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LACUNARY SERIES IN SEVERAL VARIABLES

by
John Dieterich Pesek, Jr.

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
1975

Doctoral Committee:

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To my Grandfather

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In writing this dissertation, we have accumulated mathematical debts to three people. First I thank my advisor Peter Duren for much help and encouragement. Next I am grateful to Harold Shapiro whose lectures on Harmonic Analysis reawakened my interest in Mathematics. Finally I thank Eric Bedford for many helpful conversations about Several Complex Variables.

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CHAPTER I

FOURIER ANALYSIS PRELIMINARIES

In this chapter we shall do two things. First we shall state the conventions and notations that we shall use in the second and third chapters. Second we shall define the Lipschitz classes and give the Boman-Shapiro characterization of these classes which we shall use in Chapter Two.

1.1. Conventions and Notations

T^n will be the n-dimensional torus. R^n will be n-dimensional Euclidean space. Z^n will be the integer lattice points of R^n.

$L^p(T^n)$ will be the space of p-summable functions on T^n. $C^k(T^n)$ will be the space of continuous functions on T^n with continuous partial derivatives of order k on T^n. By the identification $T^n \simeq R^n/(2\pi Z)^n$, we can (and usually will) consider functions on the n-torus T^n as functions on R^n which have a period of 2π in each of the n coordinate variables. Thus if $1 \leq p < \infty$, the norm of a function $f \in L^p(T^n)$ is denoted by $\|f\|_p$ and equals

$$\|f\|_p = \left(\frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} |f(x)|^p \, dx \right)^{1/p}$$

If $p = \infty$,

$$\|f\|_p = \operatorname*{ess\,sup}_{x \in R^n} |f(x)|$$

$\mathcal{S}(R^n)$ will denote the Schwartz class of rapidly decreasing C^∞ functions.

$C_0^\infty(R^n)$ will denote the class of C^∞ functions with compact support.

$M(R^n)$ shall denote the space of bounded measures on R^n. By a theorem of Riesz, we can identify $M(R^n)$ with the dual of $C_0(R^n)$, the space of continuous functions on R^n vanishing at ∞. If $\sigma \in M(R^n)$, we shall denote the norm of σ by $\|\sigma\|$. We can identify $L^1(R^n)$ with the members of $M(R^n)$ which are absolutely continuous with respect to Lebesgue measure.

If $f \in L^1(R^n)$, we shall denote its Fourier transform by $\hat{f}$ and define it by

$$\hat{f}(x) = \int_{R^n} f(t)\, e^{-ixt}\, dt$$

If $\sigma \in M(R^n)$, we shall denote its Fourier transform by $\hat{\sigma}$ and define it by

$$\hat{\sigma}(x) = \int_{R^n} e^{-ixt}\, d\sigma(t)$$

(Here $x = (x_1,\ldots,x_n)$, $t = (t_1,\ldots,t_n)$ and $xt = \sum_{j=1}^{n} x_j t_j$.)

If $\hat{f} \in L^1(R^n)$, the inverse Fourier transform of f is given by

$$f(x) = \frac{1}{(2\pi)^n} \int_{R^n} \hat{f}(t)\, e^{ixt}\, dt \tag{1.1}$$

We shall not need a more general inversion formula.

If $f \in L^1(T^n)$ and $m \in Z^n$, the Fourier coefficient of f at m is denoted by $\hat{f}(m)$ and given by

$$\hat{f}(m) = \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} f(t)\, e^{-imt}\, dt$$

(where $mt = \sum_{j=1}^{n} m_j t_j$).

The series $\sum \hat{f}(m)\, e^{imx}$ is called the Fourier series of f. The class $A(T^n)$ is the class of functions with absolutely convergent Fourier series. If $f \in A(T^n)$, then

$$f(x) = \sum \hat{f}(m)\, e^{imx} .$$

If $f \in L^1(R^n)$ and $a > 0$, the dilation of f by a is denoted by $f_{(a)}$ and defined by

$$f_{(a)}(x) = a^{-n} f(x/a) \tag{1.2}$$

$$\hat{f}_{(a)}(x) = \hat{f}(ax)$$

If $\sigma \in M(R^n)$, the dilation of σ by a is denoted by $\sigma_{(a)}$ and defined by

$$\sigma_{(a)}(\phi) = \int_{R^n} \phi(at)\, d\sigma(t) \quad \text{for } \phi \in C_0(R^n)$$

$$\hat{\sigma}_{(a)}(x) = \hat{\sigma}(ax)$$

If f and g are in $L^1(R^n)$, the convolution $f*g$ is defined by

$$f*g(x) = \int_{R^n} f(x-y)g(y)dy.$$

If σ and τ are in $M(R^n)$, the convolution $\sigma*\tau$ is defined by

$$\sigma*\tau(\phi) = \iint \phi(x+y)d\sigma(y)d\tau(x) \quad \text{for} \quad \phi \in C_0(R^n).$$

If $f \in L^p(R^n)$ and $\sigma \in M(R^n)$, the convolution $f*\sigma$ is in $L^p(T^n)$ and is defined by

$$f*\sigma(x) = \int_{R^n} f(x-y)d\sigma(y)$$

$$\|f*\sigma\|_p \leq \|f\|_p \|\sigma\|$$

If $f \in L^p(T^n)$ and $g \in L^1(R^n)$, the convolution $f*g$ is in $L^p(T^n)$ and is given by

$$f*g(x) = \int_{R^n} f(x-y)g(y)dy = \int_{R^n} f(y)g(x-y)dy \tag{1.3}$$

$$\|f*g\|_p \leq \|f\|_p \|g\|_1$$

If $f \in L^p(T^n)$ and $g \in L^1(R^n)$,

$$\widehat{f*g}(m) = \hat{f}(m)\hat{g}(m) \tag{1.4}$$

where $\hat{f}(m)$ is the m-th Fourier coefficient of f and $\hat{g}(m)$ is the Fourier transform of g evaluated at m.

For $x \in R^n$ and $m \in Z^n$ we define

$$|x| = \sqrt{\sum |x_j|^2} \quad \text{and} \quad |m| = \sqrt{\sum |m_j|^2} .$$

An exception to this rule of notation will occur in Chapter 3 when we discuss multi-orders.

Most of this material is available in Stein and Weiss [29], Chapters I and VII, with adjustments for the different conventions used. See also Shapiro [27], Chapter 9.

1.2. Lipschitz classes and the method of Boman and Shapiro

In Chapter 2 our results are proved by a method developed first by Shapiro and later in collaboration with Boman. (See Shapiro [25], [26], and [27], Chapter 9, and Boman and Shapiro [1]). Unfortunately the results we use have not appeared in print. In this section we shall give the Boman-Shapiro definition of the Lipschitz classes. Then we show how to apply the theory of Boman and Shapiro to prove that this definition is equivalent to two classical definitions of Lipschitz classes.

DEFINITION 1. Let $g \in L^1(R^n)$. We call g a Shapiro-Boman kernel if

(i) $\hat{g} \in C_0^\infty(R^n)$.

(ii) $\hat{g}(x) = 0$ in a neighborhood of the origin.

(iii) For each $v \in R^n \setminus \{0\}$, $\hat{g}(tv) \neq 0$ for some $t > 0$.

Now we define the Lipschitz classes.

DEFINITION 2. Let $\alpha > 0$, $f \in L^\infty(T^n)$. Let g be a Boman-Shapiro kernel. Then f is in the Lipschitz class of order α which we will denote by Λ_α if

$$\|f*g_{(b)}\|_\infty = O(b^\alpha) \quad .$$

Next we need a lemma.

LEMMA. The class Λ_α does not depend on the choice of the Boman-Shapiro kernel g.

This is an easy consequence of LEMMA 9.4.4.1, Shapiro [27], p. 221.

Next we define the L^p Lipschitz classes to be denoted by Λ_α^p for $1 \leq p < \infty$. (Λ_α could be written Λ_α^∞.) (In Stein [28], Chapter V, $\Lambda_\alpha^p = \Lambda_\alpha^{p,\infty}$.)

DEFINITION 3. Let g be a Shapiro-Boman kernel. $f \in \Lambda_\alpha^p$ if and only if $f \in L^p(T^n)$ and

$$\|f*g_{(b)}\|_p = O(b^\alpha) \tag{1.5}$$

LEMMA. The class Λ_α^p does not depend on the choice of Shapiro-Boman kernel g.

Again this is a consequence of Lemma 9.4.4.1, p. 221, Shapiro [27].

These definitions are quite different from the classical definitions of the Lipschitz classes. We will now state a result that asserts the equivalence of two definitions of the Lipschitz classes to the one we have just given.

THEOREM. The following are equivalent for $1 \leq p \leq \infty$, $\alpha > 0$.

A. $f \in \Lambda_\alpha^p$

B. Let r be an integer greater than α. For $1 \leq j \leq n$,

let e_j be the vector in R^n whose j-th coordinate is one and the rest zero. The condition is $f \in L^p(T^n)$ and

$$(1.6) \qquad \Big\| \sum_{j=0}^{r} (-1)^j \binom{r}{j} f(x+jte_v) \Big\|_p \leq C\,|t|^{\alpha}, \quad 1 \leq v \leq n$$

for $t > 0$.

C. Let k be the greatest integer less than α. Then the condition is: $f \in L^p(T^n)$ and f has partial derivatives of all orders less than or equal to k in $L^p(T^n)$ and if β has order k and if $\alpha-k < 1$, then

$$\|D^{\beta}f(x+t) - D^{\beta}f(x)\|_p \leq C\,|t|^{\alpha-k} .$$

If β has order k and $\alpha-k = 1$, then

$$\|D^{\beta}f(x+t) - 2D^{\beta}f(x) + D^{\beta}f(x-t)\|_p \leq C|t| .$$

Definition B uses finite differences.

Definition C is the most classical. This is especially evident if $\alpha < 1$. Note that if $p = \infty$, the derivatives are actually continuous. Otherwise we take derivatives in the sense of the theory of distributions.

Let us see how these might be shown to be equivalent. First B and C can be seen to be equivalent by considering the direct and converse approximation theorems in Timan [30], Sections 6.3 and 7.3. Since conditions B and C characterize classes with the same approximation properties by trigonometric polynomials, these must be the same classes. Now it will suffice to show conditions A and B are equivalent. To do this, we will use Theorem 9.4.4.5

and Lemma 9.4.4.1 of Shapiro [27], Chapter 9.

Let $\beta_j = \delta_o - \delta_{e_j}$ where e_j is defined in condition B. (δ_x is the measure which has a mass of one at x.) Let $\beta_{j,r} = \beta_j * \beta_j * \ldots * \beta_j$ (r times).

Then

$$\|f*\beta_{j,r(t)}\|_p = O(t^\alpha) \quad j=1\ldots n$$

is equivalent to (1.6).

$$\|f*\beta_{j,r(t)}\|_p = O(t^\alpha)$$

is equivalent to

$$\omega_{\beta_{j,r,p}}(f;a) = \sup_{0 < t \leq a} \|f*\beta_{j,r(t)}\|_p = O(a^\alpha)$$

$\|f*g_{(b)}\|_p = O(b^\alpha)$ is equivalent to

$$\omega_g(f;a) = \sup_{0 < b \leq a} \|f*g_{(b)}\|_p = O(a^\alpha)$$

$\hat{\beta}_{j,r}(x)$ has a zero of order r at the origin. We may now apply Theorem 9.4.4.5 of Shapiro [27], p. 226, with $\tau = \beta_{j,r}$ and $\sigma = g\,dx$. From this theorem it is not hard to see that

$\omega_g(f;a) = O(a^\alpha)$ implies

$$\omega_{\beta_{j,r}}(f;a) = O(a^\alpha) \quad \text{for} \quad j=1\ldots n$$

This shows that Condition A implies Condition B. (Strictly speaking (9.4.4.5) does not cover the case $p = \infty$. But this case is covered by Boman and Shapiro [1], Theorem 4.4, p. 106.)

To show that Condition B implies Condition A, we use LEMMA 9.4.4.1 in Shapiro [27], p. 221, and the modifications mentioned in text of Shapiro on page 226.

We shall end this section by defining what it means to be Lipschitz of order α at a point.

DEFINITION 4. Let $\alpha > 0$. Let $f \in L^1(T^n)$. Let k be the greatest integer less than α. Let $x_o \in T^n$. Then we say that f is <u>Lipschitz of order α at x_o</u> if there is a polynomial P of degree at most k and numbers $\delta > 0$, $B > 0$ such that

$$|f(x) - P(x-x_o)| \leq B|x-x_o|^{\alpha} \quad \text{for} \quad |x-x_o| < \delta.$$

CHAPTER II

LIPSCHITZ CLASSES, LACUNARY FOURIER SERIES, AND DECAY OF FOURIER COEFFICIENTS

In this chapter we shall consider the problem of how much the Fourier coefficients must decay and how lacunary a Fourier series must be in order for it to belong to a given Lipschitz class. In the first section we will consider the classes Λ_α. In the second section we will be concerned with the classes Λ_α^p. Previous results are due to Lorentz [18] and Kokilašvili [15].

2.1. The L^∞ case.

First we shall need two definitions.

DEFINITION 5. $F \subseteq Z^n$ is called a Hadamard set if it can be described as a sequence $F = \{\lambda_k\}_0^\infty$ such that there is a number $q > 1$ and

$$|\lambda_{k+1}| > q|\lambda_k| \quad \text{for} \quad k=0,1,2,\ldots$$

DEFINITION 6. Let $q > 1$. Then

$$V(q,r) = \{x \in R^n | r \leq |x| \leq qr\} \tag{2.1}$$

We are ready to state our first result.

THEOREM 1. Let $f \in L^1(T^n)$, $E \subseteq Z^n$, $\alpha > 0$, $0 \leq \tau \leq n$, and $q > 1$. Assume

(2.2) $$|\hat{f}(m)| \leq C|m|^{-(\alpha+\tau)}$$

Then supp $\hat{f} \subseteq E$ implies $f \in \Lambda_\alpha$ if and only if

(2.3) $$\#(E \cap V(q,r)) \leq C(q,E)r^\tau$$

(The symbol '#' stands for set cardinality.)

Thus our lacunarity condition concerns counting the number of elements where $\hat{f}(m) \neq 0$ in a spherical shell of fixed proportions but variable radius.

We will state the cases $\tau = 0$ and $\tau = n$ as corollaries.

COROLLARY 1. Let $f \in L^1(T^n)$ and $E \subseteq Z^n$. Assume

(2.4) $$|\hat{f}(m)| \leq C|m|^{-\alpha}$$

Then supp $\hat{f} \subseteq E$ implies that $f \in \Lambda_\alpha$ if and only if E is contained in the finite union of Hadamard sets.

For $n = 1$, the 'if' side of this corollary is due to Lorentz [18]. For $n = 2$, the 'if' side is due to Kokilašvili [15].

COROLLARY 2. Let $f \in L^1(T^n)$. Assume

(2.5) $$|\hat{f}(m)| \leq C|m|^{-(\alpha+n)}$$

Then $f \in \Lambda_\alpha$.

The case $n = 1$ is due to Lorentz [17].

Before we prove these results, we will give some examples of sets E to which they apply.

a. $E = \{2^m\}$ where $n = 1$ and $\tau = 0$.

b. $E = \{(2^m, 3^m)\}$ where $n = 2$ and $\tau = 0$.

c. $E = \{m^2\}$ where $n = 1$ and $\tau = 1/2$.

d. $E = \{(m, m^2)\}$ where $n = 2$ and $\tau = 1/2$.

e. $E = \{(m^2, k^2)\}$ where $n = 2$ and $\tau = 1$.

f. $E = \{m^3\}$ where $n = 1$ and $\tau = 1/3$.

g. $E = \{(2^k, 2^m)\}$ where $n = 2$ and $\tau = \varepsilon > 0$ but $\tau \neq 0$.

Note that the choice of the number q is not mentioned. In fact, the choice of q, once it is known to be larger than one is not important as the following lemma shows.

LEMMA 1. Let $p, q > 1$. If

$$\#(E \cap V(q,r)) \leq C(q,E) r^{\tau}, \tag{2.3}$$

then

$$\#(E \cap V(p,r)) \leq C(p,E) r^{\tau}$$

where $C(p,E)$ depends on p, q, and $C(q,E)$.

Proof. Since $q > 1$, there is a positive integer j such that $q^j > p$.

Thus

$$V(p,r) \subseteq \bigcup_{k=0}^{j-1} V(q, q^k r)$$

And so

$$E \cap V(p,r) \subseteq \bigcup_{k=0}^{j-1} (E \cap V(q, q^k r))$$

Thus

$$\#(E \cap V(p,r)) \leq \sum_{k=0}^{j-1} \#(E \cap V(q,q^k r))$$

$$\leq C(q,E) \sum_{k=0}^{j-1} (q^k r)^\tau \quad \text{by} \quad (2.3)$$

$$= (C(q,E) \sum_{k=0}^{j-1} q^{k\tau}) r^\tau$$

Hence we can choose $C(p,E) = C(q,E) \sum_{k=0}^{j-1} q^{k\tau}$.

Lemma 1 is proven.

Now we shall prove Theorem 1.

Proof. First we will assume supp $\hat{f} \subseteq E$ and (2.3) and then prove $f \in \Lambda_\alpha$. Then we shall show the converse by means of a counterexample.

We shall choose a Shapiro-Boman kernel g satisfying

(2.6) $$\text{supp } \hat{g} \subseteq \{x \mid 1 \leq |x| \leq 2\}$$

(2.7) $$\|\hat{g}\|_\infty = 1$$

(See Chapter 1, Definition 1 for the concept of Shapiro-Boman kernel.)

Note that it follows from (1.2), (2.3) and supp $\hat{f} \subseteq E$ that the Fourier series of f is absolutely convergent. Thus $f \in L^\infty$. By Definition 2 $f \in \Lambda_\alpha$ if $\|f*g_{(b)}\|_\infty = O(b^\alpha)$ for some Boman-Shapiro kernel. We shall prove it for the one mentioned above.

We estimate $\|f*g_{(b)}\|_\infty$

$$\|f*g_{(b)}(x)\|_\infty = \| \sum_{\frac{1}{b} \leq |m| \leq \frac{2}{b}} \hat{f}(m)\hat{g}(bm)e^{imx} \|_\infty$$

by (2.6) and (1.4).

By the triangle inequality and (2.7) this is less than or equal to

$$\sum_{1/b \le |m| \le 2/b} |\hat{f}(m)|$$

$$\le C \sum_{1/b \le |m| \le 2/b,\ m \in E} |m|^{-(\alpha+\tau)}$$

by (2.2) and supp $\hat{f} \subseteq E$

$$\le C \#(E \cap V(2,1/b))(1/b)^{-(\alpha+\tau)}$$

$$\le C\, C(2,E)(1/b)^{\tau}\,(1/b)^{-(\alpha+\tau)} \quad \text{by (2.3)}$$

$$= C\, C(2,E) b^{\alpha}$$

Therefore $\|f*g_{(b)}\|_\infty = O(b^\alpha)$ and $f \in \Lambda_\alpha$ as desired. Half of Theorem 1 is proven.

We proceed to prove the other half. We shall assume that E does not satisfy (2.3). We shall then construct a function f such that supp $\hat{f} \subseteq E$ and f satisfies (2.2) but $f \notin \Lambda_\alpha$. An obvious candidate is

$$f(x) = \sum_{m \in E} |m|^{-(\alpha+\tau)}\, e^{imx}$$

However, we cannot guarantee that this series is the Fourier series of an L^∞ function. To overcome this difficulty, we will choose a subset E' of E such that the series

$$\sum_{m \in E'} |m|^{-(\alpha+\tau)}\, e^{imx}$$

is absolutely convergent but the function defined by this series is not in Λ_α.

We define E' as follows

If $\#(E \cap V(2,2^k)) \leq 2^{k(\tau+\alpha/2)}$, let $E_k' = E \cap V(2,2^k)$

If $\#(E \cap V(2,2^k)) > 2^{k(\tau+\alpha/2)}$, choose any $[2^{k(\tau+\alpha/2)}]$ elements of $E \cap V(2,2^k)$ and let them constitute E_k'.

Let

$$E' = \bigcup_{k=0}^{\infty} E_k'$$

Let

$$f(x) = \sum_{m \in E'} |m|^{-(\alpha+\tau)} e^{imx} \tag{2.8}$$

$$\sum_{m \in E'} |m|^{-(\alpha+\tau)} = \sum_{k=0} \sum_{m \in E_k'} |m|^{-(\alpha+\tau)}$$

$$\leq \sum_{0}^{\infty} \frac{2^{k(\tau+\alpha/2)}}{2^{k(\alpha+\tau)}} = \sum_{0}^{\infty} 2^{-k\alpha/2} < \infty.$$

Therefore f has an absolutely convergent Fourier series. f satisfies (2.2) trivially. It remains to show $f \notin \Lambda_\alpha$. That is

$$\|f*g_{(b)}\|_\infty \neq O(b^\alpha) \quad \text{where}$$

g is some Shapiro-Boman kernel.

We choose a Shapiro-Boman kernel g with the following properties .

$$\text{supp } \hat{g} = \{x|\ 7/8 \leq |x| \leq 17/8\} \tag{2.9}$$

$$\|\hat{g}\|_\infty = 1 \tag{2.10}$$

$$\hat{g} \geq 0 \tag{2.11}$$

$$\hat{g}(x) = 1 \quad \text{if} \quad 1 \leq |x| \leq 2. \tag{2.12}$$

Next we estimate $\|f*g_{(b)}\|_\infty$.

$$\|f*g_{(b)}\|_\infty = \| \sum_{7/8b \le |m| \le 17/8b} \hat{f}(m)\hat{g}(bm)e^{imx} \|_\infty$$

by (2.9)

$$\ge \sum_{\substack{7/8b \le |m| \le 17/8b \\ m \in E'}} |m|^{-(\alpha+\tau)}\hat{g}(bm) \qquad \text{by (2.8) and (2.11)}$$

and setting $x = 0$.

$$\ge \sum_{\substack{1/b \le |m| \le 2/b \\ m \in E'}} |m|^{-(\alpha+\tau)} \qquad \text{by (2.10), (2.11), and (2.12)}$$

$$\ge \#(E' \cap V(2,1/b))(2/b)^{-(\alpha+\tau)}$$

$$= (1/2)^{\alpha\tau}(\#(E' \cap V(2,1/b))/(1/b)^{\tau})b^{\alpha}$$

We can show $\|f*g_{(b)}\|_\infty \neq O(b^\alpha)$ and be finished if we can show that

$$\overline{\lim_{b\to 0}} \ \#(E' \cap V(2,1/b))/(1/b)^{\tau} = \infty$$

By considering the subsequence $b = 1/2^k$, we see that it suffices to show

$$\overline{\lim_{k\to\infty}} \ \#(E' \cap V(2,2^k))/2^{k\tau} = \infty$$

Recalling that $E' \cap V(2,2^k) = E'_k$ and recalling the construction of the E'_k, we see that this equals

$$\lim_{k\to\infty} \min(\#E \cap V(2,2^k), [2^{k(\tau+\alpha/2)}])/2^{k\tau}$$

If $[2^{k(\tau+\alpha/2)}]$ is the minimum an infinite number of times, then this limit superior is infinity. If $[2^{k(\tau+\alpha/2)}]$ is the minimum a finite number of times, then this limit superior is equal to

$$\overline{\lim_{k\to\infty}} \ \#(E \cap V(2,2^k))/2^{k\tau}$$

It now suffices to show that this quantity is ∞.

If $r \geq 1$, then $2^k \leq r \leq 2^{k+1} \leq 2r \leq 2^{k+2}$ for some $k \geq 0$. Thus

$$V(2,r) \subseteq V(2,2^k) \cup V(2,2^{k+1})$$

and

$$E \cap V(2,r) \subseteq (E \cap V(2,2^k)) \cup (E \cap V(2,2^{k+1}))$$

$$\#(E \cap V(2,r))/r^\tau \leq \#(E \cap V(2,2^k))/r^\tau +$$

$$\#(E \cap V(2,2^{k+1}))/r^\tau$$

$$\leq \#(E \cap V(2,2^{k+1}))/2^{k\tau} + 2^\tau \#(E \cap V(2,2^{k+1}))/2^{(k+1)\tau}$$

This implies that

$$\overline{\lim_{r}} \ \#(E \cap V(2,r))/r^\tau \leq (1+2^\tau) \ \overline{\lim_{k}} \ \#(E \cap V(2,2^k))/2^{k\tau}$$

But $\overline{\lim_{r}} \ \#(E \cap V(2,r))/r^\tau = \infty$ since we have assumed that (2.3) fails. Thus

$$\overline{\lim_{k}} \ \#(E \cap V(2,2^k))/2^{k\tau} \geq \infty$$

Thus $f \notin \Lambda_\alpha$. The proof of Theorem 1 is complete.

In order to prove Corollary 1, we shall need the following geometric lemma.

LEMMA 2. Let $E \subseteq Z^n$ and $q > 1$. Then E is contained in the finite union of Hadamard sets if and only if

$$\#(E \cap V(q,r)) \leq C(q,E) \quad \text{for all} \quad r > 0. \tag{2.13}$$

Proof. By Lemma 1 it suffices to consider the case $q = 2$. First we assume (2.13). Without loss of generality we may assume $C(2,E)$ is an integer. We will construct $2C(2,E)$ Hadamard sequences, $H_1,\ldots,H_{2C(2,E)}$ whose union contains E.

First we construct $H_k = \{h_{k,j}\}_{j=0}^{\infty}$ for $k = 1,\ldots,C(2,E)$. If possible, choose

$$h_{k,j} \in (E \cap V(2,2^{2j})) \setminus \bigcup_{v<j} H_v .$$

If this is not possible, choose $h_{k,j} \in V(2,2^{2j})$.

Next we construct $H_k = \{h_{k,j}\}_{j=0}^{\infty}$ for $k = C(2,E)+1,\ldots,2C(2,E)$. If possible, choose

$$h_{k,j} \in (E \cap V(2,2^{2j+1})) \setminus \bigcup_{C(2,E)<v<k} H_v$$

If this is not possible, choose $h_{k,j} \in V(2,2^{2j+1})$. If $0 \in E$, let $h_{1,-1} = 0$.

Let us show that these sets H_k have the desired properties. First the H_k are Hadamard sets.

If $k \leq C(2,E)$, $h_{k,j} \in V(2,2^{2k})$ and $h_{k,j+1} \in V(2,2^{2k+2})$. From (2.1) we see that $|h_{k,j}| \leq 2^{2k+1}$ and $|h_{k,j+1}| \geq 2^{2k+2}$. Thus $|h_{k,j+1}| \geq 2|h_{k,j}|$. H_k satisfies Definition 5 and is therefore a Hadamard set. The case of H_k, when $C(2,E) < k \leq 2C(2,E)$,

is similar.

It remains to show

$$E \subseteq \bigcup_{k=1}^{2C(2,E)} H_k$$

If $m \in E$, either $m = 0$ or $m \in V(2,2^k)$ for some $k \geq 0$. The case $m = 0$ has been taken care of. By construction, the H_k will exhaust up to $C(2,E)$ elements of E in $V(2,2^k)$. But by (2.13) there are at most $C(2,E)$ elements of E in $V(2,2^k)$. This proves one half of Lemma 2.

To prove the other half, we assume that E is contained in the union of Hadamard sets H_k, $k=1,\ldots,L$. Then we shall show that (2.13) holds for E with some $q > 1$. By Lemma 1 this shall suffice.

Since the H_k are Hadamard sets, there is a number $q > 1$ such that $|h_{k,j+1}| > q|h_{k,j}|$ for $k = 1\ldots L$ and $j \geq 0$. We claim that

$$\#(H_k \cap V(q,r)) \leq 1 \quad \text{for } k = 1,\ldots,L \text{ and } r > 0.$$

If $h_{k,j} \in V(q,r)$, $|h_{k,j}| \geq r$. Then

$|h_{k,j+1}| > q|h_{k,j}| \geq qr$ and $q|h_{k,j-1}| < |h_{k,j}| \leq r$ or

$|h_{k,j-1}| < r/q$, but

$V(q,r) = \{x|\ r \leq |x| \leq qr\}$ by (2.1)

Thus $h_{k,j-1}$ and $h_{k,j+1}$ do not belong to $V(q,r)$ and all other elements of H_k are either larger or smaller than these.

Thus at most one element of H_k can belong to $V(q,r)$.

Since $E \subseteq \bigcup_{j=0}^{L} H_j$,

$$\#(E \cap V(q,r)) \leq \sum_{k=1}^{L} \#(H_k \cap V(q,r)) \leq L.$$

Thus (2.13) is valid with $C(q,E) = L$. The proof of Lemma 1 is complete.

Now we can prove Corollary 1.

Proof. Condition (2.4) is just condition (2.2) with $\tau = 0$. Condition (2.14) is just (2.3) with $\tau = 0$. By Theorem 1, $\operatorname{supp} \hat{f} \subseteq E$ implies $f \in \Lambda_\alpha$, if and only if (2.13) is valid. But by Lemma 2, (2.13) is valid if and only if E is contained in the finite union of Hadamard sets. This proves Corollary 1.

We complete the section by proving Corollary 2.

Proof. We apply Theorem 1. Let $\tau = n$ and $E = Z^n$. Then $\#(E \cap V(q,r)) \leq C(q,E)r^n$. Thus (2.5) implies $f \in \Lambda_\alpha$. We have now proven Theorem 1 and its corollaries.

2.2. The case $p \neq \infty$.

This section is concerned with extending the results of the previous section to the L^p Lipschitz classes Λ_α^p for $1 < p < \infty$. We have had only partial success. We have an 'if and only if' theorem for the case $p = 2$ and a sufficient condition when $2 < p < \infty$. We consider the case $p = 2$ first.

THEOREM 2. Let $f \in L^2(T^n)$, $\alpha > 0$, $0 \leq \tau \leq n$, $q > 1$ and $E \subseteq Z^n$. Assume that

(2.14) $$|\hat{f}(m)| \leq C|m|^{-(\alpha+\tau/2)}$$

Then supp $\hat{f} \subseteq E$ implies $f \in \Lambda^2_\alpha$ if and only if

(2.3) $$\#(E \cap V(q,r)) \leq C(q,E)r^\tau$$

Proof. First we assume that supp $\hat{f} \subseteq E$ and (2.3). Then we show that $f \in \Lambda^2_\alpha$.

Choose a Boman-Shapiro kernel g meeting conditions (2.6) and (2.7). By Definition 3 in Chapter 1, it shall suffice to prove $\|f*g_{(b)}\|_2 = O(b^\alpha)$.

We estimate $\|f*g_{(b)}\|_2$

$$\|f*g_{(b)}\|_2 = \Big\| \sum_{1/b \leq |m| \leq 2/b} \hat{f}(m)\hat{g}(bm)e^{imx} \Big\|_2$$

$$= \left(\sum_{1/b \leq |m| \leq 2/b} |\hat{f}(m)|^2 \, |\hat{g}(bm)|^2 \right)^{1/2}$$

by Plancherel's theorem.

$$\leq \left(C \sum_{\substack{1/b \leq |m| \leq 2/b \\ m \in E}} |m|^{-(\alpha+\tau/2)2} \right)^{1/2}$$

by (2.14), supp $\hat{f} \subseteq E$, and (2.7).

$$\leq C^{1/2} \left(\#(E \cap V(2,1/b))b^{2\alpha+\tau} \right)^{1/2}$$

$$\leq C^{1/2} \left(C(2,E)(1/b)^\tau b^{2\alpha+\tau} \right)^{1/2}$$

$$= C^{1/2} \, C(2,E)^{1/2} b^\alpha$$

Thus $\|f*g_{(b)}\|_2 = O(b^\alpha)$ as required.

We shall prove the converse in a manner similar to the way we proved it in Theorem 1. We assume that (2.3) fails for E. We shall find an L^2 function f satisfying (2.14) such that supp $\hat{f} \subseteq E$

but $f \notin \Lambda^2_\alpha$.

We construct a subset E' of E.

If $\#(E \cap V(2,2^k)) \leq 2^{k(\tau+\alpha/2)}$, let $E'_k = E \cap V(2,2^k)$.

If $\#(E \cap V(2,2^k)) > 2^{k(\tau+\alpha/2)}$, choose any $[2^{k(\tau+\alpha/2)}]$ elements of $E \cap V(2,2^k)$ and let them constitute E'_k. Let

$$E' = \bigcup_{k=0}^{\infty} E'_k$$

Let

$$f(x) = \sum_{m \in E'} (1/|m|)^{\alpha+\tau/2} e^{imx} \tag{2.15}$$

$$\sum_{m \in E'} (|m|^{-(\alpha+\tau/2)})^2 = \sum_{k=0}^{\infty} \sum_{m \in E'_k} (1/|m|)^{2\alpha+\tau}$$

$$\leq \sum_{k=0}^{\infty} \frac{\#E'_k}{2^{k(2\alpha+\tau)}} \qquad \text{since} \quad E'_k \subseteq V(2,2^k)$$

$$\leq \sum \frac{2^{k(\tau+\alpha/2)}}{2^{(2\alpha+\tau)}} = \sum 2^{-3k\alpha/2} < \infty$$

So $f \in L^2(T^n)$.

It remains to show $f \notin \Lambda^2_\alpha$. To do this, we must show $\|f*g_{(b)}\|_2 \neq O(b^\alpha)$ where g is any Shapiro-Boman kernel. Let g be a Shapiro-Boman kernel having the properties (2.9) thru (2.12). We estimate $\|f*g_{(b)}\|_2$.

$$\|f*g_{(b)}\|_2 = \Big\| \sum_{7/8b \leq |m| \leq 17/8b} \hat{f}(m)\hat{g}(bm)e^{imx} \Big\|_2$$

$$= \left(\sum_{7/8b \leq |m| \leq 17/8b} |\hat{f}(m)|^2 \, |\hat{g}(bm)|^2 \right)^{1/2}$$

$$\geq \left(\sum_{1/b \leq |m| \leq 2/b} |\hat{f}(m)|^2 \, |\hat{g}(bm)|^2 \right)^{1/2}$$

$$= \left(\sum_{\substack{1/b \leq |m| \leq 2/b \\ m \in E'}} (1/|m|)^{2\alpha+\tau} \right)^{1/2} \quad \text{by (2.11) and (2.17)}$$

$$\geq \left(\#(E' \cap V(2,1/b)) b^{2\alpha+\tau} \right)^{1/2}$$

$$= \left(\#(E' \cap V(2,1/b))/(1/b)^{\tau} \right)^{1/2} b^{\alpha}$$

It remains to show that

$$\overline{\lim_{b \to 0}} \left(\#(E' \cap V(2,1/b))/(1/b)^{\tau} \right)^{1/2} = \infty.$$

But this certainly follows from

$$\overline{\lim_{b \to 0}} \; \#(E' \cap V(2,1/b))/(1/b)^{\tau} = \infty.$$

And now the proof can be concluded in exactly the same manner as the proof of Theorem 1. We note that the set E' is the same in each proof. Theorem 2 has been proven.

We now state Theorem 3.

THEOREM 3. Let $f \in L^p(T^n)$, $\alpha > 0$, $0 \leq \tau \leq n$, $q > 1$ and $E \subseteq Z^n$. Assume

$$|\hat{f}(m)| \leq C \, |m|^{-(\alpha+(1-1/p)\tau)} \tag{2.16}$$

Then if $2 < p < \infty$, supp $\hat{f} \subseteq E$, and

$$\#(E \cap V(q,r)) \leq C(q,E) r^{\tau} \tag{2.3}$$

imply that $f \in \Lambda^p_\alpha$.

Proof. Choose a Shapiro-Boman kernel g such that (2.6) and (2.7) are valid. Then by Definition 3, $f \in \Lambda_\alpha^p$ if and only if $\|f*g_{(b)}\|_p = O(b^\alpha)$.

We shall estimate $\|f*g_{(b)}\|_p$ by estimating $\|f*g_{(b)}\|_\infty$ and $\|f*g_{(b)}\|_2$ and then interpolating.

$$\|f*g_{(b)}\|_\infty = \Big\| \sum_{1/b \le |m| \le 2/b} \hat{f}(m)\hat{g}(bm)e^{imx} \Big\|_\infty$$

$$\le \sum_{1/b \le |m| \le 2/b} |\hat{f}(m)| \qquad \text{by (2.7)}$$

$$\le C \sum_{\substack{1/b \le |m| \le 2/b \\ m \in E}} |m|^{-(\alpha+(1-1/p)\tau)} \qquad \text{by (2.16)}$$

$$\le C \ \#(E \cap V(2,1/b))b^{\alpha+(1-1/p)\tau}$$

$$\le C \ C(2,E)(1/b)^\tau b^{\alpha+(1-1/p)\tau} \qquad \text{by (2.3)}$$

So we have

$$\|f*g_{(b)}\|_\infty \le C \ C(2,E)b^{\alpha-\tau/p} \tag{2.17}$$

Next we estimate

$$\|f*g_{(b)}\|_2 = \Big\| \sum_{1/b \le |m| \le 2/b} \hat{f}(m)\hat{g}(bm)e^{imx} \Big\|_2$$

$$= \left(\sum_{1/b \le |m| \le 2/b} |\hat{f}(m)|^2 \ |\hat{g}(bm)|^2 \right)^{1/2}$$

$$\le \left(C \sum_{\substack{1/b \le |m| \le 2/b \\ m \in E}} \left(\frac{1}{|m|}\right)^{(\alpha+(1-1/p)2} \right)^{1/2}$$

$$\le \left(C \ \#(E \cap V(2,1/b))b^{(\alpha+(1-1/p)\tau)2} \right)^{1/2}$$

$$\leq C^{1/2}\, C(2,E)^{1/2} \left((1/b)^{\tau} b^{(\alpha+(1-1/p)\tau)2}\right)^{1/2}$$

$$= C^{1/2}\, C(2,E)^{1/2} b^{\alpha+(1/2-1/p)}$$

So we have

$$\|f*g_{(b)}\|_2 \leq C^{1/2}\, C(2,E)^{1/2} b^{\alpha+(1/p-1/2)\tau} \tag{2.18}$$

The L^p norms are logarithmically convex. Thus we may interpolate (2.17) and (2.18) to get an estimate for $\|f*g_{(b)}\|_p$

$$\frac{1}{p} = \frac{1}{\infty}\left(1 - \frac{2}{p}\right) + \frac{1}{2}\left(\frac{2}{p}\right)$$

So

$$\|f*g_{(b)}\|_p \leq \|f*g_{(b)}\|_\infty^{1-2/p}\, \|f*g_{(b)}\|_2^{2/p}$$

By (2.17) and (2.18) this is no more than

$$\left(C\cdot C(2,E) b^{\alpha-\tau/p}\right)^{1-2/p} \left(C^{1/2}\, C(2,E)^{1/2} b^{\alpha+(1/2-1/p)\tau}\right)^{2/p}$$

$$= C^{1-1/p}\, C(2,E)^{1-1/p}\, b^{\alpha}.$$

Thus Theorem 3 has been proven.

It is very easy to conjecture a theorem containing Theorems 1,2 and 3 as special cases, covering $1 < p \leq \infty$. We have not succeeded in proving it. A point of interest is that the excess of decay over $|m|^{-\alpha}$ is less than it is for the case $p = \infty$.

CHAPTER III

ONE POINT REGULARITY PROPERTIES OF MULTIPLE FOURIER SERIES WITH GAPS

In this chapter we will extend results of Hsieh Xie-Fan [11] Izumi, Izumi and Kahane [13], and G. Freud [5] to several variables. The main result in one variable is that if a summable function has a lacunary Fourier series of a certain type and satisfies a Lipschitz condition of order α at some point then it is Lipschitz of order α everywhere. We extend this and other results to several variables and get some refinements in one variable. The work is done in Section 1. We give a coefficient estimate that results from lacunarity and one point of regularity. In Section 2 we use this result and Theorem 1 of Chapter 2 to give various one point regularity results. An application is made to certain partial differential equations. In Section 3 we extend a counterexample of Izumi, Izumi and Kahane to several variables.

3.1. The coefficient estimate.

The object of this section is to prove the following theorem.

THEOREM 4. Let $f \in L^1(T^n)$, $\alpha > 0$, $E \subseteq Z^n$, $\gamma > 0$, $0 < \delta < 1$, $B > 0$, and $0 < \theta \leq 1$. Assume

(3.1) $\quad \operatorname{supp} \hat{f} \subseteq E$

(3.2) $\quad$ For $m \in E$, $\operatorname{dist}(E, E \setminus \{m\}) > \gamma |m|^{\theta}$

(3.3) There is a point x_o and polynomial P of degree at most k (where k is the greatest integer less than α) such that

$$|f(x) - P(x-x_o)| \leq B|x-x_o|^{\alpha} \quad \text{if} \quad |x-x_o| < \delta$$

Then

(3.4) $|\hat{f}(m)| \leq C\,|m|^{-\theta\alpha}$ where C depends on α, E, γ, B, θ and f but not on m.

dist $(m, E \setminus \{m\})$ is the distance from m to the other points in E. (3.1) and (3.2) give the lacunarity condition. The non-vanishing coefficients are separated by a distance proportional to a power of their distance from the origin. (3.3) is a Lipschitz condition of order α at the point x_o. In fact, it is the one given in Definition 4 of Chapter 1.

The proof depends on Lemmas 3 and 4 to be stated below. We shall state these lemmas and use them to prove Theorem 4. Then we will prove the lemmas.

LEMMA 3. Let $\{a_\beta\}_{0 \leq |\beta| \leq k}$ be complex numbers indexed by the multi-orders of total order at most k. There exists a trigonometric polynomial T of degree at most nk (where n is the dimension) such that

$$D^{\beta}T(0) = a_\beta \qquad 0 \leq |\beta| \leq k$$

(Here $|\beta|$ is not the Euclidean norm but the total order of β. This equals $\sum_1^n \beta_j$ if $\beta = (\beta_1,\ldots,\beta_n)$. This is the only time we will make this exception in Chapters 1, 2 and 3. From Chapter 4 onwards it will be the rule.)

LEMMA 4. Let $f \in L^1(T^n)$, $\alpha > 0$, $B > 0$, $0 < \delta < 1$, $N > 0$, and $m \in Z^n$. Assume

(3.5) If $p \in Z^n$ and $0 < |p-m| < N$, then $\hat{f}(p) = 0$

(3.6) $|f(x)| \leq B|x|^{\alpha}$ for $|x| < \delta$

(3.7) $N > \delta^{-1/2}$

Then

$$|\hat{f}(m)| \leq C_1 N^{-\alpha} \quad \text{where}$$

C_1 depends on n, f, B, α and δ but not on N and m.

Proof of Theorem 4. Without loss of generality, we assume $x_o = 0$. By Lemma 3 there is a trigonometric polynomial $T(x)$ with total degree at most nk such that

(3.9) $$D^{\beta}T(0) = D^{\beta}P(0) \quad \text{for } 0 \leq |\beta| \leq k$$

So

(3.10) $$|f(x) - T(x)| \leq |f(x) - P(x)| + |P(x) - T(x)|$$

$$\leq B|x|^{\alpha} + A|x|^{k+1} \leq B'|x|^{\alpha} \quad \text{if } |x| < \delta.$$

We estimate the first term by (3.3) and the second by (3.9) and Taylor's theorem. Recall that k is as in (3.3).

Define the function $F(x) = f(x) - T(x)$ supp $\hat{F} \subseteq E'$ where $E \subseteq E'$ and $\#(E' \setminus E) < \infty$.

Thus E' satisfies (3.2) with γ replaced by some $\gamma' > 0$. We have

For $m \in E'$, $\operatorname{dist}(m, E' \setminus \{m\}) > \gamma' |m|^{\theta}$

By (3.10)

$$|F(x)| \leq B'|x|^{\alpha} \qquad \text{for} \quad |x| < \delta$$

If we let $m \in E'$, F be f, B' be B, δ be δ, α be α and $N = \gamma'|m|^{\theta}$, Lemma 4 applies when $\gamma'|m|^{\theta} > \delta^{-1/2}$. We obtain

$$|\hat{f}(m)| < C_1(\gamma'|m|^{\theta})^{-\alpha} = C_1(\gamma')^{-\alpha}\,|m|^{-\theta\alpha}$$

for all but a finite number of $m \in E'$ by (3.8). Since $\hat{F}(m) = \hat{f}(m)$ for all but a finite number of $m \in E$. We have

$$|\hat{f}(m)| \leq C\,|m|^{-\alpha\theta} \quad \text{for} \quad m \in E$$

where the constant C has been adjusted to cover the finite number of exceptions. If $m \notin E$, the inequality is still true since $\hat{f}(m) = 0$. Thus we have (3.4) which proves Theorem 4.

Now we must prove the lemmas.

Proof of Lemma 3. First we restrict our attention to the case $n = 1$. We must show there is

$$T(x) = \sum_{j=0}^{k} b_j e^{ijx} \qquad \text{such that}$$

$$D^{v}T(0) = a_v \qquad \text{for} \quad 0 \leq v \leq k.$$

We see by computation that this is equivalent to

$$\sum_{j=0}^{k} (ij)^{v} b_j = a_v \qquad \text{for} \quad 0 \leq v \leq n.$$

The matrix of this linear system is a Vandermonde matrix. The values of the determinants of Vandermonde matrices are well known: In this case, the determinant is not zero. Thus the

matrix is invertible, and the linear system has a solution. The lemma has been proven for $n = 1$.

From this we see that there are trigonometric polynomials $T_{kv}(x)$ where $0 \leq v \leq k$ and the degree of T_{kv} is at most k such that

$$D^j T_{kv}(0) = \delta_{jv} \quad \text{for} \quad 0 \leq j \leq k \quad \text{and} \quad 0 \leq v \leq k \tag{3.11}$$

(δ_{jv} is the Kronecker delta .)

Now we shall consider the general case.

Let $$T(x) = \sum_{0 \leq |\beta| \leq k} a_\beta \prod_{j=1}^{n} T_{k\beta_j}(x_j)$$

$T(x)$ has degree at most nk.

$$D^\gamma T(0) = \sum_{0 \leq |\beta| \leq k} a_\beta D^\gamma \prod_{j=1}^{n} T_{k,\beta_j}(x_j)$$

$$= \sum_{0 \leq |\beta| \leq k} a_\beta \prod_{j=1}^{n} D^{\gamma_j} T_{k,\beta_j}(x_j)$$

$$= \sum_{0 \leq |\beta| \leq k} a_\beta \prod_{j=1}^{n} \delta_{\gamma_j \beta_j} \qquad \text{by (3.11)}$$

$$= a_\gamma \quad \text{if} \quad 0 \leq |\gamma| \leq k.$$

So T is a trigonometric polynomial with the required properties. Lemma 3 has been proven.

In order to prove Lemma 4, we shall need a function g with certain properties. The next lemma asserts the existence of g.

LEMMA 5. There is a function $g \in L^1(R^n)$ with the following properties.

(3.12) $\int_{R^n} g = 1$

(3.13) $g \geq 0$

(3.14) g is even

(3.15) $\hat{g} \in C_0^\infty(R^n)$

(3.16) $\operatorname{supp} \hat{g} \subseteq \{x| \ |x| \leq 1\}$

(3.17) $g \in \mathcal{S}$

Proof. Let $K \in C_0^\infty(R^n)$ with $K \geq 0$, $\operatorname{supp} K = \{x| \ |x| \leq 1/2\}$ and K even. Let

$$\hat{g}(x) = K*K(x)/K*K(0) \qquad (K \neq 0)$$

Then $\hat{g}$ is even since the convolution of two even functions is even.

$\hat{g} \in C_0^\infty(R^n)$ and $\operatorname{supp} \hat{g} \subseteq \{x| \ |x| \leq 1\}$ since the support of the convolution is contained in the vector sum of the supports of the factors.

Finally $\hat{g}(0) = K*K(0)/K*K(0) = 1.$

Let $g(x) = \frac{1}{(2\pi)^n} \int \hat{g}(t) e^{ixt} dt.$

Then $\hat{g}$ is the Fourier transform of g. (See (1.5).) We have established (3.15) and (3.16). (3.17) follows from (3.15) since it shows that g is in the Schwartz class. Since $\int g = \hat{g}(0) = 1$, we have (3.12). Since Fourier and inverse Fourier transforms of even functions are even, we have (3.14), g is even. It remains to prove (3.13).

$$g(x) = \frac{1}{(2\pi)^n} \int \frac{K*K(t)}{K*K(0)} e^{itx} dt$$

$$= \frac{1}{(2\pi)^n} \frac{1}{K*K(0)} (\hat{K}(-x))^2$$

$K*K(0) > 0$ since $K \geq 0$ and not equal to 0.

$\hat{K}$ is real valued since K is even and real valued. So $g \geq 0$. Lemma 5 has been proven.

We give the proof of Lemma 4.

Proof. We claim that

$$\hat{f}(m) = \int_{R^n} f(x) e^{-imx} N^n g(Nx) dx \tag{3.18}$$

Let $f \in L^1(T^n)$, $\phi \in \mathcal{A}$. The v-th Fourier coefficient of $f*\phi$ is $\hat{f}(v)\hat{\phi}(v)$. (See (1.4).) Since ϕ is rapidly decreasing $f*\phi$ has an absolutely convergent Fourier series.

Thus $$f*\phi(0) = \sum \hat{f}(v)\hat{\phi}(v)$$

$$= \int f(x)\phi(-x)dx$$

(See (1.3).)

Since $g \in \mathcal{A}$, $N^n g(Nx) \in \mathcal{A}$.

So

$$\int f(x) e^{-imx} N^n g(Nx) dx$$

$$= \int f(x) e^{-imx} N^n g(-Nx) dx \quad \text{by (3.14)}$$

$$= \sum \hat{f}(v+m)\hat{g}(v/N) \quad \text{(See (1.2).)}$$

$$= \sum_{|k| < N} \hat{f}(v+m)\hat{g}(v/N) \quad \text{by (3.15)}$$

$$= \hat{f}(m)\hat{g}(0) \quad \text{by (3.6)}$$

$$= \hat{f}(m) \quad \text{by (3.12)}$$

We have proven (3.18).

By (3.18) and (3.13)

$$|\hat{f}(m)| \leq \int_{R^n} |f(x)| N^n g(Nx)\,dx$$

(This method if estimating $\hat{f}(m)$ is originally due to Noble [20].)

$$= \left(\int_{|x| \leq N^{-1}} + \int_{N^{-1} \leq |x| \leq N^{-\frac{1}{2}}} + \int_{N^{-\frac{1}{2}} \leq |x| \leq 1} + \int_{|x| \leq 1} |f(x)| N^n g(Nx)\ dx \right)$$

$= I + II + III + IV$ respectively.

We will prove Lemma 4 by getting a satisfactory estimate on each of the four pieces, I, II, III and IV.

$$I = \int_{|x| \leq N^{-1}} f(x)\ N^n g(Nx)\,dx$$

$$\leq B \int_{|x| \leq N^{-1}} |x|^{\alpha}\ N^n g(Nx)\,dx \qquad \text{by (3.6) and (3.7)}$$

$$\leq BN^{-\alpha} \int_{|x| \leq N^{-1}} N^n g(Nx)\,dx$$

$$\leq BN^{-\alpha} \int_{R^n} N^n g(Nx)\,dx = BN^{-\alpha} \qquad \text{by (3.12)}$$

This takes care of I.

$$II = \int_{N^{-1} \leq |x| \leq N^{-\frac{1}{2}}} |f(x)| N^n g(Nx)\,dx$$

$$\leq B \int_{N^{-1} \leq |x| \leq N^{-\frac{1}{2}}} |x|^{\alpha} N^n g(Nx)\,dx \qquad \text{by (3.6) and (3.7)}$$

By (3.17), $g(x) \leq c_1|x|^{-(n+\alpha+1)}$

Thus

$$II \leq c_1 B \int_{N^{-1} \leq |x| \leq N^{-\frac{1}{2}}} |x|^{\alpha} N^n (N|x|)^{-(n+\alpha+1)} dx$$

$$= c_1 B N^{-\alpha-1} \int_{N^{-1} \leq |x| \leq N^{-\frac{1}{2}}} |x|^{-(n+1)} dx$$

By changing to spherical coordinates this equals

$$c_1 B \omega_{n-1} N^{-\alpha-1} \int_{N^{-1}}^{N^{-\frac{1}{2}}} r^{-2} dr$$

where ω_{n-1} is the area of the $n-1$ sphere.

$$= c_1 B \omega_{n-1} N^{-\alpha-1} (N - N^{\frac{1}{2}})$$

$$= c_1 B \omega_{n-1} N^{-\alpha}$$

This handles II.

$$III = \int_{N^{-\frac{1}{2}} \leq |x| \leq 1} |f(x)| N^n g(nx) dx$$

By (3.17) $g(x) \leq c_2|x|^{-2(n+\alpha)}$

Thus

$$III \leq c_2 \int_{N^{-\frac{1}{2}} \leq |x| \leq 1} |f(x)| N^n (N|x|)^{-2(n+\alpha)} dx$$

$$\leq c_2 \int_{N^{-\frac{1}{2}} \leq |x| \leq 1} |f(x)| N^{-n} (NN^{-\frac{1}{2}})^{-2(n+\alpha)} dx$$

$$= c_2 N^{-\alpha} \int_{N^{-\frac{1}{2}} \leq |x| \leq 1} |f(x)| dx$$

$$\leq c_2 \|f\|_1 N^{-\alpha}$$

This handles III.

Before we estimate IV, let us note that there is a constant c_3 such that

$$\int_{|x|\leq R} |f(x)|\,dx \leq c_3 R^n \tag{3.19}$$

since $f \in L^1(T^n)$

There is a constant c_4 such that

$$g(x) \leq c_4 |x|^{-(n+\alpha)} \qquad \text{by (3.17)}$$

Thus

$$IV \leq c_4 \int_{|x|>1} |f(x)| N^n (N|x|)^{-(n+\alpha)}\,dx$$

$$= c_4 N^{-\alpha} \int_{|x|>1} |f(x)|\ |x|^{-(n+\alpha)}\,dx$$

It will suffice to show that

$$\int_{|x|>1} |f(x)|\ |x|^{-(n+\alpha)}\,dx < \infty$$

$$\int_{|x|>1} |f(x)|\ |x|^{-(n+\alpha)}\,dx = \sum_{j=0}^{\infty} \int_{2^j \leq |x| \leq 2^{j+1}} |f(x)|\ |x|^{-(n+\alpha)}\,dx$$

$$\leq \sum_{j=0}^{\infty} \int_{2^j \leq |x| \leq 2^{j+1}} |f(x)|\ 2^{-j(n+\alpha)}\,dx$$

$$\leq c_3 \sum_{0}^{\infty} 2^{(j+1)n}\ 2^{-j(n+\alpha)} \qquad \text{by (3.19)}$$

$$= c_3\, 2^n \sum_{0}^{\infty} 2^{-\alpha j} < \infty$$

This handles IV.

Thus $|\hat{f}(m)| \leq C_1 N^{-\alpha}$ as desired.

The proof of Lemma 4 is complete.

3.2. Applications and examples.

In this section we shall give results that are relatively easy consequences of Theorems 1 and 4. We state Theorem 5.

THEOREM 5. Let $f \in L^1(T^n)$, $\alpha > 0$, $E \subseteq Z^n$, $\gamma > 0$, $0 < \delta < 1$, $B > 0$, and $0 < \theta \leq 1$. Assume

(3.1) $\operatorname{supp} \hat{f} \subseteq E$

(3.2) For $m \in E$, $\operatorname{dist}(E, E \setminus \{m\}) > \gamma |m|^{\theta}$

(3.3) There is a point x_o such that f satisfies a Lipschitz condition of order α at x_o.

Then $f \in \Lambda_{\theta\alpha+n(\theta-1)}$ if $\alpha > n(1-\theta)/\theta$

COROLLARY 3. Let $f \in L^1(T^n)$, $\operatorname{supp} \hat{f} \subseteq E$. Assume

(3.20) For $m \in E$, $\operatorname{dist}(m, E \setminus \{m\}) > \gamma |m|$ and f satisfies a Lipschitz condition of order α at some point of T^n.

Then f is in the Lipschitz class Λ_α.

In the case $n = 1$, the corollary is the result of Hsieh Xie-Fan [11], and Izumi, Izumi and Kahane [13]. A special case is due to G. Freud [5]. (His result anticipated the others.) It will be an easy consequence of the next lemma and Lemma 2.

Theorem 5 and its corollary are consequences of Theorems 1 and 4 and the following geometric lemma.

LEMMA 6. Let $0 < \theta \leq 1$, $\gamma > 0$, $q > 1$, and $E \subseteq Z^n$. Assume

(3.2) For $m \in E$, $\text{dist}(m, E \setminus \{m\}) > \gamma |m|^{\theta}$

Then

$$\#(E \cap V(q,r)) \leq C(q,E) r^{n(1-\theta)}$$

Thus the gap condition of Theorem 4 implies the gap condition of Theorem 1 with $\tau = n(1-\theta)$.

Proof. We wish to estimate $\#(E \cap V(q,r))$. Let $m_1, m_2 \in E \cap V(q,r)$. Then

$$\text{dist}(m_1,m_2) > \gamma |m_1|^{\theta} \geq \gamma r^{\theta} \quad \text{by (3.2) and (2.1).}$$

We can conclude that $\#(E \cap V(q,r))$ is no more than the largest number of points that can be placed in a cube of side $2qr$ so that the distance between any two of these points is at least γr^{θ}.

Since

$$([2qr/(\gamma r^{\theta}/2\sqrt{n})] + 1)(\gamma r^{\theta}/2\sqrt{n}) \geq 2qr,$$

we shall still be estimating $\#(E \cap V(q,r))$ if we estimate the largest number of points separated pairwise by a distance at least γr^{θ} in a n-cube whose side has this larger length. Subdivide this n-cube into n-cubes with side $\gamma r^{\theta}/2\sqrt{n}$. The diameter of each of these smaller n-cubes is $\gamma r^{\theta}/2$. At most one point of a set of points separated by γr^{θ} can lie in any of the small n-cubes. Therefore

$$\#(E \cap V(q,r)) \leq \text{number of small cubes}$$

$$= ([2qr/(\gamma r^{\theta}/2\sqrt{n})] + 1)^n$$

$$= \left(\left[4\sqrt{n}\, q r^{1-\theta} \right] + 1 \right)^n \leq 2^n (4\sqrt{n})^n q^n r^{n(1-\theta)}$$

since $[x] + 1 \leq 2x$ if $x > 1$.

This gives a correct estimate for large r. We can complete the estimate by adjusting the constant for small r. (We know that the points of E are separated by the distance one.) Lemma 6 has been proven.

Now we shall prove Theorem 5 and Corollary 3.

Proof. Theorem 5 has the same hypotheses as Theorem 4. Thus we have

$$|\hat{f}(m)| \leq c_1 |m|^{-\theta\alpha}$$

Using this estimate and Lemma 6, we can apply Theorem 1 with $\theta\alpha - n(1-\theta)$ as α and $n(1-\theta)$ as τ. This gives

$$f \in \Lambda_{\theta\alpha - n(1-\theta)} \quad \text{if} \quad \theta\alpha - n(1-\theta) > 0$$

which it is if $\alpha > n(1-\theta)/\theta$.

Corollary 3 is just the special case $\theta = 1$ of Theorem 5.

We can combine Theorems 1 and 4 to get a further result.

THEOREM 6. Let $f \in L^1(T^n)$, $E \subseteq Z^n$, $\gamma > 0$, $0 < \delta < 1$, $B > 0$, $0 < \theta \leq 1$, $0 \leq \tau \leq n$, and $q > 1$. Assume

(3.1) $\quad \operatorname{supp} \hat{f} \subseteq E,$

(3.2) $\quad$ For $m \in E$, $\operatorname{dist}(E, E \setminus \{m\}) > \gamma |m|^{\theta}$

(2.3) $\quad \#(E \cap V(q,r)) \leq C(q,E) r^{\tau}$

(3.3) $\quad$ There is a point x_o such that f satisfies a Lipschitz condition of order α at x_o.

Then $f \in \Lambda_{\theta\alpha - \tau}$ if $\theta\alpha > \tau$.

Proof. Apply Theorem 4. Then apply Theorem 1 with $\theta\alpha-\tau$ as α and τ as τ.

The next result is a consequence of Theorem 4.

THEOREM 7. Let $f \in L^1(T^n)$ and suppose that (3.1) and (3.2) hold for f. Suppose also that at some point f has differentials of all orders.

Then f is a C^∞ function.

Proof. Since f has differentials of all orders at some point, it satisfies Lipschitz conditions of all orders α at this point.

By Theorem 3 we have

$$|\hat{f}(m)| \leq C_\alpha |m|^{-\theta\alpha} \quad \text{for} \quad \alpha > 0.$$

so

$$|\hat{f}(m)| \leq C'_k |m|^{-k} \quad \text{for} \quad k=1,2,\ldots$$

This implies $C^\infty(T^n)$.

We shall devote the rest of this section to examples.

Ex. 1. Let $n = 1$, and let $E = \{4^m\}_1^\infty \cup \{4^m + 2^m\}_1^\infty$.

Then if f is Lipschitz of order α at some point, $f \in \Lambda_{\alpha/2}$.

Ex. 2. Let $n = 1$ and let $E = \{27^m\} \cup \{27^m + 3^m\}$.

Then if f is Lipschitz of order $\alpha > 0$ at some point, $f \in \Lambda_{\alpha/3}$.

Ex. 3. Let $n = 1$, and let $E = \{m^2\}$, then if f is Lipschitz of order α at some point and $\alpha > 1$, then $f \in \Lambda_{(\alpha-1)/2}$.

Ex. 4. Let $n = 3$ and $E = \{(m, m^2, 3^m)\}$. Then if f is Lipschitz of order α at some point, $f \in \Lambda_\alpha$.

Ex. 5. Let $n = 3$ and $E = \{(m, m^2, m^3)\}$. Then if f is Lipschitz of order α at some point and $\alpha > 1/2$, $f \in \Lambda_{2\alpha/3-1/3}$.

The examples given above exhibit both polynomial and geometric growth.

We will conclude with two examples concerning partial differential equations.

Ex. 6. Consider the equation $u_{xx} = iu_t$ and assume $u \in L^1(T^2)$. Then

$$u(x,t) = \sum_{m \in Z} a_m e^{i(mx+m^2t)}$$

Thus $\operatorname{supp} \hat{u} \subseteq \{(m, m^2)\}_1^\infty$.

Our results apply. We can conclude

1. If u has differentials of all orders at some point, then $u \in C^\infty(T^2)$.

2. If u satisfies a Lipschitz condition of order α at some point and $\alpha > 1$, then $f \in \Lambda_{(\alpha-1)/2}$.

Next we give an example of a partial differential equation with a stronger property.

Ex. 7. Consider the equation $u_{xx} - 2u_{tt} + u = 0$ with $u \in L^1(T^2)$.

$$\operatorname{supp} \hat{u} \subseteq E = \{(m_1, m_2) \mid m_1^2 - 2m_2^2 = 1\}.$$

We claim that if u satisfies a Lipschitz condition of order α

at some point, then u is in the Lipschitz class Λ_α. To show this, it will suffice by Corollary 3 to show

(3.20) For $m \in E$, $\operatorname{dist}(m, E \setminus \{m\}) > \gamma |m|$.

E is the set of solutions to Pell's equation with $d = 2$. (See, for instance Nagell [19], p. 195.) The solutions with positive entries are given by (a_k, b_k) where $a_k + b_k\sqrt{2} = (3 + 2\sqrt{2})^k$. (We equate rational and irrational parts.) We also have the point $(1,0)$. The other solutions are given by changes of sign. We note that the positive solutions grow geometrically.

With a little care we see that solutions in different quadrants are sufficiently far apart. Thus (3.20) is satisfied and our claim is valid.

There are, of course, many more partial differential equations with constant coefficients to which our results apply. But given the role of Diophantine equations, it would be very difficult and probably impossible to characterize this class of equations.

3.3. Some Counterexamples.

In this final section of Chapter 3, we shall construct counterexamples to Theorem 4 in the case that $\alpha \leq 1$.

THEOREM 8. Let $0 < \alpha \leq 1$, let everything be as in Theorem 4, except that we assume (3.2) is false. Then there is an $f \in L^1$ such that f satisfies a Lipschitz condition of order α at 0, but $|\hat{f}(m)| \neq O(|m|^{-\theta\alpha})$.

To prove this we shall need a lemma.

LEMMA 7. Let $a_m > 0$, $b_m > 0$ and $b_m = o(a_m)$. $\exists c_m \geq 0$ such that $\overline{\lim}\, c_m a_m = 0$ and $\sum c_m b_m < \infty$.

Proof. $\exists c'_m > 0$ so that $\lim c'_m a_m = \infty$ and $\lim c'_m b_m = 0$. Indeed, let $c'_m = 1/\sqrt{a_m b_m}$.

Since $\lim c'_m b_m = 0$, for every $k \geq 0$ there exists j_k such that $c'_{j_k} b_{j_k} \leq 2^{-k}$.

Choose $c_m = c'_{j_k}$ if $m = j_k$ for some k. Choose $c_m = 0$ otherwise.

Then $\sum c_m b_m \leq \sum 2^{-k} < \infty$ and $\overline{\lim}\, c_m a_m = \infty$. The proof of Lemma 7 is complete.

We prove Theorem 8.

Proof. Assume $\operatorname{dist}(E, E \setminus \{m\}) \geq \gamma |m|^{\theta}$ fails for all $\gamma > 0$. Then there exists $\{m_k\}_1^{\infty} \subseteq E$ such that

$$\lim_{k\to\infty} |m_k|^{\theta} / |m_k - m_{k+1}| = \infty$$

or

$$|m_k - m_{k+1}| = o(|m_k|^{\theta}).$$

Thus

$$|m_k - m_{k+1}| = o(|m_k|^{\theta\alpha}) \text{ since } \alpha > 0.$$

We apply Lemma 7 to $|m_{2k}|^{\theta\alpha}$ and $|m_{2k} - m_{2k+1}|^{\alpha}$. There exists $c_k \geq 0$ such that $\sum c_k |m_{2k} - m_{2k+1}|^{\alpha} < \infty$ and $\overline{\lim}\, c_k |m_{2k}|^{\theta\alpha} = \infty$.

$$\sum c_k |m_{2k} - m_{2k+1}|^{\alpha} < \infty \text{ implies } \sum c_k < \infty .$$

This implies $f(x) = \sum c_k (e^{im_{2k}x} - e^{im_{2k+1}x}) \in A(T^n)$.

$\overline{\lim}\, c_k |m_{2k}|^{\theta\alpha} = \infty$ implies $f(m) \neq O(|m|^{-\theta\alpha})$.

$$|f(x) - f(0)| = |\sum c_k (e^{im_{2k}x} - e^{im_{2k+1}x}) - \sum c_k 0|$$

$$\leq \sum c_k \max (|m_{2k} - m_{2k+1}||x|, 2)$$

$$\leq 2 \sum c_k |m_{2k} - m_{2k+1}|^{\alpha} |x|^{\alpha}$$

$$= B|x|^{\alpha} \quad \text{where}$$

$$B = 2 \sum c_k |m_{2k} - m_{2k+1}|^{\alpha} \quad \text{which is finite.}$$

We would like to extend the argument to larger α but we have not succeeded. In the case $n = 1$ and $\theta = 1$, this argument can be generalized to larger α by taking primitives. This technique is awkward in higher dimensions.

CHAPTER IV

EXTENSIONS OF THE HADAMARD AND FABRY GAP THEOREMS TO SEVERAL VARIABLES

From this chapter till the end of the dissertation we shall be concerned with several complex variables rather than several real variables. Accordingly the first section of this chapter is devoted to a summary of facts and notations for power series in several variables. In the next section we will give a generalization of Hadamard's natural boundary theorem to several variables. This result is due in large part to Faber [3] and Lösch [17], but since they only worked with two variables and since our method is different from Faber's at a critical point, we feel justified in including an account here. The generalization of the Fabry theorem is new. The proof is given in the third section of this chapter.

4.1. Facts and notations for power series in several variables.

Let $z = (z_1,\ldots,z_n) \in C^n$. Let $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n) \in (Z^+)^n$, the n-fold Cartesian product of the non-negative integers.

Let

$$z^\alpha = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}.$$

Then $\sum_{\alpha \in (Z^+)^n} a_\alpha z^\alpha$ and $\sum_{\alpha \in (Z^+)^n} a_\alpha (z-b)^\alpha$ are power series centered about 0 and b respectively.

A power series converges <u>normally</u> at $\eta \in C^n$ if it converges absolutely in a neighborhood of η.

The domain of convergence of the power series $\sum a_\alpha z^\alpha$ is the set of points in C^n where $\sum a_\alpha z^\alpha$ converges normally.

An open set D is the domain of convergence of some power series if

1. $\xi \in D$ implies $z \in D$ if $|z_j| < |\xi_j|$ for $j=1...n$.
2. The set $\{\eta \in R^n | (e^{\eta_1},..., e^{\eta_n}) \in D\}$ is convex.

(This last property is often stated 'D is logarithmically convex'.)

An open set D is called a Reinhardt domain if $z \in D$ implies $(e^{i\phi_1}z_1,..., e^{i\phi_n}z_n)$ for $(\phi_1,...,\phi_n) \in R^n$.

Let $\rho = (\rho_1,..., \rho_n)$, $\rho_j > 0$.

$P(b,\rho) = \{z | |z_j-b_j| < \rho_j, j=1...n\}$

is called the polydisc with center b and polyradius ρ. If ρ is a positive number, we shall sometimes write $P(b,\rho)$ for $P(b, (\rho,...,\rho))$.

Let $U^n = P(0,1) = \{z | |z_j| < 1\}$ be the unit polydisc.

$T^n = \{z | |z_j| = 1\}$ is called the distinguished boundary of U^n. We shall call the set of points on the boundary of U^n not in T^n, the undistinguished boundary of U^n.

We shall sometimes use more than one copy of T^n, U^n and C^n. In that case we shall mark them with a complex variable. For instance, we write $U^n_z = \{z | |z_j| < 1\}$ and $U^n_w = \{w | |w_j| < 1\}$. We do the same thing for T^n and C^n.

If $\alpha \in (Z^+)^n$, the order of α, $|\alpha| = \sum_{j=1}^{n} \alpha_j$.

If $\alpha \in (Z^+)^n$, α factorial, $\alpha! = \alpha_1!\, \alpha_2! \ldots \alpha_n!$.

In order to be able to subscript multi-orders $\alpha \in (Z^+)^n$ and not confuse these subscripts with the coordinate subscripts, we shall use the notation $\alpha_{(v)}$ were $\alpha_{(v)} = (\alpha_{1,v}, \alpha_{2,v}, \ldots, \alpha_{n,v})$.

$D^\alpha f(z)$ is the partial derivative of f at z of order α.

$$D^\alpha f(z) = \left(\frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial z_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}} \right) f(z)$$

For properties of power series in several variables, one may consult Hörmander [10], p. 34, and Ronkin [24], p. 17.

4.2. The extension of Hadamard's theorem.

Hadamard's theorem [8] says that if $f(z) = \sum a_k z^k$ is an analytic function and there is a number $q > 1$ such that $\lambda_{k+1} > q\lambda_k$, then the disc of convergence of $\sum a_k z^k$ is the domain of existence of $f(z)$. We state a generalization.

THEOREM 9. Let $E \subseteq (Z^+)^n$. Let $f(z) = \sum_{\alpha \in E} a_\alpha z^\alpha$ define a holomorphic function. Suppose that E satisfies the following condition. There is a number $q > 1$ such that for each $\alpha \in E$ the set

$$\{x \in R^n \mid \alpha_j/q \leq x \leq q\alpha_j, \quad j=1\ldots n\}$$

contains no member of E other than α.
Then the domain of convergence of $\sum_{\alpha \in E} a_\alpha z^\alpha$ is the domain of existence of $f(z)$.

COROLLARY 4. Let $E \subseteq (Z^+)^n$. Let $f(z) = \sum_{\alpha \in E} a_\alpha z^\alpha$ define a holomorphic function. Suppose that $E = \prod_1^n E_j$ where $E_j \subseteq Z^+$,

$E_j = \{\lambda_{j,k}\}_{k=1}^{\infty}$ and there is a number $q > 1$ such that

$$\lambda_{j,k+1} > q\lambda_{j,k} \qquad j=1...n, \quad k=1,2,...$$

Then the domain of convergence of $\sum_{\alpha \in E} a_\alpha z^\alpha$ is the domain of existence of $f(z)$.

If $\alpha \in (Z^+)^n$ let $\|\alpha\| = (\sum \alpha_j^2)^{1/2}$.

COROLLARY 5. Let $f(z) = \sum a_v z^{\beta(v)}$ define a holomorphic function. Assume that there is a number $q > 1$ such that

$$\|\beta_{(v+1)}\| > q\|\beta_{(v)}\| \qquad v=1,2,...$$

Then the domain of convergence of the power series is the domain of existence of $f(z)$.

(For $n = 2$, this is Faber's result.)

Both the corollaries are proved by checking that they satisfy the hypotheses of Theorem 9.

Examples of sets E to which these results apply are $\{(2^m,3^k)\}$, $\{(2^k+3^m,3^m)\}$, and $\{(m,2^m)\}$.

We now give the proof of Theorem 9.

Proof. Let ξ be any point on the boundary of the domian of convergence of $\sum_{\alpha \in E} a_\alpha z^\alpha$. We wish to show that ξ is a singular point of $f(z)$. To do this, we will assume the opposite and then show that ξ is in the domain of convergence which is a contradiction.

By a simple change of variables, we can without loss of generality assume that $\xi = (1,1,...,1)$. Then f is analytic and

the power series $\sum_{\alpha \in E} a_\alpha z^\alpha$ converges normally in the polydisc U^n, and we are assuming that f is analytic at $\xi = (1,\ldots,1)$.

Our proof depends on a certain mapping g. Let s be a positive integer such that

(4.1) $\qquad 1 + 1/s < q$

Define $g: C_w^n \to C_z^n$ by

(4.2) $\qquad z_j = g_j(w) = \frac{1}{2}(w_j^s + w_j^{s+1}) \quad j = 1\ldots n.$

Let $F(w)$ be the analytic function defined by

(4.3) $\qquad F(w) = f(g(w))$

We shall need a lemma. We forget for the moment our definition of f.

LEMMA 8. Let f be analytic in U_z^n and at one point of the distinguished boundary T_z^n. Then if $F(w)$ is defined by (4.3), it is analytic in U_w^n and a neighborhood of the distinguished boundary T_w^n.

We shall use Lemma 8 again in Chapter 5.

Proof of Lemma 8. Without loss of generality assume that the point of T_z^n where f is analytic is $\xi = (1,\ldots,1)$.

If f is analytic at $g(w)$, then F is analytic at w. $g(U_w^n) \subseteq U_z^n$ by (4.2). Thus F is analytic in U_w^n. $g((1,\ldots,1)) = (1,\ldots,1)$. So F is analytic at $(1,\ldots,1)$. If $w \in T_w^n$ and $w_j \neq 1$ for $j=1\ldots n$, then $g(w) \in U_w^n$, and F is

analytic at such points. The case remaining is $w \in T_w^n$ with $w_j = 1$ for some j and $w_j \neq 1$ for some other j. Then $g(w)$ is on the undistinguished boundary of U_z^n and we do not know whether $f(z)$ is analytic there. We shall use the strong 'disk' theorem to show that f is indeed analytic at the points where we require it to be analytic. For the strong 'disk' theorem, see Vladimirov [31], Section 17.3.

Let z be such that $|z_j| < 1$ or $z_j = 1$ for $j = 1...n$. We shall show that f is analytic at z. Let k be the number of coordinates of z not equal to one. We shall use induction on k. By assumption our claim is valid for $k = 0$. Suppose it is valid for $k-1$. By relabelling coordinates if necessary, we may write.

$$z = (z_1, \ldots z_k, 1, \ldots, 1) \quad \text{where}$$

$|z_j| < 1$ if $j \leq k$. Using the notation of the strong 'disk' theorem as found in Vladimirov [31] we let

$$\tilde{u}(t) = \tilde{u}^o + b\lambda(t)$$

where

$$b = (\underset{2}{0}, \ldots, \underset{k}{0}, \underset{k+1}{1}, \ldots, 1) \in C^{n-1}$$

$$\tilde{u}^o = (z_2, \ldots z_k, 0, \ldots, 0) \in C^{n-1}$$

and $\lambda(t) = 1-t$.

(We substitute the complex variable u for z.)

Let $D(t) = \{u_1 \mid |u_1| < 1\}$.

Since $D(t)$ does not depend on t, the assumption in the strong 'disk' theorem regarding compact subsets of the $D(t)$ is trivially satisfied.

The discs

$$\{u|\ u_1 \in D(t),\ \tilde{u} = \tilde{u}(t)\} \quad \text{for } 0 < t \leq 1$$

$$= \{u|\ u = (u_1, z_2, \ldots, z_k, 1-t, \ldots, 1-t),\ |u_1| < 1\}$$

are contained in U^n. Thus f is analytic on these discs.

The 'limit disc'

$$\{u|\ u_1 \in D(0),\ \tilde{u} = \tilde{u}(0)\}$$

$$= \{u \in C^n|\ u = (u_1, z_2, \ldots, z_k, 1, \ldots, 1),\ |u_1| < 1\}$$

The point $(1, z_2, \ldots, z_k, 1, \ldots, 1)$ is on the boundary of the 'limit disc'. Since it has only $k-1$ coordinates different from one, by inductive hypothesis we know f is analytic at this point. This implies that f is analytic at some point of the 'limit disc'. Now we can apply the 'strong disk' theorem. We conclude that f is analytic in the limit disc. In particular, it is analytic at z. Our argument is complete by induction.

Thus if $w \in T_w^n$ and $g(w)$ is on the undistinguished boundary of U_z^n, f is analytic at $g(w)$ and F is analytic at w. The proof of Lemma 8 is complete.

We shall also need a lemma connecting the power series representations of $F(w)$ and $f(z)$. We shall again regard $f(z)$ as more general than it is in Theorem 9.

LEMMA 9. Let $f(z) = \sum a_\alpha z^\alpha$ be analytic in U^n. Then $F(w) = f(g(w))$ can be represented by

$$F(w) = \sum a_\alpha g(w)^\alpha \quad \text{when} \quad w \in U^n.$$

If $g(w)^\alpha = \sum_\beta c_{\alpha,\beta} w^\beta$ and we set

$$b_\beta = \sum_\alpha a_\alpha c_{\alpha,\beta},$$ then the series $\sum b_\beta z^\beta$ converges absolutely to $F(w)$ in U_w^n and is therefore the power series expansion of $F(w)$.

Proof. Let $w \in U_w^n$.

$$\sum_\alpha a_\alpha g(w)^\alpha = \sum_\alpha \sum_\beta a_\alpha c_{\alpha,\beta} w^\beta. \tag{4.4}$$

If we can show that this series is absolutely convergent, the Fubini-Tonelli theorem will allow us to reverse the order of summation. We take absolute values

$$\sum_\alpha \sum_\beta |a_\alpha| \; c_{\alpha,\beta} |w|^\beta$$

($c_{\alpha,\beta} \geq 0$ by (4.2).)

$$= \sum_\alpha |a_\alpha| \sum_\beta c_{\alpha,\beta} (|w_1|, \ldots, |w_n|)^\beta$$

$$= \sum_\alpha |a_\alpha| \; g((|w_1|, \ldots, |w_n|))^\alpha < \infty$$

since the series (4.4) is absolutely convergent if the argument is in U^n and $(|w_1|, \ldots, |w_n|) \in U^n$ if $w \in U^n$.

Thus by Fubini

$$\sum_\beta b_\beta w^\beta = \sum_\beta \sum_\alpha a_\alpha c_{\alpha,\beta} w^\beta$$

is absolutely convergent for $w \in U^n$ and equals

$$\sum a_\alpha g(w)^\alpha = F(w).$$

$b_\beta = \sum_\alpha a_\alpha c_{\alpha,\beta}$ is actually a finite sum since $s\alpha_j \leq \beta_j$ for $j = 1...n$ if $a_\alpha g(w)^\alpha$ makes a contribution to $b_\beta w^\beta$ by (4.2).

We shall use Lemma 9 again in Chapter 5.

Let us resume the assumption that $f(z) = \sum_{\alpha \in E} a_\alpha z^\alpha$ as in Theorem 9.

By Lemma 8, $F(w)$ is analytic in U_w^n and a neighborhood of T_w^n. Thus F is analytic in a Reinhardt domain containing $U_w^n \cup T_w^n$ which is connected and contains the origin. By a standard result (see Hörmander [10], p. 42), $\sum b_\beta w^\beta$ converges normally to $F(w)$ in a neighborhood of T_w^n. In particular, it converges normally at $(1,...,1)$.

We will show that $\sum_{\alpha \in E} a_\alpha g(w)^\alpha$ is a grouping of the terms of $\sum b_\beta w^\beta$. That is, $(Z^+)^n = \bigcup_{\alpha \in E} H_\alpha$ where $H_\alpha \cap H_{\alpha'} = \phi$ if $\alpha \neq \alpha'$ and

$$F(w) = \sum b_\beta w^\beta = \sum_{\alpha \in E} \sum_{\beta \in H_\alpha} b_\beta w^\beta = \sum_{\alpha \in E} a_\alpha g(w)^\alpha.$$

Once we have done this, we are done since then $\sum b_\beta w^\beta$ converges normally at $(1,...,1)$ implies $\sum_{\alpha \in E} a_\alpha g(w)^\alpha$ converges normally at $(1,...,1)$. Because g is bijective near $(1,...,1)$, $\sum_{\alpha \in E} a_\alpha z^\alpha$ converges normally at $g((1,...,1)) = (1,...,1) = \xi$. This is the desired contradiction.

It remains to show $\sum_{\alpha \in E} a_\alpha g(w)^\alpha$ is a grouping of the terms of $\sum b_\beta w^\beta$. It is at this point that we shall finally use our hypothesis of lacunarity.

Let α and α' belong to E, $\alpha \neq \alpha'$

$$g(w)^{\alpha} = \sum c_{\alpha,\beta} w^{\beta}$$

$$g(w)^{\alpha'} = \sum c_{\alpha'\beta} w^{\beta}$$

We shall show that $g(w)^{\alpha}$ and $g(w)^{\alpha'}$ contain no non-zero terms of the same degree.

Suppose there is a non-zero term of the form $d_{\gamma} w^{\gamma}$ in $g(w)^{\alpha}$.

Then

$$s\alpha_j \leq \gamma_j \leq (s+1)\alpha_j \quad \text{for } j = 1...n \quad \text{by (4.2).}$$

Suppose that there is also a non-zero term of the form $d'_{\gamma} w^{\gamma}$ in $g(w)^{\alpha'}$.

Then

$$s\alpha'_j \leq \gamma_j \leq (s+1)\alpha'_j \qquad j = 1...n \quad \text{by (4.2)}$$

Thus $$\alpha'_j \leq ((s+1)/s)\alpha_j \leq q\alpha_j \qquad \text{by (4.1)}$$

and $$\alpha_j \leq ((s+1)/s)\alpha_j \leq q\alpha'_j \qquad \text{by (4.1).}$$

So $$\alpha_j/q \leq \alpha'_j \leq q_j \qquad \text{for } j = 1...n.$$

But this contradicts the hypothesis that α is the only member of E in the set

$$\{x \in R^n \mid \alpha_j/q \leq x \leq q\alpha_j \quad \text{for } j = 1...n\}.$$

Thus $g(w)^{\alpha}$ and $g(w)^{\alpha'}$ have no non-zero terms of the same degree.

By Lemma 9, $b_{\beta} = \sum_{\alpha} a_{\alpha} c_{\alpha,\beta}$

By the argument above, we see that at most one term of

$\sum_{\alpha} a_\alpha c_{\alpha,\beta}$ can be non-zero.

We define sets $\{H_\alpha\}_{\alpha \in E}^{\infty}$ as follows. If $b_\beta \neq 0$, $b_\beta = a_\alpha c_{\alpha,\beta}$ for some $\alpha \in E$. Let $\beta \in H_\alpha$. If $b_\beta = 0$, choose any $\alpha \in E$ and let $\beta \in H_\alpha$. Thus

$$(Z^+)^n = \sum_{\alpha \in E} H_\alpha \quad \text{and} \quad H_\alpha \cap H_{\alpha'} = \phi \quad \text{if} \quad \alpha \neq \alpha'.$$

$$\sum b_\beta w^\beta = \sum_{\alpha \in E} \sum_{\beta \in H_\alpha} b_\beta w^\beta$$

$$= \sum_{\alpha \in E} \sum_{\beta} a_\alpha c_{\alpha,\beta} w^\beta = \sum_{\alpha \in E} a_\alpha g(w)^\alpha$$

Thus $\sum_{\alpha \in E} a_\alpha g(w)^\alpha$ is a grouping of the terms of $\sum b_\beta w^\beta$. The proof of Theorem 9 is complete.

4.2. An extension of Fabry's theorem.

Fabry [4] proved in 1896 that if $f(z) = \sum a_k z^{\lambda_k}$ defines a holomorphic function and $k/\lambda_k \to 0$ as $k \to \infty$ then the disc of convergence is the natural domain of $f(z)$. Our result is that if we take an n-fold product of one-dimensional sets each satisfying Fabry's condition then the same conclusion can be drawn. We state the result formally.

THEOREM 10. Let $E \subseteq (Z^+)^n$. Let $f(z) = \sum_{\alpha \in E} a_\alpha z^\alpha$ define a holomorphic function. Suppose that

$$E = \prod_{j=1}^{n} E_j, \quad E_j \subseteq Z^+$$

$E_j = \{\lambda_{j,k}\}_{k=1}^{\infty}$ and $k/\lambda_{j,k} \to 0$ as $k \to \infty$ for $j = 1 \ldots n$.

Then the domain of convergence of $\sum_{\alpha \in E} a_\alpha z^\alpha$ is the domain of existence of $f(z)$.

We give some examples of sets E for which Theorem 10 is valid.

a. $E = \{(m^4, k^2)\}$

b. $E = \{(m^2, k^3)\}$

c. $E = \{(m^2, k^2, s^2)\}$

d. $E = (\{m^2\} \cup \{m^2+1\}) \times (\{k^2+1\} \cup \{k^4\})$

Of course, the conclusion of the theorem is also valid for a set E which is the subset of a product set satisfying the hypotheses of the theorem.

By result of Pólya [23] for the one-dimensional case, it is easy to see that the condition of the theorem is the best possible for a product set E.

Another extension of the Fabry gap theorem to several variables is given in Ehrenpreis [2], p. 432, in the context of Dirichlet series. We remark that although the sets E for which Ehrenpreis' result is valid do not have to satisfy as rigid a geometric condition as being a product set they do have to be much 'thinner' than the sets in our theorem have to be. For instance, examples c. and d. will not satisfy the hypotheses of Ehrenpreis' theorem.

To prove Theorem 10, we shall reduce it to Theorem 9 by means of a differential operator of infinite order. This operator is defined by an entire function of zero type. This is a classical method of proving the one-dimensional theorem. An account of this method may be found in Hille [9], p. 87. We shall use Hille as our guide and quote results when necessary.

Before we prove Theorem 10, we shall state a lemma concerning differential operators of infinite order together with the required definitions and notations.

DEFINITION 7. Let $G(z)$ be an entire holomorphic function defined on C^n. We say that G is of <u>zero type</u> if for each $\varepsilon > 0$, there is a constant $M(\varepsilon) > 0$ such that

$$|G(z)| \leq M(\varepsilon) e^{\varepsilon \|z\|_1}$$

$$(\|z\|_1 = \sum_{j=1}^{n} |z_j|.)$$

Let

$$\Psi = \left(z_1 \frac{\partial}{\partial z_1}, z_2 \frac{\partial}{\partial z_2}, \cdots, z_n \frac{\partial}{\partial z_n} \right)$$

If $\alpha \in (Z^+)^n$,

$$\Psi^\alpha = \left(z_1 \frac{\partial}{\partial z_1} \right)^{\alpha_1} \left(z_2 \frac{\partial}{\partial z_2} \right)^{\alpha_2} \cdots \left(z_n \frac{\partial}{\partial z_n} \right)^{\alpha_n} \tag{4.5}$$

LEMMA 10. Let f be a holomorphic function in the domain $D \subseteq C^n$. Let G be an entire function of zero type. Suppose $G(z) = \sum (g_\alpha/\alpha!) z^\alpha$ is the power series expansion of G. Then the operator $G(\Psi)$ defined by

$$G(\Psi) f(z) = \sum (g_\alpha/\alpha!) \Psi^\alpha f(z) \tag{4.6}$$

is well defined for f and z in D, and $G(\Psi)f(z)$ is a holomorphic function in D. In fact, the series on the right hand side of (4.6) is absolutely convergent on compact subsets of D.

Furthermore, if $f(z) = \sum a_\beta z^\beta$ is the power series expansion

of $f(z)$, then

$$(4.7) \qquad G(\Psi)f(z) = \sum a_\beta G(\beta) z^\beta$$

whenever $\sum a_\beta z^\beta$ converges normally. ($G(\beta)$ is the value of G at β.)

We shall prove Theorem 10 assuming this lemma. Then we will prove Lemma 10.

Proof of Theorem 10. The idea of the proof is to observe, using (4.7), that if $G(\beta) = 0$ the β-coefficient of $G(\Psi)f(z)$ vanishes. If G is chosen properly, then we may be able to annihilate so many coefficients that $G(\Psi)f(z)$ has a power series to which Theorem 9 applies. This is what we will do, but many technical considerations will be necessary in order to accomplish this.

Let ξ be a point on the boundary of the domain of convergence of $\sum_{\alpha \in E} a_\alpha z^\alpha$. We wish to show that ξ is a singular point of $f(z)$. Without loss of generality, we may assume that $\xi = (1,\ldots,1)$. Then $\sum_{\alpha \in E} a_\alpha z^\alpha$ converges normally in U^n. Since ξ is on the boundary of the domain of convergence, we have

$$\overline{\lim_{|\alpha| \to \infty}} |a_\alpha|^{1/|\alpha|} = 1$$

(See Ronkin [24] p. 20, where the analog of the limit supremum formula for the radius of convergence for several variables is given.)

There exists a sequence $\{\beta_{(s)}\} \subseteq E$ such that

$$\lim_{s\to\infty} |a_{\beta_{(s)}}|^{1/|\beta_{(s)}|} = 1 \tag{4.8}$$

Let P be the set of j, $1 \le j \le n$ such that

$$\overline{\lim_{s\to\infty}}\ \beta_{j,s} = \infty,$$

and let Q be the set of j, $1 \le j \le n$ such that

$$\overline{\lim_{s\to\infty}}\ \beta_{j,s} < \infty.$$

Choose $q > 1$.

We shall several times extract a subsequence from $\{\beta_{(s)}\}$. Each time, we shall call this subsequence $\{\beta_{(s)}\}$, and the sets P and Q shall be defined relative to the current $\{\beta_{(s)}\}$.

P is not empty by (4.8).

Take $j \in P$ and choose a subsequence $\{\beta_{(s)}\}$ of $\{\beta_{(s)}\}$ such that

$$\beta_{j,s+1} > q\beta_{j,s} \quad \text{for } s = 1,2,3,... \tag{4.9}$$

(4.8) is, of course, still valid.

If there is a $j \in P$ such that (4.9) is false, choose a subsequence $\{\beta_{(s)}\}$ of $\{\beta_{(s)}\}$ such that (4.9) is valid for this j. (4.10) is still valid for any $j \in P$ that it was previously, provided j is still in P. (4.8) is valid for the new $\{\beta_s\}$. After a finite number of subsequence extractions, (4.9) is valid for all $j \in P$.

For $j = 1...n$, let

$$A_j = \{m \in Z^+ \mid m = \beta_{j,s} \text{ for some } s\}$$

A_j is finite if $j \notin P$.

Then

$$A = \prod_{j=1}^{n} A_j \subseteq \prod_{j=1}^{n} E_j = E$$

By (4.9) we see that if A is substituted for E in Theorem 9, the hypotheses of the theorem are valid for A.

The argument above was designed to give us an opportunity to invoke Theorem 9, and at the same time (4.8) will let us control convergence.

Let

$$B_j = E_j \setminus A_j \qquad j = 1...n$$

Then either B_j is finite or

$$B_j = \{\mu_{j,s}\}_{s=1}^{\infty} \quad \text{and} \quad s/\mu_{j,s} \to 0 \quad \text{as} \quad s \to \infty$$

since

$$0 \leq s/\mu_{j,s} \leq s/\lambda_{j,s} \quad \text{and} \quad s/\lambda_{j,s} \to 0.$$

Define $g_j(w)$ for $j = 1...n$ to be

(4.10)
$$\prod_{m \in B_j} \left(1 - \frac{w^2}{m^2}\right) \quad \text{if} \quad 0 \notin B_j,$$

or

$$w \prod_{\substack{m \notin B_j \\ m \neq 0}} \left(1 - \frac{w^2}{m^2}\right) \quad \text{if} \quad 0 \in B_j$$

(An empty product is identically one.)

By Levinson [16], Theorem XXX, we have g_j is an entire function of zero type for $j = 1...n$.

(4.11) For $\varepsilon > 0$, there is a $M_1(\varepsilon) > 0$ such that

$$|g_j(w)| \leq M_1(\varepsilon)e^{\varepsilon|w|} \quad j = 1...n.$$

Also, if $|w-m| \geq 1/2$ for $m \in B_j$, then for $\varepsilon > 0$ there is an $M_2(\varepsilon) > 0$ such that

(4.12) $$|g_j(w)| \geq M_2(\varepsilon)e^{-\varepsilon|w|}$$

Let

(4.13) $$G(z) = \prod_{j=1}^{n} g_j(z_j)$$

By (4.11) and (4.13), G is an entire function of zero type. By Lemma 10, we can apply $G(\Psi)$ to $f(z) = \sum_{\alpha \in E} a_\alpha z^\alpha$ and obtain

$$G(\Psi)f(z) = \sum_{\alpha \in E} a_\alpha G(\alpha) z^\alpha \quad \text{for } z \in U^n \text{ by (4.7)}$$

$G(\alpha) = \prod g_j(\alpha_j)$ by (4.13) which vanishes if $\alpha \notin A$ by (4.10).

So

$$G(\Psi)f(z) = \sum_{\alpha \in A} a_\alpha G(\alpha) z^\alpha$$

Since f is analytic in U^n, $G(\Psi)f$ is analytic in U^n by Lemma 10 and so

(4.14) $$\overline{\lim_{|\alpha|\to\infty}} |a_\alpha G(\alpha)|^{1/|\alpha|} \leq 1$$

(See Ronkin [24], p. 20.)

By (4.11), (4.12) and (4.13) we have for $\varepsilon > 0$ and $\alpha \in A$

$$|a_\alpha|(M_2(\varepsilon))^n e^{-\varepsilon|\alpha|} \leq |a_\alpha G(\alpha)| \leq |a_\alpha|(M_1(\varepsilon))^n e^{\varepsilon|\alpha|}$$

Thus

$$|a_\alpha|^{1/|\alpha|}(M_2(\varepsilon))^{n/|\alpha|}e^{-\varepsilon} \leq |a_\alpha G(\alpha)|^{1/|\alpha|} \leq |a_\alpha|^{1/|\alpha|}(M_1(\varepsilon))^{n/|\alpha|}e^{\varepsilon}$$

Restricting consideration to the sequence $\beta_{(s)}$ we have

$$|a_{\beta_{(s)}}|^{1/|\beta_{(s)}|}(M_2(\varepsilon))^{n/|\beta_{(s)}|}e^{-\varepsilon} \leq |a_{\beta_{(s)}}G(\beta_{(s)})|^{1/|\beta_{(s)}|} \leq |a_{\beta_{(s)}}|^{1/|\beta_{(s)}|}(M_1(\varepsilon))^{n/|\beta_{(s)}|}e^{\varepsilon}.$$

By (4.8)

$$e^{-\varepsilon} \leq \underline{\lim}_{s\to\infty} |a_{\beta_{(s)}}G(\beta_{(s)})|^{1/|\beta_{(s)}|} \leq \overline{\lim}_{s\to\infty} |a_{\beta_{(s)}}G(\beta_{(s)})|^{1/|\beta_{(s)}|} \leq e^{\varepsilon}.$$

Since ε is arbitrary, we have

$$\overline{\lim}_{s\to\infty} |a_{\beta_{(s)}}G(\beta_{(s)})|^{1/|\beta_{(s)}|} = 1$$

This and (4.14) yield

$$\overline{\lim}_{|\alpha|\to\infty} |a_\alpha G(\alpha)|^{1/|\alpha|} = 1.$$

By Ronkin [24], p. 20, we see that T^n is on the boundary of the domain of convergence of $\sum_{\alpha\in A} a_\alpha G(\alpha) z^\alpha$. Since A satisfies the hypotheses of Theorem 9, $G(\Psi)f$ is singular on T^n and, in particular, it is singular at $(1,1,\ldots,1)$.

If f were analytic at $\xi = (1,\ldots,1)$, Lemma 10 would imply that $G(\Psi)f$ is analytic at $(1,\ldots,1)$. Since it is not, f is singular at $\xi = (1,\ldots,1)$. Theorem 10 has been proven.

We shall now give the proof of Lemma 10. We need a lemma.

LEMMA 11. Let $G(z) = \sum(g_\alpha/\alpha!)z^\alpha$ be an entire function of zero type. Then for $\varepsilon > 0$, there exists $V(\varepsilon) > 0$ such that

$$|g_\alpha| \leq V(\varepsilon)\varepsilon^{|\alpha|} \tag{4.15}$$

Proof. Consider the polydisc $P(0,\rho)$ for $\rho > 0$. By Cauchy's inequalities

$$|g_\alpha| \leq \alpha! \sup_{P(0,\rho)} |G(z)|\rho^{-|\alpha|}$$

Since G is of zero-type,

$$|g_\alpha| \leq \alpha! M(\varepsilon)e^{n\ \varepsilon\ \rho}\rho^{-|\alpha|}$$

Let $\rho = |\alpha|/\varepsilon$. We have

$$|g_\alpha| \leq M(\varepsilon)\alpha! e^{n|\alpha|}|\alpha|^{-|\alpha|}\varepsilon^{|\alpha|}$$

The quantity $\alpha! e^{n|\alpha|}|\alpha|^{-|\alpha|} \leq |\alpha|! e^{n|\alpha|}|\alpha|^{-|\alpha|}$

This last quantity tends to zero as $|\alpha| \to \infty$. Thus we can absorb $\alpha! e^{n|\alpha|}|\alpha|^{-|\alpha|}$ into the constant.

We have

$$|g_\alpha| \leq V(\varepsilon)\varepsilon^{|\alpha|}.$$

This proves Lemma 11.

We now prove Lemma 10.

Proof. From Lemma 11.2.1 and its corollary in Hille [9], pp. 47-48, we have that if functions $P_k(w,a)$ are defined by

$$\left(w\frac{d}{dw}\right)^k \frac{1}{a-w} = \frac{P_k(w,a)}{(a-w)^{k+1}} \tag{4.16}$$

Then $P_k(w,a)$ is a polynomial of degree k and

(4.17) $$|P_k(w,a)| \leq k!(\max(|w|,|a|)^k.$$

Suppose f is holomorphic in $D \subset C^n$. Let $b \in D$. Then there is a $\rho > 0$ such that the polydisc $P(b,2\rho)$ is relatively compact in D.

Let $z \in P(b,\rho)$, and let E be a finite subset of $(Z^+)^n$. Then

$$\sum_{\alpha \in E} \frac{g_\alpha}{\alpha!} \psi^\alpha f(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_j - b_j| = 2\rho} f(\zeta) \left(\sum_{\alpha \in E} \frac{g_\alpha}{\alpha!} \psi^\alpha \frac{1}{\Pi(\zeta_j - z_j)} \right) d\zeta$$

by applying Cauchy's formula and differentiating under the integral sign.

We estimate this quantity and show that it is dominated by an absolutely convergent series uniformly for $z \in P(b,\rho)$. By the standard compactness argument, this is enough to establish the first conclusion of Lemma 10.

By (4.5) the integral in the last formula equals

$$\frac{1}{(2\pi i)^n} \int_{|\zeta_j - b_j| = 2\rho} f(\zeta) \left(\sum_{\alpha \in E} \frac{g_\alpha}{\alpha!} \prod_1^n \left(z_j \frac{\partial}{\partial z_j} \right)^{\alpha_j} \frac{1}{(\zeta_j - z_j)} \right) d\zeta$$

By (4.16) this is

$$\frac{1}{(2\pi i)^n} \int_{|\zeta_j - b_j| = 2\rho} f(\zeta) \left(\sum_{\alpha \in E} \frac{g_\alpha}{\alpha!} \prod_1^n \frac{P_{\alpha_j}(z_j,\zeta_j)}{(\zeta_j - z_j)^{\alpha_j + 1}} \right) d\zeta$$

Using (4.17), we see that the magnitude of the last expression is no more than

$$2^n M_f(b,2\rho) \sum_{\alpha \in E} \frac{g_\alpha}{\alpha!} \prod_1^n \alpha_j! \cdot \max_{|\zeta_j - b_j| = 2\rho} (|z_j|, |\zeta_j|))^{\alpha_j} \rho^{-\alpha_j}$$

where $M_f(b,2\rho) = \sup_{P(b,2\rho)} |f(\zeta)|$

Let $R = \sup_{\substack{z \in P(0,\rho) \\ |\zeta_j - b_j| = 2\rho}} \max_{1 \le j \le n} (\max(|z_j|, |\zeta_j|)$

Then the last quantity is no more than

$$2^n M_f(b,2\rho) \sum_{\alpha \in E} |g_\alpha| R^{|\alpha|} \rho^{-|\alpha|}$$

Applying (4.15) with $\varepsilon = \rho/2R$, we see that the last quantity is no more than

$$2^n M_f(b,2\rho)\ V(\rho/2R) \sum_{\alpha \in E} \left(\frac{1}{2}\right)^{|\alpha|} \cdot$$

$\sum_{\alpha \in (Z^+)^n} (1/2)^{|\alpha|}$ is an absolutely convergent series. We see that the finite partial sums of (4.6) are dominated by the finite partial sums of an absolutely convergent series for $z \in P(b,\rho)$. This gives us the first conclusion of Lemma 10.

It remains to demonstrate (4.7). Let $f(z) = \sum a_\beta z^\beta$. (The series converges normally at z.)

$$\begin{aligned} G(\Psi)f(z) &= \sum_\alpha \frac{g_\alpha}{\alpha!} \Psi^\alpha \left(\sum_\beta a_\beta z^\beta \right) \\ &= \sum_\alpha \sum_\beta \frac{g_\alpha}{\alpha!} \Psi^\alpha a_\beta z^\beta \\ &= \sum_\alpha \sum_\beta \frac{g_\alpha}{\alpha!} \beta^\alpha a_\beta z^\beta \qquad \text{by (4.5)} \end{aligned}$$

We wish to justify reversing the order of summation.

We estimate

$$\sum_{\beta}\sum_{\alpha}\left|\frac{g_\alpha}{\alpha!}\beta^\alpha a_\beta z^\beta\right|$$

By (4.15), this is no more than

$$\sum_{\beta}\sum_{\alpha}\left|\frac{M(\varepsilon)\varepsilon^{|\alpha|}}{\alpha!}\beta^\alpha a_\beta z^\beta\right|$$

$$= \sum_{\beta} M(\varepsilon)e^{\varepsilon|\beta|}|a_\beta||z^\beta|$$

$$= \sum_{\beta} M(\varepsilon)\ |a_\beta||(e^\varepsilon z)^\beta|$$

Since $\sum a_\beta z^\beta$ converges normally in a neighborhood of z, it converges normally (and hence absolutely) for $e^\varepsilon z$ if ε is sufficiently small. Thus

$$M(\varepsilon)\sum_{\beta}|a_\beta||(e^\varepsilon z)^\beta| < \infty.$$

By Fubini's theorem, we have

$$G(\Psi)f(z) = \sum_{\beta}\sum_{\alpha}\frac{g_\alpha}{\alpha!}\beta^\alpha a_\beta z^\beta$$

$$= \sum_{\beta} G(\beta)a_\beta z^\beta$$

which is (4.7). Lemma 10 has been proven.

CHAPTER V

OVERCONVERGENCE THEOREMS FOR SEVERAL VARIABLES

In this chapter we will prove generalizations of two of Ostrowski's overconvergence results. The first is due to Lösch [17] for two variables. We give a new proof and extend the result to any number of variables. The second is due to Ostrowski in one variable. A good reference for the one variable results is Ostrowski [22].

5.1. Overconvergence.

We will generalize the following theorem to several variables.

THEOREM. Let $f(z) = \sum a_n z^n$ define a holomorphic function. Let $(m_k, m_k')_{k=1}^{\infty}$ be a sequence of pairs of integers and let $q > 1$ such that $m_k'/m_k \geq q$ and $m_k' \leq m_{k+1}$. Assume that $a_n = 0$ if $m_k < n < m_k'$ for some k. Then the sequence of partial sums $s_{m_k}(z) = \sum_0^{m_k} a_n z^n$ converges uniformly to $f(z)$ in a neighborhood of any point on the boundary of the disc of convergence where f is analytic.

Now we state the generalization.

THEOREM 11. Let $f(z) = \sum a_\alpha z^\alpha$ define a holomorphic function. Let $(\beta_{(v)}, \beta'_{(v)})_{v=1}^{\infty}$ be a sequence of pairs of multi-orders and

let $q > 1$ such that $\beta'_{j,v} > q\beta_{j,v}$ for $j = 1...n$ and $\beta_{(v+1)} \geq \beta'_{(v)}$. Assume that $a_\alpha = 0$ if $\beta_{j,v} < \alpha_j < \beta'_{j,v}$ for some j and some v. Then at any point ξ of the boundary of the domain of convergence where f is analytic the sequence of partial sums

$$s_{\beta_{(v)}}(z) = \sum_{\alpha \leq \beta_{(v)}} a_\alpha z^\alpha$$

converges uniformly to $f(z)$ in a neighborhood of ξ.

By $\alpha \leq \beta$ we mean that $\alpha_j \leq \beta_j$ for $j = 1...n$.

The proof is similar to that of Theorem 9.

Proof. Let s be a positive integer such that

(4.1) $\quad 1 + 1/s < q.$

We shall use the mapping g defined by (4.2) and the function $F(w)$ defined by (4.3).

Let f be holomorphic at ξ on the boundary of the domain of convergence. Without loss of generality, we may assume $\xi = (1,...,1)$. Then $\sum a_\alpha z^\alpha$ converges normally in U^n.

Since f is analytic in U^n_z and at the point $(1,...,1) \in T^n_z$, Lemma 9 shows that $F(w) = f(g(w))$ is analytic on U^n_w and in a neighborhood of T^n_w. Thus $F(w) = \sum b_\gamma w^\gamma$ converges normally in neighborhood of T^n_w and, in particular, at $(1,...,1)$. This follows by the same Reinhardt domain argument used in the proof of Theorem 9.

To complete the proof of Theorem 11, it will suffice to show that

$$(5.1) \qquad \sum_{\alpha < \beta_{(v)}} a_\alpha g(w)^\alpha = \sum_{\gamma \leq \beta_{(v)}(s+1)} b_\gamma w^\gamma .$$

This is so because $\sum b_\gamma w^\gamma$ converges absolutely near $(1,\ldots,1)$ which implies the sequence of partial sums

$$\sum_{\gamma \leq \beta_{(v)}(s+1)} b_\gamma w^\gamma = \sum_{\alpha \leq \beta_{(v)}} a_\alpha g(w)^\alpha$$

converges uniformly near $(1,\ldots,1)$. Then since g is bijective near $(1,\ldots,1)$, $\sum_{|\alpha| \leq \beta_{(v)}} a_\alpha z^\alpha$ converges uniformly near $g((1,\ldots,1)) = (1,\ldots,1) = \xi$ to $f(z)$. This is the desired conclusion.

We must show (5.1). By Lemma 10 the following computation is valid for $w \in U^n$

$$\sum a_\alpha g(w)^\alpha = \sum_\alpha a_\alpha \sum_\gamma c_{\alpha\gamma} w^\gamma$$

$$= \sum_\gamma (\sum_\alpha a_\alpha c_{\alpha\gamma}) w^\gamma = \sum_\gamma b_\gamma w^\gamma .$$

$b_\gamma = \sum a_\alpha c_{\alpha\gamma}$ and the right hand side is a finite sum. Since both sides of (5.1) are polynomials, (5.1) valid for all $w \in U^n$ implies (5.1) valid for all $w \in C^n$.

If $\alpha \leq \beta_{(v)}$, $g(w)^\alpha = \sum c_{\alpha\gamma} w^\gamma$ has terms $c_{\alpha,\gamma} w^\gamma$ of order at most

$$(s+1)\alpha \leq (s+1)\beta_{(v)} \quad \text{by (4.2).}$$

Thus every term of $\sum_{\alpha \leq \beta_{(v)}} a_\alpha g(w)^\alpha$ contributes only to $\sum_{\gamma \leq (s+1)\beta_{(v)}} b_\gamma w^\gamma$.

If $\gamma \leq \beta_{(v)}(s+1)$, then for $a_\alpha g(w)^\alpha$ to make a contribution to $b_\gamma w^\gamma$, it is necessary that

$$\alpha_j s \leq \beta_{j,v}(s+1) \quad \text{for } j = 1...n.$$

Thus

$$\alpha_j \leq (1+1/s)\beta_{j,v} < q\beta_{j,v} < \beta'_{j,v} \quad \text{by (4.1)}$$

and the hypotheses of Theorem 11. Since $a_\alpha = 0$ if $\beta_{j,v} < \alpha_j < \beta'_{j,v}$ for some j, we must have $\alpha \leq \beta_{(v)}$ in order for $a_\alpha g(w)^\alpha$ make a contribution to $b_\gamma w^\gamma$.

Since the only terms of $\sum a_\alpha g(w)^\alpha$ that make contributions to $\sum_{\gamma \leq (s+1)\beta_{(v)}} b_\gamma w^\gamma$ are in $\sum_{\alpha \leq \beta_{(v)}} a_\alpha g(w)^\alpha$ and the terms of $\sum_{\alpha \leq \beta_{(v)}} a_\alpha g(w)^\alpha$ only make contributions to $\sum_{\gamma \leq \beta_{(v)}(s+1)} b_\gamma w^\gamma$, these two sums must be equal. Theorem 11 has been proven.

5.2 Overconvergence and Runge domains

In 1926 Ostrowski [23] proved the following theorem.

THEOREM. Let $f(z) = \sum a_n z^n$ be a power series converging in the unit disc. Let $(m_k, m'_k)_{k=1}^{\infty}$ be a sequence of pairs of integers such that $m'_k/m_k \to \infty$ and $m_k < m'_k \leq m_{k+1}$. Assume that $a_n = 0$ if $m_k < n < m'_k$ for some k. Let $p:S \to C$ be the Riemann surface (possibly many-sheeted) of $f(z)$. Then the sequence $s_{m_k}(z) = \sum_{n=0}^{m_n} a_n z^n$ converges uniformly to $f(z)$ on compact sets in S, and S is single-sheeted and simply-connected.

In this section we shall prove the following generalization.

THEOREM 12. Let $f(z) = \sum a_\alpha z^\alpha$ define a holomorphic function in the unit polydisc U^n. Let $(m_k, m_k')_{k=1}^{\infty}$ be a sequence of pairs of integers such that $m_k'/m_k \to \infty$ and $m_k < m_k' \leq m_{k+1}$. Assume that $a_\alpha = 0$ if $m_k < |\alpha| < m_k'$ for some k. Let $p:\Omega \to C^n$ be the (possibly many-sheeted) domain of existence of $f(z)$. Then the sequence $s_{m_k}(z) = \sum_{|\alpha| \leq m_k} a_\alpha z^\alpha$ converges uniformly to $f(z)$ on compact subsets in Ω.

COROLLARY 6. Ω is single-sheeted.

COROLLARY 7. Ω is a Runge domain.

Ω is a Runge domain if every function in Ω can be approximated by polynomials on compact sets. See, for instance, Hörmander [10], p. 52, for more information about Runge domains.

By $s_{m_k}(z)$ we mean $s_{m_k}(p(z))$.

We give an example. The series

$$f(z_1,z_2) = \sum_{k=1}^{\infty} \left(\frac{z_2(1-z_1)}{2} \right)^{k!}$$

has the domain of existence

$$\{(z_1,z_2) \mid |z_2(1-z_1)| < 2\}$$

$m_k = 2k!$ and $m_k' = (k+1)!$.

Our proof follows that of Ostrowski.

(The proof of Theorem 12 has been greatly shortened due to a suggestion of H. Alexander.)

Proof of Theorem 12. Let $\eta \in \Omega$. It will suffice to show $s_{m_k}(z)$ converges uniformly to $f(z)$ in a neighborhood of η since η is arbitrary.

Let D_1 be a subdomain of Ω such that $D_1 \subset\subset \Omega$, $\eta \in D_1$ and the origin is in D_1. ('$\subset\subset$' means 'relatively compact'.) (The function element at the origin is defined by $\sum a_\alpha z^\alpha$.)

Let D_2 be an open neighborhood of η such that $D_2 \subset\subset D_1$.

Let D_3 be a polydisc $P(0,\rho)$ such that $D_3 \subset\subset D_1$ and $\rho \leq 1/4$.

We need two estimates which we state as lemmas.

LEMMA 12. There exists $R \geq 1$ such that

$$|f(z) - s_{m_k}(z)| \leq C_1(m_k+2)^n R^{m_k} \quad \text{for} \quad z \in D_1$$

where C_1 depends on D_3, D_1 and f and R depends on D_1 and D_3.

LEMMA 13. There exists an absolute constant θ, $0 < \theta < 1$ such that

$$|f(z) - s_{m_k}(z)| \leq C_2(m_k' + n)^{n+1} \theta^{m_k'} \quad \text{for} \quad z \in D_3$$

where C_2 depends on f, D_3 and n.

Assuming these two estimates, we will apply the three domains theorem. (See Narasimhan [20], p. 130.) (Our sets do not satisfy the hypotheses in Narasimhan's statement of the theorem, but it is easy to see from the proof that the theorem is valid in this case.) So there exists τ, $0 \leq \tau \leq 1$ such that

$$|f(z) - s_{m_k}(z)| \leq \left(C_2(m_k'+n)^{n+1}\ \theta^{m_k'}\right)^{\tau} \left(C_1(m_k+2)^n R^{m_k}\right)^{1-\tau} \quad \text{for} \quad z \in D_2.$$

Since $0 < \theta < 1$ and $m_k'/m_k \to \infty$, we see that

$$\sup_{z \in D_2} |f(z) - s_{m_k}(z)| \to 0 \quad \text{as} \quad k \to \infty.$$

Thus the proof of Theorem 12 is complete once we have proven Lemmas 12 and 13.

Proof of Lemma 12. Recall that $D_3 = P(0,\rho)$. Applying Cauchy's inequalities we obtain

$$|a_\alpha| \leq M(f,\rho)\rho^{-|\alpha|} \tag{5.2}$$

where $M(f,\rho) = \sup_{z \in P(0,\rho)} |f(z)|$

Let $V = \sup_{z \in D_1} |f(z)|$. Since $M(f,\rho) \leq V$,

$$|a_\alpha| \leq V\rho^{-|\alpha|} \quad \text{by (5.2)}$$

Let

$$s = \sup_{z \in D_1} \left(\max_{1 \leq j \leq n} |z_j| \right).$$

Then

$$|f(z) - s_{m_k}(z)| \leq V + \sum_{|\alpha| \leq m_k} V\rho^{-|\alpha|}\, s^{|\alpha|}$$

If we set $R = \max(s/\rho, 1)$, then this quantity is no more than

$$VR^{|\alpha|} + \sum_{|\alpha| \leq m_k} VR^{|\alpha|} \leq V(m_k+2)^n R^{m_k}$$

The proof of Lemma 12 is complete.

Proof of Lemma 13. By hypothesis of Theorem 12

$$f(z) - s_{m_k}(z) = f(z) - s_{m_k'-1}(z). \tag{5.3}$$

By hypothesis f is analytic in U^n, and we have chosen $\rho \leq 1/4$. Therefore $P(0,2\rho) \subset\subset U^n$. By this discussion and (5.3)

$$f(z) - s_{m_k}(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_j|=2\rho} \frac{f(\zeta)}{\prod_1^n \zeta_j} \sum_{|\alpha| \geq m_k'} \frac{z^\alpha}{\zeta_\alpha} \, d\zeta$$

for $z \in D_3$.

The magnitude of this quantity is no more than

$$M(f,2\rho) \sum_{|\alpha| \geq m_k'} \left(\frac{1}{2}\right)^\alpha$$

where $M(f,2\rho) = \sup_{|\zeta_j|=2\rho} |f(\zeta)|$

$$= M(f,2\rho) \sum_{v=m_k'}^{\infty} \binom{v+n-1}{n-1} \left(\frac{1}{2}\right)^v$$

$$= M(f,2\rho) \sum_{j=0}^{n-1} \binom{m_k'+n}{j} \frac{\left(\frac{1}{2}\right)^{m_k'+n-j}}{\left(1-\frac{1}{2}\right)^{n-j-1}}$$

(See computation below.)

$$= M(f,2\rho) \sum_{0}^{n-1} \binom{m_k'+n}{j} \left(\frac{1}{2}\right)^{m_k'+1}$$

$$\leq M(f,2\rho) \cdot n \cdot (m_k'+n)^n \, (1/2)^{m_k'}$$

So

$$|f(z) - s_{m_k}(z)| \leq M(f,2\rho) n (m_k'+n)^n \left(\frac{1}{2}\right)^{m_k'}$$

for $z \in D_3$.

This is an estimate of the desired type.

As noted above, one step in the argument requires more clarification.

$$\sum_{v=0}^{\infty} \binom{v+n-1}{n-1} x^v = \frac{1}{(n-1)!} \frac{d^{n-1}}{dx^{n-1}} \left(\frac{1}{1-x} \right) = \frac{1}{(1-x)^n}$$

$$\sum_{v \geq m} \binom{v+n-1}{n-1} x^v = \frac{1}{(n-1)!} \frac{d^{n-1}}{dx^{n-1}} \left(\frac{x^{m+n}}{1-x} \right)$$

for $|x| < 1$.

We obtain this by using the formula for the remainder of the geometric series and allowing for the backwards shift caused by differentiation.

By applying the Leibniz formula to

$$\frac{d^{n-1}}{dx^{n-1}} \left(x^{m+n} \frac{1}{1-x} \right) \quad \text{and}$$

simplifying we have the formula

$$\sum_{v \geq m} \binom{v+n-1}{n-1} x^v = \sum_{j=0}^{n-1} \binom{m+n}{j} \frac{x^{m+n-j}}{(1-x)^{n-j-1}}$$

which is the one used in the estimate. Lemma 13 is proven.

It remains to prove the corollaries.

Proof of Corollary 6. Suppose Ω is not single-sheeted. Then there are points z' and z'' in Ω such that $p(z') = p(z'')$ and $f(z') \neq f(z'')$. By Theorem 12, $s_{m_k}(z') \to f(z')$ and $s_{m_k}(z'') \to f(z'')$. However, $s_{m_k}(z') = s_{m_k}(p(z')) = s_{m_k}(p(z'')) = s_{m_k}(z'')$. Thus $f(z') = f(z'')$. This is a contradiction. Ω is single-sheeted.

Proof of Corollary 7. Ω is the domain of uniform convergence (on compact sets) of a sequence of polynomials. By a result of Cartan and Thullen, it follows that Ω is a Runge domain. (See Vladimirov [31], p. 216.)

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