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PERIODOGRAMS OF GRADUATION AND SUBTABULATION OPERATORS

by
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CHAPTER I

THE GENERAL NOTION OF A PERIODOGRAM

1. Introduction

The purpose of this chapter is to provide background material and motivation for the work of the succeeding chapters. Such a plan of presentation necessarily requires the immediate introduction of many concepts which are considered in detail in later sections. It would be awkward, lengthy, and at odds with our purposes to attempt to define precisely at this stage all of the terms encountered in this discussion. Therefore, for the moment we shall engage in the mathematically suspect practice of defining a minimum number of terms, while allowing intuition and common language usage to guide the reader. Definitions appearing in this chapter will be repeated at the appropriate time in the later development. In the one instance where there are two different statements defining the same term (periodogram), there is no inconsistency since the two statements are logically equivalent. The second form is to be regarded as the mathematical definition.

2. Graduation

The starting concepts in this study were those of graduation. More specifically, we had in mind graduation as it occurs in actuarial practice, and it is well to bear in mind throughout the discussion exactly where the investigation has its roots.

M. D. Miller [15; p.4] defines graduation as "the proc-

ess of securing from an irregular series of observed values of a continuous variable a smooth regular series of values consistent in a general way with the observed series of values." For instance, an actuary constructing a mortality table calculates graduated rates of mortality q_x for a domain of integral ages x from the mortality rates measured among an observed group of persons.

In general, if a function is to be constructed to represent some physical phenomenon, the normal procedure begins with assignment to each element of a necessarily discrete domain a real number based on some observations of the phenomenon. In case there is reason to believe that this process has produced intuitively undesirable irregularities in the assigned values, a systematic revision of the function values can reduce the irregularity of the function. This revision can at the same time produce a function with a new domain, if such is wanted.

For a great many of the graduation techniques in use, the revision involves taking as the graduated value at x a symmetrically weighted average of observed values at neighboring arguments and it is this approach which we wish to abstract.

It is clear that if the original, observed function is defined for integral arguments then the symmetrically weighted average of an odd number of neighboring values yields a function with the same domain of definition. On the other hand, an even number of nearby values gives rise to a function defined on the so-called <u>half</u>-<u>integers</u>; i.e.

numbers of the form $t+\frac{1}{2}$ where t is an integer.

As an example to justify our consideration of the latter situation, an actuary with data tabulated at "age nearest birthday", might want a function which yields values on an "age last birthday" basis. In this example, data would have been gathered and recorded using the convention that any individual between exact ages $t-\frac{1}{2}$ and $t+\frac{1}{2}$, where t is an integer, is t years old, whence the actuary would have a function defined for integers. If he now were to graduate this function using a symmetrically weighted average of an even number of neighboring values to calculate the revised function, he would have a function defined on the half-integers. The argument $t+\frac{1}{2}$ would now give the appropriate function value for any individual between ages t and t+1. Of course, the natural translation of one half unit would restore the original domain (but not the original function).

3. Notational Conventions

The functional notation u_x will be used in accordance with wide actuarial practice. The combinations v_x and w_x will also be used to denote functions of x. More specifically, in this chapter only, u_x will mean a trivial extension of the function of observed values. The revised function will be denoted as v_x. Thus, in the problem at hand, v_m is a symmetrically weighted average of values of u_x at arguments near m.

A particular function which will be needed shortly and frequently thereafter has for any real number x a value equal

to the greatest integer not larger than x and is usually denoted by [x]. However, we want to reserve the square bracket notation for a special operator which will be introduced in Chapter II. Therefore, we shall use $\underline{\langle x \rangle}$ for the greatest integer not exceeding x.

4. Basic Symmetric Linear Operators

In actuarial problems of graduation, the domain of u_x is the integers. It will be convenient to extend the domain of u_x by defining $u_x = 0$ for x not an integer. Now we can write

$$v_m = \sum_t p_t u_{m+t/2}$$

$$= \ldots + p_{-1} u_{m-\frac{1}{2}} + p_0 u_m + p_1 u_{m+\frac{1}{2}} + \ldots \qquad (1)$$

where the p_t's are symmetric weights and thus $p_t = p_{-t}$. We note that if m is an integer, then those terms of formula (1) in which the coefficient p_t has an odd index are zero; similarly if m is a half-integer, those terms in which p_t has an even index are zero.

If we define the basic symmetric linear operator G_m by

$$G_m w_x = \begin{cases} w_{x+m/2} + w_{x-m/2} & m \neq 0 \\ w_x & m = 0, \end{cases} \qquad (2)$$

where w_x is any real valued function defined on the set of real numbers, we can express (1) in either the form

$$v_m = \left(\sum_{t=0} p_t G_t\right) u_m \qquad (3)$$

or the form

$$v_m = P\, u_m \qquad (4)$$

where

$$P = \sum_{t=0} p_t G_t . \tag{5}$$

The descriptive phrase "basic symmetric linear operator" is applied to G_m for rather transparent reasons. Just as obviously, P is called a symmetric linear operator. We shall defer further discussion of these points until Chapter II in which we consider symmetric linear operators in detail.

Another convention should be noted here. For an operator P the expression $P\, u_m$ means $P\, u_x$ evaluated at x=m; that is, P is applied to the function u_x prior to setting x=m.

5. Periodogram of P

We shall now restrict our attention to the finite situation in which all the p_t's are zero for some index n and beyond. Thus, v_m is based on the sequence

$$\left\{u_{m-(n-1)/2},\ u_{m-(n-3)/2},\cdots,\ u_{m+(n-3)/2},\ u_{m+(n-1)/2}\right\}.$$

Consider the following Fourier sum, adapted from Whittaker and Robinson [23; p.264].

$$w_x = a_0 + \sum_{k=1}^{\langle n/2 \rangle} \left(a_k \cos\frac{2\pi k}{n}(x-m) + b_k \sin\frac{2\pi k}{n}(x-m)\right) \tag{6}$$

where

$$a_0 = (1/n)\sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} u_{m+j} ,$$

$$a_k = (2/n)\sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} u_{m+j}\cos\frac{2j\pi k}{n} \qquad k \neq 0, n/2 ,$$

$$a_{n/2} = 0 ,$$

$$b_k = (2/n)\sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} u_{m+j}\sin\frac{2j\pi k}{n} \qquad k \neq n/2,$$

and

$$b_{n/2} = (1/n)\sum_{j=-\frac{n-1}{2}}^{\frac{n-1}{2}} u_{m+j}\sin j\pi .$$

Now if x is in $\{m-(n-1)/2,\ m-(n-3)/2,\ldots,\ m+(n-1)/2\}$ then $w_x = u_x$; that is, we have a sum of sinusoidal curves which reproduces those values of u_x upon which v_m is based. This fact enables us to find v_m by applying the operator P to w_x and setting x = m. Thus

$$v_m = Pu_m = Pw_m = \left(\sum_{t=0}^{n-1} p_t G_t\right)\left(a_0 + \sum_{k=1}^{\langle n/2\rangle}\left(a_k \cos\frac{2\pi k}{n}(x-m) + b_k \sin\frac{2\pi k}{n}(x-m)\right)\right)\Bigg|_{x=m}.$$

If we apply G_t to $(a\cos\alpha x + b\sin\alpha x)$ we have

$$G_t(a\cos\alpha x + b\sin\alpha x) = \begin{cases} (2\cos(\alpha t/2))(a\cos\alpha x + b\sin\alpha x) & t \neq 0 \\ (a\cos\alpha x + b\sin\alpha x) & t = 0, \end{cases} \qquad (7)$$

and hence

$$v_m = a_0\left(p_0 + 2\sum_{t=1}^{n-1} p_t\right) + \sum_{k=1}^{\langle n/2\rangle}\left(p_0 + 2\sum_{t=1}^{n-1} p_t \cos\frac{\pi k t}{n}\right)\left(a_k \cos\frac{2\pi k}{n}(x-m) + b_k \sin\frac{2\pi k}{n}(x-m)\right)\Bigg|_{x=m}. \qquad (8)$$

The term $a_k\cos\frac{2\pi k}{n}(x-m) + b_k\sin\frac{2\pi k}{n}(x-m)$ in formula (6) represents a sinusoidal curve with period n/k. Application of P to w_x multiplies this term by the factor $\left(p_0 + 2\sum_{t=1}^{n-1} p_t \cos\frac{\pi k t}{n}\right)$. This factor is a function of k, or equivalently a function of n/k, the wave length of the sinusoidal term it multiplies. This function takes on values for only $k = 1,2,\ldots,\langle n/2\rangle$ as a factor in formula(8), but we shall extend the domain and denote the resulting function by

$$\mathcal{P}_P(\beta) = p_0 + 2\sum_{t=1}^{n-1} p_t \cos(\pi t/\beta) \qquad \beta \geq 1. \qquad (9)$$

The function $\mathcal{P}_P(\beta)$ will be called the <u>periodogram function</u> or simply the <u>periodogram</u> of P. When no confusion can

arise, the subscript denoting the operator will be suppressed.

The role of the function $P(\beta)$ is best seen in formula (8). In obtaining v_m from a series of values of u_x, the sinusoidal terms of wave length n/k are multiplied by $P(n/k)$, and we have a measure of the extent to which waves of length n/k are retained by the graduation. It must be borne in mind that this is a relative measure in that the actual waves in w_x for a specific point m are determined by the data set $\{u_{m-(n-1)/2}, \cdots, u_{m+(n-1)/2}\}$. As the graduation process moves to the next point, the data set changes to $\{u_{m+1-(n-1)/2}, \cdots, u_{m+1+(n-1)/2}\}$ and there will be some changes in the waves. The waves should not vary drastically for nearby values of m because of the many common data values; but for widely separated m, there may be considerable differences.

The periodogram function as we have defined it is patterned after the characteristic function used by Elphinstone in his treatment of the foundations of the theory of graduation (4), but we shall be able to use it on a larger class of operators. We shall turn in Chapter II to a close examination of symmetric linear operators, in preparation for a presentation in Chapters III and IV of applications of the periodogram function to graduation and interpolation formulae.

CHAPTER II

SYMMETRIC LINEAR OPERATORS

1. Introduction

Symmetric linear operators play an important role in the graduation methods used by actuaries. Indeed, almost all the graduation techniques of actuaries can be expressed in terms of symmetric linear operators. Exceptions to be noted are the graphic method and curve fitting techniques. In this chapter we shall first define symmetric linear operators precisely and study the structure of a certain class of them, namely those which are finite and discrete. Then we shall formalize the concept of the periodogram of a symmetric linear operator and seize upon a few consequences of this idea for use in later applications.

2. The Space of Symmetric Linear Operators

If T is the set of real valued functions defined on the set of real numbers, then an operator is a function of T into T. The sum and product of two operators O_1 and O_2 is defined by:

$$(O_1 + O_2)u_x = O_1u_x + O_2u_x \tag{1}$$

and

$$(O_1O_2)u_x = O_1(O_2u_x) \tag{2}$$

respectively, where u_x is in T.

Recall that G_m is defined by

$$G_mu_x = \begin{cases} u_{x+m/2} + u_{x-m/2} & m \neq 0 \\ u_x & m = 0. \end{cases} \tag{3}$$

We shall restrict m to integral values, since such

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We shall restrict m to integral values, since such

restriction still allows sufficient latitude to deal with the problems of actuarial graduation. Notational convenience is the sole reason for using the operator G_m as the cornerstone of our work instead of employing the familiar translation operator E^t or Elphinstone's

$$F_t = \begin{cases} E^t + E^{-t} & t \neq 0 \\ 1 & t = 0. \end{cases}$$

With this foundation we avoid dealing with non-integer indices.

It is clear that G_m is a linear operator, and hence so is any real linear combination of the operators $G_0, G_1, \ldots$. In addition, there is a certain symmetry about the operator $G_m = E^{m/2} + E^{-m/2}$, and, in fact, $G_m = G_{-m}$. More generally, we shall be interested in operators of the form

$$P = \sum_{t=0} p_t G_t = \sum_{t=0} p_t (E^{t/2} + E^{-t/2})$$

where $E^{t/2}$ and $E^{-t/2}$ have the same weight p_t. Such an operator will be called a symmetric linear operator (SLO). Since for our purpose G_m is defined on a discrete set, it is normal to speak of G_m, and hence P, as discrete operators.

A SLO is finite if $p_t = 0$ for all t at least as large as some non-negative integer, and the smallest such number is the length, d(P), of P. For example, the length of $\sum_{t=0}^{n-1} p_t G_t$, where $p_{n-1} \neq 0$, is n. It is not hard to see, inductively or otherwise, that if $P_1, P_2, \ldots, P_k$ are SLO's, then

$$d(P_1 P_2 \ldots P_k) = \sum_{i=1}^{k} d(P_i) - (k-1). \qquad (4)$$

Let $V_n = \left\{ \sum_{t=0}^{n-1} p_t G_t : p_t \text{ is a real number} \right\}$; that is, V_n

is the set of all finite symmetric linear operators of length less than or equal to n. One readily verifies that V_n is a vector space over the reals. The operations of vector addition and scalar multiplication in V_n are just operator addition and a special case of operator multiplication respectively.

The set $\{G_0, G_1, \ldots, G_{n-1}\}$ obviously spans V_n. Furthermore, the set is linearly independent, so that V_n is n-dimensional. For, assume there is relation $\sum_{t=0}^{n-1} p_t G_t = 0$. This means that

$$(\sum_{t=0}^{n-1} p_t G_t) u_x = 0 u_x = 0$$

for all u_x in T. But if we let

$$u_x^{(t)} = \begin{cases} 1 & x = t/2 \\ 0 & x \neq t/2 \end{cases} \qquad t = 0,1,\ldots,n-1,$$

then we have

$$0 = (\sum_{t=0}^{n-1} p_t G_t) u_0^{(t)} = p_t \qquad t = 0,1,\ldots,n-1.$$

One purpose of this chapter is to examine this space. To this end we will set forth several bases of V_n consisting of those finite SLO's which are used widely. The transformations relating these bases can facilitate computations of periodograms later on, and as an interesting sidelight can be used to develop trigonometric identities.

3. Some Special Operators

We frequently shall find it convenient in what follows to use the function $\mathcal{E}(t)$, which is defined for t an integer as the remainder upon division of t+1 by 2; that is,

$$\mathcal{E}(t) = \begin{cases} 0 & t \text{ odd} \\ 1 & t \text{ even.} \end{cases}$$

In the expressions with which we deal and in which one of the functions $\langle x \rangle$ or $\mathcal{E}(t)$ appears, we can usually restate the expression in such a way as to use only the other function. For example, the finite SLO, summation m, is defined by either

$$[m] = \begin{cases} \sum_{t=0}^{\langle \frac{m-1}{2} \rangle} G_{m-(2t+1)} & m > 0 \\ 0 & m = 0 \end{cases}$$

or

$$[m] = \sum_{t=0}^{m} \mathcal{E}(m-t+1) G_t .$$

Our expressions will typically appear with whichever function yields the simpler final form.

The mean operator,

$$\mu = \tfrac{1}{2} G_1 ,$$

derives its name from the fact that μu_x is the average of $u_{x+\frac{1}{2}}$ and $u_{x-\frac{1}{2}}$. A more general mean operator,

$$\mu_m = \tfrac{1}{2} G_m ,$$

will be called the m-interval mean operator. Another finite SLO, average m, is defined by

$$\overline{[m]} = [m]/m .$$

We shall have occasion to do some computing with the operator G_m, and for this purpose display the multiplication table:

$$G_k G_m = \begin{cases} G_k & m = 0 \\ G_m & k = 0 \\ G_{2m} + 2G_0 & m = k \neq 0 \\ G_{m+k} + G_{m-k} & 0 \neq m \neq k \neq 0 . \end{cases} \tag{5}$$

An induction argument establishes

$$(G_m)^k = \sum_{t=0}^{k} \varepsilon(k-t) \binom{k}{(k-t)/2} G_{tm}. \tag{6}$$

4. Bases for V_n

The operator $[m]$ for $m = 1,2,\ldots,n$ is in V_n by definition. The definition of μ and formula (6) insure that μ^m is in V_n for $m = 0,1,2,\ldots,n-1$. If the sets $\{[1],[2],\ldots,[n]\}$ and $\{\mu^0,\mu^1,\ldots,\mu^{n-1}\}$ are linearly independent then each is a basis for V_n.

To check the linear independence of the first set, assume we are given some relation $\sum_{t=1}^{n} p_t [t] = 0$, and consider the function

$$u_x = \begin{cases} 1 & x = (n-1)/2 \\ 0 & \text{otherwise.} \end{cases}$$

$$0 = \sum_{t=1}^{n} p_t [t] u_0 = p_n [n] u_0 + \sum_{t=1}^{n-1} p_t [t] u_0$$

$$= p_n \cdot 1 + 0 = p_n .$$

Now assume $p_n = p_{n-1} = \cdots = p_{n-k} = 0$ and let

$$v_x = \begin{cases} 1 & x = (n-(k+2))/2 \\ 0 & \text{otherwise.} \end{cases}$$

$$0 = \sum_{t=1}^{n} p_t [t] v_0 = \sum_{t=1}^{n-(k+1)} p_t [t] v_0$$

$$= p_{n-(k+1)} [n-(k+1)] v_0 + \sum_{t=1}^{n-(k+2)} p_t [t] v_0$$

$$= p_{n-(k+1)} \cdot 1 + 0 = p_{n-(k+1)} .$$

Therefore, by induction, $p_t = 0$ for $t = 1,2,\ldots,n$ and $\{[1],[2],\ldots,[n]\}$ is a linearly independent set. If we choose

$$u_x = \begin{cases} 2^{n-1} & x = (n-1)/2 \\ 0 & \text{otherwise,} \end{cases}$$

$$v_x = \begin{cases} 2^{n-(k+1)} & x = (n-(k+1))/2 \\ 0 & \text{otherwise,} \end{cases}$$

and follow the pattern of proof used above, we see that $\{\mu^0, \mu^1, \ldots, \mu^{n-1}\}$ is linearly independent.

Each of the bases $\{\mu_0, \mu_1, \ldots, \mu_{n-1}\}$, $\{[\overline{1}], [\overline{2}], \ldots, [\overline{n}]\}$, and $\{[2]^0, [2]^1, \ldots, [2]^{n-1}\}$ is so closely related to one of the three bases already considered that we will do no more than give them this passing mention.

5. Transformations of Bases

At this point we can focus upon the problem of exhibiting the transformations relating each pair of these bases by expressing a typical element of one basis in terms of the other basis of the pair. This can be handled best by first noting the recurrence relation of the following lemma and the operator identity of Theorem 1.

Lemma 1: $[m+1] = [2][m] - [m-1]$

Proof: $$[2][m] = G_1 \sum_{t=0}^{m} \varepsilon(m-t+1) G_t$$

$$= \varepsilon(m+1)G_1 + \varepsilon(m)(G_2+2G_0) + \sum_{t=2}^{m} \varepsilon(m-t+1)(G_{t+1} + G_{t-1})$$

$$= \sum_{t=0}^{m+1} \varepsilon(m-t) G_t + \sum_{t=0}^{m-1} \varepsilon(m-t) G_t$$

$$= [m+1] + [m-1] \ .$$

If we consider the subspace V_m, $m \leq n$, defined in a manner analogous to that for V_n, and note that $\{[2]^0, [2]^1, \ldots, [2]^{m-1}\}$ is a basis for V_m, and $[m]$ is an element of V_m, we conclude that $[m]$ has an expansion in powers of $[2]$ up to the power

m-1. Alternatively, Lemma 1, together with the relationships

$$[1] = 1\cdot[2]^0 \tag{7}$$

and

$$[2] = 0\cdot[2]^0 + 1\cdot[2]^1 \tag{8}$$

shows by induction that such an expansion exists. We shall now state the expansion explicitly in the form of a theorem.

Theorem 1: $[m] = \sum_{t=0}^{m-1} \varepsilon(m-t-1)(-1)^{(m-t-1)/2} \binom{(m+t-1)/2}{t} [2]^t$.

Proof: Let $[m] = \sum_{t=0}^{m-1} a_t^m [2]^t$, and assume similar relations for $[m+1]$ and $[m-1]$. Expanding each side of the equation of Lemma 1 in powers of $[2]$, and equating the coefficients we get

$$\begin{aligned}
a_0^{m+1} &= -a_0^{m-1}, \\
a_t^{m+1} &= a_{t-1}^{m} - a_t^{m-1}, \qquad t = 1,2,\ldots,m-2 \\
a_{m-1}^{m+1} &= a_{m-2}^{m}, \\
\text{and} \\
a_m^{m+1} &= a_{m-1}^{m}.
\end{aligned} \tag{9}$$

Relations (7) and (8) show that $a_0^1 = 1$, $a_0^2 = 0$, and $a_1^2 = 1$. It follows that

$$a_0^{m+1} = -a_0^{m-1} \text{ implies } a_0^m = \varepsilon(m-1)(-1)^{(m-1)/2}, \tag{10}$$

$$a_{m-1}^{m+1} = a_{m-2}^{m} \text{ implies } a_{m-2}^{m} = 0, \tag{11}$$

and

$$a_m^{m+1} = a_{m-1}^{m} \text{ implies } a_{m-1}^{m} = 1. \tag{12}$$

Statements (10), (11), and (12) show that the coefficients given in the theorem are correct for t = 0, m-2, and m-1 respectively. In particular, the theorem holds for m = 1 and 2.

Now, assuming that the theorem holds for the coefficients a_t^{k-1} and a_t^k appearing in the expansions of $[k-1]$ and $[k]$ and considering the coefficients a_t^{k+1} for $[k+1]$ we have from formula (9)

$$a_t^{k+1} = a_{t-1}^{k} - a_t^{k-1}$$

$$= \varepsilon(k-t)(-1)^{(k-t)/2}\binom{(k+t-2)/2}{t-1}$$

$$- \varepsilon(k-t-2)(-1)^{(k-t-2)/2}\binom{(k+t-2)/2}{t}$$

$$= \varepsilon(k-t)(-1)^{(k-t)/2}\left(\binom{(k+t-2)/2}{t-1} + \binom{(k+t-2)/2}{t}\right)$$

$$= \varepsilon(k+1-t-1)(-1)^{(k+1-t-1)/2}\binom{(k+1+t-1)/2}{t}$$

$$t = 1,2,\ldots,k-2.$$

Thus, by induction we have

$$a_t^m = \varepsilon(m-t-1)(-1)^{(m-t-1)/2}\binom{(m+t-1)/2}{t}$$

$$t = 1,2,\ldots,m-3.$$

Since the cases t = 0, m-2, and m-1 were previously verified, the theorem is proved.

The recursion relation (9) provides us with a sort of "Pascal triangle" method for calculating the coefficients once we have a_0^1, a_0^2, and a_1^2. However, zeros appear alter-

nately and the relation involves coefficients from 3 rows. This is illustrated by the following table.

TABLE 1

VALUES OF a_t^m

m	a_0^m	a_1^m	a_2^m	a_3^m	a_4^m	a_5^m	a_6^m	a_7^m
1	1							
2	0	1						
3	-1	0	1					
4	0	-2	0	1				
5	1	0	-3	0	1			
6	0	3	0	-4	0	1		
7	-1	0	6	0	-5	0	1	
8	0	-4	0	10	0	-6	0	1

The explicit relationships between the bases can now be given, based on definitions, Theorem 1, and some incidental computations. We will proceed by considering the pairs of bases in turn.

A. $\{G_0, G_1, \ldots, G_{n-1}\}$ and $\{[1], [2], \ldots, [n]\}$

$$[m] = \sum_{t=0}^{m} \mathcal{E}(m-t+1)G_t \qquad m = 1,2,\ldots,n \text{ by definition.} \tag{13}$$

$$G_m = \begin{cases} [m+1] - [m-1] & m = 1,2,\ldots,n-1 \\ [1] & m = 0 \end{cases} \tag{14}$$

since

$$[m+1] - [m-1] = \sum_{t=0}^{m+1} \mathcal{E}(m+1-t+1)G_t - \sum_{t=0}^{m-1} \mathcal{E}(m-1-t+1)G_t$$

$$= \mathcal{E}(1)G_{m+1} + \mathcal{E}(2)G_m + \sum_{t=0}^{m-1} (\mathcal{E}(m-t+2) - \mathcal{E}(m-t))G_t$$

$$= G_m \qquad m = 1,2,\ldots,n-1.$$

B. $\{G_0, G_1, \ldots, G_{n-1}\}$ and $\{\mu^0, \mu^1, \ldots, \mu^{n-1}\}$

$$\mu^m = (1/2^m)(G_1)^m = (1/2^m)\sum_{t=0}^{m} \varepsilon(m-t) \binom{m}{(m-t)/2} G_t \qquad (15)$$

$m = 0,1,\ldots,n-1$ by formula (6).

$$G_m = [m+1] - [m-1]$$

$$= \sum_{t=0}^{m} \varepsilon(m+1-t-1)(-1)^{(m+1-t-1)/2} \binom{(m+1+t-1)/2}{t} 2^t \mu^t$$

$$- \sum_{t=0}^{m-2} \varepsilon(m-1-t-1)(-1)^{(m-1-t-1)/2} \binom{(m-1+t-1)/2}{t} 2^t \mu^t$$

$$= \sum_{t=0}^{m} \varepsilon(m-t)(-1)^{(m-t)/2} \left(\binom{(m+t)/2}{t} + \binom{(m+t-2)/2}{t} \right) 2^t \mu^t \qquad (16)$$

$m = 1,2,\ldots,n-1$ by Theorem 1,

while

$$G_0 = \mu^0 .$$

C. $\{[1],[2],\ldots,[n]\}$ and $\{\mu^0, \mu^1, \ldots, \mu^{n-1}\}$

$$[m] = \sum_{t=0}^{m-1} \varepsilon(m-t-1)(-1)^{(m-t-1)/2} \binom{(m+t-1)/2}{t} 2^t \mu^t \qquad (17)$$

$m = 1,2,\ldots,n$ by Theorem 1.

$$\mu^m = (1/2^m)\sum_{t=1}^{m+1} \varepsilon(m+t-1) \left(\binom{m}{(m+t-1)/2} - \binom{m}{(m+t+1)/2} \right) [t] \qquad (18)$$

$m = 0,1,\ldots,n-1$

since

$$\sum_{t=1}^{m+1} \varepsilon(m+t-1) \left(\binom{m}{(m+t-1)/2} - \binom{m}{(m+t+1)/2} \right) [t]$$

$$= \sum_{t=1}^{m+1} \varepsilon(m+t-1) \left(\binom{m}{(m+t-1)/2} - \binom{m}{(m+t+1)/2} \right) \sum_{j=0}^{t} \varepsilon(t-j+1) G_j$$

$$= \sum_{t=0}^{m} \mathcal{E}(m-t) \binom{m}{(m-t)/2} G_t$$

$$= (G_1)^m .$$

The calculation justifying formula (18) has been presented in deceptively shortened form. The expansion and simplification of the sums involved are straightforward, but quite tedious, and therefore an alternate proof, paralleling that of Theorem 1, will be proposed. The problem is essentially that of showing that if

$$b_t^m = \mathcal{E}(m+t-1) \left(\binom{m}{(m+t-1)/2} - \binom{m}{(m+t+1)/2} \right)$$

then

$$[2]^m = \sum_{t=1}^{m+1} b_t^m [t] .$$

From the relation $[2]^m = [2] \cdot [2]^{m-1}$ one establishes the recurrence relation

$$b_t^m = b_{t-1}^{m-1} + b_{t+1}^{m-1} \qquad t = 2,3,\ldots,m-1 \quad (19)$$

and the end term formulae

$$b_1^m = b_2^{m-1} ,$$

$$b_m^m = b_{m-1}^{m-1} ,$$

and

$$b_{m+1}^m = b_m^{m-1} .$$

Direct computation of b_t^m for small values of m enables one to verify the coefficients for t = 1, m, and m+1. With this information and relation (19) one completes the proof by induction. Thus we have

$$[2]^m = \sum_{t=1}^{m+1} \varepsilon(m+t-1)\left(\binom{m}{(m+t-1)/2} - \binom{m}{(m+t+1)/2}\right)[t]$$

$$m = 0,1,\ldots,n-1$$

and μ^m can be expressed as in formula (18). It should be noted that the coefficients for small values of m can be easily calculated by a "triangle" technique based on the recurrence relation (19).

The frequent appearance in graduation and interpolation formulae of Sheppard's central difference operator, δ, defined by $\delta u_x = u_{x+\frac{1}{2}} - u_{x-\frac{1}{2}}$, makes some further analysis desirable. The operator δ is not in V_n, but $\delta^2 = G_2 - 2G_0$ is in V_n. In fact, $\delta^{2m} = \sum_{t=0}^{m}(-1)^{m-t}\binom{2m}{m+t}G_{2t}$ so δ^{2m} is in V_n for $m = 0,1,\ldots,\langle(n-1)/2\rangle$.

The fact that this expansion of δ^{2m} involves only even indices for G_m brings up a point heretofore ignored. The phenomenon of indices having the same parity in an expansion is present in all of our basis transformations. Thus V_n splits naturally into the direct sum of two subspaces V_n' and V_n'' with bases $\{G_1, G_3, \ldots, G_{2\langle n/2\rangle - 1}\}$ and $\{G_0, G_2, \ldots, G_{2\langle(n-1)/2\rangle}\}$ respectively. Even powers of δ form a basis for V_n'' while operators of the form $\mu\delta^{2m}$ form a basis for V_n'. Corresponding basis transformations could be displayed, but we shall present only G_{2m} as a linear combination of even powers of δ.

$$G_{2m} = \begin{cases} 2\delta^0 + \sum_{t=1}^{m}(m/t)\binom{m+t-1}{m-t}\delta^{2t} & m \neq 0 \\ \delta^0 & m = 0. \end{cases} \qquad (20)$$

We shall find it convenient, particularly in Chapter IV, to have available those infinite dimensional spaces, V' and V'',

spanned by $\{G_1, G_3, \ldots\}$ and $\{G_0, G_2, \ldots\}$.

6. Periodograms of Symmetric Linear Operators

Throughout this section we shall let

$$u_x = a \cos(2\pi x/\beta) + b \sin(2\pi x/\beta) \qquad \beta \geq 1.$$

With this convention, the <u>periodogram function</u>, or simply <u>periodogram</u>, $\mathcal{P}_P(\beta)$, of a symmetric linear operator, $P = \sum_{t=0} p_t G_t$, is the real valued function of β alone, defined by the relation

$$\mathcal{P}_P(\beta) u_x = P u_x \,. \tag{21}$$

The statement

$$\begin{aligned} P u_x &= \left(\sum_{t=0} p_t G_t\right)(a \cos(2\pi x/\beta) + b \sin(2\pi x/\beta)) \\ &= \left(p_0 + 2\sum_{t=0} p_t \cos(\pi t/\beta)\right) \\ &\qquad (a \cos(2\pi x/\beta) + b \sin(2\pi x/\beta)) \,, \end{aligned}$$

which depends upon formula (7) of Chapter I, shows that such a function exists. However, we have seen that the operator P can be written in many ways corresponding to the several basis choices for V_n. Indeed, different forms of P may lead to different forms of $\mathcal{P}_P(\beta)$ but nevertheless the forms are equivalent, and the periodogram function is well defined.

An immediate consequence of the definition of the periodogram of a SLO is the following theorem.

<u>Theorem 2</u>: For two symmetric linear operators P and Q,

$$\mathcal{P}_{P+Q}(\beta) = \mathcal{P}_P(\beta) + \mathcal{P}_Q(\beta) \,, \tag{22}$$

and

$$\mathcal{P}_{PQ}(\beta) = \mathcal{P}_P(\beta)\mathcal{P}_Q(\beta) \,. \tag{23}$$

Proof: The linearity of P and Q and definitions (1), (2), and (21) justify writing the relations

$$(\mathcal{P}_{P+Q}(\beta))u_x = (P+Q)u_x = Pu_x + Qu_x$$
$$= (\mathcal{P}_P(\beta))u_x + (\mathcal{P}_Q(\beta))u_x$$
$$= (\mathcal{P}_P(\beta) + \mathcal{P}_Q(\beta))u_x ,$$

and

$$(\mathcal{P}_{PQ}(\beta))u_x = (PQ)u_x = P(Qu_x)$$
$$= P(\mathcal{P}_Q(\beta)u_x)$$
$$= \mathcal{P}_Q(\beta)(Pu_x)$$
$$= (\mathcal{P}_P(\beta)\mathcal{P}_Q(\beta))u_x$$

which prove statements (22) and (23) respectively.

Formula (23) provides us with the useful corollary,

$$\mathcal{P}_{P^m}(\beta) = (\mathcal{P}_P(\beta))^m . \tag{24}$$

The periodogram for the operator G_m has already been displayed in formula (7) of Chapter I although in somewhat disguised form. The periodograms for the other basis operators are either special cases of $\mathcal{P}_{G_m}(\beta)$ or else can be computed directly. Elphinstone [4] gives $\mathcal{P}_{[m]}(\beta)$ and $\mathcal{P}_{\delta 2m}(\beta)$. We shall summarize the results for reference purposes.

$$\mathcal{P}_{G_m}(\beta) = \begin{cases} 2\cos(\pi m/\beta) & m \neq 0 \\ 1 & m = 0, \end{cases} \tag{25}$$

$$\mathcal{P}_{[m]}(\beta) = \begin{cases} (\sin(\pi m/\beta))/(\sin(\pi/\beta)) & \beta \neq 1 \\ 1 & \beta = 1, \end{cases} \tag{26}$$

$$\mathcal{P}_{\mu}(\beta) = \cos(\pi/\beta) , \tag{27}$$

$$\mathcal{P}_{\mu^m}(\beta) = (\cos(\pi/\beta))^m , \tag{28}$$

and

$$\mathcal{P}_{\delta 2m}(\beta) = (-4\sin^2(\pi/\beta))^m . \tag{29}$$

The symmetric linear operator P was defined as a linear combination of the operators $G_0, G_1, \ldots$. It is clear from our discussion of V_n that if P is finite then it can be

written in terms of another basis of V_n. For example, we can write

$$P = \sum_{t=0}^{n-1} p_t G_t = \sum_{t=0}^{n-1} p'_t \mu^t = \sum_{t=1}^{n} p''_t [t] \ .$$

Since it will always be explicitly stated exactly which form of P is under discussion, it will not cause any confusion if we suppress any distinguishing primes. Thus, we may write $P = \sum_{t=0}^{n-1} p_t \mu^t$, and it is not to be inferred from this that P is the same linear combination of powers of the operator μ as it is of the operators $G_0, G_1, \ldots, G_{n-1}$. The convention is, then, that a general symmetric linear operator will be denoted by upper case Latin letters and no matter which basis of V_n is selected, the coefficients of the basis elements will be the corresponding lower case characters, with appropriate subscripts, of course.

The periodogram of a finite SLO characterizes the SLO completely. This is so important that we state it as a theorem.

Theorem 3: If P and Q are finite symmetric linear operators, say $P = \sum_{t=0}^{n-1} p_t \mu^t$ and $Q = \sum_{t=0}^{n'-1} q_t \mu^t$, then

$$P = Q \iff \mathcal{P}_P(\beta) = \mathcal{P}_Q(\beta). \tag{30}$$

Proof: The implication from left to right follows from the definition of the periodogram of a SLO.

For the converse, we may assume without loss of generality that $n \geq n'$. Then we can write

$$\begin{aligned} 0 &= \mathcal{P}_P(\beta) - \mathcal{P}_Q(\beta) \\ &= (p_0 - q_0) + \sum_{t=1}^{n-1} (p_t - q_t)(\cos(\pi/\beta))^t \end{aligned} \tag{31}$$

where $q_t = 0$ for $t \geq n'$. In particular, relation (31) holds for $\beta = (i+1)/i$ where $i = 1,2,\ldots,n$. This yields a system of n equations in the n unknowns $(p_t - q_t)$, $t = 0,1,\ldots,n-1$. The determinant of the matrix of the coefficients of this system,

$$\begin{vmatrix} 1 & \cos(\pi/2) & (\cos(\pi/2))^2 & \ldots & (\cos(\pi/2))^{n-1} \\ 1 & \cos(2\pi/3) & (\cos(2\pi/3))^2 & \ldots & (\cos(2\pi/3))^{n-1} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & \cos(n\pi/n+1) & (\cos(n\pi/n+1))^2 & \ldots & (\cos(n\pi/n+1))^{n-1} \end{vmatrix},$$

is a Vandermonde determinant, and is equal to

$$\prod_{i<j} (\cos(j\pi/j+1) - \cos(i\pi/i+1)). \tag{32}$$

Since the cosine function is strictly decreasing between $\pi/2$ and π, each factor of the product (32) is different from zero. It follows that the entire product (32) does not equal 0, and hence the only solution to the system of n equations is the trivial one. Therefore, $p_t = q_t$, $t = 0,1,\ldots,n-1$, which is to say $P = Q$ as claimed.

7. The Factor Theorem

We shall now turn our attention to deriving some results which will be of assistance in computing periodograms of specific symmetric linear operators. We shall begin by finding a factored form of $P_{[m]}(\beta)$.

<u>Lemma 2</u>: $$P_{[m]}(\beta) = 2^{m-1} \prod_{i=1}^{m-1} (\cos(\pi/\beta) - \cos(\pi i/m)). \tag{33}$$

Proof: We know from formula (26) that

$$P_{[m]}(\beta) = 0 \qquad \beta = m, m/2, \ldots, m/(m-1). \tag{34}$$

If we write $[m] = \sum_{t=0}^{m-1} p_t \mu^t$, then

$$\mathcal{P}_{[m]}(\beta) = \sum_{t=0}^{m-1} p_t (\cos(\pi/\beta))^t \qquad (35)$$

and

$$p_{m-1} = 2^{m-1} .$$

From statement (34), it follows that the polynomial (35) in $\cos(\pi/\beta)$ has the roots $\cos(\pi i/m)$, $i = 1,2,\ldots,m-1$, and hence we can write formula (33).

The factored form of $\mathcal{P}_{[m]}(\beta)$ can be used to characterize a set of zeros of the periodogram of a general finite SLO.

Theorem 4: If P is a finite SLO, then a necessary and sufficient condition for P to have a factor $[m]$, i.e. $P = [m]Q$ for some SLO Q, is that the periodogram $\mathcal{P}_P(\beta)$, of P is zero for $\beta = m, m/2, \ldots, m/(m-1)$.

Proof: The condition is necessary since $P = [m]Q$ implies

$$\begin{aligned} \mathcal{P}_P(\beta) &= \mathcal{P}_{[m]}(\beta)\mathcal{P}_Q(\beta) \\ &= \frac{\sin(m\pi/\beta)}{\sin(\pi/\beta)}\mathcal{P}_Q(\beta) \\ &= 0 \qquad\qquad \beta = m, m/2, \ldots, m/(m-1). \end{aligned}$$

The condition is also sufficient. For let $P = \sum_{t=0}^{n-1} p_t \mu^t$ where $p_{n-1} \neq 0$. Then

$$\begin{aligned} \mathcal{P}_P(\beta) &= \sum_{t=0}^{n-1} p_t (\cos(\pi/\beta))^t \\ &= 2^{m-1} \prod_{i=1}^{m-1} (\cos(\pi/\beta) - \cos(\pi i/m)) \\ &\qquad \sum_{t=0}^{n-m} q_t (\cos(\pi/\beta))^t . \end{aligned}$$

Setting $Q = \sum_{t=0}^{n-m} q_t \mu^t$, we have

$$\mathcal{P}_P(\beta) = \mathcal{P}_{[m]}(\beta)\,\mathcal{P}_Q(\beta) = \mathcal{P}_{[m]Q}(\beta)$$

by Theorem 2. Therefore, by Theorem 3

$$P = [m]Q \ .$$

Corollary 1: For any finite symmetric linear operator P, $\mathcal{P}_P(\beta)$ has zeros of multiplicity r at $\beta = m, m/2, \ldots, m/m-1$ if and only if $P = [m]^r Q$ for some SLO Q.

If we write $P = P' + P''$ where P' is in V_n' and P'' is in V_n'', then $\mathcal{P}_P(\beta) = \mathcal{P}_{P'}(\beta) + \mathcal{P}_{P''}(\beta)$ by formula (12). Now we can state the following result.

Corollary 2: If $m \geq 2$, then

$$\mathcal{P}_P(\beta) = 0 \quad \beta = m, m/2, \ldots, m/(m-1) \iff$$
$$\mathcal{P}_{P'}(\beta) = \mathcal{P}_{P''}(\beta) = 0 \quad \beta = m, m/2, \ldots, m/(m-1).$$

Proof: The non-trivial implication is established as follows. If $\mathcal{P}_P(\beta) = 0$ for $\beta = m, m/2, \ldots, m/(m-1)$, then by the theorem there exists an $r \geq 1$ such that

$$P = [m]^r Q = [m]^r (Q' + Q'')$$
$$= [m]^r Q' + [m]^r Q''$$

where Q' is in V' and Q'' is in V''. Furthermore, exactly one of $[m]^r Q'$ and $[m]^r Q''$ is in V', the other being in V''. But the sum is direct, and thus P has a unique representation $P = P' + P''$. Therefore, the conclusion follows upon noting that

$$\mathcal{P}_{[m]^r Q'}(\beta) = \mathcal{P}_{[m]^r Q''}(\beta) = 0$$

for $\beta = m, m/2, \ldots, 2/(m-1)$.

8. An Application

Each of the formulae (13) through (18) gives two forms of a SLO. If we write down in turn the periodograms of both sides of these relations, a trigonometric identity results in every instance. The identities assume a neater form if we replace (π/β) by α in the expressions.

We shall list the identities which arise, identifying each by the formula number, distinguished by a prime, of the basis transformation from which it is derived.

$$\frac{\sin m\alpha}{\sin\alpha} = \varepsilon(m+1) + 2\sum_{t=1}^{m}\varepsilon(m-t+1)\cos t\alpha. \qquad (13')$$

$$2\cos m\alpha = \frac{\sin(m+1)\alpha - \sin(m-1)\alpha}{\sin\alpha}. \qquad (14')$$

$$\cos^m\alpha = (1/2^m)\left(\varepsilon(m)\binom{m}{\frac{m}{2}} + 2\sum_{t=1}^{m}\varepsilon(m-t)\binom{m}{\frac{m-t}{2}}\cos t\alpha\right). \qquad (15')$$

$$2\cos m\alpha = \sum_{t=0}^{m}\varepsilon(m-t)(-1)^{(m-t)/2}\left(\binom{\frac{m+t}{2}}{t} + \binom{\frac{m+t-2}{2}}{t}\right)2^t\cos^t\alpha. \qquad (16')$$

$$\frac{\sin m\alpha}{\sin\alpha} = \sum_{t=0}^{m-1}\varepsilon(m-t-1)(-1)^{(m-t-1)/2}\binom{\frac{m+t-1}{2}}{t}2^t\cos^t\alpha. \qquad (17')$$

$$\cos^m\alpha = \frac{1}{2^m\sin\alpha}\sum_{t=1}^{m+1}\varepsilon(m+t-1)\left(\binom{m}{\frac{m+t-1}{2}} - \binom{m}{\frac{m+t+1}{2}}\right)\sin t\alpha. \qquad (18')$$

These identities can, of course, be obtained by other means. For instance, (14') is a special case of $\sin(\theta+\phi) - \sin(\theta-\phi) = 2\cos\theta\sin\phi$, while (16') and (17') are essentially the explicit formulations of Tchebysheff polynomials (in $\cos\alpha$) of the first and second kind respectively (5; p.185).

This completes our investigation of V_n. We shall make use of certain of the results in the succeeding work as we examine periodograms of graduation formulae and subtabulators which arise from interpolation formulae.

CHAPTER III

PERIODOGRAMS OF GRADUATORS

1. Introduction

The concept of graduation was described briefly in Chapter I for the purpose of motivating the investigation of the periodogram function of a symmetric linear operator. In the present chapter we shall again direct our attention to graduation. In particular we shall define precisely "graduation operator" or "graduator" as a suitably restricted SLO. Since a graduation operator is, among other things, a SLO, it will make sense to talk about the periodogram of a graduator. We shall see that the additional properties of a graduator show up characteristically in the periodogram.

Elphinstone [4] has presented the periodograms of several graduators. In Chapter II we built up a certain amount of machinery for the purpose of computing periodograms of symmetric linear operators. We shall make use of this material in computing the periodograms of some graduators which Elphinstone did not consider. In Chapter I we noted some characteristics that the periodogram of a "satisfactory" graduator should have. We shall elaborate upon these remarks and see how well the graduators we have discussed meet these requirements.

2. Reproduction Conditions

In one approach to graduation of actuarial data, the sequence of observed values $\{u_{x+t}\}$ is considered as a sum sequence $\{w_{x+t} + e_{x+t}\}$ where $\{w_{x+t}\}$ denotes the "true" underlying sequence and $\{e_{x+t}\}$ an error sequence. It is fur-

ther considered that subsequences of moderate length from the $\{w_{x+t}\}$ sequence can be approximated well by a polynomial. Objectives of the graduation are to reproduce the $\{w_{x+t}\}$ sequence and to minimize and smooth out the error sequence. With this in mind, a commonly used restriction in graduation is that of requiring the graduation process to reproduce polynomials of a certain degree, often the third; that is, if one were to graduate a sequence of such polynomial values, the values of the polynomial at the points of application would result. For graduators of odd length, such sequences are in what Greville [10] calls the "smooth space" of the graduation process, and are left invariant by it.

Such a restriction will be called a reproduction condition. A reproduction condition on a symmetric linear operator, $P = \sum_{t=0} p_t G_t$ takes the form of restrictions on the coefficients p_t of P. For example, if P is to reproduce polynomials of degree 2r+1 or less, the relations

$$p_0 + 2\sum_{t=1} p_t = 1 \qquad (1)$$

and

$$\sum_{t=0} t^2 p_t = \sum_{t=0} t^4 p_t = \ldots = \sum_{t=0} t^{2r} p_t = 0 \qquad (2)$$

must hold.

We have seen that P can be expressed in many ways. In each form, reproduction conditions appear as restrictions on the coefficients. For instance, if P is in V", say

$$P = p_0' + p_2'\delta^2 + p_4'\delta^4 + \ldots ,$$

the reproduction condition of degree 2r+1 is equivalent to the requirements

$$p_0' = 1 \quad \text{and} \quad p_t' = 0 \qquad t = 2,4,\ldots,2r.$$

3. Graduation Operators

A graduation operator, Q, is a symmetric linear operator $\sum_t q_t G_t$ having both a reproduction condition and the property that the indices of all non-zero coefficients are either all odd or all even. The latter condition provides that Q is applicable to data spaced at unit intervals.

We shall now take a look at some characteristics of the periodogram of a graduator which arise because of the defining conditions for the graduator.

Theorem 1: If $Q = \sum_{t=0}^{n-1} q_t G_t$ is a graduation operator then $\mathcal{P}_Q(\beta)$ approaches 1 asymptotically as β increases without bound.

Proof: Since Q must reproduce polynomials of some degree, we have the relation

$$\lim_{\beta\to\infty} \mathcal{P}_Q(\beta) = \lim_{\beta\to\infty}(q_0 + 2\sum_{t=1}^{n-1} q_t \cos(\pi t/\beta))$$

$$= q_0 + 2\sum_{t=0}^{n-1} q_t = 1 ,$$

by formula (1).

Although the values of β actually occurring in the Fourier sum which was used in the development of the periodogram in Chapter I are never less than 2, we have defined the periodogram function for all real numbers greater than or equal to 1. This extension was due in part to the fact that at $\beta = 1$, the order of contact of the periodogram with the horizontal reflects the reproduction condition of the graduation operator.

Theorem 2: The highest order of polynomials reproduced by a graduator Q is equal to the order of contact of its periodogram with the horizontal at $\beta = 1$.

Proof: Let $Q = \sum_{t=0} q_t G_t$, in which case $\mathcal{P}_Q(\beta) = q_0 + 2\sum_{t=1} q_t \cos(\pi t/\beta)$.

If we define recursively

$$f_1(t,\beta) = 0 ,$$
$$g_1(t,\beta) = (\pi t/\beta^2) ,$$
$$f_{k+1}(t,\beta) = \frac{\partial}{\partial\beta} f_k(t,\beta) - (\pi t/\beta^2) g_k(t,\beta) \qquad k > 1, \quad (3)$$

and

$$g_{k+1}(t,\beta) = \frac{\partial}{\partial\beta} g_k(t,\beta) + (\pi t/\beta^2) f_k(t,\beta) \qquad k > 1,$$

then we can write

$$\mathcal{P}^{(k)}(\beta) = 2\sum_{t=1} q_t (f_k(t,\beta)\cos(\pi t/\beta) + g_k(t,\beta)\sin(\pi t/\beta)) \quad (4)$$

where $\mathcal{P}^{(k)}(\beta)$ is a standard notation for the k^{th} derivative, with respect to β, of $\mathcal{P}(\beta)$. From formula (4) it follows that

$$\mathcal{P}^{(k)}(1) = \pm 2\sum_{t=1} q_t f_k(t,1) \quad (5)$$

according as the non-zero coefficients of Q have even or odd indices. We shall proceed by determining that the form of $f_k(t,1)$ is

$$f_k(t,1) = a_1 t^2 + a_2 t^4 + \ldots + a_{\langle k/2\rangle} t^{2\langle k/2\rangle}, \quad (6)$$

where $a_i \neq 0$ for $i = 1,2,\ldots,\langle k/2\rangle$, since once relation (6) is established we shall have

$$\mathcal{P}^{(k)}(1) = \pm(a_1\sum_{t=1} q_t t^2 + a_2\sum_{t=1} q_t t^4 + \ldots + a_{\langle k/2\rangle}\sum_{t=1} q_t t^{2\langle k/2\rangle}) , \quad (7)$$

from which it follows that

$$\sum_{t=1} q_t (t^2)^j = 0 \qquad j = 1,2,\ldots,\langle m/2\rangle$$
$$\Longleftrightarrow \mathcal{P}^{(k)}(1) = 0 \qquad k = 1,2,\ldots,m . \quad (8)$$

Statement (8) is equivalent to the theorem.

It remains, then to justify formula (6). If we compute the first few members of the sequences $\{f_1(t,\beta), f_2(t,\beta), \ldots\}$ and $\{g_1(t,\beta), g_2(t,\beta), \ldots\}$ we find that

$$f_1(t,\beta) = 0, \qquad g_1(t,\beta) = (\pi t)/\beta^2,$$

$$f_2(t,\beta) = -\frac{(\pi t)^2}{\beta^4}, \qquad g_2(t,\beta) = -\frac{2\pi t}{\beta^3},$$

$$f_3(t,\beta) = \frac{6(\pi t)^2}{\beta^5}, \qquad g_3(t,\beta) = \frac{6\pi t}{\beta^4} - \frac{(\pi t)^3}{\beta^6}, \tag{9}$$

$$f_4(t,\beta) = -\frac{36(\pi t)^2}{\beta^6} + \frac{(\pi t)^4}{\beta^8}, \qquad g_4(t,\beta) = -\frac{24\pi t}{\beta^5} + \frac{12(\pi t)^3}{\beta^7}.$$

These and additional computations suggest that for $k > 1$, the general forms for $f_k(t,\beta)$ and $g_k(t,\beta)$, taking in order the cases k even ($k = 2r$) and k odd ($k = 2r+1$), are

$$f_{2r}(t,\beta) = -\frac{b_1(\pi t)^2}{\beta^{2r+2}} + \frac{b_2(\pi t)^4}{\beta^{2r+4}} + \ldots + (-1)^r \frac{b_r(\pi t)^{2r}}{\beta^{4r}}, \tag{10}$$

$$g_{2r}(t,\beta) = -\frac{c_1(\pi t)}{\beta^{2r+1}} + \frac{c_2(\pi t)^3}{\beta^{2r+3}} + \ldots + (-1)^r \frac{c_r(\pi t)^{2r-1}}{\beta^{4r-1}}, \tag{11}$$

$$f_{2r+1}(t,\beta) = \frac{d_1(\pi t)^2}{\beta^{2r+3}} - \frac{d_2(\pi t)^4}{\beta^{2r+5}} + \ldots + (-1)^{r+1} \frac{d_r(\pi t)^{2r}}{\beta^{4r+1}}, \tag{12}$$

and

$$g_{2r+1}(t,\beta) = \frac{e_1(\pi t)}{\beta^{2r+2}} - \frac{e_2(\pi t)^3}{\beta^{2r+4}} + \ldots + (-1)^r \frac{e_{r+1}(\pi t)^{2r+1}}{\beta^{4r+2}}, \tag{13}$$

where the coefficients b_i, c_i, d_i, and e_i are positive for all i.

Now if statements (10) and (11) do, in fact, display the form of $f_{2r}(t,\beta)$ and $g_{2r}(t,\beta)$ respectively, then we can compute forms for $f_{2r+1}(t,\beta)$ and $g_{2r+1}(t,\beta)$ using definition (3). Doing so, we get

$$f_{2r+1}(t,\beta) = \frac{(b_1(2r+2)+c_1)(\pi t)^2}{\beta^{2r+3}} - \frac{(b_2(2r+4)+c_2)(\pi t)^4}{\beta^{2r+5}} + \ldots + (-1)^{r+1} \frac{(b_r 4r + c_r)(\pi t)^{2r}}{\beta^{4r+1}} \tag{14}$$

and

$$g_{2r+1}(t,\beta) = \frac{c_1(2r+1)(\pi t)}{\beta^{2r+2}} - \frac{(c_2(2r+3)+b_1)(\pi t)^3}{\beta^{2r+4}} + \cdots + (-1)^r \frac{b_r(\pi t)^{2r+1}}{\beta^{4r+2}} \cdot \quad (15)$$

In the same manner, if statements (12) and (13) give the form of $f_{2r+1}(t,\beta)$ and $g_{2r+1}(t,\beta)$ respectively, we can compute the following forms of $f_{2(r+1)}(t,\beta)$ and $g_{2(r+1)}(t,\beta)$:

$$f_{2(r+1)}(t,\beta) = -\frac{(d_1(2r+3)+e_1)(\pi t)^2}{\beta^{2r+4}} + \frac{(d_2(2r+5)+e_2)(\pi t)^4}{\beta^{2r+6}} + \cdots + (-1)^{r+1} \frac{e_{r+1}(\pi t)^{2r+2}}{\beta^{4r+4}} \quad (16)$$

and

$$g_{2(r+1)}(t,\beta) = -\frac{e_1(2r+2)(\pi t)}{\beta^{2r+3}} + \frac{(e_2(2r+4)+d_1)(\pi t)^3}{\beta^{2r+5}} + \cdots + (-1)^{r+1} \frac{(e_{r+1}(4r+2)+d_r)(\pi t)^{2r+1}}{\beta^{4r+3}} . \quad (17)$$

We observe that expressions (14), (15), (16), and (17) are of precisely the same form as expressions (12), (13), (10), and (11) respectively. This observation, in the presence of the initial computations (9), establishes inductively that the suggested forms (10), (11), (12), and (13) are indeed correct. Upon setting $\beta = 1$, formulae (10) and (12) each yield the form of relation (6) and the proof of the theorem is complete.

Elphinstone [4; p.30] cites a result similar, but not equivalent, to Theorem 2. His characteristic function is our periodogram function viewed as a function of $\alpha = 2\pi/\beta$, and the result to which we refer is that "the highest order

of polynomials exactly reproduced (by a graduation operator) is the same as the order of contact of the characteristic function with the horizontal when $\alpha = 0$."

In connection with Theorem 2, we note that $\mathcal{P}_Q(1) = \pm 1$ according as the non-zero coefficients of Q have even or odd indices. The value of $\mathcal{P}_Q(2)$ is readily available if Q is written in terms of powers of μ, say $Q = \sum_{t=0} q_t \mu^t$, since in this case

$$\mathcal{P}_Q(2) = q_0 + \sum_{t=1} q_t (\cos(\pi/2))^t = q_0 .$$

In particular, if Q is in V', then $\mathcal{P}_Q(2) = 0$, which by Theorem 4 of Chapter II implies [2] divides Q.

4. Periodograms of Basic Operators

Graduation operators are generally expressed as linear combinations of either the basic symmetric linear operators G_m (or some similar operators) or even powers of δ, with perhaps a summation factor or two. Theorem 2 of Chapter II suggests that we can construct the periodogram of any symmetric linear operator, and hence any graduation operator from the periodograms of the basic operators. For instance, if $Q = q_0 + \sum_{t=1}^{n-1} q_t G_t$, then $\mathcal{P}_Q(\beta) = q_0 + \sum_{t=1}^{n-1} q_t \mathcal{P}_{G_t}(\beta)$. It will be of value then to have tables of these basic periodograms available, and such tables are presented in Appendix A.

5. Interpretation of Periodograms

In view of formula (8) of Chapter I, we can think of the periodogram of a SLO in the following way. If one were to fit a Fourier sum to a sequence of graduated values, the coefficients of the sum could be computed by multiplying the coefficients of the Fourier sum which fits the sequence of

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5. Interpretation of Periodograms

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ungraduated values by the appropriate periodogram values. Thus the values of the periodogram are weights and indicate to what extent waves of a given length are preserved by the graduation.

Intuitively one feels that a satisfactory graduator will suppress short waves (reflecting local irregularities) while retaining longer waves (revealing trends). Such qualities of a graduator would be indicated by its periodogram if the periodogram values were small for β small, and near 1 for β large. In addition, there seems to be merit in a graduator with a monotonically increasing periodogram having few inflection points.

Clearly, a periodogram function may assume negative values for some ranges of β. Indeed, in summation type graduators, i.e. graduation operators with factors of $[m]$, the zeros of $\rho_{[m]}(\beta)$ and the continuity of $\rho(\beta)$ imply that if an odd power of $[m]$ occurs as a factor, there must be negative values of the periodogram. In such instances, it follows in our interpretation of the periodogram of a graduator that waves of length β are inverted. There seems to be no excuse for tolerating large negative values, but small values need not be alarming.

6. Periodograms of Special Graduation Operators

We shall now compute and discuss the periodograms of several graduation operators. First, we shall mention those with which Elphinstone dealt, then carry on with some other published graduators, and finally end by examining the periodograms of some presumably unpublished graduators.

6.1 Elphinstone's Survey of Graduators

Elphinstone [4; p.38] in his survey of graduation methods, analyzes the periodograms of several graduation operators, including some which were never intended to graduate a sequence of data. Of the better known graduators, he considers seriously the formulae of Spencer [18], Hardy [12], Sheppard [17], Rhodes [16], and Whittaker [22], concluding that, of this list, the only graduator "without obviously objectional features is Whittaker's" and the best summation type graduator is Spencer's.

Elphinstone has carefully plotted the periodograms of these and other graduators, and we shall not reproduce that work except to make available, in Appendix B, his periodogram of Whittaker's formula so that the comparisons and comments to be made in section 6.3 may be followed more easily.

6.2 Periodograms of Some Linear Compound Graduators

The definition of a graduation operator does not, of course, determine the coefficients of the operator if it is of length > 1. The device used for fixing the coefficients of a particular formula provides, then, a convenient characteristic by which one can distinguish various graduators.

For finite linear compound graduators the coefficients may be determined by the technique of minimizing, under certain assumptions, the mean square error of some order, m, (including as a possibility the 0^{th} order) of differences of the graduated values. The term "minimum R_m^2 formula" is often used to denote such a formula. Commonly encountered orders are m = 0, 3, and 4. One should see, for example,

Miller [15] for a discussion of the classical treatment of the problem originally conceived and worked out by Sheppard.

Lidstone [14] and Aitken [1] utilize Tchebychef polynomials of the first kind to fit polynomials of a given order to a sequence of data by the Method of Least Squares. This technique has been shown by Sheppard [17] and Lidstone among others, to provide results identical to those obtained by the reduction of mean square error of the 0^{th} differences. Therefore, the periodogram of the 21 term minimum R_0^2 formula is exactly that of Sheppard's 21 term formula which Elphinstone presents in his survey.

Greville [7] summarizes the work of Lidstone and Aitken and cleverly employs the Tchebychef polynomials to find the coefficients for minimum R_m^2 formulae for values of $m > 0$. Using his results, we shall compute the periodograms of several minimum R_m^2 formulae. In particular, we shall consider the 21 term **minimum** R_3^2 and R_4^2 formulae reproducing third degree polynomials and the 29 term minimum R_3^2 and R_4^2 formulae with the same order of reproduction. For reference purposes we shall denote these formulae by Q_1, Q_2, Q_3 and Q_4 respectively. Greville [8] has published tables of the coefficients of minimum R_3^2 and R_4^2 formulae reproducing third degree polynomials for odd (5 to 29 inclusive) length formulae, and we shall use these tables for computing $\mathcal{P}_{Q_i}(\beta)$ for $i = 1,2,3,4$. The results of our calculations are displayed by graphs in Appendix B.

One observes that in each instance $\mathcal{P}_{Q_i}(\beta)$ has the desirable small values for $\beta \leq 8$, but the small values persist

beyond this point in the longer, i.e. 29 term, formulae. In fact, one wonders if perhaps the longer formulae do not over-graduate and emphasize smoothness to the extent that they not only smooth waves which indicate local irregularities but also distort those indicating trends.

From the general shape of the graphs of the periodograms of the graduators in our sample, it appears that the length of the graduator has more effect upon the periodogram than the smoothing criterion (i.e. order of differences for which the mean square error is minimized). If we compare the periodograms of graduators of the same length, we see that in each instance the periodogram values for large β are greater for the R_4^2 case. This phenomenon is to be expected since the minimization of the mean square error of third differences of the graduated values is the stronger smoothing condition, and for a given β a smaller value of $\mathcal{P}(\beta)$ means more suppression of waves of length β.

In summary, while minimum R_m^2 formulae are certainly not suitable for graduation for all m and all lengths, there surely are some satisfactory graduators in this class, and it is idle to dismiss such operators as "little more than the solutions of problems in elementary algebra."

6.3 Periodograms of Some Difference Equation Graduators

Difference equation graduators are so called because the technique for establishing the coefficients of the graduator involves the solution of the difference equation which arises from minimizing a function designed to measure fit and smoothness. A quite general function of this type is

Spoerl's [19],

$$\sum(v_x-u_x)^2 + g\sum(\Delta v_x)^2 + h\sum(\Delta^2 v_x)^2 + k\sum(\Delta^3 v_x)^2, \quad (18)$$

where g, h, and k are arbitrary non-negative weights, u_x is the ungraduated sequence and v_x the graduated sequence. In expression (18), the first sum is the traditional measure of fit, while the other three sums are measures of smoothness. Clearly one could have any number of additional sums for measuring smoothness involving higher orders of difference. The effect of additional sums upon the periodogram of the resultant graduator will be observed shortly.

Expression (18) is minimized upon satisfaction of the difference equation

$$u_x = (1 - g\delta^2 + h\delta^4 - k\delta^6)v_x \quad (19)$$

subject to some provision for starting values. If exactly one of g, h, or k is different from 0, we shall call the graduator a <u>difference equation graduator</u> while if more than one of g, h, or k is non-zero we shall, following Spoerl, say we have a <u>mixed difference equation graduator</u>.

Equation (19) can be rephrased to give

$$v_x = (1 - g\delta^2 + h\delta^4 - k\delta^6)^{-1}u_x \; . \quad (20)$$

It follows from a method attributed by Whittaker [23; p.308] to A. C. Aitken's thesis, and further elaborated by Spoerl, that the operator $(1 - g\delta^2 + h\delta^4 - k\delta^6)^{-1}$ has an expansion $q_0 + q_2G_2 + \ldots$, with q_{2n} converging to zero sufficiently rapidly for graduation purposes so that

$$v_x = (q_0 + q_2G_2 + \ldots)u_x \; . \quad (21)$$

Furthermore the same type of argument shows that

$$P(\beta) = (1 + g(4\sin^2(\pi/\beta)) + h(16\sin^4(\pi/\beta)) + k(64\sin^6(\pi/\beta)))^{-1}. \quad (22)$$

Whittaker [22], the originator of the difference equation method, considered only the case of $g = h = 0$. Elphinstone's survey of graduators includes the Whittaker formula with $k = 1$, and we repeat it only for comparison with other graduation operators, in particular the Whittaker formula where $k = 20$. We see that there is a remarkably similar shape for the periodograms of these two graduators, denoted by Q_5 and Q_6 respectively. Furthermore, quite expectedly, the larger value of k in Q_6, indicating greater emphasis on the measure of smoothness, has the effect of reducing the periodogram value for any β, i.e. $P_{Q_6}(\beta) < P_{Q_5}(\beta)$.

We shall now turn to Spoerl's mixed difference equation graduators. Specifically let

$$Q_7 = (1 + \delta^4 - \delta^6)^{-1}$$

and

$$Q_8 = (1 - \delta^2 + \delta^4 - \delta^6)^{-1} .$$

The periodograms $P_{Q_7}(\beta)$ and $P_{Q_8}(\beta)$ resemble $P_{Q_5}(\beta)$ and $P_{Q_6}(\beta)$. They also give evidence of the general result that if two mixed difference equation graduators Q and Q' have the same arbitrary weights for all but one of the smoothness sums and the respective weights for them are j and j', then $P_Q(\beta) \lesseqgtr P_{Q'}(\beta)$ according as $j \lesseqgtr j'$. At this point the effect of including smoothness sums involving higher orders of differences in the function to be minimized is clear. Additional sums result in smaller periodogram values.

The periodograms of difference equation graduators (including the mixed difference case) are never negative

and always have the "steady sweep" which prompted Elphinstone to single out Whittaker's formula (with k = 1) as a clearly superior graduator. I would submit that he is unduly harsh on periodograms with small negative values such as those for minimum R_m^2 graduators and a little too lenient in not restricting the k to values somewhat larger than 1 inasmuch as $P_{Q_5}(\beta)$ departs rather quickly from 0 as β increases beyond the relatively small value of 4.

6.4 The Periodogram of a Summation Graduator

Theorem 4 of Chapter II gives a set of m-1 values of β for which the periodogram of a graduator is zero if the graduator has a factor of summation m. Since it seems desirable to have small values for a periodogram of a graduation operator for small wave lengths, this theorem suggests that a satisfactory graduator can be constructed from several summation factors. Accordingly, we shall consider the graduator

$$Q_9 = (1/840)([4][5][6][7])(1 - (61/12)\delta^2) .$$

The particular choice of Q_9 has several advantages over many of the other graduators which could be built up using the summation factor criterion alone. Judicious choice of distinct m's provides a large number of zeros of $P_{Q_9}(\beta)$ in the range $\beta \leq 7$. This tends to keep the periodogram values small in that range. Furthermore, Q_9 is a 21 term formula and is therefore comparable to Spencer's 21 term formula and the two 21 term formulae of section 6.2. The graduation operator Q_9 reproduces polynomials up through the third degree, a common actuarial criterion for fit, and it is this property which necessitates the inclusion of the factor

$(1 - (61/12)\delta^2)$.

The periodogram, $\mathcal{P}_{Q_9}(\beta)$, provides justification for our choice of Q_9. The values for $\beta \leq 7$ are negligible, while at $\beta = 7$ the periodogram begins a steady rise toward the horizontal asymptote at $\mathcal{P}(\beta) = 1$. Indeed, $\mathcal{P}_{Q_9}(\beta)$ compares quite favorably with the other periodograms considered.

Although this extension of Elphinstone's survey completes our investigation of periodograms of graduation operators, we shall see in Chapter IV that from certain interpolation formulae we can construct a symmetric linear operator which is a graduator. Hence we shall be recalling the results of this chapter as we discuss the periodograms which arise.

CHAPTER IV

PERIODOGRAMS OF SUBTABULATORS

1. Introduction

Although Vaughan [20] was not the first to recognize that there exists some connection between interpolation and graduation by summation methods, he first realized how close the relationship is. His investigation was prompted by "the thought that interpolation formulae designed for smoothness might be subjected to the test customary for summation formulae." Greville [9] generalized and clarified Vaughan's work in the discrete case, and presented several applications of the methods. More recently, Greville and Vaughan [11] have published a complete treatment of the continuous situation.

In this chapter we shall deal with the discrete case exclusively, and the material is founded primarily upon Greville's work in this area. The nature of the discrete case is such that an exhaustive analysis is quite difficult. It is not our intention to attempt such a treatment. Rather we shall extend Greville's work only insofar as it affords us improved techniques for computing and interpreting periodograms of an interpolation subtabulator. Thus, in a sense, we shall be furthering Vaughan's original purpose of testing interpolation formulae.

2. Discrete Interpolation Formulae

Although in general the arguments of those function values upon which an interpolation formula is based need not be equally spaced, we shall be concerned with the case

in which the arguments are one unit apart, occurring either at integer or half-integer values, and in which there is an integer that is central to the set of arguments. This "central integer" (which we shall take equal to zero without loss of generality) may or may not be in the set of arguments.

Ordinarily, when one wishes to interpolate at a given number of equally spaced points on the interval $(-\frac{1}{2},\frac{1}{2})$, one takes for his set of points either

$$J_a = \{j : j = -(m-1)/2m,\ -(m-3)/2m,\ldots,\ (m-1)/2m\}$$

or

$$J_b = \{j : j = -1/2,\ -(m-2)/2m,\ldots,\ 1/2\}$$

where m is an integer. These choices will arise as special cases if we subdivide the unit interval centered upon zero into 2m equal intervals, each of length 1/2m. The set

$$J = \{j : j = -1/2,\ -(m-1)/2m,\ldots,\ (m-1)/2m,\ 1/2\}$$

accomplishes such a subdivision, and it is at these points that we shall want to interpolate.

If one views an interpolated value as a weighted sum of known values u_i of a function, then, upon a moment's reflection, one would expect the set of weights used to interpolate at one point, say j, of our interpolation set J, would be a function of the interval lengths (i-j) between the data points and the interpolation point. Further, one would expect this set of weights to be used in reverse order to interpolate at the symmetric counterpart -j of the point j. For the interpolation formulae with which we shall deal these statements are true except that it may happen that the weight of u_i in interpolation at $j = -\frac{1}{2}$ may differ from the weight

of u_{i+1} in interpolation at $j = \frac{1}{2}$. To take account of this rather rare exception, a more generalized notation will be used than would otherwise be necessary.

The preceding discussion of interpolation formulae touches upon the characteristics which we want a discrete interpolation formula to have, namely, that it may be written as a set of equations

$$v_j = \sum_i a_{i,j} u_i \tag{1}$$

where i runs over a finite number of integers or half-integers and j over J, or perhaps one of the subsets J_a or J_b, and the coefficients are subject to the condition

$$a_{i,j} = a_{-i,-j} \tag{2}$$

and, of course, such other conditions as were used to determine the formula.

3. Subtabulators of Discrete Interpolation Formulae

If we have the finite array $(a_{i,j})$ of coefficients of a discrete interpolation formula, A, then the <u>subtabulation operator</u> (or <u>subtabulator</u>), S_A, of A with respect to the integer m is the sum

$$S_A = \sum_t s_t E^{t/2} \tag{3}$$

where the coefficients, s_t, depend upon the entries in the array according to the rule

$$s_t = s_{2m(i-j)} = \begin{cases} a_{i,j} & |j| \neq \frac{1}{2} \\ \frac{1}{2}(a_{i,-\frac{1}{2}} + a_{i+1,\frac{1}{2}}) & j = -\frac{1}{2} \\ \frac{1}{2}(a_{i,\frac{1}{2}} + a_{i-1,-\frac{1}{2}}) & j = \frac{1}{2} \end{cases} \tag{4}$$

for t an integer. If we wish to stipulate which interpolation set J_a or J_b is involved we shall write $S_A^{(a)}$ or $S_A^{(b)}$ respectively. The absence of (a) or (b) implies interpolation at all points of J. We note that exactly one of $S^{(a)}$ and $S^{(b)}$ is in V', and the other is in V". Although the subtabulator depends upon m, we have not made any indication of this fact in our notation, because we feel that to do so would add unnecessarily to an already bulky symbol. When no confusion will arise, the subscript identifying the interpolation formula will be suppressed.

At this point a discussion of definition (4) is appropriate, for it is not altogether evident that s_t is well-defined. Specifically one might wonder whether there are different representations of an integer t in the form 2m(i-j) and if so, does each yield the same coefficient s_t. The questions are resolved by studying the solutions (2i,2mj) of the Diophantine equation

$$m(2i) + (-1)(2mj) = t$$

where m and t are arbitrary but fixed integers. We find that there are exactly two solutions $(2i_1,2mj_1)$ and $(2i_2,2mj_2)$ such that j_1 and j_2 are in J unless j_1 or j_2 equals $\frac{1}{2}$ or $-\frac{1}{2}$, in which case there is an additional solution $(2i_3,2mj_3)$. In the ordinary situation involving only two solutions, we find that precisely one of i_1 and i_2 is an integer, the other being a half-integer; since all the values of i must be of one type, we have a unique ordered pair (i,j) corresponding to a given t. In the unusual event of three solutions, we can label them in such a way that $j_1 = -\frac{1}{2}$, $j_2 = 0$, and $j_3 = \frac{1}{2}$,

in which case $i_3 = i_1 + 1$. As in the ordinary situation, either i_1 or i_2 is an integer while the other is a half-integer; thus only one of the solutions $(2i_1, 2mj_1)$ and $(2i_2, 2mj_2)$ is applicable. It remains then to show that $s_{2m(i_1-j_1)} = s_{2m(i_3-j_3)}$. To see this we note that

$$s_{2m(i_3-j_3)} = s_{2m((i_1+1)-\frac{1}{2})} = \tfrac{1}{2}(a_{i_1+1,\frac{1}{2}} + a_{i_1,-\frac{1}{2}})$$
$$= s_{2m(i_1-(-\frac{1}{2}))} = s_{2m(i_1-j_1)} .$$

The following computation, using condition (2) on the coefficients of a discrete interpolation formula, reveals that the subtabulator of such a formula is a SLO. In fact, for $|j| \neq \frac{1}{2}$,

$$s_{-t} = s_{-2m(i-j)} = s_{2m(-i-(-j))}$$
$$= a_{-i,-j} = a_{i,j}$$
$$= s_t ,$$

while for $j = -\frac{1}{2}$ (and similarly for $j = \frac{1}{2}$) we have

$$s_{-t} = s_{-2m(i-(-\frac{1}{2}))} = s_{2m(-i-\frac{1}{2})}$$
$$= \tfrac{1}{2}(a_{-i,\frac{1}{2}} + a_{-i-1,-\frac{1}{2}}) = \tfrac{1}{2}(a_{i,-\frac{1}{2}} + a_{i+1,\frac{1}{2}})$$
$$= s_{2m(i-(-\frac{1}{2}))} = s_t .$$

Thus, for a discrete interpolation formula, A we may write

$$S_A = \sum_{t=0} s_t G_t \qquad (5)$$

and this final form explains the purpose of the factor 2m in rule (4).

We can now speak of the periodogram of the subtabulator of a discrete interpolation formula but in order to apply the periodogram analysis and interpretation of Chapter III

we must first see the close connection between a graduation operator and a subtabulator of a discrete interpolation formula which Greville has called Vaughan's Principle.

4. Vaughan's Principle

Prior to stating Vaughan's principle we shall adapt one of Greville's arguments to demonstrate the equivalence of interpolation at a point j of J and application of the subtabulation operator of the interpolation formula to U_t at $t = mj$ where

$$U_t = \begin{cases} u_i & t = mi \\ 0 & t \neq mi \end{cases} .$$

For $|j| \neq \frac{1}{2}$, we can write

$$v_j = \sum_i a_{i,j} u_i = \sum_i a_{i,j} U_{mi} = \sum_i a_{i,j} U_{m(j+(i-j))}$$

$$= \sum_i s_{2m(i-j)} E^{m(i-j)} U_{mj}$$

$$= \left(\sum_{k \in J} \sum_i s_{2m(i-k)} E^{m(i-k)} \right) U_{mj} \qquad (6)$$

since $U_{m(j+i-k)} = 0$ for $k \neq j$. If we interpolate at $j = \frac{1}{2}$, we have

$$v'_{\frac{1}{2}} = \sum_i a_{i,\frac{1}{2}} u_i$$

and

$$v''_{\frac{1}{2}} = \sum_i a_{i-1,-\frac{1}{2}} u_i .$$

This situation arises only in connection with subtabulation at the points of J_b and in this case it would be unreasonable to choose an interpolation process such that the interpolation arc for the interval $(-\frac{1}{2},\frac{1}{2})$ did not meet the arc for $(\frac{1}{2},3/2)$ at $j = \frac{1}{2}$. Hence, without practical restriction, we

may assume $v'_{\frac{1}{2}} = v''_{\frac{1}{2}} = v_{\frac{1}{2}}$ and write

$$v_{\frac{1}{2}} = \sum'_i \tfrac{1}{2}(a_{i,\frac{1}{2}} + a_{i-1,-\frac{1}{2}})u_i = \sum_i s_{2m(i-\frac{1}{2})} U_{mi}$$

$$= \sum_i s_{2m(i-\frac{1}{2})} E^{m(i-\frac{1}{2})} U_{m(\frac{1}{2})}$$

$$= \left(\sum_{k \in J} \sum_i s_{2m(i-k)} E^{m(i-k)}\right) U_{m(\frac{1}{2})} \qquad (7)$$

since again

$$U_{m(\frac{1}{2}+i-k)} = 0 \qquad |k| \neq \tfrac{1}{2}.$$

Relations (6) and (7) combine to give

$$v_j = S\, U_{mj} \qquad (8)$$

for any point j of J.

Because of the equivalence implied by (8), one would, as Greville suggests, expect that a characteristic of an interpolation formula will appear as a corresponding characteristic of the subtabulator of the formula. Vaughan's principle is just such a case in point. Before stating the principle, we must make the following definition. The subtabulator S of a discrete interpolation formula is said to have <u>reproduction degree r</u> if (1/m)S reproduces a set of values lying on any polynomial of r^{th} degree or less. The factor 1/m appears since the coefficients of S relative to the $\{G_t\}$ basis constitute an array such that each of the m columns yields a set of interpolating weights with sum equal to one.

Greville expresses Vaughan's principle in two parts, but the portion which we should like to use involves a sin-

gle statement.

Vaughan's Principle: A necessary and sufficient condition that a discrete interpolation formula shall reproduce polynomials of the r^{th} degree is that its subtabulation operator S is of the form $[m]^{r+1}H$ and has reproduction degree r.

In view of Vaughan's principle, it is evident that if S is the subtabulation operator, using either J_a or J_b as the set of interpolation points, of a discrete interpolation formula which reproduces polynomials of some degree, then (1/m)S is a graduation operator. Thus the periodogram analysis, in general, and periodogram interpretations of Chapter III, in particular, have application to this situation. In fact, if we recall Theorem 4 (and its first corollary) of Chapter II and Theorem 2 of Chapter III, the condition of Vaughan's principle can be rephrased in terms of the periodograms of the subtabulation operator of the given interpolation formula. Specifically we have the following restatement of the principle.

Vaughan's Principle: A necessary and sufficient condition for a discrete interpolation formula to reproduce polynomials of the r^{th} degree is that the periodogram $\mathcal{P}(\beta)$ of the subtabulator (using J_a or J_b) of the interpolation formula has zeros of order r+1 at $\beta = m, m/2, \ldots, m/(m-1)$ and has order of contact r with the horizontal at $\beta = 1$.

5. General Problem of Computing Subtabulators

Frequently an interpolation formula assumes a form different from that of a weighted sum such as would immediately yield the array required for its subtabulator. The

interpolation formulae which we shall consider can always be rewritten as a weighted sum, but then the subtabulator is expressed relative to the $\{G_t\}$ basis, and this is not the most convenient form for computing periodograms. A factored expression is generally more desirable for such purposes. This consideration prompts us to seek an alternate means of finding a subtabulator.

Greville [9] has shown by example that given the assumptions underlying a particular interpolation formula, Vaughan's principle together with some mathematical ingenuity will produce the subtabulator of the formula. However, his technique can hardly be classed as an algorithm for finding the subtabulator, and yet this is what we really want. Therefore, we shall focus upon one large class of interpolation formulae and develop just such a procedure.

One quite general form in which an interpolation formula may appear is

$$v_j = (c_0\phi_j^0 + d_1\Theta_j^1 + c_2\phi_j^2 + d_3\Theta_j^3 + \dots)u_0 \qquad (9)$$

for an odd length formula, and

$$v_j = (d_0\Theta_j^0 + c_1\phi_j^1 + d_2\Theta_j^2 + c_3\phi_j^3 + \dots)u_0 \qquad (10)$$

for an even length formula, where we are using the central factorial notation

$$\Theta_j^n = \binom{j-(n-1)/2}{n}$$

and

$$\phi_j^n = \begin{cases} (j/n)\Theta_j^{n-1} & n \neq 0 \\ 1 & n = 0 , \end{cases}$$

and the central difference operators

$$c_i = \sum_{k=0} a_{i:i+2k}\, \delta^{i+2k} ,$$

and

$$d_i = \sum_{k=0} b_{i:i+2k} \mu \delta^{i+2k} ,$$

suggested by White [21]. Note that if $c_i = \delta^i$ and $d_i = \mu\delta^i$ for $i = 0,1,2,\ldots$ then formulae (9) and (10) are those of Stirling and Bessel, which we will denote by $\mathcal{S}_r$ and $\mathcal{B}_r$, respectively, where r is the highest power of δ appearing in the formula. We shall see that the subtabulators of these two special cases can be used to compute the subtabulators of the general forms (9) and (10).

As an initial step in the development of a technique for calculating these subtabulators we note the following lemma which summarizes some immediate consequences of the definitions involved.

Lemma 1: Let A and B identify two arrays $(a_{i,j})$ and $(b_{i,j})$ of interpolation coefficients such that the i's in each are of the same type, i.e. integers or half-integers. If we denote the arrays $(ca_{i,j})$ and $(a_{i,j} + b_{i,j})$ by cA and A + B, respectively, then

$$S_{cA} = cS_A$$

and

$$S_{A+B} = S_A + S_B .$$

With these results, we have reduced our problem to that of computing the subtabulators of the individual terms of Stirling's and Bessel's interpolation formulae and of composites of these terms with μ and δ operators. Furthermore,

if the subtabulators of Stirling's and Bessel's formulae to all orders of differences are available, then the individual term subtabulators are easily computed by subtraction. For example

$$S_{\phi_j^2 \delta^2} = S_{\delta_2} - S_{\delta_1} .$$

6. Lengths of Subtabulators

Greville has shown that in certain instances of ordinary advancing difference interpolation, the subtabulators can be computed simply by applying Vaughan's principle in order to determine what power of $[m]$ must be a factor and what the initial terms of the correcting operator must be, and then arguing that the length of the subtabulator can admit no further terms of the correcting operator. We shall be able to use this general technique for computing the subtabulators of δ_r and $\mathcal{B}_r$ for some values of r. In order to do this, we must be able to determine the length of a subtabulator of a formula of this type.

We shall assume from now on that interpolation occurs at the points of either J_a or J_b. The formula for the length of the subtabulator of a Stirling or Bessel interpolation formula depends upon which of these interpolating sets is chosen, and given J_b, whether the number, h, of data points involved in the interpolation is odd or even. The rule is that

$$d(S_{\ }) = \begin{cases} mh & j \text{ in } J_a \\ mh+1 & j \text{ in } J_b \text{ and } h \text{ odd} \\ mh-1 & j \text{ in } J_b \text{ and } h \text{ even.} \end{cases} \qquad (11)$$

One must bear in mind that rule (11) applies to the subtabulators of two specific types of interpolation formulae and does not hold in general.

The first case of formula (11) is not unexpected since it does not involve the troublesome interpolation points $j = \pm\frac{1}{2}$. An understanding of the other cases hinges upon the fact that

$$\begin{aligned} s_{mh} &= s_{2m((h-1)/2\ -(-\frac{1}{2}))} \\ &= \tfrac{1}{2}(a_{(h-1)/2,-\frac{1}{2}} + a_{(h+1)/2,\frac{1}{2}}) \\ &= \tfrac{1}{2}(a_{(h-1)/2,-\frac{1}{2}}) \end{aligned}$$

since $a_{(h+1)/2,\frac{1}{2}} = 0$. It follows then that

$$s_{mh} = \tfrac{1}{4}\Theta^{h-2}_{-\frac{1}{2}} \quad \text{or} \quad \tfrac{1}{2}(\tfrac{1}{2}\Theta^{h-2}_{-\frac{1}{2}} + \phi^{h-1}_{-\frac{1}{2}}) ,$$

depending upon whether the interpolation operator ends with a mean or with a single difference. Thus s_{mh} is or is not zero according as h is or is not even.

We should note that since one of J_a or J_b is the set of interpolation points, S is in one of the spaces V' or V". That is to say, the subscripts of non-zero coefficients are all odd or all even, and any factor of S will have this same property.

7. $S_{\mathcal{S}_r}$ and $S_{\mathcal{B}_r}$

The formulae $\mathcal{S}_r$ and $\mathcal{B}_r$ reproduce polynomials of the r^{th} degree, and thus by Vaughan's principle $S_{\mathcal{S}_r}$ and $S_{\mathcal{B}_r}$ have reproduction degree r and each assumes the form $[m]^{r+1}H$.

In those instances where the interpolation operator ends

in a single difference (as opposed to a mean difference) we shall be able to write down the correcting operator, H, after identifying the space V' or V'' to which it belongs, by using arguments generally similar to those of Greville. We can decide in which of V' and V'' the correcting operator, H, lies merely by noting its length, for d(H) is even or odd according as H is in V' or V''. This information plus the knowledge of the reproduction degree of the subtabulator enables us to write down at least the first few terms of H.

For example, let us compute $S^{(b)}_{\delta_2}$. Vaughan's principle yields

$$S^{(b)}_{\delta_2} = [m]^3 H ,$$

and $S^{(b)}_{\delta_2}$ reproduces second degree polynomials. In addition we note that using rule (11) we have

$$d(S^{(b)}_{\delta_2}) = 3m + 1$$

and using equation (4) of Chapter II we have

$$d([m]^3) = 3m - 2 ,$$

whence

$$d(H) = (3m+1) - (3m-2) - 1 = 4.$$

Therefore H is in V' and has a factor of μ. The correcting operator associated with the factor $[m]^3$ begins

$$[m]^{-3}/m^2 = (1/m^2)(1 - ((m^2-1)/8)\delta^2 + \ldots)$$

while that associated with μ begins

$$\mu^{-1} = (1 - \delta^2/8 + \ldots).$$

Thus, since $S^{(b)}_{\delta_2}$ is to have reproduction degree 2 it must begin

$$S_{\delta_2}^{(b)} = ([m]^3/m^2)(\mu)(1 - m^2\delta^2/8 + \dots). \qquad (12)$$

However, $d(H) = 4$, so $S_{\delta_2}^{(b)}$ has no more terms than actually appear in statement (12).

A similar analysis yields the subtabulators for any Stirling and Bessel formula which ends in a single difference. Unfortunately, one cannot use the same technique to find S_{δ_r} or $S_{\mathcal{B}_r}$ for all values of r. Although δ_r and $\mathcal{B}_r$ reproduce polynomials of the r^{th} degree, so that the associated subtabulation operators have a factor $[m]^{r+1}$, it happens that for those r such that δ_r or $\mathcal{B}_r$ ends in a mean difference the length of the correcting operator, H, is a function of m, and thus far we have no simple way to obtain H. We shall support the preceding remarks by summarizing in Table 2 the lengths of the symmetric linear operators involved in the arguments. We shall, however, defer the presentation (cf. Tables 3 and 4) of those subtabulators which we can now compute, until we have a method for obtaining the rest.

TABLE 2

LENGTHS OF THE CORRECTING OPERATOR H IN S_{δ_r} and $S_{\mathcal{B}_r}$

r	$d(S_{\delta}^{(a)})$	$d(S_{\delta}^{(b)})$	$d(S_{\mathcal{B}}^{(a)})$	$d(S_{\mathcal{B}}^{(b)})$	$d([m]^{r+1})$	$d(H_{\delta}^{(a)})$	$d(H_{\delta}^{(b)})$	$d(H_{\mathcal{B}}^{(a)})$	$d(H_{\mathcal{B}}^{(b)})$
0	m	m+1	2m	2m-1	m	1	2	m+1	m
1	3m	3m+1	2m	2m-1	2m-1	m+2	m+3	2	1
2	3m	3m+1	4m	4m-1	3m-2	3	4	m+3	m+2
3	5m	5m+1	4m	4m-1	4m-3	m+4	m+5	4	3
4	5m	5m+1	6m	6m-1	5m-4	5	6	m+5	m+4
5	7m	7m+1	6m	6m-1	6m-5	m+6	m+7	6	5

In order to attack the remaining half of our problem, let us review our position and remove, for the moment, one of the conditions affecting the coefficients of a subtabulator.

The notion of a subtabulation operator as defined by relations (3) and (4) assumes an array of interpolation coefficients, and thus the subtabulator coefficients are indirectly restricted by condition (2). While the individual terms of formulae (9) and (10) yield arrays with entries satisfying condition (2), certain factors of these terms, for example $\theta_j^1 u_0$, do not. The only property of a subtabulator encountered thus far which depends upon condition (2) is that of being a SLO, so if we relax this condition we shall still have the sum (3) available, and we shall continue to refer to it as a subtabulator. Moreover, an array $(a_{i,j})$ which has all the properties of an array of interpolation coefficients except condition (2) will be called a subtabulation array.

If A identifies a subtabulation array, $(a_{i,j})$, then we shall write $\underline{\delta A}$ to denote the subtabulation array $(a'_{i,j}) = (a_{i-\frac{1}{2},j} - a_{i+\frac{1}{2},j})$, and define $\underline{\delta^n A}$ to mean $\delta(\delta^{n-1}A)$ for $n > 1$. Analogously, we shall use $\underline{\mu A}$ to denote the array $(a''_{i,j}) = \frac{1}{2}(a_{i-\frac{1}{2},j} + a_{i+\frac{1}{2},j})$, and define $\underline{\mu^n A}$ to mean $\mu(\mu^{n-1}A)$ for $n > 1$. We note that if A represents an interpolation operator, then δA represents the composition of the two operators.

With these changes and additions, we have the following result which improves our overall position substantially.

Theorem 1: If A identifies a subtabulation array $(a_{i,j})$ then

$$S_{\delta A} = \delta_m S_A \tag{13}$$

and

$$S_{\mu A} = \mu_m S_A \ . \tag{14}$$

Proof: In order to prove statement (13), we merely calculate both sides of the equation and identify the results. On the one hand

$$S_{\delta A} = \sum_t s'_t E^{t/2} \ ,$$

where

$$s'_{2m(i-j)} = \begin{cases} a'_{i,j} & |j| \neq \frac{1}{2} \\ \frac{1}{2}(a'_{i,-\frac{1}{2}} + a'_{i+1,\frac{1}{2}}) & j = -\frac{1}{2} \\ \frac{1}{2}(a'_{i-1,-\frac{1}{2}} + a'_{i,\frac{1}{2}}) & j = \frac{1}{2} \end{cases}$$

$$= \begin{cases} a_{i-\frac{1}{2},j} - a_{i+\frac{1}{2},j} & |j| \neq \frac{1}{2} \\ \frac{1}{2}(a_{i-\frac{1}{2},-\frac{1}{2}} - a_{i+\frac{1}{2},-\frac{1}{2}} + a_{i+\frac{1}{2},\frac{1}{2}} - a_{i+3/2,\frac{1}{2}}) & j = -\frac{1}{2} \\ \frac{1}{2}(a_{i-3/2,-\frac{1}{2}} - a_{i-\frac{1}{2},-\frac{1}{2}} + a_{i-\frac{1}{2},\frac{1}{2}} - a_{i+\frac{1}{2},\frac{1}{2}}) & j = \frac{1}{2}. \end{cases} \tag{15}$$

On the other hand

$$\delta_m S_A = (E^{m/2} - E^{-m/2}) \sum_t s_t E^{t/2}$$

$$= \sum_t (s_{t-m} - s_{t+m}) E^{t/2} \ .$$

$$= \sum_t s^*_t E^{t/2}$$

where

$$s^{*}_{2m(i-j)} = s_{2m(i-j)-m} - s_{2m(i-j)+m}$$

$$= s_{2m((i-\frac{1}{2})-j)} - s_{2m((i+\frac{1}{2})-j)}$$

$$= \begin{cases} a_{i-\frac{1}{2},j} - a_{i+\frac{1}{2},j} & |j| \neq \frac{1}{2} \\ \frac{1}{2}(a_{i-\frac{1}{2},-\frac{1}{2}} + a_{i+\frac{1}{2},\frac{1}{2}} - a_{i+\frac{1}{2},-\frac{1}{2}} - a_{i+3/2,\frac{1}{2}}) & j = -\frac{1}{2} \\ \frac{1}{2}(a_{i-3/2,-\frac{1}{2}} + a_{i-\frac{1}{2},\frac{1}{2}} - a_{i+\frac{1}{2},\frac{1}{2}} - a_{i-\frac{1}{2},-\frac{1}{2}}) & j = \frac{1}{2}. \end{cases} \quad (16)$$

Relations (15) and (16) show that $s'_t = s^{*}_t$ for all values of t and the proof is complete. An entirely similar computation establishes statement (14).

Corollary 1: Using the notation of the theorem,

$$S_{\delta^r A} = \delta^r_m S_A = \delta^r [m]^r S_A .$$

To complete our procedure for computing the subtabulator of an interpolator of the form (9) or (10) we must obtain the subtabulators of Stirling and Bessel formulae which end in mean differences, or equivalently find the subtabulator of the individual mean difference terms. Theorem 1 and its corollary enable us to do this.

As an example of the technique, let us calculate

$$S^{(b)}_{\theta^1_j \mu\delta} = S^{(b)}_{\ell_1} - S^{(b)}_{\ell_0} .$$

By Theorem 1 we have

$$S^{(b)}_{\theta^1_j \mu\delta} = \mu_m \delta_m S^{(b)}_{\theta^1_j} .$$

If we use formula (4) to compute $S^{(b)}_{\theta^1_j}$ we get

$$S^{(b)}_{\theta^1_j} = \tfrac{1}{2}\theta^1_{\frac{1}{2}}E^{-m/2} + \theta^1_{(m-2)/2m}E^{-\frac{1}{2}(m-2)} + \cdots + \theta^1_{-(m-2)/2m}E^{(m-2)/2} + \tfrac{1}{2}\theta^1_{-\frac{1}{2}}E^{m/2},$$

and the computation of $S^{(b)}_{\theta^1_j\mu\delta} = \mu_m[m]\delta S^{(b)}_{\theta^1_j}$ continues in the following manner.

$$\begin{aligned} S^{(b)}_{\theta^1_j} &= \mu_m[m]\left(-\tfrac{1}{2}(\tfrac{1}{2})E^{-\frac{1}{2}(m+1)} + (-(m-2)/2m + \tfrac{1}{2}\cdot\tfrac{1}{2})E^{-\frac{1}{2}(m-1)} + \right. \\ &\qquad \left. \cdots + (\tfrac{1}{2}(\tfrac{1}{2}) - (m-2)/2m)E^{\frac{1}{2}(m-1)} - \tfrac{1}{2}\cdot\tfrac{1}{2}E^{\frac{1}{2}(m+1)}\right) \\ &= \mu_m[m]\left(-(\tfrac{1}{4})G_{m+1} - (\tfrac{1}{4})G_{m-1} + [m]/m\right) \\ &= ([m]^2/m)\mu_m - (\tfrac{1}{4})\mu_m[m](G_{m+1} + G_{m-1}) . \end{aligned} \tag{17}$$

Expression (17) can be simplified further by noting that

$$G_{m+1} + G_{m-1} = 4\mu\mu_m .$$

Thus we have

$$\begin{aligned} S^{(b)}_{\theta^1_j\mu\delta} &= ([m]^2/m)\mu_m - \mu_m^2[m]\mu \\ &= ([m]^2/m)\mu_m - (1 + \delta_m^2/4)[m]\mu \\ &= ([m]^2/m)(\mu_m - (m/4)\mu\delta^2[m]) - \mu[m] , \end{aligned}$$

and since $S^{(b)}_{\delta_0} = \mu[m]$, it follows that

$$S^{(b)}_{\delta_1} = ([m]^2/m)\mu_m - (m/4)\mu\delta^2[m]). \tag{18}$$

The example we have just presented is a relatively sim-

ple illustration of the labor involved in calculating the subtabulator of an interpolation formula which ends in a mean difference. The complexity increases with r in a non-linear manner. However, the technique is straightforward, and provides us with an algorithm for computing those subtabulators which we could not handle with the simple length arguments.

We shall conclude this section with a tabular presentation of $S^{(a)}_{\delta_r}$, $S^{(b)}_{\delta_r}$, $S^{(a)}_{\beta_r}$, and $S^{(b)}_{\beta_r}$ for small values of r. Where possible the entries have been worked out by both of the methods of this section.

TABLE 3

$S^{(a)}_{\delta_r}$ AND $S^{(b)}_{\delta_r}$

r	$S^{(a)}_{\delta_r}$	$S^{(b)}_{\delta_r}$
0	$[m]$	$\mu[m]$
1	$\frac{[m]^2}{m}\left(\mu\mu_m - \frac{m}{4}\delta^2[m]\right)$	$\frac{[m]^2}{m}\left(\mu_m - \frac{m}{4}\mu\delta^2[m]\right)$
2	$\frac{[m]^3}{m^2}\left(1 - \frac{m^2-1}{8}\delta^2\right)$	$\frac{\mu[m]^3}{m^2}\left(1 - \frac{m^2}{8}\delta^2\right)$
3	$\frac{[m]^4}{m^3}\left(\mu\mu_m\left(1-\frac{m^2-1}{24}\delta^2\right) - \frac{m}{4}\delta^2[m]\left(1-\frac{m^2-1}{8}\delta^2\right)\right)$	$\frac{[m]^4}{m^3}\left(\mu_m\left(1-\frac{m^2-4}{24}\delta^2\right) - \mu\delta^2[m]\frac{m}{4}\left(1-\frac{m^2}{8}\delta^2\right)\right)$
4	$\frac{[m]^5}{m^4}\left(1 - \frac{5(m^2-1)}{24}\delta^2 + \frac{(m^2-1)(9m^2-1)}{384}\delta^4\right)$	$\frac{\mu[m]^5}{m^4}\left(1 - \frac{5m^2-2}{24}\delta^2 + \frac{3m^4}{128}\delta^4\right)$

TABLE 4

$S^{(a)}_{\mathfrak{B}_r}$ AND $S^{(b)}_{\mathfrak{B}_r}$

r	$S^{(a)}_{\mathfrak{B}_r}$	$S^{(b)}_{\mathfrak{B}_r}$
0	$\mu_m[m]$	$\mu\mu_m[m]$
1	$\frac{\mu[m]^2}{m^1}$	$\frac{[m]^2}{m^1}$
2	$\frac{[m]^3}{m^2}\left(\mu_m(1+\frac{\delta^2}{8})-\mu\frac{m}{4}\delta^2[m]\right)$	$\frac{[m]^3}{m^2}\left(\mu\mu_m-\frac{m}{4}[m]\delta^2\right)$
3	$\mu\frac{[m]^4}{m^3}\left(1-\frac{4m^2-1}{24}\delta^2\right)$	$\frac{[m]^4}{m^3}\left(1-\frac{m^2-1}{6}\delta^2\right)$
4	$\frac{[m]^5}{m^4}\left(\mu_m\left(1-\frac{2m^2-5}{24}-\frac{4m^2-1}{384}\delta^4\right)-\mu\frac{m}{4}\delta^2[m]\left(1-\frac{4m^2-1}{24}\delta^2\right)\right)$	$\frac{[m]^5}{m^4}\left(\mu\mu_m\left(1-\frac{m^2-1}{12}\delta^2\right)-\frac{m}{4}\delta^2[m]\left(1-\frac{m^2-1}{6}\delta^2\right)\right)$
5	$\mu\frac{[m]^6}{m^5}\left(1-\frac{2m^2-1}{8}\delta^2+\frac{(4m^2-1)(16m^2-1)}{1920}\delta^4\right)$	$\frac{[m]^6}{m^5}\left(1-\frac{m^2-1}{4}\delta^2+\frac{(m^2-1)(4m^2-1)}{120}\delta^4\right)$

8. Periodograms of Interpolation Subtabulators

As we said in the introductory remarks of this chapter, our ultimate goal is to evaluate the periodograms of interpolation subtabulators and analyze them much as we did for graduation operators in Chapter III. Indeed, as already noted, except for a factor of 1/m, subtabulators of interpolation formulae reproducing some order of polynomials are graduators.

If the interpolation formula appears as an array, then the periodogram is computed in the same way as in the case

of the linear compound graduation of Chapter III. If the formula is expressed as in statement (9) or (10) then the development of Section 7 provides the equipment for calculation. Of course, the subtabulation operators vary with the values of m, but m = 5 is so often used, that we shall tabulate in Appendix A the periodogram values of the subtabulation operators of the interpolation formulae for this value. Of course, periodogram values of subtabulators of individual terms can be obtained by differencing these tables.

9. Periodograms of Specific Subtabulators

We shall conclude with a brief survey of the periodograms of subtabulation operators of some well-known interpolation formulae which can be used to interpolate on one of the interpolation sets J_a or J_b.

9.1 Osculatory Interpolation

As an example of osculatory interpolation, wherein the interpolating arcs are not only required to meet at the ends of two successive interpolation intervals, but also to have equal derivatives to some order at the points of juncture, we shall use Jenkins' [13] modified fifth-difference osculatory interpolation formula. In this formula, the derivatives through the second order are equal for adjacent interpolating arcs. Boyer [3] has presented the subtabulation array for Jenkins' formula where m = 5 and the interpolation set J_b is used. This is the natural set of interpolation points inasmuch as the formula is based on an even number

of given values.

We have used Boyer's coefficients to compute the periodogram of the subtabulator of Jenkins' formula, and the results are displayed in Appendix B. If we apply the interpretations of Chapter III, we see that the periodogram is very satisfactory; a possible criticism might be that longer waves are suppressed somewhat more by this formula than by others yet to be considered, but the distinction is little more than hair-splitting.

There is an alternate form for the subtabulator of Jenkins' formula which can be used to calculate the periodogram. To display it we first observe that for $m = 5$, $S^{(b)}$ is of length 29. Therefore we may write

$$S^{(b)} = S^{(b)}_{B_3} + ([5]^4/125)(a_0\delta^4 + a_2\delta^6 + \ldots + a_8\delta^{12})$$

$$= S^{(b)}_{B_3} + ([5]^4\delta^4/125)(a_0 + a_2\delta^2 + \ldots + a_8\delta^8)$$

$$= S^{(b)}_{B_3} + ((6G_0 - 4G_{10} + G_{20})/125)(b_0 + b_2G_2 + \ldots + b_8G_8) \quad (19)$$

$$= S^{(b)}_{B_3} + (6b_0 - (6b_2 + 4b_8)G_2 + \ldots + b_0G_{20} + b_2G_{22} + \ldots + b_8G_{28})/125 \quad (20)$$

From Boyer's array we know the coefficients, s_t, such that

$$S^{(b)} = \sum_{t=0}^{28} s_t G_t \,. \quad (21)$$

Noting that $d(S^{(b)}_{B_3}) = 19$, we see that if we equate the coefficients of expressions (20) and (21) we have, among other

things,

$$b_t = s_{20+t} \qquad t = 0,2,4,6,8.$$

Hence we can rewrite line (19) with the numerical values of b_t, and use this form to compute the periodogram.

9.2 Minimized Difference Formulae

Beers [2] has criticized osculatory interpolation as concentrating excessively on the behavior of interpolating arcs at the juncture points. He has suggested that a more reasonable approach would be to minimize under certain assumptions the sum of squares of some order of differences of the interpolated values. Although Beers considered specific cases initially, Greville [6] has generalized the method.

The particular example we shall consider reproduces polynomials of degree 3 or less and minimizes the sum of squares of the fourth differences of the interpolated values. This is one of the formulae published by Beers and he compiles the results in the form of a subtabulation array with m = 5 and interpolation on J_b, which we have used to compute the periodogram of the subtabulator of the formula. The periodogram curve is quite satisfactory. The values for small wave lengths are slightly greater in absolute value than those of the Jenkins example, and moreover they are negative, but the general shape is not undesirable.

9.3 Interlocking Formulae

White [21] has answered Beers' criticism of osculatory

interpolation techniques in a different way. His alternative to setting orders of derivatives equal at the junction points of interpolating arcs is to equate orders of differences. The term "interlocking" is descriptive of the curves which arise; White notes that they resemble interwoven wires.

We shall take as examples of the interlocking technique, White's formulae (28) and (32), which are respectively the analogues of Sprague's osculatory formula and Jenkins' fourth difference modified formula. The specifications of formula (28) are: (1) fifth degree formula, (2) number of data points, h, equal to 6, (3) reproduction degree, r, equal to 4, and (4) three interlocking points. It is in Everett form and translation to form (10) yields,

$$v_j = (\mu + \varphi_j^1 + \theta_j^2 \mu\delta^2 + \varphi_j^3 \delta^3 + \theta_j^4 \mu\delta^4 + 29.285714\varphi_j^5\delta^5 + 2.262857\varphi_j^3\delta^5)u_0 , \quad (22)$$

and we can compute the subtabulation operator by the methods of Section 7. It happens that Greville [9] has computed this subtabulator and provides a check on our work.

Formula (32) has as specifications: (1) fourth degree formula, (2) number of data points, h, equals 5, (3) reproduction degree r equal to 3, and (4) three interlocking points. After computation of the numerical coefficients we have

$$v_j = (\varphi_j^0 + \theta_j^1 \mu\delta + \varphi_j^2\delta^2 + \theta_j^3\mu\delta^3 + (-0.0146289\varphi_j^0 + 0.0896875\varphi_j^2 + 3.5625\varphi_j^4)\delta^4)u_0 . \quad (23)$$

The periodogram of each of these subtabulation operators has a generally satisfactory shape, but formula (28) gives the most outstanding results, for the periodogram of its subtabulator rises more sharply than the periodograms of the subtabulators of any of the other three interpolation formulae considered in our brief survey.

This survey is not intended to cover all types of interpolation formulae. Such a treatment would be almost impossible. Rather we have chosen a few formulae to illustrate how the periodograms can be computed and the formulae tested using the periodogram methods which we have been developing throughout this investigation.

10. Conclusion

In this chapter, we have been exploring the concept of the subtabulation operator of a discrete interpolation formula. We have seen that the derivation and final form of a subtabulator depend directly upon the various available conditions. We now know that, in addition to those assumptions underlying the interpolation formula itself, such as the reproduction degree, the number of data points, and the basic type of formula (interlocking, osculatory, etc.), the choice of interpolating sets, J_a or J_b, and the specification of the value of m definitely affect the situation. This range of possibilities complicates the analysis, and although a full treatment might parallel the development of the continuous case by Greville and Vaughan [11], distinctly new problems arise; for example, in the continuous case, there is

neither a choice of interpolating set nor of m.

Our somewhat limited objective then has been to develop and use techniques such that, given an interpolation formula, we can obtain a reduced form of the subtabulator and thus enable ourselves to compute the associated periodogram and see some of its properties. The algorithm for determining the subtabulation operator from an interpolation formula somewhat complements the work of Greville [9] where, from the underlying specifications of an interpolation formula, he finds the subtabulation operator and then the discrete interpolation formula itself. Vaughan's principle has played a major role in our development, and we have noted that it can be restated in terms of properties of the periodogram instead of the subtabulators.

Finally, as in the case of the general symmetric linear operators and graduation operators, so also for the subtabulators, the interpretation of a periodogram provides a criterion for judging the effect of the subtabulation process.

APPENDIX A

SUPPLEMENTARY TABLES

$$\mathcal{P}_{G_m}(\beta) = 2\cos(m\pi/\beta)$$

m	β=2	2.5	3	3.5	4	4.5	5
1	0	.6180	1	1.2470	1.4142	1.5321	1.6180
2	-2	-1.6180	-1	-0.4450	.0000	.3473	.6180
3	0	-1.6180	-2	-1.8019	-1.4142	-1.0000	-0.6180
4	2	.6180	-1	-1.8019	-2.0000	-1.8794	-1.6180
5	0	2.0000	1	-0.4450	-1.4142	-1.8794	-2.0000
6	-2	.6180	2	1.2470	.0000	-1.0000	-1.6180
7	0	-1.6180	1	2.0000	1.4142	.3473	-0.6180
8	2	-1.6180	-1	1.2470	2.0000	1.5321	.6180
9	0	.6180	-2	-0.4450	1.4142	2.0000	1.6180
10	-2	2.0000	-1	-1.8019	.0000	1.5321	2.0000
11	0	.6180	1	-1.8019	-1.4142	.3473	1.6180
12	2	-1.6180	2	-0.4450	-2.0000	-1.0000	.6180
13	0	-1.6180	1	1.2470	-1.4142	-1.8794	-0.6180
14	-2	.6180	-1	2.0000	.0000	-1.8794	-1.6180
15	0	2.0000	-2	1.2470	1.4142	-1.0000	-2.0000
16	2	.6180	-1	-0.4450	2.0000	.3473	-1.6180
17	0	-1.6180	1	-1.8019	1.4142	1.5321	-0.6180
18	-2	-1.6180	2	-1.8019	.0000	2.0000	.6180
19	0	.6180	1	-0.4450	-1.4142	1.5321	1.6180
20	2	2.0000	-1	1.2470	-2.0000	.3473	2.0000
21	0	.6180	-2	2.0000	-1.4142	-1.0000	1.6180
22	-2	-1.6180	-1	1.2470	.0000	-1.8794	.6180
23	0	-1.6180	1	-0.4450	1.4142	-1.8794	-0.6180
24	2	.6180	2	-1.8019	2.0000	-1.0000	-1.6180
25	0	2.0000	1	-1.8019	1.4142	.3473	-2.0000
26	-2	.6180	-1	-0.4450	.0000	1.5321	-1.6180
27	0	-1.6180	-2	1.2470	-1.4142	2.0000	-0.6180
28	2	-1.6180	-1	2.0000	-2.0000	1.5321	.6180
29	0	.6180	1	1.2470	-1.4142	.3473	1.6180
30	-2	2.0000	2	-0.4450	.0000	-1.0000	2.0000

$$P_{G_m}(\beta) = 2\cos(m\pi/\beta)$$

m	β= 5.5	6	6.5	7	7.5	8
1	1.6825	1.7321	1.7709	1.8019	1.8271	1.8478
2	.8308	1.0000	1.1361	1.2470	1.3383	1.4142
3	-0.2846	.0000	.2411	.4450	.6180	.7654
4	-1.3097	-1.0000	-0.7092	-0.4450	-0.2091	.0000
5	-1.9190	-1.7321	-1.4970	-1.2470	-1.0000	-0.7654
6	-1.9190	-2.0000	-1.9419	-1.8019	-1.6180	-1.4142
7	-1.3097	-1.7321	-1.9419	-2.0000	-1.9563	-1.8478
8	-0.2846	-1.0000	-1.4970	-1.8019	-1.9563	-2.0000
9	.8308	.0000	-0.7092	-1.2470	-1.6180	-1.8478
10	1.6825	1.0000	.2411	-0.4450	-1.0000	-1.4142
11	2.0000	1.7321	1.1361	.4450	-0.2091	-0.7654
12	1.6825	2.0000	1.7709	1.2470	.6180	.0000
13	.8308	1.7321	2.0000	1.8019	1.3383	.7654
14	-0.2846	1.0000	1.7709	2.0000	1.8271	1.4142
15	-1.3097	.0000	1.1361	1.8019	2.0000	1.8478
16	-1.9190	-1.0000	.2411	1.2470	1.8271	2.0000
17	-1.9190	-1.7321	-0.7092	.4450	1.3383	1.8478
18	-1.3097	-2.0000	-1.4970	-0.4450	.6180	1.4142
19	-0.2846	-1.7321	-1.9419	-1.2470	-0.2091	.7654
20	.8308	-1.0000	-1.9419	-1.8019	-1.0000	.0000
21	1.6825	.0000	-1.4970	-2.0000	-1.6180	-0.7654
22	2.0000	1.0000	-0.7092	-1.8019	-1.9563	-1.4142
23	1.6825	1.7321	.2411	-1.2470	-1.9563	-1.8478
24	.8308	2.0000	1.1361	-0.4450	-1.6180	-2.0000
25	-0.2846	1.7321	1.7709	.4450	-1.0000	-1.8478
26	-1.3097	1.0000	2.0000	1.2470	-0.2091	-1.4142
27	-1.9190	.0000	1.7709	1.8019	.6180	-0.7654
28	-1.9190	-1.0000	1.1361	2.0000	1.3383	.0000
29	-1.3097	-1.7321	.2411	1.8019	1.8271	.7654
30	-0.2846	-2.0000	-0.7092	1.2470	2.0000	1.4142

$$P_{G_m}(\beta) = 2\cos(m\pi/\beta)$$

m	β = 9	10	11	12	13	14
1	1.8794	1.9021	1.9190	1.9319	1.9419	1.9499
2	1.5321	1.6180	1.6825	1.7321	1.7709	1.8019
3	1.0000	1.1756	1.3097	1.4142	1.4970	1.5637
4	.3473	.6180	.8308	1.0000	1.1361	1.2470
5	-0.3473	.0000	.2846	.5176	.7092	.8678
6	-1.0000	-0.6180	-0.2846	.0000	.2411	.4450
7	-1.5321	-1.1756	-0.8308	-0.5176	-0.2411	.0000
8	-1.8794	-1.6180	-1.3097	-1.0000	-0.7092	-0.4450
9	-2.0000	-1.9021	-1.6825	-1.4142	-1.1361	-0.8678
10	-1.8794	-2.0000	-1.9190	-1.7321	-1.4970	-1.2470
11	-1.5321	-1.9021	-2.0000	-1.9319	-1.7709	-1.5637
12	-1.0000	-1.6180	-1.9190	-2.0000	-1.9419	-1.8019
13	-0.3473	-1.1756	-1.6825	-1.9319	-2.0000	-1.9499
14	.3473	-0.6180	-1.3097	-1.7321	-1.9419	-2.0000
15	1.0000	.0000	-0.8308	-1.4142	-1.7709	-1.9499
16	1.5321	.6180	-0.2846	-1.0000	-1.4970	-1.8019
17	1.8794	1.1756	.2846	-0.5176	-1.1361	-1.5637
18	2.0000	1.6180	.8308	.0000	-0.7092	-1.2470
19	1.8794	1.9021	1.3097	.5176	-0.2411	-0.8678
20	1.5321	2.0000	1.6825	1.0000	.2411	-0.4450
21	1.0000	1.9021	1.9190	1.4142	.7092	.0000
22	.3473	1.6180	2.0000	1.7321	1.1361	.4450
23	-0.3473	1.1756	1.9190	1.9319	1.4970	.8678
24	-1.0000	.6180	1.6825	2.0000	1.7709	1.2470
25	-1.5321	.0000	1.3097	1.9319	1.9419	1.5637
26	-1.8794	-0.6180	.8308	1.7321	2.0000	1.8019
27	-2.0000	-1.1756	.2846	1.4142	1.9419	1.9499
28	-1.8794	-1.6180	-0.2846	1.0000	1.7709	2.0000
29	-1.5321	-1.9021	-0.8308	.5176	1.4970	1.9499
30	-1.0000	-2.0000	-1.3097	.0000	1.1361	1.8019

$$P_{[m]}(\beta) = (\sin(m\pi/\beta))/(\sin(\pi/\beta))$$

m	β=2	2.25	2.5	2.75	3	3.25	3.5	3.75
1	1	1.0000	1.0000	1.0000	1	1.0000	1.0000	1.0000
2	0	.3473	.6180	.8308	1	1.1361	1.2470	1.3383
3	-1	-0.8794	-0.6180	-0.3097	0	.2908	.5550	.7910
4	0	-0.6527	-1.0000	-1.0882	-1	-0.8058	-0.5550	-0.2798
5	1	.6527	.0000	-0.5944	-1	-1.2062	-1.2470	-1.1654
6	0	.8794	1.0000	.5944	0	-0.5647	-1.0000	-1.2798
7	-1	-0.3473	.6180	1.0882	1	.5647	.0000	-0.5473
8	0	-1.0000	-0.6180	.3097	1	1.2062	1.0000	.5473
9	1	.0000	-1.0000	-0.8308	0	.8058	1.2470	1.2798
10	0	1.0000	.0000	-1.0000	-1	.2908	.5550	1.1654

	β = 4	4.5	5	5.5	6	6.5
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.4142	1.5321	1.6180	1.6825	1.7321	1.7709
3	1.0000	1.3473	1.6180	1.8308	2.0000	2.1361
4	.0000	.5321	1.0000	1.3979	1.7321	2.0120
5	-1.0000	-0.5321	.0000	.5211	1.0000	1.4269
6	-1.4142	-1.3473	-1.0000	-0.5211	.0000	.5150
7	-1.0000	-1.5321	-1.6180	-1.3979	-1.0000	-0.5150
8	.0000	-1.0000	-1.6180	-1.8308	-1.7321	-1.4269
9	1.0000	.0000	-1.0000	-1.6825	-2.0000	-2.0120
10	1.4142	1.0000	.0000	-1.0000	-1.7321	-2.1361

	β = 7	8	9	10	12	14
1	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
2	1.8019	1.8478	1.8794	1.9021	1.9319	1.9499
3	2.2470	2.4142	2.5321	2.6180	2.7321	2.8020
4	2.2470	2.6131	2.8794	3.0777	3.3461	3.5135
5	1.8019	2.4142	2.8794	3.2360	3.7321	4.0489
6	1.0000	1.8478	2.5321	3.0777	3.8637	4.3813
7	.0000	1.0000	1.8794	2.6180	3.7321	4.4940
8	-1.0000	.0000	1.0000	1.9021	3.3461	4.3813
9	-1.8019	-1.0000	.0000	1.0000	2.7321	4.1489
10	-2.2470	-1.8478	-1.0000	.0000	1.9319	3.5135

$$P_{\delta 2m}(\beta) = (-4 \sin^2(\pi/\beta))^m$$

m	$\beta=2$	2.25	2.5	2.75	3	3.25
1	-4	-3.8794	-3.6179	-3.3097	-3	-2.7090
2	16	15.0500	13.0895	10.9543	9	7.3389
3	-64	-58.3855	-47.3572	-36.2558	-27	-19.8815
4	256	226.5028	171.3359	119.9968	81	53.8598
5	-1024	-878.7029	-619.8840	-397.1569	-243	-145.9087

	$\beta=3.5$	3.75	4	4.5	5	5.5	6
1	-2.4450	-2.2092	-2	-1.6526	-1.3819	-1.1692	-1
2	5.9783	4.8807	4	2.7310	1.9096	1.3670	1
3	-14.6172	-10.7825	-8	-4.5132	-2.6389	-1.5982	-1
4	35.7397	23.8209	16	7.4584	3.6467	1.8686	1
5	-87.3853	-52.6258	-32	-12.3257	-5.0394	-2.1847	-1

	$\beta=6.5$	7	8	9	10	12	14
1	-.8638	-.7530	-.5858	-.4680	-.3821	-.2680	-.1981
2	.7461	.5670	.3431	.2190	.1460	.0718	.0392
3	-.6445	-.4270	-.2010	-.1025	-.0558	-.0192	-.0078
4	.5567	.3215	.1178	.0480	.0213	.0052	.0015
5	-.4809	-.2421	-.0690	-.0225	-.0081	-.0014	-.0003

$P_{S\beta_r(a,(\beta)}$ m = 5

r	β=2	3	4	5	6	7	8	9	10	12	14
0	0	-.500	.707	0	-.866	-1.123	-.924	-.500	.000	.966	1.757
1	0	.100	.141	0	.173	.585	1.077	1.558	1.992	2.691	3.197
2	0	.063	.092	0	.013	.225	.720	1.356	1.993	3.031	3.636
3	0	.054	.052	0	.036	.031	.876	1.515	2.149	3.156	3.809
4	0	.037	.030	0	.053	.148	.631	1.369	2.150	3.391	4.023
5	0	.033	.022	0	.008	.170	.682	1.435	2.217	3.440	4.151

$P_{S\beta_r(b)(\beta)}$ m = 5

0	.000	-.250	.500	0	-.750	-1.012	-.854	-.470	.000	.933	1.713
1	.200	.200	.200	0	.200	.650	1.166	1.659	2.513	2.786	3.283
2	.200	.140	.120	0	.020	.266	.796	1.453	2.094	3.119	3.790
3	.136	.104	.072	0	.040	.338	.908	1.579	2.218	3.215	3.853
4	.136	.075	.040	0	.006	.165	.674	1.433	2.218	3.446	4.166
5	.109	.063	.030	0	.009	.184	.718	1.486	2.248	3.285	4.191

$P_{S\delta_r(a)(\beta)}$ m = 5

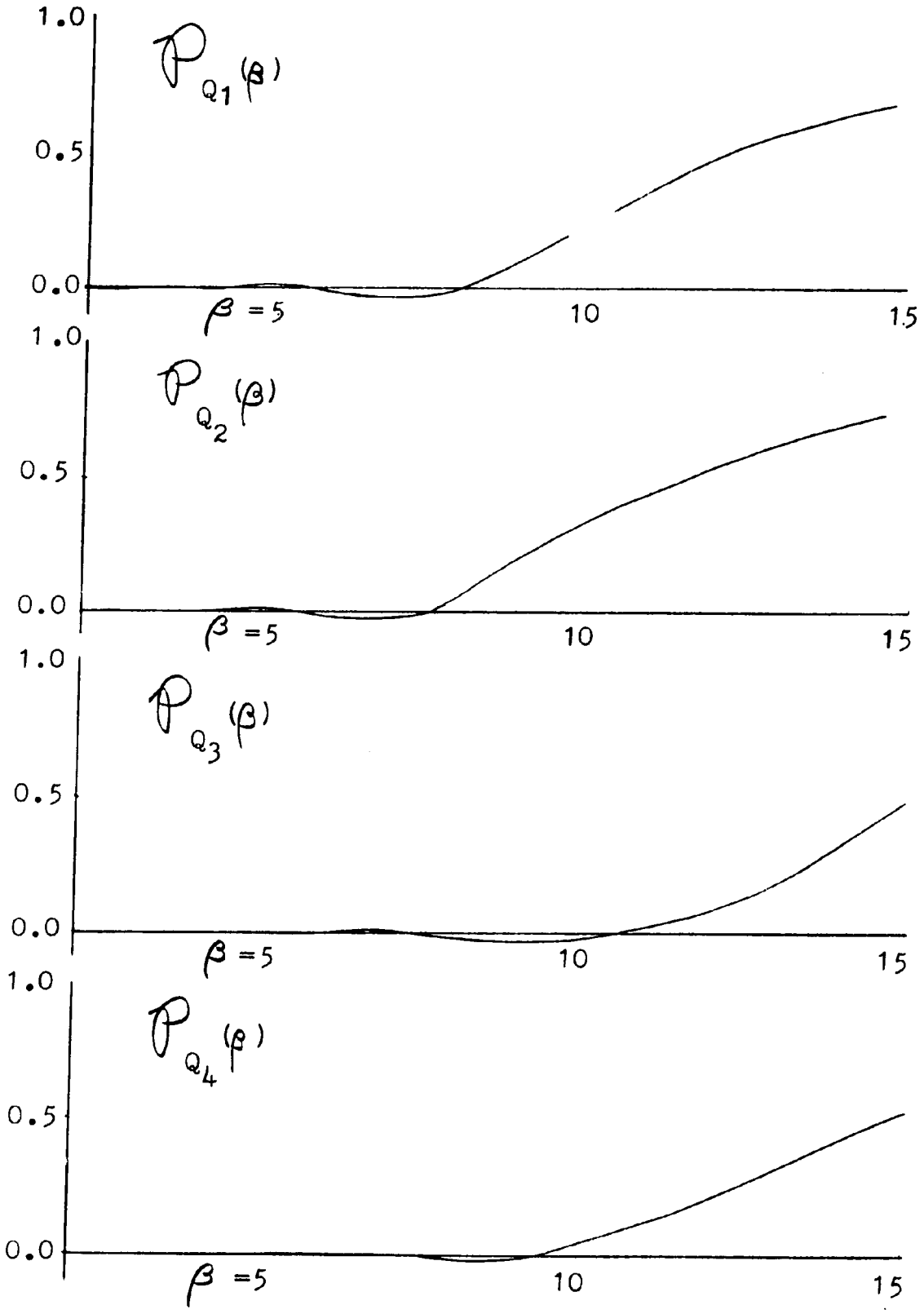
0	1.000	-1.000	-1.000	0	1.000	1.802	2.414	2.879	3.236	3.732	4.049
1	1.000	-.700	-.600	0	.100	.737	1.648	2.523	3.236	4.179	4.677
2	.520	-.400	-.280	0	.160	.762	1.552	2.397	2.908	3.751	4.233
3	.520	-.292	-.152	0	.028	.384	1.171	2.095	2.909	3.992	4.533
4	.392	-.227	-.107	0	.032	.388	1.144	2.030	2.813	3.877	4.187

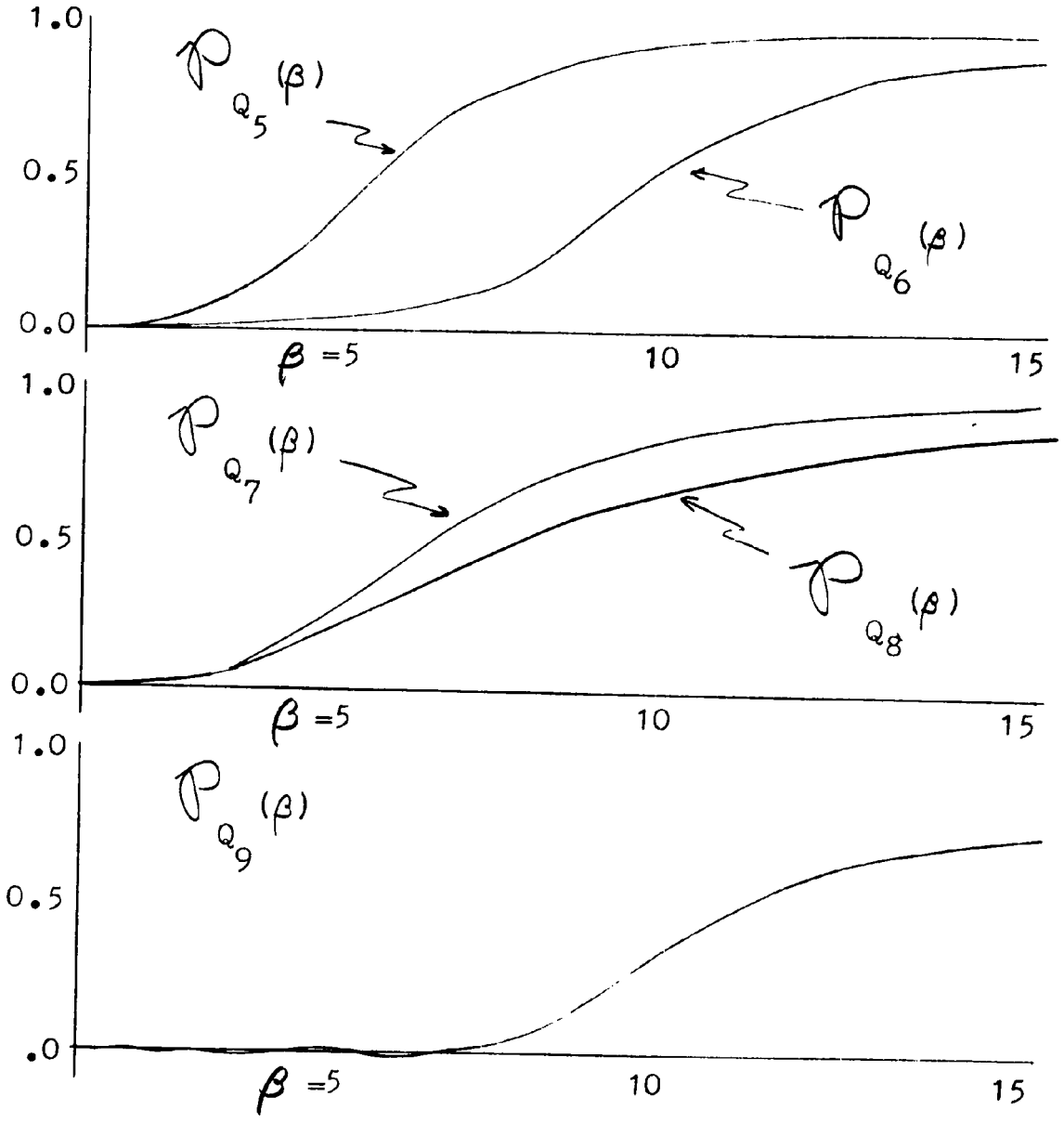
$P_{S\delta_r(b)(\beta)}$ m = 5

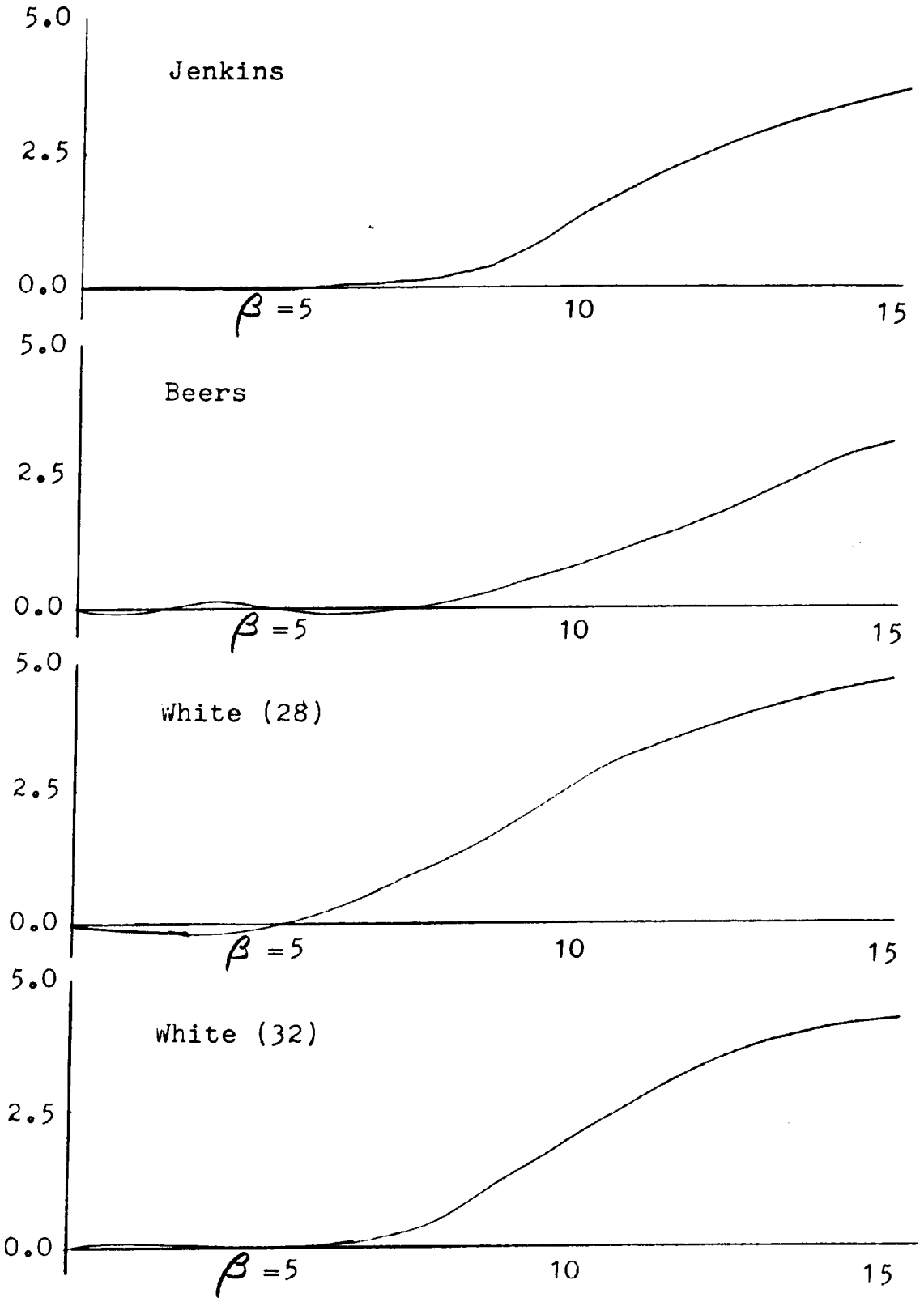
0	0	-.500	-.707	0	.866	1.623	2.230	2.706	3.078	3.605	3.947
1	0	-.275	-.495	0	.043	.587	1.458	2.336	3.078	4.084	4.627
2	0	-.208	-.205	0	.143	.707	1.472	2.210	2.829	3.691	4.193
3	0	-.141	-.118	0	.023	.345	1.099	2.009	2.829	3.940	4.389
4	0	-.123	-.083	0	.031	.396	1.216	2.204	3.104	4.345	4.969

APPENDIX B

GRAPHS OF PERIODOGRAMS




1.0
0.5
0.0
$P_{Q_5}(\beta)$
$P_{Q_6}(\beta)$
β =5
10
15
1.0
0.5
0.0
$P_{Q_7}(\beta)$
$P_{Q_8}(\beta)$
β =5
10
15
1.0
0.5
.0
$P_{Q_9}(\beta)$
β =5
10
15


5.0
Jenkins
2.5
0.0
β =5
10
15
5.0
Beers
2.5
0.0
β =5
10
15
5.0
White (28)
2.5
0.0
β =5
10
15
5.0
White (32)
2.5
0.0
β =5
10
15

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