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TITLE SOME EFFECTS OF SURFACE CURVATURE ON LAMINAR BOUNDARY-LAYER FLOW

AUTHOR JAMES SIDNEY MURPHY

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SOME EFFECTS OF SURFACE CURVATURE ON LAMINAR BOUNDARY-LAYER FLOW

by

James S. Murphy

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Committee in charge:

Professor Arnold M. Kuethe, Cochairman
Associate Professor Robert C. F. Bartels, Cochairman
Associate Professor Paul F. Chenea
Associate Professor Julius D. Schetzer
Dr. John R. Sellars



PREFACE

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CHAPTER I

INTRODUCTION

The flow of a viscous fluid has been the subject of a large amount of investigation during the past fifty years. Until recently, most work on the subject has dealt with the incompressible case, but the high speeds of today's aircraft have brought about the necessity of determining the effects of compressibility on the flow of a viscous fluid. Some of these effects have been treated theoretically, but due to the nonlinear nature of the equations governing the motion of both compressible and incompressible flows, complete analyses are limited to a few special cases.

Hamel showed that if the streamlines of a viscous flow are restricted to coincide with those of a potential flow, the resulting viscous motion has streamlines in the form of logarithmic spirals (Ref. 1). The special cases for these flows are:

(a) radial flow through converging or diverging two-dimensional channels; and

(b) flow in concentric circles.

Solutions of complete equations of motion have been obtained for both types of motion. The general solution for type (a) can be found for converging walls, but the solution for diverging walls is restricted to small angular openings.

Another solution of complete equations of motion was obtained by von Karman for the case of steady motion of a viscous, incompressible fluid about a rotating disk (Ref. 2).

In addition, approximations to complete solutions have been obtained for flows of more general interest. The first successful attempt at finding such approximations was made by Prandtl. In 1904, on the basis of physical experience, he introduced the concept of boundary-layer flow (Ref. 3). By means of an order-of-magnitude analysis, he proceeded to show which terms in the complete equations of motion were negligibly small for practical purposes, and thus obtained a set of equations which have been solved for certain flows. These equations have become known as the boundary-layer equations of motion. The first solution of the boundary-layer equations was obtained by Blasius for the steady motion of a viscous, incompressible fluid past a flat surface (Ref. 4).

Prior to the work of Prandtl and Blasius, Reynolds had shown experimentally that there are two regimes of viscous fluid flow. He designated them as laminar and turbulent, and noted that the laminar flow always preceded the turbulent flow; consequently, he called the region of change from one type to the other the transition region. Reynolds also was the first to find the similarity parameter for viscous flows. As a result, this parameter is known as the Reynolds number. Small Reynolds numbers indicate laminar flow of a viscous fluid; larger values indicate turbulent flow. Because turbulent flow is unsteady, with local fluctuations about some mean velocity, it is more difficult to treat theoretically than is the laminar flow. However, considerable progress has been made since the beginning of this century towards understanding both regimes of viscous fluid flow.

As would be expected, more results have been obtained for laminar than for turbulent motions. Solutions of the boundary-layer equations for laminar flow over two-dimensional curved surfaces have been formulated which take into consideration the pressure gradient along the surface, but

which neglect the other effects of surface curvature. One contribution to the literature on such flows was made by Howarth, who showed the influence of surface pressure gradient on laminar motions over the general curved surface, and summarized the state of knowledge on the subject up to the year 1934 (Ref. 5).

The most important effect of surface pressure gradient is the phenomenon of separation, which occurs for positive surface pressure gradient. When separation occurs, the fluid initially adjacent to the surface moves out into the main body of fluid, and the assumptions involved in the boundary-layer equations of motion are thereby invalidated. In addition, separation alters the surface pressure distribution from that predicted by the potential flow about the body. As a consequence, a solution of the boundary-layer equations using the potential flow surface pressure distribution does not accurately predict the location of separation. Nevertheless, successful prediction of the location of separation is obtained with these equations for blunt bodies, provided the potential flow pressure distribution is replaced by an experimentally observed distribution. However, the corresponding semi-empirical results for streamline bodies with separation do not in general yield a satisfactory prediction of the location of the point of separation. A more careful analysis of the general problem of flow over two-dimensional curved surfaces is therefore necessary.

In the belief that the effects of surface curvature, other than the surface pressure gradient, might prove to have a bearing on separation, and in order to determine how these effects influence the wall stress, the work presented here was initiated.

Recent observations on the flow in supersonic nozzles having large convex surface curvature near the throat indicate that the laminar

boundary-layer separates there even though the pressure gradient is favorable. While results of this investigation do not apply directly to compressible flow, the general trends for incompressible flow will probably apply to compressible flow. Hence some of the results presented here may be of assistance in the solution of certain engineering problems.

CHAPTER II

DERIVATION OF EQUATIONS OF MOTION

Section 1. Two-Dimensional Motion

We consider the motion of a viscous incompressible fluid. In general vector notation, the equation governing this motion is

$$\frac{\partial \bar{V}}{\partial t} - \bar{V} \times \bar{w} = - \operatorname{grad}\left(\frac{p}{\rho} + \frac{1}{2}\bar{V}^2\right) - \nu \operatorname{curl} \bar{w} , \qquad (2.1)$$

where $\bar{V}$ = velocity vector, and

$\bar{w}$ = curl $\bar{V}$.

The condition of continuity must be satisfied in addition. Thus we have the equation

$$\operatorname{div} \bar{V} = 0 . \qquad (2.2)$$

Let x_i $(i = 1,2,3)$ be general curvilinear coordinates, and let u_i be the components of $\bar{V}$ in directions of increasing x_i. Then the components of $\bar{w}$ are,

$$\begin{aligned} w_1 &= \frac{1}{h_2 h_3}\left\{\frac{\partial}{\partial x_2}(h_3 u_3) - \frac{\partial}{\partial x_3}(h_2 u_2)\right\} \\ w_2 &= \frac{1}{h_3 h_1}\left\{\frac{\partial}{\partial x_3}(h_1 u_1) - \frac{\partial}{\partial x_1}(h_3 u_3)\right\} \qquad (2.3) \\ w_3 &= \frac{1}{h_1 h_2}\left\{\frac{\partial}{\partial x_1}(h_2 u_2) - \frac{\partial}{\partial x_2}(h_1 u_1)\right\} , \end{aligned}$$

where

$$h_1\, dx_1 , \qquad h_2\, dx_2 , \qquad \text{and} \qquad h_3\, dx_3 \qquad (2.4)$$

are elements of length in the directions of increasing x_i, respectively. The components of grad are

$$\frac{1}{h_1}\frac{\partial}{\partial x_1}\ , \qquad \frac{1}{h_2}\frac{\partial}{\partial x_2}\ , \qquad \text{and} \qquad \frac{1}{h_3}\frac{\partial}{\partial x_3}\ ; \tag{2.5}$$

and the components of curl $\bar{w}$ are

$$\begin{aligned}
\theta_1 &= \frac{1}{h_2 h_3}\left\{\frac{\partial}{\partial x_2}(h_3 w_3) - \frac{\partial}{\partial x_3}(h_2 w_2)\right\} \\
\theta_2 &= \frac{1}{h_3 h_1}\left\{\frac{\partial}{\partial x_3}(h_1 w_1) - \frac{\partial}{\partial x_1}(h_3 w_3)\right\} \qquad (2.6) \\
\theta_3 &= \frac{1}{h_1 h_2}\left\{\frac{\partial}{\partial x_1}(h_2 w_2) - \frac{\partial}{\partial x_2}(h_1 w_1)\right\} \ .
\end{aligned}$$

Finally, the components of $\bar{V} \times \bar{w}$ are

$$u_2 w_3 - u_3 w_2\ ; \qquad u_3 w_1 - u_1 w_3\ ; \qquad \text{and} \qquad u_1 w_2 - u_2 w_1\ . \tag{2.7}$$

The general equations will now be specialized to those governing the two-dimensional motion past a cylindrical body which has a curved surface. The derivation below closely follows that presented in Ref. 6. Take x_3 as the coordinate along the cylinder axis, which is at right angles to the plane of motion. Then $u_3 = 0$ and all quantities are independent of x_3 for the motion under consideration.

Let x_1 = constant be curves normal to the surface and x_2 = constant be curves parallel to the cylinder surface which intersect the normals at equal distances from the surface as shown in Fig. 1.

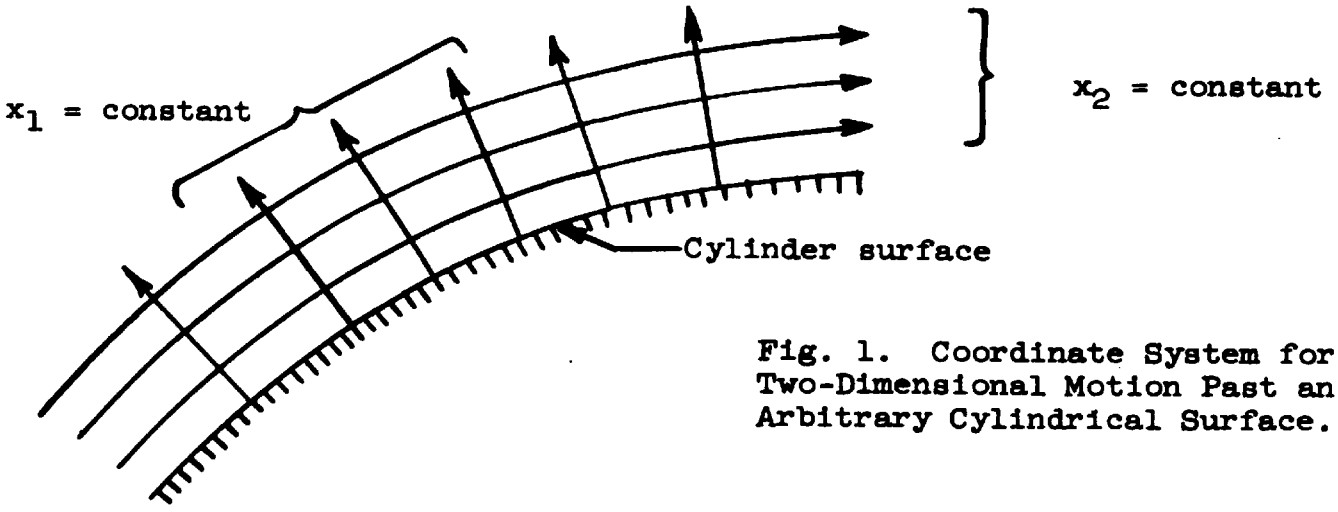


Fig. 1. Coordinate System for Two-Dimensional Motion Past an Arbitrary Cylindrical Surface.

Replace x_1 by x and x_2 by y, and take $u_1 = u$, and $u_2 = v$; also, denote the curvature of the wall by the function, $K = K(x)$. We shall suppose that $K(x)$ is a continuous function of x and that it is positive for convex surfaces and negative for concave surfaces. Then the elements of length in the directions parallel and normal to the surface are, respectively,

$$(1 + Ky)\,dx \qquad \text{and} \qquad dy \;; \tag{2.8}$$

and from (2.4), we have

$$h_1 = 1 + Ky \qquad \text{and} \qquad h_2 = 1 \;. \tag{2.9}$$

Therefore, by (2.3), the components of $\bar{w}$ are

$$w_1 = w_2 = 0 \;; \quad w_3 = \frac{1}{1 + Ky}\left\{\frac{\partial v}{\partial x} - \frac{\partial}{\partial y}\left[(1 + Ky)u\right]\right\} \;. \tag{2.10}$$

From (2.5), the components of grad $(\frac{1}{2}\bar{V}^2)$ are

$$\frac{1}{1 + Ky}\left[u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right] \qquad \text{and} \qquad \left[u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\right] \;; \tag{2.11}$$

and, by (2.6), the components of curl $\bar{w}$ are

$$\theta_1 = -\frac{K(\partial v/\partial x)}{(1+Ky)^2} + \frac{1}{1+Ky}\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} + \frac{K^2 u}{(1+Ky)^2} - \frac{K}{1+Ky}\frac{\partial u}{\partial y}$$

$$\theta_2 = -\frac{1}{1+Ky}\left[\frac{-y(\partial K/\partial x)}{(1+Ky)^2}\frac{\partial v}{\partial x} + \frac{1}{1+Ky}\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} - \frac{u}{1+Ky}\frac{\partial K}{\partial x} + \frac{Kyu}{(1+Ky)^2}\frac{\partial K}{\partial x} - \frac{K}{1+Ky}\frac{\partial u}{\partial x}\right] ; \tag{2.12}$$

while, by (2.7), the components of $\bar{V} \times \bar{w}$ are

$$\left[\frac{v}{1+Ky}\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial y} - \frac{Kuv}{1+Ky}\right] \quad \text{and} \quad \left[-\frac{u}{1+Ky}\frac{\partial v}{\partial x} + u\frac{\partial u}{\partial y} + \frac{u^2 K}{1+Ky}\right] . \tag{2.13}$$

The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\left[(1 + Ky)\,v\right] = 0 \;. \tag{2.14}$$

The first component of the equation of motion (2.1) is

$$\frac{\partial u}{\partial t} - \frac{v}{1+Ky}\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial y} + \frac{K u v}{1+Ky} + \frac{1}{1+Ky}\left[u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right]$$

$$= -\frac{1}{1+Ky}\frac{1}{\rho}\frac{\partial p}{\partial x} - \nu\left[\frac{-K}{(1+Ky)^2}\frac{\partial v}{\partial x} + \frac{1}{1+Ky}\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} + \frac{K^2 u}{(1+Ky)^2} - \frac{K}{1+Ky}\frac{\partial u}{\partial y}\right] \tag{2.15}$$

From (2.14), it follows that

$$-\frac{1}{1+Ky}\frac{\partial^2 v}{\partial x \partial y} = -\frac{y}{(1+Ky)^3}\frac{\partial K}{\partial x}\frac{\partial u}{\partial x} + \frac{1}{(1+Ky)^2}\frac{\partial^2 u}{\partial x^2} + \frac{v}{(1+Ky)^2}\frac{\partial K}{\partial x} - \frac{Kyv}{(1+Ky)^3}\frac{\partial K}{\partial x} + \frac{K}{(1+Ky)^2}\frac{\partial v}{\partial x} \quad . \tag{2.16}$$

Substitution of (2.16) into (2.15) yields the following for the first component of the equation of motion:

$$\frac{\partial u}{\partial t} + \frac{1}{1+Ky}u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{K u v}{1+Ky} = -\frac{1}{\rho}\frac{1}{1+Ky}\frac{\partial p}{\partial x} + \nu\left[\frac{1}{(1+Ky)^2}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{y}{(1+Ky)^3}\frac{\partial K}{\partial x}\frac{\partial u}{\partial x} + \frac{K}{1+Ky}\frac{\partial u}{\partial y} - \frac{K^2 u}{(1+Ky)^2} + \frac{2K}{(1+Ky)^2}\frac{\partial v}{\partial x} + \frac{v}{(1+Ky)^3}\frac{\partial K}{\partial x}\right]. \tag{2.17}$$

Similarly, the second component of the equation of motion is

$$\frac{\partial v}{\partial t} + \frac{1}{1+Ky}u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} - \frac{Ku^2}{1+Ky} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left[\frac{1}{(1+Ky)^2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} - \frac{y}{(1+Ky)^3}\frac{\partial K}{\partial x}\frac{\partial v}{\partial x} + \frac{K}{1+Ky}\frac{\partial v}{\partial y} - \frac{K^2 v}{(1+Ky)^2} - \frac{u}{(1+Ky)^3}\frac{\partial K}{\partial x} - \frac{2K}{(1+Ky)^2}\frac{\partial u}{\partial x}\right]. \tag{2.18}$$

Equations (2.14), (2.17), and (2.18) determine the two-dimensional motion past a cylinder whose axis is perpendicular to the plane of the motion. The shape of the cylinder is determined by $K = K(x)$, a known function of x.

Section 2. Three-Dimensional Motion

Now let us consider the motion which occurs when the cylinder ends in a surface lying in the plane of the motion, as shown in Fig. 2.

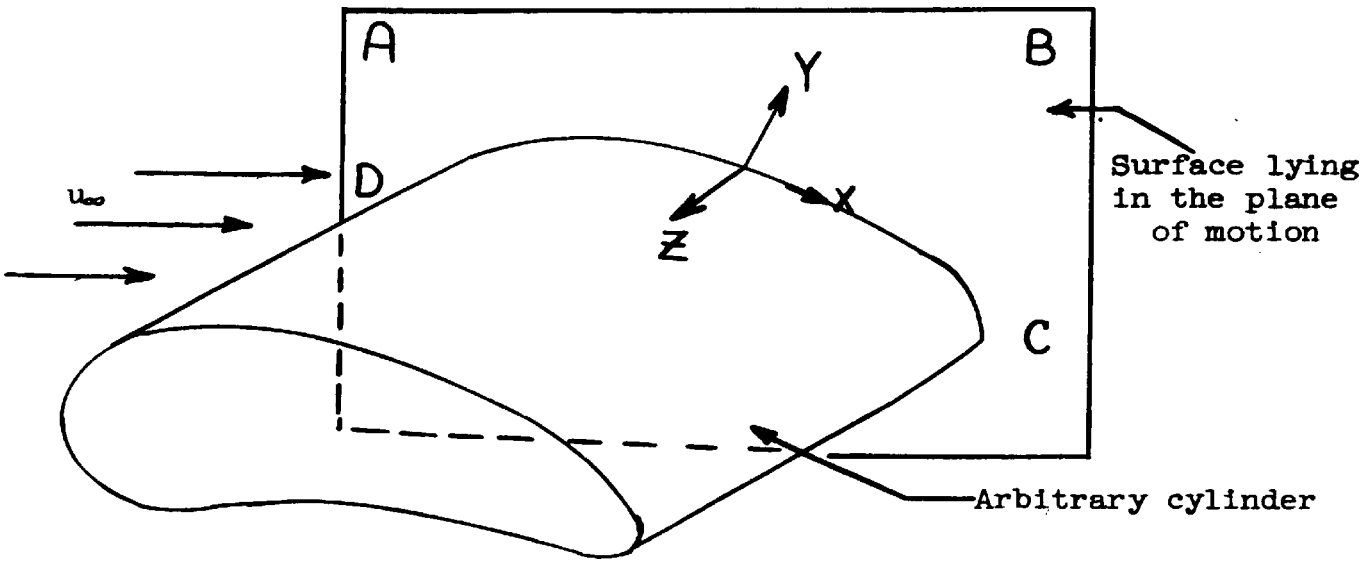


Fig. 2.

Again take x = constant as curves normal to the surface and y = constant as curves parallel to the surface, both of which lie in the plane z = constant. The coordinate system is shown in Fig. 2.

Then the elements of length in directions x, y, and z, respectively, are

$$(1+Ky)\,dx \quad , \qquad dy \quad , \qquad \text{and} \qquad dz \quad . \tag{2.19}$$

Now denote the velocity components in the directions of x, y, and z, respectively, by u, v, and w. Then the components of $\bar{w}$ are

$$\begin{aligned} w_1 &= \left[\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right] \\ w_2 &= \frac{1}{1+Ky}\left[(1+Ky)\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right] \\ w_3 &= \frac{1}{1+Ky}\left[\frac{\partial v}{\partial x} - (1+Ky)\frac{\partial u}{\partial y} - Ku\right] . \end{aligned} \tag{2.20}$$

The components of grad $(\frac{1}{2}\bar{V}^2)$ are

$$\frac{1}{1+Ky}\left[u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}+w\frac{\partial w}{\partial x}\right],$$

$$\left[u\frac{\partial u}{\partial y}+v\frac{\partial v}{\partial y}+w\frac{\partial w}{\partial y}\right], \qquad (2.21)$$

and

$$\left[u\frac{\partial u}{\partial z}+v\frac{\partial v}{\partial z}+w\frac{\partial w}{\partial z}\right];$$

and the components of curl $\bar{w}$ are

$$\theta_1 = \frac{-K}{(1+Ky)^2}\frac{\partial v}{\partial x}+\frac{1}{1+Ky}\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 u}{\partial y^2}+\frac{K^2u}{(1+Ky)^2}-\frac{K}{1+Ky}\frac{\partial u}{\partial y}-\frac{\partial^2 u}{\partial z^2}+\frac{1}{1+Ky}\frac{\partial^2 w}{\partial x\partial z}$$

$$\theta_2 = \frac{\partial^2 w}{\partial z\partial y}-\frac{\partial^2 v}{\partial z^2}-\frac{1}{1+Ky}\left[\frac{-y}{(1+Ky)^2}\frac{\partial K}{\partial x}\frac{\partial v}{\partial x}+\frac{1}{1+Ky}\frac{\partial^2 v}{\partial x^2}-\frac{\partial^2 u}{\partial x\partial y}\right.$$

$$\left.-\frac{u}{1+Ky}\frac{\partial K}{\partial x}+\frac{Kyu}{(1+Ky)^2}\frac{\partial K}{\partial x}-\frac{K}{1+Ky}\frac{\partial u}{\partial x}\right] \qquad (2.22)$$

$$\theta_3 = \frac{1}{1+Ky}\left[\frac{y}{(1+Ky)^2}\frac{\partial K}{\partial x}\frac{\partial w}{\partial x}+\frac{\partial^2 u}{\partial x\partial z}-\frac{1}{1+Ky}\frac{\partial^2 w}{\partial x^2}-K\frac{\partial w}{\partial y}+K\frac{\partial v}{\partial z}-(1+Ky)\frac{\partial^2 w}{\partial y^2}\right.$$

$$\left.+(1+Ky)\frac{\partial^2 v}{\partial z\partial y}\right].$$

The components of $\bar{V}\times\bar{w}$ are

$$\left[\frac{v}{1+Ky}\frac{\partial v}{\partial x}-v\frac{\partial u}{\partial y}-\frac{Kuv}{1+Ky}-w\frac{\partial u}{\partial z}+\frac{w}{1+Ky}\frac{\partial w}{\partial x}\right],$$

$$\left[w\frac{\partial w}{\partial y}-w\frac{\partial v}{\partial z}-\frac{u}{1+Ky}\frac{\partial v}{\partial x}+u\frac{\partial u}{\partial y}+\frac{Ku^2}{1+Ky}\right], \qquad (2.23)$$

and

$$\left[u\frac{\partial u}{\partial z}-\frac{u}{1+Ky}\frac{\partial w}{\partial x}-v\frac{\partial w}{\partial y}+v\frac{\partial v}{\partial z}\right].$$

Then the components of the equation of motion (2.1) are

$$\frac{1}{1+Ky}u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+\frac{Kuv}{1+Ky} = -\frac{1}{1+Ky}\frac{\partial p}{\partial x}\frac{1}{\rho}-\nu\left[\frac{-K}{(1+Ky)^2}\frac{\partial v}{\partial x}\right.$$

$$\left.+\frac{1}{1+Ky}\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 u}{\partial y^2}+\frac{K^2u}{(1+Ky)^2}-\frac{K}{1+Ky}\frac{\partial u}{\partial y}-\frac{\partial^2 u}{\partial z^2}+\frac{1}{1+Ky}\frac{\partial^2 w}{\partial x\partial z}\right] \qquad (2.24)$$

and

$$\frac{u}{1+Ky}\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} - \frac{Ku^2}{1+Ky} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left[-\frac{\partial^2 w}{\partial z \partial y} + \frac{\partial^2 v}{\partial z^2}\right.$$

$$-\frac{y}{(1+Ky)^3}\frac{\partial K}{\partial x}\left\{\frac{\partial v}{\partial x} - (1+Ky)\frac{\partial u}{\partial y} - Ku\right\} + \frac{1}{(1+Ky)^2}\left\{\frac{\partial^2 v}{\partial x^2} - \frac{\partial K}{\partial x}\, y\frac{\partial u}{\partial y}\right.$$

$$\left.\left. - (1+Ky)\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial K}{\partial x}\, u - K\frac{\partial u}{\partial x}\right\}\right] \quad ; \tag{2.25}$$

and

$$\frac{u}{1+Ky}\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left[-\frac{y}{(1+Ky)^3}\frac{\partial K}{\partial x}\frac{\partial w}{\partial x} - \frac{1}{1+Ky}\frac{\partial^2 u}{\partial x \partial z}\right.$$

$$\left. + \frac{1}{(1+Ky)^2}\frac{\partial^2 w}{\partial x^2} + \frac{K}{1+Ky}\frac{\partial w}{\partial y} - \frac{K}{1+Ky}\frac{\partial v}{\partial z} + \frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 v}{\partial z \partial y}\right] . \tag{2.26}$$

The equation of continuity becomes

$$\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\left[(1+Ky)v\right] + \frac{\partial}{\partial z}\left[(1+Ky)w\right] = 0 ,$$

or

$$\frac{\partial u}{\partial x} + (1+Ky)\frac{\partial v}{\partial y} + Kv + (1+Ky)\frac{\partial w}{\partial z} = 0 . \tag{2.27}$$

From (2.27), it follows that

$$\left.\begin{aligned} -\frac{1}{1+Ky}\frac{\partial^2 v}{\partial x \partial y} &= -\frac{y}{(1+Ky)^3}\frac{\partial K}{\partial x}\frac{\partial u}{\partial x} + \frac{1}{(1+Ky)^2}\frac{\partial^2 u}{\partial x^2} + \frac{v}{(1+Ky)^2}\frac{\partial K}{\partial x} \\ &\quad - \frac{Kyv}{(1+Ky)^3}\frac{\partial K}{\partial x} + \frac{K}{(1+Ky)^2}\frac{\partial v}{\partial x} \quad \frac{1}{1+Ky}\frac{\partial^2 w}{\partial z \partial x} , \\ -\frac{\partial^2 w}{\partial z \partial y} &= -\frac{K}{1+Ky}\frac{\partial u}{\partial x} + \frac{1}{1+Ky}\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} + \frac{K}{1+Ky}\frac{\partial v}{\partial y} - \frac{K^2 v}{(1+Ky)^2} , \\ \text{and} \\ -\frac{1}{1+Ky}\frac{\partial^2 u}{\partial x \partial z} &= \frac{\partial^2 v}{\partial y \partial z} + \frac{K}{1+Ky}\frac{\partial v}{\partial z} + \frac{\partial^2 w}{\partial z^2} . \end{aligned}\right\} \tag{2.28}$$

Substitution of (2.28) in (2.25), (2.26), and (2.27), respectively, yields the following components of the equation of motion

$$\frac{1}{1+Ky}\,u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + w\,\frac{\partial u}{\partial z} + \frac{K u v}{1+Ky} = -\frac{1}{1+Ky}\,\frac{1}{\rho}\,\frac{\partial p}{\partial x} + \nu\left[\frac{1}{(1+Ky)^2}\,\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right.$$

$$\left. + \frac{\partial^2 u}{\partial z^2} - \frac{y}{(1+Ky)^3}\,\frac{\partial K}{\partial x}\,\frac{\partial u}{\partial x} + \frac{K}{1+Ky}\,\frac{\partial u}{\partial y} - \frac{K^2 u}{(1+Ky)^2} + \frac{2K}{(1+Ky)^2}\,\frac{\partial v}{\partial x} + \frac{v(\partial K/\partial x)}{(1+Ky)^3}\right]; \quad (2.29)$$

$$\frac{1}{1+Ky}\,u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + w\,\frac{\partial v}{\partial z} - \frac{K u^2}{1+Ky} = -\frac{1}{\rho}\,\frac{\partial p}{\partial y} + \nu\left[\frac{1}{(1+Ky)^2}\,\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right.$$

$$\left. + \frac{\partial^2 v}{\partial z^2} - \frac{y}{(1+Ky)^3}\,\frac{\partial K}{\partial x}\,\frac{\partial v}{\partial x} + \frac{K}{1+Ky}\,\frac{\partial v}{\partial y} - \frac{K^2 v}{(1+Ky)^2} - \frac{u}{(1+Ky)^3}\,\frac{\partial K}{\partial x} - \frac{2K}{(1+Ky)^2}\,\frac{\partial u}{\partial x}\right]; \quad (2.30)$$

and

$$\frac{1}{1+Ky}\,u\,\frac{\partial w}{\partial x} + v\,\frac{\partial w}{\partial y} + w\,\frac{\partial w}{\partial z} = -\frac{1}{\rho}\,\frac{\partial p}{\partial z} + \nu\left[\frac{1}{(1+Ky)^2}\,\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right.$$

$$\left. - \frac{y}{(1+Ky)^3}\,\frac{\partial K}{\partial x}\,\frac{\partial w}{\partial x} + \frac{K}{1+Ky}\,\frac{\partial w}{\partial y}\right]. \quad (2.31)$$

Equations (2.27), (2.29), (2.30), and (2.31) govern the motion along the plane surface ABCD in Fig. 2.

CHAPTER III

ORDER OF MAGNITUDE ANALYSIS OF EQUATIONS OF MOTION

Section 1. Nondimensional Form of Equations

We consider the flow of a viscous fluid over a two-dimensional surface having a curvature distribution denoted by the function $K = K(x)$. The complete equations of motion, developed in the preceding chapter, are nonlinear and extremely difficult to solve. However, as will be shown, certain terms in the complete equations are small for motions of interest, and may be neglected in comparison to other terms, so that a set of equations more amenable to solution may be obtained. As a first step in determining the relative size or "order of magnitude" of the terms in the complete equations of motion, we desire to make the equations nondimensional. Therefore, let us introduce the following nondimensional quantities:

$$u' = \frac{u}{U_1}, \quad v' = \frac{v}{U_1}, \quad y' = \frac{y}{L}, \quad x' = \frac{x}{L},$$
$$K' = KL, \quad p' = \frac{p}{\rho U_1^2}, \quad \text{and} \quad t' = \frac{U_1 t}{L}, \tag{3.1}$$

where U_1 and L are a characteristic velocity and length, respectively, in the flow under consideration.

In terms of these nondimensional quantities, the components of the equation of motion, (2.17) and (2.18), become respectively,

$$\frac{\partial u'}{\partial t'} + \frac{1}{1+K'y'} u' \frac{\partial u'}{\partial x'} + v' \frac{\partial u'}{\partial y'} + \frac{K'v'u'}{1+K'y'} = -\frac{1}{1+K'y'} \frac{\partial p'}{\partial x'}$$
$$+ \frac{1}{R}\left[\frac{1}{(1+K'y')^2} \frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2} - \frac{y'}{(1+K'y')^3} \frac{\partial K'}{\partial x'} \frac{\partial u'}{\partial x'} + \frac{K'}{1+K'y'} \frac{\partial u'}{\partial y'}\right.$$
$$\left. - \frac{K'^2 u'}{(1+K'y')^2} + \frac{2K'}{(1+K'y')^2} \frac{\partial v'}{\partial x'} + \frac{v'}{(1+K'y')^3} \frac{\partial K'}{\partial x'}\right], \tag{3.2}$$

and

$$\frac{\partial v'}{\partial t'} + \frac{1}{1+K'y'} u' \frac{\partial v'}{\partial x'} + v' \frac{\partial v'}{\partial y'} - \frac{K' u'^2}{1+K'y'} = - \frac{\partial p'}{\partial y'}$$

$$+ \frac{1}{R} \left[\frac{1}{(1+K'y')^2} \frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2} - \frac{y'}{(1+K'y')^3} \frac{\partial K'}{\partial x'} \frac{\partial v'}{\partial x'} + \frac{K'}{1+K'y'} \frac{\partial v'}{\partial y'} \right.$$

$$\left. - \frac{K'^2 v'}{(1+K'y')^2} - \frac{u'(\partial K'/\partial x')}{(1+K'y')^3} - \frac{2K'}{(1+K'y')^2} \frac{\partial v'}{\partial x'} \right] \quad ; \qquad (3.3)$$

where

$$R = \frac{\rho U_1 L}{\mu} = \text{Reynolds number} \quad . \qquad (3.4)$$

The equation of continuity becomes:

$$\frac{\partial u'}{\partial x'} + \frac{\partial}{\partial y'} \left[(1+K'y')v' \right] = 0 \quad . \qquad (3.5)$$

Section 2. Definition of Order of Magnitude

We shall take, as a basis for determination of the orders of magnitude of the terms in the complete equations of motion, the following fact from physical experience. Experiment indicates that when a fluid having small viscosity flows over a curved surface, the effects of viscosity, for a large portion of the flow, are appreciable only in a thin layer near the solid boundary. This thin layer is known as the boundary layer. Its thickness may be characterized by the condition that the coordinate y' of a point in the layer is very small in comparison to unity; that is,

$$|y'| << 1 \quad . \qquad (3.6a)$$

This may be expressed in terms of a thickness parameter δ', which is small in comparison to unity and is such that for all points in the boundary layer

$$|y'| \leq \delta' << 1 \quad . \qquad (3.6b)$$

The quantity y' is then said to be of order δ', which is written as

$$y' = \mathcal{O}(\delta') \ , \qquad (3.7)$$

the implication being that y' satisfies the inequality (3.6a).

In general the quantity α is said to be of order δ'^q at points within the boundary layer if

$$|\alpha| \leq C\,\delta'^q \ , \qquad \text{for some number } C << \frac{1}{\delta'} \ .$$

In this case we write

$$\alpha = \mathcal{O}(\delta'^q) \ ,$$

and in particular

$$\alpha = \mathcal{O}(1) \ ,$$

for q = 0. Hence, if α and β are quantities such that

$$\alpha = \mathcal{O}(\delta'^q) \qquad \text{and} \qquad \beta = \mathcal{O}(\delta'^r) \ ;$$

then

$$\alpha\beta = \mathcal{O}(\delta'^{q+r}) \qquad \text{and} \qquad \alpha/\beta = \mathcal{O}(\delta'^{q-r}) \ .$$

Restricting q to integral values or zero, we obtain a hierarchy of orders of magnitude, with the characteristic thickness parameter as a base, which have the relative orders indicated by the inequalities

$$\ldots << \mathcal{O}(\delta'^2) << \mathcal{O}(\delta') << \mathcal{O}(1) << \mathcal{O}\left(\frac{1}{\delta'}\right) << \mathcal{O}\left(\frac{1}{\delta'^2}\right) << \ldots$$

Section 3. Order of Magnitude Analysis for Surface with Moderate Curvature

Consider flow over a surface with a curvature distribution which may, in the spirit of the foregoing definition of order of magnitude, be characterized by

$$K'(x') = \mathcal{O}(1) \ , \qquad (3.8)$$

and call this a "moderate" curvature. Equation (3.8) states that the curvature of the surface under consideration is no greater than unity.

As shown on the basis of physical experience, the boundary-layer flow over such a surface may be characterized by the fact that

$$y' = \mathcal{O}(\delta')$$

for any point in the boundary layer. Physical experience also indicates that we may assume

$$u' = \mathcal{O}(1), \quad \frac{\partial u'}{\partial x'} = \mathcal{O}(1), \quad \frac{\partial^2 u'}{\partial x'^2} = \mathcal{O}(1), \quad \frac{\partial u'}{\partial t'} = \mathcal{O}(1), \quad \frac{\partial p'}{\partial x'} = \mathcal{O}(1) \; . \qquad (3.9)$$

These assumptions, when applied to the equation of continuity, (3.5), will indicate the orders of magnitude of the velocity component v' and its derivatives in the boundary-layer flow. Since v' equals zero along the wall, integration of (3.5) with respect to y' gives

$$(1+K'y')v' = -\int_0^{y'} \frac{\partial u'}{\partial x'} \, dy' \; . \qquad (3.10)$$

Hence, if the orders of magnitude in (3.9) hold, we may write for any point in the boundary layer

$$|v'| \leq \frac{1}{|1+K'y'|} \int_0^y \left|\frac{\partial u'}{\partial x'}\right| dy' \leq \frac{1}{|1+K'y'|} \int_0^{y'} dy' \leq \frac{y'}{|1+K'y'|} \; . \qquad (3.11)$$

Now for $K' > 0$ (convex curvature),

$$\frac{1}{|1+K'y'|} \leq \frac{1}{1+|K'y'|} \leq \frac{1}{1+\delta'} \; .$$

On the other hand, if $K' < 0$ (concave curvature),

$$\frac{1}{|1+K'y'|} \leq \frac{1}{1-|K'y'|} \leq \frac{1}{1-\delta'} \; .$$

Therefore,

$$\text{for } K' > 0 \ , \qquad |v'| \leq \frac{1}{1+\delta'} \ ;$$
$$\text{and for } K' < 0 \ , \qquad |v'| \leq \frac{1}{1-\delta'} \ . \tag{3.12}$$

But $\delta' << 1$, so that

$$\frac{\delta'}{1+\delta'} \leq \delta' << 1$$

and

$$\frac{\delta'}{1-\delta'} << 1 \ ,$$

and we may write, in terms of the foregoing orders of magnitude,

$$v' = \mathcal{O}(\delta') \ , \tag{3.13}$$

for any point in the boundary layer on a convex or concave surface having the property (3.8).

Now, rewrite the equation of continuity in the form

$$\frac{\partial u'}{\partial x'} + (1+K'y')\frac{\partial v'}{\partial y'} + K'v' = 0 \ , \tag{3.14}$$

and obtain

$$\left|\frac{\partial v'}{\partial y'}\right| \leq \frac{1}{|1+K'y'|}\left|\frac{\partial u'}{\partial x'}\right| + \frac{|K'||v'|}{|1+K'y'|} \ . \tag{3.15}$$

Hence, for positive surface curvature, it follows that at all points in the boundary layer in which (3.9) holds

$$\left|\frac{\partial v'}{\partial y'}\right| \leq 1+\delta' \qquad (K' > 0) \ ; \tag{3.16a}$$

while for negative curvature, using the results above,

$$\left|\frac{\partial v'}{\partial y'}\right| \leq \frac{1}{1-|K'y'|}\left|\frac{\partial u'}{\partial x'}\right| + \frac{|K'|\cdot|y'|}{(1-|K'y|)^2} \leq \frac{1}{1-\delta'} + \frac{1}{(1-\delta')^2} \quad (K' < 0) \ . \tag{3.16b}$$

Thus, for $\delta' << 1$, we may write

$$\frac{\partial v'}{\partial y'} = \mathcal{O}(1) \ . \tag{3.17}$$

Returning again to the equation of continuity in the form (3.10), we obtain by differentiation with respect to x',

$$\frac{\partial v'}{\partial x'} = -\frac{1}{1+K'y'}\int_0^{y'}\frac{\partial^2 u'}{\partial x'^2}\,dy' - \frac{v'y'}{1+K'y'}\frac{\partial K'}{\partial x'} \quad . \tag{3.18}$$

Therefore, making use of (3.7), (3.9), and (3.13), we have

$$\frac{\partial v'}{\partial x'} = \mathcal{O}(\delta') \quad , \tag{3.19}$$

provided that $\partial K'/\partial x'$ is at most of order $1/\delta'$. Let us therefore assume that

$$\frac{\partial K'}{\partial x'} = \mathcal{O}\left(\frac{1}{\delta'}\right) \quad . \tag{3.20}$$

Finally, differentiation of the equation of continuity in the form (3.14) with respect to x' yields

$$\frac{\partial^2 u'}{\partial x'^2} + (1+K'y')\frac{\partial^2 v'}{\partial x'\partial y'} + y'\frac{\partial K'}{\partial x'}\frac{\partial v'}{\partial y'} + v'\frac{\partial K'}{\partial x'} + K'\frac{\partial v'}{\partial x'} = 0 \quad . \tag{3.21}$$

Hence, by applying (3.7) and (3.9) and the above results, we have

$$\frac{\partial^2 v'}{\partial x'\partial y'} = \mathcal{O}(1) \quad . \tag{3.22}$$

Thus, we have shown that if we assume the following orders of magnitude:

$$u' = \mathcal{O}(1), \quad \frac{\partial u'}{\partial x'} = \mathcal{O}(1), \quad \frac{\partial^2 u'}{\partial x'^2} = \mathcal{O}(1), \quad \frac{\partial u'}{\partial t'} = \mathcal{O}(1),$$

$$y' = \mathcal{O}(\delta'), \quad K' = \mathcal{O}(1), \quad \text{and} \quad \frac{\partial K'}{\partial x'} = \mathcal{O}\left(\frac{1}{\delta'}\right) ;$$

then we can prove from the continuity equation that

$$v' = \mathcal{O}(\delta'), \quad \frac{\partial v'}{\partial y'} = \mathcal{O}(1), \quad \frac{\partial v'}{\partial x'} = \mathcal{O}(\delta'), \quad \frac{\partial^2 v'}{\partial x'\partial y'} = \mathcal{O}(1) \quad . \tag{3.23}$$

It is observed from the orders of magnitude of the velocity component v' and its derivatives that, at least for the cases considered,

differentiation of a function with respect to y' raises the order by one order of magnitude. It appears reasonable, therefore, to make the additional assumptions:

$$\frac{\partial u'}{\partial y'} = \mathcal{O}\left(\frac{1}{\delta'}\right), \quad \frac{\partial^2 u'}{\partial y'^2} = \mathcal{O}\left(\frac{1}{\delta'^2}\right), \quad \frac{\partial^2 v'}{\partial x'^2} = \mathcal{O}(\delta'), \quad \frac{\partial^2 v'}{\partial y'^2} = \mathcal{O}\left(\frac{1}{\delta'}\right) \quad ; \qquad (3.24)$$

for the boundary-layer flow.

We shall now determine the form of the equations of motion which is consistent with the above orders of magnitude. Inspection of the orders of magnitude of the terms in square brackets on the right side of the first component of the equation of motion (3.2) reveals that all terms may be neglected in comparison to $\partial^2 u'/\partial y'^2$. Also, the term $K'v'u'/(1+K'y')$ on the left side may be neglected in comparison to the others. All the remaining terms are $\mathcal{O}(1)$ except the term on the right involving the Reynolds number, which has an unknown order of magnitude. Hence, we conclude that this term can be of order no greater than unity, and find that

$$\frac{1}{R} = \mathcal{O}(\delta'^2) \quad . \qquad (3.25)$$

This result, and the orders of magnitude of the velocity components and their derivatives from (3.9), (3.23), and (3.24), show that all terms in the second component of the equation of motion, (3.3), may be neglected in comparison to the term $K'u'^2/(1+K'y')$ on the left side. Since this term is $\mathcal{O}(1)$, we conclude that

$$\frac{\partial p'}{\partial y'} = \mathcal{O}(1) \quad . \qquad (3.26)$$

Hence, the equations which govern the motion in the boundary layer on a two-dimensional curved surface are, in dimensional form:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}$$

$$Ku^2 = \frac{1}{\rho}\frac{\partial p}{\partial y} \quad , \qquad (3.27)$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \ ; \tag{3.28}$$

when the surface curvature satisfies the conditions

$$K' = \mathcal{O}(1) \ , \qquad \frac{\partial K'}{\partial x'} = \mathcal{O}\left(\frac{1}{\delta'}\right) \ ; \tag{3.29}$$

and the Reynolds number, the condition (3.25).

Section 4. Order of Magnitude Analysis for Surface with Large Curvature

In this section, we consider flow over a surface which has a curvature distribution such that the maximum curvature may be characterized by

$$K'(x') = \mathcal{O}\left(\frac{1}{\delta'}\right) \ , \tag{3.30}$$

and call this a "large" curvature. Let us again assume

$$\begin{aligned} &u' = \mathcal{O}(1) \ , \quad \frac{\partial u'}{\partial x'} = \mathcal{O}(1) \ , \quad \frac{\partial^2 u'}{\partial x'^2} = \mathcal{O}(1) \ , \\ &\frac{\partial p'}{\partial x'} = \mathcal{O}(1) \ , \quad \frac{\partial u'}{\partial t'} = \mathcal{O}(1) \ , \quad y' = \mathcal{O}(\delta') \ . \end{aligned} \tag{3.31}$$

Since $v' = 0$ at the surface, it follows from the continuity equation that for any point in the boundary layer

$$(1+K'y')v' = -\int_0^{y'} \frac{\partial u'}{\partial x'} \, dy' \ .$$

Therefore

$$|v'| \leq \frac{|y'|}{|1+K'y'|} \ . \tag{3.32}$$

Thus if $K' > 0$ (convex curvature),

$$\frac{1}{|1+K'y'|} \leq \frac{1}{1+|K'y'|} \leq 1 \ ,$$

for any point in the boundary layer. Hence for positive curvature

$$v' = \mathcal{O}(\delta') \quad . \tag{3.33}$$

On the other hand, if $K' < 0$ (concave curvature), then

$$|v'| \leq \frac{|y'|}{|1+K'y'|} \leq \frac{|y'|}{1 - |K'y'|} \quad .$$

Since $K'y' = \mathcal{O}(1)$, it follows that the expression for $|v'|$ does not have a finite upper bound. As a consequence, we restrict this analysis to convex curvatures such that

$$K' = \mathcal{O}(\frac{1}{\delta'}) \quad , \qquad K' > 0 \quad . \tag{3.34}$$

Consider the equation of continuity in the form (3.14), and obtain

$$\left|\frac{\partial v'}{\partial y'}\right| \leq \frac{|K'||v'|}{|1+K'y'|} + \frac{1}{|1+K'y'|}\left|\frac{\partial u'}{\partial x'}\right| \leq 1 + 1 = 2 \quad ,$$

so that

$$\frac{\partial v'}{\partial y'} = \mathcal{O}(1) \quad . \tag{3.35}$$

Returning to the continuity equation in the form (3.10), and differentiating with respect to x', yields

$$\left|\frac{\partial v'}{\partial x'}\right| \leq \frac{|y'|}{|1+K'y'|} + \frac{|y'||v'|}{|1+K'y'|}\left|\frac{\partial K'}{\partial x'}\right| \leq \delta + \delta^2\left|\frac{\partial K}{\partial x}\right| .$$

Therefore,

$$\frac{\partial v'}{\partial x'} = \mathcal{O}(\delta') \quad , \tag{3.36}$$

provided that $\partial K'/\partial x'$ is at most of order $1/\delta'$. We shall therefore restrict the rate of change of curvature by assuming

$$\frac{\partial K'}{\partial x'} = \mathcal{O}\left(\frac{1}{\delta'}\right) .$$

Finally, differentiating Equation (3.14) with respect to x' yields

$$\left|\frac{\partial^2 v'}{\partial x' \partial y'}\right| \leq \frac{1}{|1+K'y'|}\left\{\left|\frac{\partial^2 u'}{\partial x'^2}\right| + |y'|\left|\frac{\partial v'}{\partial y'}\right|\left|\frac{\partial K'}{\partial x'}\right| + \left|\frac{\partial K'}{\partial x'}\right||v'| + |K'|\left|\frac{\partial v'}{\partial x'}\right|\right\} .$$

Hence,

$$\frac{\partial^2 v'}{\partial x' \partial y'} = \mathcal{O}(1) . \tag{3.38}$$

On the basis of these results, we again make the additional assumptions,

$$\frac{\partial u'}{\partial y'} = \mathcal{O}\left(\frac{1}{\delta'}\right), \quad \frac{\partial^2 u'}{\partial y'^2} = \mathcal{O}\left(\frac{1}{\delta'^2}\right), \quad \frac{\partial^2 v'}{\partial x'^2} = \mathcal{O}(\delta'), \quad \frac{\partial^2 v'}{\partial y'^2} = \mathcal{O}\left(\frac{1}{\delta'}\right). \tag{3.39}$$

Then it is evident that a solution satisfying the orders of magnitude, (3.31) to (3.39), for the case of large convex curvature is consistent with the two components of the equation of motion, provided that the Reynolds number of the flow is of order

$$R = \mathcal{O}\left(\frac{1}{\delta'^2}\right) , \tag{3.40}$$

and the normal pressure gradient is of order

$$\frac{\partial p'}{\partial y'} = \mathcal{O}\left(\frac{1}{\delta'}\right) . \tag{3.41}$$

Solutions satisfying these conditions may be found by solving the equations of motion

$$\frac{1}{1+Ky} u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{K u v}{1+Ky} = -\frac{1}{\rho(1+Ky)} \frac{\partial p}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial y^2} + \frac{K}{1+Ky}\frac{\partial u}{\partial y} - \frac{K^2 u}{(1+Ky)^2}\right]$$

$$\frac{K u^2}{1+Ky} = \frac{1}{\rho}\frac{\partial p}{\partial y} \tag{3.42}$$

in conjunction with the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\left[(1+Ky)v\right] = 0 , \qquad (3.43)$$

with the restrictions on K that

$$K' = \mathcal{O}\left(\frac{1}{\delta'}\right) , \qquad \frac{\partial K'}{\partial x'} = \mathcal{O}\left(\frac{1}{\delta'}\right) , \qquad K' > 0 . \qquad (3.44)$$

Section 5. Review of Literature on Order of Magnitude Analyses

Several analyses of the order of magnitude of the terms in the equations of motion for flow over a curved two-dimensional surface, (2.14), (2.17), and (2.18), can be found in the literature. Tollmien indicates that the first component of the equation of motion is the same as that for boundary-layer flow over a flat surface, and that the second equation of motion may be neglected as long as the radius of curvature is not too small and does not change in value suddenly along the surface (Ref. 7). Goldstein obtains the results given in (3.27) and (3.28) above with the same restrictions on K' and its derivatives as those in (3.29) (Ref. 6).

Schmidt and Schröder give one of the most complete order of magnitude analyses of the exact equations of motion (Ref. 8). The basic assumptions involved in their work are

$$u' = \mathcal{O}(1), \quad \frac{\partial u'}{\partial x'} = \mathcal{O}(1), \quad \frac{\partial^2 u'}{\partial x' \partial t'} = \mathcal{O}(1), \quad \frac{\partial^2 u'}{\partial x'^2} = \mathcal{O}(1), \quad \frac{\partial p'}{\partial x'} = \mathcal{O}(1); \qquad (3.45)$$

which are similar to those made in the analysis given above. In addition, they make the assumption

$$\frac{\partial^2 v'}{\partial x'^2} = \mathcal{O}\left(\frac{1}{\delta'}\right) . \qquad (3.46)$$

Using these assumptions, together with a rigorous analysis of the equations, they show that the equations of motion reduce to

$$\frac{\partial u'}{\partial t'} + u' \frac{\partial u'}{\partial x'} + v' \frac{\partial u'}{\partial y'} + \frac{\partial p'}{\partial x'} = \frac{1}{R} \frac{\partial^2 u'}{\partial y'^2}$$

$$Ku'^2 - (1+Ky) \frac{\partial p'}{\partial y'} = \frac{1}{R} \frac{\partial^2 u'}{\partial y' \partial x'} \quad . \tag{3.47}$$

The term on the right side of the second component of the equation of motion results from the weak assumption (3.46), which is less restrictive than the results presented in the foregoing sections. If this assumption had not been made, the equations obtained by these authors would have been the same as (3.27) and (3.28). It is suggested that the equations (3.27) and (3.28) determine the motion in the boundary layer on moderately curved surfaces with sufficient accuracy for practical purposes.

An order of magnitude analysis for surfaces with large curvature does not appear in the literature to the writer's knowledge.

CHAPTER IV

BOUNDARY CONDITIONS

In the preceding chapter, the boundary-layer equations of motion of a viscous, incompressible fluid past a two-dimensional curved surface were developed for the cases of moderate and large surface curvature. Now we shall determine the boundary conditions which must be satisfied for these flows.

Section 1. Inner Boundary Condition

The no-slip condition at the surface requires that the velocity vanish at the surface, $y = 0$. Thus the following two conditions must hold:

$$u = v = 0 \qquad \text{at} \qquad y = 0 \ . \tag{4.1}$$

Section 2. Outer Boundary Condition

In addition to the foregoing conditions, a smooth transition is required between the motion of the fluid in the boundary layer and the motion represented by the potential flow in the region exterior to the layer. Thus, let us require that at the "outer edge" of the boundary layer the solution of the boundary-layer equations tends asymptotically to an expression representing approximately the potential flow in the neighborhood of the surface. This approximate representation characterizing the nature of the potential flow near the surface depends on the type of surface curvature and therefore will be different for the two cases considered.

It will be supposed, as in the case of the flow within the boundary layer, that in a neighborhood of the surface the potential flow is such

that

$$u' = \mathcal{O}(1), \qquad \frac{\partial u'}{\partial x'} = \mathcal{O}(1), \qquad \frac{\partial^2 u'}{\partial x'^2} = \mathcal{O}(1), \tag{4.2}$$

where the nondimensional quantities x', y', u', and v' are introduced as in (3.1). Then it follows as before from the equation of continuity that in a neighborhood of a surface having moderate or large positive curvature in which $y' = \mathcal{O}(\delta')$, the potential flow satisfies the further order relations

$$v' = \mathcal{O}(\delta'), \qquad \frac{\partial v'}{\partial x'} = \mathcal{O}(\delta'), \qquad \frac{\partial v'}{\partial y'} = \mathcal{O}(1) . \tag{4.3}$$

We therefore seek an approximate representation of the potential flow that is consistent with these order relationships.

Since the potential flow is irrotational, the components of velocity satisfy a condition which, in terms of the curvilinear coordinate system introduced in Chapter II, can be written in the form

$$\frac{1}{1+K'y'} \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'} - \frac{K'u'}{1+K'y'} = 0 . \tag{4.4}$$

Then we may write for surfaces of moderate curvature or large convex curvature

$$\left|\frac{\partial u'}{\partial y'}\right| \leq \frac{1}{|1+K'y'|} \left\{ \left|\frac{\partial v'}{\partial x'}\right| + |K'|\,|u'| \right\} \leq |K'| \tag{4.5}$$

for all points in the neighborhood of the surface. Therefore, in this neighborhood,

$$\frac{\partial u'}{\partial y'} = \mathcal{O}(K') . \tag{4.6}$$

For the case of moderate curvature, if we neglect terms of $\mathcal{O}(\delta')$ in (4.4), the condition of irrotationality may be expressed in dimensional form as

$$\frac{1}{u} \frac{\partial u}{\partial y} = -K . \tag{4.7}$$

This differential equation requires that

$$u = C(x)\, e^{-Ky} \quad . \tag{4.8}$$

where the arbitrary function $C(x)$ is determined by the value of u prescribed by the potential flow along the surface. If, on the surface, $u > 0$ and $u = u_o(x)$, then an approximate representation of the potential flow in the neighborhood of a surface with moderate curvature is given by

$$u = u_o(x)\, e^{-Ky} \qquad \text{and} \qquad v' = \mathcal{O}(\delta') \quad . \tag{4.9}$$

Hence, for the case of surfaces of moderate curvature, the outer boundary condition for the solution of the boundary-layer equations is taken as

$$\begin{aligned} u &= u_o\, e^{-Ky} \cong u_o \left[1 - Ky\right] + \mathcal{O}(\delta'^2) \\ \text{and} \\ \frac{\partial u}{\partial y} &= -Ku_o e^{-Ky} \cong -Ku_o \left[1 - Ky\right] + \mathcal{O}(\delta'^2) \quad . \end{aligned} \tag{4.10}$$

Thus for convex curvature ($K > 0$),

$$\left.\frac{\partial u}{\partial y}\right]_{y=\delta} < 0 \quad ;$$

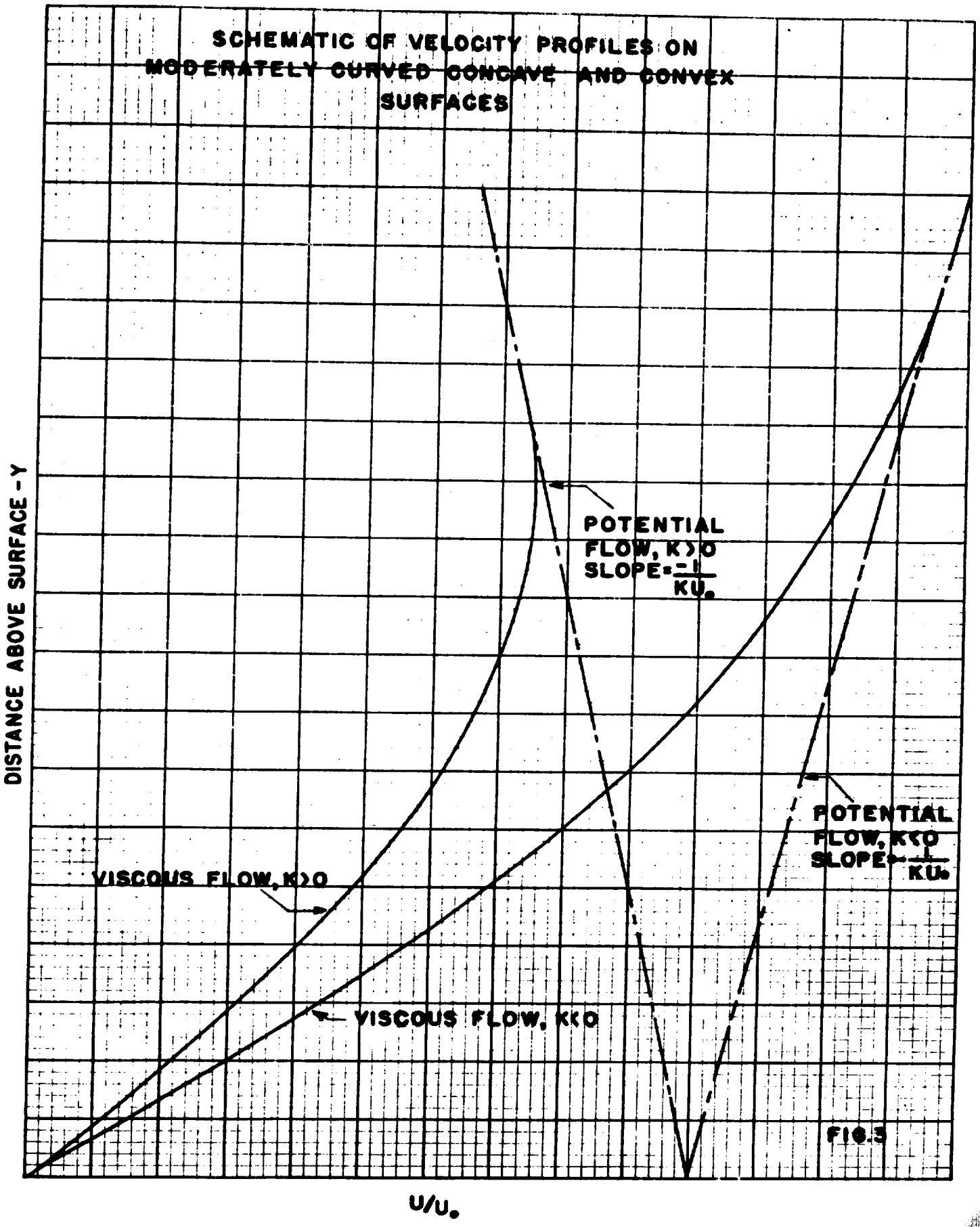
while for concave curvature ($K < 0$),

$$\left.\frac{\partial u}{\partial y}\right]_{y=\delta} > 0 \quad .$$

The velocity profiles must follow the general trend shown schematically in Fig. 3.

In the case of large positive curvature, if we neglect terms of order δ', the condition of irrotationality (4.4), when expressed in dimensional form, becomes:

$$\frac{1}{u}\frac{\partial u}{\partial y} = -\frac{K}{1+Ky} \quad . \tag{4.11}$$


SCHEMATIC OF VELOCITY PROFILES ON MODERATELY CURVED CONCAVE AND CONVEX SURFACES
DISTANCE ABOVE SURFACE - Y
U/U∞
POTENTIAL FLOW, K>0 SLOPE = -1/KU∞
POTENTIAL FLOW, K<0 SLOPE = -1/KU∞
VISCOUS FLOW, K>0
VISCOUS FLOW, K<0
FIG. 3

This differential equation requires that

$$u = \frac{C(x)}{1+Ky} , \tag{4.12}$$

where the arbitrary function $C(x)$ is determined by the value of u prescribed by the potential flow along the surface. If, as before, $u = u_o(x) > 0$ on the surface, then an approximate representation of the potential flow near the surface is

$$u = \frac{u_o(x)}{1+Ky} \qquad \text{and} \qquad v = \mathcal{O}(\delta) . \tag{4.13}$$

Hence, for the case of large positive curvature, the outer boundary condition is taken as

$$u = \frac{u_o}{1+Ky}$$

and

$$\frac{\partial u}{\partial y} = - \frac{Ku_o}{(1+Ky)^2} . \tag{4.14}$$

Thus for convex curvature ($K > 0$),

$$\left.\frac{\partial u}{\partial y}\right]_{y=\delta} < 0 .$$

In this case, the velocity profile must follow the trend shown schematically in Fig. 4.

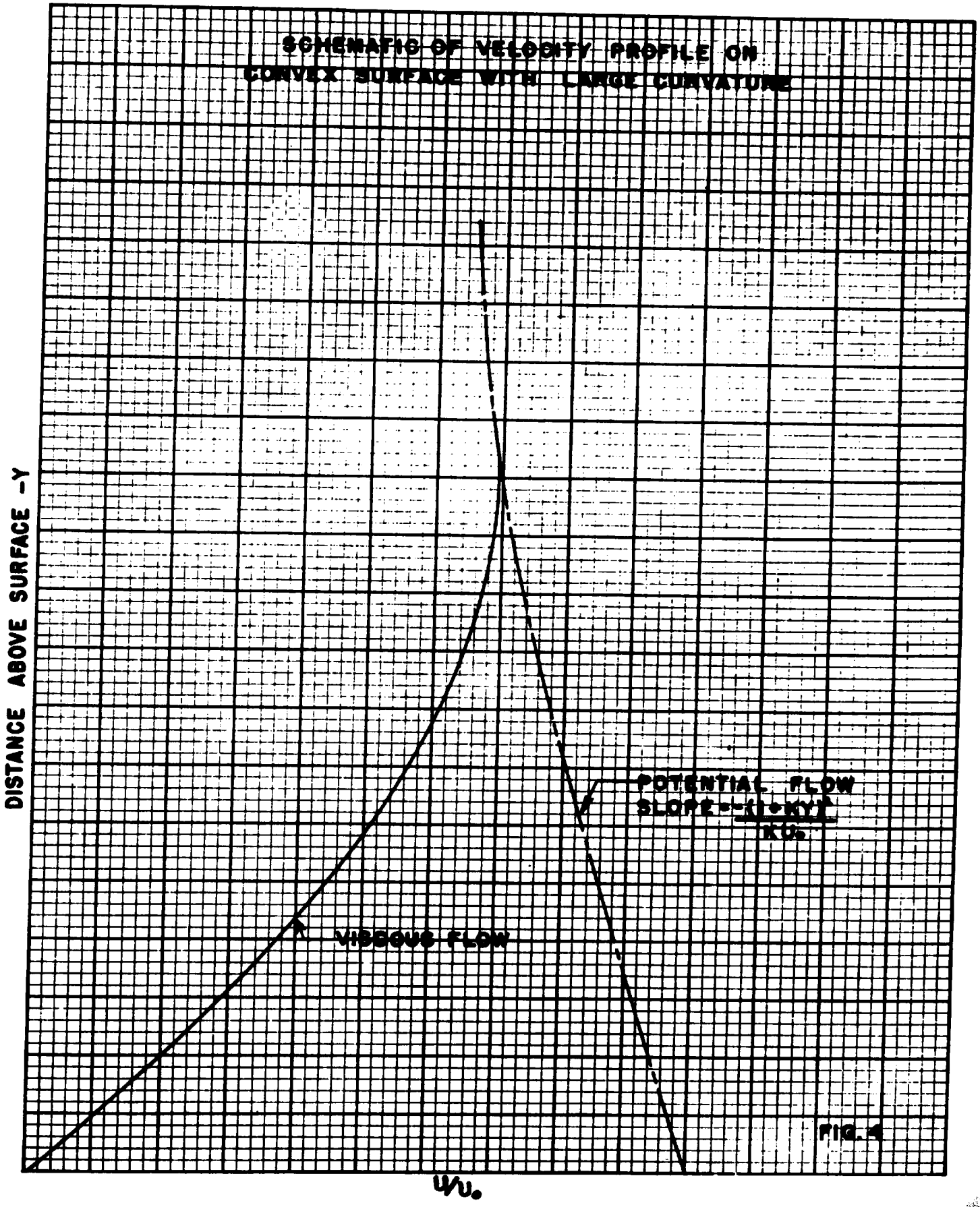

SCHEMATIC OF VELOCITY PROFILE ON
CONVEX SURFACE WITH LARGE CURVATURE
DISTANCE ABOVE SURFACE -Y
POTENTIAL FLOW
SLOPE = -(1+KY)/Ku
VISCOUS FLOW
u/u.

FIG. 4

CHAPTER V

CONSIDERATIONS ON FLOW ALONG A PARTICULAR SURFACE

Section 1. Flow Along a Surface with Moderate Curvature

We shall now solve the boundary-layer equations of motion for an idealized flow past a particular surface having moderate curvature. The idealized flow is chosen in such a manner that the pressure gradient is zero. However, the pressure gradient normal to the surface is included, and the outer boundary condition developed in Chapter IV for curved surfaces is satisfied. The solution will demonstrate the trend of the above effects of surface curvature on the wall stress for laminar flow.

Consider the equations derived in Chapter III for flow along a moderately curved surface. They are

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} , \tag{5.1a}$$

$$Ku^2 = \frac{1}{\rho} \frac{\partial p}{\partial y} , \tag{5.1b}$$

and

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 . \tag{5.1c}$$

The equation of continuity (5.1c) may be satisfied by introduction of a stream function Ψ, such that

$$u = \frac{\partial \Psi}{\partial y} \quad \text{and} \quad v = - \frac{\partial \Psi}{\partial x} . \tag{5.2}$$

In order to seek a solution of Equations (5.1) for flow along a particular surface, let us define a change of independent variables

$$\eta = \frac{y}{2} \left(\frac{U_1}{\nu x} \right)^{1/2} \tag{5.3a}$$

and

$$\xi = x \ , \tag{5.3b}$$

where U_1 is a constant having the dimensions of velocity. Then

$$\frac{\partial}{\partial y} = \frac{1}{2}\left(\frac{U_1}{\nu\xi}\right)^{1/2} \frac{\partial}{\partial \eta}$$

and

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} - \frac{\eta}{2\xi} \frac{\partial}{\partial \eta} \ . \tag{5.4}$$

Let us seek solutions for which the stream function has the form,

$$\Psi = (U_1 \nu \xi)^{1/2} f(\eta) \ ; \tag{5.5}$$

where $f(\eta)$ is a function of η alone. By (5.2), (5.4), and (5.5), we obtain

$$u = \frac{1}{2} U_1 f' \ , \qquad \frac{\partial u}{\partial x} = -\frac{1}{4} \frac{U_1}{\xi} \eta f'' \ ,$$

$$\frac{\partial u}{\partial y} = \frac{U_1}{4}\left(\frac{U_1}{\nu\xi}\right)^{1/2} f'' \ , \qquad \frac{\partial^2 u}{\partial y^2} = \left(\frac{U_1}{8}\right)\left(\frac{U_1}{\nu\xi}\right) f''' \ , \tag{5.6}$$

$$v = \frac{1}{2}\left(\frac{U_1 \nu}{\xi}\right)^{1/2} (f'\eta - f) \ .$$

Substitution of (5.6) in (5.1a) yields

$$-\frac{1}{8}\frac{U_1^2}{\xi} f' \eta f'' + \frac{1}{2}\left(\frac{U_1\nu}{\xi}\right)^{1/2} (f'\eta - f) \frac{U_1}{4}\left(\frac{U_1}{\nu\xi}\right)^{1/2} f''$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial \xi} + \frac{1}{\rho}\frac{\eta}{2\xi}\frac{\partial p}{\partial \eta} + \nu\left(\frac{U_1}{8}\right)\left(\frac{U_1}{\nu\xi}\right) f''' \ ; \tag{5.7}$$

and (5.1b) becomes

$$K \rho \left(\frac{1}{2} U_1 f'\right)^2 = \frac{1}{2}\left(\frac{U_1}{\nu\xi}\right)^{1/2} \frac{\partial p}{\partial \eta} \ . \tag{5.8}$$

If we let the curvature vary with ξ in the following manner:

$$K = A\left(\frac{U_1}{\nu\xi}\right)^{1/2} \ , \tag{5.9}$$

where A is a constant of proportionality having the approximate value

$$A \sim \left(\frac{\nu\xi}{U_1}\right)^{1/2} = \mathcal{O}(\delta) , \tag{5.10}$$

Equation (5.8) becomes

$$\frac{\partial p}{\partial \eta} = \frac{1}{2} A \rho U_1^2 f'^2 . \tag{5.11}$$

Substitution of (5.11) in (5.7) yields

$$f f'' + f''' + 2A\eta f' f' - \frac{8\xi}{\rho U_1^2}\frac{\partial p}{\partial \xi} = 0 . \tag{5.12}$$

Now assume that $\partial p/\partial \xi = 0$, so that the equation of motion becomes

$$f f'' + f''' + 2A\eta f' f' = 0 . \tag{5.13}$$

The inner boundary conditions which must be satisfied are

$$f = f' = 0 \qquad \text{at} \qquad \eta = 0 . \tag{5.14}$$

The outer boundary condition is

$$u \longrightarrow U_1 (1 - Ky) ,$$

where U_1 is a constant. By (5.9) and (5.3a), this condition becomes

$$f' \longrightarrow 2 - 4A\eta \qquad \text{for } \eta \text{ large} . \tag{5.15}$$

Let us now assume a solution of (5.13) of the form

$$f = \sum_{i=0}^{\infty} \frac{c_i \eta^i}{i!} . \tag{5.16}$$

The two conditions (5.14) indicate that

$$c_o = c_1 = 0 . \tag{5.17}$$

Substituting the function (5.16) in (5.13), collecting powers of η, and equating the coefficient of each power to zero gives the coefficients C_i $(i > 2)$ in terms of C_2 as

$$
\begin{aligned}
&C_2 && C_3 = 0 && C_4 = 0\\
&C_5 = -C_2^2 && C_6 = -12\,AC_2^2 && C_7 = 0\\
&C_8 = 11\,C_2^3 && C_9 = 312\,AC_2^3 && C_{10} = 2{,}016\,A^2C_2^3\\
&C_{11} = -375\,C_2^4 && C_{12} = -18{,}492\,AC_2^4 && C_{13} = -295{,}776\,A^2C_2^4\\
&C_{14} = 27{,}897\,C_2^5 - 798{,}336\,A^3C_2^4 && C_{15} = 2{,}114{,}256\,AC_2^5 && C_{16} = 59{,}325{,}312\,A^2C_2^5\\
&C_{17} = -3{,}817{,}742\,C_2^6 + 736{,}394{,}816\,A^3C_2^5 && C_{18} = -435{,}669{,}862\,AC_2^6 + 3{,}576{,}465{,}280\,A^4C_2^5 &&\\
&\quad\vdots && \quad\vdots && \quad\vdots
\end{aligned}
\qquad (5.18)
$$

Hence the function f may be written in the form

$$
f = C_2^{1/3} F_o(C_2^{1/3}\eta) - AF_1(C_2^{1/3}\eta) + \frac{A^2}{C_2^{1/3}} F_2(C_2^{1/3}\eta) + \ldots + \frac{(-1)^n A^n F_n(C_2^{1/3}\eta)}{C_2^{\frac{n-1}{3}}} + \ldots \,, \qquad (5.19)
$$

where

$$
\begin{aligned}
F_o &= \frac{(C_2^{1/3}\eta)^2}{2!} - \frac{(C_2^{1/3}\eta)^5}{5!} + \frac{11(C_2^{1/3}\eta)^8}{8!} - \frac{375(C_2^{1/3}\eta)^{11}}{11!}\\
&\quad + \frac{27{,}897(C_2^{1/3}\eta)^{14}}{14!} - \frac{3{,}817{,}742(C_2^{1/3}\eta)^{17}}{17!} + \ldots
\end{aligned}
\qquad (5.20)
$$

$$
\begin{aligned}
F_1 &= \frac{12(C_2^{1/3}\eta)^6}{6!} - \frac{312(C_2^{1/3}\eta)^9}{9!} + \frac{18{,}492(C_2^{1/3}\eta)^{12}}{12!}\\
&\quad - \frac{2{,}114{,}256(C_2^{1/3}\eta)^{15}}{15!} + \frac{435{,}669{,}862(C_2^{1/3}\eta)^{18}}{18!} - \ldots
\end{aligned}
$$

$$F_2 = \frac{2016(C_2^{1/3}\eta)^{10}}{10!} - \frac{295,776(C_2^{1/3}\eta)^{13}}{13!} + \frac{59,325,312(C_2^{1/3}\eta)^{16}}{16!} - \cdots$$

$$F_3 = \frac{798,336(C_2^{1/3}\eta)^{14}}{14!} - \frac{736,394,816\ (C_2^{1/3}\eta)^{17}}{17!} + \cdots \qquad (5.20)$$

$$F_4 = \frac{3,576,465,280(C_2^{1/3}\eta)^{18}}{18!} - \cdots \ .$$

Note that the series F_o is that obtained by Blasius for flow over a flat surface, i.e., a surface for which $K = A = 0$. If we consider the solution in the neighborhood of the wall, we find

$$f''_{\eta=0} = C_2^{2/3}$$

and (5.21)

$$f'''_{\eta=0} = 0 \ .$$

Hence, no inflection point appears in the velocity profile. Since

$$\tau_o = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu \frac{U_1}{4}\left(\frac{U_1}{\nu\xi}\right)^{1/2} f''\Big]_{\eta=0}$$

$$= \mu \frac{U_1}{4}\left(\frac{U_1}{\nu x}\right)^{1/2} C_2^{2/3} \ , \qquad (5.22)$$

we have to determine the manner in which C_2 varies with A in order to find the effect of moderate surface curvature on wall stress for laminar flow along the particular surface defined by (5.9). The dependence of C_2 on A is given by the outer boundary condition (5.15). This condition is valid up to quantities of $\mathcal{O}(\delta^2)$. Hence, the solution we seek must be valid up to $\mathcal{O}(\delta^2)$. Then, by (5.10), we may write the solution (5.19) in the form

$$f = C_2^{1/3} F_o(C_2^{1/3}\eta) - A F_1(C_2^{1/3}\eta) + \mathcal{O}(\delta^2) \ . \qquad (5.23)$$

Neglecting the higher-order quantities and using the condition (5.15) yields

$$C_2^{2/3} F_o'(C_2^{1/3}\eta) - A C_2^{1/3} F_1'(C_2^{1/3}\eta) \rightarrow 2 - 4A\eta , \quad \eta \text{ large} . \quad (5.24)$$

This can be written in the form

$$C_2^{2/3} \left[F_o'(C_2^{1/3}\eta) - A\eta H_1'(C_2^{1/3}\eta)\right] \rightarrow 2 - 4A\eta , \quad \eta \text{ large} , \quad (5.25)$$

where

$$H_1'(C_2^{1/3}\eta) = \frac{F_1'(C_2^{1/3}\eta)}{C_2^{1/3}\eta} . \quad (5.26)$$

Making the substitution

$$\zeta = C_2^{1/3}\eta, \quad (5.27)$$

the left side of (5.25) becomes

$$f' = C_2^{2/3}\left[F_o'(\zeta) - A\eta H_1'(\zeta)\right] \rightarrow 2 - 4A\eta . \quad (5.28)$$

It has been shown in Ref. 8 that the value of $F_o'(\zeta)$ for large is

$$F_o'(\zeta) = 1.65562 \qquad (\zeta > 3.3) . \quad (5.29)$$

The approximate form of $H_1'(\zeta)$ for large ζ will be determined here by use of the series expansion (5.20) along with (5.26). This series has been summed for various values of ζ and the results are shown on Fig. 5. The approximate value of the function for $\zeta > 2$ is determined from the tangent to the curve at $\zeta = 2$. The equation of the tangent is

$$H_1'(\zeta) = 1.29\zeta - 1.613 \qquad (\zeta > 2) . \quad (5.30)$$

Substitution of the expressions (5.29) and (5.30) in (5.28) yields

$$f' \rightarrow C_2^{2/3}\left[1.65562 - A\eta(1.29\, C_2^{1/3}\eta - 1.613)\right], \quad \eta > 3 ; \quad (5.31)$$

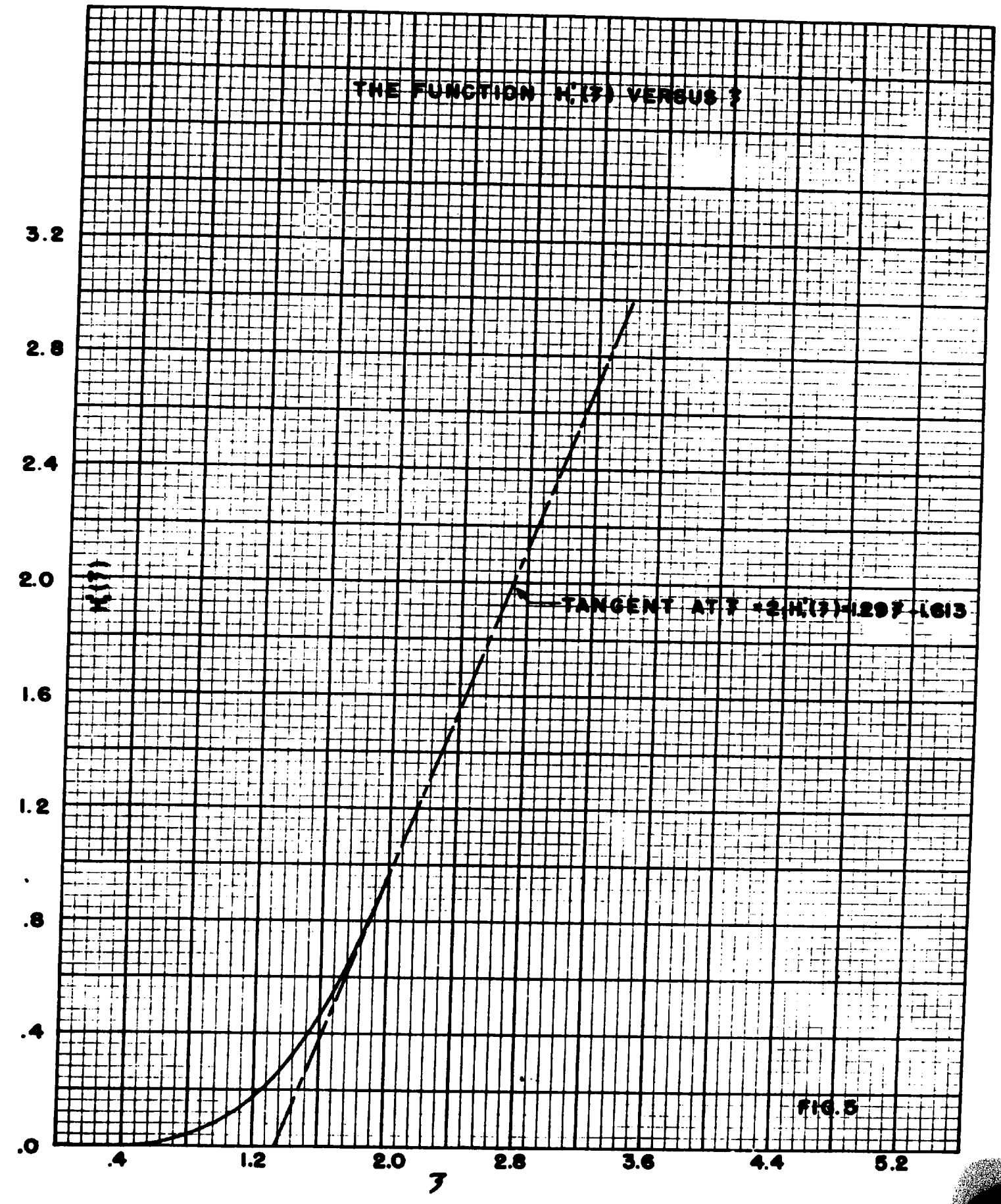

THE FUNCTION H'(ξ) VERSUS ξ
H'(ξ)
ξ
TANGENT AT ξ = 2, H'(ξ) = 1.29ξ - .1613
3.2
2.8
2.4
2.0
1.6
1.2
.8
.4
.0
.4
1.2
2.0
2.8
3.6
4.4
5.2

FIG. 5

and making use of this expression in (5.25) gives

$$C_2^{2/3}\left[1.65562 - 1.29\,A\,C_2^{1/3}\eta^2 + 1.613\,A\eta\right]_{\eta\cong 3} = \left[2 - 4\,A\eta\right]_{\eta\cong 3} . \quad (5.32)$$

This expression is approximately valid for $\eta \cong 3$. Applying it at $\eta = 3$ yields

$$C_2^{2/3}\left[1.65562 - 11.61\,A\,C_2^{1/3} + 4.839\,A\right] = 2 - 12\,A . \quad (5.33)$$

Setting

$$C_2^{1/3} = \alpha , \quad (5.34)$$

Equation (5.33) becomes

$$11.61\,A\,\alpha^3 - (1.65562 + 4.839\,A)\alpha^2 + 2 - 12\,A = 0 . \quad (5.35)$$

This equation has been solved with A as a parameter. The results are shown graphically in Fig. 6, as coefficient C_2 for various values of A. The results indicate that

$$(C_2)_{A>0} < (C_2)_{A=0} \quad \text{and} \quad (C_2)_{A<0} > (C_2)_{A=0} . \quad (5.36)$$

Therefore, using (5.22), it is evident that the wall stress for convex surfaces is less than the wall stress for concave surfaces. Convex surface curvature appears to have a stronger effect on wall stress than concave curvature for the particular flow considered here. The results of this solution indicate that for

$$A = +0.05 \qquad (\tau_o) < (\tau_o)_{\substack{\text{flat}\\ \text{surface}}} \quad \text{by } 8\% \quad \text{(convex curvature)}$$

$$A = -0.05 \qquad (\tau_o) > (\tau_o)_{\substack{\text{flat}\\ \text{surface}}} \quad \text{by } 3\% \quad \text{(concave curvature)} \quad (5.37)$$

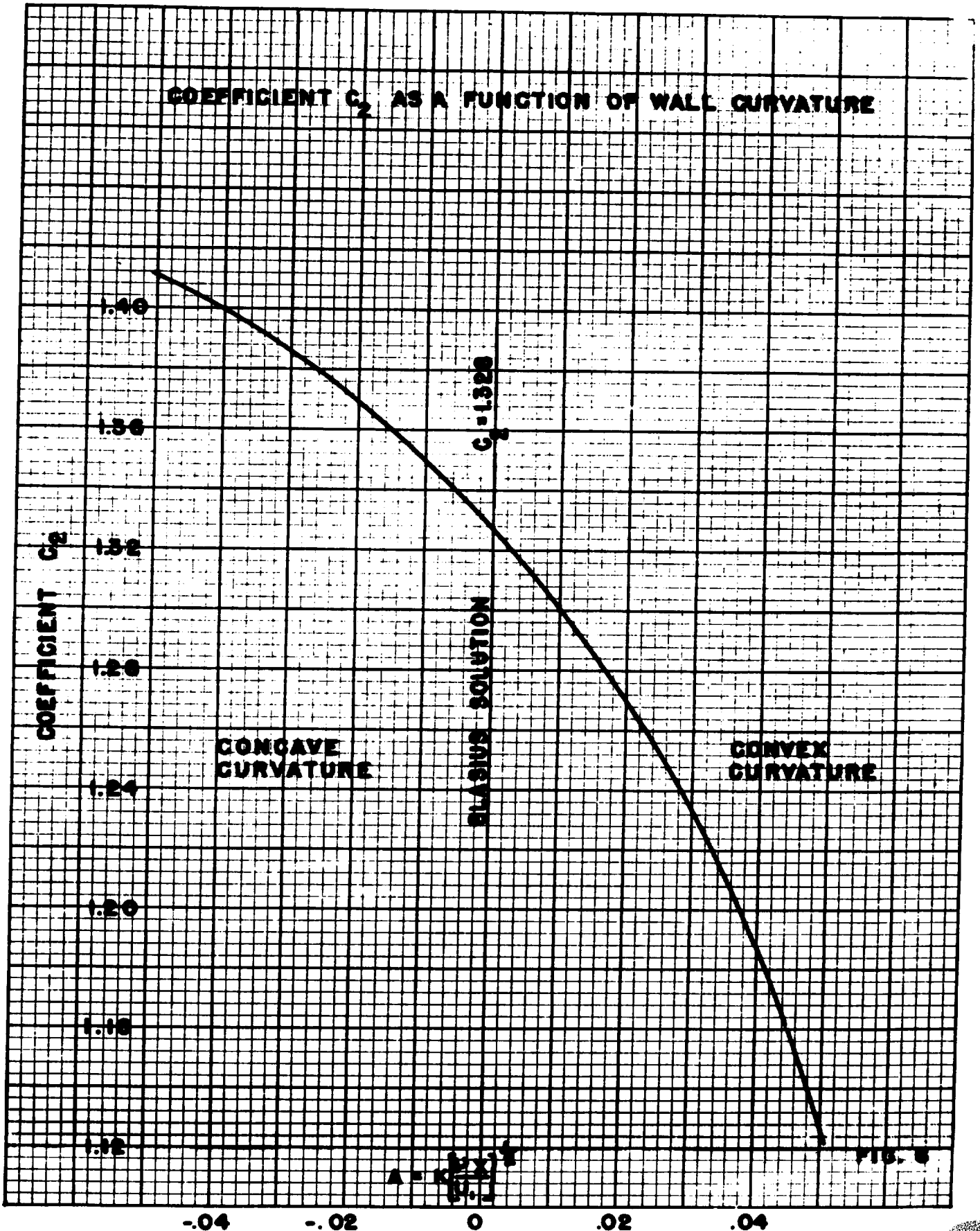

COEFFICIENT C_2 AS A FUNCTION OF WALL CURVATURE
COEFFICIENT C_{f2}
1.40
1.36
1.32
1.28
1.24
1.20
1.16
1.12
C_{f2} = 1.328
BLASIUS SOLUTION
CONCAVE CURVATURE
CONVEX CURVATURE
$A = K \frac{\delta^*}{r}$
-.04
-.02
0
.02
.04

FIG. 6

A comparison between approximate velocity profiles for flow over convex and concave surfaces is given in Fig. 7. The profile for concave surfaces is fuller than that for convex surfaces. Also, it should be noted that the surface curvature influences the outer portion of the profile more than the portion near the wall, due to the modification of the outer boundary condition.

Section 2. Flow Along a Surface with Large Curvature

Consider the equations derived in Chapter III for flow along a surface having large curvature. They are

$$\frac{1}{1+Ky}\,u\,\frac{\partial u}{\partial x}+v\,\frac{\partial u}{\partial y}+\frac{K\,u\,v}{1+Ky} = -\frac{1}{1+Ky}\,\frac{1}{\rho}\,\frac{\partial p}{\partial x}+\nu\left[\frac{\partial^2 u}{\partial y^2}+\frac{K}{1+Ky}\,\frac{\partial u}{\partial y}-\frac{K^2 u}{(1+Ky)^2}\right] , \quad (5.38a)$$

$$\frac{K\,u^2}{1+Ky} = \frac{1}{\rho}\,\frac{\partial p}{\partial y} , \quad (5.38b)$$

and

$$\frac{\partial u}{\partial x}+\frac{\partial}{\partial y}\left[(1+Ky)v\right] = 0 . \quad (5.38c)$$

Introduce a stream function Ψ which satisfies (5.38c), i.e., such that

$$u = \frac{\partial \Psi}{\partial y} \quad \text{and} \quad v = -\frac{1}{1+Ky}\,\frac{\partial \Psi}{\partial x} . \quad (5.39)$$

Now let us make the transformation of independent variables

$$\eta = \frac{y}{2}\left(\frac{U_1}{\nu x}\right)^{1/2}$$

and (5.40)

$$\xi = x ,$$

where U_1 is a constant having the dimensions of velocity. Let us again seek solutions for which Ψ has the form

$$\Psi = (U_1\,\nu\,\xi)^{1/2}\,f(\eta) ; \quad (5.41)$$

so that

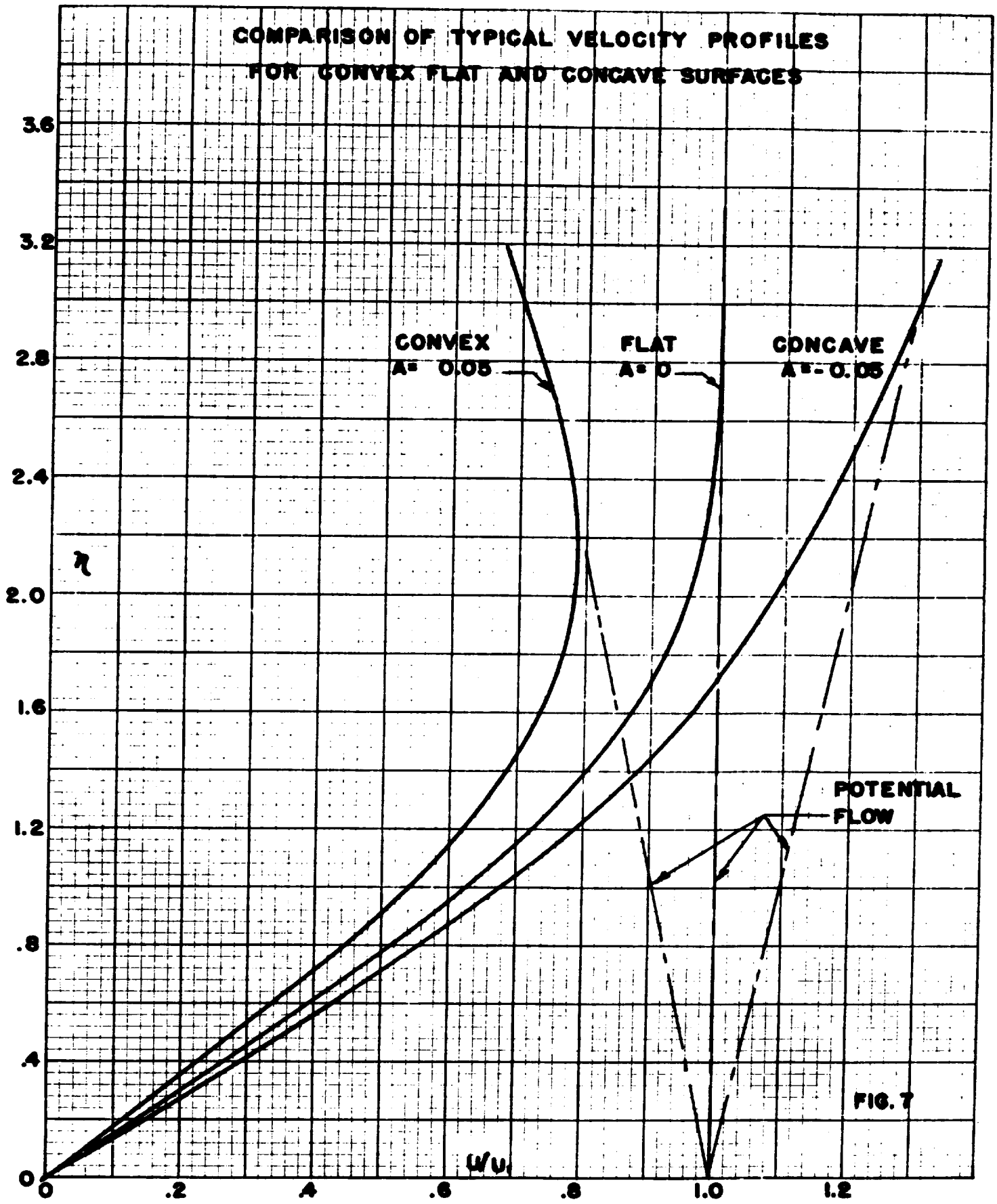

COMPARISON OF TYPICAL VELOCITY PROFILES
FOR CONVEX FLAT AND CONCAVE SURFACES
CONVEX
A= 0.05
FLAT
A= 0
CONCAVE
A=-0.05
POTENTIAL
FLOW
η
u/U
0
.2
.4
.6
.8
1.0
1.2
1.6
2.0
2.4
2.8
3.2
3.6

FIG. 7

$$u = \frac{1}{2} U_1 f' , \qquad \frac{\partial u}{\partial x} = -\frac{1}{4}\frac{U_1}{\xi}\eta f'' , \qquad \frac{\partial u}{\partial y} = \frac{U_1}{4}\left(\frac{U_1}{\nu\xi}\right)^{1/2} f''$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{U_1}{8}\left(\frac{U_1}{\nu\xi}\right) f''' , \quad \text{and} \quad v = \frac{1}{2(1+2A\eta)}\left(\frac{U_1\nu}{\xi}\right)^{1/2}(f'\eta - f) . \tag{5.42}$$

Then let the curvature vary with ξ as

$$K = A\left(\frac{U_1}{\nu\xi}\right)^{1/2} , \qquad A = \mathcal{O}(1) ; \tag{5.43}$$

so that

$$Ky = Ay\left(\frac{U_1}{\nu x}\right)^{1/2} = 2A\eta . \tag{5.44}$$

Substitution of (5.44) and (5.42) into (5.38a) and (5.38b) yields

$$(1+2A\eta)\left(\frac{1}{2}U_1 f'\right)\left(-\frac{1}{4}\frac{U_1}{\xi}\eta f''\right) + \left(\frac{1+2A\eta}{2}\right)\left(\frac{U_1\nu}{\xi}\right)^{1/2}(f'\eta - f)\frac{U_1}{4}\left(\frac{U_1}{\nu\xi}\right)^{1/2} f''$$

$$+ A\left(\frac{U_1}{\nu\xi}\right)^{1/2}\left(\frac{1}{2}U_1 f'\right)\frac{1}{2}\left(\frac{U_1\nu}{\xi}\right)^{1/2}(f'\eta - f) = -\frac{1+2A\eta}{\rho}\left(-\frac{\eta}{2\xi}\frac{\partial p}{\partial \eta} + \frac{\partial p}{\partial \xi}\right)$$

$$+ \nu\left[(1+2A\eta)^2\frac{U_1}{8}\frac{U_1}{\nu\xi} f''' + A\left(\frac{U_1}{\nu\xi}\right)^{1/2}(1+2A\eta)\frac{U_1}{4}\left(\frac{U_1}{\nu\xi}\right)^{1/2} f'' - A^2\left(\frac{U_1}{\nu\xi}\right)\frac{U_1 f'}{2}\right] \tag{5.45}$$

and

$$\frac{\partial p}{\partial \eta} = \frac{\rho A U_1^{\,2} f'^2}{2(1+2A\eta)} . \tag{5.46}$$

Now, using (5.46) in (5.45) and assuming that

$$\frac{\partial p}{\partial \xi} = 0 \tag{5.47}$$

yields

$$f''' + ff'' + 2Aff' + 2A\eta ff'' + 4A\eta f''' + 4A^2\eta^2 f''' + 2Af'' + 4A^2\eta f'' - 4A^2 f' = 0 . \tag{5.48}$$

Boundary conditions are

$$f = f' = 0 \qquad \text{at} \qquad \eta = 0$$

and

$$f' \longrightarrow \frac{2}{1 + 2A\eta} \qquad \text{for } \eta \text{ large} \; . \tag{5.49}$$

Again, assume a solution of the form

$$f = \sum_{i=0}^{\infty} \frac{c_i \eta^i}{i!} \; . \tag{5.50}$$

In view of (5.49),

$$c_0 = c_1 = 0 \; . \tag{5.51}$$

Substitution of (5.50) in (5.48), expanding, and equating the coefficients of powers of η to zero gives

$$\begin{array}{lll}
c_2 & c_3 = -2\,Ac_2 & c_4 = 12\,A^2c_2 \\
c_5 = -c_2^2 - 96\,A^3c_2 & c_6 = 10\,Ac_2^2 & c_7 = -128\,A^2c_2^2 - 11{,}520\,A^5c_2 \\
c_8 = 11c_2^2 + 1832\,A^3c_2^2 + 161{,}280\,A^6c_2 & c_9 = -324\,Ac_2^3 - 29{,}520\,A^4c_2^2 - 2{,}580{,}480\,A^7c_2 & c_{10} = 8{,}184\,A^2c_2^3 + 531{,}936\,A^5c_2^2 + 46{,}448{,}640\,A^8c_2 \\
c_{11} = -375\,c_2^4 - 203{,}784\,A^3c_2^3 - 10{,}623{,}744\,A^6c_2^2 - 928{,}972{,}800\,A^9c_2 & \vdots & \vdots \\
\vdots & \vdots & \vdots
\end{array} \tag{5.52}$$

Hence the function f takes the form

$$f = c_2^{1/3} F_o(c_2^{1/3}\eta) - AG_1(c_2^{1/3}\eta) + \frac{A^2}{c_2^{1/3}} G_2(c_2^{1/3}\eta) + \ldots + \frac{(-1)^n A^n}{c_2^{(n-1)/3}} G_n(c_2^{1/3}\eta) + \ldots \; , \tag{5.53}$$

$$F_o = \frac{(c_2^{1/3}\eta)^2}{2!} - \frac{(c_2^{1/3}\eta)^5}{5!} + \frac{11(c_2^{1/3}\eta)^8}{8!} - \frac{375(c_2^{1/3}\eta)^{11}}{11!}$$

$$+ \frac{27,897(c_2^{1/3}\eta)^{14}}{14!} - \cdots ,$$

$$G_1 = \frac{2(c_2^{1/3}\eta)^3}{3!} - \frac{10(c_2^{1/3}\eta)^6}{6!} + \frac{324(c_2^{1/3}\eta)^9}{9!} - \cdots ,$$

$$G_2 = \frac{12(c_2^{1/3}\eta)^4}{4!} - \frac{128(c_2^{1/3}\eta)^7}{7!} + \frac{8184(c_2^{1/3}\eta)^{10}}{10!} - \cdots ,$$

$$G_3 = \frac{96(c_2^{1/3}\eta)^5}{5!} - \frac{1832(c_2^{1/3}\eta)^8}{8!} + \frac{203,784(c_2^{1/3}\eta)^{11}}{11!} - \cdots ,$$

$$G_4 = \frac{960(c_2^{1/3}\eta)^6}{6!} - \frac{29,520(c_2^{1/3}\eta)^9}{9!} + \cdots , \qquad (5.54)$$

$$G_5 = \frac{11,520(c_2^{1/3}\eta)^7}{7!} - \frac{531,936(c_2^{1/3}\eta)^{10}}{10!} + \cdots ,$$

$$G_6 = \frac{161,280(c_2^{1/3}\eta)^8}{8!} - \frac{10,623,744(c_2^{1/3}\eta)^{11}}{11!} ,$$

$$G_7 = \frac{2,580,480(c_2^{1/3}\eta)^9}{9!} - \cdots ,$$

$$G_8 = \frac{46,448,640(c_2^{1/3}\eta)^{10}}{10!} - \cdots ,$$

$$G_9 = \frac{928,972,800(c_2^{1/3}\eta)^{11}}{11!} + \cdots ,$$

Again the series F_o is that obtained by Blasius for flow along a flat plate. The outer boundary condition (5.49) requires that

$$C_2^{2/3} F_o'(C_2^{1/3}\eta) - A\, C_2^{1/3}\, G_1'(C_2^{1/3}\eta) + A^2\, G_2'(C_2^{1/3}\eta)$$

$$+ \ldots + \frac{(-1)^n A^n G_n'(C_2^{1/3}\eta)}{C_2^{(n-2)/3}} + \ldots \rightarrow \frac{2}{1 + 2A\eta}\,, \quad \eta \text{ large}\,. \qquad (5.55)$$

This can be written in the simpler form

$$C_2^{2/3}\left[F_o'(C_2^{1/3}\eta) - A\eta E_1'(C_2^{1/3}\eta) + \eta^2 A^2 E_2'(C_2^{1/3}\eta)\right.$$

$$\left. + \ldots + (-1)^n A^n \eta^n E_n'(C_2^{1/3}\eta)\right] \rightarrow \frac{2}{1 + 2A\eta}\,, \quad \eta \text{ large}\,, \qquad (5.56)$$

where

$$E_n' = \frac{G_n'(C_2^{1/3}\eta)}{C_2^{n/3}\,\eta^n}\,. \qquad (5.57)$$

By solution of (5.56), the unknown C_2 may be evaluated in terms of A. This solution will not be completed here. However, it is possible to show from (5.53) that the slope of the velocity profile at the wall is

$$\left.\frac{\partial u}{\partial y}\right]_{y=0} \sim f''_{\eta=0} = C_2 > 0\,, \qquad (5.58)$$

and the second derivative of the profile at the wall is

$$\left.\frac{\partial^2 u}{\partial y^2}\right]_{y=0} \sim f'''_{\eta=0} = -2AC_2 < 0\,. \qquad (5.59)$$

Section 3. Discussion of Solutions for Large and Moderate Surface Curvature

A comparison of the solution for large curvature (5.53) with that for moderate curvature (5.23) shows that for equal values of C_2, the velocity profile for large convex curvature is less full near the surface than that for moderate convex curvature. It is expected that C_2 for large convex curvature will be less than C_2 for moderate curvature on the basis of the solution given in Section 1. Therefore, the conclusion that large convex surface curvature gives a less full velocity profile than does modera

convex surface curvature is valid. No inflection point occurs in the profile for large curvature, as shown by (5.59) (see Ref. 9).

Thus, for the flows over the particular surfaces considered here, separation does not occur, no matter how large the surface curvature. However, the velocity profiles obtained for moderately convex surfaces would appear to be more sensitive to an adverse surface pressure gradient ($\partial p/\partial x > 0$) than the profiles for moderately concave surfaces.

It should be noted that the numerical values obtained in the solution for moderately curved surfaces are approximate. These values could be improved upon by numerical integration of (5.13), subject to the boundary conditions (5.14) and (5.15). The solution for large curvature also can be completed accurately by numerical integration of (5.48) with the boundary conditions (5.49).

CHAPTER VI

DISCUSSIONS CONCERNING GENERAL SOLUTIONS

Section 1. Considerations from Integral Form of Equations of Motion

We are concerned with developing a method for solving the general problem of flow over a surface with arbitrary moderate curvature. Before proceeding with this, however, it is possible to make some qualitative remarks concerning the effects of surface curvature on flow in a boundary layer by writing the equations in a partially integrated form. Consider the equations for moderate curvature,

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \tag{6.1a}$$

$$Ku^2 = \frac{1}{\rho} \frac{\partial p}{\partial y} \tag{6.1b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad , \tag{6.1c}$$

with the boundary conditions

$$\begin{aligned} u &= v = 0 \qquad \text{for} \qquad y = 0 \\ u &= U(x)\, e^{-Ky} \qquad \text{for} \qquad y = \delta \quad . \end{aligned} \tag{6.2}$$

Integrate (6.1b) to obtain

$$\int_y^\delta \frac{\partial p}{\partial y} \, dy = \rho \int_y^\delta Ku^2 \, dy = p_\delta - p \quad . \tag{6.3}$$

Differentiation with respect to x yields

$$- \frac{1}{\rho} \frac{\partial p}{\partial x} = - \frac{1}{\rho} \frac{\partial p_\delta}{\partial x} + \frac{\partial}{\partial x} \int_y^\delta Ku^2 \, dy$$

$$= - \frac{1}{\rho} \frac{\partial p_\delta}{\partial x} + \frac{\partial K}{\partial x} \int_y^\delta u^2 dy - 2K \int_y^\delta u \frac{\partial u}{\partial x} \, dy + KU^2 e^{-2K\delta} \frac{d\delta}{dx} \quad . \tag{6.4}$$

Substitution of (6.4) in (6.1a) yields

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p_\delta}{\partial x}+2K\int_y^\delta u\frac{\partial u}{\partial x}\,dy+\frac{\partial K}{\partial x}\int_y^\delta u^2dy+KU^2e^{-2K\delta}\frac{d\delta}{dx}+\nu\frac{\partial^2 u}{\partial y^2} \quad (6.5)$$

as the equation of motion.

If now the two surface boundary conditions (6.2) are applied, (6.5) becomes

$$\frac{1}{\rho}\frac{\partial p_\delta}{\partial x}-2K\int_0^\delta u\frac{\partial u}{\partial x}\,dy-\frac{\partial K}{\partial x}\int_0^\delta u^2dy-KU^2e^{-2K\delta}\frac{d\delta}{dx} = \nu\frac{\partial^2 u}{\partial y^2}\Big]_{y=0} \quad (6.6)$$

Then, setting

$$\left(\frac{\partial p}{\partial x}\right)_{\text{effective}} = \nu\frac{\partial^2 u}{\partial y^2}\Big]_{y=0}, \quad (6.7)$$

interpretation of the effects of surface curvature on boundary-layer flow in terms of alterations to the pressure gradient is possible.

When K is zero,

$$\frac{\partial p_\delta}{\partial x} = \left(\frac{\partial p}{\partial x}\right)_{\text{effective}}. \quad (6.8)$$

However, (6.6) shows that when K and $\partial K/\partial x$ are different from zero, the effective pressure gradient is more or less favorable than $\partial p_\delta/\partial x$, depending on the sign of K and $\partial K/\partial x$ as well as on the sign of $\partial u/\partial x$ and $d\delta/dx$. Let us assume that $d\delta/dx > 0$ for all possible cases. Then, for $K < 0$, the term $KU^2(d\delta/dx)e^{-2K\delta}$ makes the effective pressure gradient less favorable (more positive or less negative); and for $K > 0$, this term causes the effective pressure gradient to be more favorable. The contribution to $(\partial p/\partial x)_{\text{effective}}$ of the other terms on the left side of (6.6) may be found in a similar manner. These contributions are given in Table I.

TABLE I

CONTRIBUTION OF TERMS IN (6.6) TO EFFECTIVE PRESSURE GRADIENT

Contribution of Term:	
$-2K \int_0^\delta u \frac{\partial u}{\partial x} dy$	more favorable for $K > 0$ less favorable for $K < 0$ } if $\frac{\partial u}{\partial x} > 0$ more favorable for $K < 0$ less favorable for $K > 0$ } if $\frac{\partial u}{\partial x} < 0$
Contribution of Term: $-\frac{\partial K}{\partial x} \int_0^\delta u^2 dy$	more favorable for $\partial K/\partial x > 0$ less favorable for $\partial K/\partial x < 0$
Contribution of Term: $-KU^2 \frac{d\delta}{dx} e^{-2K\delta}$	more favorable for $K > 0$ less favorable for $K < 0$

Now let us consider the orders of magnitude of the respective terms in Equation (6.6). The terms $\partial p_\delta/\partial x$ and $\nu(\partial^2 u/\partial y^2)_{y=0}$ are $\mathcal{O}(1)$. The term $-2K \int_0^\delta u(\partial u/\partial x)\, dy$ is $\mathcal{O}(\delta)$ and $-KU^2(d\delta/dx)e^{-2K\delta}$ is $\mathcal{O}(\delta)$. Thus these terms will not influence the pressure gradient appreciably. However, in Chapter III, we developed an analysis which showed that $\partial K/\partial x$ could be as large as $\mathcal{O}(1/\delta)$ and the equation of motion (6.5) would remain valid. Applying this fact indicates that the term $(\partial K/\partial x) \int_0^\delta u^2 dy$ can be of $\mathcal{O}(1)$. Hence, it is the term which can affect the pressure gradient most strongly, and by the results in Table I we see that the important case arises when $\partial K/\partial x < 0$. Physical considerations indicate that $\partial p_\delta/\partial x$ should be a smooth function of x. Then the above reasoning indicates the possibility of a reversal in sign of the pressure gradient $\partial p/\partial x$ within the boundary layer at a point along a surface where the curvature changes rapidly in such a manner that $\partial K/\partial x < 0$. For example, consider

the surface illustrated in Figure 8.

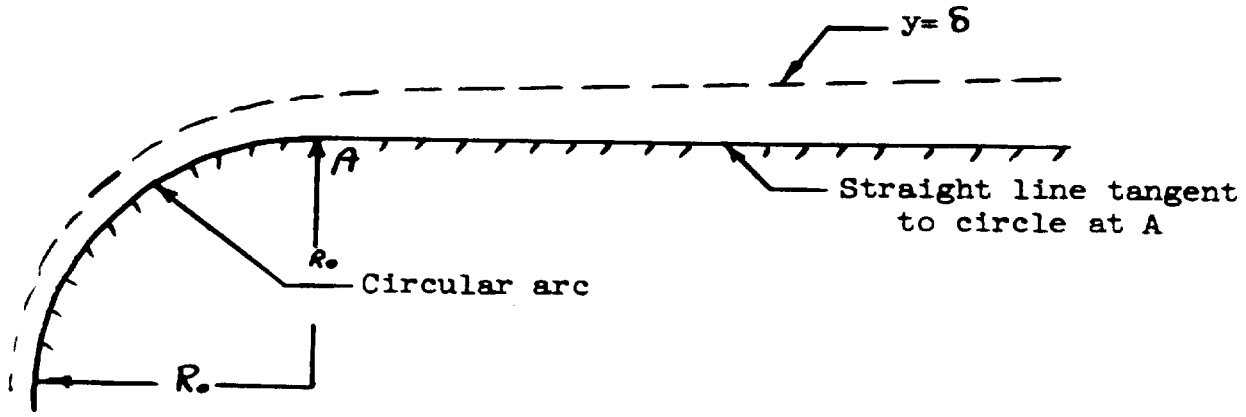


Fig. 8. Illustrative Surface.

At point A, a circular arc having curvature $1/R_o$ of $\mathcal{O}(1)$ goes into a tangent to the arc. At this point $\partial K/\partial x < 0$ and is infinitely large. Our equations are not valid then, because we require $\partial K/\partial x$ to be at most of $\mathcal{O}(1/\delta)$. However, by a slight modification of the surface, we can satisfy the condition $\partial K/\partial x = \mathcal{O}(1/\delta)$ at this point. Then the equations we have developed are valid and we see that the surface pressure distribution may change from a favorable pressure gradient upstream of A to an unfavorable gradient at A while $\partial p_\delta/\partial x$ is favorable both upstream and downstream of A. This reversal in sign of the surface pressure gradient could lead to separation at point A. When separation occurs, the expected pressure increase does not take place and the conjectures based on the boundary-layer equations of motion are no longer valid, since the assumptions used in deriving them are not met.

Let us now consider the equations developed in Chapter III for a surface with large curvature. They are

$$\frac{1}{1+Ky}\,u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} + \frac{K\,u\,v}{1+Ky} = -\frac{1}{1+Ky}\,\frac{1}{\rho}\,\frac{\partial p}{\partial y} + \nu\left[\frac{\partial^2 u}{\partial y^2} + \frac{K}{1+Ky}\,\frac{\partial u}{\partial y} - \frac{K^2 u}{1+Ky}\right], \quad (6.9)$$

and $$\frac{Ku^2}{1+Ky} = \frac{1}{\rho}\frac{\partial p}{\partial y} \quad . \tag{6.10}$$

Integration of (6.10) with respect to y yields

$$\frac{1}{\rho}(p_\delta - p) = \int_y^\delta \frac{K}{1+Ky} u^2 \, dy \quad ,$$

which becomes upon differentiation with respect to x

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} = -\frac{1}{\rho}\frac{\partial p_\delta}{\partial x} + \frac{\partial}{\partial x}\int_y^\delta \frac{K}{1+Ky} u^2 \, dy \quad . \tag{6.11}$$

Substitution of (6.11) in (6.9) gives

$$\frac{1}{1+Ky} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{Kuv}{1+Ky} = -\frac{1}{\rho(1+Ky)}\frac{\partial p_\delta}{\partial x} + \frac{1}{1+Ky}\frac{\partial}{\partial x}\int_y^\delta \frac{K}{1+Ky} u^2 \, dy$$

$$+ \nu\left\{\frac{\partial^2 u}{\partial y^2} + \frac{K}{1+Ky}\frac{\partial u}{\partial y} - \frac{K^2 u}{(1+Ky)^2}\right\} . \tag{6.12}$$

Applying the conditions $u = v = 0$ at $y = 0$ gives

$$\frac{1}{\rho}\frac{\partial p_\delta}{\partial x} - \frac{\partial}{\partial x}\int_0^\delta \frac{K}{1+Ky} u^2 dy - \nu K \frac{\partial u}{\partial y}\Big]_{y=0} = \nu \frac{\partial^2 u}{\partial y^2}\Big]_{y=0} = \frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_{\text{effective}} . \tag{6.13}$$

The upper limit of the term involving the integration is obtained by placing $y = 0$ under the integral sign. Then (6.13) becomes

$$\frac{1}{\rho}\frac{\partial p_\delta}{\partial x} - 2K\int_0^\delta u\frac{\partial u}{\partial x}\, dy - \frac{\partial K}{\partial x}\int_0^\delta u^2 dy - \frac{KU^2}{(1+K\delta)^2}\frac{d\delta}{dx} - \nu K\frac{\partial u}{\partial y}\Big]_{y=0} = \frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_{\text{effective}} . \tag{6.14}$$

Comparison of (6.14) with (6.6) shows that the effective pressure gradient is the same for both large and moderate curvature, with the exception of the term $-\nu K\,(\partial u/\partial y)_{y=0}$ in (6.14). The contribution of this additional term to the effective pressure gradient depends only on the sign

of K, since $(\partial u/\partial y)_{y=0} > 0$, except at separation. Then if $K > 0$, the additional term causes the effective pressure gradient to be more favorable.

Table III shows how each term in (6.14) contributes to the effective pressure gradient for the cases

$$\left.\begin{array}{ll} \text{(a)} & K \sim x^n \\ \text{(b)} & K \sim \dfrac{1}{x^n} \\ \text{(c)} & K = 0\,, \quad \dfrac{\partial K}{\partial x} \neq 0 \\ \text{(d)} & K = \text{const.}, \quad \dfrac{\partial K}{\partial x} = 0 \end{array}\right\} \quad K = \mathcal{O}\left(\frac{1}{\delta}\right) > 0 \ .$$

The solution obtained in Chapter V belongs to class (b). The sign of $\partial u/\partial x$ in this solution is obtained by consideration of the equation

$$\frac{\partial u}{\partial x} = -\frac{1}{4}\frac{U_1}{\xi}\,\eta\, f'' \ .$$

The sign of f" near the surface is

$$f''_{\eta=0} = C_2 > 0 \ .$$

Hence, $\partial u/\partial x < 0$ for this solution, and Table II indicates that the terms $(\partial K/\partial x)\int_0^\delta u^2 dy$ and $2K\int_0^\delta u(\partial u/\partial x)dy$ cause the effective pressure gradient to be less favorable. This should result in a velocity profile which is less full than that on a flat plate, which is in agreement with the results obtained in Chapter V.

Section 2. Formulation of General Solution for Arbitrary Moderate Surface Curvature

The equations of motion in the form (6.5) and (6.12) are satisfactory for a qualitative understanding of the effects of surface curvatu

TABLE II

CONTRIBUTIONS OF TERMS IN (6.14) TO EFFECTIVE PRESSURE GRADIENT

Case	Contribution of Term: $\frac{\partial K}{\partial x}\int_0^\delta u^2 dy$	Contribution of Term: $2K\int_0^\delta u\frac{\partial u}{\partial x}dy$	Contribution of Term: $\nu K\frac{\partial u}{\partial y}\Big]_{y=0}$	Contribution of Term: $K\frac{d\delta}{dx}\frac{U^2}{(1+K\delta)^2}$
(a) $K \sim x^n$ $\frac{\partial K}{\partial x} \sim nx^{n-1}$ $K > 0$	More favorable	More favorable if $\frac{\partial u}{\partial x} > 0$ Less favorable if $\frac{\partial u}{\partial x} < 0$	More favorable	More favorable
(b) $K \sim \frac{1}{x^n}$ $\frac{\partial K}{\partial x} \sim -\frac{n}{x^{n+1}}$	Less favorable	More favorable if $\frac{\partial u}{\partial x} > 0$ Less favorable if $\frac{\partial u}{\partial x} < 0$	More favorable	More favorable
(c) $K = 0$ $\frac{\partial K}{\partial x} \neq 0$ $\frac{\partial K}{\partial x} > 0$	More favorable	None	None	None
(c') $K = 0$ $\frac{\partial K}{\partial x} < 0$	Less favorable	None	None	None
(d) $K = \text{const.}$ $K > 0$ $\frac{\partial K}{\partial x} = 0$	None	More favorable if $\frac{\partial u}{\partial x} > 0$ Less favorable if $\frac{\partial u}{\partial x} < 0$	Move favorable	More favorable

on boundary-layer flow in the general case, but are difficult to handle mathematically for quantitative evaluation of these effects. In this section a method of solution of the general problem of flow over a two-dimensional surface with arbitrary moderate surface curvature will be presented. The approach is similar to that of Howarth (1934) for flows in which the second equation of motion is neglected. Consider the equations of motion and the continuity equation (6.1a - 6.1c) with the boundary conditions (6.2). The pressure, p, may be eliminated from (6.1a) and (6.1b) to obtain one equation of motion

$$u\frac{\partial^2 u}{\partial x \partial y} + v\frac{\partial^2 u}{\partial y^2} + 2Ku\frac{\partial u}{\partial x} + u^2\frac{\partial K}{\partial x} = \nu\frac{\partial^3 u}{\partial y^3} \quad . \tag{6.15}$$

The equation of continuity may be satisfied by introduction of a stream function Ψ, such that

$$u = \frac{\partial \Psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \Psi}{\partial x} \quad , \tag{6.16}$$

and in terms of Ψ the equation of motion (6.15) becomes:

$$\frac{\partial \Psi}{\partial y}\frac{\partial^3 \Psi}{\partial y^2 \partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial^3 \Psi}{\partial y^3} + 2K\frac{\partial \Psi}{\partial y}\frac{\partial^2 \Psi}{\partial y \partial x} + \left(\frac{\partial \Psi}{\partial y}\right)^2\frac{\partial K}{\partial x} = \nu\frac{\partial^4 \Psi}{\partial y^4} \quad . \tag{6.17}$$

In terms of the stream function Ψ, the boundary conditions may be expressed as

$$\frac{\partial \Psi}{\partial y} = \frac{\partial \Psi}{\partial x} = 0 \qquad \text{at} \qquad y = 0$$

$$\lim_{y \to \infty} \frac{\partial \Psi}{\partial y} \longrightarrow U_o(x)\left[1 - K(x)\, y\right] \tag{6.18}$$

$$\lim_{y \to \infty} \frac{\partial^2 \Psi}{\partial y^2} \longrightarrow -K\, U_o(x)\left[1 - K(x)\, y\right] \quad ,$$

where $U_o(x)$ is the surface velocity determined by the potential flow over the surface.

The surface curvature of an arbitrary two-dimensional surface may be expressed in the form

$$K = K_o + K_1 x + K_2 x^2 + K_3 x^3 + \ldots , \tag{6.19}$$

and we shall assume that the stream function may be expressed in the form

$$\Psi = F_1(y)\, x + F_2(y)\, x^2 + F_3(y)\, x^3 + \ldots \tag{6.20}$$

Substituting (6.20) in (6.17) and equating the coefficients of the powers of x to zero yields the differential equations

$$\text{(a)}\quad F_1'F_1'' - F_1F_1''' + 2K_oF_1'F_1' = \nu F_1'''' ,$$

$$\text{(b)}\quad 2F_2''F_1' + F_1''F_2' - F_1F_2''' - 2F_2F_1''' + 2K_1F_1'F_1' + 6K_oF_2'F_1' + K_1F_1''F_1' = \nu F_2'''' ,$$

$$\text{(c)}\quad 3F_3''F_1' + 2F_2''F_2' + F_1''F_3' - F_1F_3''' - 2F_2F_2''' - 3F_3F_1''' + 2K_o(4F_1'F_3' + 2F_2'F_2') + 4K_2F_1'F_1' + 8K_1F_1'F_2' = \nu F_3'''' , \tag{6.21}$$

$$\text{(d)}\quad 4F_4''F_1' + 3F_3''F_2' + 2F_2''F_3' + F_1''F_4' - F_1F_4''' - 2F_2F_3''' - 3F_3F_2''' - 4F_4F_1''' + 2K_o(5F_1'F_4' + 5F_2'F_3') + 5K_3F_1'F_1' + 10K_2F_2'F_1' + K_1(5F_2'F_2' + 10F_1'F_3') = \nu F_4'''' ,$$

and

$$\text{(e)}\quad 5F_5''F_1' + 4F_4''F_2' + 3F_3''F_3' + 2F_2''F_4' + F_1''F_5' - F_1F_5''' - 2F_2F_4''' - 3F_3F_3''' - 4F_4F_2''' - 5F_5F_1''' + 2K_o(6F_1'F_5' + 6F_2'F_4' + 3F_3'F_3') + 12K_1(F_1'F_4' + F_2'F_3') + 6K_2(F_2'F_2' + 2F_1'F_3') + 12K_3F_1'F_2' + 6K_4F_1'F_1' = \nu F_5'''' ,$$

where the primes indicate differentiation with respect to y.

Let us assume that the velocity, U_o, determined by the potential flow may be expressed in the form

$$U_o(x) = \beta_1 x + \beta_2 x^2 + \beta_3 x^3 + \ldots ; \tag{6.22}$$

then the outer boundary conditions (6.18) may be written as

$$\begin{aligned} F_1'(\infty) &= \beta_1(1 - K_o y) \\ F_2'(\infty) &= \beta_2(1 - K_o y) - K_1 y \beta_1 \\ F_3'(\infty) &= \beta_3(1 - K_o y) - K_1 y \beta_2 - K_2 y \beta_1 \\ F_4'(\infty) &= \beta_4(1 - K_o y) - K_1 y \beta_3 - K_2 y \beta_2 - K_3 y \beta_1 \end{aligned} \tag{6.23a}$$

and

$$\begin{aligned} F_1''(\infty) &= -(1 - K_o y)\, K_o \beta_1 \\ F_2''(\infty) &= -(1 - K_o y)(K_o \beta_2 + K_1 \beta_1) + K_o K_1 \beta_1 y \\ F_3''(\infty) &= -(1 - K_o y)(K_o \beta_3 + K_1 \beta_2 + K_2 \beta_1) + K_1(K_o \beta_2 + K_1 \beta_1) y + K_2 K_o \beta_1 y \\ F_4''(\infty) &= -(1 - K_o y)(K_o \beta_4 + K_1 \beta_3 + K_2 \beta_2 + K_3 \beta_1) + K_1(K_o \beta_3 + K_1 \beta_2 \\ &\quad + K_2 \beta_1) y + K_2(K_o \beta_2 + K_1 \beta_1) y + K_o K_3 \beta_1 y \quad . \end{aligned} \tag{6.23b}$$

The inner boundary conditions require that

$$F_1(0) = F_1'(0) = F_2(0) = F_2'(0) = F_3(0) = F_3'(0) = F_4(0) = F_4'(0) = 0 \ . \tag{6.24}$$

The series of equations (6.21) may be made partially nondimensional by introduction of the functions

$$\begin{aligned} \eta &= \left(\frac{\beta_1}{\nu}\right)^{1/2} y \ , \\ F_1 &= (\beta_1 \nu)^{1/2} f_1 \ , \\ F_2 &= 3\beta_2 \left(\frac{\nu}{\beta_1}\right)^{1/2} f_2 \ , \\ F_3 &= 4 \left(\frac{\nu}{\beta_1}\right)^{1/2} \left[\beta_3 g_3 + \frac{\beta_2^2}{\beta_1} h_3 \right] \ , \end{aligned} \tag{6.25a}$$

$$F_4 = 5\left(\frac{\nu}{\beta_1}\right)^{1/2}\left[\beta_4 g_4 + \frac{\beta_2\beta_3}{\beta_1} h_4 + \frac{\beta_2^3}{\beta_1^2} j_4\right] ,$$

$$K_o = k_o\left(\frac{\beta_1}{\nu}\right)^{1/2} , \qquad K_2 = \frac{k_2\beta_3}{\beta_1}\left(\frac{\beta_1}{\nu}\right)^{1/2} , \quad (6.25b)$$

$$K_1 = \frac{k_1\beta_2}{\beta_1}\left(\frac{\beta_1}{\nu}\right)^{1/2} , \quad \text{and} \quad K_3 = \frac{k_3\beta_4}{\beta_1}\left(\frac{\beta_1}{\nu}\right)^{1/2} .$$

Equations (6.21) then become

(a) $f_1'f_1'' - f_1f_1''' + 2k_of_1'f_1' = f_1^{iv}$,

(b) $6f_2''f_1' + 3f_1''f_2' - 3f_1f_2''' - 6f_2f_1''' + 3k_1f_1'f_1' + 18k_of_2'f_1' = 3f_2^{iv}$,

(c) $12g_3''f_1' + 4g_3'f_1'' - 4f_1g_3''' - 12g_3f_1''' + 32k_of_1'g_3' + 4k_2f_1'f_1' = 4g_3^{iv}$,

(d) $12h_3''f_1' + 6f_2''f_2' + 4f_1''h_3' - 4f_1h_3'' - 18f_2f_2''' - 12h_3f_1'''$
$+ 32k_oh_3'f_1' + 36k_of_2'f_2' + 24k_1f_2'f_1' = 4h_3^{iv}$, (6.26)

(e) $20g_4''f_1' + 5g_4'f_1'' - 5g_4'''f_1 - 20g_4f_1''' + 50\,k_og_4'f_1' + 5k_3f_1'f_1' = 5g_4^{iv}$,

(f) $20h_4''f_1' + 36g_3''f_2' + 24g_3'f_2'' + 5f_1''h_4' - 5h_4'''f_1 - 24f_2g_3''' - 36g_3f_2'''$
$- 20h_4f_1''' + 50k_oh_4'f_1' + 120k_og_3'f_2' + 40k_1f_1'g_3' + 30k_2f_2'f_1' = 5h_4^{iv}$,

and

(g) $20j_4''f_1' + 36h_3''f_2' + 24f_2''h_3' + 5f_1''j_4' - 5f_1j_4''' - 24f_2h_3''' - 36h_3f_2'''$
$- 20j_4f_1''' + 50k_oj_4f_1' + 120k_oh_3'f_2' + 15k_1f_2'f_2' + 40k_1f_1'h_3' = 5j_4^{iv}$,

where the primes denote differentiation with respect to η.

The boundary conditions then become

$$f_1(0) = f_1'(0) = f_2(0) = \ldots\ldots = h_4'(0) = j_4(0) = j_4'(0) = 0 \ ,$$

$$f_1'(\infty) = 1 - k_o\eta \ , \qquad f_2'(\infty) = \tfrac{1}{3}(1 - k_o\eta) - \tfrac{1}{3}k_1\eta \ ,$$

$$g_3'(\infty) = \tfrac{1}{4}(1 - k_o\eta) - \frac{k_2\eta}{4} \ , \qquad h_3' = -\tfrac{1}{4}k_1\eta \ ,$$

$$g_4'(\infty) = \tfrac{1}{5}(1 - k_o\eta) - \frac{k_3\eta}{5} \ , \quad h_4'(\infty) = -\tfrac{1}{5}k_1\eta - \tfrac{1}{5}k_2\eta \ , \quad j_4'(\infty) = 0 \ ,$$

$$f_1''(\infty) = -(1 - k_o\eta)k_o \ , \qquad f_2''(\infty) = -\tfrac{1}{3}k_o(1 - k_o\eta) - \tfrac{1}{3}k_1 \ ,$$

$$g_3''(\infty) = \tfrac{1}{4}(k_o + k_2)(1 - k_o\eta) + \tfrac{1}{4}k_o k_2\eta \ , \qquad (6.27)$$

$$h_3''(\infty) = -\tfrac{1}{4}(k_1)(1 - k_o\eta) + \frac{k_1}{4}(k_o + k_1)\eta \ ,$$

$$g_4''(\infty) = -\tfrac{1}{5}(k_o + k_3)(1 - k_o\eta) + \tfrac{1}{5}k_o k_3\eta \ ,$$

$$h_4''(\infty) = -\tfrac{1}{5}(k_1 + k_2)(1 - k_o\eta) + \tfrac{1}{5}(k_o + k_2)k_1\eta + \tfrac{1}{5}(k_o + k_1)k_2\eta \ , \quad \text{and}$$

$$j_4''(\infty) = \tfrac{1}{5}k_1 k_1 \eta \ .$$

Thus the function $f_n(\eta)$ can be determined by numerical integration of the equation (6.26) subject to the boundary conditions (6.27). When this has been accomplished, the pressure can be found by solution of Equations (6.1a) and (6.1b).

REFERENCES

1. Hamel, George, "Spiralförmige Bewegungen Zäher Flüssigkeiten", Jahresbericht der Deutschen Mathematiker-Vereinigung, Band 25, 34-60 (1917).

2. von Karman, Th., "Über laminare und turbulente Reibung", Zeitschr. für Angewandte Math. und Mech., 1, 244-247 (1921).

3. Prandtl, L., "Über Flüssigkeitbewegung bei sher kleiner Reibung", Verhandlungen des Dritten Internationalen Mathematiker-Kongress, Heidelberg, 484-491 (1904).

4. Blasius, H., "Grenzschichten in Flüssigkeiten mit kleiner Reibung", Zeitschr. für Mathematik und Physik, Band 16, 1 (1908).

5. Howarth, L., "On the Calculation of Steady Flow in the Boundary Layer Near the Surface of a Cylinder in a Stream", Gt. Brit. Aero. Res. Comm. R and M No. 1632, July, 1934.

6. Goldstein, S., Modern Developments in Fluid Dynamics, Oxford Univ. Press, Vol. 1, 1938, pp. 101-120.

7. Tollmien, W., "Grenzschichttheorie", Beitrag in Wein und Harms, Handbuch der Experimental Physik, Band IV, Part 1, 1931, pp. 241-281.

8. Schmidt, Harry, and Schröder, Kurt, "Laminar Grenzschichten", Luftfahrt-Forschung, Band 19, 26 April, 1942.

9. Kuethe, A.M., and Schetzer, J.D., Foundations of Aerodynamics, John Wiley and Sons, Inc., New York, 1950, p. 231.