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TRANSONIC ROTATIONAL FLOW AROUND A CONVEX CORNER
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by
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ABSTRACT

TRANSONIC ROTATIONAL FLOW AROUND A CONVEX CORNER WITH APPLICATION TO A SUPERSONIC BOUNDARY LAYER

by

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A systematic method of solution for the expansion of inviscid rotational transonic flows around a sharp corner is presented. Solutions are obtained for a flow configuration corresponding to a supersonic boundary layer over a plate which ends at a sharp corner with a pressure drop of the order of the upstream static pressure. Viscous effects near the corner are relegated to a negligibly thin sublayer and the sonic line is assumed to intersect the corner. The solution scheme assumes a perturbation from a uniform sonic stream and treats the resulting equations.

The sonic line is found to intersect the corner at an acute angle with respect to the upstream wall. The angle varies between 0° and 90° depending on the rotation in the incoming flow. This dependence is given by the solution.

The effects of an axisymmetric geometry are systematically treated in the context of the solution and the highest order effects are presented.

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LIST OF SYMBOLS

a	sonic speed
$\tilde{A}_1$	constant to be determined by matching with a sublayer solution
$A, \tilde{A}$	$\hat{x}$ -dependent coefficients in the upstream boundary layer
a_0	constant determined by the boundary conditions
A_1, A_2	constants of given value
$B, \tilde{B}$	$\hat{x}$ -dependent coefficients in the upstream boundary layer
B_0	constant determined by the boundary conditions
b_0	constant determined by the boundary conditions
$\bar{C}_p$	specific heat at constant pressure (dimensional)
C_f	skin friction coefficient
$C_0, C_{1/2}, C_1$	$\hat{x}$ -dependent coefficients in the upstream boundary layer
κ	constant determined by the boundary conditions
D_1, D_2	constants determined by the boundary conditions
E_1	small parameters ordering the asymptotic expansion for U
$f_1(\eta)$	η -dependence of $\phi^{(n)}$
$f_2(\eta)$	η -dependence of $\phi^{(n)}$
$\tilde{F}$	rescaled value of ε_1
ε_1	scaled η -dependence of $\phi^{(n)}$
ε_2	scaled η -dependence of $\phi^{(n)}$
G	rescaled value of the homogenous part of ε_2
$\bar{h}_t$	total enthalpy
$\bar{h}$	enthalpy
h.o.t.	abbr. for "higher order terms"

i	index with integer values
$\bar{k}$	coefficient of heat conductivity
$\bar{K}_0, \bar{K}_{1/2}, \bar{K}_1$	$\hat{x}$ -dependent coefficients in the upstream boundary layer
$\tilde{K}_0, \tilde{K}_{1/2}$	$\hat{x}$ -dependent coefficients in the upstream boundary layer
K	constant to be determined by matching with a sublayer solution
$\bar{l}_0$	length of the forebody
$\bar{L}$	entropy length scale
M_e	external Mach number
$\bar{n}$	distance normal to a streamline
$\bar{P}_b$	base pressure
P	pressure
P_i	components of the asymptotic expansion for P
Pr	Prandtl number
R_b	velocity along the body
q	magnitude of the velocity
q_w	wall heat transfer
R_b	body radius of curvature
$\bar{R}$	lateral coordinate
R_0	radius of the base of the axisymmetric body
Re	Reynolds number
$\bar{s}$	distance along a streamline
s	phase plane variable
t	phase plane variable
T	temperature
T_w	wall temperature in the boundary layer
U	longitudinal velocity (X-component of velocity)

u_i	components of the asymptotic expansion for U
u'	second order longitudinal velocity
U_i	components of the asymptotic expansion of U for the axisymmetric problem
$\overline{u}_i$	components of the asymptotic expansion of U_1
V	lateral velocity (Y-component of velocity)
v_i	components of the asymptotic expansion for V
V_i	components of the asymptotic expansion of V for the axisymmetric problem
$\overline{v}_i$	components of the asymptotic expansion of V_1
X	longitudinal coordinate
$\overset{A}{X}$	longitudinal coordinate in the upstream boundary layer
x	stretched longitudinal coordinate in the transonic corner region
x_0, x_1, x_2	variables of the conical transformation
Y	lateral coordinate from the corner
$\overset{A}{Y}$	lateral coordinate in the upstream boundary layer
y	stretched lateral coordinate in the transonic corner region
z	modified similarity variable
α	cone half angle
β	constant with value $\frac{\gamma-1}{\gamma+1}$
γ	ratio of specific heats (assumed constant)
Γ	constant with value $\frac{\gamma}{\gamma-1}$
δ_0	incoming boundary layer thickness
δ	boundary layer thickness
Δ_i	small parameters ordering the U_i
ε	small parameter ordering the lateral extent of the transonic corner region

$\mathcal{J}_b$	vorticity at the body streamline
$\bar{\mathcal{J}}$	vorticity
$\mathcal{J}$	scaled similarity variable
η	similarity variable
η_{c+}	value of η associated with the left running characteristic from the corner
η_{c-}	value of η associated with the right running characteristic intersecting the corner
Θ	turning angle of the flow
λ_i	small parameter ordering the asymptotic expansion for P
μ	viscosity coefficient
μ	Mach angle
ν_i	parameters ordering the asymptotic expansion for V
$\mathcal{F}$	representation for $(\mathcal{F}-1)$
ρ	density
$\bar{\rho}_i$	components of the asymptotic expansion for the axisymmetric correction to the density
σ_i	small parameters ordering the V_i
τ	small parameter ordering the longitudinal extent of the transonic corner region
τ_w	wall shear stress
$\phi^{(1)}$	first order modified velocity potential
$\phi^{(2)}$	second order modified velocity potential
ϕ	turning angle for the Prandtl-Meyer solution
$\bar{\Psi}$	stream function (dimensional)
Ψ	modified dimensionless stream function
$\hat{\Psi}$	stream function in the upstream boundary layer (dimensionless)

$$\left(\phi_r^{(n)} = \frac{\partial \phi^{(n)}}{\partial x} \right)$$

ω_b	angle between the sonic line and the body downstream of the sonic line
ω	exponent in the viscosity variation relation
$\bar{R}$	gas constant
π_i	small parameters ordering the $\bar{u}_i$
$\tilde{\pi}_i$	small parameters ordering the $\bar{v}_i$
π_i	small parameters ordering the $\bar{\rho}_i$

Superscripts

$()^*$	critical value (value at sonic velocity)
$(\bar{})$	denotes a dimensional quantity (except with constants)
$(\wedge)$	denotes variables in the upstream boundary layer
$(\sim)$	denotes variables in the inner region
$(\bar{})$	denotes variables in the first axisymmetric correction

Subscripts

$()_0$	denotes the reference values in the upstream boundary layer
$()_e$	denotes the value in the external flow

Ordering symbols

$$f(\varepsilon) = \mathcal{O}(g(\varepsilon)) \quad \text{as } \varepsilon \rightarrow 0 \quad \text{if} \quad \lim_{\varepsilon \rightarrow 0} \frac{f(\varepsilon)}{g(\varepsilon)} < \infty$$

$$f(\varepsilon) = o(g(\varepsilon)) \quad \text{as } \varepsilon \rightarrow 0 \quad \text{if} \quad \lim_{\varepsilon \rightarrow 0} \frac{f(\varepsilon)}{g(\varepsilon)} = 0$$

I INTRODUCTION

Transonic rotational flow about a sharp convex corner arises in several common flow problems. These fall into two general categories: (1) inviscid flows expanding about a corner where the rotation in the flow is produced by a curved shock wave upstream of the corner and (2) supersonic boundary layers expanding about a sharp corner with the rotation produced by the action of viscosity in the boundary layer.

The obvious example of the first situation is the flight of an axisymmetric blunt body at supersonic speeds. The detached shock wave gives a finite entropy gradient normal to the wall of the body (Lin and Rubinov (1948)). The subsonic flow with finite entropy gradient produced by the shock, accelerates along the wall and typically expands to supersonic speeds around a sharp corner connecting the blunt front surface to its afterbody. The situation at the corner is that of a subsonic flow with finite entropy gradient at the wall, accelerating to sonic speed at the corner and then negotiating the sharp corner through a centered expansion wave while expanding to supersonic speeds.

The knowledge that the sonic line is a line of constant velocity, at least in isoenergetic flow, allows an expression to be written for the angle between the sonic line and the body surface. If ω_b is the angle between the sonic line and the body, q_b is the velocity along the body, $\mathcal{J}_b$ is the vorticity at the body and R_b is the body radius of curvature in a meridian plane, then

$$\tan \omega_b = \left[\frac{-\left(\frac{\partial \xi}{\partial s}\right)_b}{-f_b - \frac{\xi_b}{R_b}} \right] \quad (1.1)$$

(see Hayes and Probstein, 1966, p.393). Clearly ω_b depends on the entropy gradient at the wall through f_b . In the present configuration the direct application of (1.1) is difficult, since the proper R_b is unknown. However, it seems clear that the sonic line shape must depend on the upstream flow through the vorticity variation in the flow. To make a more definitive statement about the sonic line at a sharp corner requires the solution of the transonic flow region there, which is the subject of this work.

Of course, when speaking of inviscid flow about a sharp corner, one is really speaking of the inviscid aspects of a real flow in which the corner appears sharp to the scale of the inviscid flow. The analogy between this problem and that of a supersonic, laminar boundary layer negotiating a sharp corner must be developed.

Typical situations where one finds a supersonic boundary layer expanding around a sharp corner are the flow about the lip of an underexpanded supersonic nozzle and the flow about the rear edge of a cone or wedge in supersonic flight. In either case, the boundary layer turns a corner which has a large pressure decrease across it, such that the base pressure is less than the sonic pressure based on upstream conditions near the wall.

In the usual boundary layer theory the variations of the variables in the longitudinal direction are small compared with variations in the lateral direction. However, this is not true near a corner at which the pressure decrease is of order one. Experiments (Hama (1966)) on a wedge at supersonic velocity show that the difference between the static pressure several boundary layer thicknesses upstream of the corner and that several boundary layer thicknesses below the corner on the base, is of the order of the upstream static pressure. Further, the pressure along the upstream wall begins to fall a few boundary layer thicknesses ahead of the corner. The large pressure gradient at the corner has components in both the longitudinal and lateral directions which are of the same order, at least in some neighborhood of the corner. Hence, it seems that near the corner variations in the flow properties in the longitudinal direction are of the order of those in the lateral direction. Thus a corner region where both the longitudinal and lateral directions are scaled with respect to the incoming boundary layer thickness, $\delta_0 (\sim R_e^{-1/2} \bar{l}_0)$, where $\bar{l}_0$ is the length of the forebody), is defined in the model of Matveeva and Neyland (1967). (See also Olsson and Messiter (1969).) With this scaling the governing equations for this region are, to first order, the Euler equations. That is, in this corner region the large pressure drop causes the pressure and longitudinal velocity gradients to increase greatly from those in the undisturbed, upstream boundary layer, while the viscous shear stress remains of the same order.

Hence viscous effects are of lower order and the inviscid equations of motion approximately describe the flow.

To satisfy the no slip condition on the upstream wall it is necessary to consider a sublayer near the wall in which the convective and leading viscous terms in the Navier-Stokes equations have the same order. Thus the longitudinal extent of this region is of the order of a boundary layer thickness (the same as the inviscid corner region) while its lateral extent is of the order of $Re^{-\frac{1}{4}} \bar{l}_0$ (see Matveeva and Neyland (1967)). This ordering of the Navier-Stokes equations generally results in the Prandtl boundary layer equations. Notice that in this scaling the lateral velocity component at the outer edge must be of order $Re^{-\frac{1}{4}}$ which is considerably larger than the usual velocity component at the outer edge of a boundary layer.

The solutions for the longitudinal velocity component and vorticity in these two regions must match. Thus, following Matveeva and Neyland (1967) the matching condition is

$$\bar{U}_{\text{viscous sublayer}}(\tilde{x}, \tilde{y}) \Big|_{\tilde{y} \rightarrow \infty} \longrightarrow \bar{U}_{\text{inviscid corner region}}(x, y) \Big|_{y \rightarrow 0} \quad (1.2)$$

$$\bar{J}_{\text{viscous sublayer}}(\tilde{x}, \tilde{y}) \Big|_{\tilde{y} \rightarrow \infty} \longrightarrow \bar{J}_{\text{inviscid corner region}}(x, y) \Big|_{y \rightarrow 0} \quad (1.3)$$

where $\bar{U}$ is the longitudinal velocity component, $\bar{J}$ is the vorticity, $\tilde{x}, \tilde{y}$ are respectively the longitudinal and lateral coordinates scaled for the viscous sublayer and x, y are the corresponding coordinates scaled for the inviscid corner region.

The thin sublayer can be considered analogous to a boundary layer with favorable pressure gradient subject to the given matching requirements (Olsson and Messiter (1969)). Since the imposed pressure gradient is large, the sublayer displacement thickness decreases towards the corner. However, this description of the flow can be expected to break down on the order of a Stokes radius (R_e^{-1}) from the corner (Weinbaum (1968)). That is, very near the corner the pressure gradient is balanced by viscous forces as well as the inertia of the fluid. According to Weinbaum (1968), the velocity should be continuous on the Stokes length scale while the pressure and density should be singular at the origin to accommodate the rapid changes in these variables on a non-continuum, Knudsen length scale. To the scale of the incoming boundary layer thickness, the entire Stokes region at the corner should appear as a singular point. Thus the velocity as well as the pressure and density are singular at the corner for the inviscid corner region. Also, the sublayer is infinitesimally thin to this scale so the wall upstream of the corner appears flat to the inviscid corner region.

Qualitative evidence supporting this two layer model for the flow near the rear corner of a supersonic wedge was found by Hama (1966). His shadowgraphs of the flow clearly show two positive maxima of the second derivative of the density in the separated free shear layer downstream of the corner. He speculates that these should correspond to the outer edge of the inner shear layer and to the outer edge of the entire

shear layer.

Experiments show that the pressure at an incoming boundary layer thickness below the corner on the base wall is very close to the base pressure. In fact, it is slightly less than the pressure further down on the base wall. This indicates that even the flow very near the surface downstream of the corner must have obtained supersonic velocity. Hama's (1966) observations indicate that there is no appreciable pressure gradient across the free shear layer that separates from the corner region and continues downstream from the rear corner of the wedge. Thus the pressure above the free shear layer is essentially the base pressure. The sonic line cannot be expected to intersect the free shear layer downstream of the corner, since this would require a pressure gradient along the free shear layer where there is none. Hence the sonic line must be located within the inviscid corner region.

If one thinks of gradually accelerating the wedge to its final steady state velocity, the back pressure, $\bar{P}_b$, is gradually lowered to its value in the final flow configuration. As $\bar{P}_b$ is lowered the flow near the corner has increased velocity. Eventually $\bar{P}_b$ becomes low enough that the inviscid corner region concept becomes valid. If $\bar{P}_b$ is lowered further a point is reached where the velocity at the corner is sonic (to the scale of the inviscid corner region). Further lowering of the back pressure will no longer effect the flow in the inviscid corner region ahead of the leading characteristic from the corner, except through the sublayer. This effect is only

noticed in the sublayer a sublayer thickness, $Re^{-\frac{3}{4}}$, upstream of the corner which is essentially zero to the scale of the inviscid corner region. Hence one is led to hypothesize that the sonic line intersects the corner to the scale of the inviscid corner region (see figure (1.1)).

In the inviscid corner region the acceleration produced by the large longitudinal pressure gradient causes the subsonic stream tubes to contract and the supersonic stream tubes to expand. Initially the narrowing of the subsonic region dominates. The boundary layer upstream of the corner becomes thinner initially. The fluid close to the sublayer is accelerated more than that further away since $d\bar{u} \sim \bar{u}' d\bar{P}$ and $\bar{u}$ is smaller near the sublayer. Thus the velocity profile in the inviscid region is somewhat flattened although still rotational. This effect causes the second derivative of the velocity to become very large at the outer edge of the sublayer giving the appearance of slip velocity there. Of course, since the Euler equations are used in the inviscid corner region, a slip velocity along the boundary is necessary for the only boundary condition that can be met is that there be no flow normal to the boundary.

The preceding discussion indicates that the corner flow can be modeled as an inviscid flow turning a sharp corner with large pressure decrease. The sonic line should intersect the corner where a singularity of the velocity should exist. The incoming flow has finite vorticity at the wall and is partly subsonic and partly supersonic. The lateral velocity component at the

outer edge of the viscous sublayer is small, that is, of order $Re^{-\frac{1}{4}}$, so the proper boundary condition for the inviscid corner region is that there be no flow normal to the wall.

The flow downstream of the sonic line is turned sharply by a centered expansion (to the scale of the inviscid region) issuing from the supersonic edge of the corner. The curved characteristics of this expansion are tangent to the radial lines of a Prandtl-Meyer expansion arbitrarily close to the corner on the supersonic edge. Thus the solution in the inviscid transonic corner region must match with the Prandtl-Meyer solutions as the corner is approached from a position downstream of the corner.

In retrospect, both categories of flow delineated at the beginning of this discussion lead to the same flow model. A diagram of this model is shown in figure (1.2). The viscous sublayer is not visible to the scale of this diagram. The transonic region shown will be explained in more detail in Section II.

Vaglio-Laurin (1960) treated this problem originally. He assumed a small perturbation superposed on a uniform sonic stream in the x-direction (along the subsonic wall) and obtained first order equations that were irrotational. That is, at the subsonic upstream wall his first order equations give $\frac{\partial \bar{u}}{\partial y}|_{y=0} = 0$. Since the leading equations were irrotational, a velocity potential was introduced so that the governing equation became the usual transonic potential equation (see, for example, Guderley, 1962, p.30). He assumed a similarity solution for

the velocity potential of the form $\phi = y^{3p-2} f(\frac{x}{y^p})$ and discovered from an analysis of the phase plane of the resulting ordinary differential equation and from numerical calculations that $p = (5/4)$ gives the solution satisfying the boundary conditions. No other value of p achieved this. In his first order solution the sonic line was given by $(\frac{x}{y^4}) = \eta_{sonic}$ where η_{sonic} is a constant greater than zero. This result agrees with the sonic line obtained by Guderley (1948) from a hodograph solution for planar, irrotational flow over a corner. However, as pointed out by Friedman (1962), Vaglio-Laurin failed to include rotational effects in his higher order equations so his higher order solutions are incomplete. A more important question here is whether the first order equations should be irrotational or not.

C.C. Lin and S. I. Rubinov (1948) established that in the inviscid flow over a blunt body, the entropy gradient (vorticity gradient) generated by the curved shock preceding the body in supersonic flight vanishes with distance from the wall for a planar, two dimensional body, but is finite for an axisymmetric body. Thus Vaglio-Laurin's (1960) second order equations could be appropriate for axisymmetric flow, but not for the two dimensional case. Friedman (1962) pointed this out in his discussion, but still insisted on irrotational first order equations for both cases with proper account of the entropy gradient in second order terms. As a result, he too obtained Vaglio-Laurin's sonic line, at least to lowest order. Thus, again the sonic line at the wall only depended on the vorticity at the

wall through higher order effects. He also pointed out that the extra term in the substantial derivative due to the axisymmetric geometry generates a term in the solution of the form

$$\bar{U} = 0, \quad \bar{V} = -(r+1) \frac{\sin \alpha}{\bar{R}_0} \bar{Y}$$

where α is the half angle of the conical body and $\bar{R}_0$ is the radius of the base. However, the question of whether or not the first order equations should be irrotational is unresolved. In particular, the sonic line obtained to second order in these solutions is perpendicular to the subsonic, upstream wall regardless of vorticity effects. From analogy with the case where the sonic line intersects a wall that has finite, non-zero radius of curvature, the angle between the wall and the sonic line may be affected by vorticity at the wall even to first order. It seems that one must require the first order equations to contain rotational effects to accomplish this.

Adamson (1968) considered the problem of planar, inviscid, supersonic, rotational flow about a corner. His coordinates were the left running characteristics from the corner and the streamlines. He obtained asymptotic solutions valid near the leading left running characteristic from the corner and expansions valid near the corner. When the incoming velocity on the wall at the corner is sonic, a similarity solution valid in the transonic region very near the corner was found that matched with the other solutions farther away from the corner. When the similarity solution is expanded near the

leading characteristic it is found that the lateral velocity only has terms of the form

$$V \sim \sum_{i=1}^{\infty} y^{\frac{3}{2}i} \left(\frac{x}{y^{\frac{1}{2}}} \right)^{S_i}$$

where the exponent S_i becomes smaller with each successive term so each succeeding term is of lower order for x and y small. The leading term has $S_1 = 3$ and the first three terms are

$$V \sim -A \left(\frac{x^3}{y^3} \right) - Bx + Cx^{\frac{3}{2}}y^{\frac{3}{2}} + \dots$$

where A, B and C are positive constants. This expression agrees with the expansion Vaglio-Laurin (1960) gives for V valid in the limit as $y \rightarrow 0, (\frac{x}{y})$ fixed but small, with $p = (3/2)$. This seems to raise doubt as to whether Vaglio-Laurin's result that $p = (5/4)$ is the only possibility for a similarity solution of the transonic corner problem.

It will be shown in Section III that p can have either value, depending on whether the first order equations are considered irrotational or not.

II DERIVATION OF EQUATIONS

A. GOVERNING EQUATIONS

The dimensional governing equations for a two dimensional or axisymmetric, chemically inert, laminar, inviscid flow can be written as

(gasdynamic equation)

$$\bar{a}^2 \left[\frac{\partial \bar{V}}{\partial \bar{R}} + j \frac{\bar{V}}{\bar{R}} + \frac{\partial \bar{U}}{\partial \bar{X}} \right] = \left[\bar{U} \frac{\partial}{\partial \bar{X}} + \bar{V} \frac{\partial}{\partial \bar{R}} \right] \left(\frac{\bar{U}^2 + \bar{V}^2}{2} \right) \quad (2.1)$$

(Crocco equation)

$$\bar{U} \left[\frac{\partial \bar{V}}{\partial \bar{X}} - \frac{\partial \bar{U}}{\partial \bar{R}} \right] = \bar{T} \frac{\partial \bar{S}}{\partial \bar{R}} - \frac{\partial \bar{h}_t}{\partial \bar{R}} \quad (2.2)$$

$$\bar{V} \left[\frac{\partial \bar{V}}{\partial \bar{X}} - \frac{\partial \bar{U}}{\partial \bar{R}} \right] = \bar{T} \frac{\partial \bar{S}}{\partial \bar{X}} - \frac{\partial \bar{h}_t}{\partial \bar{X}} \quad (2.3)$$

(Bernoulli equation)

$$\bar{a}^2 = (\gamma - 1) \left[\bar{h}_t - \frac{\bar{U}^2 + \bar{V}^2}{2} \right] \quad (2.4)$$

where $\bar{X}$ is the coordinate along the upstream wall with corresponding velocity component $\bar{U}$, $\bar{R}$ is the normal coordinate with velocity component $\bar{V}$, $\bar{S}$ is the entropy and $\bar{h}_t$ is the total enthalpy as defined for a calorically perfect gas. The local speed of sound is denoted by $\bar{a}$. Here $j=0$ for the two dimensional case while $j=1$ for the axisymmetric case. With the substitution of equation (2.4) into equation (2.1) and the

introduction of the stream function defined by

$$\frac{\partial \bar{\Psi}}{\partial \bar{R}} = \bar{r} \bar{U} \bar{R}^j \quad (2.5.a)$$

$$\frac{\partial \bar{\Psi}}{\partial \bar{X}} = -\bar{r} \bar{V} \bar{R}^j, \quad (2.5.b)$$

the governing equations can be written as

$$\left[(\gamma-1) \bar{h}_t - \frac{\gamma+1}{2} \bar{U}^2 - \frac{\gamma-1}{2} \bar{V}^2 \right] \frac{\partial \bar{U}}{\partial \bar{X}} + \left[(\gamma-1) \bar{h}_t - \frac{\gamma-1}{2} \bar{U}^2 - \frac{\gamma+1}{2} \bar{V}^2 \right] \frac{\partial \bar{V}}{\partial \bar{R}}$$

$$- \bar{U} \bar{V} \left[\frac{\partial \bar{V}}{\partial \bar{X}} + \frac{\partial \bar{U}}{\partial \bar{R}} \right] + j(\gamma-1) \left[\bar{h}_t - \frac{\bar{U}^2 + \bar{V}^2}{2} \right] \frac{\bar{V}}{\bar{R}} = 0 \quad (2.6)$$

$$\bar{U} \left[\frac{\partial \bar{V}}{\partial \bar{X}} - \frac{\partial \bar{U}}{\partial \bar{R}} \right] = \left[\bar{T} \frac{d\bar{S}}{d\bar{\Psi}} - \frac{d\bar{h}_t}{d\bar{\Psi}} \right] \frac{\partial \bar{\Psi}}{\partial \bar{R}} \quad (2.7)$$

where the fact that entropy and total enthalpy depend only on the stream function in inviscid flow has been utilized.

Equation (2.3) adds no necessary information over that contained in equation (2.7). It is convenient at this juncture to transform the coordinates to the corner in the axisymmetrical case by setting

$$\bar{R} = j \bar{R}_0 + \bar{Y}, \quad (2.8)$$

where $\bar{R}_0$ is the body radius in the axisymmetric flow. Note that $\bar{R}_0$ is a real physical length in the problem which is not present in the planar case. The only remaining physical length in the problem is that associated with the incoming entropy gradient, that is, the length over which the entropy change is of order one. This would correspond to the thickness of the incoming boundary layer for the rear facing step problem. Choosing the latter entropy length for the length scale, dimensionless variables are defined by

$$U = \frac{\bar{u}}{\bar{a}^*} \quad (2.9.a)$$

$$V = \frac{\bar{v}}{\bar{a}^*} \quad (2.9.b)$$

$$\frac{1}{2} \left(\frac{\gamma+1}{\gamma-1} \right) \bar{a}^{*2} = \bar{h}_t \quad (2.9.c)$$

$$T = \frac{\bar{T}}{\bar{T}^*} = \frac{\bar{T}}{\left(\frac{\bar{a}^{*2}}{\gamma \bar{R}} \right)} = a^2 \quad (2.9.d)$$

$$S = \frac{\bar{S}}{\bar{R}} \quad (2.9.e)$$

$$a = \frac{\bar{a}}{\bar{a}^*} \quad (2.9.f)$$

$$X = \frac{\bar{X}}{\bar{L}} \quad ; \quad Y = \frac{\bar{Y}}{\bar{L}} \quad (2.9.g)$$

$$R_o = \frac{\bar{R}_o}{\bar{L}} \quad (2.9.h)$$

where $\bar{L}$ denotes the entropy length for the problem and superscript asterisk indicates conditions at the sonic speed. Observe that if the total enthalpy varies from streamline to streamline, the critical speed of sound, $\bar{a}^*$, varies similarly (see (2.9.c)). The velocities and state variables are made dimensionless with respect to a sliding scale in this scheme. Also note that R_o is a parameter for the axisymmetric problem giving the ratio of the body diameter to the entropy length.

In order to obtain the desired form of the governing equations in terms of this variable scaling, it must be recognized that the critical speed depends only upon the stream function so that

$$\frac{\partial \bar{a}^*}{\partial \bar{X}} = \frac{d \bar{a}^*}{d \bar{\Psi}} \frac{\partial \bar{\Psi}}{\partial \bar{X}} = - \bar{r} \bar{V} (\bar{R}_o + \bar{Y})^j \frac{d \bar{a}^*}{d \bar{\Psi}} \quad (2.10.a)$$

and

$$\frac{\partial \bar{a}^*}{\partial \bar{Y}} = \frac{d \bar{a}^*}{d \bar{\Psi}} \frac{\partial \bar{\Psi}}{\partial \bar{Y}} = \bar{r} \bar{U} (\bar{R}_o + \bar{Y})^j \frac{d \bar{a}^*}{d \bar{\Psi}} \quad (2.10.b)$$

The gasdynamic equation (2.6) becomes

$$\begin{aligned} & \left[\frac{\gamma+1}{2} (1-u^2) - \frac{\gamma-1}{2} V^2 \right] \frac{\partial U}{\partial \bar{X}} + \left[\frac{\gamma+1}{2} (1-V^2) - \frac{\gamma-1}{2} U^2 \right] \frac{\partial V}{\partial \bar{Y}} \\ & - \left[\frac{\partial V}{\partial \bar{X}} + \frac{\partial U}{\partial \bar{Y}} \right] UV + j \frac{a^2 V}{R_o + \bar{Y}} = 0 \end{aligned} \quad (2.11)$$

while the Crocco equation (2.7) becomes

$$\frac{\partial V}{\partial X} - \frac{\partial U}{\partial Y} = \frac{\bar{\rho} a^2}{\gamma-1} \left(1 + j \frac{Y}{R_0}\right) \left[\frac{\gamma-1}{\gamma} \frac{dS}{d\bar{\Psi}} - \frac{1}{\bar{a}^{*2}} \frac{d\bar{a}^{*2}}{d\bar{\Psi}} \right] \quad (2.12)$$

where a modified dimensionless stream function had been introduced by

$$d\bar{\Psi} = \left[\bar{\rho}^* \bar{a}^* \left[\bar{R}_0^j \right]^{-1} \right] d\bar{\Psi} \quad (2.13)$$

Note that the expression in square brackets on the right hand side of equation (2.12) is a function of the stream function alone. That is, since dimensionless entropy, critical speed, critical density, etc. are constant along a streamline in inviscid flow, this bracketed term is a fixed function of the stream function throughout the inviscid corner region. This function is assumed to be given as the upstream boundary condition. That is, it is given by solution of the laminar, compressible boundary layer leading up to the corner. Observe in passing that the coordinate system in the axisymmetric problem can be inclined at angle α to the axis of symmetry by changing the axisymmetric term in the gasdynamic equation (2.11) to

$$j a^2 \left[\frac{U \sin \alpha + V \cos \alpha}{R_0 + Y \cos \alpha + X \sin \alpha} \right] \quad (2.14)$$

where the coordinates are as shown in figure (2.1).

B. TRANSONIC APPROXIMATION

The assumption is made that, in the neighborhood of the sonic corner, the velocities can be considered as perturbations from a parallel sonic stream. Since the incoming flow is rotational and accelerating, the region in which this approximation can be made is strictly limited (see figure (1.2)). The geometric extent of this transonic corner region is intimately related to the allowed perturbation of the sonic stream and must be obtained as part of the solution. To this end it is necessary to introduce stretched coordinates as follows

$$\begin{aligned}x &= \frac{X}{\tau} = \mathcal{O}(1) \\y &= \frac{Y}{\varepsilon} = \mathcal{O}(1) \\ \tau, \varepsilon &\ll 1\end{aligned}\tag{2.15}$$

where τ and ε are related to the magnitude of the velocity perturbation and respectively order the longitudinal and lateral extent of the transonic region under consideration. The velocity components are assumed to be of the form

$$u = 1 + \sum_{j=1}^{\infty} E_j(\varepsilon) u_j(x, y)\tag{2.16}$$

$$v = \sum_{j=1}^{\infty} v_j(\varepsilon) v_j(x, y)\tag{2.17}$$

where

$$E_i, v_i \ll 1$$

and
$$E_{i+1} = o(E_i) \quad , \quad v_{i+1} = o(v_i) \quad (2.18)$$

while
$$u_i(x, y) = O(1) \quad \quad v_i(x, y) = O(1) \quad .$$

Before substituting these asymptotic representations for the velocity components into the governing equations, an expression for the entropy and total enthalpy variation in the Crocco equation must be found. It is this term that sets the ordering of the equations and is of paramount importance in determining the solution. In particular, the asymptotic representation for the transonic corner region must be found so that various orders of the entropy and total enthalpy variation can be appropriately balanced against suitable terms in the velocity expansions. This function of the streamlines is assumed given as the upstream boundary condition. Thus, in order to proceed farther, the entropy and total enthalpy variation in a compressible boundary layer must be examined.

C. PROPERTIES OF A COMPRESSIBLE BOUNDARY LAYER NEAR A WALL

In order to calculate the incoming entropy and total enthalpy distribution, it is necessary to consider the laminar, supersonic boundary layer ahead of the corner. To this end let dimensionless variables in the boundary layer be defined by

$$\hat{u} = \frac{\bar{u}}{\bar{a}_0^*} \quad (2.19.a)$$

$$\hat{v} = \frac{\bar{v}}{\delta_0 \bar{a}_0^*} \quad (2.19.b)$$

$$\hat{p} = \frac{\bar{p}}{\bar{p}_0^* \bar{a}_0^{*2}} \quad (2.19.c)$$

$$\hat{x} = \frac{\bar{x}}{\bar{l}_0} ; \quad \hat{y} = \frac{\bar{y}}{\delta_0 \bar{l}_0} ; \quad \hat{\mu} = \frac{\bar{\mu}}{\bar{a}_0^* \bar{p}_0^* \bar{l}_0 \delta_0^2} \quad (2.19.d)$$

where $\bar{a}_0^*$ is the reference critical speed with associated density $\bar{p}_0^*$ and temperature $\bar{T}_0^*$ and

$$\delta_0 = \left[\frac{\bar{p}_0^* \bar{a}_0^* \bar{l}_0}{\bar{\mu}_0} \right]^{-\frac{1}{2}} = \frac{1}{\sqrt{Re_0^*}}$$

is the order of the boundary layer thickness. Here $\bar{\mu}_0$ is the reference coefficient of viscosity defined by $\bar{\mu} = \bar{\mu}_0$ when

$$\bar{T} = \bar{T}_r = \bar{a}_0^{*2} \frac{\gamma-1}{\gamma} \bar{R}^{-1} = (\gamma-1) \bar{T}_0^*$$

and $\bar{l}_0$ is the reference length taken to be the length from the start of the boundary layer on the forebody to the corner (see figure (2.2)). The dimensionless temperature and enthalpy have been defined by

$$\hat{T} = \frac{\bar{T}}{\left(\frac{\bar{a}_0^*{}^2}{\Gamma \bar{R}} \right)} \quad (2.19.g)$$

with

$$\Gamma = \frac{\gamma}{\gamma-1}$$

and

$$\hat{h} = \frac{\bar{h}}{\bar{a}_0^*{}^2} \cdot \quad (2.19.h)$$

It is assumed that γ , $\bar{R}$ and P_r are constants while $\bar{\mu}$ and $\bar{k}$ are considered functions of temperature.

If the boundary layer thickness, $\bar{L}$, is small compared to $\bar{R}_0$, then the Mangler transformation (Young (1953), p.375) can be used to reduce the axisymmetric boundary layer equations to the planar boundary layer equations. If the body is a cone rather than a wedge, the solution for the wedge holds for the cone with the transformation

$$\left[\hat{\psi}(\hat{x}^3, \sqrt{3} \hat{x} \hat{y}) \right]_{\substack{\text{axisymmetric} \\ \text{stream function} \\ \text{for cone}}} = \left[\sqrt{3} \hat{\psi}(\hat{x}, \hat{y}) \right]_{\substack{\text{planar} \\ \text{stream function} \\ \text{for wedge}}} \cdot$$

For a given value of $\hat{x}$, the $\hat{y}$ dependence is the same in either case. In particular, at the entrance to the inviscid corner region at the rear corner of the cone or wedge, $\hat{x} = 1$, so the only difference in the stream functions are the factors of the square root of three. The functional dependence is the same. For this reason it is sufficient to consider only the planar boundary layer equations with the restriction that $\bar{L} / \bar{R}_0 \ll 1$.

In these variables the boundary layer equations for two dimensional flow become

$$\frac{\partial}{\partial \hat{x}} (\hat{\rho} \hat{u}) + \frac{\partial}{\partial \hat{y}} (\hat{\rho} \hat{v}) = 0 \quad (2.20)$$

$$\begin{aligned} \hat{\rho} \left(\hat{u} \frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{u}}{\partial \hat{y}} \right) &= - \frac{d\hat{P}}{d\hat{x}} + \frac{\partial}{\partial \hat{y}} \left(\hat{\mu} \frac{\partial \hat{u}}{\partial \hat{y}} \right) \\ &= - \frac{d\hat{P}}{d\hat{x}} + \hat{T}^\omega \left\{ \frac{\partial^2 \hat{u}}{\partial \hat{y}^2} + \frac{\omega}{\hat{T}} \frac{\partial \hat{T}}{\partial \hat{y}} \frac{\partial \hat{u}}{\partial \hat{y}} \right\} \end{aligned} \quad (2.21)$$

$$\hat{\rho} \left(\hat{u} \frac{\partial \hat{h}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{h}}{\partial \hat{y}} \right) = \hat{u} \frac{d\hat{P}}{d\hat{x}} + \frac{\partial}{\partial \hat{y}} \left(\frac{\hat{\mu}}{Pr} \frac{\partial \hat{h}}{\partial \hat{y}} \right) + \hat{\mu} \left(\frac{\partial \hat{u}}{\partial \hat{y}} \right)^2 \quad (2.22)$$

where $Pr = \frac{\bar{\mu} \bar{C}_p}{\bar{k}} = \text{Prandtl number}$. In addition, the equation of state and viscosity variation with temperature are given by

$$\Gamma \hat{P} = \hat{\rho} \hat{T} \quad (2.23)$$

$$\hat{\mu} = \hat{T}^\omega \quad \left(\frac{1}{2} < \omega \leq 1 \right). \quad (2.24)$$

Notice that in these variables the relation between the enthalpy and the temperature is given by

$$\hat{h} = \hat{T} . \quad (2.25)$$

For free stream velocities greater than twice the free stream sonic speed, sonic velocity occurs within 0.3δ of the wall (Schlichting, 1960, p.311). Thus expansions of the boundary layer variables in terms of the vertical coordinate seem appropriate for the entire subsonic portion of the incoming boundary layer. In the current model it is assumed that even streamlines very near the wall in the incoming boundary layer are accelerated to sonic velocity at the corner. Thus streamlines entering the inviscid transonic corner region are those that pass through the lower region of the subsonic portion of the incoming boundary layer. Certainly, then, expansions in y will be sufficient to describe these streamlines in the upstream boundary layer. Thus let

$$\hat{u} = \sum_{i=0}^{\infty} \hat{u}_i(\hat{x}) \hat{y}^{\alpha_i} \quad (2.26.a)$$

$$\hat{v} = \sum_{i=0}^{\infty} \hat{v}_i(\hat{x}) \hat{y}^{\alpha_i} \quad (2.26.b)$$

$$\hat{T} = \sum_{i=0}^{\infty} \hat{T}_i(\hat{x}) \hat{y}^{\beta_i} \quad (2.26.c)$$

where certainly $\beta_0 = 0$. After substitution of these expansions into the boundary layer equations (2.20), (2.21) and (2.22) with cognizance that the imposed pressure gradient should be balanced by viscous forces to lowest order at the wall, the expansions (2.26) are found to be power series so that

$$\hat{u} = \sum_{j=1}^{\infty} \hat{u}_j(\hat{x}) \hat{y}^j \quad (2.27)$$

$$\hat{v} = \sum_{j=2}^{\infty} \hat{v}_j(\hat{x}) \hat{y}^j \quad (2.28)$$

$$\hat{T} = \sum_{j=0}^{\infty} \hat{T}_j(\hat{x}) \hat{y}^j \quad (2.29)$$

The equations governing the functions of $\hat{x}$ are

$$\text{(continuity)} \quad \frac{\partial}{\partial \hat{x}} \left[\frac{\hat{P}}{\hat{T}_0} \hat{u}_1 \right] + 2 \frac{\hat{P}}{\hat{T}_0} \hat{v}_2 = 0 \quad (2.30)$$

$$\text{(momentum)} \quad 2 \hat{u}_2 \frac{\hat{T}_0}{\hat{T}_0} + \omega \hat{u}_1 \frac{\hat{T}_0}{\hat{T}_0} = \frac{d\hat{P}}{d\hat{x}} \quad (2.31)$$

$$\hat{u}_3 = -\frac{1}{6} \omega(\omega-1) \hat{u}_1 \frac{\hat{T}_1^2}{\hat{T}_0^2} - \frac{1}{3} \omega \hat{u}_1 \frac{\hat{T}_2}{\hat{T}_0} - \frac{2}{3} \omega \hat{u}_2 \frac{\hat{T}_1}{\hat{T}_0} \quad (2.32)$$

$$\text{(energy)} \quad 2 \hat{T}_2 + \omega \frac{\hat{T}_1^2}{\hat{T}_0} = -P_r \hat{U}_1^2 \quad (2.33)$$

$$\begin{aligned} \hat{T}_3 = \frac{1}{6} \frac{P_r}{\hat{T}_0^2} \left[P_r \hat{U}_1 \frac{1}{\hat{T}_0} \frac{\partial \hat{T}_0}{\partial \hat{x}} - \hat{U}_1 \frac{dP}{d\hat{x}} - 4 \hat{U}_1 \hat{U}_2 \frac{\hat{T}_0}{\hat{T}_0} - \omega \hat{U}_1^2 \frac{\hat{T}_1}{\hat{T}_0} \frac{\hat{T}_0}{\hat{T}_0} \right] \\ - \omega \frac{\hat{T}_1 \hat{T}_2}{\hat{T}_0} - \frac{\omega(\omega-1)}{6} \frac{\hat{T}_1^3}{\hat{T}_0^2} . \end{aligned} \quad (2.34)$$

Of course, explicit solutions for the $\hat{x}$ dependent coefficients cannot be obtained directly; however, it is sufficient to know the form of the expansions. Observe from (2.31) that if there is no heat transfer ($\frac{\partial \hat{T}}{\partial y} = \hat{T}_1 = 0$) at the wall (or if the viscosity is independent of the temperature) and if there is in addition no pressure gradient, then $\hat{u}_2 = 0$.

For convenience the heat transfer and shear stress at the wall can be written in terms of the temperature gradient at the wall ($\hat{T}_1$), the velocity gradient at the wall ($\hat{u}_1$) and the wall temperature ($\hat{T}_0 = \hat{T}_w$),

$$\tau_w = \hat{T}_w^\omega \hat{u}_1 \quad (2.35)$$

$$q_w = - \hat{T}_w^\omega \frac{\hat{T}_1}{P_r} , \quad (2.36)$$

where $\bar{\tau}_w$ and $\bar{q}_w$ have been made dimensionless and of order one

by dividing by $(\delta \bar{a}_0^{*2} \bar{p}_0^*)$ and $(\delta \bar{a}_0^{*3} \bar{p}_0^*)$ respectively.

With these substitutions the energy and momentum equations give

$$\hat{T}_2 = -\frac{1}{2} R \hat{T}_w^{-2\omega} \left[\tau_w^2 + \omega \frac{1}{\hat{T}_w} R q_w^2 \right] \quad (2.37.a)$$

$$\hat{U}_2 = \frac{1}{2} \hat{T}_w^{-\omega} \left[\frac{d\hat{P}}{d\hat{x}} + \omega \frac{1}{\hat{T}_w^{\omega+1}} R q_w \tau_w \right], \quad (2.37.b)$$

and the continuity equation gives

$$\hat{V}_2 = -\frac{1}{2} \frac{\partial}{\partial \hat{x}} \left(\frac{\tau_w}{\hat{T}_w} \right) - \frac{1}{2} \frac{\tau_w}{\hat{T}_w} \left[\frac{1}{\hat{P}} \frac{\partial \hat{P}}{\partial \hat{x}} - \frac{1}{\hat{T}_w} \frac{\partial \hat{T}_w}{\partial \hat{x}} \right]. \quad (2.37.c)$$

Also the third order velocity is

$$\begin{aligned} \hat{U}_3 = & \frac{1}{6} \omega (1+2\omega) \hat{T}_w^{-(2+3\omega)} \tau_w R^2 q_w^2 \\ & + \frac{1}{6} \omega \hat{T}_w^{-(1+3\omega)} R \tau_w^3 \\ & + \frac{1}{3} \omega \hat{T}_w^{-(1+2\omega)} R q_w \frac{dP}{d\hat{x}}. \end{aligned}$$

(2.38)

Our purpose here is to calculate the initial entropy and total enthalpy distributions for the beginning of the transonic corner region. As the boundary layer begins to feel the pressure gradient from the corner and accelerate, essentially inviscidly, entropy and total enthalpy distributions become frozen to the streamlines. To find these distributions the stream function must be introduced:

$$\frac{\partial \bar{\Psi}}{\partial \bar{Y}} = \bar{\rho} \bar{U} \quad \frac{\partial \bar{\Psi}}{\partial \bar{X}} = -\bar{\rho} \bar{V} \quad (2.39)$$

or with
$$\hat{\Psi} = \frac{\bar{\Psi}}{\bar{\rho}_0^* \bar{a}_0^* \bar{l}_0 \delta_0}$$

as the appropriately stretched stream function, the definition becomes

$$\frac{\partial \hat{\Psi}}{\partial \hat{Y}} = \hat{\rho} \hat{U} \quad , \quad \frac{\partial \hat{\Psi}}{\partial \hat{X}} = -\hat{\rho} \hat{V} \quad (2.40)$$

To write $\hat{\Psi}$ as a function of the stretched vertical coordinate, use

$$\hat{\rho} \hat{U} = \Gamma \frac{\hat{P}}{\hat{T}} \hat{U} = \Gamma \hat{P} \frac{(\hat{U}, \hat{Y} + \hat{U}_1 \hat{Y}^2 + \dots)}{(\hat{T}_0 + \hat{T}_1 \hat{Y} + \hat{T}_2 \hat{Y}^2 + \dots)} \quad (2.41)$$

so that

$$\frac{\partial \hat{\Psi}}{\partial \hat{Y}} = \Gamma \frac{\hat{P}}{\hat{T}_0} \left\{ \hat{U}_1 \hat{Y} + \left[\hat{U}_2 - \frac{\hat{T}_1}{\hat{T}_0} \hat{U}_1 \right] \hat{Y}^2 + \left[\hat{U}_3 - \frac{\hat{T}_1}{\hat{T}_0} (\hat{U}_2 - \hat{U}_1 \frac{\hat{T}_1}{\hat{T}_0}) - \hat{U}_1 \frac{\hat{T}_2}{\hat{T}_0} \right] \hat{Y}^3 + \dots \right\} \quad (2.42)$$

or

$$\hat{\psi} = \frac{1}{2} \Gamma \frac{\hat{P}}{\hat{T}_0} \hat{u}_1 \hat{y}^2 + \frac{1}{3} \Gamma \frac{\hat{P}}{\hat{T}_0} \left(\hat{u}_2 - \frac{\hat{T}_1}{\hat{T}_0} \hat{u}_1 \right) \hat{y}^3 + \dots \quad (2.43)$$

where $\hat{\psi} = 0$ along $\hat{y} = 0$ has been chosen.

Inversion of this series gives

$$\hat{y} = \left(\frac{1}{2} \Gamma \frac{\hat{P}}{\hat{T}_0} \hat{u}_1 \right)^{-\frac{1}{2}} \hat{\psi}^{\frac{1}{2}} - \frac{2}{3} \left(\frac{\hat{u}_2 - \frac{\hat{T}_1}{\hat{T}_0} \hat{u}_1}{\Gamma \frac{\hat{P}}{\hat{T}_0} \hat{u}_1^2} \right) \hat{\psi} + \mathcal{O}(\hat{\psi}^{\frac{3}{2}}) \quad (2.44)$$

$$= A^{-\frac{1}{2}} \hat{\psi}^{\frac{1}{2}} - \frac{2}{3} B \hat{\psi} + \mathcal{O}(\hat{\psi}^{\frac{3}{2}})$$

where $\hat{y}$ and $\hat{\psi}$ are both stretched by the suitably defined boundary layer thickness. Since the expansions are in terms of $\hat{x}$ and $\hat{y}$, a relation between derivatives with respect to the stream function and derivatives with respect to $\hat{x}$ and $\hat{y}$ must be found. Generally it can be shown that

$$\frac{\partial}{\partial \bar{\Psi}} = - \frac{1}{\bar{P}} \frac{1}{\bar{u}^2 + \bar{v}^2} \left[\bar{v} \frac{\partial}{\partial \bar{x}} - \bar{u} \frac{\partial}{\partial \bar{y}} \right]$$

or

$$\frac{\partial}{\partial \hat{\psi}} = - \frac{1}{\hat{P}} \frac{1}{\hat{u}^2 + \delta_o^2 \hat{v}^2} \left[\delta_o^2 \hat{v} \frac{\partial}{\partial \hat{x}} - \hat{u} \frac{\partial}{\partial \hat{y}} \right] \quad (2.45)$$

Substitution of the expansions for $\hat{u}$ and $\hat{v}$ and the observation that δ_o^2 terms properly belong in a second order boundary layer solution gives

$$\begin{aligned}
\frac{\partial}{\partial \hat{\psi}} = & \left(\Gamma \frac{\hat{P}}{\hat{T}_0} \hat{u}_1 \right)^{-1} \left\{ \frac{1}{\hat{y}} \frac{\partial}{\partial \hat{y}} + \left[\frac{\hat{T}_1}{\hat{T}_0} - \frac{\hat{u}_2}{\hat{u}_1} \right] \frac{\partial}{\partial \hat{y}} \right. \\
& + \left[\frac{\hat{u}_2^2}{\hat{u}_1^2} - \frac{\hat{u}_3}{\hat{u}_1} - \frac{\hat{u}_2}{\hat{u}_1} \frac{\hat{T}_1}{\hat{T}_0} + \frac{\hat{T}_2}{\hat{T}_0} \right] \hat{y} \frac{\partial}{\partial \hat{y}} \\
& \left. + \left[2 \frac{\hat{u}_2}{\hat{u}_1} \frac{\hat{u}_3}{\hat{u}_1} - \frac{\hat{u}_4}{\hat{u}_1} - \frac{\hat{u}_2^3}{\hat{u}_1^3} - \frac{\hat{u}_2}{\hat{u}_1} \frac{\hat{T}_1}{\hat{T}_0} + \frac{\hat{u}_2^2}{\hat{u}_1^2} \frac{\hat{T}_1}{\hat{T}_0} - \frac{\hat{u}_2}{\hat{u}_1} \frac{\hat{T}_2}{\hat{T}_0} + \frac{\hat{T}_3}{\hat{T}_0} \right] \hat{y}^2 \frac{\partial}{\partial \hat{y}} + O(\hat{y}^3) \right\}. \quad (2.46)
\end{aligned}$$

Observe that the derivative across the streamlines of any quantity that approaches zero at the wall slower than $\hat{y}^2$ has a singularity at the wall. This singularity is not a problem with the physical variables, but is introduced by the fact that $\hat{y}$ is not analytic in $\hat{\psi}$. It will be shown that the entropy and total enthalpy variations have the same singular behavior at the wall so their difference displays no singularity.

The thermodynamic relations are now used to find the entropy and total enthalpy. From thermodynamics

$$\bar{T} d\bar{S} = d\bar{h} - \frac{1}{\bar{p}} d\bar{P}.$$

Let

$$\hat{S} = \frac{\bar{S}}{\frac{\bar{y}}{\bar{r}-1} \bar{R}} = \frac{\bar{S}}{\bar{C}_P};$$

then

$$d\hat{S} = \frac{d\hat{T}}{\hat{T}} - \frac{d\hat{P}}{\hat{P}} \quad (2.47)$$

in dimensionless variables. The dimensionless total enthalpy is given by

$$d\hat{h}_t = d\hat{T} + d\left(\frac{\hat{u}^2}{2}\right) \quad (2.48)$$

The temperature variation $\frac{\partial \hat{T}}{\partial \hat{\psi}}$ displays singular behavior at the wall since temperature is not analytic in the stream function. However, the same singularity occurs in the total enthalpy variation so that the function required for the Crocco equation of part B. has no singularity. Here two cases are considered: homoeenergetic flow where $d\hat{h}_t=0$ and the more general case where both $\hat{S}$ and $\hat{h}_t$ are allowed to vary.

First consider homoeenergetic flow. Here $d\hat{h}_t=0$ so the temperature differential is given by

$$d\hat{T} = - d\left(\frac{\hat{u}^2}{2}\right) \quad (2.49)$$

This relation requires that $\hat{T}_1=0$, that is, there can be no heat transfer at the wall if the total enthalpy is to be constant. Substitution in equation (2.47) gives

$$\frac{\partial \hat{S}}{\partial \hat{\psi}} = - \frac{1}{\hat{T}} \frac{\partial \left(\frac{\hat{u}^2}{2}\right)}{\partial \hat{\psi}} - \frac{1}{\hat{P}} \frac{\partial \hat{P}}{\partial \hat{\psi}} \quad (2.50)$$

Insertion of the appropriate expansions into (2.50) gives

$$\frac{\partial \hat{S}}{\partial \hat{\psi}} = - \left(\Gamma \frac{\hat{P}}{\hat{T}_0} \hat{u}_1 \right)^{-1} \left\{ \frac{\hat{u}_1^2}{\hat{T}_0} + \left[2 \frac{\hat{u}_1 \hat{u}_2}{\hat{T}_0} \right] \hat{y} + \left[3 \frac{\hat{u}_1 \hat{u}_3}{\hat{T}_0} \right] \hat{y}^2 + O(\hat{y}^3) \right\} \quad (2.51)$$

Thus, since $\hat{u}_1 > 0$, the entropy gradient at the wall is negative;

$$\left(\frac{\partial \hat{S}}{\partial \hat{\psi}} \right)_{\hat{y}=0} < 0 \quad (2.52)$$

Equation (2.51) can be written in terms of the shear stress at the wall (τ_w) as

$$\frac{\partial \hat{S}}{\partial \hat{\psi}} = -\frac{1}{\Gamma \hat{P}} \left\{ \frac{\tau_w}{\hat{T}_w} + \frac{1}{\hat{T}_w} \frac{d\hat{P}}{d\hat{x}} \hat{y} + \frac{\omega}{2} \frac{P_r \tau_w^3}{\hat{T}_w^{3\omega+1}} \hat{y}^2 + O(\hat{y}^3) \right\} \quad (2.53)$$

or ultimately in terms of the stream function as

$$\begin{aligned} \frac{\partial \hat{S}}{\partial \hat{\psi}} = & -\frac{1}{\Gamma \hat{P}} \left\{ \frac{\tau_w}{\hat{T}_w} + \frac{1}{\hat{T}_w} \frac{d\hat{P}}{d\hat{x}} \left(\frac{1}{2} \Gamma \frac{\hat{P} \tau_w}{\hat{T}_w^{\omega+1}} \right)^{-\frac{1}{2}} \hat{\psi}^{\frac{1}{2}} \right. \\ & \left. - \frac{1}{\Gamma \hat{P}} \left[\frac{1}{3} \left(\frac{d\hat{P}}{d\hat{x}} \right)^2 \frac{\hat{T}_w}{\tau_w^2} - \omega \frac{P_r \tau_w^2}{\hat{T}_w^{2\omega}} \right] \hat{\psi} + O(\hat{\psi}^{\frac{3}{2}}) \right\} \\ & = -\frac{1}{\Gamma \hat{P}} \left\{ \bar{K}_0 + \bar{K}_{\frac{1}{2}} \hat{\psi}^{\frac{1}{2}} + \bar{K}_1 \hat{\psi} + O(\hat{\psi}^{\frac{3}{2}}) \right\} \quad (2.54) \end{aligned}$$

Note that if $\frac{d\hat{P}}{d\hat{x}} = 0$, then the $\hat{\psi}^{\frac{1}{2}}$ term is zero and the entropy variation becomes

$$\frac{\partial \hat{S}}{\partial \hat{\psi}} = -\frac{1}{\Gamma \hat{P}} \left\{ \frac{\tau_w}{\hat{T}_w} + \omega \frac{P_r \tau_w^2}{\Gamma \hat{P} \hat{T}_w^{2\omega}} \hat{\psi} + O(\hat{\psi}^2) \right\} \quad (2.55)$$

If the reference conditions are taken to be the values in the external flow, then for $\omega = 1$

$$\tau_w = \frac{1}{2} C_f \frac{\bar{t}_e}{\bar{p}_0^*} M_e^2 \frac{\bar{a}_e^2}{\bar{a}_0^*} \left[\frac{\bar{a}_0^* \bar{p}_0^* \bar{l}_0}{\bar{\mu}_0^*} \right]^{\frac{1}{2}} \quad (2.56)$$

where M_e is the Mach number of the external flow, C_f is the local skin friction coefficient ($\tau_w = C_f \frac{1}{2} \bar{\rho}_e \bar{u}_e^2$) and $\bar{l}_o$ is the length of the forebody (see figure(2.2)). The friction coefficient and wall temperature for an adiabatic flat plate with no pressure gradient and $\omega = 1$ are given by

$$C_f = 1.328 \left[\frac{\bar{u}_e \bar{\rho}_e \bar{l}_o}{\bar{\mu}_e} \right]^{\frac{1}{2}} = 1.328 Re_{\bar{l}_o}^{\frac{1}{2}} \quad (2.57)$$

$$\frac{\hat{T}_w}{\hat{T}_e} = 2.6 \quad (Pr = 0.723, Me = 2.8) \quad (2.58)$$

from Schlichting (1960), p.344. As an example, consider a cone flying at supersonic speed through air ($\gamma = 1.4$). The expression for τ_w depends only on the flight Mach number and the cone half angle. For a Mach number of 3 and a half angle of 6° ,

$$\tau_w = 2.42 \quad (2.59)$$

If $\hat{T}_e$ in (2.58) is chosen to correspond to the free stream conditions of standard temperature and density, then the dimensionless wall temperature is

$$\hat{T}_w = 0.834 \quad (2.60)$$

Thus
$$\frac{\partial \hat{S}}{\partial \hat{\psi}} = - \frac{\gamma-1}{\gamma} \frac{1}{\hat{P}} (2.91) \quad (2.61)$$

at the wall. The dimensionless pressure with these same free stream conditions can be shown to be

$$\hat{P} = 1.82$$

Thus
$$Pr \tau_w^2 (\Gamma \hat{P} \hat{T}_w^2)^{-1} = 5.60$$

so that in this case,

$$\frac{\partial \hat{S}}{\partial \hat{\psi}} = - \frac{1}{\Gamma \hat{P}} \left[2.91 + 5.60 \hat{\psi} + O(\hat{\psi}^2) \right] \quad (2.62)$$

is the entropy variation near the wall. Here, all the $\hat{\psi}^{\frac{1}{2}}$ terms have vanished giving $\frac{\partial \hat{S}}{\partial \hat{\psi}}$ as an analytic function of $\hat{\psi}$.

Next, the more general case where both the total enthalpy and entropy are varying must be treated. In this case, the heat transfer at the wall need not be zero and both the entropy and total enthalpy can have a singularity at the wall. It will be shown (see(2.74)) that it is necessary to calculate the function

$$F(\hat{\psi}) = \frac{\partial \hat{S}}{\partial \hat{\psi}} - \frac{1}{\hat{T} + \frac{\hat{u}^2}{2}} \frac{\partial \hat{h}_t}{\partial \hat{\psi}} \quad (2.63)$$

In view of (2.47) and (2.48) this can be written as

$$F(\hat{\psi}) = \frac{\frac{\hat{u}^2}{2}}{\hat{T}(\hat{T} + \frac{\hat{u}^2}{2})} \frac{\partial \hat{T}}{\partial \hat{\psi}} - \frac{1}{\hat{T} + \frac{\hat{u}^2}{2}} \frac{\partial (\frac{\hat{u}^2}{2})}{\partial \hat{\psi}} - \frac{1}{\hat{P}} \frac{\partial \hat{P}}{\partial \hat{\psi}} \quad (2.64)$$

Note that $\hat{u}^2 \frac{\partial \hat{T}}{\partial \hat{\psi}}$ does not have a singularity at the wall. Thus

$F(\hat{\psi})$ does not possess a singularity. Substitution of the appropriate expansions into (2.64) allows $F(\hat{\psi})$ to be written as

$$F(\hat{\psi}) = -\frac{1}{\Gamma \hat{P}} \left\{ \hat{u}_1 + \left(2\hat{u}_2 - \frac{\hat{u}_1 \hat{T}_1}{2 \hat{T}_0} \right) \hat{\gamma} - \left(\hat{u}_1 \frac{\hat{T}_1}{\hat{T}_0} - \frac{\hat{u}_2 \hat{T}_1}{2 \hat{T}_0} - \frac{\hat{u}_1 \hat{T}_1^2}{2 \hat{T}_0^2} - 3\hat{u}_3 + \frac{\hat{u}_1^3}{\hat{T}_0} \right) \hat{\gamma}^2 + O(\hat{\gamma}^3) \right\} \quad (2.65)$$

Introduction of the stream function gives

$$\begin{aligned} F(\hat{\psi}) &= -\frac{1}{\Gamma \hat{P}} \left\{ \frac{\tau_w}{\hat{T}_w^\omega} + \left(\frac{2}{\Gamma \hat{P}} \frac{\hat{T}_w^{1-\omega}}{\tau_w} \right)^{\frac{1}{2}} \left[\frac{d\hat{P}}{d\hat{x}} + \left(\omega + \frac{1}{2} \right) \frac{Pr \tau_w \tau_w}{\hat{T}_w^{\omega+1}} \right] \hat{\psi}^{\frac{1}{2}} + O(\hat{\psi}) \right\} \\ &= -\frac{1}{\Gamma \hat{P}} \left\{ \tilde{K}_0 + \tilde{K}_{\frac{1}{2}} \hat{\psi}^{\frac{1}{2}} + O(\hat{\psi}) \right\} \quad (2.66) \end{aligned}$$

This relation is equivalent to (2.54) if $q_w = 0$; however, it shows that if $\frac{d\hat{P}}{d\hat{x}} = 0$ but $q_w \neq 0$, the half powers of the stream function remain, that is

$$F(\hat{\psi}) = -\frac{1}{\Gamma\hat{P}} \left[\frac{\tau_w}{\hat{T}_w^\omega} + \left(\frac{2}{\Gamma\hat{P}} \tau_w^{-1} \hat{T}_w^{1-\omega} \right)^{\frac{1}{2}} \frac{Pr q_w \tau_w}{\hat{T}_w^{\omega+1}} \hat{\psi}^{\frac{1}{2}} + O(\hat{\psi}) \right]$$

$$\text{if } \frac{d\hat{P}}{d\hat{x}} = 0. \quad (2.67)$$

Thus the $\hat{\psi}^{\frac{1}{2}}$ term is necessary to account for heat transfer at the wall or an external pressure gradient. Note that from (2.54) and (2.66) the functions of $\hat{x}$ are related by

$$\bar{K}_0 = \tilde{K}_0 = \frac{\tau_w}{\hat{T}_w^\omega}, \quad (2.68.a)$$

$$\bar{K}_{\frac{1}{2}} = \tilde{K}_{\frac{1}{2}} \quad \text{if } q_w = 0 \quad (2.68.b)$$

and so on.

The function to be calculated at the entrance of the inviscid transonic region, when written in the inviscid variables (2.9), is

$$\left(\frac{\gamma-1}{\gamma} \frac{dS}{d\Psi} - \frac{1}{\bar{a}^{*2}} \frac{d\bar{a}^{*2}}{d\Psi} \right).$$

Using the definitions of the dimensionless variables (2.9) and (2.19) it can easily be shown that

$$\left(\frac{dS}{d\Psi}\right)_{\text{inviscid variables}} = \bar{p}^* \bar{a}^* \bar{L} \frac{\bar{R}_o^j}{\bar{R}} \frac{d\bar{S}}{d\bar{\Psi}}, \quad (2.69)$$

$$\left(\frac{d\bar{a}^{*2}}{d\Psi}\right)_{\text{inviscid variables}} = 2 \frac{\gamma-1}{\gamma+1} \bar{p}^* \bar{a}^* \bar{L} \bar{R}_o^j \frac{d\bar{h}_t}{d\bar{\Psi}}, \quad (2.70)$$

$$\left(\frac{\partial \hat{S}}{\partial \hat{\Psi}}\right)_{\text{viscous variables}} = \frac{\bar{p}_o^* \bar{a}_o^* \delta_o \bar{l}_o \bar{R}_o^j}{\frac{\gamma}{\gamma-1} \bar{R}} \frac{\partial \bar{S}}{\partial \bar{\Psi}}, \quad (2.71)$$

$$\left(\frac{\partial \hat{h}_t}{\partial \hat{\Psi}}\right)_{\text{viscous variables}} = \frac{\bar{p}_o^* \delta_o \bar{l}_o \bar{R}_o^j}{\bar{a}_o^*} \frac{\partial \bar{h}_t}{\partial \bar{\Psi}}. \quad (2.72)$$

The boundary layer thickness at the corner is given by $\delta_o \bar{l}_o$ where $\bar{l}_o$ is the dimensional length of the forebody (see figure (2.2)). This is also the scaling length for the unstretched variables in the inviscid corner region since it represents the thickness of the boundary layer entering the inviscid corner region and thus is the entropy length for the corner region, that is,

$$\bar{L} = \delta_o \bar{l}_o. \quad (2.73)$$

At the entrance to the inviscid transonic region the flow properties are found by evaluating the solutions for the compress-

ible boundary layer at the corner. Thus,

$$\left(\frac{\gamma-1}{\gamma} \frac{dS}{d\Psi} - \frac{1}{\bar{a}^{*2}} \frac{d\bar{a}^{*2}}{d\Psi} \right)_{\text{inviscid region}} \left[\frac{\bar{p}^* \bar{a}^*}{\bar{p}_0^* \bar{a}_0^*} \left(\frac{\partial \hat{S}}{\partial \hat{\Psi}} - (\hat{T} + \frac{\hat{u}^2}{2})^{-1} \frac{\partial \hat{h}_t}{\partial \hat{\Psi}} \right) \right]_{\text{boundary layer at } \bar{x} = \bar{x}_0} \quad (2.74)$$

In the boundary layer,

$$\frac{\bar{p}^* \bar{a}^*}{\bar{p}_0^* \bar{a}_0^{*2}} = \gamma \frac{\bar{P}^*}{\bar{p}_0^* \bar{a}_0^{*2}} = \gamma \hat{P}^* \quad (2.75)$$

If equation (2.47) is written in terms of stagnation conditions and the fact that $\hat{P}^* = (\text{constant})(\hat{P}_t)$ is used, the behavior of $\hat{P}^*$ in the boundary layer can be shown to be

$$\hat{P}^* = \left[1 + \frac{\hat{u}_1}{\hat{P}} \hat{\Psi} + O(\hat{\Psi}^{\frac{3}{2}}) \right] \quad (2.76)$$

Also, in general $\bar{a}^{*2} = 2 \frac{\gamma-1}{\gamma+1} \bar{h}_t = 2\beta \bar{h}_t$ or

$$\frac{\bar{a}^{*2}}{\bar{a}_0^{*2}} = 2\beta \hat{h}_t \quad \text{where } \beta = \frac{\gamma-1}{\gamma+1} \quad (2.77)$$

From the definition (2.48) $\hat{h}_t$ can be calculated as a function of $\hat{\Psi}$. Thus it can be shown that

$$\begin{aligned} \frac{\bar{p}^* \bar{a}^*}{\bar{p}_0^* \bar{a}_0^*} &= \gamma (2\beta \hat{T}_w)^{-\frac{1}{2}} \left\{ 1 - \frac{1}{2} A^{\frac{1}{2}} \frac{\hat{T}_1}{\hat{T}_0} \hat{\Psi}^{\frac{1}{2}} + \left(\frac{1}{3} B \frac{\hat{T}_1}{\hat{T}_w} + \frac{2}{3} A^{-1} \frac{\hat{T}_1^2}{\hat{T}_w^2} - \frac{1}{2} A^{-1} \frac{\hat{T}_2}{\hat{T}_w} - \frac{1}{4} A^{-1} \frac{\hat{u}_1^2}{\hat{T}_w} + \frac{\hat{u}_1}{\hat{P}} \right) \hat{\Psi} \right. \\ &\quad \left. + O(\hat{\Psi}^{\frac{3}{2}}) \right\} \\ &= \gamma (2\beta \hat{T}_w)^{-\frac{1}{2}} \left[1 + \tilde{A} \hat{\Psi}^{\frac{1}{2}} + \tilde{B} \hat{\Psi} + O(\hat{\Psi}^{\frac{3}{2}}) \right] \quad (2.78) \end{aligned}$$

where $A, B, \hat{T}_1, \hat{u}_1$ and $\hat{T}_2$ are functions of τ_w and q_w defined

by (2.44) and (2.35)-(2.37).

Finally it should be noted that the relation between the dimensionless stream function in the boundary layer solution and that in the inviscid transonic corner region is given by

$$\frac{d\Psi}{d\hat{\Psi}} = \frac{\bar{p}_0^* \bar{a}_0^*}{\bar{p}^* \bar{a}^*} \quad (2.79)$$

The right hand side of (2.79) has been expressed as a function of half powers of $\hat{\Psi}$, that is, it is the reciprocal of (2.78). This series can be integrated and inverted to give $\hat{\Psi}$ as a function of Ψ ,

$$\hat{\Psi}^{\frac{1}{2}} = \left(\frac{\gamma}{\sqrt{2\beta} \hat{T}_w} \right)^{\frac{1}{2}} \Psi^{\frac{1}{2}} + \frac{1}{6} \frac{\gamma}{\sqrt{2\beta} \hat{T}_w} \frac{P_r \hat{g}_w}{A^{\frac{1}{2}} \hat{T}_w^{\omega+1}} \Psi + O(\Psi^{\frac{3}{2}}) \quad (2.80.a)$$

$$\hat{\Psi} = \frac{\gamma}{\sqrt{2\beta} \hat{T}_w} \Psi + \frac{1}{3} \left[\frac{\gamma}{\sqrt{2\beta} \hat{T}_w} \right]^{\frac{3}{2}} \frac{P_r \hat{g}_w}{A^{\frac{1}{2}} \hat{T}_w^{\omega+1}} \Psi^{\frac{3}{2}} + O(\Psi^{\frac{5}{2}}). \quad (2.80.b)$$

This completes the necessary knowledge of the entropy and total enthalpy distributions along the streamlines entering the inviscid transonic corner region. It has been demonstrated that generally,

$$\begin{aligned} & \left(\frac{\gamma-1}{\gamma} \frac{dS}{d\Psi} - \frac{1}{\bar{a}^{*2}} \frac{d\bar{a}^{*2}}{d\Psi} \right)_{\text{inviscid region}} \\ &= -(\gamma-1) \left[C_0 + C_{\frac{1}{2}} \Psi^{\frac{1}{2}} + C_1 \Psi + O(\Psi^{\frac{3}{2}}) \right] \quad (2.81) \end{aligned}$$

with

$$C_0 = (2\beta \hat{T}_w)^{-\frac{1}{2}} \tilde{K}_0 = (2\beta \hat{T}_w)^{-\frac{1}{2}} \frac{\tau_w}{\hat{T}_w},$$

$$C_{\frac{1}{2}} = \frac{\gamma^{\frac{1}{2}} \hat{P}^{-1}}{(2\beta \hat{T}_w)^{\frac{3}{2}}} [\tilde{K}_{\frac{1}{2}} + \tilde{K}_0 \tilde{A}] ,$$

$$C_1 = \frac{\gamma \hat{P}^{-1}}{(2\beta \hat{T}_w)^{\frac{5}{2}}} \left[\tilde{K}_0 \left(\tilde{B} + \frac{1}{6} \frac{Pr q_w}{A^{\frac{1}{2}} \hat{T}_w^{\omega+1}} \tilde{A} \right) + \tilde{K}_{\frac{1}{2}} \left(\tilde{A} + \frac{1}{6} \frac{Pr q_w}{A^{\frac{1}{2}} \hat{T}_w^{\omega+1}} \right) + \tilde{K}_1 \right]$$

where q_w , τ_w , $\hat{P}$ and $\hat{T}_w$ are to be evaluated at the corner, $\bar{x}=\bar{l}_0(x=1)$. Clearly C_0 is positive as long as $\frac{\partial \bar{u}}{\partial \bar{y}} > 0$. The sign of $C_{\frac{1}{2}}$ can be either positive or negative depending on the signs and relative magnitudes of $\frac{d\hat{P}}{d\hat{x}}$, $\hat{T}_w^{\omega}$ and $(q_w \tau_w)$. If $q_w=0$, then $C_{\frac{1}{2}}$ has the sign of $\frac{d\hat{P}}{d\hat{x}}$ which can be expected to be negative. If $\frac{d\hat{P}}{d\hat{x}}=0$, then $C_{\frac{1}{2}}$ has the same sign as q_w . Thus, if there is heat transfer to the wall, $q_w < 0$ and $C_{\frac{1}{2}} < 0$. If both the external pressure gradient and the heat transfer at the wall are zero, then $C_{\frac{1}{2}}=0$. In this case $C_1 > 0$ and has significance since it becomes the second term in the expansion. It can be shown using this expansion scheme to write $\left(\frac{\partial \hat{h}_t}{\partial \hat{\psi}}\right)$ directly in terms of $\frac{d\hat{P}}{d\hat{x}}$ and q_w that $\frac{d\hat{P}}{d\hat{x}}=0$ and $q_w=0$ does not imply $\left(\frac{\partial \hat{h}_t}{\partial \hat{\psi}}\right)=0$ unless the Prandtl number is unity. Thus $\frac{d\hat{P}}{d\hat{x}}=0$, $q_w=0$ and $Pr=1$ gives $\left(\frac{\partial \hat{h}_t}{\partial \hat{\psi}}\right)=0$ to all orders in $\hat{\psi}$; the total enthalpy is constant in the boundary layer.

D. TRANSONIC PRANDTL-MEYER EXPANSION

If the solutions are to match with a Prandtl-Meyer expansion in the limit as the corner is approached along radial lines just downstream of the left running characteristic from the corner, the ordering of the velocity asymptotic expansions (2.16) and (2.17) must approach that of the Prandtl-Meyer transonic expansions. The Prandtl-Meyer transonic expansions (that is, for small turning angle) are

$$U = 1 + \frac{1}{\gamma+1} \underline{\phi}^2 - \frac{2\gamma+1}{6} \frac{1}{(\gamma+1)^2} \underline{\phi}^4 + O(\underline{\phi}^6) \quad (2.82)$$

$$V = -\frac{2}{3} \frac{1}{\gamma+1} \underline{\phi}^3 + \frac{2\gamma}{15} \frac{1}{(\gamma+1)^2} \underline{\phi}^5 + O(\underline{\phi}^7) \quad (2.83)$$

where $\underline{\phi} \ll 1$ for transonic flow. Thus essentially these are velocity expansions in powers of $\frac{X}{Y} = \tan \phi \approx \underline{\phi}$ where $\frac{X}{Y}$ is small. This means that near the corner where both X and Y are approaching zero, the expansions are valid for a small sector downstream from the normal to the corner.

In the inviscid transonic corner region the perturbation from Mach one in the horizontal velocity is ordered by E_1 while the vertical velocity is ordered by V_1 . For matching with (2.82) and (2.83) one must require that:

$$V_1 = (\gamma+1)^{\frac{1}{2}} E_1^{\frac{3}{2}}$$

where the $(\gamma+1)^{\frac{1}{2}}$ is added for convenience later. Further, (2.82) and (2.83) imply that the velocity asymptotic expansion for U must contain a sequence of integral power multiples of

E_1 while any other intermediate order terms must vanish at the corner, that is, in the limit $Y \rightarrow 0$, $\frac{X}{Y}$ fixed. The evaluation of these orders should come from the equations, particularly Crocco's equation. It will be observed that requiring $\gamma_1 = (\gamma+1)^{\frac{1}{2}} E_1^{\frac{3}{2}}$ leads to a nonlinear lowest order equation of the mixed type necessary to describe transonic flow as well as being necessary to satisfy the downstream matching condition.

Further information can be gleaned from this condition by observing that $(\frac{X}{Y})$ is of order $(\frac{\tau}{\varepsilon})$ in the transonic corner region. Thus the transonic corner region extends to an angle $\phi = \mathcal{O}(\frac{\tau}{\varepsilon})$ downstream of the corner. Since E_1 is associated with ϕ^2 from the horizontal velocity expansion, it is clear that

$$\frac{\tau}{\varepsilon} = (\gamma+1)^{\frac{1}{2}} E_1^{\frac{1}{2}}$$

defines the relative extent of the transonic corner region (again the $(\gamma+1)^{\frac{1}{2}}$ is added for later convenience).

Substitution of $\tan \phi = \frac{\tau}{\varepsilon} \frac{X}{Y}$ into (2.82) and (2.83) leads to the final expressions for the required behavior at the downstream corner:

$$U = 1 + E_1 \left(\frac{X}{Y}\right)^2 - E_1^2 (\gamma + \frac{5}{6}) \left(\frac{X}{Y}\right)^4 + \dots \quad (2.84)$$

$$V = (\gamma+1)^{\frac{1}{2}} \left[-\frac{2}{3} E_1^{\frac{3}{2}} \left(\frac{X}{Y}\right)^3 + E_1^{\frac{5}{2}} \left(\frac{4}{5}\gamma + \frac{2}{3}\right) \left(\frac{X}{Y}\right)^5 + \dots \right] \quad (2.85)$$

The asymptotic expansions (2.16) and (2.17) must match with these in the limit $\{Y \rightarrow 0, \frac{X}{Y} = \text{constant} \ll 1\}$.

III PERTURBATION EQUATIONS

Substitution of the entropy and total enthalpy variation term from the upstream matching condition into the Crocco equation completes the governing equations

(gasdynamic equation)

$$(a^2 - U^2) \frac{\partial U}{\partial X} - UV \left(\frac{\partial V}{\partial X} + \frac{\partial U}{\partial Y} \right) + (a^2 - V^2) \frac{\partial V}{\partial Y} + j a^2 \left[\frac{U \sin \alpha + V \cos \alpha}{R_0 + Y \cos \alpha + X \sin \alpha} \right] = 0, \quad (3.1)$$

(Crocco equation)

$$\frac{\partial U}{\partial Y} - \frac{\partial V}{\partial X} = -\epsilon a^2 \left(1 + j \frac{Y}{R_0} \right) \left(C_0 + C_{\frac{1}{2}} \Psi^{\frac{1}{2}} + C_1 \Psi + \dots \right) \quad (3.2)$$

which are valid for small Ψ . From the definition of the stream function,

$$\begin{aligned} \frac{\partial \Psi}{\partial Y} &= \epsilon U = \left[1 - E_1 U_1 - E_2 U_2 + O(E_i^2) \right] \left[1 + E_1 U_1 + E_2 U_2 + O(E_i^2) \right] \\ &= 1 + O(E_i^2, E_2) \end{aligned} \quad (3.3)$$

where the continuity equation and the fact that the dimensionless density is unity at the sonic line have been employed to find the expression for the density. This implies that

$$\Psi = Y + O(E_i^2, E_2), \quad (3.4)$$

where Ψ and Y are scaled with the incoming boundary layer

thickness and $\Psi = 0$ when $Y=0$. Thus one expects that the stream function can be expanded in a Taylor's series from the wall which will be valid for distances from the wall small compared to the thickness of the incoming boundary layer. That is,

$$\Psi = \Psi_0 + \left(\frac{\partial \Psi}{\partial Y}\right)_0 Y + \left(\frac{\partial^2 \Psi}{\partial Y^2}\right)_0 Y^2 + \dots$$

where $\Psi_0 = 0$ (the zeroth streamline),

$$\left(\frac{\partial \Psi}{\partial Y}\right)_0 = (\rho u)_0 ,$$

and
$$\left(\frac{\partial^2 \Psi}{\partial Y^2}\right)_0 = \left[\frac{\partial}{\partial Y} (\rho u)\right]_0 = O\left(\frac{E_1^2}{\varepsilon}\right) .$$

Then it is assumed that

$$\Psi = Y + O\left(\frac{E_1^2}{\varepsilon} Y^2\right) , \quad (3.5)$$

that is, the flow is essentially a uniform flow with small perturbations. This expansion is valid for small X and Y and thus is valid throughout the transonic corner region. In transonic corner variables the dimensionless stream function is given by

$$\Psi = \varepsilon Y + O(E_1^2 \varepsilon) \quad (3.6)$$

where y is of order one.

Now the governing equations can be written in terms of the stretched variables as

(gasdynamic equation)

$$(a^2 - u^2) \frac{\partial u}{\partial x} - uv \left(\frac{\partial v}{\partial x} + \frac{\tau}{\varepsilon} \frac{\partial u}{\partial y} \right) + \frac{\tau}{\varepsilon} (a^2 - v^2) \frac{\partial v}{\partial y} + \frac{j}{R_0} \tau a^2 (u \sin \alpha + v \cos \alpha) \left[1 + \frac{\varepsilon \gamma \cos \alpha + \tau x \sin \alpha}{R_0} \right]^{-1} = 0, \quad (3.7)$$

(Crocco equation)

$$\frac{\partial v}{\partial x} - \frac{\tau}{\varepsilon} \frac{\partial u}{\partial y} = -\tau a^2 \left(1 + j \frac{\varepsilon \gamma}{R_0} \right) \left[C_0 + \varepsilon^{\frac{1}{2}} C_{\frac{1}{2}} \gamma + \varepsilon C_1 \gamma + h.o.t. \right]. \quad (3.8)$$

The introduction of the asymptotic expansions for the velocities and the use of the Bernoulli equation (2.4) for the dimensionless sonic speed allows the governing equations to be written to various orders as

$$\begin{aligned} & -(\gamma+1)E_1^2 u_1 \frac{\partial u_1}{\partial x} + \frac{\tau}{\varepsilon} v_1 \frac{\partial v_1}{\partial y} - \frac{\gamma+1}{2} E_1^3 u_1^2 \frac{\partial u_1}{\partial x} - (\gamma+1)E_1 E_2 u_2 \frac{\partial u_1}{\partial x} \\ & - (\gamma+1)E_1 E_2 u_1 \frac{\partial u_2}{\partial x} + \frac{\tau}{\varepsilon} v_2 \frac{\partial v_2}{\partial y} - (\gamma+1) \frac{\tau}{\varepsilon} E_1 v_1 u_1 \frac{\partial v_1}{\partial y} - v_1^2 v_1 \frac{\partial v_1}{\partial x} \\ & - \frac{\tau}{\varepsilon} E_1 v_1 v_1 \frac{\partial u_1}{\partial y} + \frac{j}{R_0} \tau \sin \alpha + h.o.t. = 0, \quad (3.9) \end{aligned}$$

$$\begin{aligned} & v_1 \frac{\partial v_1}{\partial x} - \frac{\tau}{\varepsilon} E_1 \frac{\partial u_1}{\partial y} + v_2 \frac{\partial v_2}{\partial x} - \frac{\tau}{\varepsilon} E_2 \frac{\partial u_2}{\partial y} + h.o.t. \\ & = - \left[\tau C_0 + \tau \varepsilon^{\frac{1}{2}} C_{\frac{1}{2}} \gamma^{\frac{1}{2}} + \tau \varepsilon C_1 \gamma - \tau E_1 C_0 u_1 \right. \\ & \quad \left. + \frac{j}{R_0} \tau \varepsilon (\gamma-1) C_0 \gamma + h.o.t. \right]. \quad (3.10) \end{aligned}$$

From the Prandtl-Meyer matching at the downstream corner (see Section II),

$$\frac{\tau}{\varepsilon} = (\gamma+1)^{\frac{1}{2}} E_1^{\frac{1}{2}} \quad (3.11)$$

$$\nu_1 = (\gamma+1)^{\frac{1}{2}} E_1^{\frac{3}{2}} \quad (3.12)$$

Thus (3.10) becomes

$$\begin{aligned} E_1^{\frac{3}{2}} \left\{ \frac{\partial u_1}{\partial Y} - \frac{\partial v_1}{\partial X} \right\} + \frac{\nu_2}{(\gamma+1)^{\frac{1}{2}}} \left\{ \frac{\varepsilon^{\frac{1}{2}} E_2}{\nu_2} \frac{\partial u_2}{\partial Y} - \frac{\partial v_2}{\partial X} \right\} + h. o. t. \\ = \varepsilon^{\frac{1}{2}} C_0 + \varepsilon^2 C_1 Y^{\frac{1}{2}} + h. o. t. \quad (3.13) \end{aligned}$$

The Crocco equation (3.13) must provide the final link between the parameters. There is some difficulty in deducing this relation. One requires that the vorticity is of order one when written in the unstretched variables for this is exactly the definition of the entropy length, $\bar{L}$. However, this is not enough since, for example, one could put $\frac{\varepsilon^{\frac{1}{2}} E_2}{\nu_2} = 1$ and $\nu_2 = (\gamma+1)^{\frac{1}{2}} \varepsilon^{\frac{1}{2}}$ giving

$$\frac{\partial u_1}{\partial Y} - \frac{\partial v_1}{\partial X} = 0$$

$$\frac{\partial u_2}{\partial Y} - \frac{\partial v_2}{\partial X} = C_0$$

and still

$$\frac{\partial U}{\partial Y} - \frac{\partial V}{\partial X} = C_0$$

is of order one. The reason for this, of course, is that the irrotational terms in the asymptotic expansions contribute nothing

to the vorticity, while the size of the corner region (given by ε) can be adjusted to make the velocity variation (E_1) as large or as small as desired. It is necessary to bring in other considerations to settle the ordering.

If ε were small compared to E_1 so that $\frac{\partial u_1}{\partial y} - \frac{\partial v_1}{\partial x} = 0$, then at the wall, where v_1 and $\frac{\partial v_1}{\partial x}$ are zero, $\frac{\partial u_1}{\partial y}$ must also be zero. This would be inconsistent with the postulated incoming boundary layer profile. Clearly, then, rotational effects should be felt in the first order terms for the problem at hand.

That is

$$\left(\frac{\partial U}{\partial Y}\right)_{Y=0} = \left(\frac{\partial u_1}{\partial y}\right)_{y=0} = C_0 \neq 0$$

is a requirement of the current analysis. Thus,

$$E_1 = \varepsilon \quad (3.14)$$

$$\frac{\partial u_1}{\partial y} - \frac{\partial v_1}{\partial x} = C_0 \quad (3.15)$$

is the proper ordering of equation (3.13). Notice that this choice allows the most curvature in the incoming velocity profile.

If it were desired to treat the first order velocities as being irrotational, but to include the effects of rotation on the second order velocities, then it would be necessary to have $E_1 = \varepsilon^{\frac{1}{2}}$. This choice leads to the equations treated by Vaglio-Laurin (1960) as will become apparent. Evidently this allows only a second order perturbation from an irrotational incoming velocity profile. Vaglio-Laurin's results show the first order sonic line normal to the subsonic upstream wall. The desired solution in the present problem must have a sonic line that,

even at the wall, depends on the incoming vorticity at the wall. Rotational effects in the first order equations seem necessary to achieve this.

Other choices must be discounted because of the results obtained by Vaglio-Laurin (1960). His numerical solution shows that the solution of the first order irrotational equations satisfying the proper boundary conditions must have $E_1 = \epsilon^{\frac{1}{2}}$. Since his results corroborate Guderley's (1948) irrotational solution, it seems that his ordering is the only correct one for the problem, except the one treated here. As will become apparent, the success of the present ordering is due to a singularity of the governing equation which exists along a singular solution of the equation. No other ordering allows this circumstance to occur.

With (3.14), the second order Crocco equation is

$$\frac{\partial u_1}{\partial y} - \frac{\partial v_1}{\partial x} = C_{\frac{1}{2}} \gamma^{\frac{1}{2}} \quad \left(C_{\frac{1}{2}} \neq 0 \right) \quad (3.16)$$

where
$$v_2 = (\gamma+1)^{\frac{1}{2}} \epsilon^2 \quad (3.17)$$

and
$$E_2 = \epsilon^{\frac{3}{2}} \quad (3.18)$$

This immediately settles the first and second order gasdynamic equations as

$$u_1 \frac{\partial u_1}{\partial x} - \frac{\partial v_1}{\partial y} = 0 \quad (3.19)$$

and

$$u_2 \frac{\partial u_1}{\partial x} + u_1 \frac{\partial u_2}{\partial x} - \frac{\partial v_2}{\partial y} = 0 \quad (3.20)$$

where the axisymmetric terms have been left to higher orders (see Section VI).

Equations (3.15) and (3.19) can be combined to give

$$u_1 \frac{\partial^2 u_1}{\partial x^2} - \frac{\partial^2 u_1}{\partial y^2} = - \left(\frac{\partial u_1}{\partial x} \right)^2 \quad (3.21)$$

which clearly demonstrates the mixed nature of the leading equations. This non-linearity is essential in describing mixed, transonic flows.

The form of the first order Crocco equation (3.15) suggests introducing a modified velocity potential by

$$u_1 = \phi_x^{(1)} \quad (3.22)$$

$$v_1 = \phi_y^{(1)} - C_0 x \quad (3.23)$$

Thus (3.15) is identically satisfied while the gasdynamic equation becomes

$$\phi_x^{(1)} \phi_{xx}^{(1)} - \phi_{yy}^{(1)} = 0 \quad (3.24)$$

which is the same transonic potential equation found in irrotational flow, but with a modified potential. A solution of (3.24) that satisfies the imposed boundary conditions is desired.

The fact that this modified transonic potential is the first

order term of an asymptotic expansion expressed in terms of an artificial stretching parameter, imposes a further important condition on its functional form. In particular, it must be that the solution of this problem, when written in terms of coordinates made dimensionless with the entering boundary layer thickness (the only physical length in the problem) is independent of the artificial stretching parameter ϵ . This is the "principle of eliminability" as discussed by Chang (1961). Generally, a similarity solution for the problem is necessary, for it must be that $\epsilon u_1(x, y)$ is not a function of ϵ . Thus there exists a similarity parameter independent of ϵ such that

$$\epsilon u_1(x, y) = \epsilon u_1(x, y) = \epsilon y f_1'(\eta) \quad (3.25)$$

and
$$\eta = \frac{x}{y^m} = \frac{X}{Y^m} \frac{\epsilon^m}{\gamma} = \frac{X}{Y^m} \frac{\epsilon^m}{\epsilon^{\frac{1}{2}}} (\gamma+1)^{-\frac{1}{2}} \quad (3.26)$$

so
$$m = \frac{3}{2} \quad . \quad (3.27)$$

Observe that if the first order solutions are required to be irrotational so that $E_1 = \epsilon^{\frac{1}{2}}$, then $\gamma = (\gamma+1)^{\frac{1}{2}} \epsilon^{\frac{1}{2}}$ and the above reasoning leads to the similarity parameter $\eta = \frac{x}{y^{\frac{3}{2}}}$. This is the similarity parameter found by Vaglio-Laurin from the numerical solution of the first order transonic potential equation (3.24), but with the modified velocity potential replaced by the usual velocity potential. That is, a first order irrotational solution satisfying the given boundary conditions requires that the similarity parameter be $\eta = \frac{x}{y^{\frac{3}{2}}}$. This allows the

entropy gradient at the wall to be of the order of the second order velocity perturbation.

With this information it can be shown using equation (3.24) that the potential (3.22) must be of the form

$$\phi^{(1)} = \gamma^{\frac{5}{2}} f_1(\eta) \quad (3.28)$$

and
$$u_1 = \gamma \frac{df_1}{d\eta} \quad , \quad (3.29)$$

$$v_1 = \gamma^{\frac{3}{2}} \left[\frac{5}{2} f_1 - \frac{3}{2} \eta \frac{df_1}{d\eta} - C_0 \eta \right] \quad . \quad (3.30)$$

The transonic potential equation (3.24) reduces to the ordinary, non-linear differential equation

$$\left(\frac{df_1}{d\eta} - \frac{9}{4} \eta^2 \right) \frac{d^2 f_1}{d\eta^2} + \frac{15}{4} \eta \frac{df_1}{d\eta} - \frac{15}{4} f_1 = 0 \quad . \quad (3.31)$$

This is equivalent to the equation given by Vaglio-Laurin (1960) with $p=3/2$. The condition that there be no flow through the upstream wall $\{ \eta=0, \chi < 0 \}$ requires that

$$\frac{5}{2} f_1 - \frac{3}{2} \eta \frac{df_1}{d\eta} - C_0 \eta \rightarrow 0 \quad \text{as } \eta \rightarrow -\infty \quad . \quad (3.32)$$

The requirement that the flow match with the Prandtl-Meyer transonic expansion in the limit as the corner is approached implies matching as $\eta = \frac{\chi}{\gamma^{\frac{1}{2}}} \rightarrow +\infty$. The matching requires that f_1 have the form

$$f_1 = \frac{\eta^3}{3} + \dots \quad \text{as } \eta \rightarrow +\infty \quad (3.33)$$

so the velocity components agree with the transonic Prandtl-Meyer expansions (2.84) and (2.85).

It is well known (Germain (1962)) that this differential equation (3.31) has a group property, so that with suitable new variables the second order ordinary differential equation becomes a first order ordinary differential equation. Thus we put

$$S = \frac{f_1}{\eta^3} \quad (3.34)$$

$$t = \frac{f_1'}{\eta^2} \quad (3.35)$$

and equation (3.31) becomes

$$\frac{dt}{ds} = \frac{\frac{15}{4}s + \frac{3}{4}t - 2t^2}{(t-3s)(t-\frac{9}{4})} \quad (3.36)$$

with the inverse transformation given by

$$\ln \eta = \int \frac{ds}{t-3s} \quad (3.37)$$

Much of the analysis of the solution curves of (3.36) in the (s,t) plane has been carried out by Germain (1962) and Vaglio-Laurin (1960). The analysis provides the location of the singular points, physical significance of these points and the behavior of the solution curves in their neighborhood. Equation (3.36) has three singularities in the finite (s,t) plane and

and two singularities at infinity. Examination of each singularity follows.

A. THE SINGULAR POINT AT THE ORIGIN, $s=0$, $t=0$

In the neighborhood of the singularity at the origin equation (3.36) can be simplified to

$$\frac{dt}{ds} = \frac{\frac{15}{4}s + \frac{3}{4}t}{\frac{9}{4}(3s-t)} \quad (3.38)$$

with integral curves,

$$\frac{(t - \frac{5}{3}s)^2}{(t-s)^3} = \text{constant} . \quad (3.39)$$

The point is a node with all integral curves except one, tangent to the line $t = \frac{5}{3}s$. The inverse transformation gives

$$\eta = \frac{\text{constant}}{(t - \frac{5}{3}s)^{\frac{1}{3}}} \quad (3.40)$$

and the asymptotic expansion for f_1 in the neighborhood of the singularity is

$$f_1 = A_0 (-\eta)^{\frac{5}{3}} + A_1 (-\eta) - \frac{25}{27} A_0^2 (-\eta)^{\frac{1}{3}} - \frac{5}{27} A_1 A_0 (-\eta)^{-\frac{1}{3}} + O(\eta^{-1}) \quad (3.41)$$

where A_0 and A_1 are arbitrary constants. The asymptotic expansions for the velocities can be written directly;

$$u_1 = - \left[\frac{5}{3} A_0 (-x)^{\frac{2}{3}} + A_1 y - \frac{25}{81} A_0^2 (-x)^{-\frac{2}{3}} y^2 + \frac{5}{81} A_1 A_0 (-x)^{-\frac{4}{3}} y^3 + O(y^4) \right] \quad (3.42.a)$$

and

$$V_1 = (A_1 + C_0)(-x) - \frac{50}{27} A_0^2 (-x)^{\frac{1}{3}} y - \frac{5}{9} A_1 A_0 (-x)^{\frac{1}{3}} y^2 + O(y^3) . \quad (3.42.b)$$

At this singularity η approaches infinity. Since the velocities do not have the desired Prandtl-Meyer behavior it must be that this represents the subsonic edge of the inviscid transonic region.

To require v_1 to vanish at the upstream wall is equivalent to requiring $\left(\frac{\partial u_1}{\partial y}\right)_{y=0} = C_0$, that is,

$$A_1 = -C_0 . \quad (3.43)$$

If $A_0 > 0$ the flow is subsonic at the wall. If $A_0 < 0$ it is supersonic. Vaglio-Laurin (1960) obtained a similar asymptotic expansion for his rotationless first order equations, but with $A_1 = 0$.

B. THE SINGULAR POINT AT $s = \frac{1}{3}$, $t = 1$

In the neighborhood of this singularity the differential equation can be simplified to

$$\frac{d\bar{\tau}}{d\bar{\sigma}} = \frac{15\bar{\sigma} - 13\bar{\tau}}{-5(\bar{\tau} - 3\bar{\sigma})} \quad (3.44)$$

where $\bar{\tau} = S - \frac{1}{3}$ and $\bar{\sigma} = t - 1$. The integral curves of (3.44) are given by

$$\left[\bar{\sigma} - 5\bar{\tau}\right]^5 \left[\bar{\sigma} - \frac{3}{5}\bar{\tau}\right]^6 = \text{constant} . \quad (3.45)$$

The singularity is a saddle point with two principal directions

given by $\bar{\sigma} = 5 \bar{\tau}$ and $\bar{\sigma} = \frac{3}{5} \bar{\tau}$. Along the principal direction given by $\bar{\sigma} = 5 \bar{\tau}$ the inverse mapping (3.37) gives

$$\eta = C \bar{\sigma}^{\frac{1}{2}}$$

with C as the integration constant. Thus passing through the singularity in this direction must correspond to the y-axis. The leading terms for the horizontal velocity in the neighborhood of the y-axis would then be

$$u_1 \sim \gamma \eta^2 = \left(\frac{x}{y}\right)^2.$$

That is, u_1 would be symmetrical in x, a situation not allowed in the current problem. Along the principal direction given by $\bar{\sigma} = \frac{3}{5} \bar{\tau}$ (3.37) gives

$$\eta = C \bar{\tau}^{-\frac{5}{12}} \quad (3.46)$$

Passing through the singularity in this direction corresponds to $|\eta| \rightarrow \infty$ and the asymptotic expansion and corresponding velocity components are

$$f_1 = \frac{\eta^3}{3} + B_0 \eta^{\frac{3}{5}} - \frac{3}{425} B_0^2 \eta^{-\frac{3}{5}} - \frac{108}{244375} B_0^3 \eta^{-\frac{21}{5}} + O(\eta^{-\frac{17}{5}}), \quad (3.47)$$

$$u_1 = \left(\frac{x}{y}\right)^2 + \frac{3}{5} B_0 y^{\frac{8}{5}} x^{-\frac{2}{5}} + O\left(y^{\frac{26}{5}} x^{-\frac{14}{5}}\right) \quad (3.48)$$

and
$$V_1 = -\frac{2}{3} \left(\frac{X}{Y} \right)^3 - C_0 X + \frac{8}{5} B_0 Y^{\frac{2}{5}} X^{\frac{2}{5}} + O \left(Y^{\frac{4}{5}} X^{-\frac{2}{5}} \right) \quad (3.49)$$

Clearly this is the correct behavior for $\eta \rightarrow +\infty$ as it matches with the Prandtl-Meyer transonic expansions (2.84) and (2.85) near the downstream corner. The $C_0 x$ term in equation (3.49) matches with a corresponding term found by Adamson (1968) in expanding his transonic solutions valid close to the corner for small turning angle (his solution is valid near the corner for any expansion angle).

C. THE SINGULAR POINT AT $s = \frac{9}{4}$, $t = \frac{9}{4}$

Notice that the integral curve $t = \frac{9}{4}$ represents a limiting line in the (s, t) plane except at the singular point (see Vaglio-Laurin (1960)). That is, $\frac{ds}{dt} = 0$ if $t = \frac{9}{4}$ and $s \neq \frac{9}{4}$ so that $d(\ln \eta) = 0$ showing that the line represents an extremum of η . Thus any solution intersecting the line $t = \frac{9}{4}$ except at $s = \frac{9}{4}$ must be rejected. It is left to show that a valid solution curve can pass through the singular point and that, as Vaglio-Laurin (1960) observed, the singular point corresponds to a characteristic.

The substitution of the velocity expansions into the general expression for the characteristic slope,

$$\left(\frac{dY}{dX} \right)_{\text{characteristic}} = \tan (\oplus \mp \mu)$$

gives in the transonic region variables

$$\left(\frac{dY}{dX} \right)_{\text{characteristic}} = \mp U_1^{-\frac{1}{2}} \left\{ 1 - \varepsilon^{\frac{1}{2}} \frac{U_2}{U_1} + \varepsilon \left(\frac{1-2\varepsilon}{4} U_1 - \frac{1}{2} \frac{U_3}{U_1} + \frac{3}{8} \frac{U_2^2}{U_1^2} + \frac{V_1}{U_1^{3/2}} \right) + h.at. \right\} \quad (3.50)$$

The compatibility condition along a characteristic,

$$\cot \mu \frac{dq}{q} \pm d\Theta + \sin \mu \cos \mu \frac{dS}{\gamma R} = 0$$

similarly becomes

$$\begin{aligned} & \varepsilon^{\frac{2}{3}}(x) \left\{ u_1^{\frac{1}{3}} du_1 \pm dv_1 - u_1^{\frac{1}{3}} C_0 dy \right\} \\ & + \varepsilon^{\frac{2}{3}}(x) \left\{ \frac{1}{2} \frac{u_2}{u_1} u_1^{\frac{1}{3}} [du_1 - C_0 dy] + u_1^{\frac{1}{3}} [du_2 - C_{\frac{1}{2}} y^{\frac{1}{2}} dy] \pm dv_2 \right\} \\ & + h. o. t. = 0 \end{aligned} \quad (3.51)$$

where the entropy variation has been written as an expansion in y as previously discussed in Section II. This expression can be simplified by writing the velocities in terms of the modified velocity potential,

$$u_1 = \phi_x^{(1)} \quad (3.22)$$

$$v_1 = \phi_y^{(1)} - C_0 x \quad (3.23)$$

The first order compatibility condition becomes

$$\left[\phi_x^{(1)} \right]^{\frac{1}{2}} d\phi_x^{(1)} \pm d\phi_y^{(1)} + C_0 dx - \left[\phi_x^{(1)} \right]^{\frac{1}{2}} C_0 dy = 0.$$

Using the first order slope along the characteristic from (3.50) to relate dx and dy the above expression becomes simply

$$\left[\phi_x^{(1)} \right]^{\frac{1}{2}} d\phi_x^{(1)} \pm d\phi_y^{(1)} = 0 \quad (3.52)$$

which has exactly the form as for irrotational flow, but with the modified velocity potential instead of the usual velocity potential.

The singular point is a node with integral curves in its neighborhood given by

$$\frac{(t-s)^6}{\left(t - \frac{5}{6}s - \frac{3}{8}\right)^5} = \text{constant} \quad (3.53)$$

All integral curves through this point have slope $\frac{dt}{ds} = \frac{5}{6}$ except for the singular solution $t = s$. The inverse transformation gives

$$\eta = \eta_c e^{[-\frac{2}{9}(s-\frac{3}{4}) + h.o.t.]} \quad (3.54)$$

where η_c is the value of η associated with the singularity. The absolute value of η is decreasing as the singularity is approached from below the line $t = \frac{9}{4}$ and increasing when approached from above. Presumably, then, η_c will be negative for solutions approaching from below and positive for solutions approaching from above since the singularities representing $\eta \rightarrow +\infty$ and $\eta \rightarrow -\infty$ are both below the line $t = \frac{9}{4}$. Now, the slope of a curve $\eta = \eta_c$ with η_c a constant value is

$$\left(\frac{d\gamma}{dx}\right)_{\eta=\eta_c} = \mp \frac{2}{3} \eta_c^{-1} \gamma^{-\frac{1}{2}} \quad (3.55)$$

Along any characteristic the slope is

$$\frac{d\gamma}{dx} = \mp u_1^{-\frac{1}{2}} \quad (3.56)$$

to first order from (3.50). If $t = \frac{9}{4}$, then $u_1 = \frac{9}{4} \gamma_c^2 \gamma$ and (3.56) becomes identical to (3.55). The negative sign refers to a left running characteristic while the positive sign indicates a right running characteristic. Thus the curve $\gamma = \gamma_c$ has everywhere the slope of a first order characteristic.

If the first order compatibility condition (3.52) is written in terms of s and t and $t = \frac{9}{4}$ is imposed, the only value of s that satisfies the relation is $s = \frac{9}{4}$. This is true for either the left running or right running characteristic slope. Thus only the point $s = \frac{9}{4}$ along the line $t = \frac{9}{4}$ can represent a characteristic in the flow. Clearly, then, the singular point represents characteristics in the flow. Note that $(t-3s)$ is negative in the neighborhood of this singularity so that near this point

$$\frac{d(\ln \gamma)}{dt} < 0 \quad (3.57)$$

as pointed out by Vaglio-Laurin. Since

$$t = \gamma^{-2} \left(\frac{df_1}{d\gamma} \right),$$

the inverse transformation (3.37) implies

$$\frac{1}{\gamma} \frac{d}{d\gamma} \left(\frac{df_1}{d\gamma} \right) = 2t + (t-3s) \frac{dt}{ds}, \quad (3.58)$$

so that for t and s near $\frac{9}{4}$ and with $\frac{dt}{ds} \leq 1$ for all integral curves through the singular point, the expression on the right hand side of (3.58) is positive. The behavior of u_1 is like

that of $\frac{df_1}{d\eta}$, suggesting in view of (3.58) that u_1 is increasing if η is positive or decreasing if η is negative. If $\frac{dt}{ds} = 1$ there is no acceleration in the neighborhood of the characteristic represented by this singular point.

This discussion leads to the conclusion that indeed a valid solution curve can pass through this singular point and that two $\eta = \text{constant}$ lines may be associated with characteristics in the flow; one to be identified with the right running characteristic which intersects the corner and the other the first left running characteristic associated with the corner expansion fan. The asymptotic expansions valid in the neighborhood of these characteristics for solution curves passing through the singular point with slope (5/6) are

$$f_1 = \frac{9}{4} \eta_c^3 \left[1 + \left(\frac{\eta}{\eta_c} - 1 \right) + \frac{1}{6} \left(\frac{\eta}{\eta_c} - 1 \right)^2 + \mathcal{C} \left(\frac{\eta}{\eta_c} - 1 \right)^{\frac{11}{5}} + \mathcal{O} \left(\frac{\eta}{\eta_c} - 1 \right)^{\frac{12}{5}} \right], \quad (3.59)$$

$$u_1 = \frac{9}{4} \eta_c^3 \left[1 + \frac{1}{3} \left(\frac{1}{\eta_c} \frac{x}{y^{\frac{1}{2}}} - 1 \right) + \frac{11}{5} \mathcal{C} \left(\frac{1}{\eta_c} \frac{x}{y^{\frac{1}{2}}} - 1 \right)^{\frac{6}{5}} + \mathcal{O} \left(\frac{1}{\eta_c} \frac{x}{y^{\frac{1}{2}}} - 1 \right)^{\frac{7}{5}} \right] \quad (3.60)$$

and

$$V_1 = \frac{9}{4} \eta_c^3 y^{\frac{3}{2}} \left[\left[\frac{5}{2} - \left(\frac{3}{2} + \frac{C_0}{4\eta_c} \right) \frac{1}{\eta_c} \frac{x}{y^{\frac{1}{2}}} \right] + \frac{1}{2} \left(5 - \frac{1}{\eta_c} \frac{x}{y^{\frac{1}{2}}} \right) \left(\frac{1}{\eta_c} \frac{x}{y^{\frac{1}{2}}} - 1 \right) - \frac{33}{10} \mathcal{C} \frac{1}{\eta_c} \frac{x}{y^{\frac{1}{2}}} \left(\frac{1}{\eta_c} \frac{x}{y^{\frac{1}{2}}} - 1 \right)^{\frac{6}{5}} + \mathcal{O} \left(\frac{1}{\eta_c} \frac{x}{y^{\frac{1}{2}}} - 1 \right)^{\frac{7}{5}} \right] \quad (3.61)$$

with $\mathcal{C}$ an undetermined constant.

To examine the behavior of (3.31) at infinity, it is usual to map the (s,t) plane onto a hemisphere so that points at infinity are mapped onto the equator. This is accomplished by the conical transformation,

$$S = \frac{X_1}{X_0} \quad (3.62.a)$$

$$t = \frac{X_2}{X_0} \quad (3.62.b)$$

The projections of the integral curves on the hemisphere are in turn projected on a plane perpendicular to the (s,t) plane so the singularities at infinity which are mapped on the equator can be studied with two independent variables (see figure (3.2)). Towards this end choose $x_1 = 1$ so that the equation describing the projections of the solution curves onto the $x_1 = 1$ plane is

$$\frac{dx_2}{dx_0} = \frac{X_2 (X_2 - 3) (X_2 - \frac{3}{4} X_0) + 2 X_2^2 - \frac{3}{4} X_2 X_0 - \frac{15}{4} X_0}{X_0 (X_2 - 3) (X_2 - \frac{3}{4} X_0)} \quad (3.63)$$

There are two singularities of (3.64) occurring at $x_0 = 0$, $x_2 = 0$ and at $x_0 = 0$, $x_2 = 1$.

D. THE SINGULAR POINT AT (0, 0)

As observed by Vaglio-Laurin, all curves enter this singular point nearly tangent to the x_2 axis. That is, the right hand side of (3.63) becomes infinite at the singular point as long as

$x_2^2 x_0^{-1}$ does not approach $-(15/4)$ as x_2 and x_0 approach zero. The hemispherical projection requires that x_0 be non-negative, so this condition is always met. Solution curves passing through this singular point thus have $x_2 \gg x_0$ in its neighborhood. Vaglio-Laurin shows that the alternate assumption $x_0 \gg x_2$ leads to contradictory results. Using this approximation (3.63) can be reduced and integrated to give the integral curves in the neighborhood of this node as

$$x_2 = x_0^{\frac{1}{3}} \left[\bar{K} + \left(\frac{15}{2} - \frac{4}{3} \bar{K}^{\frac{3}{2}} \right) x_0^{\frac{1}{3}} \right]^{\frac{1}{2}} \quad (3.64)$$

where $\bar{K}$ is an arbitrary constant. The inverse transformation gives

$$\eta = \text{constant } S^{-\frac{1}{3}} \quad (3.65)$$

Traversing an integral curve through the singularity causes η to go through zero and change sign. The result (3.65) differs slightly from that given by Vaglio-Laurin (1960) which appears to be in error. This difference would have no effect on the numerical calculations reported by him in his paper.

E. THE SINGULAR POINT AT (0, 1)

The singular point at $x_0 = 0, x_2 = 1$ also offers a possible solution path for η passing through zero. Linearization of (3.63) in the neighborhood of the singular point and integration give the integral curves

$$(x_2 - 1)(x_0)^{\frac{1}{2}} = \text{constant} \quad (3.66)$$

The singularity is a saddle point with principal directions

$x_0 = 0$ and $x_2 = 1$. The principal direction $x_0 = 0$ is, of course, the projection of the equator of the hemisphere and is the representation of the line $s = \infty$ in the (s, t) plane. This does not correspond to a solution curve of the equation in the (s, t) plane. The principal direction $x_2 = 1$ corresponds to the limit as $s \rightarrow \infty$ of the solution curve $s = t$. The inverse transformation shows the behavior of η near this singularity to be

$$\eta = \text{constant } s^{-\frac{1}{2}}. \quad (3.67)$$

Thus a solution entering this singular point along the principal direction given by $x_2 = 1$ corresponds to $\eta \rightarrow 0$. Observe that according to (3.67) both f_1 and f_1' depend only on the square of the constant making them independent of its sign. Further, s depends on the square of η so that the integral curve in the (s, t) plane approaches the singular point and retreats from it along the same path as η increases through zero.

The singularities in the finite (s, t) plane are shown in figure (3.1). The integral curves in the neighborhood of each singularity are shown and typical integral curves are demonstrated. All the singularities are shown in the spherical projection of the (s, t) plane of figure (3.2).

Following the discussion of the singular points, it is possible to describe qualitatively the desired solution curve in the (s, t) plane. Clearly the desired solution curve originates at A. ($\eta \rightarrow -\infty$) and eventually ceases at B. ($\eta \rightarrow +\infty$). Further, η can only change sign at D. or E. ($\eta \rightarrow 0$). This is true since η cannot display any of the above behavior

at any regular point of the (s, t) plane.

Far upstream at the wall (3.42.a) indicates that $A_0 > 0$ for the flow to be subsonic. With this condition the integral curves emerge from A. in the direction of negative t and break towards the positive s direction. All solutions leave A. with slope $(5/3)$, but those with $C_0 > 0$ have slopes that increase after leaving A. (figure(3.1), type i) while those with $C_0 < 0$ have slopes decreasing after leaving A. (figure (3.1), type ii). Since $s=t$ is an integral curve of the equation, no other integral curve may intersect it except at one of the singularities A., C. or E. which lie along it. Any integral curve passing through the region $\{t < \frac{9}{4}, t < s\}$ cannot intersect point C. without first meeting the limiting line, $t = \frac{9}{4}$. Any such solution curve must be rejected on physical grounds. The horizontal velocity reaches sonic speed at the intersection of the solution curve with the s -axis. Thus no physically acceptable solution leaving A. with $t < s$ and intersecting the s -axis (sonic line) is possible. However, it is possible for integral curves leaving A. with $t < s$ to reach B. by passing through the singular point D. before crossing the sonic line. Since point D. corresponds to $\eta = 0$ in the physical plane, this solution has its sonic line downstream from the vertical axis ($\eta = 0$).

The integration of (3.36) numerically, starting at A. with the asymptotic expansion (3.41), gives solutions entering D. only for C_0 sufficiently less than zero. All curves with positive C_0 have a limit line. Of course, negative C_0 implies

that $\frac{\partial u_1}{\partial y} < 0$ at the wall, so these solutions are not of current interest. However, this discussion does point out that a subsonic incoming velocity profile with $\frac{\partial u_1}{\partial y} < 0$ has its sonic line downstream of the y-axis. The slower moving fluid farther from the wall takes a greater distance to be accelerated to sonic velocity. This reinforces the hypothesis that the desired incoming velocity profile, with $\frac{\partial u_1}{\partial y} > 0$, should have its sonic line upstream from the vertical axis.

Solutions from A. with $t > s$ are also unacceptable since they correspond to supersonic incoming flows. That is for $t > s$ in the neighborhood of A., $A_0 < 0$ and $u_1 > 0$.

The only other other integral curve leaving A. still to be considered is the line $t=s$. This integral curve proceeds from A. ($\eta \rightarrow -\infty$) through C. ($\eta = \eta_{c-}$) to E. ($\eta = 0$). As η continues to increase the solution curve retraces its path to C. ($\eta = \eta_{c+}$) and then to B. ($\eta \rightarrow +\infty$) as shown in figure (3.1), type iii. Here η_{c+} and η_{c-} refer respectively to the left running and right running characteristics which intersect the corner. The solution curve $t=s$ corresponds to f_1 proportional to η . Substitution into (3.29) and (3.30) gives

$$f_1 = C_0 \eta, \quad (3.68)$$

$$u_1 = C_0 \eta, \quad (3.69)$$

$$v_1 = 0. \quad (3.70)$$

Since both s and t have the value $(9/4)$ at the characteristics which intersect the corner, the values of η_{c+} and η_{c-} depend only on C_0 . Thus from the definition of t ,

$$C_0 = \frac{9}{4} \eta_c^2$$

so that (3.67) gives

$$\eta_{c+} = -\eta_{c-} = \frac{2}{3} C_0^{\frac{1}{2}} \quad . \quad (3.71)$$

The solution described by this integral of (3.31) is a parallel, vortical flow with constant $\left(\frac{\partial u_1}{\partial y}\right)$ for $-\infty < \eta \leq \eta_{c+}$. The sonic line lies along the upstream wall. Downstream of the leading (η_{c+}) characteristic from the corner the solution describes a transonic vortical flow expanding about a corner. This problem has been treated by Adamson (1968) using left running characteristics and streamlines as coordinates. The asymptotic expansions valid near the leading characteristic (3.59) and near the corner (3.47) give velocity components that agree with those in that paper.

If $\eta_{c+} = 1$ then $C_0 = (9/4)$ and (3.31) can be numerically integrated from C to B using the asymptotic expansion valid near $\eta \rightarrow +\infty$, (3.47), to start the integration. The value of B_0 that gives $\eta_{c+} = 1$ is found by numerical iteration to be

$$B_0 = 1.94852 \quad . \quad (3.72)$$

Further, this numerical integration can be employed to give the required value of the constant α in the asymptotic expansion (3.59).

At this juncture the only possible solution of the desired type has been shown to be a parallel incoming flow with the

sonic line along the upstream wall and $\frac{\partial u_1}{\partial y} = C_0$. This flow has no subsonic region and lacks the curved boundary layer profile typical of a boundary layer. However, abandonment of this method of solution is not justified for it will be shown that the difficulty is inherent in the first order equations.

It is clear from examination of equation (3.21),

$$u_1 \frac{\partial^2 u_1}{\partial x^2} + \left(\frac{\partial u_1}{\partial x} \right)^2 = \frac{\partial^2 u_1}{\partial y^2}, \quad (3.21)$$

that in the neighborhood of the sonic line where $u_1=0$,

$$\frac{\partial^2 u}{\partial y^2} - \text{h.o.t} = \frac{\partial^2 u_1}{\partial y^2} \approx \left(\frac{\partial u_1}{\partial x} \right)^2 > 0. \quad (3.73)$$

Although a supersonic boundary layer with $\frac{\partial^2 u}{\partial y^2}$ positive near the sonic line is conceivable, it should be generally expected that $\frac{\partial^2 u}{\partial y^2}$ is negative. Typically, in fact, $\frac{\partial^2 u}{\partial y^2}$ is negative throughout a boundary layer. It is not possible to obtain this condition from a solution of the first order equations. The most boundary layer like solution obtainable from this equation is the one presented where $\frac{\partial u_1}{\partial y} = C_0$ and $\frac{\partial^2 u_1}{\partial y^2} = 0$ for $x < 0$. It seems then, that this first order solution must be used and variations from the constant slope upstream velocity profile are due to the second order terms in the velocity asymptotic expansion. Physically this condition implies that although perturbations from sonic velocity can be used to treat an irrotational flow, it is necessary to perturb from a rotational

velocity profile in a basically rotational flow. That is, the perturbation from the incoming flow should be written as

$$U = 1 + C_0 Y + u' \quad (3.74)$$

with u' the perturbation of the velocity profile. The fact that the transonic corner region currently under consideration is small requires $C_0 Y$ to be small, thus lending validity to the present approach. It seems clear that second order terms are necessary to describe the flow being contemplated.

From (3.74) it is immediately apparent that special care is necessary as $Y \rightarrow 0$ to insure the uniformity of the expansion. That is, a region can exist where the $C_0 Y$ term and the perturbation term, u' , may be of the same order. Here another expansion scheme would need to be considered. This turns out to be the case; solutions will be obtained for both regions and matched.

In taking account of second order terms, it becomes necessary to distinguish between two situations. The entropy and total enthalpy variation has the form

$$(\gamma+1) \left[C_0 + \varepsilon^{\frac{1}{2}} C_{\frac{1}{2}} Y^{\frac{1}{2}} + \varepsilon C_1 Y + h. o. t. \right] .$$

As discussed in Section II, $C_{\frac{1}{2}} = 0$ if there is no wall heat transfer or pressure gradient in the boundary layer leading into the inviscid transonic corner region. The ordering of the second order equations depends on whether $C_{\frac{1}{2}}$ is zero or non-zero. The governing equations are different for each case so each case must be treated independently. Section IV treats the case where $C_{\frac{1}{2}} \neq 0$. The results for $C_{\frac{1}{2}} = 0$ are presented in Section V.

IV THE SECOND ORDER SOLUTIONS FOR $C_{\frac{1}{2}} \neq 0$

A second order modified velocity potential defined by

$$u_2 = \phi_x^{(2)} + \frac{2}{3} C_{\frac{1}{2}} \gamma^{\frac{2}{3}} \quad (4.1.a)$$

$$V_2 = \phi_y^{(2)} \quad (4.1.b)$$

is introduced into the second order equations (3.16) and (3.20).

Thus the second order Crocco equation is identically satisfied

while the second order gasdynamic equation becomes

$$- \left[\phi_x^{(2)} + \frac{2}{3} C_{\frac{1}{2}} \gamma^{\frac{2}{3}} \right] \phi_{xx}^{(2)} - \phi_x^{(2)} \phi_{xy}^{(2)} + \phi_{yy}^{(2)} = 0 \quad (4.2)$$

As is usual with asymptotic expansion problems of this sort,

the second order equation is linear in the second order variables

and it must have the same similarity variable as the first order.

Thus the potential must have the form

$$\phi^{(2)} = \gamma^3 f_2(\eta) \quad (4.3.a)$$

$$\phi_x^{(2)} = \gamma^{\frac{2}{3}} \frac{df_2}{d\eta} \quad (4.3.b)$$

$$\phi_y^{(2)} = 3\gamma^2 \left[f_2 - \frac{1}{2} \eta \frac{df_2}{d\eta} \right] \quad (4.3.c)$$

The linear ordinary differential equation for f_2 is then

$$\left(\frac{df_1}{d\eta} - \frac{9}{4}\eta^2\right)\frac{d^2f_2}{d\eta^2} + \left(\frac{d^2f_1}{d\eta^2} + \frac{21}{4}\eta\right)\frac{df_2}{d\eta} - 6f_2 = -\frac{2}{3}C_{\frac{1}{2}}\frac{d^2f_1}{d\eta^2} \quad (4.4)$$

Since $C_0 = \frac{9}{4}\eta_c^2$ it is convenient to normalize (4.4) by defining new variables

$$f = \frac{\eta}{\eta_c} \quad , \quad (4.5.a)$$

$$\eta_c g_2(f) = f_2(\eta) \quad , \quad (4.5.b)$$

and
$$\frac{9}{4}\eta_c^3 g_1(f) = f_1(\eta) \quad , \quad (4.5.c)$$

to obtain

$$\left[\frac{dg_1}{df} - f^2\right]\frac{d^2g_2}{df^2} + \left[\frac{d^2g_1}{df^2} + \frac{7}{3}f\right]\frac{dg_2}{df} - \frac{8}{3}g_2 = -\frac{2}{3}C_{\frac{1}{2}}\frac{d^2g_1}{df^2} \quad (4.6)$$

To reduce this equation to a recognizable form it is necessary to rescale the independent variable and shift the origin of coordinates. To accomplish this we define

$$z = \frac{1}{2}(1+f) \quad , \quad (4.7.a)$$

$$g_2(f) = G(z) \quad , \quad (4.7.b)$$

and

$$g_1(f) = \tilde{F}(z) \quad .$$

Then (4.6) becomes

$$\left[\frac{1}{8}\left(\frac{d\tilde{F}}{dz} - 2\right) + z(z+1)\right]\frac{d^2G}{dz^2} + \left[\frac{1}{8}\frac{d^2\tilde{F}}{dz^2} - \frac{2}{6} + \frac{7}{3}z\right]\frac{dG}{dz} - \frac{8}{3}G = -\frac{1}{6}C_{\frac{1}{2}}\frac{d^2\tilde{F}}{dz^2} \quad (4.8)$$

Upstream of the leading characteristic from the corner the analytical form of the first order solution, F , was found to be

$$\tilde{F} = (2z-1) \quad \text{for } z \leq 1 \quad (\eta \leq \eta_{c+}) \quad (4.9)$$

Thus

$$z(z-1) \frac{d^2 G}{dz^2} + \left[-\frac{2}{6} + \frac{2}{3} z \right] \frac{dG}{dz} - \frac{8}{3} G = 0 \quad (4.10)$$

for $z \leq 1$. This is the Gauss Hypergeometric equation which has three regular singular points, here chosen at $0, 1, -\infty$. These correspond to $\eta = \eta_c^-$, $\eta = \eta_{c+}$, $\eta = -\infty$ in the physical plane. That is, the second order solution can also have mild discontinuities at the two first order characteristics hitting the corner. This suggests that the first order characteristics must also be the characteristics to second order and this can be shown to be the case.

The general solution to (4.10) for $|z| < 1$ is

$$G = \hat{D}_1 \left[1 - \frac{16}{7} z(1-z) \right] + \hat{D}_2 \left[z(1-z) \right]^{\frac{13}{6}} F\left(3, \frac{2}{3}; \frac{13}{6}; z\right) \quad (4.11)$$

where $F\left(3, \frac{2}{3}; \frac{13}{6}; z\right)$ is the Gauss Hypergeometric function,

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{\frac{\Gamma(a+n)}{\Gamma(a)} \frac{\Gamma(b+n)}{\Gamma(b)}}{\frac{\Gamma(c+n)}{\Gamma(c)} \Gamma(n+1)} z^n, \quad (4.12)$$

with $\Gamma(a)$ representing the usual Gamma function while $\hat{D}_1, \hat{D}_2$

are arbitrary constants. The hypergeometric function has an analytic continuation for $z \leq -1$ in terms of $\frac{z}{z-1}$,

$$F\left(3, \frac{7}{3}; \frac{19}{6}; z\right) = (1-z)^{-\frac{7}{3}} F\left(\frac{7}{3}, \frac{1}{6}; \frac{19}{6}; \frac{z}{z-1}\right), \quad (4.13)$$

where, of course, equals means identically equal in domains where both converge. Observe that $F\left(\frac{7}{3}, \frac{1}{6}; \frac{19}{6}; \frac{z}{z-1}\right)$ converges at $\frac{z}{z-1} = 1$ ($z \rightarrow -\infty$). Then for $z \leq -1$ the solution is

$$G = \hat{D}_1 \left[1 - \frac{16}{7} z(1-z)\right] + \hat{D}_2 e^{i\frac{19}{6}\pi} \left[-z(1-z)\right]^{\frac{19}{6}} (1-z)^{-\frac{7}{3}} F\left(\frac{7}{3}, \frac{1}{6}; \frac{19}{6}; \frac{z}{z-1}\right). \quad (4.14)$$

However, to clarify the behavior of G as $z \rightarrow -\infty$ the solution can be represented in terms of z^{-1} for $z^{-1} < (-1)$ as

$$F\left(\frac{7}{3}, \frac{1}{6}; \frac{19}{6}; \frac{z}{z-1}\right) = A_1 \left(\frac{z-1}{z}\right)^{\frac{7}{3}} F\left(\frac{7}{3}, \frac{1}{6}; \frac{1}{3}; \frac{1}{z}\right) + A_2 \left(\frac{z-1}{z}\right)^{\frac{5}{6}} (1-z)^{-\frac{1}{3}} F\left(\frac{5}{6}, -\frac{4}{3}; \frac{5}{3}; \frac{1}{z}\right),$$

where

$$A_1 = \frac{\Gamma(\frac{19}{6})\Gamma(\frac{2}{3})}{\Gamma(3)\Gamma(\frac{5}{6})} \quad \text{and} \quad A_2 = \frac{\Gamma(\frac{19}{6})\Gamma(-\frac{2}{3})}{\Gamma(\frac{7}{3})\Gamma(\frac{1}{6})} \quad (4.15)$$

Notice that $G(z)$ cannot be written directly in terms of (z^{-1})

since there is no overlapping of the domains of convergence. The $\frac{z}{z-1}$ representation is necessary to tie the constants together. With this, the behavior of G as $z \rightarrow -\infty$ can be seen to be

$$\begin{aligned} G = & \left(\frac{16}{7} \hat{D}_1 + \bar{D}_2 A_2 \right) (-z)^2 + A_1 \bar{D}_2 (-z)^{\frac{4}{3}} \\ & + \left(\frac{16}{7} \hat{D}_1 + \bar{D}_2 A_2 \right) (-z) + \frac{2}{3} A_1 \bar{D}_2 (-z)^{\frac{1}{3}} \\ & + \left(\hat{D}_1 + \frac{2}{16} \bar{D}_2 A_2 \right) + \dots \end{aligned} \quad (4.16)$$

where $\bar{D}_2 = \hat{D}_2 (-1)^{(13/6)}$.

The derivatives of the second order velocity potential can be written in terms of G as

$$\phi_x^{(1)} = -\frac{1}{2} y^{\frac{3}{2}} \frac{dG}{d(-z)} \quad (4.17.a)$$

$$\phi_y^{(1)} = 3y^{\frac{3}{2}} \left[G - \frac{1}{2} (-z) \frac{dG}{d(-z)} - \frac{1}{4} \frac{dG}{d(-z)} \right] \quad (4.17.b)$$

Relation (4.16) then gives the behavior of the velocities as $z \rightarrow -\infty$, written here in terms of x and y , as

$$\phi_x^{(1)} = -\frac{1}{2\eta_{c+}} \left(\frac{16}{7} \hat{D}_1 + \bar{D}_2 A_2 \right) (-x) - \frac{2}{3} A_1 \bar{D}_2 \left(\frac{1}{2\eta_{c+}} \right)^{\frac{1}{3}} (-x)^{\frac{1}{3}} y + O(-z)^{-\frac{2}{3}}, \quad (4.18.a)$$

$$\phi_y^{(u)} = \frac{1}{2} A_1 \bar{D}_2 \left(\frac{1}{2\eta_{c+}} \right)^{\frac{1}{3}} (-x)^{\frac{4}{3}} + 3\eta_{c+} \left(\frac{3}{7} \hat{D}_1 + \frac{3}{16} \bar{D}_2 A_2 \right) y^2 + O(-x)^{-\frac{2}{3}} \quad (4.18.b)$$

Since the horizontal velocity does not vanish along the upstream wall, there must exist a region where this second order solution exceeds the first order solution already obtained. In fact, writing the horizontal velocity in terms of unstretched variables gives

$$U = 1 + C_0 Y - \frac{1}{2\eta_{c+}} \left(\frac{16}{7} \hat{D}_1 + \bar{D}_2 A_2 \right) (-X) + \dots \quad (4.19)$$

as $Y \rightarrow 0$ or $X \rightarrow -\infty$. Thus, this form of the solution may not be uniformly valid when Y is smaller than a constant times X . It is necessary to define an inner region with a different stretching of coordinates so that this nonuniformity does not arise.

The region where the outer solution fails is defined by

$$Y = O(X) \quad (4.20)$$

To look at the detail in this region, new stretching parameters must be introduced by

$$Y = \tilde{\varepsilon} \tilde{Y} \quad , \quad \tilde{Y} = O(1) \quad (4.21.a)$$

$$X = \tilde{\tau} \tilde{X} \quad , \quad \tilde{X} = O(1) \quad (4.21.b)$$

In this inner region, the relation between the stretching parameters is given by (4.20) as

$$\tilde{\tau} = (\delta+1)^{\frac{1}{2}} \tilde{\varepsilon} \quad (4.22)$$

where the factor $(\gamma+1)^{\frac{1}{2}}$ is added for convenience. Notice that this inner region is a wedge shaped region and cannot be considered "thin" in the sense of small γ alone. In fact, this inner region includes the limit $X \rightarrow -\infty, \gamma > 0$ as well as the limit $\gamma \rightarrow 0, X < 0$.

Consider new velocity asymptotic expansions valid in the inner region to be defined by

$$U = 1 + \sum_{j=1}^{\infty} \tilde{E}_j(\tilde{\epsilon}) \tilde{U}_j(\tilde{x}, \tilde{y}) \quad (4.23.a)$$

$$V = \sum_{j=1}^{\infty} \tilde{V}_j(\tilde{\epsilon}) \tilde{V}_j(\tilde{x}, \tilde{y}) \quad (4.23.b)$$

Thus $\tilde{E}_1$ should be identified with E_1 of the outer region since both represent the perturbation from sonic speed.

The gasdynamic equation becomes

$$\begin{aligned} & -(\gamma+1) \tilde{E}_1^2 \tilde{U}_1 \frac{\partial \tilde{U}_1}{\partial \tilde{x}} + \frac{\tilde{\epsilon}}{\tilde{E}_1} \tilde{V}_1 \frac{\partial \tilde{V}_1}{\partial \tilde{y}} + \frac{\gamma+1}{2} \tilde{E}_1^3 \tilde{U}_1^2 \frac{\partial \tilde{U}_1}{\partial \tilde{x}} - (\gamma+1) \tilde{E}_1 \tilde{E}_2 \tilde{U}_2 \frac{\partial \tilde{U}_1}{\partial \tilde{x}} \\ & - (\gamma+1) \tilde{E}_1 \tilde{E}_2 \tilde{U}_1 \frac{\partial \tilde{U}_2}{\partial \tilde{x}} + \frac{\tilde{\epsilon}}{\tilde{E}_1} \tilde{V}_2 \frac{\partial \tilde{V}_2}{\partial \tilde{y}} - (\gamma+1) \frac{\tilde{\epsilon}}{\tilde{E}_1} \tilde{E}_1 \tilde{V}_1 \tilde{U}_1 \frac{\partial \tilde{V}_1}{\partial \tilde{y}} \\ & - \tilde{V}_1^2 \tilde{V}_1 \frac{\partial \tilde{V}_1}{\partial \tilde{x}} - \frac{\tilde{\epsilon}}{\tilde{E}_1} \tilde{E}_1 \tilde{V}_1 \tilde{V}_1 \frac{\partial \tilde{U}_1}{\partial \tilde{y}} + h. o. t. = 0 \quad (4.24) \end{aligned}$$

and the Crocco equation becomes

$$\tilde{v}_1 \frac{\partial \tilde{v}_1}{\partial \tilde{x}} - \frac{\tilde{v}_1}{\tilde{\epsilon}} \tilde{E}_1 \frac{\partial \tilde{u}_1}{\partial \tilde{y}} + \text{h.o.t.} = (\gamma+1)^{\frac{1}{2}} \tilde{\epsilon} C_0 - \text{h.o.t.} \quad (4.25)$$

The lowest order terms of (4.24) are

$$-(\gamma+1) \tilde{E}_1^2 \tilde{u}_1 \frac{\partial \tilde{u}_1}{\partial \tilde{x}} + \frac{\tilde{v}_1}{\tilde{\epsilon}} \tilde{v}_1 \frac{\partial \tilde{v}_1}{\partial \tilde{y}} = 0 \quad .$$

Generally neither $\tilde{u}_1 \frac{\partial \tilde{u}_1}{\partial \tilde{x}}$ nor $\frac{\partial \tilde{v}_1}{\partial \tilde{y}}$ will be of higher order in the inner region if matching is to take place with the outer region.

This requires that

$$\tilde{v}_1 = (\gamma+1)^{\frac{1}{2}} \tilde{E}_1^2 \quad (4.26)$$

where again the $(\gamma+1)^{\frac{1}{2}}$ is included for convenience. The lowest order term in the left hand side of the Crocco equation (4.25)

is then of order $\tilde{E}_1$. Thus,

$$\tilde{E}_1 = \tilde{\epsilon} \quad (4.27)$$

and the first order inner equations are

$$\tilde{u}_1 \frac{\partial \tilde{u}_1}{\partial \tilde{x}} - \frac{\partial \tilde{v}_1}{\partial \tilde{y}} = 0 \quad (4.28.a)$$

$$\frac{\partial \tilde{u}_1}{\partial \tilde{y}} = C_0 \quad . \quad (4.28.b)$$

Equation (4.28.a) is the same equation found earlier in the

outer region. However, (4.28.b) shows that the $\frac{\partial \tilde{u}_1}{\partial \tilde{y}}$ term has

become dominant over the $\frac{\partial \tilde{v}_1}{\partial \tilde{x}}$ term which appears in irrotational

flow problems and appeared in the outer region first order equations. This should be expected, since derivatives with respect to x are of lower order in the inner region than in the outer. This situation appears in the boundary layer equations and can be anticipated here where the solution must match upstream with the subsonic portion of the incoming boundary layer. Solutions to (4.28) are easily found to be

$$\tilde{u}_1 = C_0 \tilde{y} + f(\tilde{x}) \quad (4.29.a)$$

$$\tilde{v}_1 = \left[\frac{1}{2} C_0 \tilde{y}^2 + f(\tilde{x}) \tilde{y} \right] \frac{df(\tilde{x})}{d\tilde{x}}, \quad (4.29.b)$$

where $v_1 = 0$ at $y = 0$. The unknown function can be inferred by the principle of eliminability. That is, it must be that

$$\tilde{\epsilon} [C_0 \tilde{y} + f(\tilde{x})]$$

is independent of $\tilde{\epsilon}$ when written in unstretched variables. Thus $f(\tilde{x}) = K \tilde{x}$ where K is an arbitrary constant to be determined.

With this the inner solutions are

$$\tilde{u}_1 = C_0 \tilde{y} + K \tilde{x} \quad (4.30.a)$$

$$\tilde{v}_1 = K \left(\frac{1}{2} C_0 \tilde{y}^2 + K \tilde{x} \tilde{y} \right) \quad (4.30.b)$$

Notice that the perturbation from sonic velocity is the same in both regions. Thus $\varepsilon = \tilde{\varepsilon}$ and the inner region is really a region where X is stretched less than in the outer region. The Y stretching is the same in either region. Thus the matching must occur for both limits $y \rightarrow 0$, x fixed and $x \rightarrow -\infty$, y fixed. To facilitate the matching, (4.30) are written in outer variables using the relations

$$y = \frac{Y}{\varepsilon} = \frac{\tilde{\varepsilon}}{\varepsilon} \tilde{y} = \tilde{y}, \quad x = \frac{X}{\varepsilon^{\frac{1}{2}}} = \frac{\tilde{\varepsilon}}{\varepsilon^{\frac{1}{2}}} \tilde{x} = \varepsilon^{-\frac{1}{2}} \tilde{x}$$

to give

$$u_{inner} = 1 + \tilde{\varepsilon} \tilde{u}_1 + h.o.t. = 1 + \varepsilon C_0 y + \varepsilon^{\frac{3}{2}} K x + h.o.t. \quad (4.31.a)$$

$$V_{inner} = (\gamma+1)^{\frac{1}{2}} \tilde{\varepsilon}^2 \tilde{V}_1 + h.o.t. = (\gamma+1)^{\frac{1}{2}} K \left[\varepsilon^{\frac{1}{2}} C_0 y^2 + \varepsilon^{\frac{5}{2}} K x y + h.o.t. \right] \quad (4.31.b)$$

When the asymptotic expansions (4.18) are substituted into the outer region velocity expansions there result

$$u_{outer} = 1 + \varepsilon C_0 y - \varepsilon^{\frac{3}{2}} \frac{1}{2\gamma_c} \left(\frac{16}{7} \hat{D}_1 + \bar{D}_2 A_2 \right) (-x) + O(x^{\frac{1}{2}} y) \quad (4.32.a)$$

$$V_{outer} = (\gamma+1)^{\frac{1}{2}} \varepsilon^2 \left[\frac{1}{2} A_1 \bar{D}_2 \left(\frac{1}{2\gamma_c} \right)^{\frac{1}{2}} (-x)^{\frac{3}{2}} + 3\gamma_c \left(\frac{2}{7} \hat{D}_1 + \frac{2}{16} \bar{D}_2 A_2 \right) y^2 + O(x^{\frac{1}{2}} y^3) \right] \quad (4.32.b)$$

The comparison of (4.31) with (4.32) shows that for matching

to occur,

$$\bar{D}_2 = 0 \quad (4.33)$$

$$K = \frac{8}{7} \frac{\hat{D}_1}{\eta_{c+}} \quad (4.34)$$

The constant K remains arbitrary and this matching can be carried out for any value of K . This constant would presumably be determined by matching with some solution which is valid on a larger length scale than that being considered here. However, as will be shown, K can be identified with the dimensionless pressure gradient upstream of the corner. As such its value should be expected to be of order one and $K=1$ has been used in numerical computations.

It is important to note the inner solution does not give a term of order $\epsilon^{\frac{2}{3}}$. Thus the condition requiring v_1 to vanish at the upstream wall, which led to (3.43), is correct.

From this analysis the solution for G , at least in the range $z < 0$ is simply

$$G = \frac{7}{8} \eta_{c+} K \left[1 - \frac{16}{7} z(1-z) \right] \quad (4.35)$$

The general form for the solution in the range $0 \leq z \leq 1$ is given by (see (4.11))

$$G = D_1^* \left[1 - \frac{16}{7} z(1-z) \right] + D_2^* \left[z(1-z) \right]^{\frac{13}{6}} F\left(3, \frac{7}{3}, \frac{13}{6}; z\right) \quad (4.36)$$

Continuity at $z=0$ requires

$$D_1^* = \frac{7}{8} \eta_{c+} K = \frac{7}{12} K C_0^{\frac{1}{2}} \quad (4.37)$$

However, D_2^* cannot be determined at $z=0$, for the only condition that could be imposed is a physically meaningless "boundedness" condition on the third derivative of G . That is, since $z=0$ corresponds to the right running characteristic intersecting the corner, only the velocities and thus the velocity potential can be considered as continuous. The acceleration and higher derivatives of the velocity potential may have discontinuities at $z=0$. Of course, the same must be true at the left running characteristic from the corner at $z=1$, and so D_2^* must be determined at $z=1$.

The hypergeometric function of (4.36) does not converge at $z=1$. To study the solution near $z=1$, its analytic continuation in terms of $(1-z)$ must be employed. This is given by

$$\begin{aligned} F\left(3, \frac{7}{3}; \frac{13}{6}; z\right) &= \frac{\Gamma\left(\frac{13}{6}\right) \Gamma\left(-\frac{13}{6}\right)}{\Gamma\left(\frac{1}{6}\right) \Gamma\left(\frac{5}{6}\right)} F\left(3, \frac{7}{3}; \frac{13}{6}; 1-z\right) \\ &+ \frac{\Gamma\left(\frac{13}{6}\right) \Gamma\left(\frac{13}{6}\right)}{\Gamma(3) \Gamma\left(\frac{7}{3}\right)} (1-z)^{-\frac{13}{6}} F\left(\frac{1}{6}, \frac{5}{6}; -\frac{7}{6}; 1-z\right) \end{aligned} \quad (4.38)$$

for $0 \leq (1-z) < 1$. Observe that at $z=1$ the solution (4.36) has the value,

$$G(1) = \frac{7}{8} \eta_c + K + D_2^* \frac{\Gamma\left(\frac{13}{6}\right) \Gamma\left(\frac{13}{6}\right)}{\Gamma(3) \Gamma\left(\frac{7}{3}\right)} \quad (4.39)$$

and further,

$$\left(\frac{dG}{dz}\right)_{z=1} = \frac{16}{7} \left[\frac{7}{8} \eta_c + K + D_2^* \frac{\Gamma\left(\frac{13}{6}\right) \Gamma\left(\frac{13}{6}\right)}{\Gamma(3) \Gamma\left(\frac{7}{3}\right)} \right] \quad (4.40)$$

As suspected, the derivative at $z=1$ is just a multiple of the value of the function. This must be so since there is only one constant to be determined at $z=1$.

Now it is required that the velocities match at $z=1$. To effect this, asymptotic expansions for G are needed near, but just downstream of, the characteristic at $z=1$. For convenience, equation (4.6) relating $g_2(\mathcal{F})$ to $g_1(\mathcal{F})$ will be employed where $g_1(\mathcal{F}) = \frac{4}{9}\mathcal{F}_c^{-3}f_1(\mathcal{F})$. The asymptotic expansion for f_1 valid just downstream of the characteristic is given by (3.59) so that for $(\mathcal{F}-1)$ small but positive,

$$g_1(\mathcal{F}) = 1 + (\mathcal{F}-1) + \frac{1}{6}(\mathcal{F}-1)^2 + \mathcal{O}(\mathcal{F}-1)^{\frac{11}{5}} + \mathcal{O}(\mathcal{F}-1)^{\frac{13}{5}}. \quad (4.41)$$

If $\mathcal{F}$ is substituted for $(\mathcal{F}-1)$ the equation (4.6) for g_2 becomes

$$\begin{aligned} & \left[-\frac{15}{4}\mathcal{F} + \frac{11}{5}\mathcal{G}\mathcal{F}^{\frac{6}{5}} - \frac{9}{4}\mathcal{F}^2 + \text{h.o.t.} \right] \frac{d^2 g_2}{d\mathcal{F}^2} \\ & + \left[6 + \frac{66}{25}\mathcal{G}\mathcal{F}^{\frac{1}{5}} + \text{h.o.t.} \right] \frac{d g_2}{d\mathcal{F}} \\ & - 6g_2 = -\frac{2}{3}\mathcal{C}_{\frac{1}{2}} \left[\frac{3}{4} + \frac{66}{25}\mathcal{G}\mathcal{F}^{\frac{1}{5}} + \text{h.o.t.} \right] \end{aligned} \quad (4.42)$$

where now $0 < \mathcal{F} \ll 1$. Asymptotic expansions for the two linearly independent homogeneous solutions are found to be,

$$g_2^{(1)} = a_0 \left[\mathcal{F}^{\frac{13}{5}} + \frac{572}{75}\mathcal{G}\mathcal{F}^{\frac{14}{5}} + \text{h.o.t.} \right] \quad (4.43.a)$$

$$g_1^{(2)} = b_o \left[1 + f - \frac{44}{105} f^{\frac{6}{5}} + h.o.t. \right] \quad (4.43.b)$$

where a_o and b_o are arbitrary constants. The particular solution is similarly calculated to be,

$$g_2^{(p)} = -\frac{1}{12} C_{\frac{1}{2}} f - \frac{11}{45} C_{\frac{1}{2}} \mathcal{G} f^{\frac{6}{5}} + h.o.t. \quad (4.44)$$

The general solution is the sum of (4.43.a), (4.43.b) and (4.44).

Thus

$$g_2^D = b_o + \left(b_o - \frac{1}{12} C_{\frac{1}{2}} \right) f - \left(\frac{44}{105} b_o + \frac{11}{45} C_{\frac{1}{2}} \mathcal{G} \right) f^{\frac{6}{5}} + h.o.t. \quad (4.45)$$

where the superscript "D" denotes the solution downstream of the characteristic. To insure that the velocities are continuous, g_2^D at $f=0$ must match with (4.39) and $2 \frac{dg_2}{df}$ at $f=0$ must match with (4.40);

$$b_o = \frac{7}{8} \zeta_c K + D_2^* \frac{\Gamma(\frac{19}{6}) \Gamma(\frac{13}{6})}{\Gamma(3) \Gamma(\frac{7}{3})} \quad (4.46.a)$$

$$2 \left(b_o - \frac{1}{12} C_{\frac{1}{2}} \right) = \frac{16}{7} \left[\frac{7}{8} \zeta_c K + D_2^* \frac{\Gamma(\frac{19}{6}) \Gamma(\frac{13}{6})}{\Gamma(3) \Gamma(\frac{7}{3})} \right] \quad (4.46.b)$$

Thus the constant D_2^* is given by

$$D_2^* = -\frac{7}{4} \frac{\Gamma(3)\Gamma(\frac{7}{3})}{\Gamma(\frac{19}{6})\Gamma(\frac{13}{6})} \left[\frac{1}{3} C_{\frac{1}{2}} + \frac{1}{2} \eta_c + K \right] \quad (4.47)$$

and b_0 may be found from (4.46).

As pointed out in Section III, the constant $\mathcal{K}$ is known by the numerical integration of equation (3.31) from the singular point at $\eta = +\infty$ to the characteristic singular point. The constant a_0 in (4.43) depends in a similar way on the behavior of the second order solution as $\eta \rightarrow +\infty$.

As discussed in Section II, part D., this present second order solution cannot match with the required Prandtl-Meyer behavior in the limit as the corner is approached along radial lines, downstream of the left running characteristic from the corner. This is because the ordering here, $\epsilon^{\frac{1}{2}}$ for u_2 and ϵ^2 for v_2 , does not fit in with the Prandtl-Meyer solution (2.84) and (2.85). Thus, the present second order solutions must vanish in the limit $y \rightarrow 0$, $(\frac{x}{y})$ fixed (but small) so they will not interfere with the appropriate Prandtl-Meyer solution. This condition then serves as the necessary boundary condition for the present second order solution as $\eta \rightarrow +\infty$.

Substitution of the asymptotic expansion for f_1 , valid as $\eta \rightarrow +\infty$, (3.47), into the equation for f_2 , (4.4), shows the homogeneous solutions to have the behavior

$$f_2^{(n)} \sim K_1 \eta^6 + \dots \quad (4.48.a)$$

$$f_2^{(n)} \sim K_2 \gamma^{\frac{4}{5}} + \dots \quad (4.48.b)$$

as $\gamma \rightarrow +\infty$ where K_1 and K_2 are arbitrary constants. The particular solution has the form

$$f_2^{(n)} \sim -\frac{16}{15} C_{\frac{1}{2}} \gamma + \dots \quad (4.48.c)$$

as $\gamma \rightarrow +\infty$. If $f_2 \sim K_1 \gamma^6$ as $\gamma \rightarrow +\infty$, then $u_2 = \gamma^{\frac{2}{3}} \frac{df_2}{d\gamma} \sim K_1 x^5 \gamma^{-6}$. This solution is improper so $K_1 = 0$. Since the equation for f_2 is complicated for $\gamma > \gamma_c$ due to the more complicated form of f_1 in this range, it must be integrated numerically. The integration would start with (4.48.b) plus (4.48.c) and proceed to $\gamma \rightarrow +1$. Iteration for various values of K_2 would be used to match with the solution near $\gamma = +1$. In principle this would give K_2 in (4.48.b) and a_0 in (4.42); however, owing to the complexity of the computation, it has not been carried out, for the second order solution has little effect in the range $\gamma_c < \gamma < \infty$.

Composite solutions, furnishing uniformly valid solutions throughout the solution region, are constructed by adding the inner and outer solutions and subtracting the common terms. It is particularly easy to write the solution upstream of the right running characteristic intersecting the corner. In this region ($\gamma \leq \gamma_c$)

$$u_{outer} = 1 + \varepsilon C_0 \gamma + \varepsilon^{\frac{2}{3}} \left(K x + \frac{2}{3} C_{\frac{1}{2}} \gamma^{\frac{1}{2}} \right) + o(\varepsilon^{\frac{2}{3}}), \quad (4.49)$$

$$U_{inner} = 1 + \varepsilon C_0 \gamma + \varepsilon^{\frac{2}{3}} K X + o(\varepsilon^{\frac{2}{3}}), \quad (4.50)$$

$$V_{outer} = (\gamma+1)^{\frac{1}{2}} \left[\varepsilon^{\frac{2}{3}} \frac{1}{2} C_0 K \gamma^2 + o(\varepsilon^{\frac{2}{3}}) \right], \quad (4.51)$$

$$V_{inner} = (\gamma+1)^{\frac{1}{2}} \left[\varepsilon^{\frac{2}{3}} \frac{1}{2} C_0 K \gamma^2 + \varepsilon^{\frac{5}{6}} K^2 X \gamma + o(\varepsilon^{\frac{5}{6}}) \right]. \quad (4.52)$$

The composite solution for $\gamma \leq \gamma_c^-$ must be

$$U = 1 + \varepsilon C_0 \gamma + \varepsilon^{\frac{2}{3}} \left(K X + \frac{2}{3} C_0 \frac{1}{2} \gamma^{\frac{2}{3}} \right) + o(\varepsilon^{\frac{2}{3}}), \quad (4.53)$$

$$V = (\gamma+1)^{\frac{1}{2}} \left[\varepsilon^{\frac{2}{3}} \frac{1}{2} C_0 K \gamma^2 + \varepsilon^{\frac{5}{6}} K^2 X \gamma + \dots \right]. \quad (4.54)$$

The more general form of the composite solution for $\gamma \leq \gamma_c^+$ is simply

$$U = 1 + \varepsilon C_0 \gamma + \varepsilon^{\frac{2}{3}} U_2 + o(\varepsilon^{\frac{2}{3}}),$$

$$V = (\gamma+1)^{\frac{1}{2}} \left[\varepsilon^{\frac{2}{3}} V_2 + \varepsilon^{\frac{5}{6}} K^2 X \gamma + \dots \right]$$

where u_2 and v_2 can be found from equations (4.3), (4.5) and (4.7). For $\gamma_c^- \leq \gamma \leq \gamma_c^+$ the solutions for u_2 and v_2 are relatively complex expressions; however, they can be written out explicitly after some labor.

The sonic line must occur upstream of the right running characteristic intersecting the corner. Thus the sonic line can be calculated, at least to order $\varepsilon^{\frac{2}{3}}$, by setting (4.53)

equal to one after writing it in terms of unstretched variables; that is, the sonic line is given by

$$\frac{2}{3} C_{\frac{1}{2}} Y^{\frac{3}{2}} + C_0 Y = K (-X) . \quad (4.55)$$

The slope of the sonic line is

$$\left(\frac{dY}{dX} \right)_{\text{sonic}} = -K \left[C_{\frac{1}{2}} Y^{\frac{1}{2}} + C_0 \right]^{-1} \quad (4.56)$$

near the corner ($X, Y \ll 1$). Thus the sonic line intersects the corner with slope $-\frac{K}{C_0}$ which is typically of order one. The sonic line to order $\xi^{\frac{3}{2}}$ depends on the variation of entropy and total enthalpy entering the inviscid transonic corner region through C_0 and $C_{\frac{1}{2}}$, and upon the slip velocity through K . If $C_{\frac{1}{2}}$ is negative, as expected from previous arguments, then the sonic line is slightly convex with respect to the incoming flow. Larger K means a larger slip velocity along the upstream wall, thus causing the sonic line to be less steep at the corner. Also, a smaller entropy gradient in the incoming flow causes the angle of the sonic line with respect to the upstream wall to increase.

The sonic line for an irrotational incoming flow is normal to the upstream wall, so the current result gives the proper tendency of the sonic line as the rotation in the incoming flow decreases. That is, as C_0 decreases the slope at the wall becomes more steep. In fact, in the limit as $C_0 \rightarrow 0$ the slope at the wall becomes normal; however, the functional behavior of the sonic line on Y does not agree with the result of

Vaglio-Laurin. It seems that there is danger in overextending the current result too far. Throughout this analysis, C_0 , $C_{1/2}$ and K have been taken as order one. The whole analysis becomes invalid if C_0 is allowed to shrink to a lower order. In fact, putting C_0 equal to zero leads to a quite different analysis when using the inviscid equations (Friedman (1962)).

The irrotational first order equations solved by Vaglio-Laurin (1960) lead to a sonic line normal to the upstream wall, but curved downstream. In terms of the (s,t) plane, this means that the solution curve passes through $\eta = 0$ before crossing the sonic line (s -axis). The integral curve leaves singular point A. and proceeds to singular point D., crossing the s -axis only after leaving D. In this manner the solution curve always stays below the constant t line corresponding to the occurrence of a limiting line or a characteristic. The chosen solution curve is dictated by the behavior of solutions near the singular point A. Considering Vaglio-Laurin's similarity parameter, $\eta = \frac{x}{y^{1/2}}$, the only solutions leaving A. that have no vertical velocity along the upstream wall must be along the singular tangent through A. The slope of the singular tangent for $\eta = \frac{x}{y^{1/2}}$ is $\frac{dt}{ds} = \frac{7}{5}$ rather than $\frac{dt}{ds} = 1$ as in the current work where $\eta = \frac{x}{y}$. The solution leaving in the sense of positive t corresponds to a uniform, supersonic incoming velocity profile. This, of course, does not lead to transonic flow. The solution curve chosen by Vaglio-Laurin is the only possibility by elimination.

The flaw in expecting the current solution to pass over directly into Vaglio-Laurin's, lies in the nature of these solutions.

The (s,t) plane configuration depends only on the similarity parameter and not on C_0 . However, the value of η corresponding to points along the solution curve in the (s,t) plane does depend on C_0 . As $C_0 \rightarrow 0$ so does $\eta_{c+} = -\eta_{c-}$. That is, the two characteristics have as limiting positions, the vertical axis. This implies that the flow becomes uniform with sonic velocity everywhere. In the (s,t) plane the singular points A, C and E all correspond to the same value of η ($\eta = 0$). Thus the solution completely degenerates as $C_0 \rightarrow 0$ to a uniform sonic stream turning a corner through a Prandtl-Meyer expansion (that is, B_0 in (3.48) must be zero to match at $\eta = 0$).

The ordering corresponding to Vaglio-Laurin's (1960) solution must have a term in the second order U solution behaving like $C_0 Y$. Thus, as C_0 becomes large, the second order term becomes as important as the first. The asymptotic expansion in this situation becomes invalid. After relegating rotational effects to higher order terms, they cannot become large without destroying the asymptotic solution.

In either solution the asymptotic expansions obtained are not uniformly valid in the parameter C_0 . To be able to describe the flow for any value of C_0 requires obtaining the exact solution of the governing equations (3.1) and (3.2) satisfying the boundary conditions associated with this problem. Since we are unable to obtain this exact solution, approximate asymptotic representations must suffice along with their inherent limitations.

Some verification of the present solution can be made from the previous observation that equation (3.24) is the same tran-

sonic potential equation obtained in analyzing irrotational transonic flow. The only difference is in the definition of the velocity potential. Equation (3.24) has a well known approximate solution, valid in a small neighborhood of a sonic line that is not orthogonal to the streamlines crossing it (see Landau and Lifshitz, 1959. p.442). The velocity potential is expanded in powers of the local $\bar{\psi}$ and $\bar{x}$ coordinates. This result is transformed into the hodograph plane where it can be identified with suitable solutions of the Euler-Tricomi equation. The results, presented in the book by Landau and Lifshitz, can be transformed back into physical variables to give:

$$\phi_x^{(1)} \sim K_1 \bar{\psi} + K_2 \bar{x} + \left(\begin{array}{c} \text{nonlinear order} \\ \text{terms in } \bar{\psi} \text{ \& } \bar{x} \end{array} \right) \quad (4.57.a)$$

$$\phi_y^{(1)} \sim K_1 \bar{x} + \left(\begin{array}{c} \text{nonlinear order} \\ \text{terms in } \bar{\psi} \text{ \& } \bar{x} \end{array} \right) \quad (4.57.b)$$

$$\text{(sonic line)} \quad \bar{\psi} = - \frac{K_2}{K_1} \bar{x} + \left(\begin{array}{c} \text{nonlinear order} \\ \text{terms in } \bar{\psi} \text{ \& } \bar{x} \end{array} \right) \quad (4.57.c)$$

$$\text{(characteristics)} \quad \bar{x} = \pm \frac{2}{3} K_1^{\frac{1}{2}} \bar{\psi}^{\frac{3}{2}} + \left(\begin{array}{c} \text{nonlinear order} \\ \text{terms in } \bar{\psi} \text{ \& } \bar{x} \end{array} \right). \quad (4.57.d)$$

This solution should correspond locally with the leading terms of the rotational solution at any point of the sonic line, for no streamline is normal to the sonic line. The axes $(\bar{\psi}, \bar{x})$ above are respectively defined to be normal and tangent to the streamline at its point of intersection with the sonic line. Thus at the corner, $\bar{\psi}$ corresponds to y and $\bar{x}$ corresponds to x .

The identification of k_1 with C_0 and k_2 with K quickly identifies the above with the first order terms of the present results. It is interesting to note that (4.57.a) includes a term from the present second order solution. Actually, this (KX) term is of first order in the inner region. In fact, the inner region is also governed by (3.24) to first order, with the modified potential defined by $\tilde{\phi}_x^{(1)} = \tilde{u}$, and $(\tilde{\phi}_y^{(1)} - C_0 \tilde{x}) = \tilde{v}$. In unstretched variables, both the inner and outer solution must give the same result to equivalent orders. A uniformly valid composite solution can be formed. The solution (4.57) is valid throughout a neighborhood of the corner and as such, it gives the leading terms of the composite solution which includes the (KX) term. The sonic line (4.57.c) agrees, to lowest order with (4.55) while the characteristics (4.57.d) correspond exactly with the characteristics of the present analysis.

The composite solutions for U and V were computed numerically and integration performed to find the streamlines. This was carried out with $\gamma=1.4$, $K=1$, $\eta_c=1$ and $C_{\eta}=-1$ for $\eta \leq \eta_c$. The sonic line, the characteristics which intersect the corner, streamlines and typical velocity profiles are illustrated in figures (4.1), (4.2) and (4.3) for a flow with $E_1 = \mathcal{O}(M-1)=0.1$. In reality, K must be found by matching with the viscous sublayer close to the wall.

It is informative to examine the pressure gradient found from this solution. The pressure gradient is related to the velocity through the usual Euler equations,

$$U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = -\frac{1}{\gamma} \frac{1}{\bar{P}} \frac{\partial P}{\partial X} - \frac{PV}{\gamma} \frac{1}{\bar{P}^*} \frac{d\bar{P}^*}{d\Psi} \quad (4.58)$$

$$U \frac{\partial V}{\partial X} + V \frac{\partial V}{\partial Y} = -\frac{1}{\gamma} \frac{1}{\bar{P}} \frac{\partial P}{\partial Y} - \frac{PU}{\gamma} \frac{1}{\bar{P}^*} \frac{d\bar{P}^*}{d\Psi}, \quad (4.59)$$

where the dimensionless pressure, P , is defined by $P = \frac{\bar{P}}{P^*}$ and the fact that $\bar{P}^*$ is constant along a streamline in the inviscid region has been used.

The variation of $\bar{P}^*$ across the streamlines can be found from the relations defining the entropy and critical speed in terms of critical pressure and density,

$$\gamma \frac{\bar{P}^*}{\bar{\rho}^*} = \bar{a}^{*2},$$

$$\frac{\bar{P}^*}{\bar{\rho}^{*\gamma}} \frac{\bar{\rho}_r}{\bar{P}_r} = \exp \left\{ \gamma \frac{\bar{S} - \bar{S}_r}{\bar{C}_p} \right\},$$

where subscript "r" refers to reference values. It can be shown that

$$\frac{1}{\bar{P}^*} \frac{d\bar{P}^*}{d\Psi} = -\frac{\gamma}{\gamma-1} \left[\frac{\gamma-1}{\gamma} \frac{d\bar{S}}{d\Psi} - \frac{1}{\bar{a}^{*2}} \frac{d\bar{a}^{*2}}{d\Psi} \right] \quad (4.60)$$

where Ψ is the dimensionless stream function in the inviscid region. Substitution of the result from Section II, part C. for the expression in brackets gives

$$\frac{1}{\bar{P}^*} \frac{d\bar{P}^*}{d\Psi} = \gamma C_0 + \varepsilon^{\frac{1}{2}} \gamma C_{\frac{1}{2}} \gamma^{\frac{1}{2}} + \varepsilon \gamma C_1 \gamma + \dots \quad (4.61)$$

where $\varepsilon \gamma$ has been substituted for Ψ .

Let the perturbation in the pressure be given by

$$P = 1 + \lambda_1 P_1 + \lambda_2 P_2 + \dots \quad (4.62)$$

After substitution of the appropriate expansions into (4.58) and (4.59) it can easily be shown that

$$\lambda_1 = \varepsilon \quad (4.63)$$

$$\frac{\partial P_1}{\partial x} = 0 \quad \frac{\partial P_1}{\partial y} = -\gamma C_0 \quad (4.64)$$

$$\lambda_2 = \varepsilon^{\frac{3}{2}} \quad (4.65)$$

$$\frac{\partial P_2}{\partial x} = -\gamma \frac{\partial u_2}{\partial x} \quad \frac{\partial P_2}{\partial y} = -\gamma \frac{\partial v_2}{\partial x} - \gamma^2 C_{\frac{1}{2}} \gamma^{\frac{1}{2}} \quad (4.66)$$

These pressure gradients can be easily calculated for $\gamma \leq \gamma_c$ since the modified velocity potential is simply a polynomial in z . Then for $\gamma \leq \gamma_c$ the velocity components are given by

$$u_2 = -K(-x) + \frac{2}{3} C_{\frac{1}{2}} \gamma^{\frac{3}{2}} \quad (4.67)$$

$$v_2 = \frac{9}{8} \gamma_c^2 K \gamma^2 \quad (4.68)$$

The pressure gradient upstream of the right running characteristic intersecting the corner is given by

$$\frac{\partial P}{\partial X} = -\epsilon^{\frac{2}{3}} \gamma K \quad (4.69)$$

or
$$\frac{\partial P}{\partial X} = -\gamma K, \quad X \ll 1$$

and
$$\frac{\partial P}{\partial Y} = -\epsilon \gamma C_0 - \epsilon^{\frac{2}{3}} \gamma^2 C_{\frac{1}{2}} Y^{\frac{1}{2}} \quad (4.70)$$

or
$$\frac{\partial P}{\partial Y} = -\gamma C_0 - \gamma^2 C_{\frac{1}{2}} Y^{\frac{1}{2}}, \quad Y \ll 1.$$

The fact that $\frac{\partial P}{\partial Y}$ is negative to order ϵ at first seems surprising. However, the dimensionless pressure, P , depends on the $\bar{P}^*$ variation as well as the variation of $\bar{P}$. In fact, the term in $\frac{\partial P}{\partial Y}$ that is of order ϵ comes exclusively from the $\bar{P}^*$ variation term $\frac{1}{\bar{P}^*} \frac{d\bar{P}^*}{d\psi}$ in (4.59). The term of order $\epsilon^{\frac{2}{3}}$ is also contributed by the $\bar{P}^*$ variation term since $\frac{\partial V_x}{\partial X} = 0$ for $\eta \leq \eta_c^-$.

Upstream of the right running characteristic intersecting the corner, $\frac{\partial P}{\partial X}$ is constant. For $\eta > \eta_c^-$ the x-pressure gradient begins to vary with position in the flow. This is due to the discontinuities in the acceleration along the characteristic intersecting the corner that were alluded to earlier. The fact that the discontinuity at the right running characteristic intersecting the corner is so pronounced, warrants some comment on the physical meaning of this characteristic.

Since the flow is a vortical flow, every streamline displays the reflective property of a contact discontinuity. That is,

a Mach wave is partly reflected and partly transmitted at each streamline. The character of the reflected wave may even be opposite to that of the incident wave. For example, a compressive wave will reflect as an expansion wave and be transmitted as a weaker compression wave if the initial wave is incident on a region of decreasing acoustic impedance, $(\bar{\rho} \bar{a})$. Thus to visualize the flow from a detailed wave point of view seems impossible. The wave interactions are smeared out in a continuous fashion throughout the flow. However, it seems clear that waves originating upstream of the right running characteristic will, in some sense, reflect from the sonic line and continue to influence the flow upstream of the corner. Waves that originate downstream of the right running characteristic that intersects the corner have no possibility of reflecting from the sonic line. Thus this characteristic should represent a weak discontinuity in the solution, for the character of the flow field changes there.

Of course, this entire wave pattern is set up by the fact that different streamlines cross the sonic line with different turning angles, necessitating adjustments in the supersonic region. The turning angle at which streamlines cross the sonic line is determined in the subsonic flow region which is in turn influenced by the position of the sonic line and thus by the waves intersecting it. This is typical of the complex interactions that occur in transonic flows.

It is difficult to draw any conclusions about the flow from this complex description. However, qualitatively one can

expect that the pressure gradient decreases near the left running characteristic from the corner, for the transmitted waves from upstream are continuously weakened by crossing streamlines in the flow.

Numerical computations of $\frac{\partial u_1}{\partial x}$ and $\frac{\partial v_1}{\partial x}$ were carried out with $C_0 = \frac{9}{4}$, $\gamma_{c+} = 1$, $\gamma = 1.4$, $C_{\gamma/2} = -1$ and $K=1$ for $\gamma \leq \gamma_{c+}$. The results are plotted in figure (4.4).

V THE ANALYTIC ENTROPY DISTRIBUTION ($C_{\gamma_2}=0$)

In the discussion of the second order solution, C_{γ_2} was assumed to be non-zero and of order one. However, in the discussion of the laminar compressible boundary layer upstream of the inviscid transonic corner region, it was noted that if $\frac{d\hat{p}}{d\hat{x}}=0$ and $q_w=0$ in the boundary layer solution, then $C_{\gamma_2}=0$. The consequences of this as regards the shape of the sonic line at the corner must be discussed.

The fact that $C_{\gamma_2}=0$ does not alter the form of the first order equations and the solution to these must be the same. However, the second order solution is changed significantly.

The Crocco equation (3.10) shows that with $C_{\gamma_2}=0$,

$$\nu_2 = \frac{\tau}{\varepsilon} E_2 = \tau \varepsilon . \quad (5.1)$$

Thus $\nu_2 = (\gamma+1)^{\frac{1}{2}} \varepsilon^{\frac{\gamma}{2}}$ and $E_2 = \varepsilon^2$. (5.2)

From (3.9) and (3.10) the second order equations become

$$u_1 \frac{\partial u_2}{\partial x} + \left(u_2 + \frac{u_1^2}{2}\right) \frac{\partial u_1}{\partial x} + v_1 \left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x}\right) + (\gamma-1) u_1 \frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial y} = 0 \quad (5.3)$$

and $\frac{\partial u_2}{\partial y} - \frac{\partial v_2}{\partial x} = C_1 \gamma - \gamma C_0 u_1$. (5.4)

A velocity potential can be introduced that identically satisfies (5.4) by

$$\phi_x^{(2)} = u_2 , \quad (5.5)$$

$$\phi_Y^{(2)} = V_2 + C_1 \chi Y - \gamma C_0 \phi^{(1)} . \quad (5.6)$$

The modified velocity potential can be written in terms of a similarity variable (from the principle of eliminability) as

$$\phi^{(2)} = \gamma^{\frac{3}{2}} f_2(\gamma) \quad (5.7.a)$$

$$u_2 = \gamma^2 \frac{df_2}{d\gamma} \quad (5.7.b)$$

$$V_2 = \gamma^{\frac{5}{2}} \left[\frac{7}{2} f_2 - \frac{3}{2} \gamma \frac{df_2}{d\gamma} - C_1 \gamma + \gamma C_0 f_1 \right] \quad (5.7.c)$$

where f_1 is given by (3.28). Equation (5.3) then becomes

$$\begin{aligned} & \left(\frac{df_1}{d\gamma} - \frac{9}{4} \gamma^2 \right) \frac{d^2 f_2}{d\gamma^2} + \left(\frac{d^2 f_1}{d\gamma^2} + \frac{27}{4} \gamma \right) \frac{df_2}{d\gamma} - \frac{35}{4} f_2 \\ &= \gamma C_0 \left(\frac{5}{2} f_1 - \frac{3}{2} \gamma f_1 \right) - \left(\gamma - \frac{1}{2} \right) \left(\frac{df_1}{d\gamma} \right)^2 \frac{d^2 f_1}{d\gamma^2} - C_1 \gamma \\ & - \left(\frac{5}{2} f_1 - \frac{3}{2} \gamma \frac{df_1}{d\gamma} - C_0 \gamma \right) \left[2 \left(\frac{df_1}{d\gamma} - \frac{3}{2} \gamma \frac{d^2 f_1}{d\gamma^2} \right) - C_0 \right] . \end{aligned} \quad (5.8)$$

Substitution of the solution for f_1 valid for $\gamma \leq \gamma_c$ gives the simpler differential equation,

$$(1-\gamma^2) \frac{d^2 g_2}{d\gamma^2} + 3\gamma \frac{dg_2}{d\gamma} - \frac{35}{9} g_2 = \frac{9}{4} \left(\frac{\gamma C_0^2 - C_1}{C_0^2} \right) \gamma \quad (5.9)$$

where
$$\mathcal{F} = \frac{\gamma}{\gamma_c^+} \quad (5.10)$$

and
$$g_2(\mathcal{F}) = \gamma_c^{-5} f_2(\gamma) \quad (5.11)$$

The solution to equation (5.9) may be written as

$$g_2 = G(z) - \frac{1}{2} \left(\frac{9}{4} \right)^2 \left(\frac{\gamma C_0^2 - C_1}{C_0^2} \right) (2z-1) \quad (5.12)$$

where
$$z = \frac{1}{2} (1 + \mathcal{F}) \quad (5.13)$$

and $G(z)$ is the solution to the Gauss hypergeometric equation, that is,

$$z(1-z) \frac{d^2 G}{dz^2} + (3z - \frac{3}{2}) \frac{dG}{dz} - \frac{35}{9} G = 0 \quad (5.14)$$

Once again the solution is given in terms of the hypergeometric function; however, in contrast with (4.10) the equation (5.9) is non-homogeneous so the solution contains an extra term. The general solution of (5.14) for $|z| < 1$ is given by

$$G = D_1 F\left(-\frac{7}{2}, -\frac{5}{2}; -\frac{3}{2}; z\right) + D_2 z^{\frac{5}{2}} F\left(\frac{1}{2}, \frac{5}{2}; \frac{7}{2}; z\right) \quad (5.15)$$

where $F(a, b; c; z)$ is defined by (4.12). Both series in (5.15) converge at $|z| = 1$ so their analytic continuation in terms of (z^{-1}) can be written directly giving

$$G = \bar{D}_1 (-z)^{\frac{2}{3}} F\left(-\frac{2}{3}, \frac{1}{6}; \frac{1}{3}; \frac{1}{2}\right) + \bar{D}_2 (-z)^{\frac{5}{3}} F\left(\frac{5}{6}, -\frac{5}{3}; \frac{5}{3}; \frac{1}{2}\right), \quad (5.16)$$

where the arbitrary constants D_1 , $\bar{D}_1$, D_2 and $\bar{D}_2$ are related by

$$\bar{D}_1 = 2^{-\frac{1}{3}} \left[\frac{40}{27} D_1 + \frac{405}{224} D_2 e^{i\frac{5}{3}\pi} \right], \quad (5.17.a)$$

$$\bar{D}_2 = 2^{\frac{1}{3}} \left[\frac{28}{27} D_1 - \frac{81}{64} D_2 e^{i\frac{5}{3}\pi} \right]. \quad (5.17.b)$$

The behavior of G as $z \rightarrow -\infty$ is clearly

$$G = \bar{D}_1 (-z)^{\frac{2}{3}} + \bar{D}_2 (-z)^{\frac{5}{3}} - \frac{7}{6} \bar{D}_1 (-z)^{\frac{4}{3}} - \frac{5}{6} \bar{D}_2 (-z)^{\frac{2}{3}} + \dots \quad (5.18)$$

Thus the behavior of the velocities as $z \rightarrow -\infty$ is

$$u_2 = -\gamma_{c+}^4 \left[\bar{D}_1 2^{-\frac{2}{3}} \frac{7}{3} \gamma_{c+}^{-\frac{4}{3}} (-x)^{\frac{4}{3}} + \bar{D}_2 2^{-\frac{5}{3}} \frac{5}{3} \gamma_{c+}^{-\frac{1}{3}} (-x)^{\frac{1}{3}} \gamma + \frac{1}{2} \left(\frac{9}{4}\right)^2 \left(\frac{\gamma C_0^2 - C_1}{C_0^2}\right) \gamma^2 + O(-z)^{\frac{1}{3}} \right], \quad (5.19.a)$$

$$v_2 = \gamma_{c+}^5 \left[\bar{D}_2 2^{-\frac{5}{3}} \frac{5}{3} \gamma_{c+}^{-\frac{5}{3}} (-x)^{\frac{5}{3}} + \bar{D}_1 2^{-\frac{2}{3}} \frac{7}{2} \gamma_{c+}^{-\frac{1}{3}} (-x)^{\frac{1}{3}} \gamma^2 + O(-z)^{\frac{1}{3}} \right]. \quad (5.19.b)$$

As before, writing the horizontal velocity in terms of the unstretched variables shows that the solution may not be valid when Y is smaller than a constant times $X^{(4/3)}$. Once again

it is necessary to define an inner region, only this time the region is defined by

$$Y = O(X)^{\frac{1}{3}} \quad (5.20)$$

Consider an inner coordinate stretching similar to (4.21).

Relation (5.20) shows that

$$\tilde{\varepsilon} = (x+1)^{\frac{1}{2}} \tilde{\varepsilon}^{\frac{3}{4}} \quad (5.21)$$

where the factor $(x+1)^{\frac{1}{2}}$ is added for convenience. Asymptotic expansions of the velocity components of the form (4.23) are introduced into the governing equations. As before, it is clear that in order to match with the outer solution, $\tilde{v}_1 = o(\tilde{E}_1)$ and both $\tilde{u}_1 \frac{\partial \tilde{u}_1}{\partial \tilde{x}}$ and $\frac{\partial \tilde{v}_1}{\partial \tilde{y}}$ are nonzero and should be kept in the gasdynamic equation. Thus,

$$\tilde{v}_1 = -(x+1)^{\frac{1}{2}} \tilde{E}_1^2 \tilde{\varepsilon}^{\frac{1}{4}} \quad (5.22)$$

The Crocco equation then shows that

$$\tilde{E}_1 = \tilde{\varepsilon} \quad (5.23)$$

so the first order inner equations are the same as before, (4.28);

$$\tilde{u}_1 \frac{\partial \tilde{u}_1}{\partial \tilde{x}} - \frac{\partial \tilde{v}_1}{\partial \tilde{y}} = 0 \quad (5.24)$$

$$\frac{\partial \tilde{u}_1}{\partial \tilde{y}} = C_0 \quad (5.25)$$

The solutions are

$$\tilde{u}_1 = C_0 \tilde{y} + \tilde{F}(\tilde{x}) \quad (5.26)$$

$$\tilde{V}_1 = \left[C_0 \frac{\tilde{y}^2}{2} + \tilde{y} \tilde{F}(\tilde{x}) \right] \frac{d\tilde{F}}{d\tilde{x}} + G(\tilde{x}) \quad (5.27)$$

Since $\tilde{V}_1 = 0$ at $\tilde{y} = 0$ (at the wall streamline), $G(\tilde{x}) = 0$. Further, the principle of eliminability requires that

$$\tilde{F}(\tilde{x}) = \tilde{A}_1 (-\tilde{x})^{\frac{4}{3}} \quad (5.28)$$

where $\tilde{A}_1$ is an arbitrary constant. Thus, the velocities in the inner region are

$$U = 1 + \tilde{\varepsilon} \left[C_0 \tilde{y} - \tilde{A}_1 (-\tilde{x})^{\frac{4}{3}} \right] + \dots \quad (5.29.a)$$

$$V = \tilde{\varepsilon}^{\frac{2}{3}} (\tilde{x}+1)^{\frac{1}{3}} \left[\frac{2}{3} \tilde{A}_1 C_0 \tilde{y}^2 (-\tilde{x})^{\frac{1}{3}} - \frac{4}{3} \tilde{A}_1^2 \tilde{y} (-\tilde{x})^{\frac{5}{3}} \right] + \dots \quad (5.29.b)$$

Here $\tilde{A}_1$ is an arbitrary constant to be set by matching with some solution valid on a larger length scale.

These solutions must match with the outer region as $\tilde{y} \rightarrow \infty$, $\tilde{x} < 0$ fixed or as $\tilde{x} \rightarrow 0$, $\tilde{y} > 0$ fixed. The solutions (5.29) written in outer variables become

$$U = 1 + \varepsilon C_0 y - \varepsilon^2 \tilde{A}_1 (-x)^{\frac{4}{3}} + h.o.t., \quad (5.30.a)$$

$$V = (x+1)^{\frac{1}{3}} \left[\varepsilon^{\frac{2}{3}} \frac{2}{3} \tilde{A}_1 C_0 y^2 (-x)^{\frac{1}{3}} - \varepsilon^{\frac{2}{3}} \frac{4}{3} \tilde{A}_1^2 y (-x)^{\frac{5}{3}} \right] + h.o.t. \quad (5.30.b)$$

Comparison of (5.30) with (5.19) shows that for matching to occur it is necessary that

$$\bar{D}_2 = 0 \quad (5.31.a)$$

$$\bar{D}_1 = \frac{3}{7} 2^{\frac{7}{3}} \tilde{A}_1 \gamma_{c+}^{-\frac{8}{3}} \quad (5.31.b)$$

Also, since there are no terms of order $y^{\frac{2}{3}}$ in (5.30.b) the condition used in the first order outer solution to obtain the constant, (3.43), was correct.

As discussed in Section IV, E_1 and $\tilde{E}_1$ are both of the order of the perturbation from sonic velocity. Since the solution for U is the same in both regions (in unstretched variables), it must be that $\mathcal{E} = \tilde{\mathcal{E}}$ and the Y stretching is the same in both regions, but X is stretched less in the outer region than in the inner region.

The constants D_1 and D_2 may be found from (5.17) so that the solution, $G(z)$, for $(-1 \leq z \leq 0)$ is

$$G = \frac{81}{560} \left(\frac{2}{\gamma_{c+}}\right)^{\frac{8}{3}} \tilde{A}_1 \left[F\left(-\frac{7}{3}, -\frac{5}{3}; -\frac{3}{2}; z\right) + \frac{1792}{2187} (-z)^{\frac{5}{2}} F\left(\frac{1}{6}, \frac{5}{6}; \frac{7}{2}; z\right) \right] \quad (5.32.a)$$

Notice that the second term in this solution becomes imaginary for $z > 0$. A different representation is needed for $z > 0$ that will be the proper continuation of this solution. The general solution for the range $(0 \leq z \leq 1)$ is given by:

$$G = D_1^* F\left(-\frac{7}{3}, -\frac{5}{3}; -\frac{2}{3}; z\right) + D_2^* z^{\frac{5}{2}} F\left(\frac{1}{6}, \frac{5}{6}; \frac{7}{2}; z\right). \quad (5.32.b)$$

The velocities must be continuous at the right running characteristic intersecting the corner. Thus

$$D_1^* = D_1 = \frac{81}{560} \left(\frac{2}{\eta_{c+}}\right)^{\frac{8}{3}} \tilde{A}_1. \quad (5.33)$$

The constant D_2^* must be found by matching velocities at $z=1$.

Proceeding as in Section IV a relation between D_2^* and D_1^* can be derived. Finally D_1^* can be written in terms of $\tilde{A}_1$ to give

$$D_2^* = -\frac{16}{135\sqrt{3}} \left[\left(\frac{2}{\eta_{c+}}\right)^{\frac{8}{3}} \tilde{A}_1 + 27\left(\gamma-1 + \frac{C_1}{C_0^2}\right)\right]. \quad (5.34)$$

This completes the second order solution for this case.

Composite solutions, furnishing uniformly valid solutions throughout the region of interest are constructed as before.

Written in terms of the outer variables the composite solution for $\eta \leq \eta_{c+}$ is given by

$$U = 1 + \varepsilon C_0 \gamma + \varepsilon^2 u_2 + h.o.t. \quad (5.35.a)$$

$$V = (\gamma+1)^{\frac{1}{2}} \left[\varepsilon^{\frac{5}{2}} V_2 - \varepsilon^{\frac{7}{2}} \frac{4}{3} \tilde{A}_1^2 \gamma (-x)^{\frac{5}{2}} \right] + h.o.t. \quad (5.35.b)$$

The solution for $\eta \geq \eta_{c+}$ is the same as before to lowest order and again the second order terms are relatively insignificant in this region of the flow.

The sonic line ($U=1$) of this solution is essentially within the inner region, so from the inner solutions (5.29) it is described by

$$Y = \frac{\tilde{A}_1}{C_0} (-X)^{(4/3)} \quad \text{for } X, Y \ll 1 \quad (5.36)$$

in unstretched variables. Thus the sonic line is tangent to the wall at the corner. The subsonic region decreases to zero thickness as the corner is approached.

Comparison with the result of Section IV shows a quite different flow behavior. It is interesting to notice that the previous sonic line (4.56) becomes closer to being tangent to the upstream wall as K decreases, however, if $K \rightarrow 0$ the solution of Section IV degenerates and the results are not strictly valid.

The sonic line is independent of C_1 to the order of approximation in (5.36). In fact, from (5.19.a) its dependence on C_1 only comes in with a term $(-X)^{(8/3)}$ which is quite weak. Thus the sonic line is essentially determined by the slip velocity (through $\tilde{A}_1$) and the entropy and total enthalpy variation evaluated at the zeroth streamline which describes the wall.

Numerical computations for $\gamma \leq \gamma_{c+}$ based on (5.35) are presented in figures (5.1) and (5.2) with $\gamma = 1.4, A_1 = 1, C_1 = \pm 1$ and $\gamma_{c+} = 1$. The sonic line, the characteristics which intersect the corner, streamlines and typical velocity profiles are illustrated for a transonic flow with $E_1 = \mathcal{O}(M-1) = 0.1$.

The pressure gradient for this solution can be obtained from equations (4.58) and (4.59) in the same manner as before. Again one finds that $\lambda_1 = \mathcal{E}$ and

$$\frac{\partial P_1}{\partial \gamma} = -\gamma C_0 \quad (5.37)$$

while $\lambda_2 = \mathcal{E}^2$ and

$$\frac{\partial P_2}{\partial \gamma} = -\gamma \frac{\partial \gamma_2}{\partial x} + \gamma (\gamma C_0^2 - C_1) \gamma, \quad (5.38)$$

$$\frac{\partial P_2}{\partial x} = -\gamma \frac{\partial u_2}{\partial x} \quad . \quad (5.39)$$

Of particular interest is the pressure gradient along the upstream wall. Here the inner solution can be used for $\frac{\partial u_2}{\partial \gamma}$ giving

$$\left(\frac{\partial P_2}{\partial x} \right)_{\gamma=0} = -\frac{4}{3} \tilde{A}_1 (-x)^{\frac{1}{3}} \quad . \quad (5.40)$$

Thus the pressure gradient along the wall actually weakens as the corner is approached, although it is always negative. The fact that the sonic line is tangent to the wall at the corner is consistent with this result. For the sonic line approaches a line of constant pressure as it becomes tangent to a streamline (for example, the wall streamline). This is true since the critical pressure is constant along a streamline in the inviscid transonic corner region. Thus the pressure gradient along the wall should vanish at the corner.

The quantities $\frac{\partial u_2}{\partial x}$ and $\frac{\partial \gamma_2}{\partial x}$ are plotted as a function of γ for $\gamma \leq \gamma_{c+}$ in figure (5.3) with $\gamma = 1.4, \tilde{A}_1 = 1, C_1 = +1$ and $\gamma_{c+} = 1$.

This second order solution does not add anything of interest in the range $\eta_c + < \eta < +\infty$. It is merely a small correction to the first order solution in this range. However, in contrast with Section IV, the ordering of the present second order solution does fit in with the transonic expansion of the Prandtl-Meyer solution. Thus it need not vanish in the limit $y \rightarrow 0$, $\frac{X}{Y}$ fixed, but rather must give the second terms of the transonic Prandtl-Meyer expansion. If the expansion for f_1 valid as $\eta \rightarrow +\infty$ is substituted in equation (5.8), the expansions for the two homogeneous solutions of f_2 are found to be

$$f_2^{(1)} \sim K_1 \eta^7 + \dots \quad (5.41.a)$$

$$f_2^{(2)} \sim K_2 \eta + \dots \quad (5.41.b)$$

and the behavior of the particular solution is given by

$$f_2^{(P)} \sim - \left[\frac{68-5}{30} \right] \eta^5 + \dots \quad (5.41.c)$$

The general solution is the sum of (5.41). The first solution (5.41.a) does not match with the Prandtl-Meyer behavior in the limit $y \rightarrow 0$, $(\frac{X}{Y})$ fixed, nor does it vanish. It is therefore unacceptable and $K_1=0$ is required. The solution (5.41.b) does vanish in the limit as the corner is approached along radial lines. This is acceptable and the constant K_2 must be found from matching with the asymptotic expansion valid just downstream of the left running characteristic from the corner (which also contains a free constant). So far the situation

is completely analogous to that of Section IV. However, here the particular solution does not vanish in the limit $y \rightarrow 0$, $(\frac{X}{Y})$ fixed. Rather it generates the second order terms in the transonic Prandtl-Meyer expansion (2.84) and (2.85) as was anticipated.

VI THE AXISYMMETRIC PROBLEM

The form of the entropy and total enthalpy variation is valid for an axisymmetric body by the Mangler transformation. Implicit in this assertion is the assumption that the boundary layer on the axisymmetric body is small compared with the body radius $\bar{R}_0$. Thus $R_0^{-1} = \frac{\bar{L}}{\bar{R}_0} \ll 1$ is assumed. The axisymmetric term in the gasdynamic equation is of order $\frac{r \sin \alpha}{R_0} = \varepsilon^{\frac{1}{2}} \frac{\sin \alpha}{R_0}$ or $\frac{r}{R_0} v = \varepsilon^{\frac{3}{2}} \frac{1}{R_0}$, both of which have been taken small compared to ε^3 . The axisymmetric term in the Crocco equation is of order $\frac{r \varepsilon}{R_0} = \frac{\varepsilon^{\frac{1}{2}}}{R_0}$ which is small compared to $\varepsilon^{\frac{5}{2}}$. Thus the axisymmetric effects do not change the second order terms obtained previously.

The quantity $\frac{\sin \alpha}{R_0}$ has a physical significance for the flow about the rear edge of a cone. The thickness of the boundary layer at the rear of the cone, $\bar{L}$, is of order $\bar{L} = Re^{-\frac{1}{2}} \bar{l}_0$ where Re is the Reynolds number of the flow outside the boundary layer. The geometry (see figure (2.2)) gives

$$\sin \alpha = \frac{\bar{R}_0}{\bar{l}_0} = \frac{\bar{R}_0}{\bar{L}} Re^{-\frac{1}{2}} = R_0 Re^{-\frac{1}{2}}$$

so $\frac{\sin \alpha}{R_0} = Re^{-\frac{1}{2}}$ gives the order of the quantity $(\frac{\sin \alpha}{R_0})$.

The axisymmetric problem actually has two small parameters; one artificial parameter, $E_1 = \mathcal{O}(M-1)$, and one real parameter, R_0^{-1} , depending on the radius of the body. Since these parameters are independent, solutions should actually be expansions in both parameters. That is, some of the terms in the velocity expansions should be ordered by functions of $(\frac{\sin \alpha}{R_0})$ as will be shown.

If one required the axisymmetric term to appear in, say, the first order equations of (3.9) and (3.10) by forcing

$$\frac{\sin \alpha}{R_0} = \varepsilon^{\frac{1}{2}} = Re^{-\frac{1}{2}}, \quad (6.1)$$

then the leading equations become

$$u_1 \frac{\partial u_1}{\partial x} - \frac{\partial v_1}{\partial y} = (x+1)^{-1}, \quad (6.2)$$

$$\frac{\partial u_1}{\partial y} - \frac{\partial v_1}{\partial x} = C_0. \quad (6.3)$$

For practical values of ε , say $\varepsilon = 0.1$, relation (6.1) shows that $R_0 \approx 10$ which is unrealistic for the problem being discussed. However, the object here is to show that there are also basic difficulties in the mathematical analysis.

A modified potential is introduced to satisfy (6.3) identically. Then (6.2) becomes

$$\phi_x \phi_{xx} - \phi_{yy} = 0 \quad (6.4)$$

where

$$\phi_x = u_1, \quad (6.5)$$

$$\phi_y = v_1 + C_0 x - \frac{y}{x+1}. \quad (6.6)$$

As before, a similarity solution is assumed of the form

$$\phi = y^s f(\eta) \quad (6.7)$$

$$u_1 = y^{s-m} \frac{df}{d\eta} \quad (6.8)$$

$$v_1 = y^{s-1} \left[s f - p \eta \frac{df}{d\eta} - y^m C_0 \eta + \frac{y}{s+1} \right] \quad (6.9)$$

$$\eta = \frac{x}{y^m} \quad (6.10)$$

For v_1 to be of similarity form, it must be that

$$s-m = m = 1 \quad (6.11)$$

However, in order that the transonic potential equation reduce to an ordinary differential equation in η , one must require

$$s = 3m - 2, \quad (6.12)$$

which contradicts (6.11). A similarity solution is no longer possible. Trying to include the axisymmetric term of the gasdynamic equation in higher order equations creates the same problem.

Some reflection on the nature of this difficulty shows it to be expected. If one sets $\frac{\Delta \ln \alpha}{R_0}$, a parameter with physical significance, equal to an artificial parameter, the artificial parameter is given physical significance. The necessity of a similarity solution is based on the principle of eliminability for an artificial parameter. Thus giving the artificial parameter physical significance obviates the necessity of a similarity solution. In fact, it makes a similarity solution impossible. Since the axisymmetric solution should reduce to the two di-

mensional solution as $R_0^{-1} \rightarrow 0$, one is forced into considering expansions in both ε and R_0^{-1} .

To treat the axisymmetric problem in a formal manner, the velocities must be expanded in terms of $(\frac{\sin \alpha}{R_0})$. Thus let,

$$U = \sum_{i=0}^{\infty} \Delta_i \left(\frac{\sin \alpha}{R_0} \right) U_i(X, Y)$$

$$V = \sum_{i=0}^{\infty} \sigma_i \left(\frac{\sin \alpha}{R_0} \right) V_i(X, Y) \quad (6.13)$$

The present formulation must reduce to the previous planar problem as R_0 becomes large. Therefore

$$\Delta_0 \left(\frac{\sin \alpha}{R_0} \right) = 1$$

and

$$\sigma_0 \left(\frac{\sin \alpha}{R_0} \right) = 1$$

The equations for U_0 and V_0 are (2.11) and (2.12) with $j=0$ except in the definition of the streamfunction (2.13) where $j=1$. Asymptotic expansions for U_0 and V_0 are given by (2.16) and (2.17). The solutions for U_0 and V_0 have already been presented in previous sections.

The equations governing U_1 and V_1 are considerably more involved. Generally they depend on the solutions for U_0 and V_0 . Thus asymptotic expansions in ε for U_1 and V_1 are again appropriate. The leading solutions for U_1 and V_1 can be found easily. The expansions for U_1 and V_1 are assumed to be of the form

$$U_1 = \sum_{i=1}^{\infty} \mathcal{U}_i(\varepsilon) \bar{U}_i(X, Y) \quad (6.14)$$

$$V_i = \sum_{i=1}^{\infty} \tilde{V}_i(\varepsilon) \bar{V}_i(x, y) \quad (6.15)$$

Since the axisymmetric term in the gasdynamic equation (2.11) is to be kept, one finds that

$$\Delta_i = \sigma_i = \left(\frac{\sin \alpha}{R_0} \right) \quad (6.16)$$

All the effects of axisymmetry should be required to vanish at the upstream wall, $y=0$. This is equivalent to saying that the sublayer thickness is vanishingly small compared to $\bar{R}_0$ so that the planar boundary layer equations hold for the sublayer. As per the discussion of the introduction, the sublayer thickness is of order $R_0^{-\frac{1}{2}} \sim \left(\frac{\sin \alpha}{R_0} \right)^{\frac{1}{2}}$. Thus axisymmetric terms in the sublayer are definitely of lower order than in the inviscid transonic region. The inviscid transonic region solution must match with a planar solution at $y=0$, to order $\left(\frac{\sin \alpha}{R_0} \right)$, which requires U_1, V_1 to vanish at $y=0$.

Retention of the highest order terms in ε after the substitution of the solutions for U_0 and V_0 results in the leading equations

$$\text{(gasdynamic)} \quad \pi_i \varepsilon (r+1)^{\frac{1}{2}} u_i \frac{\partial \bar{u}_i}{\partial x} - \tilde{V}_i \varepsilon^{\frac{1}{2}} \frac{\partial \bar{V}_i}{\partial y} - \varepsilon^{\frac{3}{2}} = 0 \quad (6.17)$$

$$\begin{aligned} \text{(Crocco)} \quad \tilde{V}_i \frac{\partial \bar{V}_i}{\partial x} - (r+1)^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} \pi_i \frac{\partial \bar{u}_i}{\partial y} &= \frac{r}{2} C_0 (r+1)^{\frac{1}{2}} \varepsilon^{\frac{1}{2}} \pi_i \bar{V}_i \\ &\quad - (r+1)^{\frac{1}{2}} \varepsilon^{\frac{3}{2}} \pi_i C_0 \bar{u}_i + (r+1)^{\frac{1}{2}} \frac{r}{2} \varepsilon^{\frac{3}{2}} C_0 \frac{y}{\sin \alpha} \end{aligned} \quad (6.18)$$

where the dependence of the entropy and total enthalpy on the stream function is the same as derived in Section II and does not contain R_0 explicitly. This can be used since the approximation leading to the Mangler transformation is of higher order than those being made here. It is necessary to determine $\check{H}_1$ and $\check{A}_1$ as functions of $\mathcal{E}$ in order to proceed. The axisymmetric term of the gasdynamic equation must be kept in the first order equation. If $(\gamma+1)^{\frac{1}{2}} \check{A}_1 \mathcal{E} = \mathcal{E}^{\frac{3}{2}} \gg \check{H}_1 \mathcal{E}$ is chosen, then for $\eta < \eta_{c+}$ where $u_1 = C_0 y$,

$$\bar{u}_1 = \frac{1}{C_0} \frac{x}{y} + \text{function}(x). \quad (6.19)$$

However, this does not have the proper behavior at $y=0$ and must be rejected. If $(\gamma+1)^{\frac{1}{2}} \check{A}_1 \mathcal{E} = \check{H}_1 \mathcal{E}^{\frac{1}{2}} = \mathcal{E}^{\frac{3}{2}}$, the equations become

$$-u_1 \frac{\partial \bar{u}_1}{\partial x} + \frac{\partial \bar{v}_1}{\partial y} + 1 = 0, \quad (6.20)$$

$$\frac{\partial \bar{v}_1}{\partial x} - \frac{\partial \bar{u}_1}{\partial y} = 0. \quad (6.21)$$

If one writes (6.20) in terms of the usual velocity potential, Φ and substitutes the expansion for u_1 in terms of $\mathcal{E}$, there results

$$-\gamma \frac{d\check{f}_1}{d\eta} \Phi_{xx} + \Phi_{yy} + 1 = 0. \quad (6.22)$$

Again, $\bar{\Phi}$ should have a similarity solution from the principle of eliminability. Thus,

$$\begin{aligned}\bar{\Phi} &= \gamma^2 \bar{f}(\gamma) \\ \frac{\partial \bar{\Phi}}{\partial x} &= \gamma^{\frac{3}{2}} \frac{d\bar{f}}{d\gamma} \\ \frac{\partial \bar{\Phi}}{\partial y} &= \gamma \left[2\bar{f} - \frac{3}{2} \gamma \frac{d\bar{f}}{d\gamma} \right] \\ \eta &= \frac{x}{y^{\frac{1}{2}}}\end{aligned}$$

and (6.22) becomes

$$\left[\frac{d\bar{f}}{d\gamma} - \frac{9}{4} \gamma^2 \right] \frac{d^2 \bar{f}}{d\gamma^2} + \frac{9}{4} \gamma \frac{d\bar{f}}{d\gamma} - 2\bar{f} = 1 \quad (6.23)$$

For $\gamma \leq \gamma_{c+}$, $\frac{d\bar{f}}{d\gamma} = C_0 = \frac{9}{4} \gamma_{c+}$. Replacing $\bar{f}$ by $\gamma_{c+} \bar{g}_2$ and γ by $\gamma_{c+} \gamma$ the equation becomes

$$(1-\gamma^2) \frac{d^2 \bar{g}_2}{d\gamma^2} + \gamma \frac{d\bar{g}_2}{d\gamma} - \frac{8}{9} \bar{g}_2 = \frac{4}{9} \frac{1}{\gamma_{c+}} \quad (6.24)$$

Thus, $\bar{g}_2 = \bar{G} + \frac{4}{9} \gamma$ where $\bar{G}$ is the solution of the Gauss hypergeometric equation,

$$z(1-z) \frac{d^2 \bar{G}}{dz^2} + \left(-\frac{1}{2} + z\right) \frac{d\bar{G}}{dz} - \frac{8}{9} \bar{G} = 0 \quad (6.25)$$

$$\text{with } z = \frac{1}{2}(1+\gamma)$$

$\bar{G}$ can be written in terms of the hypergeometric function as

$$\bar{G} = D_1 F_1 + D_2 z^{\frac{3}{2}} F_2$$

where

$$F_1 = F\left(-\frac{4}{3}, -\frac{2}{3}; -\frac{1}{2}; z\right)$$

$$F_2 = F\left(\frac{1}{6}, \frac{5}{6}; \frac{5}{2}; z\right)$$

for $|z| \leq 1$

and

$$F_1 = \frac{\Gamma(-\frac{4}{3})\Gamma(\frac{2}{3})}{\Gamma(-\frac{2}{3})\Gamma(\frac{5}{6})} (-z)^{\frac{4}{3}} F\left(-\frac{4}{3}, \frac{1}{6}; \frac{1}{3}; \frac{1}{z}\right)$$

$$+ \frac{\Gamma(-\frac{1}{2})\Gamma(-\frac{2}{3})}{\Gamma(-\frac{4}{3})\Gamma(\frac{1}{6})} (-z)^{\frac{2}{3}} F\left(-\frac{2}{3}, \frac{5}{6}; \frac{5}{3}; \frac{1}{z}\right)$$

$$F_2 = \frac{\Gamma(\frac{5}{6})\Gamma(\frac{2}{3})}{\Gamma(\frac{5}{6})\Gamma(\frac{2}{3})} (-z)^{-\frac{1}{6}} F\left(\frac{1}{6}, -\frac{4}{3}; \frac{1}{3}; \frac{1}{z}\right)$$

$$+ \frac{\Gamma(\frac{5}{6})\Gamma(-\frac{2}{3})}{\Gamma(\frac{1}{6})\Gamma(\frac{5}{3})} (-z)^{-\frac{5}{6}} F\left(\frac{5}{6}, -\frac{2}{3}; \frac{5}{3}; \frac{1}{z}\right)$$

for $|\frac{1}{z}| \leq 1$.

The upstream wall is described in the limit as $z \rightarrow -\infty$ ($y \rightarrow 0$, $(-x)$ fixed). For z large and negative

$$\bar{\bar{G}} \sim K_1 (-z)^{\frac{4}{3}} + K_2 (-z)^{\frac{2}{3}} + O(-z)^{\frac{1}{3}}$$

and $\frac{d\bar{\bar{G}}}{dz} \sim -\frac{4}{3} K_1 (-z)^{\frac{1}{3}} + O(-z)^{-\frac{2}{3}}$

where K_1 and K_2 are constants depending on appropriate factors of D_1 and D_2 . The vertical velocity is given by

$$\bar{\bar{V}}_1 = \frac{\partial \bar{\bar{\Phi}}}{\partial y} = \gamma \left[2 \bar{\bar{f}} - \frac{3}{2} \gamma \frac{d\bar{\bar{f}}}{d\gamma} \right]$$

$$= \gamma \left[2 f + 2 \gamma_{c+} \bar{\bar{G}} - \frac{3}{4} \gamma_{c+} \gamma \frac{d\bar{\bar{G}}}{dz} \right] .$$

Since

$$(-z)^r = \left(\frac{1}{2\gamma_{c+}}\right)^r \left\{ (-\gamma)^r - r \gamma_{c+} (-\gamma)^{r-1} + \dots \right\},$$

the vertical velocity has the form

$$\bar{\bar{v}}_1 = \gamma \left\{ -\frac{2}{\gamma_{c+}} \frac{(-x)}{y^{\frac{3}{2}}} + O\left(\frac{(-x)^{\frac{3}{2}}}{y}\right) \right\}. \quad (6.26)$$

This result is improper, since it indicates that $\bar{\bar{v}}_1$ does not vanish as the upstream wall is approached, but rather grows as $\sim \frac{1}{y^{\frac{1}{2}}}$. This solution must also be rejected.

Finally we are led to conclude that

$$\bar{\bar{u}}_1 \varepsilon^{\frac{1}{2}} = \varepsilon^{\frac{3}{2}} \gg \varepsilon \bar{\bar{u}}_1 \quad (6.27)$$

implying that

$$\frac{\partial \bar{\bar{v}}_1}{\partial y} + 1 = 0 \quad (6.28)$$

are the highest order terms in the gasdynamic equation (6.17).

Thus, the solution is simply

$$\bar{\bar{v}}_1 = -y \quad (6.29)$$

for $\bar{\bar{v}}_1 = 0$ when $y=0$. The highest order terms in the continuity equation are, after substitution for $\bar{\bar{v}}_1$

$$\bar{\bar{u}}_1 \frac{\partial \bar{\bar{p}}_1}{\partial x} + \bar{\bar{u}}_1 \frac{\partial \bar{\bar{u}}_1}{\partial x} - (\gamma+1)^{\frac{1}{2}} \varepsilon^{\frac{3}{2}} = 0$$

where $\bar{\bar{u}}_1$ is the order of the density perturbation. It is

clear that $\pi_1 = \varepsilon^{\frac{3}{2}} \gg \bar{\pi}_1$ is impossible since it leads to a solution for $\bar{\bar{p}}_1$ that does not vanish at $y=0$. The same situation occurs for $\bar{\bar{u}}_1$ if $\bar{\pi}_1 = \varepsilon^{\frac{3}{2}} \gg \pi_1$. Thus one must require $\pi_1 = \bar{\pi}_1$. However, if $\bar{\pi}_1 = \varepsilon^{\frac{3}{2}}$, then

$$\bar{\bar{p}}_1 = (\gamma+1)^{\frac{1}{2}} \chi - \bar{\bar{u}}_1 + H(\gamma).$$

Again $\bar{\bar{p}}_1$ does not vanish at $y=0$, so this possibility is also ruled out. It must be concluded that

$$\varepsilon^{\frac{1}{2}} \gg \bar{\pi}_1 \gg \varepsilon^{\frac{3}{2}}$$

and the first order continuity equation simply requires that

$$\bar{\bar{p}}_1 = -\bar{\bar{u}}_1 + \bar{\bar{F}}(\gamma),$$

With this information, the Crocco equation (6.18) reduces to

$$\frac{\partial \bar{\bar{u}}_1}{\partial \gamma} = 0 \quad (6.30)$$

which, with the requirement that $\bar{\bar{u}}_1$ vanish on the upstream wall, implies that

$$\bar{\bar{u}}_1 = 0 \quad (6.31)$$

In summary, then, the leading terms of the axisymmetric corrections for the problem are given by.

$$\left(\frac{\sin \alpha}{R_0}\right) V_1 = -\varepsilon \frac{\sin \alpha}{R_0} \gamma + h.o.t. \quad (6.32)$$

$$U_1 \sim \text{terms small compared to } \varepsilon \quad .$$

Further terms in these expansions could be generated in a similar manner. However, the complexity of the equations becomes great and no more will be presented.

The solutions (6.29) and (6.31) are the same as the terms due to axisymmetry found by Friedman (1962). However, here the significance of these terms in the overall expansion scheme has been systematized.

It is worth noting that each term of the expansion in terms of $(\frac{\sin \alpha}{R_0})$ is in turn an expansion in terms of ϵ , thus the principle of eliminability applies separately to each term of the $(\frac{\sin \alpha}{R_0})$ expansion. However, the solution must depend explicitly on $(\frac{\sin \alpha}{R_0})$ since this represents a physical parameter (for example, the inverse square root of the external Reynolds number for a cone).

The limits $\epsilon \rightarrow 0$ and $(\frac{\sin \alpha}{R_0}) \rightarrow 0$ have been kept separate in this solution scheme. To obtain numerical results, both ϵ and $(\frac{\sin \alpha}{R_0})$ must be specified. Numerical calculations including the leading axisymmetric terms were not carried out. If the axisymmetric terms have any significant effect on calculated results for $\epsilon = 0.1$, the external Reynolds number would have to be as low as 100. This case does not seem to have practical importance. However, the double expansion procedure does show how axisymmetric effects fit into the overall solution of the inviscid transonic corner region.

The physical meaning of the current solution seems clear. For a conical flow without acceleration, the stream tubes, which are conical shells, must converge as y increases

in order to satisfy continuity. That is, since the circumference of a shell grows like $\sim y$, the thickness of the shell must decrease in the same proportion if there is no acceleration of the flow. Thus, these leading terms just represent a correction to the planar flow due to the axisymmetric geometry. Of course, higher order terms in ϵ should produce terms in the velocity expansions representing corrections to the acceleration. However, these corrections will be relatively small.

VI CONCLUSION

The transonic similarity solution near the corner shows that the angle the sonic line makes with the corner can have any value between 0° and 90° depending on the condition of the flow approaching the corner. Typically, the sonic line very near the corner has a slightly convex shape with respect to the upstream direction. Presumably, since the sonic line in a supersonic boundary layer far ahead of the corner is roughly parallel to the wall, the sonic line has an inflection point upstream so the overall sonic line shape may be supposed as shown in figure (6.1).

The streamlines at first converge toward the wall and then, after crossing the sonic line, begin to curve away from the wall. Physically, the streamtubes decrease in area for subsonic flow, reaching minimum area at sonic speed. Then they increase in area as the flow continues to accelerate. The streamlines eventually turn back almost parallel to the wall before encountering the leading characteristic from the corner. Thus calculations of the expansion region can be made assuming the flow at the leading characteristic to be parallel, but with a boundary layer like velocity profile. Therefore Weinbaum's (1966) rotational characteristics calculation for the expansion of a boundary layer around a corner, in which he assumed the initial flow to be a parallel flow with a boundary layer profile, seems appropriate to continue the solution downstream.

Upstream of the leading characteristic from the corner, the transonic region must be embedded in an intermediate region

ordered by the incoming boundary layer thickness, where the velocity cannot be considered a perturbation from a sonic stream. The solution in this region must match with the current transonic region in the limit as the corner is approached.

It must also match with the flow in the incoming viscous boundary layer upstream of the corner. The governing equations in this region are the Euler equations. However, in this region the proper asymptotic representations for the variables are not clear. Rather, the full equations must be solved and matched. The behavior of the singular nature of the solutions at the corner is given by the current analysis, so there should be no essential difficulty with, say, numerically integrating the equations in the intermediate region.

Below the transonic region, there exists a highly viscous sublayer in which the velocity varies from zero at the wall, to the velocity at the base of the inviscid transonic region. As discussed in the introduction, the boundary layer equations should hold in this region except very close to the corner. Within a Stokes radius of the corner, the pressure gradients in both the x and y directions in the sublayer become large. Here, a balance between viscous forces, inertia and pressure forces is expected, in contrast to the balancing of inertial and viscous forces in the sublayer upstream. To obtain and match the solutions in all the attendant regions presents formidable difficulties. In the Stokes region at the corner, the separation streamline actually is expected to lie below the corner on the base wall from Hama's (1966)

experiments and subsequent work by Weinbaum (1968). This fact lends validity to the assumption that the inviscid transonic region is followed by a centered expansion fan arising at the corner. The inviscid flow needs to execute a sharp large angle turn to follow the streamlines in the Stokes region. This is possible through the expansion fan.

The results of this work are consistent with the sonic line slope formula (1.1) for homoenergetic flow. Relation (1.1) can be extended for flow with both entropy and total enthalpy variation to give:

$$\tan \omega_b = \left[\frac{-\bar{a}^* \frac{\partial}{\partial \bar{s}} \left(\frac{\bar{q}}{\bar{a}^*} \right)}{-\frac{\bar{a}^*}{\bar{R}_b} - \left(\bar{q} + \frac{d\bar{a}^*}{d\bar{n}} \right)} \right]_b \quad (7.1)$$

The current results show $\omega_b > \frac{\pi}{2}$ indicating that

$$\left(\frac{\bar{a}^*}{\bar{R}_b} + \bar{q} + \frac{d\bar{a}^*}{d\bar{n}} \right) < 0.$$

For a boundary layer type profile $\bar{q}$ and $\frac{d\bar{a}^*}{d\bar{n}}$ are negative so the effect of the radius of curvature of the streamlines at the sonic point on the body is less than the effect of the entropy and total enthalpy variation in the incoming flow at the wall. Further, as the entropy and total enthalpy variation at the wall decreases, both (4.56) and (7.1) show $(\frac{\pi}{2} - \omega_b)$ becoming smaller but still negative. Notice that in the case of Section V, $\frac{\partial}{\partial \bar{s}} \left(\frac{\bar{q}}{\bar{a}^*} \right) = 0$ at the sonic point. Thus no conclusion regarding the denominator of (7.1) can be made. However, with its numerator zero, (7.1) gives $\omega_b = \pi$ just as predicted by (5.36).

Although appeal to Hama's (1966) experiment has been used to generate the hypostases used in this analysis, direct experimental verification of the present results is not yet possible. The pressure measurements made by Hama (1966) are roughly 30% of a boundary layer thickness upstream and downstream from the corner. His upstream pressure measurement nearest the corner shows only a slight drop in pressure while the measurement just below the corner on the base shows a pressure just slightly below the base pressure. The current work is concerned with distances that are of the order of the perturbation from sonic speed times the boundary layer thickness from the corner. The smallness of the inviscid, transonic region makes measurements difficult. The correctness of the current solution probably will have to be inferred from whether it gives a correct result for the flow farther downstream. This implies that the solutions in the various regions are known and matched together to provide the upstream conditions for calculations farther downstream.

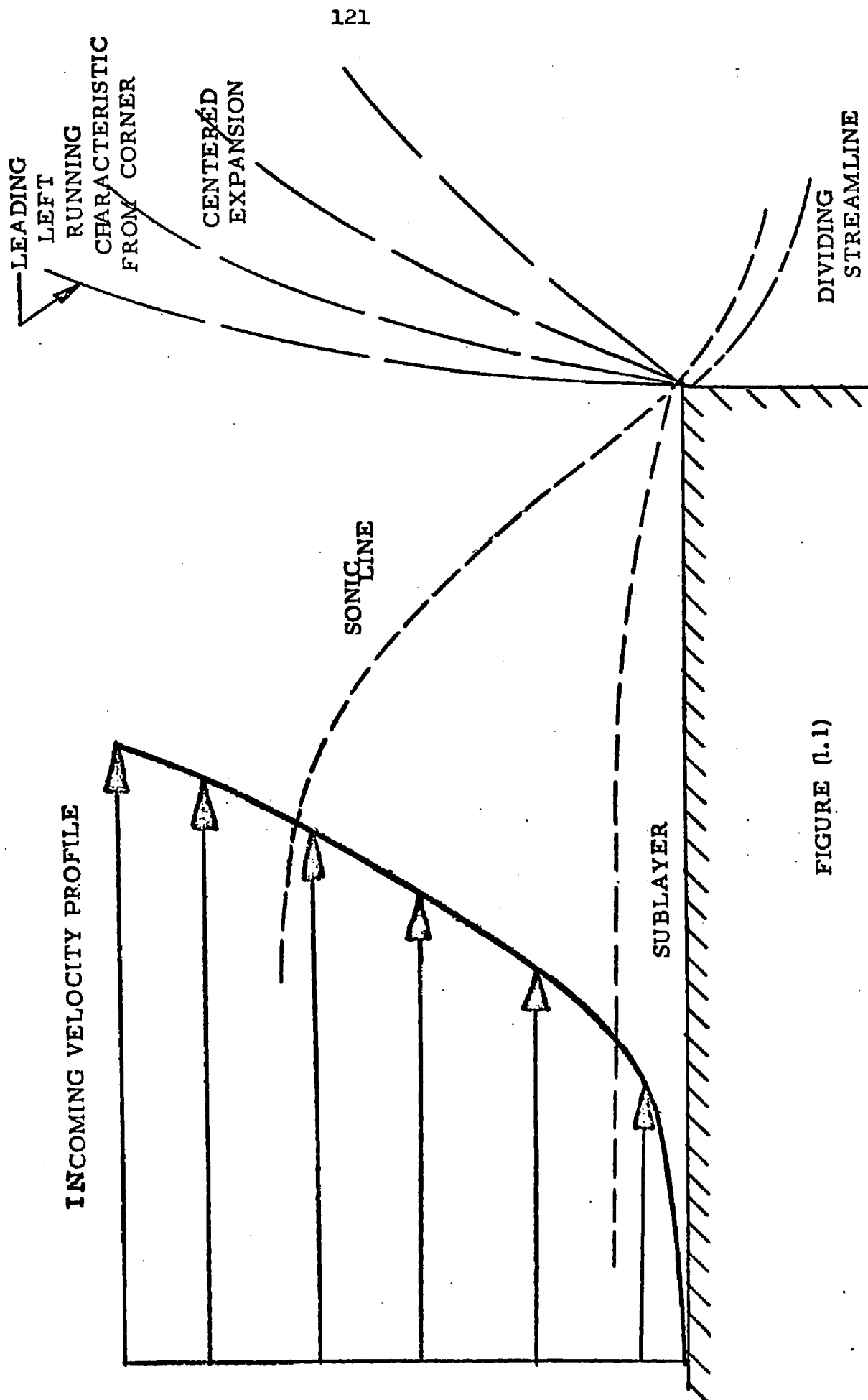


FIGURE (1.1)

FLOW STRUCTURE IN THE
INVISCID CORNER REGION

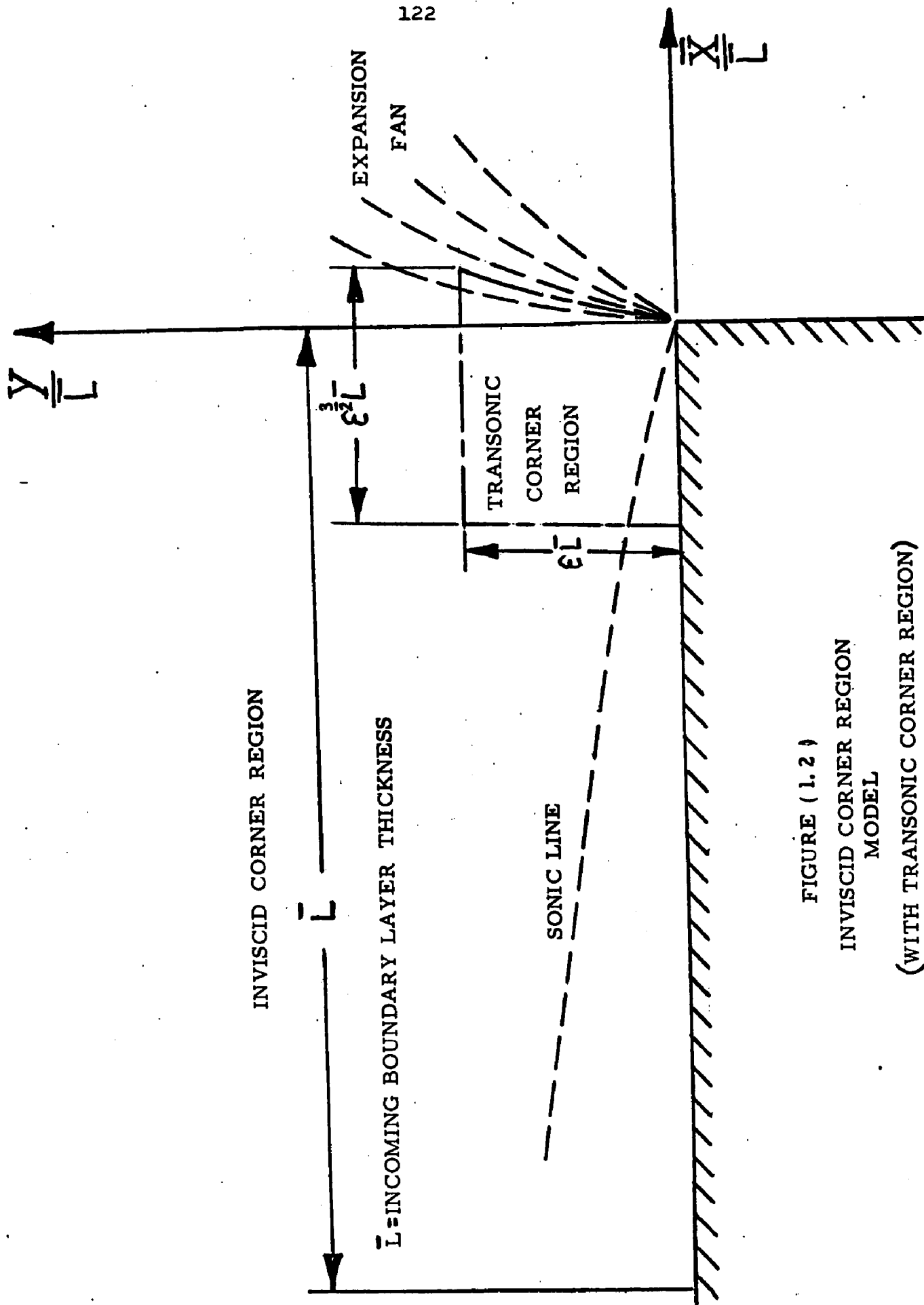


FIGURE (1.2)
 INVISCID CORNER REGION
 MODEL
 (WITH TRANSONIC CORNER REGION)

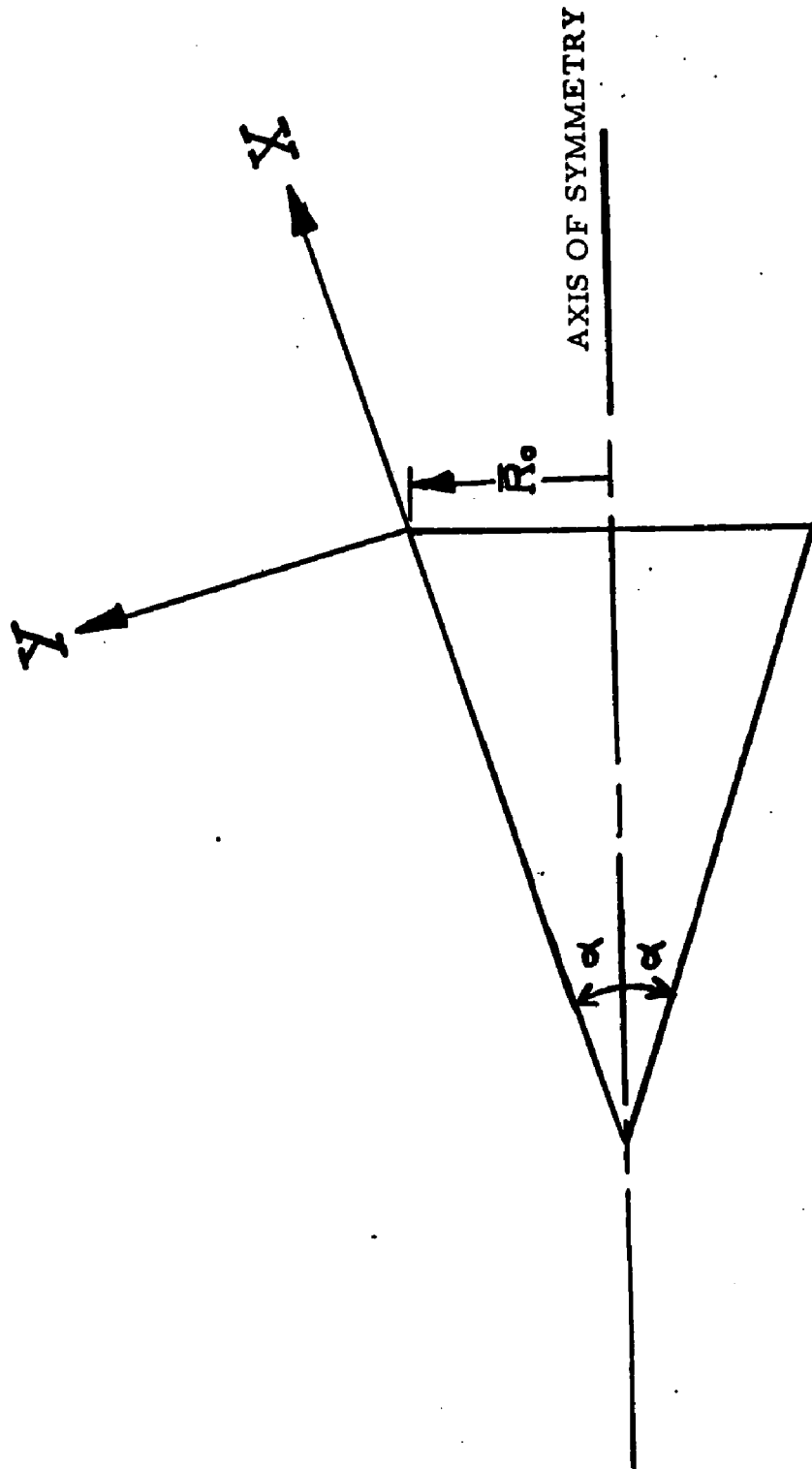


FIGURE (2.1)
 ROTATED AXISYMMETRIC COORDINATES
 (α = CONE HALF ANGLE)
 ($\bar{R}_0$ = RADIUS OF BASE)

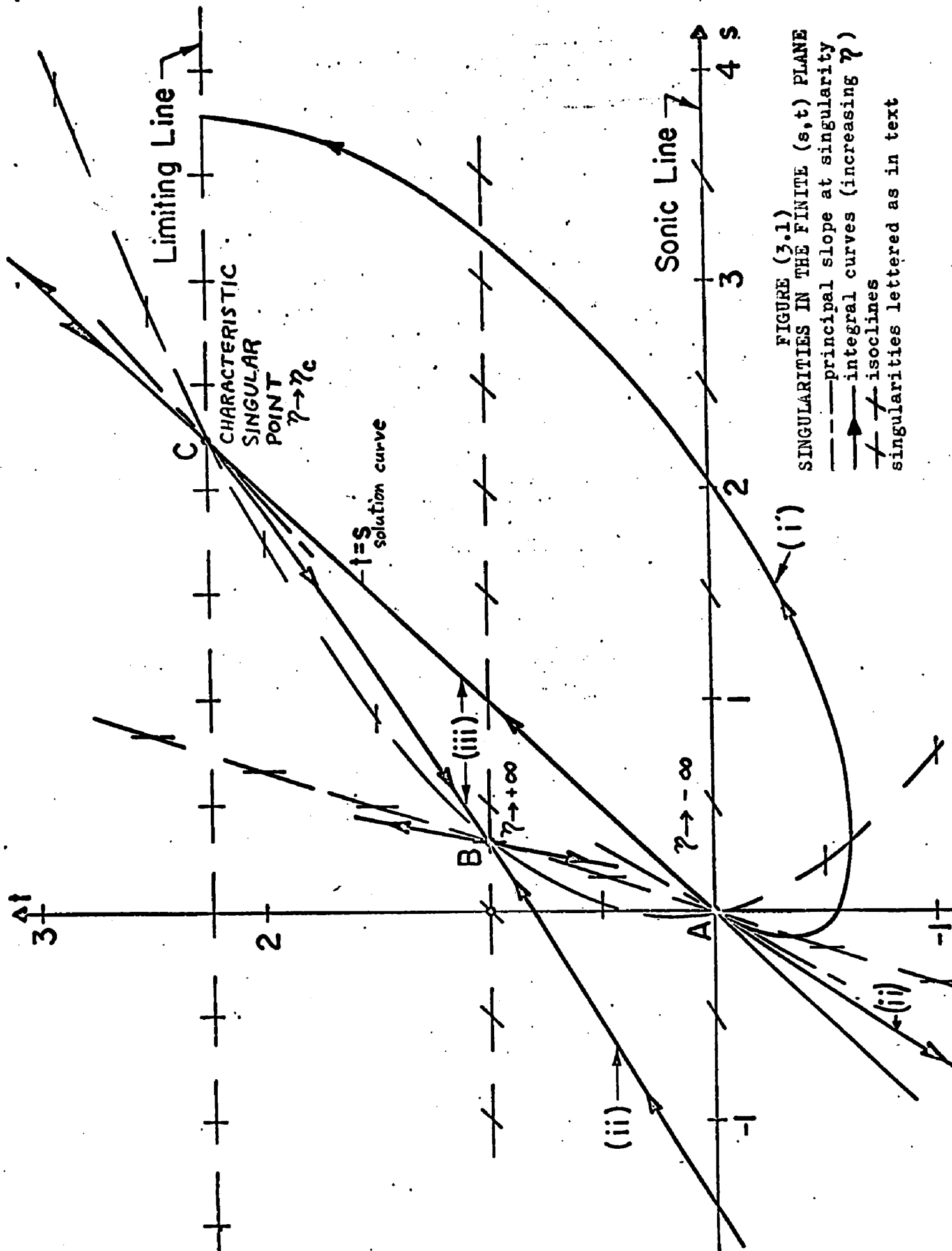


FIGURE (3.1)

SINGULARITIES IN THE FINITE (s, t) PLANE

--- principal slope at singularity
 — integral curves (increasing γ)
 - - isoclines
 singularities lettered as in text

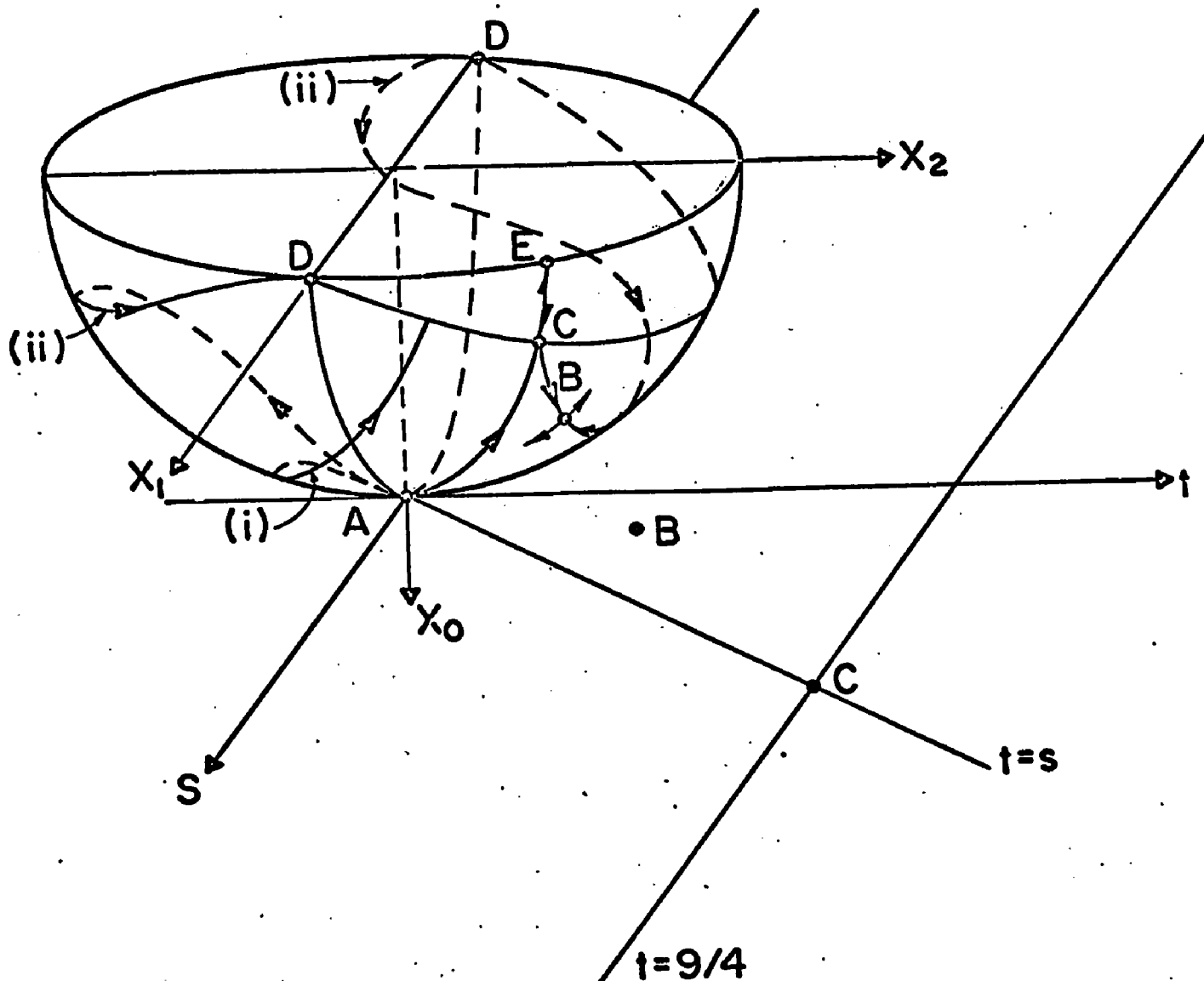


FIGURE (3.2)
 SPHERICAL PROJECTION OF (s, t) PLANE
 limiting line $D-C-D$
 sonic line $D-A-D$
 singularities lettered as in text

FIGURE (4.1)
VELOCITY PROFILES

$$C_{1/2} = -1$$

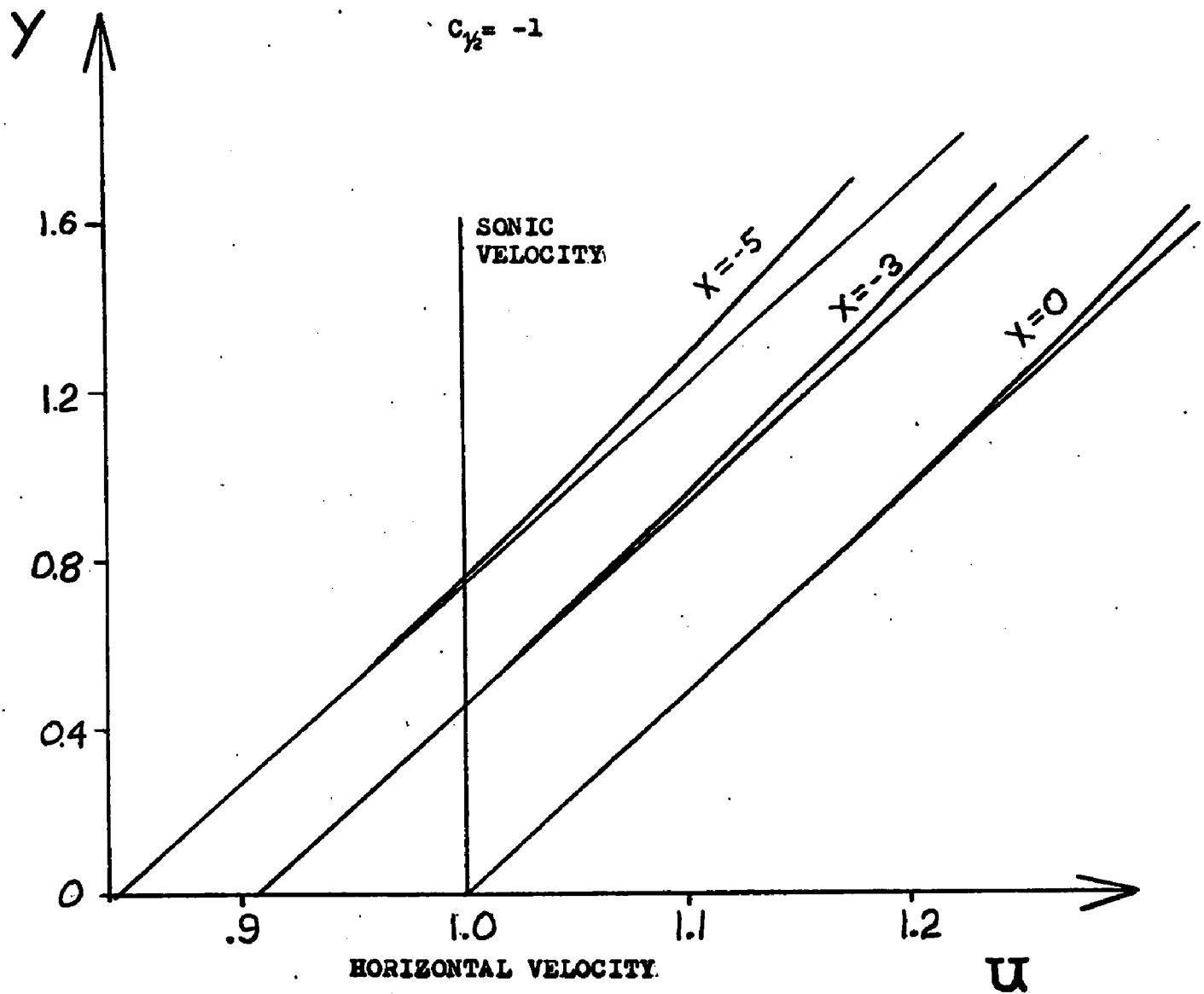
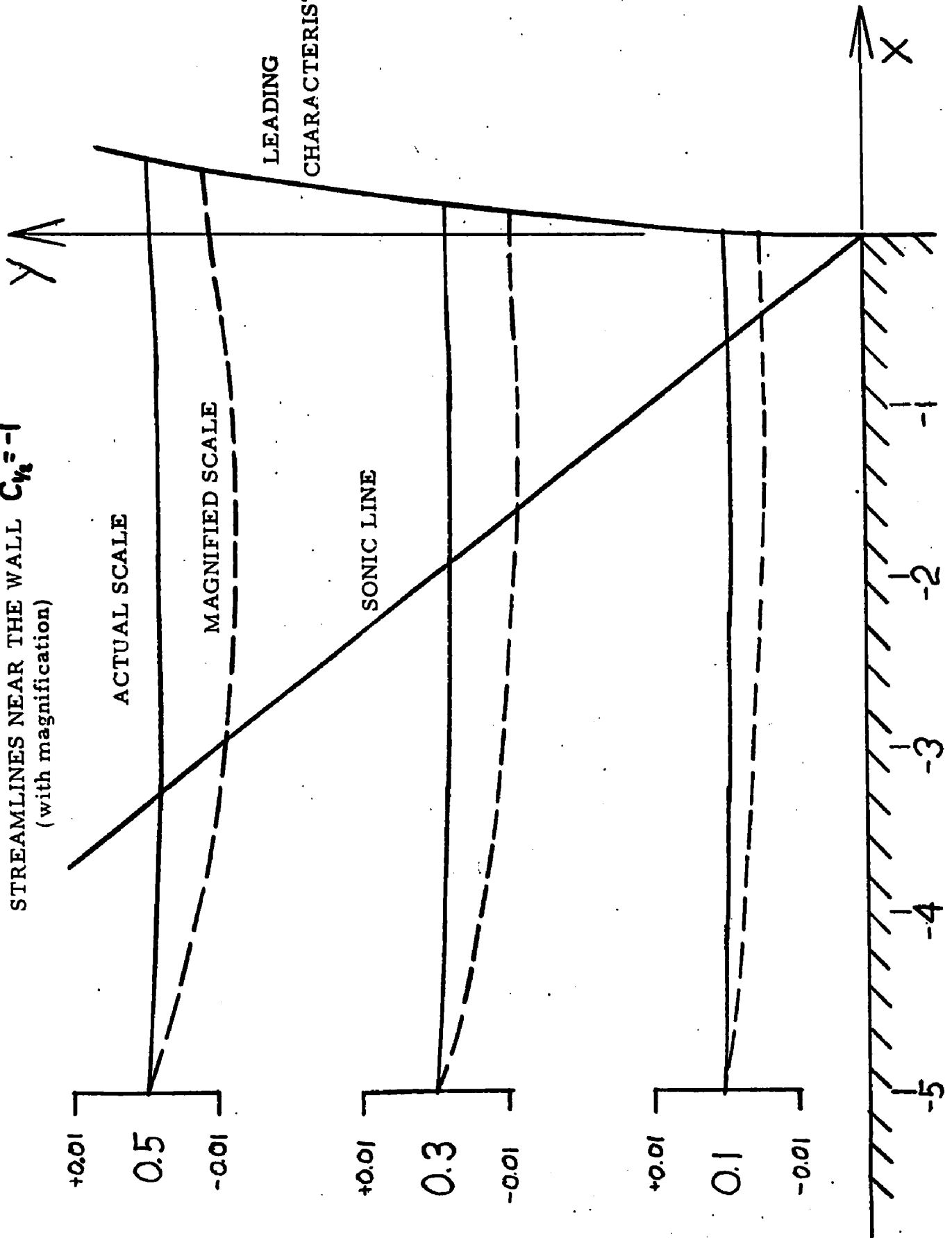


FIGURE (4.2)
STREAMLINES NEAR THE WALL $C_{\eta} = -1$
(with magnification)



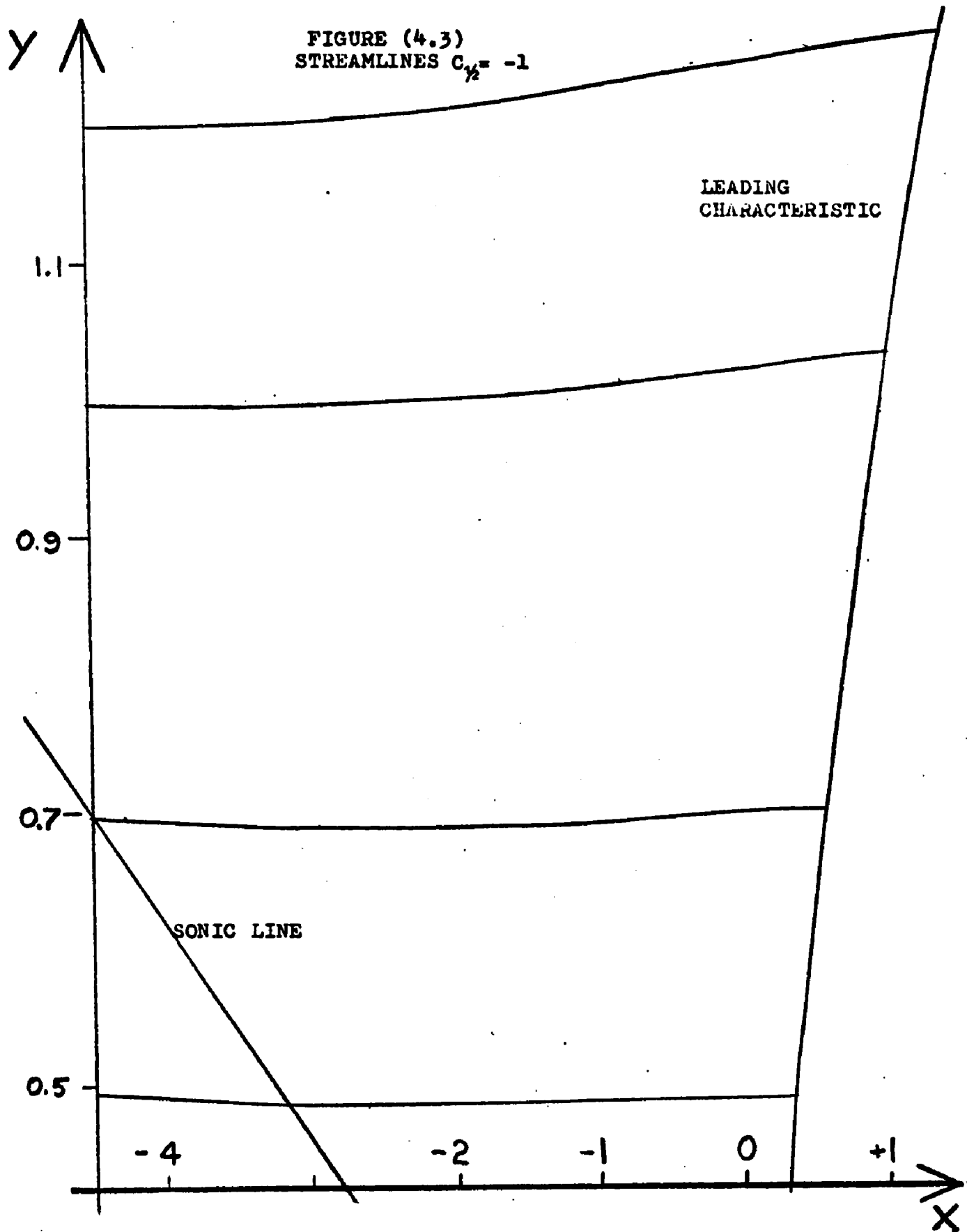


FIGURE (4.4) SECOND ORDER PRESSURE GRADIENT $C_{\frac{1}{2}} = -1$

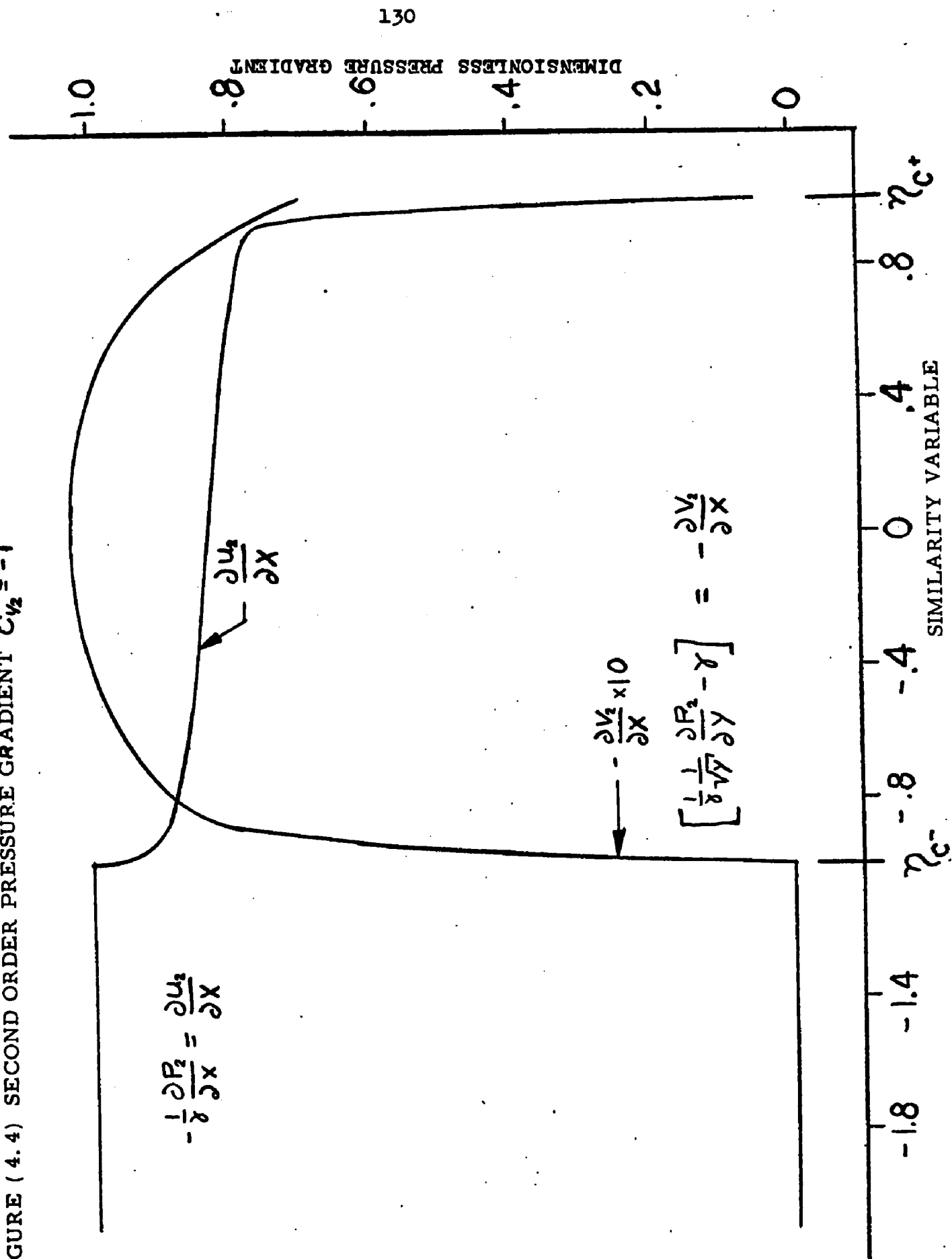
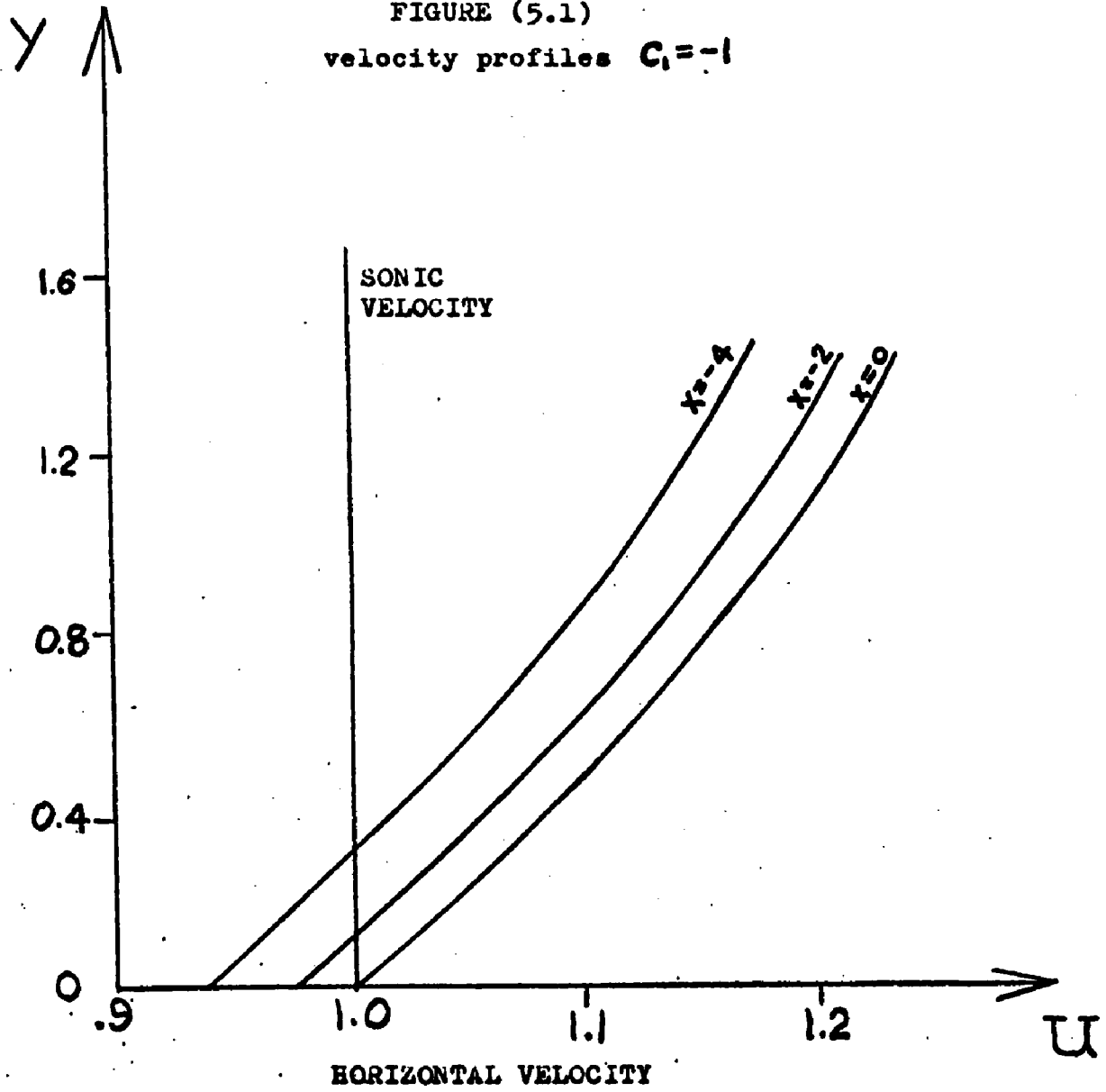
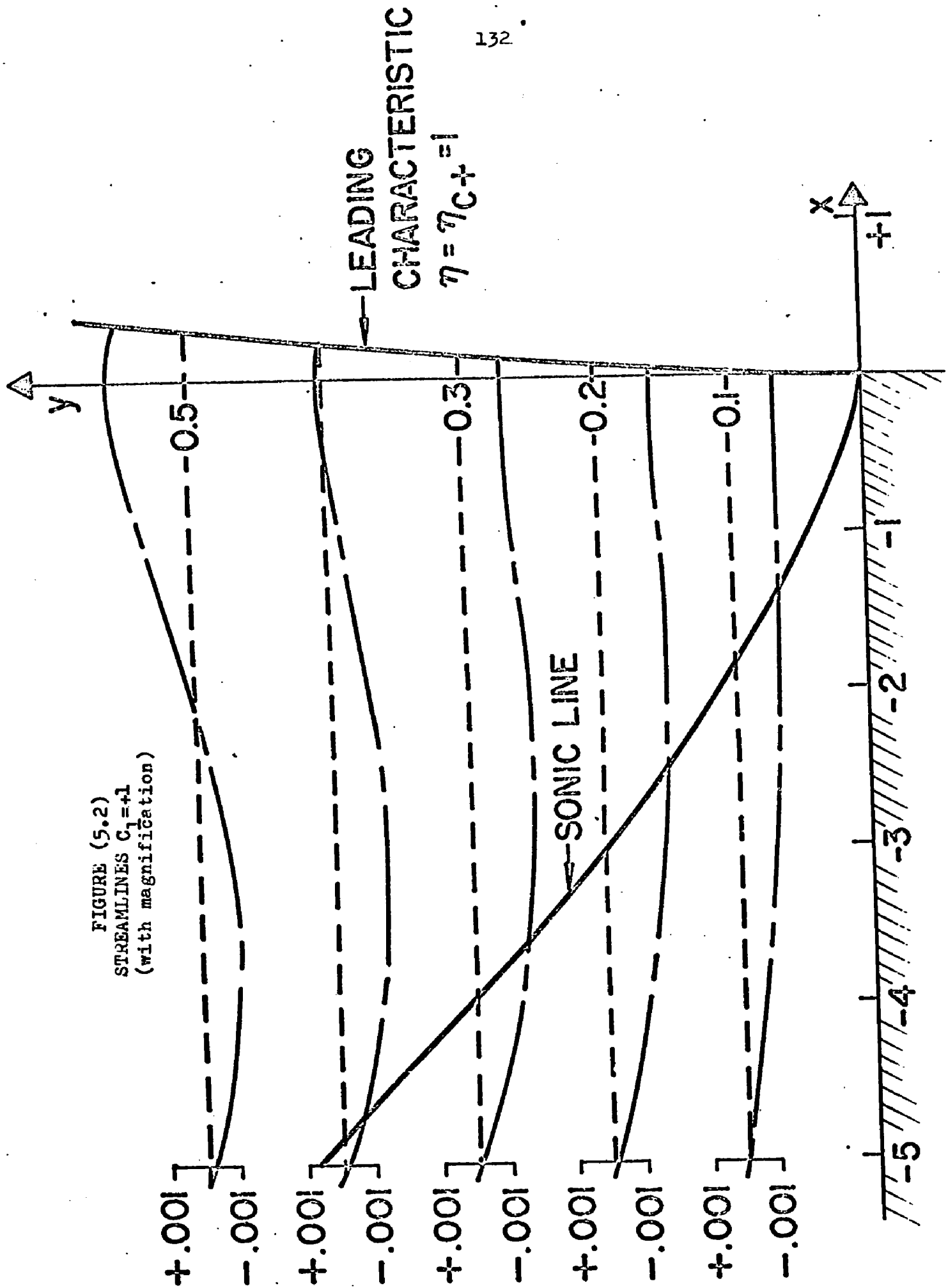


FIGURE (5.1)
velocity profiles $C_1 = -1$





LEADING CHARACTERISTIC

CHARACTERISTIC

 η_{c+}

0

 η_{c-}

-2

-3

-4

-5

-6

-7

SIMILARITY VARIABLE

FIGURE (5.3)
SECOND ORDER PRESSURE GRADIENT
 $C_1 = +1$

$$\frac{1}{\gamma} \frac{\partial P_2}{\partial \gamma} - (\gamma + 1) = -\frac{\partial u_2}{\partial x}$$

$$\frac{1}{\gamma \sqrt{\gamma}} \frac{\partial P_2}{\partial x} = -\frac{\partial u_2}{\partial x}$$

DIMENSIONLESS PRESSURE GRADIENT

INCOMING VELOCITY PROFILE

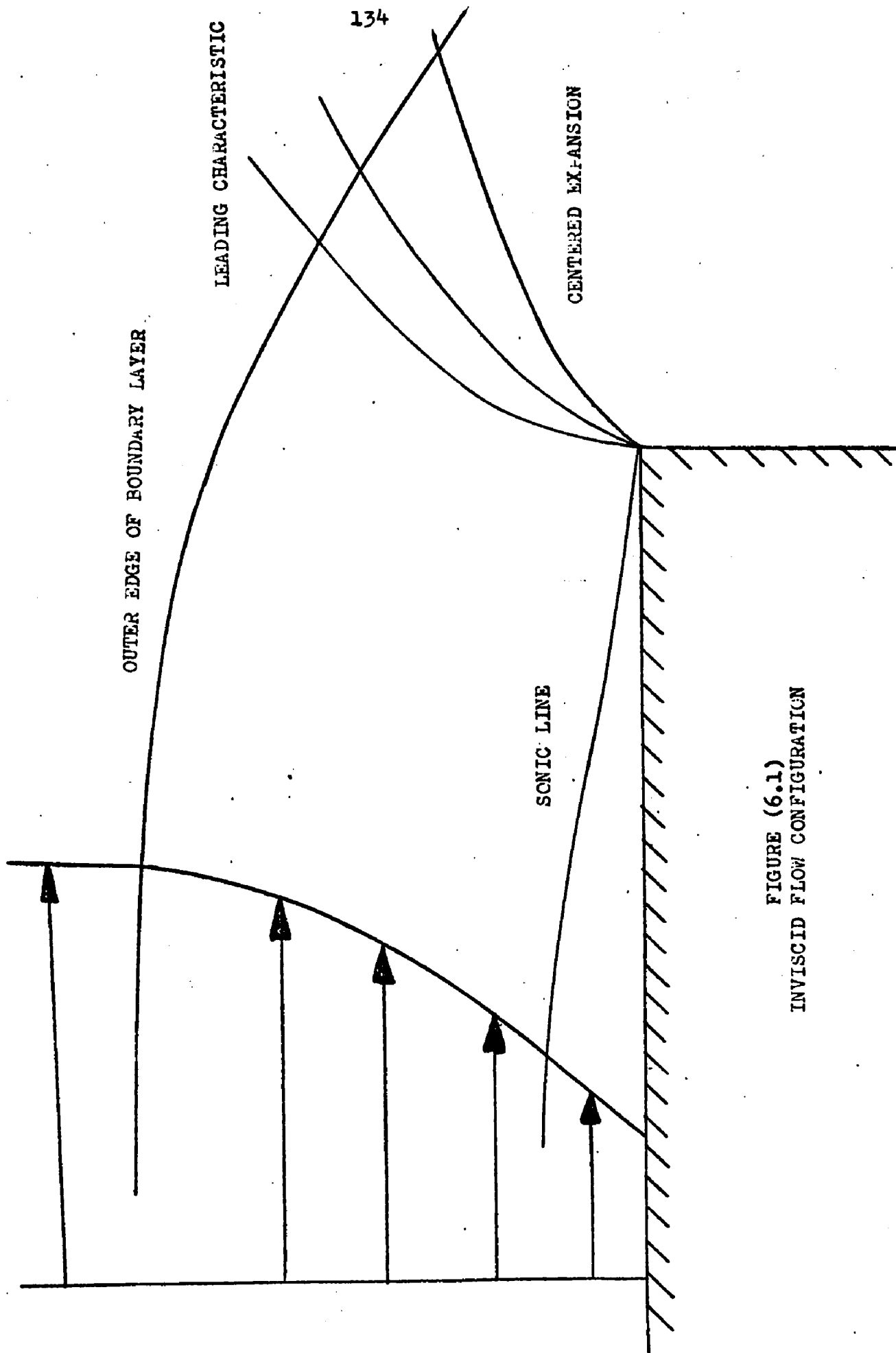


FIGURE (6.1)
INVISCID FLOW CONFIGURATION

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