_{\mathfrak{n}}/\mathfrak{m}, \quad T_{\mathfrak{q}}, T_{\mathfrak{p}} \mapsto T_{\mathfrak{q}}, -1.$$

We claim that the kernel of this morphism does the job, i.e that this action factors through the $\mathfrak{p}$-new quotient of $\mathbb{T}_{\mathfrak{pn}}$. For this it will suffice to show that V actually lies in the intersection $J_0(\mathfrak{pn})_{\mathfrak{p}\text{-old}} \cap J_0(\mathfrak{pn})_{\mathfrak{p}\text{-new}}$. A priori we know that V lies in the subgroup Δ/Σ. Hence we is actually left is to show that V is orthogonal to Σ, or equivalently that V maps to zero in $\Delta/\Sigma^{\perp}$.
The group $\Delta/\Sigma^{\perp}$ is in fact Cartier dual of Σ (in fact $\ker \alpha$). However it follows from Prop 7.1.20 that the Galois action on Σ is trivial, and thus so is the Galois action on $\Delta/\Sigma^{\perp}$. However V can not map non-trivially onto a Galois module with trival action, as we saw in Prop 6.5.14 that the semisimplification of V is the direct sum of $\rho_{\mathfrak{m}}$. □

7.2 Oort-splitting of Drinfeld-Stuhler modules

In this thesis, we use "Deuring"-type results to classify supersingular and mixed-exceptional Drinfeld-Stuhler modules. These depend on the choice of a "basepoint", that is of some $E_0 \in \Sigma_{\mathbb{F}}$ and $(E_0, C_0) \in \Sigma_{\mathbb{F}}(\mathfrak{n})$. For abelian surfaces, one can do without this choice (see for instance [Rib89]). We explain this construction for supersingular elliptic curves (following [Voi21, Section 42.4.3]): Let E be a supersingular elliptic curve over $\mathbb{F}_p^{\mathrm{alg}}$ and denote by ω a non-zero invariant differential on E. For any endomorphism $\varphi \in \mathrm{End}(E)$, the pullback $\varphi^*\omega$ is also a non-zero invariant differential. Thus there exists an $a_\varphi \in \mathbb{F}_p^{\mathrm{alg}}$ such that

$$\varphi^*\omega = a_\varphi \omega.$$

This defines a morphism

$$\mathrm{End}(E) \to \mathbb{F}_p^{\mathrm{alg}}, \quad \varphi \mapsto a_\varphi.$$

This morphism is independent of the choice of invariant differential ω. This leads to the notion of an *orientation of a maximal order* $\mathcal{O}$: an orientation of a maximal order $\mathcal{O}$ is a ring morphism

$$\mathcal{O} \to \mathbb{F}_p^{\mathrm{alg}}.$$

One can then prove that this construction induces a bijection between isomorphism classes of elliptic curves over $\mathbb{F}_p^{\mathrm{alg}}$ and oriented maximal order in a quaternion algebra of discriminant p (see [Voi21, Cor 42.4.12]).
In [Rib89], Ribet generalized this construction to classify abelian surfaces over $\bar{\mathbb{F}}_p$ with the action of a maximal order. A key ingredient in this is a phenomenon called *Oort-splitting*, which I'll briefly recall. Oort studied the question of when an abelian variety is the product of elliptic curves in the special case of algebraically closed field k of positive characteristic p. For this he defined the *Oort-invariant* of an abelian variety A

$$a(A) := \dim_k(\alpha_p, A).$$

This is an invariant in $[0, \dim A]$. In [Oor75], Oort proved that if $a(A) = \dim A$, then there exist supersingular elliptic curves $E_1, \ldots, E_{\dim A}$ over k such that

$$A \cong E_1 \times \ldots \times E_{\dim A}.$$

In [Rib89] this is used to show that supersingular abelian surfaces are the product of supersingular elliptic curves: the k-vector space $\mathrm{Hom}_k(\alpha_p, A)$ inherits an action of $\mathcal{O}_D \otimes k \cong \mathrm{Mat}_2(k)$. Hence it has to be an even dimensional vector space, and thus of dimension 2. One can similarly show ([Rib89, Prop 4.2]) that exceptional abelian surfaces are isomorphic to the product of supersingular elliptic curves. In the case of supersingular abelian surfaces, this means that

$$\mathrm{End}(A) \cong \mathrm{End}(E \times E) \cong \mathrm{Mat}_2(\mathrm{End}(E)).$$

In particular $\mathcal{O}$-abelian surfaces correspond to rank 8 $(\mathcal{O}, \mathrm{End}(E))$-bimodules satisfying certain conditions (more precisely $\mathcal{O}$-abelian surfaces correspond to rank 8 *admissible* $(\mathcal{O}, \mathrm{End}(E))$-bimodules). These admissible bimodules themselves correspond to oriented Eichler orders. The upshot of this work is the following: let M, D be coprime integers and p, q be distinct primes. Let R be a maximal order of a quaternion algebra over $\mathbb{Q}$ with discriminant D. Let $\mathcal{O}$ be a maximal order in a quaternion algebra over $\mathbb{Q}$ with discriminant pqD. In [Rib89, Thm 3.4], [Rib89, Thm 4.15], it is shown that

$$\left(\begin{array}{l}\text{supersingular } R\text{-abelian surfaces}\\ \text{over } \bar{\mathbb{F}}_q \text{with } \Gamma_0(M)\text{-level}\end{array}\right) \xrightarrow{\sim} \left(\begin{array}{l}\text{oriented Eichler order of level } M\\ \text{in quat.algebra of disc.} qD \text{ over } \mathbb{Q}\end{array}\right)$$

$$\left(\begin{array}{l}\text{mixed exceptional } \mathcal{O}\text{-abelian surfaces}\\ \text{over } \bar{\mathbb{F}}_p \text{with } \Gamma_0(M)\text{-level}\end{array}\right) \xrightarrow{\sim} \left(\begin{array}{l}\text{oriented Eichler order of level } M\\ \text{in quat.algebra of disc.} qD \text{ over } \mathbb{Q}\end{array}\right)$$

This provides a *base-pointfree* comparison between supersingular abelian surfaces over $\bar{\mathbb{F}}_q$ and mixed exceptional abelian surfaces over $\bar{\mathbb{F}}_p$. This simplifies the proof of Ribet's short exact sequence as one now does not have to take care in the choice of basepoint as one had to in [Rib90a] and this thesis.

A natural question therefor is if one can also obtain a base-point free comparision in the function field setting. A first step in this direction is that supersingular Drinfeld-modules are classified by their oriented Endomorphisms algebras:

Remark 7.2.1 *A natural analogue of this construction is given by the differential of an endomorphism of Drinfeld modules. Indeed if we denote*

$$\partial : \mathbb{F}_p^{\mathrm{alg}}[\tau] \to \mathbb{F}_p^{\mathrm{alg}}, \quad \sum_{i \geqslant 0} c_0 \tau^i \mapsto c_0.$$

Then recalling that for any Drinfeld module E we have $\mathrm{End}(E) \subset \mathbb{F}_p^{\mathrm{alg}}[\tau]$, we have

$$\partial : \mathrm{End}(E) \to \mathbb{F}_p^{\mathrm{alg}}.$$

Proposition 7.2.2 *Let E, E' be isomorphic Drinfeld-modules over $\mathbb{F}_p^{\mathrm{alg}}$. Then the orientations on their endomorphism algebras are isomorphic, that is the following diagram commutes:*

$$\begin{array}{ccc} \mathrm{End}(E) & \longrightarrow & \mathrm{End}(E') \\ & \searrow & \downarrow \\ & & \mathbb{F}_p^{\mathrm{alg}}. \end{array}$$

Proof. Let ρ be an isomorphism $E \to E'$ which we write as $c_0 \in \mathbb{F}_p^{\mathrm{alg}}\{\tau\}$. The inverse morphism is given by c_0^{-1}. The commutativity now follows from the computation

$$\partial\left(c_0 \sum_{i \geqslant 0} a_i \tau^i c_0^{-1}\right) = c_0 a_0 c_0^{-1} = a_0.$$

□

Definition 7.2.3 *Let B be a quaternion algebra. Then we denote the set of oriented maximal orders in B up to isomorphisms respecting the orientation by* $\mathrm{Max}^{\mathrm{or}}(B)$.

While the following result is certainly well known in the literature, we are unable to pin down an exact reference to it. For the convenience of the reader we shall prove it.

Proposition 7.2.4 *Let $(\mathcal{O}, \iota)$ be an oriented maximal order of a definite quaternion algebra B of discriminant $\mathfrak{p}$ over $\mathbb{F}_q(t)$. Then there exists a bijection*

$$\mathrm{Max}^{\mathrm{or}}(B) \xrightarrow{\sim} \mathrm{Pic}(\mathcal{O}).$$

Proof. For an invertible $\mathcal{O}$-module I, the associated order

$$\mathcal{O}(I) := \{x \in B \mid Ix \subset I\}$$

is a maximal order. One can use that after completion, by [Voi21, Thm 13.3.11 b)], there exists a unique maximal ideal, whose residue field lies in $\mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}}$. Thus we can define

$$\iota' : \mathcal{O}(I) \to \mathcal{O}(I)_{\mathfrak{p}} \to \mathcal{O}(I)_{\mathfrak{p}}/\mathfrak{m} \to \mathbb{F}_{\mathfrak{p}}^{\mathrm{alg}}.$$

Conversely, $(\mathcal{O}', \iota')$ be an oriented maximal order. Locally at each prime $\mathfrak{l}$, there exists a $g_\mathfrak{l} \in B_\mathfrak{l}^\times$ such that

$$\mathcal{O}'_\mathfrak{l} = g_\mathfrak{l}^{-1} \mathcal{O}_\mathfrak{l} g_\mathfrak{l}.$$

Furthermore, at $\mathfrak{p}$, we may choose $g_\mathfrak{p} \in B_\mathfrak{p}^\times$ such that

$$\iota'_\mathfrak{p}(x) = \iota_\mathfrak{p}(g_\mathfrak{p} x g_\mathfrak{p}^{-1}).$$

This gives an element of $\mathrm{Pic}(\mathcal{O}) = \mathcal{O}(\mathbb{A})^\times \backslash B(\mathbb{A})^\times / B^\times$. This locally free $\mathcal{O}$-module is well defined, as two choices $g_\mathfrak{l}, \tilde{g}_\mathfrak{l}$ must differ by an element of $\mathcal{O}_\mathfrak{l}^\times$. □

The upshot of this is that we obtain a base-point-free classification of supersingular Drinfeld modules:

Proposition 7.2.5 (Basepoint-free Deuring type result) *Let $\mathbb{F}_\mathfrak{p}^{\mathrm{alg}}$ and B be the quaternion algebra over $\mathbb{F}_p(t)$ with discriminant $\mathfrak{p} \cdot \infty$. Then we have a bijection*

$$\left(\text{sup.sing Drinfeld modules}_{\mathbb{F}_\mathfrak{p}^{\mathrm{alg}}}\right)_\sim \longrightarrow \mathrm{Max}^{\mathrm{or}}(B), \quad E \mapsto \left(\mathrm{End}(E) \xrightarrow{\partial} \mathbb{F}_\mathfrak{p}^{\mathrm{alg}}\right).$$

Proof. Let E_0 be a supersingular Drinfeld module over $\mathbb{F}_\mathfrak{p}^{\mathrm{alg}}$. Then we have a commutative diagram

$$\begin{array}{ccc} \left(\text{sup.sing Drinfeld modules}_{\mathbb{F}_\mathfrak{p}^{\mathrm{alg}}}\right)_\sim & \longrightarrow & \mathrm{Max}^{\mathrm{or}}(B) \\ & \searrow^{\sim} & \uparrow_{\sim} \\ & & \mathrm{Pic}(\mathrm{End}(E_0)) \end{array}$$

where two arrows are bijections by Thm 5.6.14 and Prop 7.2.4. □

Question 7.2.6 *Is the function field analogue of Oort-splitting true, i.e. if $\mathcal{E}$ is a Drinfeld-Stuhler module over an algebraically closed field of positive A-characteristic, is it a product of supersingular Drinfeld-modules?*

Example 7.2.7 We consider the following example of an exceptional (in fact superspecial) Drinfeld-Stuhler module (see [Pap18, Example 5.8]): let D be the unique quaternion algebra ramified over $T, T-1$ and thus

$$\mathcal{O}_D = \mathbb{F}_{q^2}[T, z]/(z^2 - T(T-1))$$

where $zT = Tz$ and for all $\alpha \in \mathbb{F}_{q^2}$ we have $T\alpha = \alpha T$. Consider the morphism $\varphi : \mathcal{O}_D \to \mathrm{Mat}_2(k[\tau])$ given by

$$\varphi(T) = \begin{pmatrix} \tau^2 & 0 \\ 0 & \tau^2 \end{pmatrix}, \quad \varphi(z) = \begin{pmatrix} 0 & \tau^2 \\ \tau^2 - 1 & 0 \end{pmatrix}, \quad \varphi(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^q \end{pmatrix}.$$

The restriction to $A \to \mathcal{O}_D \to \mathrm{Mat}_2(k[\tau])$ (i.e. forgetting the $\mathcal{O}_D$-action) gives an t-motive whose endomorphism algebra is

$$\mathrm{End}_{t\text{-mot}}(\varphi) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau^2 & 0 \\ 0 & \tau^2 \end{pmatrix} = \begin{pmatrix} \tau^2 & 0 \\ 0 & \tau^2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}$$

Hence we have $\mathrm{End}_{t\text{-mot}}(\varphi) \cong \mathrm{Mat}_2(\{a \in k[\tau] \mid a\tau^2 = \tau^2 a\})$. On the other hand, an example of a supersingular Drinfeld module is (see [KKP24, Example 2.11])

$$\psi : A \to k[\tau], \quad T \mapsto \tau^2.$$

We see that $\mathrm{End}_{t\text{-mot}}(\varphi) \cong \mathrm{Mat}_2(\mathrm{End}(\psi))$ and thus φ (as an $\mathcal{O}_D$-motive) corresponds to a morphism $\mathcal{O}_D \to \mathrm{Mat}_2(\psi)$.

One perspective of the work done by Ribet in [Rib90a] is that his short exact sequence is a geometric incarnation of the Jacquet-Langlands correspondence with integer coefficients. This idea was later adapted by Helm [Hel12] to unitary groups associated with a CM-field with signature $(1, 1)$ at all archimedean places of the totally real field of the CM-field. He considered semi-stable models of the corresponding Shimura varieties at primes of bad reduction, replacing character groups and the associated reduction theory of Jacobians by vanishing cycles and the corresponding spectral sequence on cohomology. In [Hel10] Helm considered then Shimura varieties for *arbitrary* unitary groups at primes of good reduction. On these varieties he considered a refinement of the a-number stratification. Surprisingly, the global geometry of these strata (or more precisely their desingularization) is closely related to the geometry of a Shimura variety associated to a different unitary group G'. These strata allow to give a geometric realization of the Jacquet-Langlands correspondence between these groups. This line of inquiry has been recently pursued by Tian and Xiao in [TX16] and [TX19]. In the function field case, the analogues of the a-number and therefore the Goren-Oort strata have not yet been studied extensively, and this seems an interesting avenue to pursue.

7.3 Multiple Legs and Higher Modularity

An interesting feature of shtukas is that they can have multiple legs, therefor generalizing Drinfeld modules. Shtukas with multiple legs do (at least currently) not have number analogues. However their moduli spaces play a crucial role in the work of the Lafforgues to prove the Langlands correspondence over function fields. To explain the line of inquiry we want to explain in this section, we introduce the following notation:

Definition 7.3.1 *Let X be a smooth proper curve with a geometric point $\overline{\eta}$ and corresponding function field K. Let $\mathfrak{n}$ be a divisor on X, $\mathcal{D}$ a sheaf of quaternion algebra with discriminant $\mathfrak{D}$ and denote by G the group of units of $\mathcal{D}$ over K with discriminant $\mathfrak{D}$. Let $r \geqslant 1$. Denote by* $\mathrm{Sht}^r_{\mathfrak{D}}(\mathfrak{n})$ *the* moduli space of r-legged G-shtukas with $\mathfrak{n}$-level structure. *This is a smooth scheme over X^r.*

For $r = 1$, this essentially recovers moduli spaces of Drinfeld-Stuhler modules. However for $r \geqslant 2$, the moduli spaces $\mathrm{Sht}^r_{\mathfrak{D}}(\mathfrak{n})$ are no-longer curves. In this section, we briefly review the relevant results regarding the Langlands correspondence. For a review of the current state of the art (for arbitrary general reductive groups) we refer the reader to [Laf18b]. For GL_n we refer to [Lau99], an introduction to geometric Langlands we can recommend [Fre05].

Definition 7.3.2 *A* global Langlands parameter *is a conjugacy class of morphisms*

$$\sigma : \mathrm{Gal}(K^{\mathrm{sep}}/K) \to {}^L G(\overline{\mathbb{Q}}_\ell)$$

such that σ

1. *factors through $\pi_1(U, \overline{x})$ for some open dense $U \subset X$*
2. *is defined over a finite extension of $\mathbb{Q}_\ell$*
3. *is continuous and semi-simple.*

For $G = \mathrm{GL}_n$, this simplifies to the following result:

Theorem 7.3.3 ([Laf98]) *There exists a unique bijection between isomorphism classes of global Langlands parameters and cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ pairing up the Frobenius and Hecke eigenvalues.*

The connection between these moduli space and Galois representations is for instance illustrated by the following result.

Proposition 7.3.4 ([LW22, Prop 1.3.2]) *Let X be a smooth proper curve with a geometric point $\overline{\eta}$ and corresponding function field K. For any non-isotrivial elliptic curve E of conductor $\mathfrak{n}$, denote by σ the $\pi_1(X\backslash Supp(\mathfrak{n}), \bar{\eta})$-representation $H^1(E_{\bar{K}}, \mathbb{Q}_\ell)$. Then the r-fold external tensor product $H^1(E_{\overline{K}}, \mathbb{Q}_\ell)^{\boxtimes r}$ is a quotient of $H_c^r(\mathrm{Sht}_{\mathrm{GL}_2}(\mathfrak{n})^r_{\overline{\eta}_r}, \mathbb{Q}_\ell)$.*

A natural question could be if the geometric results relating the reductions of moduli spaces of Drinfeld-Stuhler modules at different primes generalize to multiple legs. A first obstacle is that Ribet's short exact sequence is formulated in terms of character groups. It was suggested in Helm [Hel12] that the reduction theory of Jacobians which is used in [Rib90a] can be replaced with the theory of vanishing cycles and weight filtration for more general Shimura varieties. We briefly explain how to do this allowing us to pose a concrete question at the end.

Definition 7.3.5 *Let F be a non-archimedean local field with finite residue field of size q. We denote the absolute Galois group of F by G_F. A $\mathbb{Q}_\ell[G_F]$-module V is said to admit a weight filtration if it admits an increasing filtration $W_\bullet$ satisfying the following properties:*

1. *for sufficiently small i we have $W_iV = 0$ and for sufficiently large i we have $W_iV = V$*
2. *the eigenvalues of the q-th power Frobenius on $\mathrm{Gr}_i^W V$ are of complex absolute value $q^{i/2}$[1]*
3. *the inertia $I \subset G_F$ acts on all $\mathrm{Gr}_i^W V$ through a finite quotient.*

If such a filtration exists, it is indeed unique. For the purpose of this section, Weight filtrations will arise from the following example:

Example 7.3.6 *Let F be a non-archimedean local field with residue field κ and X be a smooth proper scheme over F. Then for all w, the ℓ-adic cohomology $H^w_{\text{ét}}(X_{F^{\mathrm{sep}}}, \mathbb{Q}_\ell)$ admits a weight filtration. If X admits a proper strictly semistable model $\mathcal{X}$ over $\mathcal{O}_F$, then this arises from the Rapoport-Zink spectral sequence. Denote by $\mathcal{X}^{(n)}$ the disjoint union of n-fold intersections of the irreducible components of $\mathcal{X}_\kappa$. Then we have a G_F-equivariant spectral sequence*

$$E_1^{-r,w+r} = \bigoplus_{k \geqslant \max\{0,-r\}} H^{w-r-2k}_{\text{ét}}(\mathcal{X}^{(k)}_{\kappa^{\mathrm{sep}}}, \mathbb{Q}_\ell(-r-k)) \Rightarrow H^w_{\text{ét}}(\mathcal{X}_{F^{\mathrm{sep}}}, \mathbb{Q}_\ell).$$

Furthermore, if F is of equi-characteristic, then the weight-monodromy conjecture holds*, that is there exists a natural map $E_1^{-r,n+r}(r) \to E_1^{r,n-r}$ such that for all r, w we have an isomorphism*

$$E_2^{-r,w+r}(r) \to E_2^{r,w-r}.$$

[1] after an embedding of $\bar{\mathbb{Q}}_\ell \to \mathbb{C}$

(see [Ito05, Thm 1.1]). This spectral sequence degenerates at E_2 [Ito05]. This filtration on $H^w_{\text{ét}}(X_{F^{\text{sep}}}, \mathbb{Q}_\ell)$ induces a filtration on $H^w_{\text{ét}}(X_{F^{\text{sep}}}, \mathbb{Z}_\ell)$. If we specialize X to be a curve over F admitting a proper strictly semistable model $\mathcal{C}$ over $\mathcal{O}_F$ and denote by $\text{Gr}(\mathcal{C})$ the dual graph, then we have

$$\text{Gr}_2^W H^1(C_{F^{\text{sep}}}, \mathbb{Z}_\ell) \cong \text{Gr}_2^W H^1(\mathcal{C}_{F^{\text{sep}}}, \mathbb{Z}_\ell) \cong \text{Gr}_1^W H^1(\mathcal{C}_{F^{\text{sep}}}, \mathbb{Z}_\ell)(-1)$$

$$\text{Gr}_0^W H^1(\mathcal{C}_{F^{\text{sep}}}, \mathbb{Z}_\ell) \cong H^1(\text{Gr}(\mathcal{C}), \mathbb{Z}_\ell) \cong H^1(\text{Gr}(\mathcal{C}), \mathbb{Z}) \otimes \mathbb{Z}_\ell.$$

Since $\mathcal{X}(\mathcal{C}) \cong H_1(\text{Gr}(\mathcal{C}), \mathbb{Z})$, we recover (up to duality) the ℓ-adic part of the character group and its associated monodromy pairing.

Motivated by this example, we could therefor hope that the graded pieces of $H_c^r(\text{Sht}^r_{\mathfrak{D},\overline{\eta_r}}, \mathbb{Q}_\ell)$ satisfy an analogue of Ribet's short exact sequence. However the moduli spaces of shtukas need not be proper schemes. Thus to define the weight filtration, one has to use the construction provided in [HS06].

Question 7.3.7 *For any prime $\mathfrak{p} \in |X|$, denote by $\mathfrak{p}_r \in X^r$ the corresponding point under the diagonal embedding. We then define*

$$\mathcal{X}^r(\text{Sht}^r_{\mathfrak{D}}(\mathfrak{M}), \mathfrak{p}) := \text{Gr}_0^W \left(H_c^r(\text{Sht}^r_{\mathfrak{D}}(\mathfrak{M})_{K_{\mathfrak{p}}^{\text{sep}}}, \mathbb{Z}_\ell) \right).$$

Is there a r-legged analogue of Ribet's short exact sequence, that is

$$0 \leftarrow \mathcal{X}^r(\text{Sht}^r_{\mathfrak{D}_{\mathfrak{pq}}}(\mathfrak{M}), \mathfrak{p}) \leftarrow \mathcal{X}^r(\text{Sht}^r_{\mathfrak{D}}(\mathfrak{pq}\mathfrak{M}), \mathfrak{q}) \leftarrow \mathcal{X}^r(\text{Sht}^r_{\mathfrak{D}}(\mathfrak{q}\mathfrak{M}), \mathfrak{q})^2 \leftarrow 0$$

One can also discuss the possibility of an r-legged generalization of the modular degree. For this we briefly introduce the notion of "higher modularity" as introduced by J. Weinstein and developed with A. Logan in [LW22]. The existence of modular parametrization of elliptic curves follows from the combination

- the fact that Galois representation of E occurs in the cohomology of the Drinfeld modular curve
- and the fact that the Tate conjecture is known for curves.

In particular there exists a non-zero $\pi_1((X \backslash \text{Supp}(\mathfrak{n}))^r, \overline{\eta_r})$-invariant element in

$$H_c^r \left(\text{Sht}^r_{\overline{\eta_r}}(\mathfrak{n}) \times E^r_{\overline{\eta_r}}, \mathbb{Q}_\ell \right).$$

Hence a sufficiently strong result towards the Tate conjecture would predict the existence of an algebraic correspondence Z. This leads to the notion of *higher modularity*: we say an elliptic curve E is r*-modular* if there exists an algebraic cycle realizing Prop 7.3.4. Conjecturally, all non-isotrivial elliptic curves are r-modular for all r. For $r \geqslant 2$, the only known examples are:

1. elliptic curves E of K which have a non-isotrivial extremal rational elliptic fibration model $\mathcal{E} \to X$ are 2-modular [LW22, Thm 1.2.4]

2. for $q \neq 2$,the elliptic curve E over $\mathbb{F}_q(t)$ with Weierstrass equation

$$y^2 = x(x-1)(x-t)$$

   is 3-modular [LW22, Thm 1.4.3]

From the perspective of this thesis, the following question is natural:

Question 7.3.8 *What is the "correct" notion of the "modular degree of an r-modular parametriation?*

We should point out that such a cycle is **not** unique, only unique up to equivalence, hence such a "modular degree" would have to be invariant under equivalence of cycles. If such a notion was properly defined, it would be natural to try understand its *arithmetic significance:*

Question 7.3.9 *Which data is encoded in the r-modular degree of an elliptic curve?*

There are two possible perspective on this: for one, it was shown in a series of papers [YZ17], [YZ19], by Z. Yun and W. Zhang that moduli spaces of shtukas have "higher derivative"-information about L-functions. The type of result they show is that the r-th central derivative L-function associated to an everywhere unramified cuspidal automorphic representation π of PGL_2 is related to the self-intersection product of a π-isotypic component on Sht^r (this can thus be thought of as a "higher derivative" version of the Gross-Zagier theorem). Given that the modular degree (up to well understood constants), is equal to $L(1, Sym^2\pi_E)$, a guess would be that the "r-modular degree" corresponds to $L^{(r)}(1, Sym^2\pi_E)$. Given that the modular degree computes the congruence primes, it would also be interesting to see if there exists a notion of r-congruence primes related to such a modular degree.

```
In [1]: def poly_sigma(n,k): #function computing sigma_k for polynomial n
            S = PolynomialRing(GF(5), 'x')
            sigma=S(1)
            #factorlist=n.factor_list()
            factorlist=n.factor()
            for i in range(0,len(factorlist)):
                a=S(factorlist[i][0])
                b=factorlist[i][1]
                sigma=sigma*(a^(b*k+k)-1)/(a^k-1)


            return sigma
```

```
In [2]: def partition_prime_test(b):
            R = Zp(5, prec=10)  #define 5-adics, allowing to have access to sage's expansion
            S = PolynomialRing(GF(5), 'x') # ring in which we will test prime-detection
            x= S.gen()
            for i in range(1,b): #we now test the conjecture for the i-th polynomial
                t=R(i)
                poly=S(0)
                for j in range(0,10): #this loop generates the i-th polynomial in S
                    poly=poly+S(t.expansion()[j])*x^j

                if (poly^2-poly+1)*poly_sigma(poly,1)==poly_sigma(poly,3): #this is the criterion
                       if poly.is_irreducible()==False:  #if it were to pass the criterion and not
                            # be prime, then this poly would break the conjecture
                            print(poly)
            print("done")
```

Figure VII.1: Sage-code to test partition over function fields

7.4 Partitions may know irreducibles

The following is inspired by the work of [CvIO24]. In this work, the authors prove that the partitions functions of integers are capable of detecting primes. While they create an infinite family of such conditions, to give a flavor we mention an explicit one:

Theorem 7.4.1 (Cor of [CvIO24, Thm 1.1]) *For any integer n, we have*

$$(n^2 - n + 1)\sigma_1(n) = \sigma_3(n)$$

if and only if n is prime or $n = 1$.

Notice that if f is an irreducible polynomial in $\mathbb{F}_p[t]$, then we have

$$(f^2 - f + 1)\sigma_1(f) = (f^2 - f + 1)(f + 1) = f^3 + 1 = \sigma_3(f).$$

Thus it would be natural to ask whether the function field analogue of the work of [CvIO24] holds true. I tested this for $p = 5$ using Sage (see Figure VII.1 for the code).

The criterion is valid for the first 3'000'000 polynomials, i.e. up to (and including) degree 9. This ran on a laptop for 15 minutes. It thus seems plausible to ask for the following:

Question 7.4.2 *Let $f \in \mathbb{F}_p[t]$ such that*

$$(f^2 - f + 1)\sigma_1(f) = \sigma_3(f).$$

Does this imply that f is irreducible?[2]

A key technical result of [CvIO24] is the following Lemma:

Lemma 7.4.3 ([CvIO24, Lemma 2.1]) *Let k, l be non-negative odd integers with $l > k$. For $n \geqslant 2$, the n-Fourier coefficient of the quasi-modular form*

$$f_{k,l} := (D^l + 1)E_{k+1} - (D^k + 1)E_{l+1}$$

vanishes if and only if n is prime.

The proof of this lemma is an explicit expression of the n-th Fourier coefficient. To copy this strategy, it should be mentioned that a theory of *Drinfeld quasi-modular form* was developed in [BP08]. The key would be that there exists a version of Fourier theory which makes sense of the "Fourier coefficient at an irreducible polynomial". For this one needs the notion of an A-expansion, for whose exposition we follow [Pet15]. We denote by A_+ the set of monic elements.

Definition 7.4.4 ($\mathfrak{n}$-th Goss polynomial) *Let $\tilde{\pi} \in \mathbb{C}_\infty$ the Carlitz period. For any $n \geqslant 0$, we denote by*

$$E_n(z) = \frac{1}{\tilde{\pi}} \sum_{a \in A} \frac{1}{z + a}.$$

Definition 7.4.5 *A function $f : \Omega \to \mathbb{C}_\infty$ is said to have an A-*expansion of exponent n *if there exists a positive integer n and for each monic element $a \in A_+$ an element $c_a \in \mathbb{C}_\infty$ such that*

$$f = \sum_{A_+} c_a E_a.$$

Definition 7.4.6 *Let $f : \Omega \to \mathbb{C}_\infty$ be a rigid analytic function. The n-*th unaltered hyperderivative *$\mathcal{D}_n f(z)$ is defined via*

$$f(z + \varepsilon) = \sum_{n \geqslant 0} \mathcal{D}_n f(z) \varepsilon^n.$$

If $\tilde{\pi}$ is the Carlitz period, then the n-th hyperderivative is

$$D_n(f) := \frac{(-1)^n}{\tilde{\pi}^n} \mathcal{D}_n(f).$$

[2] It seems plausible to assume that $p = 2, 3$ might produce counter-examples.

Thus a first step should be

Question 7.4.7 *Does $(D_l+1)E_{k+1}-(D_k+1)E_{l+1}$ admit an A-expansion? If so, denote its A-expansion by c_a. Does this A-expansion satisfy*

$$c_a\left((D_l+1)E_{k+1}-(D_k+1)E_{l+1}\right)=0 \overset{?}{\iff} a \text{ is irreducible.}$$

Remark 7.4.8 It is not clear to the author that more "elementary" methods are not enough to answer the question raised in this section.

7.5 Lifting Drinfeld modular forms to mixed characteristic

Using Prop 3.5.13 and Prop 3.5.9, we get a natural morphism

$$\mathcal{H}_!(\mathcal{T},\mathbb{Z})^\Gamma \to \mathcal{H}_!(\mathcal{T},\mathbb{Z}/p)^\Gamma \to H^0(X(\Gamma),\Omega^1).$$

Based on our intuition from "classical" differential forms, it would be interesting to have an interpretation of harmonic cochains as a version of differential forms. Indeed the reduction "mod p" of harmonic cochains can be seen as such. A natural way to associate a characteristic zero object to a geometric object in positive characteristic is to study its deformations to $\mathbb{Z}_p$. The group $\mathcal{H}_!(\mathcal{T},\mathbb{Z})^\Gamma$ is abelian and free of rank $g=\text{genus}(X(\Gamma))$.
Likewise, differential forms admit natural "thickenings" to $\mathbb{Z}_p/p^r$-modules, namely the de Rham-Witt differential forms $H^0(X(\Gamma),W_r\Omega^1)$. The de Rham-Witt differential forms can be thought of the differential forms of a thickening of $X(\Gamma)$ to $\mathbb{Z}_p/p^r$. For a good survey on this we refer the reader to [Ill83]. Using an integral model $\mathcal{X}(\Gamma)$ and a prime $\mathfrak{p}$ of good reduction, one can consider

$$H^0(\mathcal{X}(\Gamma)_{\mathbb{F}_\mathfrak{p}},W\Omega^1).$$

Alternatively, one can consider

$$H^0(X(\Gamma)_{\mathbb{F})q(t)^{\text{perf}}},W\Omega^1).$$

These are Hecke-modules, as Hodge-Witt cohomology gets acted upon by correspondences (see for instance [CR12]). Since $\mathcal{H}_!(\mathcal{T},-)^\Gamma$ satisfies flat base change (see [GR96, Rem 3.6.2]), we have

$$\mathcal{H}_!(\mathcal{T},W(\mathbb{F}_\mathfrak{p}))^\Gamma \cong \mathcal{H}_!(\mathcal{T},\mathbb{Z})^\Gamma \otimes_\mathbb{Z} W(\mathbb{F}_\mathfrak{p}).$$

Question 7.5.1 *Are there Hecke-equivariant isomorphism*

$$\mathcal{H}_!(\mathcal{T}, W(\mathbb{F}_\mathfrak{p}))^\Gamma \cong H^0(\mathcal{X}(\Gamma)_{\mathbb{F}_\mathfrak{p}}, W\Omega^1)$$

$$\mathcal{H}_!(\mathcal{T}, W(\mathbb{F})q(t)^{\text{perf}}))^\Gamma \cong H^0(X(\Gamma)_{\mathbb{F})q(t)^{\text{perf}}}, W\Omega^1).$$

By [Eke86, Proposition VI.3.2] we know that for smooth proper curves C we have

$$\text{rk}H^0(C, W\Omega^1) = \text{rk}H^1(C, W\mathcal{O}).$$

If C is such that for all i, j, the $W(k$-modules $H^i(C, W\Omega^j)$ are p-torsion free, then by [Ill79, Rem 4.10]

$$H^n(C, W) = \oplus_{i \geqslant 0} H^{n-i}(C, W\Omega^i).$$

Thus we would have $g = \text{rk}H^0(C, W\Omega^1)$.
For any Y/k which is smooth and proper, there exists a spectral sequence

$$E_1^{i,j} = H^j(Y, W\Omega_Y^i) \Rightarrow H^{i+j}_{\text{cris}}(Y, W(k))$$

(called the *slope spectral sequence*). In particular, this yields

$$H^j(Y, W\Omega_Y^i) \otimes \text{Quot}(W(k)) \cong \left(H^{i+j}_{\text{cris}}(Y, W(k)) \otimes \text{Quot}(W(k))\right)_{[i,i+1)}$$

or even more concretely

$$H^0(Y, W\Omega_Y^1) \otimes \text{Quot}(W(k)) \cong \left(H^1_{\text{cris}}(Y, W(k)) \otimes \text{Quot}(W(k))\right)_{[1,2)}.$$

Thus, at least rationally, $H^0(Y, W\Omega_Y^1)$ corresponds to a subspace of crystalline cohomology defined by a condition of Frobenius. However we saw that the component group is isomorphic to a graded piece of the weight filtration of ℓ-adic cohomology. One could therefor even speculate about the existence of an analogue of Raynaud's theorem

Question 7.5.2 *Let C be a curve over a DVR $\mathcal{O}$ with residue field κ. Let $\mathcal{C}$ be a regular semi-stable model of C and denote by* $\text{Gr}(\mathcal{C})$ *the dual graph thereof. Do we have an isomorphism*

$$H^1(\text{Gr}(\mathcal{C}), W(k)) \cong \left(H^1_{\text{cris}}(C, W(k)) \otimes \text{Quot}(W(k))\right)_{[1,2)}.$$

APPENDIX A

Strict Group Schemes

1.1 Strict Schemes

Definition 1.1.1 *We define the category* $\mathrm{DSch}_{\mathrm{Spec}\, R}$ *whose objects are all triples* $(\mathrm{Spec}\, S, \mathrm{Spec}\, S^\flat, \iota)$ *where*

1. S *is an augmented finite locally free* R*-algebra*
2. $S^\flat$ *is an augmented* R*-algebra*
3. $\iota : S^\flat \to S$ *is an surjeciton of augmented* R*-algebras*

such that there exists an epimorphism $R[x_1, \ldots, x_n] \xrightarrow{j} S^\flat$ *satisfying the following properties:*

1. $\ker(\iota \circ j) \subset R[x_1, \ldots, x_n]$ *is generated by a regular sequence of length* n*,*
2. $\ker j = (x_1, \ldots, x_n) \cdot \ker(\iota \circ j)$.

A morphism between $(\mathrm{Spec}\, A, \mathrm{Spec}\, A^\flat, \iota_A)$,$(\mathrm{Spec}\, B, \mathrm{Spec}\, B^\flat, \iota_B)$ *consists of morphism of augmented* R*-algebras*

$$A \xrightarrow{f} B, A^\flat \xrightarrow{f^\flat} B^\flat$$

such that the following diagram commutes

$$\begin{array}{ccc} A^\flat & \xrightarrow{\iota_A} & A \\ \downarrow{\scriptstyle f^\flat} & & \downarrow{\scriptstyle f} \\ B^\flat & \xrightarrow{\iota_B} & B \end{array}$$

Example 1.1.2 *Let* R *be a* $\mathbb{F}_q$*-algebra. Then*

$$(\mathrm{Spec}\, R[x]/(x^q), \mathrm{Spec}\, R[x]/(x^{q+1}), \iota)$$

$$(\operatorname{Spec} R[x]/(x^q - x), \operatorname{Spec} R[x]/(x^{q+1} - x^2), \iota)$$

where in both cases ι denotes the natural closed embedding, are objects in DSch_R.

Definition 1.1.3 *Let* $(\operatorname{Spec} A, \operatorname{Spec} A^\flat, \iota_A) \in \mathrm{DSch}_R$. *Denote by* $I_{A^\flat}$ *the kernel of the augmentation* $\varepsilon_{A^\flat} : A^\flat \to R$. *Then we define two* R*-modules*

$$N_{(\operatorname{Spec} A, \operatorname{Spec} A^\flat, \iota_A)} = \ker \iota_A$$

$$t^+_{(\operatorname{Spec} A, \operatorname{Spec} A^\flat, \iota_A)} := I_{A^\flat}/I^2_{A^\flat}.$$

Definition 1.1.4 *We denote by* DGr_R *the subcategory of* DSch_R *consisting of the group objects with morphisms of group objects.*

Remark 1.1.5 If $(G, G^\flat, \iota_G)$ is a group object in DSch_R, then $G = \operatorname{Spec} S$ is a a finite locally free group scheme over R. However $G^\flat$ does not need to be a group scheme over R, as the comultiplication does not need to lift

$$\begin{array}{ccc} S^\flat & \xrightarrow{\Delta^\flat} & (S \otimes_R S)^\flat \\ & \overset{??}{\searrow} & \uparrow \\ & & S^\flat \otimes_R S^\flat \end{array}$$

Definition 1.1.6 *Let* $(G, G^\flat, \iota_G) \in \mathrm{DSch}_R$. *A* strict $\mathcal{O}$-action on $(G, G^\flat, \iota_G)$ *is a homomorphism*

$$\mathcal{O} \to \operatorname{End}_{\mathrm{DSch}_R}((G, G^\flat, \iota_G))$$

such that the induced action on $\ell^\bullet_{(G,G^\flat,\iota_G)/\operatorname{Spec} R}$ *is equal to the scalar multiplication.*

Remark 1.1.7 The co-Lie complex $\ell^\bullet_{(G,G^\flat,\iota_G)/\operatorname{Spec} R}$ depends on the deformation $G^\flat$ of G. However, for another deformation $\tilde{G}^\flat$ of G such that $(G, \tilde{G}^\flat, \iota_G) \in \mathrm{DSch}_R$, we have a homotopy equivalence

$$\ell^\bullet_{(G,G^\flat,\iota_G)/\operatorname{Spec} R} \to \ell^\bullet_{(G,\tilde{G}^\flat,\iota_G)/\operatorname{Spec} R}.$$

In fact these complexes are homotopy equivalent to the co-Lie complex of G over $\operatorname{Spec} R$, that is the pullback of the cotangent complex $L_{G/\operatorname{Spec} R}$ along the unit section. This raises the question if one can weaken Def 1.1.6 to the induced action of $\mathcal{O}$ on $\ell^\bullet_{(G,G^\flat,\iota_G)/\operatorname{Spec} R}$ being homotopic to the scalar multiplication. This is true for $\mathcal{O} = \mathbb{F}_p[x]$ according to [HS19, Rem 4.5].

Definition 1.1.8 *A finite locally free $\mathcal{O}$-module scheme G over R is said to be a* strict $\mathcal{O}$-module scheme *if there exists a $(G, G^\flat, \iota_G)\mathrm{DGr}_{\mathrm{Spec}\, R}$ which admits a strict $\mathcal{O}$-action lifting the $\mathcal{O}$-action on G.*

Proposition 1.1.9 (Lemma 4.9) *The property of being a strict $\mathcal{O}$-module is Zariski local on* $\mathrm{Spec}\, R$.

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