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DIFFERENTIAL CALCULUS AND DIFFERENTIAL EQUATIONS IN MONTEL SPACES

by
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of the requirements for the degree of
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I should like to dedicate these pages to Walt Davison who convinced me that creativity was possible for a mathematician; to Prof. L. Cesari whose interest and advice has been very important to me; and most of all to Shari who inspired the non-mathematical context in which this paper was written.

Preface

This paper is an attempt to extend certain basic concepts of calculus and differential equations to a class of topological vector spaces known as Montel spaces. The definition of derivative introduced by Jèsus Gil de Lamadrid in 1955 is used to develop many of the manipulative rules of differential calculus.

The problem of Pfaff, that is, the characterization of perfect differential forms, which was solved for Banach spaces by Michael Kerner in 1933, is solved for Montel spaces.

The existence and uniqueness of solutions of differential equations in Montel spaces subject to certain boundedness conditions are demonstrated. Several examples and counterexamples showing that the assumptions are needed are given.

The existence theory is extended by an investigation of linear differential equations.

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Introduction

A topological vector space is a set which is equipped with a geometric structure and a topological structure. We may well ask if this geometry and topology is sufficient to derive an analytic structure or if one has to impose such a structure from without as is done in the case of manifolds. By an analytic structure we mean the concepts of derivative and integral, and we demand not only that these terms be defined, but that analogues of the basic theorems of differential and integral calculus be proved.

In the case of integration, it is well established that just about all of the standard results are valid in topological vector spaces. Some of the early work was done by Hildebrandt and Graves [5], and Kerner [15]. Good summaries are given by Myers [6] and Gil de Lamadrid [3]. In this paper a few extensions of these results are given. They are technical in nature and are included for their utility rather than their interest.

The situation is quite different with differentiation. The first attempts were to extend differential calculus to Banach spaces. These met with immediate success. Hildebrandt and Graves [5] gave the definitions, developed the calculus, and proved the fundamental theorem relating the derivative and the integral. Kerner [15] solved the problem of Pfaff and finally, Michal and Elconin [18] proved the classical Picard theorem (see [16]) which establishes the existence of solutions of differential equations by the method of successive approximations.

The first failure occurred when Dieudonné [17] gave an example showing that the Peano type existence theorem (see [16]) for differential

equations could not be proven. This is so even when one considers functions from the reals to a Banach space.

Shortly after the appearance of Dieudonné's counter example, Gil de Lamadrid [3] took the first step away from Banach spaces by defining the derivative for functions whose domain and range are any locally convex topological vector space.. However, Gil de Lamadrid proves only a few of the results of differential calculus.

In this paper we consider a special kind of topological vector space known as Montel space. It is shown below that Montel spaces are a generalization of finite dimensional spaces which are completely different from Banach spaces. We then attempt to show that Gil de Lamadrid's derivative is sufficient to derive an analytic structure on Montel spaces.

In Section 1 we establish the context of our theory by explaining our notation and listing the known results which will be used. Topological vector spaces in general and Montel spaces in particular are discussed with several examples given. Integration and differentiation are explained and the known theorems which give their properties are quoted. Finally a few results about function spaces are stated.

In Section 2, the development of differential calculus for Montel spaces is completed. Not all results which can be obtained are given. We consider only those which illustrate the methods, and those which are used in the sequel. We discuss the relationship between differentiability and continuity, partial differentiation, and the formulas for the chain rule and the derivative of products and reciprocals.

In Section 3, we solve the problem of Pfaff in Montel spaces. That is, we prove that if a function T is given which "looks like a derivative",

then there is a function whose derivative is T, if and only if T' satisfies a certain symmetry condition.

In Section 4, we prove existence and uniqueness theorems for differential equations in Montel spaces. The cases one dimension, finite dimensions and no dimensions for the domain (the range being a Montel space) of the solution function are considered separately. This is done first because the proofs are quite different in each case, and second because there is a crucial logical interdependence.

Finally, in Section 5 we consider the special case of linear differential equations. The theory of matrix solutions (as discussed, for example, in [16]) has its counterpart as does the notion of the exponential of a linear transformation. These are applied to obtain the variation of constants formula and an existence theorem with weaker assumptions than those used in Section 4.

Numbers in square brackets refer to the list of references at the end. Numbers with decimal points in parentheses refer to sections, and numbers without decimal points in parentheses refer to equations. In a few cases these conventions will not be used. Instead either the word "section" or "equation" will precede the number.

The symbol ∎ indicates the completion of the proof of a lemma, theorem or corollary.

1. Definitions and Known Results

In this section we introduce our notation, define certain terms and mention the known results which will be used in the sequel. Section 1.1 gives the notation. In Section 1.2, the rudiments of the theory of locally convex topological vector spaces are sketched, including geometrical and topological concepts, spaces of linear continuous functions, integration and differentiation, Montel spaces, examples, and Banach spaces. Section 1.3 lists several known results in analysis.

1.1 Notation

The notation listed below is standard and will be used throughout.

1.1.1

If f is a function with domain E and range F, then we write

$$f : E \longrightarrow F . \tag{1}$$

If $A \subset E$, then $f(A)$ is the image of A and f/A is the restriction of f to A. If

$$g : f(E) \longrightarrow G \tag{2}$$

then we have

$$g \cdot f : E \longrightarrow G \tag{3}$$

defined by

$$g \cdot f(x) = g(f(x)) . \tag{4}$$

1.1.2

$E \times F$ is the cartesian product of E and F: that is, the set of functions from a set $\{a,b\}$ to the union of E and F such that the image of a is in E and the image of b is in F.

1.1.3

If X is a collection of functions from G to H and

$$f : E \times F \longrightarrow X \tag{5}$$

and $(u,v) \in E \times F$, then $f(u,v)$ is a function from G to H. If $x \in G$, we write

$$f(u,v)(x) \in H \quad . \tag{6}$$

Through the technique of, loosely speaking, "fixing" one "variable," we can define other functions. For example, if $u \in E$, $x \in G$, we can define $f_{u,x}: F \longrightarrow H$ by

$$f_{u,x}(v) = f(u,v)(x) \quad . \tag{7}$$

1.1.4

The empty set will be denoted by $\emptyset$. The set of real numbers by R. The set of n-dimensional real vectors will be denoted by R^n. If A, B are two sets, $A - B$ will denote those elements of A which are not in B.

1.1.5

If x,y are two vectors in a vector space over the reals, then the set

$$[x,y] = \left\{ z \,/\, z = x + t(y-x) \,, \quad 0 \leq t \leq 1 \right\} \tag{8}$$

will be called the interval from x to y.

1.2 Topological vector spaces

A topological vector space is a set equipped with the structures of a vector space and a Hausdorff topology such that addition and scalar multiplication are jointly continuous operations. We will generally consider the field of scalars to be R, although most of the results are valid in the complex case. The terminology will generally be that of [1].

1.2.1 Geometry

1.2.1.1

A convex set S in a topological vector space is a set with the property that if $x, y \in S$, then $[x,y] \subset S$. Let T be any set. The intersection of all convex sets containing T is a convex set which contains T and is the smallest such. It is called the convex closure of T.

1.2.1.2

An equilibrant set S in a topological vector space is a set such that $x \in S$ implies $[-x,x] \subset S$. If S is any set, then

$$\overline{S} = \{\lambda x / x \in S , \quad |\lambda| \leq 1\} \tag{9}$$

is the smallest equilibrant set containing S and is called the equilibrant closure of S.

1.2.1.3

An absorbant set S in a topological vector space is a set with the property that if x is in the space, there exists a scalar λ such that

$$x \in \lambda S . \tag{10}$$

It follows from the continuity of scalar multiplication that every Nbd of 0 (the origin) is absorbant [1].

1.2.1.4

These geometric properties are helpful in obtaining an intuitive picture of subsets of a topological vector space and will be used throughout. We observe that the essence of this theory is contained in the connection between these properties and the topology in question.

1.2.2 Topology

We will always assume that the topology is Hausdorff. Most topological questions can be reduced to a consideration of Nbds of 0.

1.2.2.1

A Nbd base of 0 is a collection $\mathcal{V}$ of subsets of the space such that each $V \in \mathcal{V}$ is a Nbd of 0 and each Nbd of 0 contains a set which is an element of $\mathcal{V}$.

1.2.2.2

A locally convex topological vector space (LCEVT) is a topological vector space in which there exists a Nbd base of 0 consisting of convex sets. A topological vector space is a LCEVT if and only if there exists a Nbd base of 0 consisting of convex, equilibrant and absorbant sets ([1]).

1.2.2.3

A subspace of a LCEVT is a set which is closed under the operations of addition and scalar multiplication. Equipped with the usual subspace topology we obtain again a LCEVT.

1.2.2.4

If E is a LCEVT and E_0 is a subspace, there exists another subspace E_1 such that to each $x \in E$ there corresponds a unique pair x_0, x_1 with $x_0 \in E_0$, $x_1 \in E_1$ and

$$x = x_0 + x_1 \quad . \tag{11}$$

E_1 is called an algebraic supplement of E_0 ([1]). If the maps $P_0 : E \longrightarrow E_0$, $P_1 : E \longrightarrow E_1$ defined by

$$P_0(x) = x_0 \ , \quad P_1(x) = x_1 \tag{12}$$

are continuous, then E_1 is called a topological supplement of E_0. If E_0 is finite dimensional then there exists a topological supplement ([1]). In this case we say that E is the direct sum of E_0 and E_1.

1.2.2.5

If E,F are LCEVTs and $f:E \longrightarrow F$, then f is said to be uniformly continuous if for each Nbd of 0,W, in F there exists a Nbd of 0,V, in E such that if $x_1 - x_2 \in V$ then $f(x_1) - f(x_2) \in W$. If A is a set and $f:E \times A \longrightarrow F$, $x_1 \in E$, then f is said to be continuous at x_1, uniformly w.r.t. $a \in A$ if for each Nbd of 0,W, in F there exists a Nbd of 0,V, in E such that if $x \in x_1 + V$, $a \in A$ then

$$f(x,a) - f(x_1,a) \in W \quad . \tag{13}$$

If $E_0 \subset E$, $f:E_0 \longrightarrow F$, f continuous and E_0 compact, then it follows in the usual manner that f is uniformly continuous.

1.2.2.6

In defining uniform continuity, we are making use of the underlying uniform structure present in any topological vector space. With this structure we can define completeness and in fact, prove that any topological vector space can be completed. For a full discussion of this idea, see [1].

1.2.2.7

A bounded (Bdd) set S in a LCEVT is a set with the property that for each Nbd of 0,V, there is a scalar λ such that

$$B \subset \lambda V \quad . \tag{14}$$

It is easily seen that a compact set is Bdd and that the convex, equilibrant, and/or topological closure of a Bdd set is Bdd.

1.2.2.8

If E,F are LCEVTs, then E x F, equipped with the usual product topology is a LCEVT. The Bdd sets of E x F are the subsets of sets which are the product of Bdd sets in E and F respectively ([1]).

1.2.2.9

Let E be a LCEVT. The set E' of all linear continuous functions from E to R is called the dual of E. It is a vector space under the pointwise functional operations. There are several topologies which make E' a LCEVT. Our main concern will be the Hahn-Banach theorem [1] which says, among other things, that if $x \in E$, and $u(x) = 0$ for all $u \in E'$, then $x = 0$.

1.2.3 The space of linear continuous functions

1.2.3.1

If E,F are LCEVTs, then F^E is the set of all functions from E to F. $\mathcal{L}(E,F)$ is the subset of F^E consisting of linear, continuous functions. Both F^E, $\mathcal{L}(E,F)$ are vector spaces under the usual pointwise operations.

1.2.3.2

If $A \subset E$, $B \subset F$ then

$$\langle A,B\rangle = \left\{ f \in F^E \;/\; f(A) \subset B \right\} . \tag{15}$$

If G is a LCEVT and $C \subset G$, then

$$\langle A,B\rangle \langle C,A\rangle = \langle C,B\rangle . \tag{16}$$

We shall use the same notation when F^E is to be replaced by $\mathcal{L}(E,F)$ and the context will serve to eliminate ambiguity.

1.2.3.3

There are many topologies which make $\mathcal{L}(E,F)$ a LCEVT. We shall be interested only in the point-open and Bdd-open topologies. These

are sometimes called the topologies of simple convergence and uniform or Bdd convergence respectively. We now describe them.

Let $\mathcal{V}$ be a Nbd base of 0 for F. Then set

$$\mathcal{W}_p = \left\{\langle x,V\rangle / \; x \in E, \quad V \in \mathcal{V}\right\} \tag{17}$$

and

$$\mathcal{W}_B = \left\{\langle B,V\rangle / \; B \quad \text{a Bdd set in E}, \quad V \in \mathcal{V}\right\}. \tag{18}$$

These are both Nbd bases of 0 for topologies on $\mathcal{L}(E,F)$ which make it a LCEVT ([1]). $\mathcal{W}_p$ gives the point-open topology and $\mathcal{W}_B$ the Bdd-open topology. Unless otherwise stated, $\mathcal{L}(E,F)$ will be considered to have the Bdd-open topology.

1.2.3.4

The Bdd sets for the two topologies discussed in (1.2.3.3) are obtained in [1]. S is Bdd for $\mathcal{W}_p$ if and only if for each $x \in E$, $\bigcup_{u\in S} u(x)$ is a Bdd subset of F. S is Bdd for $\mathcal{W}_B$ if and only if for each Bdd set B in E, the set $\bigcup_{u\in S} u(B)$ is a Bdd subset of F. In the first case, S is said to be simply Bdd, and in the second it is said to be uniformly Bdd or, just Bdd.

1.2.3.5

A set $S \subset \mathcal{L}(E,F)$ is said to be equicontinuous if for each Nbd of 0,V, in F, there is a Nbd of 0,U, in E such that

$$\bigcup_{u\in S} u(U) \subset V \quad . \tag{19}$$

1.2.3.6

If E = R there is a natural homeomorphism, h, between F, $\mathcal{L}(E,F)$ given by

$$h(x)(t) = tx, \quad x \in F, \quad t \in E. \tag{20}$$

since all geometric and topological properties are preserved by h, we shall write

$$F = \mathcal{L}(E,F) . \tag{21}$$

1.2.4 Integration

The theory of the Riemann integral of a continuous function from an interval in R to a complete LCEVT is well known (see [2], for example) and the methods and results follow the classical situation very closely.

1.2.4.1

Let I be an interval in R, E a complete LCEVT and $f: I \longrightarrow E$. We define $f': I \longrightarrow E$ (if it exists) by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} . \tag{22}$$

The first result is the fundamental theorem of calculus.

1.2.4.2 Theorem

Let $f: I \longrightarrow E$ be continuous, $a \in I$, and define $F: I \longrightarrow E$ by

$$F(x) = \int_a^x f(t)\, dt . \tag{23}$$

Then F is continuous and $F' = f$.

pf. See [2].

1.2.4.3

The next result is an analogue of the classical theorem that the absolute value of an integral is less than or equal to the integral of the absolute value. The statement here is a slight generalization of the result in ([3]). The proof is identical and will not be given.

1.2.4.4 Theorem

Let $f: I \longrightarrow E$ and let $a,b \in I$. Let A be the convex and topological closure of the image of $[a,b]$ under f. Let f be continuous. Then

$$\int_a^b f(x)\,dx \in (b-a)\,A \quad . \tag{24}$$

1.2.5 Differentiation

1.2.5.1

The problem of defining the derivative in spaces other than finite dimensional Euclidean spaces is not new. It was first studied by Frechet (4) and Hildebrandt and Graves (5). These authors restricted themselves to normed spaces and obtained many of the results of elementary analysis. Since that time, several papers have considered the problem in non-normed spaces (see for example, Hyers [6] and Sebastião e Silva [7]). Recently, Gil de Lamadrid [3] gave a very general definition which extended both the total or Frechet derivative and the partial or Gâteaux derivative to functions from any LCEVT to any other LCEVT. We shall use a form of Gil de Lamadrid's definition.

1.2.5.2

Let E,F be LCEVTs and let E_o be an open subset of E. Let $f:E_o \longrightarrow F$. The derivative f' is a function $f':E_o \longrightarrow \mathcal{L}(E,F)$ such that for each Bdd set, B, of E and each Nbd of 0,V, in F, and each $x_o \in E_o$, there is a scalar $\lambda_o \neq 0$ such that if $0 < |\lambda| \leq \lambda_o$ then $x_o + \lambda B \subset E_o$ and if $h \in B$, then

$$\frac{f(x_o + \lambda h) - f(x_o)}{\lambda} - f'(x_o)(h) \in V \quad . \tag{25}$$

Naturally, the derivative does not always exist. Gil de Lamadrid has proven ([3]) the following results:

1.2.5.3 Theorem

If $f \in \mathcal{L}(E,F)$ then for all $x_o \in E$, $f'(x_o) = f$.

1.2.5.4 Theorem

If E,F are normed vector spaces, then f' exists if and only if f has a Frechet differential and the two are equal.

1.2.5.5 Theorem

If, in (1.2.5.2) Bdd set is replaced by finite set, then the derivative exists if and only if the Gâteaux differential exists and the two are equal.

1.2.5.6 Theorem

Let f be differentiable, E_o convex, $x_o \in E_o$. Let $I = [0,1]$ and for each $x \in E_o$ let the function $g:I \longrightarrow F$ be defined by

$$g(t) = f'(x_o + t(x-x_o))(x-x_o) \tag{26}$$

be continuous. Then

$$f(x) = f(x_o) + \int_0^1 g(t)\,dt \ . \tag{27}$$

1.2.5.7

When we say that $f':E_o \longrightarrow \mathcal{L}(E,F)$ is continuous, we consider that E_o has the topology induced by E and $\mathcal{L}(E,F)$ has the Bdd-open topology.

1.2.5.8

If f' is differentiable, then we have

$$f'':E_o \longrightarrow \mathcal{L}(E, \mathcal{L}(E,F)) \tag{28}$$

and in general

$$f^{(n)}:E_o \longrightarrow \mathcal{L}(E, \mathcal{L}(E,\ldots, \mathcal{L}(E,F))\underbrace{\ldots)}_{n \text{ copies}} \ . \tag{29}$$

A notational choice must now be made. One could consider that $f'':E_o \longrightarrow \mathcal{L}(E \times E,F)$ so that if $x_o \in E_o$, $h,k \in E$ one would write

$$f''(x_o)(h,k) \in F \ . \tag{30}$$

We will never do this. Instead, we invariably write

$$f'(x_0)(h)(k) \in F \tag{31}$$

where the order of h,k corresponds to the order of E,E in the expression $\mathcal{L}(E, \mathcal{L}(E,F))$. We will use the same notation for any function $g \in \mathcal{L}(E, \mathcal{L}(F,G))$, writing

$$g(x)(y) \in G \tag{32}$$

where $x \in E$, $y \in F$.

1.2.6 Banach spaces

1.2.6.1

One important type of LCEVT is a Banach space which is an LCEVT B which is complete and such that there is a function $N:B \longrightarrow R$ satisfying

$$\begin{aligned} &\text{i)} && N(x) = 0 \text{ if and only if } x = 0 \\ &\text{ii)} && \text{if } \lambda \in R, \text{ then } N(\lambda x) = |\lambda|\ N(x) \\ &\text{iii)} && N(x+y) \leq N(x) + N(y) \\ &\text{iv)} && \text{the collection } \mathcal{V} = \{N^{-1}([0,\epsilon])/\ \epsilon > 0\} \text{ is a Nbd base} \\ & && \text{of 0 in B.} \end{aligned} \tag{33}$$

1.2.6.2

If B' is the dual of B and we define

$$\mathcal{W} = \left\{ W_{f_1,\ldots,f_n,\epsilon} \subset B \,/\, \epsilon > 0,\ f_1, \ldots, f_n \text{ a finite subset of } B', \right.$$
$$\left. W_{f_1,\ldots,f_n,\epsilon} = \{x \in B/\ |f_i(x)| \leq \epsilon \quad i = 1, \ldots, n\} \right\}, \tag{34}$$

then it is known ([1]) that $\mathcal{W}$ is another topology in which B is a LCEVT, $\mathcal{W} \cap \mathcal{V} = \emptyset$ unless B is finite dimensional, and every Nbd of 0 w.r.t. $\mathcal{W}$ is a Nbd of 0 w.r.t. $\mathcal{V}$. The topology induced by $\mathcal{W}$ on B will be called the weak topology.

1.2.7 Montel spaces

1.2.7.1

A subset of a LCEVT which is closed, convex, equilibrant, and absorbant will be called a tonneau. Every LCEVT has a Nbd base of 0 consisting of tonneaus ([1]). On the other hand, a LCEVT will be called tonnelé if every tonneau is a Nbd of 0. It follows from the Baire category theorem that if the topology of a LCEVT is metrisable, then the space is tonnelé.

1.2.7.2

A LCEVT will be called a Montel space if it is tonnelé and if the closure of every Bdd set is compact. The following result is a generalization of the Banach-Steinhaus theorem. The proof may be found in [1].

1.2.7.3 Theorem

Let E,F be LCEVTs. Let E be tonnelé. Then every simply Bdd subset of $\mathcal{L}(E,F)$ is uniformly Bdd and equicontinuous.

1.2.7.4 Theorem

Let E,F be LCEVTS. Let E be a Montel space. Let $f:E \longrightarrow F$ be continuous. Then f maps Bdd sets into Bdd sets.

pf. Let $B \subset E$ be Bdd. Then the topological closure $\overline{B}$ of B is compact, so $f(\overline{B})$ is compact and by (1.2.2.7) it is Bdd. Since $f(B) \subset f(\overline{B})$, the result follows. ∎

1.2.7.5 Theorem

A space is both a Montel space and a Banach space if and only if it is finite dimensional.

pf. See [1].

1.2.8 Examples of Montel spaces

By (1.2.7.5) we see that Montel spaces are generalizations of finite dimensional spaces in a direction entirely different from that of Banach spaces. It remains to be seen whether there are any non-finite dimensional examples of Montel spaces. There are. In fact, the spaces $\mathcal{D}$, $\mathcal{S}$, and $\mathcal{E}$ are very important in Schwartz's theory of distributions [8].

1.2.8.1 $\mathcal{D}$

Let

$$\mathcal{D} = \{\varphi / \varphi : R \longrightarrow R,\ \varphi \text{ is infinitely differentiable and has compact support}\} . \tag{35}$$

$\mathcal{D}$ is clearly a vector space and we define a Nbd base of 0 $\mathcal{V}$ by

$$\mathcal{V} = \Big\{ V \subset \mathcal{D} / \ V = V(\{m_\nu\},\{\epsilon_\nu\}) \text{ where } \{m_\nu\} \text{ is a strictly increasing sequence of non-negative integers, } \{\epsilon_\nu\} \text{ is a decreasing sequence of positive numbers converging to 0 and} \tag{36}$$

$$V = \{\varphi \in \mathcal{D} / \text{ if } |t| \geq \nu \text{ and } p = 0,1,\ldots,m_\nu, \text{ then } |\varphi^{(p)}(t)| \leq \epsilon_\nu\}\Big\} .$$

This space is investigated in [8] where it is shown that $\mathcal{V}$ does give a LCEVT which is a Montel space. A subset $B \subset \mathcal{D}$ is Bdd if and only if all of the elements of B are supported by a single compact set and for each ν, the set of real numbers

$$\{\varphi^{\nu}(t) / \ t \in R, \quad \varphi \in B\} \tag{37}$$

is Bdd. A sequence $\{\varphi_n\}$ in $\mathcal{D}$ converges if and only if all φ_n are supported by a single compact set and for each ν, the sequence $\{\varphi_n^{\nu}\}$ of functions is uniformly convergent.

1.2.8.2 $\mathcal{D}'$

The dual $\mathcal{D}'$ of $\mathcal{D}$ equipped with the Bdd-open topology is also studied in [8] with the following results. $\mathcal{D}'$ is a Montel space. A subset $B \subset \mathcal{D}'$ is Bdd if and only if it is simply Bdd (cf. (1.2.3.4) and th. (1.2.7.3)).

In addition, if f is a continuous function, we define the distribution (element of $\mathcal{D}'$) corresponding to f by

$$T_f(\varphi) = \int_{-\infty}^{\infty} f\varphi \quad . \tag{38}$$

If $T \in \mathcal{D}'$, we define the ν^{th} derivative $T^{\nu} \in \mathcal{D}'$ by

$$T^{\nu}(\varphi) = (-1)^{\nu} T(\varphi^{\nu}) \quad . \tag{39}$$

If f is ν times continuously differentiable, then

$$T_{f^{\nu}} = (T_f)^{\nu} \quad . \tag{40}$$

1.2.8.3 The space $\mathcal{H}$ of holomorphic functions

Let $\mathcal{H}$ be the set of all complex functions which are analytic on the interior C of the unit circle and continuous on its closure, $\overline{C}$. We take for a Nbd base of 0, $\mathcal{V}$, defined by

$$\mathcal{V} = \Big\{ V(\epsilon,r) \;/\; \epsilon > 0,\ 0 < r < 1,\ V(\epsilon,r) \text{ is the set of all functions } \varphi \in \mathcal{H} \text{ such that if } |z| \le r, \text{ then } |\varphi(z)| \le \epsilon \text{ and } |\varphi'(z)| \le \epsilon \Big\}. \tag{41}$$

This space is studied in [9]. It is a Montel space. In fact, it is a Montel space which is also a Frechet space, i.e. it is complete and has a countable Nbd basis of 0.

A subset $B \subset \mathcal{H}$ is Bdd and only if for each r such that $0 < r < 1$, the set

$$\Big\{ |\varphi(z)| + |\varphi'(z)| \;/\; \varphi \in B,\ |z| \le r \Big\} \tag{42}$$

is Bdd.

1.2.8.4 $\mathcal{S}$

Let $\mathcal{S}$ be the set of all functions $\varphi: R \longrightarrow R$ such that φ is infinitely differentiable, and for each pair of integers $\nu, k \geq 0$, we have

$$\lim_{|t| \to \infty} |t^k \varphi^{\nu}(t)| = 0 \quad . \tag{43}$$

$\mathcal{S}$ is a vector space and we define a Nbd base of 0 $\mathcal{V} = \{V(m,k,\epsilon)\}$ where m,k are integers ≥ 0, and $\epsilon > 0$, and

$$V(m,k,\epsilon) = \left\{ \varphi \in \mathcal{S} \;/\; \text{for each } p = 0,1,\ldots,m \text{ and for all } t \in R,\ (1 + |t|)^k |\varphi^p(t)| \leq \epsilon \right\}. \tag{44}$$

In [8] it is shown that $\mathcal{S}$ is a LCEVT which is a Montel space.

1.3 Miscellaneous results

In this section we list a few more known results which will be used in the sequel.

1.3.1 Nets

1.3.1.1

A directed set D is a set in which there exists a relation "$\leq$" satisfying

i) for all $d \in D$, $d \leq d$

ii) if $d_1, d_2 \in D$, $d_1 \leq d_2$ and $d_2 \leq d_1$, then $d_1 = d_2$ (45)

iii) if $d_1, d_2, d_3 \in D$, $d_1 \leq d_2$, and $d_2 \leq d_3$, then $d_1 \leq d_3$

iv) for each $d_1, d_2 \in D$, there exists a $d_3 \in D$ such that $d_1 \leq d_3$, $d_2 \leq d_3$.

1.3.1.2

A net in a topological space X is a function $f: D \longrightarrow X$ where D is a directed set. A subnet of a net is a function g which is the restriction of f to a subset $D_0 \subset D$ where D_0 again satisfies Eq. (45)

and for each $d \in D$, there exists $d_o \in D_o$ such that $d \leq d_o$.

1.3.1.3

A net is said to converge to a point $x \in X$, if for each Nbd V of x there is a $d_o \in D$ such that if $d_o \leq d$, then $f(d) \in V$.

1.3.1.4

The concept of a net is a generalization of the concept of a sequence. It can be shown that any topology can be completely described by specifying which nets converge to which points ([10]). We shall use only the following result from this theory.

1.3.1.5 Theorem

A topological space X is compact if and only if each net in X has a convergent subnet.

pf. See [10].

1.3.2 The Ascoli theorem

1.3.2.1

Let X be a regular compact space and Y a LCEVT. Let C be the set of all continuous functions from X to Y. C is a vector space and we make it a LCEVT by taking a Nbd base of 0, $\mathcal{V} = \{V_U\}$, where U is a Nbd of 0 in Y and

$$V_U = \left\{ f \in C / f(X) \subset U \right\} . \tag{46}$$

In [10] it is shown that $\mathcal{V}$ gives a topology.

1.3.2.2

A subset $A \subset C$ is said to be equicontinuous if for each Nbd of 0, U, in Y and each $x \in X$, there is a Nbd, W, of x such that if $x_1 \in W$ and $f \in A$ then

$$f(x) - f(x_1) \in U . \tag{47}$$

1.3.2.3 Theorem

A subset $A \subset C$ has compact closure if and only if it is equicontinuous and for each $x \in X$, the set $\bigcup_{f \in A} f(x)$ has compact closure (in Y).

pf. See [10].

2. Differential Calculus in Montel Spaces

2.1 Introduction

Gil de Lamadrid [3] introduced a definition of derivative for functions from a LCEVT to a LCEVT, and proved that if the spaces are normable, then his derivative reduces to the Frechet differential [4]. In the same paper, Gil de Lamadrid develops certain basic theorems of the type referred to in finite dimensional spaces as differential calculus. He proves a "fundamental theorem of calculus" which we will use in a slightly altered form already mentioned (1.2.5.6). He also proves a "chain rule" restricted to the case in which the first function is linear and continuous. In this section we consider various theorems from differential calculus and obtain the following results.

2.1.1

The most striking loss in this generalization is the fact that differentiability does not imply continuity. It is for this reason that many of our theorems will include both assumptions. On the other hand, we learn where the continuity and hence the topological results which that implies, is really needed in differential calculus. We give an example of a function which is differentiable but not continuous and then prove some weaker results along this line.

2.1.2

Gil de Lamadrid's special case of the chain rule is proved with only the assumption of continuity and differentiability.

2.1.3

The concept of partial differentiation is introduced and the classical relation between partial and total derivative is obtained.

Here we make use of our results on continuity and then show how the entire problem can be avoided by demanding that our spaces be a Montel space. We retain this assumption for the remainder of the paper.

2.1.4

The product rule for differentiating ordinary functions is not easily generalized because we do not usually have a multiplication structure available. However, if we consider the ranges of the functions to be spaces of functions, then the operation of composition begins to look like multiplication and a product rule is obtained. Here the similarity stops for we do not have either the usual algebraic or the usual topological properties of multiplication.

2.1.5

In a similar vein we obtain a rule for differentiation of reciprocals. Here the result is different from the classical one because of the lack of commutativity. What we obtain is analogous to differentiation of inverses of matrices.

2.2 Selection

There are many results from differential calculus other than those given here. We have used the following criteria for selection:

i) First, of course, we prove everything that will be needed in later sections.

ii) We want to indicate that other than the restriction to Montel spaces, very few additional assumptions are needed.

iii) The methods used are different from those which suffice for the finite dimensional situation. In the latter case, considerable use is made of the local compactness of Euclidean

space, the theorem of the mean, etc. Since Montel spaces do not enjoy such properties, we must replace them by such results as th. (1.2.5.6), th.(1.2.7.3), and th.(1.2.7.4).

2.3 Differentiability and continuity

We now give an example of a function which has a derivative but which is not continuous. The basic idea is that the definition of derivative requires that a function does not distort Bdd sets too much. What we do is to take a function of a space into itself which maps Bdd sets into Bdd sets and then take two distinct topologies which give the same Bdd sets.

2.3.1 Example

Let E be the Banach space of continuous real valued functions on $[0,1]$ equipped with the sup topology. This is well known to be a Banach space ([11]). Let $\tilde{E}$ be the same set equipped with the weak topology. Let $f:\tilde{E} \longrightarrow E$ be defined by

$$f(x)(t) = x^2(t) \qquad (x(t) \in \tilde{E}) \ . \tag{1}$$

This function is not continuous at $x = 0$ for if S_1 is the unit sphere in E, that is

$$S_1 = \{x \in E \ / \ |x(t)| \leq 1 \quad \text{for all} \quad t \in [0,1]\}, \tag{2}$$

then $f^{-1}(S_1) = S_1$. But S_1 is not a Nbd of 0 in $\tilde{E}$ (see 1.2.6.2). On the other hand, the Bdd sets of E, $\tilde{E}$ are the same ([1]), i.e. the spheres and the set of squares of elements of a sphere is again contained in a sphere. Thus, if a Bdd set (i.e. a sphere) B in $\tilde{E}$ and a Nbd of 0 (again a sphere) S in E are given, we have that $x \in B$ implies

$$f(\lambda x) = \lambda^2 f(x) \in \lambda^2 r S \qquad (3)$$

where $r > 0$ is a number depending only on the radii of B,S. Hence, if $|\lambda| < \frac{1}{r}$ and $x \in B$,

$$\frac{1}{\lambda} f(\lambda x) \in S \quad . \qquad (4)$$

This means $f'(0) = 0$. Thus the desired example has been exhibited.

2.3.2

The next question that arises is what can be said about continuity from differentiability. The next two results are direct applications of th. (1.2.4.4), th. (1.2.7.3) and th. (1.2.5.6).

2.3.3 Theorem

Let E,F be LCEVTs; E_o a convex open subset of E. Let $f: E_o \longrightarrow F$ such that f' exists and is continuous and $f'(E_o)$ is an equicontinuous subset of $\mathcal{L}(E,F)$. Then f is uniformly continuous on E_o.

pf. Let V be a closed, convex Nbd of 0 in F. We must find a Nbd of 0,U, in E such that if $x_1, x_2 \in E_o$, $x_1 - x_2 \in U$, then

$$f(x_1) - f(x_2) \in V \quad . \qquad (5)$$

Since $f'(E_o)$ is equicontinuous, there is a Nbd U of 0 in E such that if $x \in E_o$,

$$f'(x)(U) \subset V \quad . \qquad (6)$$

So let $x_1, x_2 \in E_o$ and $x_1 - x_2 \in U$. From th.(1.2.5.6) we have

$$f(x_1) - f(x_2) = \int_0^1 f'(x_2 + t(x_1 - x_2))(x_2 - x_1)\, dt \quad . \qquad (7)$$

Applying (6) and th.(1.2.4.4), we have (5).

∎

2.3.4 Corollary

Let $f:E_o \longrightarrow F$ such that f' exists and is continuous and $f'(E_o)$ is a simply Bdd subset of $\mathcal{L}(E,F)$. Let E be tonnelé.

Then f is uniformly continuous on E_o.

pf. Apply th.(1.2.7.3) and th.(2.3.3). ∎

2.4 The chain rule

The proof of the chain rule in the finite dimensional case uses either a mean value theorem as in ([12]) or a Lipschitz condition as in ([13]). Neither of these techniques are available to us. We replace them by using th.(1.2.5.6) and th.(1.2.7.4). Otherwise the proof of the following theorem is essentially the same as the proof of the chain rule found in [13]. This result is also valid in normed vector spaces ([5]).

2.4.1 Theorem

Let E,F be Montel spaces. Let G be a LCEVT. Let E_o, F_o be convex open subsets of E,F respectively. Let $f:F_o \longrightarrow G$, $g:E_o \longrightarrow F$, such that g is continuous g' exists, f' is continuous and $g(E_o) \subset F_o$.

Then $(f\cdot g)'$ exists and

$$(f\cdot g)'(x) = [f'(g(x))] \cdot g'(x) \quad . \tag{8}$$

pf. Let $x \in E_o$, B a convex, equilibrant Bdd subset of E, $h \in B$ and $\lambda_o > 0$ such that $x + \lambda_o B \subset E_o$. Let $|\lambda| \leq \lambda_o$, and set

$$\Delta = \frac{f\cdot g(x+\lambda h) - f\cdot g(x)}{\lambda} - f'(g(x))(g'(x)(h)) \quad . \tag{9}$$

Now take $\lambda_1 \geq g(x) + \lambda_1 g'(x)(B) \subset F_o$. Then if $|\lambda| \leq \lambda_o, \lambda_1$ we have

$$\Delta = \frac{f(g(x+\lambda h)-f(g(x)+\lambda g'(x)(h)}{\lambda} + \frac{f(g(x)+\lambda g'(x)(h))-f(g(x))}{\lambda} -$$

$$- f'(g(x))(g'(x)(h)) \quad . \tag{10}$$

Applying th.(1.2.5.6) to the first term and using the fact that F_o is convex and $g(x+\lambda h) \in g(x+\lambda_o B) \subset g(E_o) \subset F_o$, we obtain

$$\Delta = \int_0^1 f'\Big(g(x) + \lambda g'(x)(h) + t[g(x+\lambda h)-g(x) - \lambda g'(x)(h)]\Big) \cdot$$

$$\left(\frac{g(x+\lambda h)-g(x)}{\lambda} - g'(x)(h)\right) \, dt +$$

$$+ \frac{f(g(x) + \lambda g'(x)(h)) - f(g(x))}{\lambda} - f'(g(x))(g'(x)(h)) \quad . \tag{11}$$

Let V be a convex, closed, Nbd of 0 in G. Consider the set

$$B_1 = \Big\{ g(x) + \lambda g'(x)(h) + t[g(x+\lambda h)-g(x) - \lambda g'(x)(h)] \Big/ 0 < \lambda \leq \lambda_o, \lambda_1$$

$$t \in [0,1], \; h \in B \Big\} \; .$$

This is a Bdd subset of F because g is continuous, so from th.(1.2.7.4), it follows that g(B) is Bdd; similarly, each of the other terms are Bdd, so their sum with coefficients in $[-1,1]$ is Bdd.

Since f' is continuous, again by th.(1.2.7.4), $f'(B_1)$ is a uniformly Bdd subset of $\mathcal{L}(F,G)$, so by th.(1.2.7.3) $f'(B_1)$ is equi-continuous. Therefore, there exists a Nbd of 0, U, in F such that

$$f'(B_1) \subset \langle U, \frac{1}{2} V \rangle \quad . \tag{12}$$

From the existence of g' we have $\lambda_2 > 0$ such that if $0 < |\lambda| \leq \lambda_2$ and $h \in B$, then

$$\frac{g(x+\lambda h)-g(x)}{\lambda} - g'(x)(h) \in U \quad . \tag{13}$$

Let $B_2 = g'(x)(B)$. Since $g'(x)$ is linear and continuous, B_2 is a Bdd subset of F. Thus from the existence of f', there is a $\lambda_3 > 0$ such that if $0 < |\lambda| \leq \lambda_3$ and $k \in B_2$, then

$$\frac{f(g(x)+\lambda k)-f(g(x))}{\lambda} - f'(g(x))(k) \in \frac{1}{2} V \quad . \tag{14}$$

Now if $h \in B$, then $g'(x)(h) \in B_2$, so if $0 < |\lambda| \leq \lambda_3$, then

$$\frac{f(g(x)+\lambda g'(x)(h))-f(g(x))}{\lambda} - f'(g(x))(g'(x)(h)) \in \frac{1}{2} V \quad . \tag{15}$$

Let $\lambda_4 = \min(\lambda_0,\lambda_1,\lambda_2,\lambda_3)$. Thus if $0 < |\lambda| \leq \lambda_4$ and $h \in B$, it follows from (11), (12), (13), (15) and th.(1.2.4.2) that

$$\Delta \in \frac{1}{2} V + \frac{1}{2} V \subset V \quad .$$

∎

This theorem is also valid if the spaces in question are normed ([5]).

2.5 Partial derivatives

We now wish to introduce the concept of partial differentiation. The expected connection between partial and total differentiation is obtained. We shall indicate how the assumption of a Montel space is interchangeable with an assumption of uniform continuity. Otherwise the method differs from the classical essentially as in th.(2.3.3). These results will be used throughout the remainder of the paper.

2.5.1

Let E,F,G be LCLVTs, and let E_0, F_0 be convex open subsets of E,F respectively. Let $f: E_0 \times F_0 \longrightarrow G$. We define the partial derivative of f w.r.t. E at $x,y \in E_0 \times F_0$ to be a function $\frac{\partial f}{\partial E}: E_0 \times F_0 \to \mathcal{L}(E,G)$ satisfying

$$\lim_{\lambda \to 0} \frac{f(x+\lambda h, y) - f(x,y)}{\lambda} = \frac{\partial f}{\partial E}(x,y)(h) \tag{16}$$

where this limit is to converge uniformly w.r.t. h in any Bdd subset of E. We define $\frac{\partial f}{\partial F}$ in a similar manner.

2.5.2 Theorem

If f' exists, then $\frac{\partial f}{\partial E}$, $\frac{\partial f}{\partial F}$ exist and

$$\frac{\partial f}{\partial E}(x,y)(h) = f'(x,y)(h,0)$$
$$\frac{\partial f}{\partial F}(x,y)(k) = f'(x,y)(0,k) \quad . \tag{17}$$

pf. If B is a Bdd subset of E, then $B \times \{0\} = \{(h,0) \in E \times F / h \in B\}$ is Bdd. (16) then follows from the existence of f'. ∎

2.5.3 Theorem

If $\frac{\partial f}{\partial E}, \frac{\partial f}{\partial F}$ exist and either one of them is continuous, uniformly on each Bdd subset of $E_o \times F_o$, then f' exists and

$$f'(x,y)(h,k) = \frac{\partial f}{\partial E}(x,y)(h) + \frac{\partial f}{\partial F}(x,y)(k) \quad . \tag{18}$$

pf. Let us suppose that $\frac{\partial f}{\partial E}$ is the one which is continuous. Let $x,y \in E_o \times F_o$, $B_1 \times B_2$ a convex, equilibrant, Bdd subset of $E \times F$ and $\lambda_o > 0$ such that $(x,y) + \lambda_o(B_1 \times B_2) \subset E_o \times F_o$. Let $(h,k) \in B_1 \times B_2$, $0 < |\lambda| \leq \lambda_o$ and set

$$\Delta = \frac{f(x+\lambda h, y+\lambda k) - f(x,y)}{\lambda} - \frac{\partial f}{\partial E}(x,y)(h) - \frac{\partial f}{\partial F}(x,y)(k). \tag{19}$$

Then

$$\Delta = \frac{f(x+\lambda h,y)-f(x,y)}{\lambda} - \frac{\partial f}{\partial E}(x,y)(h) + \frac{f(x,y+\lambda k)-f(x,y)}{\lambda} -$$

$$- \frac{\partial f}{\partial F}(x,y)(k) + \frac{f(x+\lambda h,y+\lambda k)-f(x,y+\lambda k)}{\lambda} - \frac{f(x+\lambda h,y)-f(x,y)}{\lambda} . \tag{20}$$

Since B_1 is equilibrant, $t\lambda h \in B_1$ for $|t| \leq 1$, $|\lambda| \leq \lambda_o$.

Applying th.(1.2.5.6) to the last two terms yields,

$$\Delta = \frac{f(x+\lambda h,y)-f(x,y)}{\lambda} - \frac{\partial f}{\partial E}(x,y)(h) + \frac{f(x,y+\lambda k)-f(x,y)}{\lambda} - \frac{\partial f}{\partial F}(x,y)(k) +$$

$$+ \int_0^1 \left\{ \frac{\partial f}{\partial E}(x+t\lambda h,\ y+\lambda k)(h) - \frac{\partial f}{\partial E}(x+t\lambda h,\ y)(h) \right\} dt . \tag{21}$$

Let V be a convex, closed, Nbd of 0 in G. From the existence of $\frac{\partial f}{\partial E}$, there exists $\lambda_1 > 0$ such that if $0 < |\lambda| \leq \lambda_1$ and $(h,k) \in B_1 \times B_2$, then

$$\frac{f(x+\lambda h,\ y) - f(x,y)}{\lambda} - \frac{\partial f}{\partial E}(x,y)(h) \in \frac{1}{3} V \tag{22}$$

and from the existence of $\frac{\partial f}{\partial F}$, there exists $\lambda_2 > 0$ such that if $0 < |\lambda| \leq \lambda_2$ and $(h,k) \in B_1 \times B_2$, then

$$\frac{f(x,y+\lambda k) - f(x,y)}{\lambda} - \frac{\partial f}{\partial F}(x,y)(k) \in \frac{1}{3} V . \tag{23}$$

Since $\frac{\partial f}{\partial E}$ is uniformly continuous, there exists a Nbd of 0, U, in F such that if $y_1 \in y + U$ and $x_1 \in x + B_1$ and $h \in B_1$ then

$$\frac{\partial f}{\partial E}(x_1,y_1)(h) - \frac{\partial f}{\partial E}(x,y)(h) \in \frac{1}{3} V . \tag{24}$$

Now there exists $\lambda_3 > 0$ such that $0 < |\lambda| \leq \lambda_3$ implies that $\lambda B_2 \subset U$. Thus if $0 < |\lambda| \leq \lambda_3$ and $(h,k) \in B_1 \times B_2$, then $y + \lambda k \in U$ and $x + t\lambda h \in x + B_1$ for all $t \in [0,1]$. So from (24) and th.(1.2.4.4) we have

$$\int_0^1 \left\{ \frac{\partial f}{\partial E}(x+t\lambda h, y+\lambda k)(h) - \frac{\partial f}{\partial E}(x+t\lambda h, y)(h) \right\} dt \in \frac{1}{3} V \quad . \tag{25}$$

Let $\lambda_4 = \min(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$. If $0 < |\lambda| \leq \lambda_4$ and $(h,k) \in B_1 \times B_2$, it then follows from (21), (22), (23) and (25) that

$$\Delta \in \frac{1}{3} V + \frac{1}{3} V + \frac{1}{3} V \subset V \quad .$$

∎

2.5.4 Corollary

In th.(2.5.3) the assumption of uniform continuity can be replaced by continuity and the assumption that E (or F) is a Montel space.

pf. If E is a Montel space, then the closure of any Bdd set is compact so we have uniform continuity.

∎

2.5.5

The above results on partial derivatives can be extended from the case of "two variables" to any finite number. The methods are identical and the whole theory of matrices and Jacobians could be developed.

2.5.6

The following lemma is a further statement of continuity type implications of differentiability. In fact, one can choose a topology for E so that the following result says just that differentiability implies continuity. We do not use this interpretation. The result is

given because it is used. The proof is essentially the same as the classical proof of the fact that differentiability implies a Lipschitz condition.

2.5.7 Lemma

Let E,F be LCEVTs, E_o an open subset of E, $f:E_o \to F$ such that f' exists.

Then $\lim_{\lambda \to 0} [f(x + \lambda h) - f(x)] = 0$ uniformly w.r.t. h in any Bdd subset of E.

pf. Let B be a Bdd subset of E, $x \in E_o$. Since $f'(x)$ is linear and continuous, $\bar{B} = f'(x)(B)$ is a Bdd subset of F. Let V be a Nbd of 0 in F. Then there exists $\lambda_o > 0$ such that $\lambda_o \bar{B} \subset \frac{1}{2} V$. From the existence of f', we have $\lambda_1 > 0$ such that if $0 < |\lambda| \leq \lambda_1$, $h \in B$, then

$$\frac{f(x + \lambda h) - f(x)}{\lambda} - f'(x)(h) \in \frac{1}{2} V \quad . \tag{26}$$

Thus if $0 < \lambda \leq \min(\lambda_o, \lambda_1)$ and $h \in B$ then

$$f(x + \lambda h) - f(x) \in \lambda f'(x)(h) + \frac{1}{2} V \in \lambda \bar{B} + \frac{1}{2} V \subset \frac{1}{2} V + \frac{1}{2} V = V \quad . \tag{27}$$

∎

2.6 Products and reciprocals

2.6.1

Now we turn to the product rule. We let E,F,G,H be LCEVTs and consider $f \in \mathcal{L}(F,G)$ and $g \in \mathcal{L}(E,F)$. Then $f \cdot g \in \mathcal{L}(E,G)$ and we write $fg = f \cdot g$, reserving "$\cdot$" for non-linear functions. This will help to reduce ambiguity, for if H_o is a convex, open subset of H, $f:H_o \to \mathcal{L}(F,G)$ and $g:H_o \to \mathcal{L}(E,F)$ then we have $fg:H_o \to \mathcal{L}(E,G)$ defined by

$$fg(x) = f(x)\, g(x) \quad , \qquad x \in H_o \quad . \tag{28}$$

Now comes the difficulty. If f', g' exist then we will define $f'g:H_o \to \mathcal{L}(H, \mathcal{L}(E,G))$, and $fg':H_o \to \mathcal{L}(H, \mathcal{L}(E,G))$ by

$$f'g(x)(h) = [f'(x)(h)]\, g(x) \tag{29}$$

and

$$fg'(x)(h) = f(x)\,[g'(x)(h)] \tag{30}$$

where $x \in H_o$, $h \in H$.

It follows immediately from the definitions that these maps have the required linearity and continuity. The interesting thing to be noted here is that if any two of E,F,G,H are the same, then the notation becomes unmanageable. Here is a case in which the more general the situation, the simpler it is.

2.6.2 Theorem

Let E,F,G,H be LCEVTs and let E,F be Montel space. Let H_o be a convex open subset of H. Let $f:H_o \to \mathcal{L}(F,G)$, $g:H_o \to \mathcal{L}(E,F)$ such that g is continuous and f', g' exist.

Then (fg)' exists and

$$(fg)' = f'g + fg' \quad . \tag{31}$$

pf. According to (29), (30) f'g and fg' have just the right domain and range.

Let

$$\Delta = \frac{fg(x+\lambda h) - fg(x)}{\lambda} - f'g(x)(h) - fg'(x)(h) \tag{32}$$

With (29), (30) we have

$$\Delta = \frac{f(x+\lambda h)g(x)-f(x)g(x)}{\lambda} - [f'(x)(h)]g(x) + \frac{f(x)g(x+\lambda h)-f(x)g(x)}{\lambda} - \tag{33}$$

$$- f(x)[g'(x)(h)] + \frac{f(x+\lambda h)g(x+\lambda h)-f(x+\lambda h)g(x)}{\lambda} - \frac{f(x)g(x+\lambda h)-f(x)g(x)}{\lambda}.$$

Let $\langle B_E, W_G \rangle$ be a Nbd of 0 in $\mathcal{L}(E,G)$, i.e. B_E is Bdd in E and W_G is a convex Nbd of 0 in G. Let B_H be a Bdd set in H. Since g is continuous and E is a Montel space, it follows from th.(1.2.7.4) that if we set

$$\mathcal{B}_F = \left\{ g(x + \lambda h) - g(x) \;/\; h \in B_H , \quad 0 < \lambda \leq 1 \right\} , \tag{34}$$

then $\mathcal{B}_F$ is a Bdd subset of $\mathcal{L}(E,F)$. This means that there is a Bdd set B_F in F such that

$$\mathcal{B}_F \subset \langle B_E, B_F \rangle \quad . \tag{35}$$

Now f'(x) is linear and continuous on H. Hence $f'(x)(B_H)$ is a Bdd and hence by th.(1.2.7.3) an equicontinuous subset of $\mathcal{L}(F,G)$. Thus there is a Nbd of 0, U_F in F such that

$$f'(x)(B_H)(U_F) \subset \frac{1}{4} W_G \quad . \tag{36}$$

By lemma (2.5.7), there is a $\lambda_1 > 0$ such that if $0 \leq |\lambda| \leq \lambda_1$ then

$$g(x + \lambda B_H) - g(x) \subset \langle B_E, U_F \rangle \quad . \tag{37}$$

We have used here the fact that for λ sufficiently small, λB_H is contained in any preassigned Nbd of 0. Since f' exists, there is a $\lambda_2 > 0$ such that if $0 < |\lambda| \leq \lambda_2$ and $h \in B_H$, then

$$\frac{f(x + \lambda h) - f(x)}{\lambda} - f'(x)(h) \in \langle B_F, \frac{1}{4} W_G \rangle \quad . \tag{38}$$

Let Δ_1 be the last two terms in (33). Then

$$\Delta_1 = \left[\frac{f(x+\lambda h)-f(x)}{\lambda}\right]\left[g(x+\lambda h)-g(x)\right] \tag{39}$$

$$= \left[f'(x)(h)\right]\left[g(x+\lambda h)-g(x)\right] + \left[\frac{f(x+\lambda h)-f(x)}{\lambda} - f'(x)(h)\right]\left[g(x+\lambda h)-g(x)\right]. \tag{40}$$

Thus if $0 < |\lambda| \leq \min(1, \lambda_1, \lambda_2)$ and $h \in B_H$, then it follows from (36), (37), (38), (34) and (35) that

$$\Delta_1 \in \langle U_F, \tfrac{1}{4} W_G\rangle\langle B_E, U_F\rangle + \langle B_F, \tfrac{1}{4} W_G\rangle\langle B_E, B_F\rangle \tag{41}$$

$$\therefore \Delta_1 \in \langle B_E, \tfrac{1}{4} W_G\rangle + \langle B_E, \tfrac{1}{4} W_G\rangle \subset \langle B_E, \tfrac{1}{2} W_G\rangle . \tag{42}$$

Now we turn to the remaining terms in Δ. Since $g(x)$ is linear and continuous, $B'_F = g(x)(B_E)$ is a Bdd subset of F. Hence there is a $\lambda_3 > 0$ such that if $0 < |\lambda| \leq \lambda_3$ and $h \in B_H$, then

$$\frac{f(x+\lambda h) - f(x)}{\lambda} - f'(x)(h) \in \langle B'_F, \tfrac{1}{4} W_G\rangle . \tag{43}$$

Therefore,

$$\frac{f(x+\lambda h)g(x) - f(x)g(x)}{\lambda} - [f'(x)(h)]g(x) = \tag{44}$$

$$= \left\{\frac{f(x+\lambda h)-f(x)}{\lambda} - f'(x)(h)\right\}g(x) \in \langle B'_F, \tfrac{1}{4} W_G\rangle\langle B_E, B'_F\rangle \subset \langle B_E, \tfrac{1}{4} W_G\rangle .$$

Finally, since $f(x)$ is a continuous function on F, there exists a Nbd of $0, V_F$, in F such that

$$f(x)(V_F) \subset \frac{1}{4} W_G . \tag{45}$$

So we choose $\lambda_4 > 0$ such that if $0 < |\lambda| \leq \lambda_4$ and $h \in B_H$ then

$$\frac{g(x+\lambda h) - g(x)}{\lambda} - g'(x)(h) \in \langle B_E, V_F \rangle \quad . \tag{46}$$

Thus it follows from (45), (46) that

$$\frac{f(x)g(x+\lambda h)-f(x)g(x)}{\lambda} - f(x)\,[g'(x)(h) \quad = \tag{47}$$

$$= f(x)\left[\frac{g(x+\lambda h)-g(x)}{\lambda} - g'(x)(h)\right] \in \langle V_F, \tfrac{1}{4} W_G \rangle \langle B_E, V_F \rangle \subset \langle B_E, \tfrac{1}{4} W_G \rangle \quad .$$

So if $0 < |\lambda| \leq \min(1, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $h \in B_H$, then it follows from (33), (42), (44), (47) that

$$\Delta \in \langle B_E, \tfrac{1}{2} W_G \rangle + \langle B_E, \tfrac{1}{4} W_G \rangle + \langle B_E, \tfrac{1}{4} W_G \rangle \subset \langle B_E, W_G \rangle \quad . \tag{48}$$

∎

2.6.3 Corollary

Let F,G,H,f be as in th.(2.6.2) and let $g: H_o \longrightarrow F$ such that g is continuous and g' exists.

Then, the map $fg: H_o \longrightarrow G$ defined by

$$fg(x) = f(g(x))(g(x)) \tag{49}$$

is such that (fg)' exists and

$$(fg)'(x)(h) = [f'(x)(h)]g(x) + f(x)[g'(x)(h)] \quad . \tag{50}$$

pf. Take E = R in th.(2.6.2).

∎

2.6.4

The last property of the derivative which we study in this section is the derivative of the reciprocal of a function. Again we must make

a special interpretation in order that such a concept make sense. However, our understanding of reciprocal will be compatible with the previous discussion of product.

2.6.5

If $u \in \mathcal{L}(F,F)$ then u^{-I} will be an alternate symbol for the usual inverse, u^{-1}. If $f:E_o \longrightarrow \mathcal{L}(F,F)$ then we define the reciprocal, f^{-I} of f, if it exists, to be a function $f^{-I}:E_o \longrightarrow \mathcal{L}(F,F)$ defined by:

$$f^{-I}(x) = [f(x)]^{-I} . \tag{51}$$

2.6.6

The following result is comparable to the situation with matrices. Suppose that A(x) is an n x n matrix-valued function of a real variable with elements $(a_{ij}(x))$. Let A'(x) be the n x n matrix-valued function of a real variable with elements $(a'_{ij}(x))$. Suppose further that A(x) has an inverse, $A^{-1}(x)$ for each x in some interval. Then it is easy to show by direct computation that

$$A^{-1'}(x) = - A^{-1}(x)\, A'(x)\, A^{-1}(x) . \tag{52}$$

Th.(2.6.7) says exactly the same thing for a more general situation. The proof uses the same techniques which have been introduced in this section.

2.6.7 Theorem

Let E be a Montel space and let F be tonnelé. Let E_o be a convex, open subset of E. Let $f:E_o \longrightarrow \mathcal{L}(F,F)$ such that f' exists and f^{-I} exists and is continuous. Then $(f^{-I})'$ exists and

$$(f^{-I})' = -f^{-I}\, f' f^{-I} .^{+} \tag{53}$$

+This symbol is defined as in (29) with $f^{-I} = g$.

pf. Let B_E be a Bdd subset of E and let $\langle B_F, V\rangle$ be a convex, equilibrant Nbd of 0 in $\mathcal{L}(F,F)$. Let $x \in E_o$ and let $\lambda_o > 0$ be such that $x + \lambda B_E \subset E_o$. Let $h \in B_E$, $\lambda \in [-\lambda_o, \lambda_o]$. Set

$$\Delta = \frac{f^{-I}(x+\lambda h) - f^{-I}(x)}{\lambda} + [f^{-I}(x)][f'(x)(h)][f^{-I}(x)] \quad . \tag{54}$$

Then

$$\Delta = [f^{-I}(x)]\left[f'(x)(h) - \frac{f(x+\lambda h) - f(x)}{\lambda}\right][f^{-I}(x)] -$$

$$- [f^{-I}(x+\lambda h) - f^{-I}(x)][f'(x)(h)][f^{-I}(x)] - \tag{55}$$

$$- [f^{-I}(x+\lambda h) - f^{-I}(x)]\left[\frac{f(x+\lambda h) - f(x)}{\lambda} - f'(x)(h)\right][f^{-I}(x)].$$

Now, $C_F = f^{-I}(x)(B_F)$ is a Bdd subset of F and thus $D_F = f'(x)(B_E)(C_F)$ is a Bdd subset of F. Since f^{-I} is continuous, there is a Nbd of 0 U in E such that if $y \in U$, then

$$f^{-I}(x+y) - f^{-I}(x) \in \langle D_F, \tfrac{1}{3} V\rangle \quad . \tag{56}$$

Also, since B_E is Bdd there is a $\lambda_1 > 0$ such that

$$\lambda_1 B_E \subset U \quad . \tag{57}$$

So it follows from (56), (57) and the definition of C_F, D_F that if $0 < |\lambda| \leq \lambda_1$ and $h \in B_E$, then

$$[f^{-I}(x+\lambda h) - f^{-I}(x)][f'(x)(h)][f^{-I}(x)] \in$$

$$\in \langle D_F, \tfrac{1}{3} V\rangle\langle C_F, D_F\rangle\langle B_F, C_F\rangle \subset \langle B_F, \tfrac{1}{3} V\rangle \quad . \tag{58}$$

Since E is a Montel space and f^{-I} is continuous, then $\{f^{-I}(x + \lambda h) - f^{-I}(x)/h \in S\}$ is, by th.(1.2.7.4) a Bdd set. Since F is tonnelé, by th.(1.2.7.3) it is then an equicontinuous subset of $\mathcal{L}(F,F)$. Therefore, there exists a Nbd of 0 W in F such that if $h \in B_E$, $0 < |\lambda| \leq 1$, $u \in W$, then

$$f^{-I}(x + \lambda h)(u) - f^{-I}(x)(u) \in \frac{1}{3} V \quad . \tag{59}$$

From the existence of f', we have $\lambda_2 \in (0,1)$ such that if $0 < |\lambda| \leq \lambda_2$ and $h \in B_E$, then

$$\frac{f(x + \lambda h) - f(x)}{\lambda} - f'(x)(h) \in \langle C_F, W \rangle \quad . \tag{60}$$

It then follows from (59), (60) that if $0 < |\lambda| \leq \lambda_2$ and $h \in B_E$, then

$$[f^{-I}(x + \lambda h) - f^{-I}(x)] \left[\frac{f(x + \lambda h) - f(x)}{\lambda} - f'(x)(h)\right] [f^{-I}(x)] \in$$

$$\in \langle W, \frac{1}{3} V \rangle \langle C_F, W \rangle \langle B_F, C_F \rangle \subset \langle B_F, \frac{1}{3} V \rangle \quad . \tag{61}$$

Finally, there is a Nbd of 0 U in F such that

$$f^{-I}(x)(U) \subset \frac{1}{3} V \tag{62}$$

and there is a $\lambda_3 > 0$ such that if $0 < |\lambda| \leq \lambda_3$ and $h \in B_E$, then

$$\frac{f(x + \lambda h) - f(x)}{\lambda} - f'(x)(h) \in \langle C_F, U \rangle \quad . \tag{63}$$

It then follows from (62), (63) that if $0 < |\lambda| \leq \lambda_3$ and $h \in B_E$, then

$$[f^{-I}(x)]\left[\frac{f(x+\lambda h)-f(x)}{\lambda}-f'(x)(h)\right][f^{-I}(x)] \in$$

$$\in \langle U, \tfrac{1}{3}V\rangle\langle C_F, U\rangle\langle B_F, C_F\rangle \subset \langle B_F, \tfrac{1}{3}V\rangle \quad . \tag{64}$$

Thus, from (55), (58), (61), (64) it follows that if $0 < |\lambda| \leq \min(1, \lambda_0, \lambda_1, \lambda_2, \lambda_3)$ and $h \in B_E$, then

$$\Delta \in \langle B_F, \tfrac{1}{3}V\rangle + \langle B_F, \tfrac{1}{3}V\rangle + \langle B_F, \tfrac{1}{3}V\rangle \subset \langle B_F, V\rangle \quad . \tag{65}$$

∎

3. The Problem of Pfaff

3.1 Statement of problem

The problem which we shall refer to as Pfaff's problem is that of finding a function once its derivative is given. Specifically, if E,F are LCEVTs, E_o a convex, open subset of E, and T a continuous function $T:E_o \longrightarrow \mathcal{L}(E,F)$, under what conditions does there exist a function $f:E_o \longrightarrow F$ such that $f' = T$?

In the case in which $E = F = R$, the problem is completely solved by the fundamental theorem of calculus. In the case of E,F two finite dimensional vector spaces, additional requirements, i.e. that T' is symmetric, are required. Specifically, if $E = R^2 = R \times R$, $F = R$, then we may write

$$T(x) = \begin{pmatrix} P(x) \\ Q(x) \end{pmatrix}' \tag{1}$$

where $x \in E_o$, $P,Q:E_o \longrightarrow R$ and the relation between linear transformations and matrices as described in [14]. Then the condition for the existence of f is that P,Q have continuous first partials and

$$\frac{\partial P}{\partial x^2} = \frac{\partial Q}{\partial x^1} \quad .^{+} \tag{2}$$

This solution of the problem has been extended to Banach spaces by Kerner [15]. In this section we extend the solution to obtain the same result for Montel spaces.

+ Superscripts will denote co-ordinates.

3.2 Known solutions

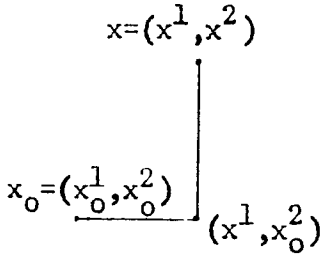
Kerner's solution is different from the one usually given for the finite dimensional case (see for example, [12]) where an easier procedure is possible. We shall compare these approaches and then describe ours which is a synthesis of the two. For simplicity, we consider, in the finite dimensional case, only the case $E = R^2$, $F = R$.

3.2.1

If $E = R^2$, $F = R$, then we choose a point $x_0 \in E_0$ and define f by

$$f(x) = \int_{x_0^1}^{x^1} P(t, x_0^2)\, dt + \int_{x_0^2}^{x^2} Q(x^1, t)\, dt \quad . \tag{3}$$

In other words, we integrate the derivative, first along one co-ordinate axis and then along the other (see Fig. 1). It follows from (2), (3) by direct computation that $f' = T$.



(Fig. 1)

3.2.2

The immediate difficulty in generalizing the classical approach is that it involves co-ordinates. Kerner made the natural replacement by integrating along the line from x_0 to x. Specifically, he defines f by

$$f(x_0) = \int_0^1 T(x_0 + t(x-x_0))(x-x_0)\, dt \quad . \tag{4}$$

This time the proof is more complicated. A considerable amount of analytic machinery is employed, including a Fubini theorem. Also the Banach space norm is strongly used.

3.2.3

In our method we define f exactly as did Kerner, but retain the co-ordinate approach in proving that f' = T. The idea is that if x_0, x lie on a line parallel to one of the co-ordinate axes, then (3), (4) are identical and both proofs are trivial. But this situation can always be obtained merely by changing co-ordinates and keeping in mind that derivatives are defined intrinsically, without mention of co-ordinates. In fact, we find that a co-ordinate system is unnecessary. All we need do is write E as a direct sum of two subspaces; a one-dimensional subspace (the line through x_0, x) and a topological supplement (see 1.2.2.4). Then (4) can be written in a form analogous to (3) and the classical computation works. The only remaining difficulty is that now a different symmetry condition is required. In lemma 3, this is proved to be equivalent to the usual one. Lemmas 1, 2 are simple results of calculus along the line of Section 2. Theorem 1 gives the final result.

3.3 Solution of proble.

The first lemma shows that we can differentiate under the integral sign.

3.3.1 Lemma

Let E be a Montel space, F a tonnelé space and E_0 a convex open subset of E. Let I = [0,1]. Let $g: I \times E_0 \longrightarrow F$ be such that g' exists and is continuous on each Bdd subset of $I \times E_0$, and g is continuous on I. Let $f: E_0 \longrightarrow F$ be defined by

$$f(x) = \int_0^1 g(t,x)\, dt \quad . \tag{5}$$

Then f' exists and

$$f'(x)(h) = \int_0^1 \frac{\partial g}{\partial E}\ (t,x)(h)\ dt\ . \tag{6}$$

pf. The continuity of g' implies the continuity of $\frac{\partial g}{\partial E}$ and thus the integral in (6) exists. Let $x \in E_0$, let B be a closed, Bdd, equilibrant subset of E_0, and let V be a closed, convex Nbd of 0 in F. Since $\frac{\partial g}{\partial E}$ is continuous on $I \times \{x + B\}$ and E is a Montel space, $\frac{\partial g}{\partial E}$ is uniformly continuous on $I \times \{x + B\}$, because this set is compact. Therefore, there exists a Nbd of 0, U, in E such that if $t \in I$, $h \in B$, $x_1, x_2 \in x + B$, $x_1 - x_2 \in U$ then

$$\frac{\partial g}{\partial E}\ (t,x_1)(h) - \frac{\partial g}{\partial E}\ (t,x_2)(h) \in V\ . \tag{7}$$

Let $\lambda_0 > 0$ be such that $\lambda_0 B \subset U$. Then if $0 < |\lambda| \leq \lambda_0$ and $h \in B$, it follows from (5), (7), th.(1.2.4.2), th.(1.2.4.4) and the fact that B is equilibrant that

$$\begin{aligned}
&\frac{f(x + \lambda h) - f(x)}{\lambda} - \int_0^1 \frac{\partial g}{\partial E}\ (t,x)(h)\ dt\ = \\
&= \int_0^1 \frac{g(t,x + \lambda h) - g(t,x)}{\lambda}\ dt - \int_0^1 \int_0^1 \frac{\partial g}{\partial E}\ (t,x)(h)\ d\sigma\ dt\ = \\
&= \int_0^1 \int_0^1 \left[\frac{\partial g}{\partial E}\ (t,x + \sigma\lambda h)(h) - \frac{\partial g}{\partial E}\ (t,x)(h)\right] d\sigma\ dt \in V\ .
\end{aligned} \tag{8}$$

∎

3.3.2

The next result is a counterpart of the "integration by parts" formula. It follows immediately from a special case of the product rule.

3.3.3 Lemma

Let E,F be Montel spaces, and E_o a convex open subset of E. Let $I = [0,1]$. Let $f:E_o \longrightarrow F$, $\alpha:I \longrightarrow R$ such that α',f' exist and are continuous.

Then for each x_o, $x \in E_o$,

$$\alpha(1)\, f(x) - \alpha(0)\, f(x_o) = \int_0^1 \alpha'(t)\, f(x_o + t(x-x_o))\, dt + \int_0^1 \alpha(t)\, f'(x_o + t(x-x_o))(x-x_o)\, dt \quad (9)$$

pf. Let x_o, $x \in E_o$. Let $\beta:I \longrightarrow F$ be defined by

$$\beta(t) = \alpha(t)\, f(x_o + t(x-x_o)) \quad (10)$$

$x_o + t(x-x_o) \in E_o$ because E_o is convex.

By th.(2.4.1) and corollary (2.6.3), β' exists and $\beta':I \longrightarrow F$ and

$$\beta'(t) = \alpha'(t)\, f(x_o + t(x-x_o)) + \alpha(t)\, f'(x_o + t(x-x_o))(x-x_o) . \quad (11)$$

It follows from (11), lemma (2.5.7), and the existence and continuity of α',f' that β' is continuous. Therefore, by th.(1.2.4.2),

$$\alpha(1)f(x) - \alpha(0)f(x_o) = \beta(1) - \beta(0) = \int_0^1 \beta'(t)\, dt = \int_0^1 \left\{\alpha'(t)f(x_o + t(x-x_o)) + \alpha(t)f'(x_o + t(x-x_o))(x-x_o)\right\} dt . \quad (12)$$

∎

3.3.4

In the next lemma we prove that the symmetry condition which will be used in the main theorem is implied by the usual symmetry condition.

The proof is surprisingly simple. The situation is, in a sense algebraic rather than analytic. That is, once the definitions are well stated, the proofs are trivial.

3.3.5

Before going into details, let us try to understand what these symmetry conditions mean. We start with a function $T:E_o \longrightarrow \mathcal{L}(E,F)$ which we want to show is the derivative of a function $f:E_o \longrightarrow F$. The usual symmetry condition merely demands that

$$T'(x)(h)(k) = T'(x)(k)(h) \qquad (13)$$

for all $x \in E_o$, $h,k \in E$. It should be recalled that $T':E_o \longrightarrow \mathcal{L}(E,\mathcal{L}(E,F))$. Our symmetry condition can be explained as follows. We restrict $T(x)$ to operate on a single vector h, take the derivative and restrict the result to a topological supplement E_h of the space generated by h. Then we repeat the process, reversing the roles of h and E_h. We demand that the two results are the same.

There are two differences between the conditions. In the first place, one differentiates first and then applies T and the other mixes them up. This difference is superficial and is only a convenience. The second is that one applies to two vectors at a time whereas the other applies to a vector and a subspace. This difference is essential and our theorem shows that the use of a co-ordinate system is only apparent in the classical solution of Pfaff's problem.

3.3.6

Now we describe our symmetry condition precisely and prove the desired result. If $h \in E$ we define $T_h:E_o \longrightarrow F$ by

$$T_h(x) = T(x)(h) \quad . \tag{14}$$

If E_h is a topological supplement of the subspace generated by h, then we define $T_{E_h}: E_o \longrightarrow \mathcal{L}(E_h, F)$ by

$$T_{E_h}(x) = T(x)/_{E_h} \quad . \tag{15}$$

It is easy to see that the existence of T' implies the existence of T'_h, T'_{E_h}. Our symmetry condition is

$$(T'_h)_{E_h} = (T'_{E_h})_h \tag{16}$$

for all $h \in E$. Now we prove our result.

3.3.7 Lemma

Let E,F be LCEVTs, E_o a convex open subset of E. Let $T: E_o \longrightarrow \mathcal{L}(E,F)$ such that T' exists.

Then

$$(15) \Longrightarrow (16) \quad .$$

pf. We assume (15) and show (16). Let us first prove that the two allegedly equal functions have the same domain and range. We have

$$T_h: E_o \longrightarrow F \qquad \text{and} \qquad T_{E_h}: E_o \longrightarrow \mathcal{L}(E_h, F) \quad . \tag{17}$$

Hence, $T'_h: E_o \longrightarrow \mathcal{L}(E,F)$ and $T'_{E_h}: E_o \longrightarrow \mathcal{L}(E, \mathcal{L}(E_h,F))$, (18)

so $(T'_h)_{E_h}: E_o \longrightarrow \mathcal{L}(E_h,F)$ and $(T'_{E_h})_h: E_o \longrightarrow \mathcal{L}(E_h,f)$ (19)

as desired.

Now let $x \in E_o$, $h \in E$, $k \in E_h$. Then

$$(T'_h)_{E_h}(x)(k) = T'_h(x)(k) = \lim_{\lambda \to 0} \frac{T_h(x + \lambda k) - T_h(x)}{\lambda} =$$

$$= \lim_{\lambda \to 0} \frac{T(x + \lambda k)(h) - T(x)(h)}{\lambda} = T'(x)(k)(h) . \quad (20)$$

On the other hand,

$$(T'_{E_h})_h(x)(k) = T'_{E_h}(x)(h)(k) = \lim_{\lambda \to 0} \frac{T_{E_h}(x + \lambda h)(k) - T_{E_h}(x)(k)}{\lambda} =$$

$$= \lim_{\lambda \to 0} \frac{T(x + \lambda h)(k) - T(x)(k)}{\lambda} = T'(x)(h)(k) . \quad (21)$$

The result then follows immediately from (13), (20), (21). ∎

The main result of this section may now be proved.

3.3.8 Theorem

Let E be a Montel space, F tonnele, E_o a convex open subset of E, $x_o \in E_o$. Let $T: E_o \longrightarrow \mathcal{L}(E,F)$ such that T' exists and is continuous.

Then there exists a function $f: E_o \longrightarrow F$ such that f' exists and

$$f' = T \quad (22)$$

if and only if for all $h, k \in E$, $x \in E_o$ we have

$$T'(x)(h)(k) = T'(x)(k)(h) . \quad (23)$$

pf. We define a function $g: I \times E_o \longrightarrow F$ by

$$g(t,x) = T(x_o + t(x-x_o))(x-x_o) . \quad (24)$$

To analyze g, we must define two other functions, $u: I \times E_o \longrightarrow E_o$ and $v: I \times E_o \longrightarrow E$ by

$$u(t,x) = x_o + t(x-x_o) \tag{25}$$

$$v(t,x) = x - x_o \ . \tag{26}$$

Note that the range of u is E_o because E_o is convex. It then follows that

$$g = (T \cdot u)v \ . \tag{27}$$

From the obvious facts that u is continuous, u' exists with

$$u'(t,x)(r,h) = t\,h + r(x-x_o) \ , \tag{28}$$

the assumption that T' exists and th.(2.4.1), it follows that $T \cdot u$ exists and

$$(T \cdot u)'(t,x)(r,h) = T'(u(t,x))(t\,h + r(x-x_o)) \ . \tag{29}$$

Now we note that v is continuous and v' exists with

$$v'(t,x)(r,h) = h \ . \tag{30}$$

Thus by corollary (2.6.3) we have g' exists and,

$$g'(t,x)(r,h) = [(T \cdot u)'(t,x)(r,h)]v(t,x) + (T \cdot u)(t,x)[v'(t,x)(r,h)] \ . \tag{31}$$

Applying (25), (26), (29), (30), we obtain

$$g'(t,x)(r,h) = T'(x_o + t(x-x_o)(th + r(x-x_o))(x-x_o) + T(x_o + t(x-x_o))(h) \ . \tag{32}$$

Thus we conclude that g' is continuous so that if we define $f: E_o \longrightarrow F$ by

$$f(x) = \int_0^1 g(t,x)\,dt \ , \tag{33}$$

then by lemma (3.3.1), f' exists and

$$f'(x)(h) = \int_0^1 \frac{\partial g}{\partial E}(t,x)(h)\, dt \quad . \tag{34}$$

Now let $k \in E$ and E_k be a topological supplement of k. Then we may define continuous linear functions $\alpha: E \longrightarrow R$, $\beta: E \longrightarrow E_k$ by the relation

$$x = \alpha(x)k + \beta(x) \qquad \text{for all } x \in E \ . \tag{35}$$

Thus we may write

$$T(x_o + t(x-x_o))(x-x_o) = \tag{36}$$

$$= \alpha(x-x_o)\, T_k(x_o + t(x-x_o)) + T_{E_k}(x_o + t(x-x_o))(\beta(x-x_o)) \ .$$

To the second term of (36) we apply the same argument as applied to g and to the first term we apply corollary (2.6.3) and th.(2.4.1) to obtain

$$\frac{\partial g}{\partial E}(t,x)(k) = g'(t,x)(0,k) = \alpha(k)\, T_k(x_o + t(x-x_o)) +$$

$$+ \alpha(x-x_o)\, T_k'(x_o + t(x-x_o))(t\,k) + \tag{37}$$

$$+ T_{E_k}'(x_o + t(x-x_o))(t\,k)(\beta(x-x_o)) + T_{E_k}(x_o + t(x-x_o))(\beta(k)) \ .$$

But $\alpha(k) = 1$ and $\beta(k) = 0$, so

$$\frac{\partial g}{\partial E}(t,x)(k) = t\left\{T_k'(x_o + t(x-x_o))(\alpha(x-x_o)k) + (T_{E_k}')_k(x_o + t(x-x_o))(\beta(x-x_o))\right\} +$$

$$+ T_k(x_o + t(x-x_o)) \ . \tag{38}$$

Now from (23) and lemma (3.3.7) we have

$$\frac{\partial g}{\partial E}(t,x)(k) = t\left\{T_k'(x_o+t(x-x_o))(\alpha(x-x_o)k) + (T_k')_{E_k}(x_o+t(x-x_o))(\beta(x-x_o))\right\}+$$

$$+ \; T_k(x_o + t(x-x_o)) \tag{39}$$

$$= \; t\, T_k'(x_o + t(x-x_o))(x-x_o) + T_k(x_o + t(x-x_o)) \quad .$$

Thus from (34), (39) we have

$$f'(x)(k) \; = \int_0^1 t\, T_k'(x_o + t(x-x_o))(x-x_o)dt + \int_0^1 T_k(x_o + t(x-x_o))\, dt \; . \tag{40}$$

Finally, from lemma (3.3.3)

$$f'(x)(k) = T_k(x) - \int_0^1 T_k(x_o + t(x-x_o))dt + \int_0^1 T_k(x_o + t(x-x_o))\, dt$$

$$= \; T(x)(k) \quad . \tag{41}$$

On the other hand, let us suppose f exists and satisfies (22). Let $h,k \in E$, $x \in E_o$. Let $u \in F'$, the dual of F. We define a function $g:R^2 \longrightarrow R$ by

$$g(s,t) \; = \; u[f(x + sh + tk)] \quad . \tag{42}$$

Then g is twice continuously differentiable and we have

$$\frac{\partial g}{\partial s}(0,t) \; = \; u[f'(x + tk)(h)] = u[T(x + tk)(h)] \tag{43}$$

and

$$\frac{\partial^2 g}{\partial s \partial t}(0,0) = u[T'(x)(h)(k)] \; . \tag{44}$$

Similarly,

$$\frac{\partial^2 g}{\partial t \partial s}(0,0) = u[T'(x)(k)(h)] \tag{45}$$

and from the classical result,

$$\frac{\partial^2 g}{\partial s \partial t} = \frac{\partial^2 g}{\partial t \partial s} \tag{46}$$

whence

$$u[T'(x)(h)(k)] = u[T'(x)(k)(h)] \tag{47}$$

and since this is true for all $u \in F'$, by sec.(1.2.2.9) we have

$$T'(x)(h)(k) = T'(x)(k)(h) \quad . \tag{48}$$

∎

4. Existence And Uniqueness Theorems For Differential Equations In Montel Spaces

4.1 Introduction

In this section we consider the following question:

Let E_1 be an open set in E, E_o an open subset of E_1

Let $T: E_1 \times F_o \longrightarrow \mathcal{L}(E,F)$, $(x_o,y_o) \in E_o \times F_o$. What are the minimum conditions under which there exists a function $f: E_o \longrightarrow F_o$ such that

$$f(x_o) = y_o \tag{1}$$

$$f'(x) = T(x,f(x)) \tag{2}$$

Furthermore, if such an f exists, is it unique? In case E=F=R the well known Cauchy-Peano Theorem (as given for example in [16]) shows that continuity of T is sufficient and if in addition T satisfies a Lipschitz condition, the solution is unique. This result can be immediately generalized to the case in which E=R, $F=R^n$ for some integer n. If however, we take $E=R^n$ then difficulties arise immediately. If (2) is replaced by the following integral equation

$$f(x) = y_o + \int_0^1 T(x_o + t(x-x_o),\ f(x_o + t(x-x_o))dt \tag{3}$$

then the problem can be solved. However (2) may not be satisfied since T(x,f(x)) may not be the derivative of any function. Thus, some kind of symmetry condition as in Section 3 must be added. Hence one is led to assume that T' exists and is symmetric in a sense to be specified below.

4.2 Banach space case

4.2.1

If we take the case E=R, F A Banach space, then continuity of T is not enough. This is shown by an example due to Dieudonne ([17]). If

we put a Lipschitz condition on T then the problem can be solved, in fact with the existence of T' and a certain symmetry condition, Michal and Elconin [18] prove the existence and uniqueness of f in a manner similar to the Picard Existence Theorem ([16]). This result is valid when E, F are any two Banach spaces.

4.3 The symmetry condition

4.3.1

Thus one cannot expect to prove the existence of f when E=R and F is a montel space merely by assuming that T is continuous. This is, in fact, the case,as the following example demonstrates. This example is suggested by that of Dieudonné [17]. He takes a sequence space and a differential equation. By looking at each co-ordinate separately a family of differential equations is obtained; the solutions of which satisfy a single inequality. But this inequality is such that the resulting sequence is not in the original space. Our problem is to replace this sequence space by a function space and then look at the values of the functions at the integers.

4.3.2 Example

Let E=R, F= $\mathcal{S}$. Let $\varphi:[0,1] \longrightarrow R$ be defined by

$$\varphi(\xi) = a \int_0^{\xi} e^{-\left(\frac{1}{1-(2\eta-1)^2}\right)} d\eta \qquad (4)$$

where a is a number such that $\varphi(1) = 1$. Then it is easily seen that φ enjoys the following properties.

$$\begin{aligned} &\text{i)} \quad \varphi \text{ is infinitely differentiable} \\ &\text{ii)} \quad \varphi^{(\nu)}(0) = 0 \text{ for all } \nu \geq 0, \\ &\text{iii)} \quad \varphi^{(\nu)}(1) = 0 \text{ for all } \nu \geq 1, \\ &\text{iv)} \quad \varphi(1) = 1 \end{aligned} \qquad (5)$$

Now we define $T: R \times \mathcal{A} \longrightarrow \mathcal{L}(R, \mathcal{L}) = \mathcal{A}$ by

$$T(x,y)(\xi) = \left(\sqrt{|y(n+1)|} - \sqrt{|y(n)|}\right)\varphi(\xi - n) + \sqrt{|y(n)|} + e^{-\xi^2}$$

$$\text{for} \quad n \leq \xi \leq n+1 \quad . \tag{6}$$

In other words, we have for each n, replaced $y(n)$ by $\sqrt{|y(n)|}$ and then connected these points infinitely smoothly by curves which look like φ. For convenience of notation, let us write

$$Z_y = T(x,y) \quad . \tag{7}$$

Let us show that $Z_y \in \mathcal{A}$. Clearly from (5), Z_y is infinite differentiable. Let

$$M_\nu = \sup_{\xi} \left|\varphi^{(\nu)}(\xi - n)\right| \quad . \tag{8}$$

M_ν is independent of n. Thus since $Z_y^{(\nu)}(n) = 0$,

$$Z_y^{(\nu)}(\xi) \leq \left(\sqrt{|y(n+1)|} - \sqrt{|y(n)|}\right) M_\nu + \sqrt{|y(n)|} + e^{-\xi^2} \tag{9}$$

$$\text{for } n \leq \xi \leq n+1 \quad .$$

Since $y \in \mathcal{A}$, for each integer $k \geq 0$,

$$\lim_{|\xi| \to \infty} \xi^k y^{(\nu)}(\xi) = 0 \quad . \tag{10}$$

It follows from (9), (10), the continuity of the square root function, and the fact that $e^{-\xi^2} \in \mathcal{A}$, that

$$\lim_{|\xi| \to \infty} \xi^k Z_y^{(\nu)}(\xi) = 0 \quad . \tag{11}$$

Thus $Z_y \in \mathcal{A}$. Now we must show that T is continuous.

If $y_1, y_2 \in \mathcal{A}$, then

$$(Z_{y_1} - Z_{y_2})(\xi) = \left(\sqrt{|y_1(n+1)|} - \sqrt{|y_2(n+1)|} - \sqrt{|y_1(n)|} + \right.$$
$$\left. + \sqrt{|y_2(n)|}\right)\Phi(\xi-n) + \sqrt{|y_1(n)|} - \sqrt{|y_2(n)|} \text{ for } n \leqslant \xi \leqslant n+1 \tag{12}$$

$$\therefore \quad |(Z_{y_1} - Z_{y_2})^{\nu}|(\xi) \leqslant \left(\sqrt{|y_1(n+1)|} - \sqrt{|y_2(n+1)|} - \sqrt{|y_1(n)|} + \right.$$
$$\left. + \sqrt{|y_2(n)|}\right)M_{\nu} + \sqrt{|y_1(n)|} - \sqrt{|y_2(n)|} \quad . \tag{13}$$

The continuity of T then follows from (13) and the continuity of the square root function. On the other hand, let I be any interval in R containing 0 and suppose there exists a function $f: I \longrightarrow \mathcal{S}$ such that $f(0) = 0$ and

$$f'(x) = T(x, f(x)) \tag{14}$$

for each $\xi \in R$. Let us define $g_{\xi}: I \longrightarrow R$ by

$$g_{\xi}(x) = f(x)(\xi) \tag{15}$$

then

$$g'_{\xi}(x) = f'(x)(\xi) = T(x, f(x))(\xi) \quad . \tag{16}$$

In particular, from (6) and (16), for $\xi = n+1$

$$g'_{n+1}(x) = \sqrt{|f(x)(n+1)|} + e^{-(n+1)^2} \tag{17}$$

that is

$$g'_{n+1}(x) = \sqrt{|g_{n+1}(x)|} + e^{-(n+1)^2} \quad . \tag{18}$$

The solutions of (18) are dominated by solutions of

$$g'_{n+1}(x) = \sqrt{|g_{n+1}(x)|} \tag{19}$$

which are

$$g_{n+1}(x) = \frac{(x+c)^2}{4} \tag{20}$$

Thus with the initial conditions, the solutions of (18) satisfy

$$\varepsilon_{n+1}(x) \geq \frac{x^2}{4} \tag{21}$$

But then we have,

$$f(x)\ (n) \geq \frac{x^2}{4} \tag{22}$$

which contradicts the fact that $f(x) \in \mathcal{S}$.

4.3.3

Thus we see that some additional assumption is required. With an eye to using some form of Ascoli's Theorem, we would like to have the range of T a compact or at least relatively compact subset of $\mathcal{L}(E,F)$. Because F will be a montel space, equicontinuity is enough, so by th.(1.3.7.3) it will suffice to assume simple boundedness, and then of course, except when E=R, we need to assume that T' exists, is continuous and satisfies a certain symmetry condition. For uniqueness we need only add, when E=R, the existence and continuity of T'.

4.3.4

In this section we prove four results and give five examples. Example (4.3.2) showing the necessity of assuming simple boundedness, appears above. Th.(4.4.1) proves existence when E=R. The proof is a generalization of the classical Cauchy-Peano Theorem as discussed, say in [16]. Th.(4.4.3) proves the uniqueness. Here the proof is different from the usual one given for E=F=R. Examples (4.4.4.1), (4.4.4.2) illustrate Th.(4.4.1), (4.4.3). Th. (4.5.14) proves the existence and uniqueness for the case in which E=RxR. To solve the integral equation (3) is no problem, but merely a routine repetition of the method used in Th.(4.4.3). But to differentiate this function is quite another matter. Th. (3.3.8) cannot be used because nothing is known about the derivative of the integr and in (3) - not even its existence, since f itself is involved. We overcome this difficulty by fixing one of the co-ordinates

of R x R and solving a one-dimensional problem. The derivative of the resulting function is then the partial of f. Finally, Th.(4.5.9) proves the existence and uniqueness for E,F Montel spaces. Again we first solve the integral equation. Then we apply Th.(4.5.14) to show that the integral of T(x,f(x)) along any polygonal path depends only on the end points. A theorem of Gil De Lamadrid gives the desired result. Examples (4.6.3), (4.6.4) illustrate Th.(4.6.5).

4.4 The one dimensional case

4.4.1 Theorem

Let E=R, F a montel space, $E_1 = [x_o, x_o+a_1] \subset E$, F_o a tonneau in F and y_o an interior point of F_o. Let $T: E_1 x F_o \longrightarrow \mathcal{L}(E,F) = F$ such that

i) $T(E_1 x F_o)$ is a Bdd subset of F

ii) T is continuous. (3)

Then there exists $E_o = [x_o, x_o + a_o]$, $0 < a_o \leq a_1$ and a function $f: E_o \longrightarrow F$ such that

1) f is continuous and differentiable,

2) $f(x_o) = y_o$

3) $f'(x) = T(x, f(x))$ (4)

pf. **First we assume that $x_o = y_o = 0$.**

By assumption i), there exists a number $a > 0$ such that if $|t| \leq a$, $x \in E_1$, $y \in F_o$, then

$$T(x,y)\ (t) \in F_o \tag{5}$$

We define $a_o = \text{Min}(a, a_1)$. Let B be the convex and topological closure of $T(E_1 \times F_o)\ (E_o)$. It again follows from i) and (1.3.2.7) that B is Bdd. Since F is a montel space, B is compact. Now E_o is also compact, so $E_o \times B$ is too. Therefore, the restriction of T to $E_o \times B$ is uniformly continuous.

Let V be a tonneau in F. There exists a tonneau W in F and a $\delta_1 > 0$ such that if $|x_1 - x_2| < \delta$, $x_1, x_2 \in E_o$, $y_1, y_2 \in B$, $y_1 - y_2 \in W$ and $t \in E_o$, then

$$T(x_1,y_1)(t) - T(x_2,y_2)(t) \in V . \qquad (26)$$

From i) there exists a $\delta_2 > 0$ such that if $|t| \leq \delta_2$, $x \in E_o$, and $y \in F_o$, then

$$T(x,y)\ (t) \in W \qquad (27)$$

We set $\delta = \text{Min}(\delta_1, \delta_2)$ and define $f_v : E_o \longrightarrow B$ by

$$f_v(x) = \begin{cases} 0 & \text{if } x = 0 \\ f_v(k\delta) + T(k\delta, f_v(k\delta))(x - k\delta) & \text{if } x \in [k\delta, (k+1)\delta] \cap E_o \\ & k = 0, 1, 2, \ldots . \end{cases} \qquad (28)$$

First we must determine that the range of f is a subset of B. Clearly $0 \in B$. Suppose that

$$f_v(k\delta) \in \frac{k\delta}{a_o} B \qquad (29)$$

then from (28), (29), and the definition of B,

$$f_v((k+1)\delta) = f_v(k\delta) + T(k\delta, f_v(k\delta))\ (\delta)$$

$$= f_v(k\delta) + \frac{\delta}{a_o} T(k\delta, f_v(k\delta))\ (a_o)$$

$$\in \frac{k\delta}{a_o} B + \frac{\delta}{a_o} B \subset \frac{(k+1)\delta}{a_o} B \qquad (30)$$

Thus by induction we may conclude that (29) is valid for $k = 0, 1, 2, \ldots$. In particular, since B is convex, the values of f at intermediate points, being line segments, are in B. (note that $k \leq \frac{a_o}{\delta}$). Furthermore, if $x \neq k\delta$ for any k, by the linearity,

$$f_v'(x)\ (t) = T(k\delta, f_v(k\delta))(t) \qquad (31)$$

Now $|x-k\delta| < \delta$ so by (27),

$$f_V(x) - f_V(k\delta) = T(k\delta, f_V(k\delta))(x-k\delta) \in W \tag{32}$$

It then follows from (26) and (32) that

$$f'_V(x)(t) - T(x,f_V(x))(t) = T(k\delta, f_V(k\delta))(t) - T(x,f_V(x))(t) \in V \tag{33}$$

Consider the family (f_V), where V ranges thru a Nbd base of o in F, as a subset of the space C of all continuous functions from E_o to B, equipped with the sup topology. Let U be a tonneau in F. By i) there exists an $\eta > 0$ such that if $x \in E_o$, $y \in F_o$, $|t| \leq \eta$ then

$$T(x,y)(t) \in U \tag{34}$$

Thus from (28), (34) if $x_1 \leq x_2$, $x_1 \in [(j-1)\delta, j\delta]$, $x_2 \in [k\delta, (k+1)\delta]$

$$\begin{aligned} f_V(x_2)-f_V(x_1) &= T(k\delta, f_V(k\delta))(x_2-k\delta) + \\ &+ T((j-1)\delta, f_V(j-1)\delta)(j\delta-x_1) + \\ &+ \sum_{i=j}^{k-1} T(i\delta, f_V(i\delta))(\delta) \in \\ &\in \frac{x_2-k\delta}{\eta} U + \frac{j\delta-x_1}{\eta} U + \sum_{i=j}^{k-1} \frac{\delta}{\eta} U \subset \\ &\subset \frac{1}{\eta}\left[x_2-k\delta + j\delta - x_1 + (k-j)\delta\right]U = \frac{x_2-x_1}{\eta} U \end{aligned} \tag{35}$$

Therefore the family (f_V) is equicontinuous. Since $f_V(E_o) \subset B$, then $\bigcup_V f_V(x) \subset B$ for all $x \in E_o$. But B is compact so by Th.(1.3.3.3), the family (f_V) has compact closure in C.

If we order our Nbd base (V) by inclusion we obtain a directed set so

that $\{f_V\}$ is a net in C. Therefore by Th.(1.3.1.5) there exist a Nbd base $\mathcal{W} = \{W\}$ and $f \in C$ such that for any V, there exists $W_1 \in \mathcal{W}$ such that $W_1 \subset V$ and if $W \in \mathcal{W}$, and $W \subset W_1$, and $x \in E_o$ then

$$f_W(x) - f(x) \in V \tag{36}$$

Now, integrating both sides of (33) we obtain

$$f_V(x) - \int_0^1 T(tx, f_V(tx))(x)\, dt \in V \tag{37}$$

Now

$$f(x) - \int_0^1 T(tx, f(tx))(x)\, dt = f(x) - f_W(x) + f_W(x) - \tag{38}$$

$$- \int_0^1 T(tx, f_W(tx))(x)\, dt + \int_0^1 \{T(tx, f_W(tx))(x) - T(tx, f(tx))(x)\}\, dt$$

From (36) and the continuity of T, given V there is a W such that

$$T(tx, f_W(tx))(x) - T(tx, f(tx))(x) \in V \tag{39}$$

Then from (36), (37), (39) and Th.(1...4.4)

$$f(x) - \int_0^1 T(tx, f(tx))(x)\, dt \in V + W + V \tag{40}$$

and this implies that

$$f(x) = \int_0^1 T(tx, f(tx))(x)\, dt \tag{41}$$

Hence

$$f(x) = \int_0^x T(t, f(t))\, dt \tag{42}$$

Therefore by Th.(1.2.4.2) we have

$$f'(x) = T(x, f(x)) \tag{43}$$

This satisfies conclusion 3). f is continuous because $f \in C$. $f(0) = 0$ because f is a uniform limit of functions f_ν with $f_\nu(0) = 0$.

Finally, in order to obtain conclusion 2), we apply the above argument to the function $S: E_1 \times F_0 \longrightarrow \mathcal{L}(E,F)$ defined by

$$S(x,y) = T(x + x_0,\ y + y_0) \tag{44}$$

Then we have a function $g: E_0 \longrightarrow F$ such that

$$g'(x) = S(x, g(x)), \quad g(0) = 0 \tag{45}$$

If we define $f: E_0 \longrightarrow F$ by

$$f(x) = g(x - x_0) + y_0 \tag{46}$$

Then $f(x_0) = y_0$ and

$$f'(x) = g'(x - x_0) = S(x - x_0, g(x - x_0))$$

$$= S(x - x_0,\ f(x) - y_0) = T(x, f(x)) \tag{47}$$

∎

4.4.2

Now we turn to the uniqueness. Let us assume that $E=F=R$ and T is constant on E. The problem then reduces to showing that if $h: E \longrightarrow F$ is continuous and non-negative, and if

$$h(x) \leq k \int_b^x h(t)\, dt \tag{48}$$

Then $h = 0$. One method of doing this is the following.

Suppose x_1 is the largest number such that $h(x) = 0$ on $[0,x_1]$. Let $x_2 > x_1$, let $x_3 \in [x_1, x_2]$ be the point at which $|h(x)|$ takes on its maximum value. From (50), for $x \in [x_0,x_2]$ we have

$$|h(x)| \leq k\int_0^x h(t)\,dt = k\int_{x_1}^x h(t)\,dt \leq |h(x_3)|\ k(x - x_1) \tag{49}$$

Thus for $x < x_1 + \frac{1}{k}$ we have

$$|h(x)| < |h(x_3)| \tag{50}$$

which is a contradiction. This argument is now in a proper form for generalization and Th.(4.4.3) does just that.

4.4.3 Theorem

Under the conditions of Th.(4.4.1), if in addition T' exists and is continuous, then the solution is unique w.r.t. the conclusions 1), 2), 3).

Pf. Suppose, f, g satisfy conclusions 1), 2), 3).of Th.(4.4.1) then

$$f(x) = y_0 + \int_{x_0}^x T(\xi,f(\xi))\,d\xi \tag{51}$$

and

$$g(x) = y_0 + \int_{x_0}^x T(\xi,g(\xi))\,d\xi \tag{52}$$

so

$$f(x) - g(x) = \int_{x_0}^x [T(\xi,f(\xi)) - T(\xi,g(\xi))]\,d\xi \tag{53}$$

Thus by (58), and Th.(1.2.5.4),

$$f(x) - g(x) = \int_{x_0}^x \int_0^1 T'(\xi,g(\xi) + \sigma\,[f(\xi) - g(\xi)\,])(0,f(\xi)-g(\xi))\,d\sigma d\xi \tag{54}$$

Let B be the convex, equilibrant, and topological closure of the range of T. Thus B is compact and hence B_1 = topological closure of B-B is a closed, convex, equilibrant Bdd set. Also by (51),(52)

$$f(x)-y_o \ \epsilon(x - x_o)\ B \tag{55}$$

and

$$g(x) - y_o \epsilon\ (x - x_o)\ B \tag{56}$$

so

$$f(x) - g(x) \epsilon\ (x-x_o)\ B_1 \tag{57}$$

We define the function $\alpha: E_o \longrightarrow E$ by

$$\alpha(x) = \inf\ \{\lambda/\lambda \geq 0,\ f(x) -g(x) \epsilon\lambda B_1\}. \tag{58}$$

From (57) we have

$$\alpha(x) \leq x - x_o \tag{59}$$

Further, if $\alpha(x) = 0$, then $f(x) -g(x) = 0$ for if $y_1 = f(x)-g(x) \neq 0$, then there exists a Nbd of 0, V, in F such that $y_1 \notin V$. Since B_1 is Bdd, there exists $\lambda_o > 0$ such that $\lambda_o B_1 \subset V$ and hence $\alpha(x) \geq \lambda_o > 0$. Finally, if $x_1, x_2 \epsilon E_o$, then we have from (53)

$$\begin{aligned} f(x_2) - g(x_2) &= \int_o^{x_1} \{T(\xi,f(\xi)) -T(\xi,g(\xi))\}\ d\xi \\ &+ \int_{x_1}^{x_2} \{T(\xi,f(\xi)) -T(\xi,g(\xi))\}\ d\xi \end{aligned} \tag{60}$$

$$= f(x_1) -g(x_1) + \int_{x_1}^{x_2} \{T(\xi,f(\xi)) -T(\xi,g(\xi))\}\ d\xi$$

So by definition of α, B_1 and Th.(1.2.4.4),

$$f(x_2) - g(x_2) \in (\alpha(x_1) + |x_2 - x_1|)\, B_1 \tag{61}$$

Hence

$$\alpha(x_2) \leq \alpha(x_1) + |x_2 - x_1| \tag{62}$$

and since the argument is symmetric in x_1, x_2 we have

$$|\, \alpha(x_2) - \alpha(x_1)\,| \leq |x_2 - x_1|$$

Thus α is continuous and hence attains a maximum on each closed subinterval of $E_o = [x_o, x_o + a]$. Let $x_1 = \text{Sup}\,\{x \mid \alpha(x) = 0\}$. Suppose $x_1 - x_o < a_o$. Let $x_2 \in (x_1, x_o + a_o]$ such that $x_2 - x_1 < 1/2$. Then there exists $x_3 \in (x_1, x_2]$ such that

$$\alpha(x_3) = \text{Sup}\,\{\alpha(x) \mid x \in [x_o, x_2]\} > 0 \tag{63}$$

Thus if $x \in [x_o, x_2]$

$$f(x) - g(x) \in \alpha(x_3)\, B_1 \tag{64}$$

We define the function $h : E_o \longrightarrow F$ by

$$h(\xi) = \int_0^1 T'(\xi, g(\xi) + \sigma\,[f(\xi) - g(\xi)]\,)\left(0, \frac{f(\xi) - g(\xi)}{\alpha(x_3)}\right) d\sigma \tag{65}$$

Since T', f, g are continuous, h is continuous. Further, by (5°), (6°) if $x \in [x_o, x_2]$

$$\int_{x_o}^{x} h(\xi)\, d\xi = \frac{f(\xi) - g(\xi)}{\alpha(x_3)} \in B_1 \tag{66}$$

We now claim that $h(x) \in 2B_1$ for all $x \in [x_0, x_2]$. Suppose $x_4 \in [x_0, x_2)$ and $h(x_4) \notin 2B_1$. Since $2B_1$ is closed and h is continuous, there exists x_5, x_6 such that

$$x_0 \leq x_5 < x_4 < x_6 < x_2 \tag{67}$$

and h(x) is in some convex open set G whose closure $\overline{G}$ is disjoint from $2B_1$ for all $x \in [x_5, x_6]$. Thus by Th.(1.2.4.2)

$$\int_{x_5}^{x_6} h(\xi)\, d\xi \in \overline{G} \quad . \tag{68}$$

However, by (66) and since B_1 is convex and equilibrant,

$$\int_{x_5}^{x_6} h(\xi)\, d\xi = \int_{x_0}^{x_6} h(\xi) d\xi - \int_{x_5}^{x_6} h(\xi) d\xi \in B_1 - B_1 \subset 2B_1 \tag{69}$$

But this contradicts the fact $\overline{G} \cap 2B_1 = \emptyset$. Hence $h(x) \in 2B_1$ for all $x \in [x_0, x_2)$ and by continuity the claim is proven. Thus for all $x \in [x_0, x_2]$,

$$f(x) - g(x) = \alpha(x_3) \int_{x_0}^{x} h(\xi)\, d\xi \tag{70}$$

and since $\alpha(x) = 0$ for $x \in [x_0, x_1]$, then $f(x) - g(x) = 0$ so by (65), $h(\xi) = 0$ for $x \in [x_0, x_1]$. Thus, by our claim,

$$f(x) - g(x) = \alpha(x_3) \int_{x_1}^{x_2} h(\xi)\, d\xi \in 2\alpha(x_3)(x_2 - x_1)\, B_1 \tag{71}$$

In particular,

$$f(x_3) - g(x_3) \in 2\alpha(x_3)(x_2-x_1)\, B_1 \tag{72}$$

and since $x_2-x_1 < 1/2$, we have $\alpha(x_3) < \alpha(x_4)$ which is a contradiction. Hence x_1 cannot be an interior point of E_o, that is $f(x) - g(x) = 0$ for all $x \in E_o$. ∎

4.4.4

We now consider two examples. In the first case, in addition to applying Th.(4.4.1),(4.4.3) we will able to exhibit the solution explicity, using the methods of ordinary differential equations. Thus we can see what kind of functions will satisfy our differential equations. In the second example, we obtain existence and uniqueness of solutions of a certain class of intergro-differential equations. Explicit solutions, if at all obtainable in this case would be very complicated.

4.4.4.1 Example

We take $E = R$, $F = \mathcal{O}$, $E_1 = R$, $F_o = V((m_\nu), (\epsilon_\nu))$ with $m_o \geq 1$. Then we define $T: E_1 \times F_o \longrightarrow F$ by

$$T(x,y) = y'(0)y_1 \tag{73}$$

where $y_1 \in F_o$ and y' is the derivative of the function y. It is easy to see that T is continuous. Further, if $y \in F_o$, M_ν is the upper bound of $| y_1^{(\nu)}(t) |$, K is the support of y_1 and $[T(x,y)]^{(\nu)}(t)$ is the νth derivative of $T(x,y)$ evaluated at t, then $T(x,y)$ is supported by K and

$$\left| [T(x,y)]^{\nu}(t) \right| = |y'(0)| \left| y_1^{\nu}(t) \right| \leq \epsilon_o M_\nu \tag{74}$$

Thus $T(E_1 x E_o)$ is a Bdd subset of F. The assumptions of (4.4.1) are then satisfied so we may assert the existence of solutions. Further, since T is linear and continuous, by Th.(1.2.5.3), we have $T' : E_1 x F_o \longrightarrow (ExF, F)$ is given by

$$T'(x,y)(t,h) = k'(0)y_1 \tag{75}$$

and this clearly continuous, so by Th.(4.4.3) the solution is unique.

We shall now obtain the explicity solution under the initial condition

$$f(0) = y_o \tag{76}$$

where $y_o \epsilon F_o$. What we are looking for is a function $f : E_1 x R \longrightarrow R$ (we abuse the language by using the same letter, f) such that $\frac{\partial^\nu f}{\partial R^\nu}$ exists for all $\nu = 0, 1, \ldots$ and for each $x \epsilon E_1$, f has compact support on R. We further demand, after (73) that

$$\frac{\partial f}{\partial x}(x,t) = \frac{\partial f}{\partial t}(x,0)\, y_1(t) \tag{77}$$

and after (76), that

$$f(0,t) = y_o(t) \tag{78}$$

If we integrate (77) w.r.t. x from 0 to x we obtain

$$f(x,t) = y_o(t) + \left(\int_o^x \frac{\partial f}{\partial t}(\xi,0)\, d\xi\right) y_1(t) \tag{79}$$

and if we define $\alpha : E_o \longrightarrow R$ by

$$\alpha(x) = \int_0^x \frac{\partial f}{\partial t}(\xi,0)\, d\xi \tag{80}$$

we have

$$f(x,t) = \alpha(x)\, y_1(t) + y_0(t) \tag{81}$$

substituting (81) back into (77) gives

$$\alpha'(x)y_1(t) + [\alpha(x)y_1'(0) + y_0'(0)\ ]\ y_1(t) \tag{82}$$

and (78) becomes

$$\alpha(0)y_1(t) = 0 \text{ for all } t. \tag{83}$$

If $y_1 = 0$ then (77) clearly has the unique solution

$$f(x,t) = y_0(t). \tag{84}$$

If $y_1 \neq 0$, (83) becomes

$$\alpha(0) = 0. \tag{85}$$

Then the problem (83), (85) represent a family of ordinary differential equations (indexed by t) which has the unique family of solutions

$$\alpha(x) = \begin{cases} \dfrac{y_0'(0)}{y_1'(0)}\left(e^{\frac{y_1'(0)x}{}} - 1\right) & \text{if } y_1'(0) \neq 0 \\ y_0'(0)x & \text{if } y_1'(0) \neq 0 \end{cases} \tag{86}$$

(81), (84) and (86) together give the following unique solution,

$$f(x) = \begin{cases} \dfrac{y_0'(0)}{y_1'(0)} \left(e^{y_1'(0)x} - 1\right) + y_1 + y_0 & \text{if } y_1'(0) \neq 0 \\ y_0'(0)\, x y_1 + y_0 & \text{if } y_1'(0) = 0 \end{cases} \tag{87}$$

4.4.4.2 Example

Let $E = R$, $F = \mathcal{D}$, $E_1 = [-1,1]$ and $F_0 = V(\{m_\nu\},\{\epsilon_\nu\})$ with $m_0 \geq 1$, $\epsilon_0 \leq 1$. Let $K: E_1 \times R \times E_1 \longrightarrow R$ such that

i) for each x, $\sigma \in E_1$ the function $K_{x,\sigma}: R \longrightarrow R$ defined by

$$K_{x,\sigma}(t) = K(x,t,\sigma)$$

is an element of $\mathcal{D}$

ii) The family $\{K_{x,\sigma}\}_{x,\sigma \in E_1}$ is a Bdd subset of $\mathcal{D}$

iii) For each $\nu = 0,1, \ldots$ the function $\dfrac{\partial^\nu K}{\partial t^\nu}$ is continuous in x,σ uniformly w.r.t. t

iv) $K(x,t,0) = 0$ for all $x \in E_1$, $t \in R$.

An example of such a function is

$$K(x,t,\sigma) \begin{cases} \sigma e^{-\frac{1}{1 - [(2+x)(2+\sigma)t]^2}} & \text{if } |(2+x)(2+\sigma)t| \leq 1 \\ 0 & \text{if } |(2+x)(2+\sigma)| \geq 1 \end{cases} \tag{89}$$

Then we may define $T: E_1 \times F_0 \longrightarrow F$ by

$$T(x,y)(t) = \int_{-\infty}^{\infty} K(x,t,y'(\xi))\, d\xi \tag{90}$$

The condition of Th.(4.4.1) are satisfied. For Th.(4.4.3) we have to assume in addition,

v) $\frac{\partial K}{\partial x}$ exists and is continuous

Then we have the existence and uniqueness of a function $f:E_1 \times R \to R$ such that

$$\frac{\partial f}{\partial x}(x,t) = \int_{-\infty}^{\infty} K(x,t, \frac{\partial f}{\partial t}(x,\xi))\, d\xi \qquad (91)$$

4.5 The Finite Dimensional Case

4.5.1

We now turn to the case in which E=RxR. At first glance, one might expect to be able to ignore this situation. Certainly, for F=R or any Banach space, it is ignored. The reason is that one must certainly assume some sort of symmetry for T', and thus we have already the existence and continuity of T'. This ensures that a Lipschitz condition is satisfied and then one can give a "Picard" type proof as is done in [18]. After some study, we are forced to conclude that the Lipschitz condition depends irrevocably upon the fact that all Nbds of 0 are scalar multiples of a single Nbd, which is sufficient for F to be a Normed space.

Thus we are led on a more circuitous path. First we present a series of lemmas which develop a calculus for functions which are continuous and or differentiable except for jumps. These are generalizations of previous results, and except for lemma (4.5.7) are rather trivial and unsurprising. We give them here and call them lemmas because without them our proof of Th.(4.5.14) would be rather vague. They will not be used elsewhere. Th. (4.5.14) on the other hand is crucial for the proof of Th.(4.6.2).

4.5.2

We begin with some definitions. Let F be a LCEVT and let $I = [0,1] \subset R$. Let $S = (a_1,\ldots,a_p)$ be a finite subset of I. Let $f:I \to F$. We say that f is continuous except for jumps on S if f is continuous on each subinterval whose interior is disjoint from S. We also define the integral by

$$\int_0^1 f(t)dt = \sum_{i=1}^{p+1} \int_{a_{i-1}}^{a_i} f(t)dt \text{ where } a_o = 0,\ a_{p+1} = 1 \tag{92}$$

The first lemma is an easy generalization of half of the fundamental theorem of calculus.

4.5.3 Lemma

Let $f:I \longrightarrow F$ be continuous except for jumps on S and define $g:I \longrightarrow F$ by

$$g(x) = \int_0^x f(t)\ dt \tag{93}$$

then g' exists and is continuous except for jumps on S and if $x \in I \sim S$, then

$$g'(x) = f(x)$$

Pf. If $x \in (a_k, a_{k+1})$ then by (92),

$$g(x) = \sum_{i=1}^{k-1} \int_{a_{i-1}}^{a_i} f(t)\ dt + \int_{a_k}^{x} f(t)\ dt \tag{94}$$

Hence if $0 < |h| < \text{Min}(x-a_k,\ a_{k+1}-x)$,

$$\frac{1}{h}[g(x+h) - g(x)] - f(x) = \frac{1}{h}\int_x^{x+h} f(t)dt - f(x)$$

$$= \frac{1}{h}\int_x^{x+h} [f(t) - f(*)]\, dt \qquad (95)$$

Let V be a convex, closed Nbd of 0 in F. Since f is continuous at x, there exists $\delta > 0$ such that if $|t-x| \leq \delta$ and $t\epsilon[a_k,\ a_{k+1}]$, then

$$f(t) - f(x)\ \epsilon V \qquad (96)$$

Thus if $|h| \leq \delta$, it follows from (95), (96) and Th.(1.2.4.4) that

$$\frac{1}{h}[g(x+h) - g(x)] - f(x)\ \epsilon \frac{1}{h}hV = V \qquad (97)$$

The next lemma is easily recognized as a classical result. The use of the Hahn-Banach Theorem is also a standard trick.

4.5.4 Lemma

Let f' exist on I~S and let $f'(x) = 0$ for all $x\epsilon$I~S.

Then f is constant on any interval whose interior is contained in I~S.

Pf. Let u be an element of the topological and algebraic dual F* of F. Then $u\cdot f: R \longrightarrow R$ and $u\cdot f$ is continuous and has vanishing derivative on each $[a_i, a_{i+1}]$. Therefore $u\cdot f$ is constant on each $[a_i, a_{i+1}]$, thus if $t_1, t_2 \epsilon [a_i, a_{i+1}]$, then

$$u(f(t_1) - f(t_2)) = u(f(t_1)) - u(f(t_2)) = 0 \qquad (98)$$

But this is true for all $u \in F^*$. Hence by the Hahn-Banach Th.(1.2.2.9) $f(t_1) = f(t_2)$. ∎

The next lemma gives us the other half of the fundamental theorem of calculus. It is mildly surprising that we obtain (99) which is free of any mention of the jumps. They are eliminated by assuming that f is continuous.

4.5.5 Lemma

Let $f: I \longrightarrow F$ be continuous such that f' is continuous except for jumps on S.

Then

$$f(x) = f(0) + \int_0^x f'(t)\, dt \tag{99}$$

Pf. Let $x \in I \sim S$. By lemma (4.5.3)

$$(f(x) - \int_0^x f'(t)\, dt)' = f'(x) - f'(x) = 0 \tag{100}$$

From lemma (4.5.4), if $x \in [a_k, a_{k+1})$,

$$f(x) - \int_0^x f'(t)\, dt = f(a_k) - \int_0^{a_k} f'(t)\, dt \tag{101}$$

For $k = 0$, this means, for $x \in [0, a_1)$,

$$f(x) - \int_0^x f'(t)\, dt = f(0) \tag{102}$$

and since f is continuous,

$$f(a_1) - \int_0^{a_1} f'(t)\,dt = f(0) \tag{103}$$

Hence, by (101), (103) for $x \in [a_1, a_2)$,

$$f(x) - \int_0^{x} f'(t)\,dt = f(0) + \int_0^{a_1} f'(t)dt - \int_0^{a_1} f'(t)dt = f(0) \tag{104}$$

and so on for the other intervals. ∎

4.5.6

Now we turn to the case, $f: I \times I \longrightarrow F$. We take S to be a subset of $I \times I$ of the form

$$S = \{(t,x) \mid t = y_i,\ i=1,\ldots,m \text{ or } tx = x_j,\ j = 1,\ldots,n\}$$

Then as before, if f is continuous on the closure of every connected subset of $I \times I \sim S$, we say that f is continuous except for jumps on S. In lemma (4.5.7) we prove an analogue of lemma (3.3.1) concerning differentiation under the integral sign. This result is not free from the influence of the jumps, although it is not easy to see on which set the result fails. It turns out to be just the set $x_1, \ldots, x_n$ in S. The proof of lemma (4.5.7) is essentially the same as that given for lemma (3.3.1) except that careful record must be kept of the contribution from the jumps.

4.5.7 Lemma

Let $f: I \times I \longrightarrow F$ be continuous and have a derivative which is continuous except for jumps on S. Let $g: I \longrightarrow F$ be defined by

$$g(x) = \int_0^1 f(t,x)\,dt \tag{105}$$

Then g is continuous and has a derivative which is continuous except for jumps on the set $S_x = \{x_1,\ldots,x_n\}$ and for $x \in I \sim S$,

$$g'(x) = \int_0^1 \frac{\partial f}{\partial t}(t,x)\,dt \quad . \tag{106}$$

pf. Let us define functions $t_\nu : I \longrightarrow I$ by

$$t_\nu(x) = y_i (i = 0,\ldots,m+1) \quad \text{or}$$

$$t_\nu(x) = \begin{cases} 1 & \text{if} \quad a \le x \le x_j \\ x_j & \text{if} \quad x_j \le x \le 1 \\ x & \end{cases} \quad . \tag{107}$$

Thus $\nu = 0,\ldots, m+n+1$ and for each x, we reorder the numbers $t_0(x), t_1(x),\ldots$ according to magnitude with duplicates being dropped. (See Fig. 1).

Now let $\bar{x} \in I \sim S_x$ and we define the function $g_\nu : I \sim S_x \longrightarrow F$ by

$$g_\nu(x) = \int_{t_\nu(x)}^{t_{\nu+1}(x)} f(t,x)dt \quad . \tag{108}$$

Let h be such that the interval $[\bar{x} - |h|, \bar{x} + |h|]$ does not meet S_x. Then we have f' is continuous on each of the compact sets,

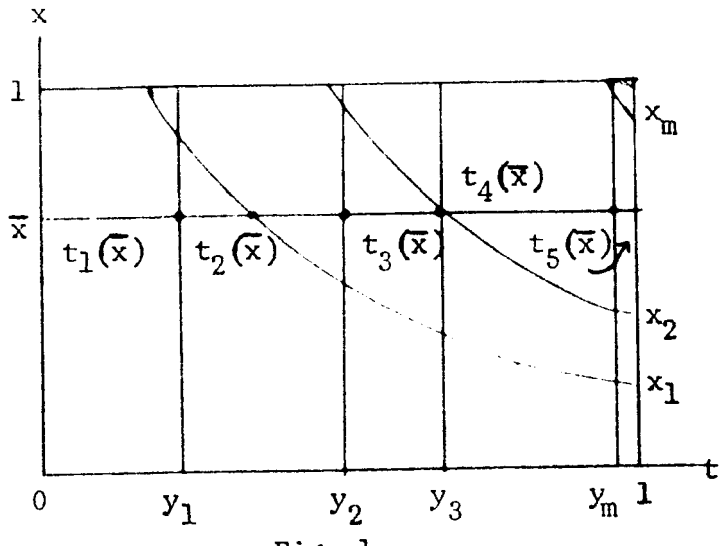


Fig. 1

$$C_\nu = \{(t,x)/\left|x-\bar{x}\right| \leq \left|h\right| \ , \ t_\nu(x) \leq t \leq t_{\nu+1}(x)\} \ . \tag{109}$$

Let

$$\Delta = \frac{g_\nu(\bar{x}+h) - g_\nu(\bar{x})}{h} \ . \tag{110}$$

By (108),

$$\Delta = \frac{1}{h}\int_{t_\nu(\bar{x}+h)}^{t_{\nu+1}(\bar{x}+h)} \{f(t,\bar{x}+h) - f(t,\bar{x})\}\,dt + \frac{1}{h}\left\{\int_{t_\nu(\bar{x}+h)}^{t_{\nu+1}(\bar{x}+h)} f(t,\bar{x})\,dt - \int_{t_\nu(\bar{x})}^{t_{\nu+1}(\bar{x})} f(t,\bar{x})\,dt\right\}$$

$$= \frac{1}{h}\int_{t_\nu(\bar{x})}^{t_{\nu+1}(\bar{x})} \{f(t,\bar{x}+h)-f(t,\bar{x})\}\,dt + \frac{1}{h}\int_{t_\nu(\bar{x}+h)}^{t_\nu(\bar{x})} \{f(t,\bar{x}+h)-f(t,\bar{x})\}\,dt +$$

$$+ \frac{1}{h}\int_{t_{\nu+1}(\bar{x})}^{t_{\nu+1}(\bar{x}+h)} \{f(t,\bar{x}+h) -f(t,\bar{x})\}\,dt + \frac{1}{h}\int_{t_\nu(\bar{x}+h)}^{t_\nu(\bar{x})} f(t,\bar{x})\,dt + \frac{1}{h}\int_{t_{\nu+1}(\bar{x})}^{t_{\nu+1}(\bar{x}+h)} f(t,\bar{x})\,dt \ . \tag{111}$$

Applying lemma (4.5.5) to (111) we obtain

$$\Delta - \left\{\int_{t_\nu(\bar{x})}^{t_{\nu+1}(\bar{x})} \frac{\partial f}{\partial x}(t,\bar{x})\,dt + t'_{\nu+1}(\bar{x}) f(t_{\nu+1}(\bar{x}),\bar{x}) - t_\nu(\bar{x}) f(t'_\nu(\bar{x}),\bar{x})\right\} =$$

$$= \int_{t_\nu(\bar{x})}^{t_{\nu+1}(\bar{x})} \frac{1}{h}\int_{\bar{x}}^{\bar{x}+h} \left(\frac{\partial f}{\partial x}(t,\xi) - \frac{\partial f}{\partial x}(t,\bar{x})\right) d\xi\,dt + \left\{\frac{t_{\nu+1}(\bar{x}+h)-t_{\nu+1}(\bar{x})}{h} - t'_{\nu+1}(\bar{x})\right\}$$

$$+\frac{1}{h}\int_{t_\nu(\bar{x}+h)}^{t_\nu(\bar{x})}\int_{\bar{x}}^{\bar{x}+h}\frac{\partial f}{\partial x}(t,\xi)\,d\xi\,dt+\frac{1}{h}\int_{t_{\nu+1}(\bar{x})}^{t_{\nu+1}(\bar{x}+h)}\int_{\bar{x}}^{\bar{x}+h}\frac{\partial f}{\partial x}(t,\xi)\,d\xi\,dt\,. \tag{112}$$

The functions $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial t}$ are continuous on the compact set C_ν so their ranges are compact and hence Bdd subset of F. Let V be a closed, convex Nbd of 0 in F. Then there exists λ_1 such that

$$\frac{\partial f}{\partial x}(C_\nu)\subset\lambda_1 V \quad\text{and}\quad \frac{\partial f}{\partial t}(C_\nu)\subset\lambda_1 V\,. \tag{113}$$

Further, since Nbds of 0 absorb points, there is a λ_2 such that

$$f(t_\nu(\bar{x}),\bar{x})\in\lambda_2 V \quad\text{and}\quad f(t_{\nu+1}(\bar{x}),\bar{x})\in\lambda_2 V\,. \tag{114}$$

Let $\lambda=\max(\lambda_1,\lambda_2)$. We choose $\delta>0$ satisfying the following six conditions. If (t_1,x_1) and (t_2,x_2) are in C_ν and $|t_1-t_2|\le\delta$, $x_1-x_2\le\delta$ then

$$\frac{\partial f}{\partial x}(t_1,x_1)-\frac{\partial f}{\partial x}(t_2,x_2)\in\frac{\mu}{6}V \tag{115}$$

where

$$\mu=\min\left\{1,\frac{1}{|t_{\nu+1}(\bar{x})-t_\nu(\bar{x})|}\right\}$$

and if $0<|h|\le\delta$, then

$$\left|\frac{t_{\nu+1}(\bar{x}+h)-t_{\nu+1}(\bar{x})}{h}-t'_{\nu+1}(\bar{x})\right|\le\frac{1}{6\lambda} \tag{116}$$

and

$$\left| \frac{t_\nu(\bar{x}+h) - t_\nu(\bar{x})}{h} - t'_\nu(\bar{x}) \right| \leq \frac{1}{6\lambda} \tag{117}$$

and

$$\left| \frac{t_{\nu+1}(\bar{x}+h) - t_{\nu+1}(\bar{x})}{h} \right|^2 \leq \frac{1}{6\lambda} \tag{118}$$

and

$$\left| \frac{t_\nu(\bar{x}+h) - t_\nu(\bar{x})}{h} \right|^2 \leq \frac{1}{6\lambda} \tag{119}$$

and

$$\left| t_{\nu+1}(\bar{x}+h) - t_{\nu+1}(\bar{x}) \right| + \left| t_\nu(\bar{x}+h) - t_\nu(\bar{x}) \right| \leq \frac{1}{6\lambda} \ . \tag{120}$$

The existence of δ follows from the uniform continuity of $\frac{\partial f}{\partial x}$ on C_ν, and continuity and differentiability of $t_{\nu+1}$, t_ν at $\bar{x}$.

Thus for $0 < |h| \leq \delta$, we have from (112),...(120) and Th.(1.2.4.4)

$$\Delta - \left\{ \int_{t_\nu(\bar{x})}^{t_{\nu+1}(\bar{x})} \frac{\partial}{\partial x} f(t,\bar{x}) \, dt + t'_{\nu+1}(\bar{x}) f(t_{\nu+1}(\bar{x}),\bar{x}) - t'_\nu(\bar{x}) f(t_\nu(\bar{x}),\bar{x}) \right\} \in$$

$$\in [t_{\nu+1}(\bar{x}) - t_\nu(\bar{x})] \ \frac{\mu}{6} V + \frac{1}{6\lambda}\lambda V + \frac{1}{6\lambda}\lambda V + \frac{1}{6\lambda}\lambda V + \frac{1}{6\lambda}\lambda V + \frac{1}{6\lambda}\lambda V \in$$

$$\in \frac{1}{6}V + \frac{1}{6}V + \frac{1}{6}V + \frac{1}{6}V + \frac{1}{6}V + \frac{1}{6}V \subset V \ . \tag{121}$$

Thus,

$$g_\nu'(\bar{x}) = \int_{t_\nu(\bar{x})}^{t_{\nu+1}(\bar{x})} \frac{\partial f}{\partial x}(t,\bar{x})\ dt + t_{\nu+1}'(\bar{x}) f(t_{\nu+1}(\bar{x}),\bar{x}) - t_\nu'(\bar{x}) f(t_\nu(\bar{x}),\bar{x}) . \quad (122)$$

On the other hand,

$$g(\bar{x}) = \sum_{\nu=0}^{p} g_\nu(\bar{x}) \quad (123)$$

so

$$g'(\bar{x}) = \sum_{\nu=0}^{p} g_\nu'(\bar{x}) \quad (124)$$

and by (122)

$$g'(\bar{x}) = \sum_{\nu=0}^{p} \int_{t_\nu(\bar{x})}^{t_{\nu+1}(\bar{x})} \frac{\partial f}{\partial x}(t,\bar{x})\ dt + \sum_{\nu=0}^{p} \left(t_{\nu+1}'(\bar{x})\ f(t_{\nu+1}(\bar{x}),\bar{x}) - \right.$$

$$\left. - t_\nu'(\bar{x}) f(t_\nu(\bar{x})\bar{x}) \right) = \int_0^1 \frac{\partial f}{\partial x}(t,\bar{x})dt + t_{p+1}'(\bar{x}) f(t_{p+1}(\bar{x}),\bar{x}) -$$

$$- t_o'(\bar{x})\ f(t_o(\bar{x}),\bar{x}) . \quad (125)$$

And because $t_{p+1}(x) = 1$, $t_o(x) = 0$ we have

$$g'(\bar{x}) = \int_0^1 \frac{\partial f}{\partial t}\ (t,x)\ dt . \qquad \blacksquare$$

4.5.8

Before going on, we indicate why we have taken our special form for S. What we have in mind is a function h: I x I $\longrightarrow$ F which is obtained from a function P: I $\longrightarrow$ F by setting

$$h(t,x) = P(tx) \quad . \tag{126}$$

Then if P (or P') has jumps at $x_1,\ldots,x_n$, then h has jumps on a set S of the kind discussed above. This will be done in lemma (4.5.12) but first a pair of simple results which are special cases of theorems in Section 2.

4.5.9 Lemma

Let $f:I \longrightarrow F$ such that f' is continuous except for jumps on S and let $\alpha:I \longrightarrow I$ such that α' is continuous.

Then $g = f \cdot \alpha$ has a derivative which is continuous except for jumps on S and if $\alpha(t) \in I \sim S$,

$$g'(t) = f'(\alpha(t))\,\alpha'(t) \quad . \tag{127}$$

Also, $h = \alpha f$ has a derivative which is continuous except for jumps on S and if $t \in I \sim S$,

$$h'(t) = \alpha'(t)\,f(t) + \alpha(t)\,f'(t) \quad . \tag{128}$$

pf. If we restrict to any subinterval of $I \sim S$, then these results follow immediately from th.(2.4.1) and corollary (2.6.3) . ∎

4.5.10 Lemma

Let $f:I \longrightarrow F$ such that f' is continuous except for jumps on S_1 and let $\alpha:I \longrightarrow I$ such that α is continuous except for jumps on S_2 and let

$$\beta:I \longrightarrow I \quad \text{by} \quad \beta(t) = \int_0^t \alpha(\tau)\,d\tau \quad .$$

Then

$$\int_0^1 \alpha(t)\,f(t)dt = \beta(1)\,f(1) - \beta(0)\,f(0) - \int_0^1 \beta(t)\,f'(t)dt \quad . \tag{129}$$

pf. With lemmas (4.5.7), (4.5.9), the proof can be done in exactly the same manner as the classical proof of the integration by parts formula. ∎

4.5.11

Now we prove lemma (4.5.12) which is actually a generalized form of th.(2.3.3), for the case E = R x R. Here we replace the symmetry condition by demanding that the cross partials differ by a certain amount. In return we get a function whose partials differ from the original functions by the same amount. With the previous six lemmas, the proof of lemma (4.5.12) is formally the same as the classical result with the error carried along.

4.5.12 Lemma

Let $S = \{(x,y) \in I \times I \,/\, x = x_i,\ i = 1,\ldots,n \text{ or } y = y_j,\ j = 1,\ldots,n\}$. Let $P,Q: I \times I \longrightarrow F$ be two continuous functions which have derivatives continuous except for jumps on S. Let $f: I \times I \longrightarrow F$ be defined by

$$f(x,y) = \int_0^1 \{x\, P(tx,ty) + y\, Q(tx,ty)\}\, dt \quad . \tag{130}$$

For each x,y let $A_{x,y}$ be a closed, convex, equilibrant subset of F with the property that for all t such that $(tx,ty) \in I \times I - S$

$$\frac{\partial Q}{\partial x}(tx,ty) - \frac{\partial P}{\partial y}(tx,ty) \in A_{x,y} \quad . \tag{131}$$

Then f is continuous, f' is continuous except for jumps on S and

$$\frac{\partial f}{\partial x}(x,y) - P(x,y) \in \frac{y}{2} A_{x,y} \tag{132}$$

$$\frac{\partial f}{\partial y}(x,y) - Q(x,y) \in \frac{x}{2} A_{x,y} \quad . \tag{133}$$

pf. Suppose y is different from $y_1,\ldots,y_n$. By lemma 4, $\frac{\partial f}{\partial x}$ exists except for $x = x_i$ and from (106), (127) and (128), it follows that for $x \neq x_i$

$$\frac{\partial f}{\partial x}(x,y) = \int_0^1 \left\{ P(tx,ty) + t\left[x\frac{\partial P}{\partial x}(tx,ty) + y\frac{\partial Q}{\partial x}(tx,ty)\right]\right\} dt$$

$$= \int_0^1 P(tx,ty)\,dt + \int_0^1 t\left[x\frac{\partial P}{\partial x}(tx,ty) + y\frac{\partial P}{\partial y}(tx,ty)\right] dt +$$

$$+ y\int_0^1 t\left[\frac{\partial Q}{\partial x}(tx,ty) - \frac{\partial P}{\partial y}(tx,ty)\right] dt \quad . \qquad (134)$$

Applying lemma (4.5.10) to the second integral and (131) to the third, we obtain

$$\frac{\partial f}{\partial x}(x,y) \in \int_0^1 P(tx,ty)\,dt + t\,P(tx,ty)\Big]_{t=0}^{t=1} - \int_0^1 P(tx,ty)dt + \frac{y}{2}A_{x,y}$$

$$= P(x,y) + \frac{y}{2}A_{x,y} \quad . \qquad (135)$$

(133) is obtained similarly. ∎

4.5.13

Now we are ready for the two-dimensional existence and uniqueness theorem. The proof can be divided into four parts. First, we break the square into subsquares and define an approximating function by making it linear on each subsquare with "slope" equal to the value of $T(x,y,z)$ at the lower left vertex. A separate argument which invokes lemma 7 and the symmetry assumption is required to show that this function is an approximate solution of the corresponding integral equation. The second step uses Ascoli's theorem to obtain a limit solution of the approximate solutions. This gives a solution to the integral equation. Unfortunately, as will be seen in the proof, we cannot simply differentiate. Instead, for the third step we consider the convergent subnet of approximate

solutions, fix y and construct, for each of these, an approximate solution of a certain one-dimensional problem. We take a second subnet, obtaining a limit which is a solution of the one-dimensional problem, which is differentiable, and which is equal to the solution of the two-dimensional problem. This gives us one partial derivative and the other is obtained similarly. The fourth and final step of the proof is the uniqueness. This follows easily from th.(4.4.3).

4.5.14 Theorem

Let F be a Montel space, F_o a tonneau in F. Let E = R x R, $I_x = [x_o, x_o+a]$, $I_y = [y_o, y_o+b]$, $E_1 = I_x \times I_y$. $T: E_1 \times F_o \longrightarrow \mathcal{L}(E,F)$ such that T,T' are continuous and

$$\frac{\partial T}{\partial E}(x,y,z)(h)(k) + \frac{\partial T}{\partial F}(x,y,z)\,(T(x,y,z)(k))\,(h) =$$

$$= \frac{\partial T}{\partial E}(x,y,z)(k)(h) + \frac{\partial T}{\partial F}(x,y,z)\,(T(x,y,z)(h))\,(k) \qquad (136)$$

where $x \in I_x$, $y \in I_y$, $z \in F_o$, $h \in E$, $k \in E$.

Assume further that

$$T(E_1 \times F_o) \text{ is simply Bdd.} \qquad (137)$$

Then there exists $I_o = [x_o, x_o+a_o] \subset I_x \cap I_y$ and a function $f: (x_o,y_o) + I_o \times I_o \longrightarrow F_o$ such that

i) $f(x_o,y_o) = z_o$

ii) f' exists and $f'(x,y) = T(x,y,f(x,y))$ (138)

iii) f is unique w.r.t. i), ii).

pf. First we assume that $x_o = y_o = z_o = 0$. The more general situation is obtained by the same type of transformation as used in th.(4.4.1). We define $P,Q: E_1 \times F_o \longrightarrow F$ by

$$P(x,y,z) = T(x,y,z)\,(1,0) \tag{139}$$

$$Q(x,y,z) = T(x,y,z)\,(0,1)\ . \tag{140}$$

From (136) it follows that

$$\frac{\partial P}{\partial y}(x,y,z) + \frac{\partial P}{\partial F}(x,y,z)(Q(x,y,z)) = \frac{\partial Q}{\partial x}(x,y,z) + \frac{\partial Q}{\partial F}(x,y,z)(P(x,y,z))\ . \tag{141}$$

Since the range of T is simply Bdd and E is tonnelé, the range of T is an equicontinuous subset of $\mathcal{L}(E,F)$ according to th.(1.2.7.3). Hence there exist $M > 0$ such that if $|s| \leq M$, $|t| \leq M$, $(x,y,z) \in E_1 \times F_o$, then

$$T(x,y,z)\,(s,t) \in F_o\ . \tag{142}$$

Let $a_o = \min(\frac{1}{2}M, a)$. Let B be the equilibrant, convex, and topological closure of the set

$$U\{\, T(x,y,z)(s,t) \;/\; (x,y) \in E_1,\quad s \in I_x,\ t \in I_y,\ z \in F_o \,\}\ .$$

Then B is a closed, Bdd and hence compact subset of the Montel space F. Thus the restriction of T to $E_1 \times B$ is uniformly continuous. Also, $\frac{\partial P}{\partial E}(E_1 \times B)$, $\frac{\partial Q}{\partial F}(E_1 \times B)$ are compact, hence Bdd, hence equicontinuous subsets of $\mathcal{L}(E \times F, F)$.

Let V be a tonneau in F. We choose $\delta > 0$ and U,W tonneaus in F such that

$$U < \frac{1}{3} V\ . \tag{143}$$

If $|x_1-x_2| \le \delta$, $|y_1-y_2| \le \delta$, $z_1-z_2 \in W$, $z_1,z_2 \in B$, $s \in I_x$, $t \in I_y$, then

$$T(x_1,y_1,z_1)(s,t) - T(x_2,y_2,z_2)(s,t) \in U \tag{144}$$

and if $(x,y) \in E_1$, $z \in B$, $v \in U$, then

$$\frac{\partial P}{\partial F}(x,y,z)(v) \in \frac{1}{3} V \tag{145}$$

$$\frac{\partial Q}{\partial F}(x,y,z)(v) \in \frac{1}{3} V \quad . \tag{146}$$

(145), (146) express the equicontinuity, with U chosen to satisfy (143). W, δ are then chosen to satisfy (144) according to the uniform continuity of T on $E_1 \times B$. We define a partition $\pi(V)$ on $E_o = I_o \times I_o$ by

$$0 = x_o < \ldots < x_{m+1} = a_o, \quad 0 = y_o < \ldots < y_{n+1} = a_o \tag{147}$$

such that

$$|\pi(V)| = \sup_{i,j}\{|x_{i+1}-x_i|, |y_{j+1}-y_j|\} \le \min(\delta, \frac{\lambda}{2}) \tag{148}$$

where $\lambda > 0$ is such that $\lambda B \subset W$. We also define the expressions

$$[x] = x_i, \quad [y] = y_j \quad \text{where} \quad x_i \le x < x_{i+1}, \quad y_j \le y < y_{j+1}. \tag{149}$$

Now we can define the function $f_\pi : E_o \longrightarrow B \cap F_o$ by

$$f_\pi(0,0) = 0$$

$$f_\pi(x,y) = f_\pi([x],[y]) + T([x],[y], f_\pi([x],[y]))(x-[x],y-[y]) \ . \tag{150}$$

Let us assume that

$$f_\pi([x],[y]) \in \frac{1}{M}[x]\, B \cap F_o + \frac{1}{M}[y]\, B \cap F_o \subset B \cap F_o \ . \tag{151}$$

Then from (150), (142) and the definition of B,

$$f_\pi(x,y) \in \frac{1}{M}[x]\, B \cap F_o + \frac{1}{M}[y]\, B \cap F_o + \frac{1}{M}(x-[x])\, B \cap F_o +$$

$$+ \frac{1}{M}(y-[y])B \cap F_o \subset \frac{x}{M} B \cap F_o + \frac{y}{M} B \cap F_o \subset B \cap F_o \quad . \qquad (152)$$

Thus the range of f_π is truly $B \cap F_o$.

Let $(x,y) \in E_o$, $(x^*,y^*) \in E_o$. We define

$$\bar{x} = \min(x,x^*), \qquad \bar{y} = \min(y,y^*) \quad . \qquad (153)$$

Clearly

$$f_\pi(x,y) - f_\pi(x^*,y^*) = f_\pi(x,y) - f_\pi(\bar{x},\bar{y}) + f_\pi(\bar{x},\bar{y}) - f_\pi(x^*,y^*). \qquad (154)$$

Suppose that $\bar{x} = x$, $\bar{y} = y$. Then

$$f_\pi(x,y) - f_\pi(x^*,y^*) = f_\pi(\bar{x},\bar{y}) - f_\pi(x^*,y^*) = f_\pi(\bar{x},\bar{y}) - f_\pi(x^*,\bar{y}) +$$

$$+ f_\pi(x^*,\bar{y}) - f_\pi(x^*,y^*) \quad . \qquad (155)$$

To each term of (155) we apply the argument used to obtain equation (35) in the proof of th.(4.4.1). We obtain

$$f_\pi(x,y) - f_\pi(x^*,y^*) \in (|x^*-x| + |y^*-y|)\, B \quad . \qquad (156)$$

The other three cases are handled similarly so that (156) is generally valid. This completes the first step of the proof. Now we must analyze the partials of f_π. Suppose $x \neq [x]$ and $y \neq [y]$. From (139), (140), (150) th.(1.2.5.3) we have

$$\begin{aligned} f_\pi'(x,y)(s,t) &= T([x],[y],f_\pi([x],[y]))(s,t) \\ &= P([x],[y],f_\pi([x],[y]))s + Q([x],[y],f_\pi([x],[y]))t \\ &= P(x,y,f_\pi(x,y))s + Q(x,y,f_\pi(x,y))t + \Delta P s + \Delta Q t \end{aligned} \qquad (157)$$

where

$$\Delta P = P([x],[y], f_\pi([x],[y])) - P(x,y,f_\pi(x,y))$$

$$= T([x],[y],f_\pi([x],[y])) (1,0) - T(x,y,f_\pi(x,y)) (1,0) \qquad (158)$$

and ΔQ has a similar expression.

By (148), $x-[x] \leq |\pi(V)|$, $y-[y] \leq |\pi(V)|$. Hence by (156),

$$f_\pi([x],[y]) - f_\pi(x,y) \in 2|\pi(V)| \; B \subset \lambda B \subset W \; . \qquad (159)$$

Thus by (144), we have

$$\Delta P \in U \; . \qquad (160)$$

Similarly, it may be shown that

$$\Delta Q \in U \; . \qquad (161)$$

Now we make the following definitions

$$\tilde{P}(x,y) = P(x,y, f_\pi(x,y)) \qquad (162)$$

$$\tilde{Q}(x,y) = Q(x,y, f_\pi(x,y)) \qquad (163)$$

$$\tilde{f}_\pi(x,y) = \int_0^1 \{x \tilde{P}(tx,ty) + y \tilde{Q}(tx,ty)\} \, dt \; . \qquad (164)$$

Applying th.(2.4.1), (157), (136), (161), (143) and (146), if (tx,ty) is not one of the mesh points, we have

$$\frac{\partial \tilde{P}}{\partial y}(tx,ty) = t \frac{\partial P}{\partial y}(tx,ty, f_\pi(tx,ty)) +$$

$$+ t \frac{\partial P}{\partial F}(tx,ty, f_\pi(tx,ty)) \left(\frac{\partial f_\pi}{\partial y}(tx,ty) \right)$$

$$= t\frac{\partial P}{\partial y}(tx,ty, f_\pi(tx,ty)) +$$

$$+ t\frac{\partial P}{\partial x}(tx,ty, f_\pi(tx,ty))\left(Q(tx,ty, f_\pi(tx,ty))\right)$$

$$+ t\frac{\partial P}{\partial z}(tx,ty, f_\pi(tx,ty))(\Delta Q)$$

$$= t\left\{\frac{\partial Q}{\partial x}(tx,ty, f_\pi(tx,ty)) + \right. \tag{165}$$

$$\left. + \frac{\partial Q}{\partial z}(tx,ty, f_\pi(tx,ty))\left(P(tx,ty, f_\pi(tx,ty))\right)\right\}$$

$$+ t\frac{\partial P}{\partial F}(tx,ty, f_\pi(tx,ty))(\Delta Q)$$

$$= \frac{\partial \tilde{Q}}{\partial x}(tx,ty) + t\frac{\partial P}{\partial F}(tx,ty, f_\pi(tx,ty))(\Delta Q) \in \frac{\partial \tilde{Q}}{\partial x}(tx,ty) + \frac{1}{3}V .$$

Thus by lemma (4.5.12), $\tilde{f}_\pi$ is continuous, its derivative is continuous except for jumps at the mesh points and

$$\frac{\partial \tilde{f}_\pi}{\partial x}(x,y) - \tilde{P}(x,y) \in \frac{1}{6}V \tag{166}$$

$$\frac{\partial \tilde{f}_\pi}{\partial y}(x,y) - \tilde{Q}(x,y) \in \frac{1}{6}V \quad . \tag{167}$$

Thus, from (166), (157), (160), (143)

$$\frac{\partial f_\pi}{\partial x}(x,y) - \frac{\partial \tilde{f}_\pi}{\partial x}(x,y) = \frac{\partial f_\pi}{\partial x}(x,y) - \tilde{P}(x,y) + \tilde{P}(x,y) - \frac{\partial \tilde{f}_\pi}{\partial x}(x,y)$$

$$\in U + \frac{1}{6}V \subset \frac{1}{2}V \tag{168}$$

and similarly,

$$\frac{\partial f_\pi}{\partial y}(x,y) - \frac{\partial \tilde{f}_\pi}{\partial y}(x,y) \in \frac{1}{2}V \quad . \tag{169}$$

Applying lemma (4.5.5) then gives

$$f_\pi(x,y) - \tilde{f}_\pi(x,y) = \int_0^x \left\{ \frac{\partial f_\pi}{\partial x}(t,y) - \frac{\partial \tilde{f}_\pi}{\partial x}(t,y) \right\} dt + \int_0^y \left\{ \frac{\partial f_\pi}{\partial y}(0,t) - \frac{\partial \tilde{f}_\pi}{\partial y}(0,t) \right\} dt \in \frac{1}{2} V + \frac{1}{2} V \subset V . \tag{170}$$

Thus by (164), (162), (163)

$$f_\pi(x,y) - \int_0^1 \left\{ x\, P(tx,ty,\, f_\pi(tx,ty)) + y\, Q(tx,ty,f_\pi(tx,ty)) \right\} dt =$$

$$= f_\pi(x,y) - \int_0^1 T(tx,ty,\, f_\pi(tx,ty))(x,y)\, dt \in V . \tag{171}$$

It follows from (156) that $\{f_\pi\}$ is an equicontinuous family. Now we apply th.(1.3.1.5) exactly as in the proof of th.(4.4.1) to obtain a continuous function $f: I_o \times I_o \longrightarrow F$ such that

$$f(x,y) = \int_0^1 T(tx,ty,\, f(tx,ty))(x,y)\, dt . \tag{172}$$

Unfortunately, none of our previous results (such as lemma (4.5.7)) will permit us to differentiate (172) to obtain (138) since the derivative of f would appear on both sides and we do not know that this exists. What we must do then is show that f, for fixed y is the solution of a one-dimensional problem. We will then be able to differentiate.

Let $\bar{y} \in I_o$ and let $\mathcal{V} = \{V\}$ be a Nbd base such that

$$\lim_{V \in \mathcal{V}} f_\pi(V) = f \tag{173}$$

where this limit is in the space $\mathcal{C}_2$ of all continuous functions from

$I_o \times I_o$ to F, equipped with the sup topology. Let $\mathcal{C}_1$ be the space of all continuous functions from I_o to F, equipped with the sup topology. For each $V \in \mathcal{V}$, let $\pi = \pi(V)$ and we define $\bar{f}_\pi : I_o \longrightarrow B_\cap F_o$ by

$$\bar{f}_\pi(x) = f_\pi(x,[\bar{y}]) . \tag{174}$$

From (156) we have

$$\bar{f}_\pi(x_1) - \bar{f}_\pi(x_2) \in |x_1 - x_2| B \tag{175}$$

and by (157), (160), (143), we have if x is not one of the mesh points,

$$\bar{f}'_\pi(x) = \frac{\partial f_\pi}{\partial x}(x,[\bar{y}]) = f'_\pi(x,[\bar{y}])(1,0) = P(x,[\bar{y}], f_\pi(x,[\bar{y}])) + \Delta P$$

$$\in P(x,[\bar{y}], \bar{f}_\pi(x)) + \frac{1}{3} V \tag{176}$$

from which it follows by lemma 3,

$$\bar{f}_\pi(x) - \int_0^x P(t,[\bar{y}], \bar{f}_\pi(t))dt \in \frac{1}{3} V . \tag{177}$$

Now we apply th.(1.3.1.5) for the family $\{\bar{f}_{\pi(V)}\}$ in $\mathcal{C}_1$ to obtain another Nbd basis $\mathcal{W} = \{W\} \subset \mathcal{V}$ and $\bar{f} \in \mathcal{C}_1$ such that

$$\lim_{W \in} \bar{f}_{\pi(W)} = \bar{f} \tag{178}$$

and

$$\bar{f}(x) = \int_0^x P(t,\bar{y}, \bar{f}(t))\, dt . \tag{179}$$

But (173), (174), and the fact that $\mathcal{W} \subset \mathcal{V}$ implies

$$f(x,y) = \bar{f}(x) . \tag{180}$$

Hence by (179), (180),

$$\frac{\partial f}{\partial x}(x,y) = \bar{f}'(x) = P(x,y,\bar{f}(x)) = P(x,y,f(x,y)) \quad . \tag{181}$$

Similarly, it is shown that

$$\frac{\partial f}{\partial y} = Q(x,y,f(x,y)) \tag{182}$$

so by th.(2.5.3),

$$f'(x,y) = T(x,y,f(x,y)) \quad . \tag{183}$$

The final step of the theorem is the uniqueness. This will be a simple consequence of th.(4.4.3).

Suppose that f_1, f_2 satisfy i), ii) of (138). Then

$$f_i(x,y) = \int_0^1 T(\tau x, \tau y, f_i(\tau x, \tau y))(x,y)\, d\tau \qquad i = 1,2. \tag{184}$$

Let $x,y \in I_o \times I_o$ and define $g_i : I \longrightarrow F_o$ by

$$g_i(t) = f_i(tx,ty) \qquad i = 1,2 \tag{185}$$

and define $\tilde{T}: I \times F_o \longrightarrow F$ by

$$\tilde{T}(t,z) = T(tx, ty, z)(x,y) \quad . \tag{186}$$

Then we have

$$g_i(t) = \int_0^1 T(\tau tx, \tau ty, f_i(\tau tx, \tau ty))(tx,ty)\, d\tau$$

$$= \int_0^1 \tilde{T}(\tau t, g_i(\tau t))\, t\, d\tau = \int_0^t \tilde{T}(\tau, g_i(\tau))\, d\tau \quad . \tag{187}$$

Hence by lemma (4.5.3),

$$g_i'(t) = \tilde{T}(t, g_i(t)) \quad . \tag{188}$$

Further, $g_1(0) = f_1(0,0) = f_2(0,0) = g_2(0)$, so by th.(4.4.3), $g_1 = g_2$. In particular, $g_1(1) = g_2(1)$ or $f_1(x,y) = f_2(x,y)$.

4.6 The non-dimensional case

4.6.1

We are now ready for the main result which is the most general form of the existence and uniqueness theorem. Before going into the proof, let us compare th.(4.4.1) and th.(4.5.14) to see how th.(4.6.2) relates to each of these. One main difference between the one-dimensional and two-dimensional theorems lies in the definition of the approximating function (see eq. (28), (150)). In the first case we approximate along a straight line (which is the only possibility). We could, in the two-dimensional case, approximate along rays which would be simpler. This would lead to a function satisfying (172) with no trouble at all. However, the argument which obtains $\bar{f}$ satisfying (179) and (180) would not be possible. Thus we define f_π by approximating along each coordinate axis separately. We then go to some trouble and use the symmetry condition to obtain (172).

In the case of E an arbitrary Montel space, neither method works. Approximating along rays leads to the same difficulty of obtaining the differential equation from the integral equation. Coordinate-wise approximation is hopeless in such a non-dimensional space. We are led to a combination of the two methods. First we approximate along rays and obtain a function f which satisfies the integral equation. Then we apply th.(4.5.14) to every two-dimensional subspace of E and obtain another function which by uniqueness is essentially f again. This permits us to assert that the integral of $T(x,f(x))$ around any triangle is zero and we are then able to compute the derivative of f and show that the differential

equation is satisfied. Finally, the uniqueness follows easily from th.(4.4.3).

4.6.2 Theorem

Let E,F be Montel spaces, E_1, F_o tonneaus in E,F respectively, and $x_o \in E_1$, $y_o \in F_o$. Let the function $T:(x_o,y_o) + E_1 \times F_o \longrightarrow \mathcal{L}(E,F)$ be continuous, differentiable, and such that

$$T((x_o,y_o) + E_1 \times F_o) \quad \text{is a simply Bdd subset of } \mathcal{L}(E,F) \tag{189}$$

and for all $x \in x_o + E_1$, $y \in y_o + F_o$, $h \in E$, $k \in E$,

$$\frac{\partial T}{\partial E}(x,y)(h)(k) + \frac{\partial T}{\partial F}(x,y)\,(T(x,y)(h))(k) =$$

$$= \frac{\partial T}{\partial E}(x,y)(k)(h) + \frac{\partial T}{\partial F}(x,y)\,(T(x,y)(k))(h)\ ; \tag{190}$$

then there exists a tonneau $E_o \subset E_1$ and a function $f:x_o + E_o \longrightarrow F_o$ such that

$$\begin{aligned} &\text{i)} && f(x_o) = y_o \\ &\text{ii)} && f'(x) = T(x,f(x)) \quad \text{for all } x \in x_o + E_o \\ &\text{iii)} && f \text{ is unique w.r.t. i), ii).} \end{aligned} \tag{191}$$

pf. We may assume that $x_o = y_o = 0$ and having proved the theorem in this case, we may apply the usual transformation mentioned in th.(4.4.1) to obtain the result for arbitrary x_o, y_o.

Th.(1.2.73) and (189) imply that $T(E_1 \times F_o)$ is an equicontinuous subset of $\mathcal{L}(E,F)$. Thus if we define

$$\tilde{E}_o = E_1 \cap T^{-1}(E_1 \times F_o)\ , \tag{192}$$

then $\tilde{E}_o$ is a tonneau in E. For reasons which will become clear much later on, we set $E_o = \frac{1}{4}\tilde{E}_o$. Let $x \in \tilde{E}_o$ and let $t_o,\ldots,t_n$ be real numbers such that

$$0 = t_o < t_1 < \ldots < t_n \leq 1 \quad . \tag{193}$$

We define the points $y_o,\ldots,y_n$ recursively by

$$y_o = 0$$

$$y_{i+1} = y_i + T(t_i x, y_i)(t_{i+1}x - t_i x) \quad . \tag{194}$$

If we assume $y_i \in t_i F_o$, then from (192), (194)

$$y_{i+1} \in t_i F_o + (t_{i+1} - t_i)F_o \subset t_{i+1} F_o \quad , \tag{195}$$

so by induction and the fact F_o is convex, we may conclude that the polygonal line connecting $y_o,\ldots,y_n$ in order is contained in F_o.

Let B be a closed, convex, equilibrant, Bdd subset of $\tilde{E}_o$. By equicontinuity, the set

$$\tilde{T}_B = U\{T(x,y)(B) \,/\, x \in \tilde{E}_o,\ y \in F_o\} \tag{196}$$

is a convex, equilibrant, Bdd subset of F_o and so is its closure T_B. We prove this as follows. Let V be a tonneau in F. Then by equicontinuity there is a tonneau W in E such that, if $x \in \tilde{E}_o$ and $y \in F_o$, then

$$T(x,y)(W) \subset V \quad . \tag{197}$$

Since B is Bdd, there is a number $\lambda > 0$ such that

$$\lambda B \subset W \quad . \tag{198}$$

We may then conclude from (196), (197), (198) that

$$\lambda \tilde{T}_B \subset V \quad . \tag{199}$$

But since V is closed, then

$$\lambda T_B \subset V \quad . \tag{200}$$

Hence T_B is Bdd. The other properties are immediate from (196).

Let $y \in [y_i, y_{i+1}]$. Then by the linearity of $T(x,y)$, there is a $t \in [t_i, t_{i+1}]$ such that

$$y = y_i + T(t_i x, y_i)(tx - t_i x) \quad . \tag{201}$$

Now if $x \in B$, then by (201)

$$y - y_i = T(t_i x, y_i)(tx - t_i x) = (t - t_i)\, T(t_i x, y_i)(x) \tag{202}$$

so by (196),

$$y - y_i \in (t - t_i)\, \tilde{T}_B \subset (t - t_i)\, T_B \quad . \tag{203}$$

Since E,F are Montel spaces, the set $B \times T_B$ is compact, so the restriction of T to $B \times T_B$ is uniformly continuous. Thus if V is a tonneau in F, there exist tonneaus U in E and W in F such that if $x_1 - x_2 \in U$, $x_1, x_2 \in B$, $y - y_2 \in W$, $y_1, y_2 \in T_B$, and $h \in B$, then

$$T(x_1, y_1)(h) - T(x_2, y_2)(h) \in V \quad . \tag{204}$$

Since B, T_B are Bdd sets, there exists $\delta_o > 0$ such that if $|\delta| \leq \delta_o$, then

$$\delta B \subset U \quad , \qquad \delta T_B \subset W \quad . \tag{205}$$

Now we further restrict the numbers $t_o, \ldots, t_n$ so that $t_n = 1$ and

$$\left| t_{i+1} - t_i \right| \leq \delta_o \qquad i = 0, \ldots, n-1 \quad . \tag{206}$$

We define $f: \tilde{E}_o \longrightarrow F_o$ by defining it along each interval $[0, \bar{x}]$, where $\bar{x}$ is an element of the boundary of $\tilde{E}_o$, as follows:

$$f(0) = 0$$

$$f(tx) = f(t_i \bar{x}) + T(t_i \bar{x}, f(t_i \bar{x}))(t\bar{x} - t_i \bar{x}) \quad , \quad t_i \leq t \leq t_{i+1} \quad . \tag{207}$$

By (195) we conclude that $f(\tilde{E}_o) \subset F_o$. Also, if $\alpha \in [0,1]$ and if $x = \alpha \bar{x} \in B$, and $t \in [t_i, t_{i+1}]$, then from (206), (205) and (203),

$$(t-t_i)x \in (t-t_i)B \subset U \tag{208}$$

and

$$f(tx) - f(t_i x) \in (t-t_i)\, T_B \subset W \ . \tag{209}$$

Thus from (204) we have

$$T(tx,\ f(tx))(x) - T(t_i x,\ f(t_i x))(x) \in V \ . \tag{210}$$

If $\alpha \in [t_i, t_{i+1}]$, then from (207)

$$f(x) - \int_0^1 T(tx,\ f(tx))(x)dt = \sum_{\nu=0}^{i-1} T(t_\nu \bar{x},\ f(t_\nu \bar{x}))(t_{\nu+1}\bar{x} - t_\nu \bar{x}) +$$

$$+ T(t_i \bar{x},\ f(t_i \bar{x}))(\alpha \bar{x} - t_i \bar{x}) - \int_0^1 T(tx,\ f(tx))(x)dt$$

$$= \sum_{\nu=0}^{i-1} \int_{\frac{t_\nu}{\alpha}}^{\frac{t_{\nu+1}}{\alpha}} \left\{ T\left(\frac{t_\nu}{\alpha} x,\ f\left(\frac{t_\nu}{\alpha} x\right)\right)(x) - T(tx,\ f(tx))(x) \right\} dt +$$

$$+ \int_{\frac{t_i}{\alpha}}^{1} \left\{ T(t_i \bar{x},\ f(t_i \bar{x}))(x) - T(tx,\ f(tx))(x) \right\} dt \ . \tag{211}$$

From (210) it follows that each integrand in (211) is an element of V, so by th.(1.2.4.4) and the convexity of V,

$$f(x) - \int_0^1 T(tx,\ f(tx))(x)dt \in \sum_{\nu=0}^{i-1} \frac{t_{\nu+1} - t_\nu}{\alpha} V + \left(1 - \frac{t_i}{\alpha}\right) V \subset V \ . \tag{212}$$

Recall that (212) has only been shown for $x \in B$. We shall refer to f as a $\langle B,V \rangle$ - approximate solution and write $f_{\langle B,V \rangle}$ where necessary to

avoid confusion. f is continuous on each set $t_{i+1}\tilde{E}_o \sim t_i\tilde{E}_o$ because T is continuous and since f is single valued on the intersection, f is continuous.

We cannot now apply the Ascoli theorem because $\tilde{E}_o$ is not compact. We shall apply it for each ray and then show that the functions patch together properly.

Let I = [0,1] and let $x \in \tilde{E}_o$. We define the function $T_x : I \times F_o \longrightarrow F$ by

$$T_x(t,y) = T(tx,y)(x) \quad . \tag{213}$$

For each tonneau V in F we define the function $g_{V,x} : I \longrightarrow F_o$ by

$$g_{V,x}(t) = f_{\langle B,V\rangle}(tx) \tag{214}$$

where $B = [-\bar{x},\bar{x}]$, $\bar{x}$ being on the boundary of $\tilde{E}_o$ and colinear with o,x. Then it follows from (212), (213) and (214) that

$$g_{V,x}(t) - \int_0^t T_x(\tau, g_{V,x}(\tau))d\tau = f_{\langle B,V\rangle}(tx) - \int_0^t T(\tau x, f_{\langle B,V\rangle}(\tau x))(x)d\tau$$

$$= f_{\langle B,V\rangle}(tx) - \int_0^1 T(\tau tx, f_{\langle B,V\rangle}(\tau tx))(tx)d\tau \in V \quad . \tag{215}$$

Now let $t,t^* \in I$ and say $t_i \le t \le t_{i+1}$, $t_j \le t^* \le t_{j+1}$, $i \le j$.

Then from (209),

$$\begin{aligned} g_{V,x}(t^*) - g_{V,x}(t) &= g_{V,x}(t^*) - g_{V,x}(t_j) + \sum_{k=i}^{j-1} [g_{V,x}(t_{k+1}) - g_{V,x}(t_k)] - \\ &\quad - g_{V,x}(t) + g_{V,x}(t_i) \\ &\in (t^*-t_j)T_B + \sum_{k=i}^{j-1} (t_{k+1}-t_k)T_B + (t-t_i)T_B \\ &\subset (t^*-t)T_B \quad . \end{aligned} \tag{216}$$

Now T_B is a Bdd subset of F which depends only on x and not on V. Thus we have shown that for each x, $\{g_{V,x}\}_V$ is equicontinuous so we may again apply th.(1.3.2.3) to assert the existence of a continuous function $g_x: I \longrightarrow F_o$ such that

$$g_x(t) = \int_0^t T_x(\tau, g_x(\tau))\, d\tau \quad . \tag{217}$$

If $\alpha \in [0,1]$, then x, αx will be co-linear, so B will be the same. Thus, from (214)

$$g_{V,\alpha x}(t) = f_{(B,V)}(t\alpha x) = g_{V,x}(\alpha t) \quad . \tag{218}$$

Hence,

$$g_{\alpha x}(g) = g_x(\alpha t) \quad . \tag{219}$$

Therefore, we may define $f: \tilde{E}_o \longrightarrow F_o$ by

$$f(x) = g_x(1) \tag{220}$$

and conclude from (217), (213), (219), that

$$f(x) = \int_0^1 T_x(\tau, g_x(\tau))d\tau = \int_0^1 T(\tau x, g_x(\tau))(x)d\tau$$

$$= \int_0^1 T(\tau x, g_{\tau x}(1))(x)d\tau = \int_0^1 T(\tau x, f(\tau x))(x)d\tau \quad . \tag{221}$$

Thus we see that f satisfies the integral equation. We now turn to the problem of differentiating (221).

Let $E_o = \frac{1}{4}\tilde{E}_o$ and let $x_1, x_2 \in E_o$. We define the function $\check{T}_{1,2}: I \times I \times F_o \longrightarrow \mathcal{L}(R \times R, F)$ by

$$T_{1,2}(s,t,y)(u,v) = T(sx_1 + tx_2, y)(ux_1 + vx_2) \tag{222}$$

and we define the function $h: I \times I \longrightarrow F_o$ by

$$h(s,t) = f(sx_1 + tx_2) \quad . \tag{223}$$

From (221), (222), (223) we have

$$h(s,t) = \int_0^1 T(\tau sx_1 + \tau tx_2,\ f(\tau sx_1 + \tau tx_2))(sx_1 + tx_2)d\tau$$

$$= \int_0^1 T_{1,2}(\tau s,\ \tau t,\ h(\tau s,\ \tau t))(s,t)d\tau \quad . \tag{224}$$

We leave h for a moment and apply theorem (4.5.14) to $T_{1,2}$. It is clear that $T_{1,2}$, $T'_{1,2}$ are continuous, for T, T' are continuous. Similarly, the range of $T_{1,2}$ is a simply Bdd subset of $\mathcal{L}(R \times R, F)$. Finally, we turn to the symmetry condition. We have, as in th.(3),

$$P(s,t,y) = T_{1,2}(s,t,y)(1,0) = T(sx_1 + tx_2, y)(x_1) \tag{225}$$

$$Q(s,t,y) = T_{1,2}(s,t,y)(0,1) = T(sx_1 + tx_2, y)(x_2) \quad . \tag{226}$$

Therefore, from the definition derivative and from (190),

$$\frac{\partial P}{\partial t}(s,t,y) + \frac{\partial P}{\partial F}(s,t,y)(Q(s,t,y)) =$$

$$= \lim_{\lambda \to 0} \frac{1}{\lambda} \left[T(sx_1 + tx_2 + \lambda x_2, y)(x_1) - T(sx_1 + tx_2, y)(x_1)\right] +$$

$$+ \lim_{\lambda \to 0} \frac{1}{\lambda} \left[T(sx_1 + tx_2, y + \lambda T(sx_1 + tx_2, y)(x_2))(x_1) - T(sx_1 + tx_2, y)(x_1)\right]$$

$$= \frac{\partial T}{\partial E}(sx_1 + tx_2, y)(x_2)(x_1) + \frac{\partial T}{\partial F}(sx_1 + tx_2, y)(T(sx_1 + tx_2, y)(x_2))(x_1)$$

$$= \frac{\partial T}{\partial E}(sx_1 + tx_2, y)(x_1)(x_2) + \frac{\partial T}{\partial F}(sx_1 + tx_2, y)(T(sx_1 + tx_2, y)(x_1))(x_2)$$

$$= \lim_{\lambda \to 0} \frac{1}{\lambda} \{T(sx_1 + tx_2 + \lambda x_1, y)(x_2) - T(sx_1 + tx_2, y)(x_2)\}$$

$$+ \lim_{\lambda \to 0} \frac{1}{\lambda} \{T(sx_1 + tx_2, y + \lambda T(sx_1 + tx_2, y)(x_1))(x_2) - T(sx_1 + tx_2, y)(x_2)\}$$

$$= \frac{\partial Q}{\partial s}(s,t,y) + \frac{\partial Q}{\partial F}(s,t,y)(P(s,t,y)) \quad . \tag{227}$$

Thus the conditions of th.(4.5.14) are satisfied and we may assert the existence of $I_o = [0, a_o]$ and a continuous function $\tilde{h}: I_o \times I_o \longrightarrow F_o$ such that

$$\tilde{h}(0,0) = 0 \tag{228}$$

and

$$\tilde{h}'(s,t) = T_{1,2}(s,t,\tilde{h}(s,t)) \quad . \tag{229}$$

Hence,

$$\tilde{h}(s,t) = \int_0^1 T_{1,2}(\tau s, \tau t, \tilde{h}(\tau s, \tau t))dt \quad . \tag{230}$$

Thus by (224) and (230) and the uniqueness, we conclude that $h = \tilde{h}$, at least on their common domain. However, let us look closer into the proof of th.(3) and compute a_o. We have $a_o = \min\{1, \frac{1}{2} M\}$. On the other hand, since $x_1, x_2 \in E_o = \frac{1}{4} \tilde{E}_o$, if $|u| \leq 2$ and $|v| \leq 2$, then by convexity,

$$ux_1 + vx_2 = 4(\frac{u}{4} x_1 + \frac{v}{4} x_2) \in 4E_o = \tilde{E}_o \quad . \tag{231}$$

Hence,

$$T_{1,2}(s,t,v)(u,v) = T(sx_1 + tx_2, y)(ux_1 + vx_2) \in T(sx_1 + tx_2, y)(\tilde{E}_o) \subset F_o \tag{232}$$

so by (140), $M \geq 2$. Thus $a_o = 1$ so $I_o = I$ and $h = \tilde{h}$. Therefore h' exists and

$$h'(s,t) = T_{1,2}(s,t, h(s,t)) \quad . \tag{233}$$

This means that the integral of (233) around any triangle is 0. More specifically, if $s_1, s_2, t_1, t_2 \in I$, by th.(1.2.5.6),

$$\int_0^1 T_{1,2}(\sigma s_2, \sigma t_2, h(\sigma s_2, \sigma t_2))(s_2,t_2)d\sigma = \int_0^1 h'(\sigma s_2, \sigma t_2)(s_2,t_2)d\sigma$$

$$= h(s_2,t_2) - h(0,0) = h(s_2,t_2) \tag{234}$$

and

$$\int_0^1 T_{1,2}(\sigma s_1, \sigma t_1, h(\sigma s_1, \sigma t_1))(s_1,t_1)d\sigma = h(s_1,t_1) \tag{235}$$

and

$$\int_0^1 T_{1,2}(s_1 + \sigma(s_2-s_1), t_1 + \sigma(t_2-t_1), h(s_1 + \sigma(s_2-s_1), t_1 + \sigma(t_2-t_1))$$

$$(s_2-s_1,t_2-t_1)d\sigma = \int_0^1 h'(s_1 + \sigma(s_2-s_1), t_1 + \sigma(t_2-t_1))(s_2-s_1,t_2-t_1)d\sigma$$

$$= h(s_2,t_2) - h(s_1,t_1) \tag{236}$$

and adding these three, and substituting $s_2 = t_1 = 0$, $s_1 = t_2 = 1$, we obtain,

$$\int_0^1 T_{1,2}(0,\sigma, h(0,\sigma))(0,1)d\sigma = \int_0^1 T_{1,2}(\sigma,0, h(\sigma,0))(1,0)d\sigma +$$

$$+ \int_0^1 T_{1,2}(1-\sigma,\sigma, h(1-\sigma,\sigma))(-1,1)d\sigma \quad . \tag{237}$$

With the relations (222), (223), we obtain from (237),

$$\int_0^1 T(\sigma x_2,\ f(\sigma x_2))(x_2)d\sigma = \int_0^1 T(\sigma x_1,\ f(x_1))(\sigma x_1)d\sigma +$$

$$+ \int_0^1 T(x_1 + \sigma(x_2-x_1),\ f(x_1 + \sigma(x_2-x_1))(x_2-x_1)d\sigma \qquad (238)$$

and (238) is valid for all $x_1, x_2 \in E_o$.

We can now compute f' from (221), (238). Let x be an interior point of E_o and let B be a closed, convex, equilibrant, Bdd subset of E. Let $\lambda_o > 0$ such that $x + \lambda_o B \subset E_o$. Let $h \in B$, $|\lambda| \leq \lambda_o$. Then with $x_1 = x$, $x_2 = x + \lambda h$, from (221), (238)

$$\frac{f(x + \lambda h)-f(x)}{\lambda} = \frac{1}{\lambda}\int_0^1 T(x + \sigma\lambda h,\ f(x + \sigma\lambda h))(\lambda h) \quad . \qquad (239)$$

Therefore,

$$\frac{f(x + \lambda h)-f(x)}{\lambda} - T(x,\ f(x))(h) = \int_0^1 \Big\{T(x + \sigma\lambda h,\ f(x + \sigma\lambda h))(h) -$$

$$- T(x,\ f(x))(h)\Big\}d\sigma$$

$$= \int_0^1\int_0^1 T'(x + \tau\sigma\lambda h,\ f(x) + \tau[f(x + \sigma\lambda h)-f(x)])(\sigma\lambda h,\ f(x + \sigma\lambda h)-f(x))$$

$$(h)d\tau\ d\sigma$$

$$= \int_0^1\int_0^1 T'(x + \tau\sigma\lambda h,\ f(x) + \tau[f(x + \sigma\lambda h)-f(x)])$$

$$\Big(\sigma\lambda h,\ \int_0^1 T(x + \rho\sigma\lambda h, f(x + \rho\sigma\lambda h))(\sigma\lambda h)d\rho\Big)(h)d\tau\ d\sigma. \qquad (240)$$

Let $\overline{B}$ be a closed, convex, equilibrant, Bdd set such that $x + \lambda_0 B \subset \overline{B} \subset E_0$. Then as we have seen, $\overline{B} \times T_B$ is a compact subset of $E \times F$. Since T' is continuous, $T'(\overline{B} \times T_B)$ is a compact subset of $\mathcal{L}(E \times F, \mathcal{L}(E,F))$ and hence it is an equicontinuous set. Thus if V is a tonneau in F, there exists a tonneau $U \times W$ in $E \times F$ such that

$$T'(\overline{B} \times T_B)(U \times W)(B) \subset V \quad . \tag{241}$$

On the other hand, since the range of T is equicontinuous, there is a tonneau U_0 in E such that

$$T(E_0 \times F_0)(U_0) \subset W \quad . \tag{242}$$

Thus, if $0 < \lambda_1 \leq \lambda_0$ and $\lambda_1 B \subset U \cap U_0$, then if $|\lambda| \leq \lambda_1$ and $h \in B$, it follows from (240), (241), (242) and th.(1.2.4.4) that

$$\frac{f(x + \lambda h) - f(x)}{\lambda} - T(x, f(x))(h) \in V \quad . \tag{243}$$

Thus we have shown that

$$f'(x) = T(x, f(x)) \quad . \tag{244}$$

This completes the existence part of the proof. The uniqueness can be shown in several ways. Perhaps the simplest is to remark that the function h has been shown to be unique. ∎

4.6.2

We conclude this section with two examples. The first considers the situation in which $F = R$ so the range of T is the dual space of E, and the second is an example of the most general situation with $E = F = \mathcal{D}$.

4.6.3 Example

Let $E = \mathcal{D}$, $F = R$, $E_1 = V(\{m_\nu\}, \{\epsilon_\nu\})$ with $m_1 \geq 1$, $F_0 = [a,b]$. Then $T: E_1 \times F_0 \longrightarrow \mathcal{L}(E,R) = \mathcal{D}'$, and specifically, we consider

$$T(x,y)(z) = \alpha(y)\left\{a\int_{-\infty}^{\infty} x'(\sigma)z'(\sigma)d\sigma + b\int_{-\infty}^{\infty} x(\sigma)z(\sigma)d\sigma\right\} \qquad (245)$$

where $\alpha: F_o \longrightarrow R$ is a differentiable function. Using the concepts introduced in Section (1.2.8.2), we can describe this situation in a more compact notation. For each $x \in \mathcal{D}$, $\nu = 0,1,\ldots$, let $D_x^{\nu} \in \mathcal{D}'$ be the ν^{th} derivative of the distribution corresponding to the function x. Then we may write,

$$T(x,y) = \alpha(y)\{-a\, D_x' + b\, D_x\} \quad . \qquad (246)$$

It follows directly from the definitions that T,T' are continuous. Let $z \in E_1 \times F_o$. Then for all $x \in E_1$, $y \in F_o$, we have

$$|T(x,y)(z)| \leq \sup_{y \in F_o} |\alpha(y)| \left\{ |a|\, \epsilon_o \int_{-\infty}^{\infty} |z'(\sigma)|\, d\sigma + |b|\, \epsilon_o \int_{-\infty}^{\infty} |z(\sigma)|\, d\sigma \right\} \qquad (247)$$

and since the right-hand side of (247) does not depend on x,y we have shown that $T(E_1 \times F_o)$ is simply Bdd. Let us turn to the symmetry condition. We have

$$\frac{\partial T}{\partial E}(x,y)(h)(k) = \alpha(y)\left\{a\int_{-\infty}^{\infty} h'(\sigma)k'(\sigma)d\sigma + b\int_{-\infty}^{\infty} h(\sigma)k(\sigma)d\sigma\right\} \qquad (248)$$

and

$$\frac{\partial T}{\partial F}(x,y)(T(x,y)(h))(k) = \alpha'(y)\alpha(y)\left\{a\int_{-\infty}^{\infty} x'(\sigma)k'(\sigma)d\sigma + \right. \qquad (249)$$

$$\left. + b\int_{-\infty}^{\infty} x(\sigma)k(\sigma)d\sigma\right\}\cdot\left\{a\int_{-\infty}^{\infty} x'(\sigma)h'(\sigma)d\sigma + b\int_{-\infty}^{\infty} x(\sigma)h(\sigma)d\sigma\right\}.$$

Since both (248), (249) are symmetric in h,k, equation (190) is satisfied.

Thus the conditions of theorem (4.6.2) are satisfied and we may assert the existence and uniqueness of a function $f:E_o \longrightarrow F$ such that $f(0)$ is prescribed and

$$f'(x)(z) = \alpha(f(x)) \left\{ a \int_{-\infty}^{\infty} x'(\sigma)z'(\sigma)d\sigma + b \int_{-\infty}^{\infty} x(\sigma)z(\sigma)d\sigma \right\} \qquad (250)$$

or, according to (246), we have a non-linear, real valued function f, defined on a Nbd in $\mathcal{D}$ such that

$$f'(x) = \alpha(f(x)) \{-a\, D_x'' + b\, D_x\} \quad . \qquad (251)$$

It is easy to see how we can do the same thing for various other combinations of derivatives of D_x. The only thing that must be checked is the symmetry condition. That this condition will not always be satisfied is readily seen by considering

$$T(x,y) = D_x' \quad . \qquad (252)$$

4.6.4 Example

Let $E = F = \mathcal{H}$ (see Section (1.2.8.3)). Let $E_1 = F_o = V(1, \frac{1}{2})$ and we define $T:E_1 \times F_o \longrightarrow \mathcal{L}(E,F)$ by

$$\left[T(x,y)(z)\right](\xi) = x(\tfrac{\xi}{2})\, y(\tfrac{\xi}{2})\, z(\tfrac{\xi}{2}) \quad . \qquad (253)$$

Again the continuity of T,T' are easily verified. Let $z \in \mathcal{H}$, $x \in E_1$, $y \in F_o$, $|\xi| \leq 1$. Then

$$\left|\left[T(x,y)(z)\right]'(\xi)\right| = \frac{1}{2}\left| x'(\tfrac{\xi}{2})y(\tfrac{\xi}{2})z(\tfrac{\xi}{2}) + x(\tfrac{\xi}{2})y'(\tfrac{\xi}{2})z(\tfrac{\xi}{2}) + \right.$$

$$\left. + \; x(\tfrac{\xi}{2})y(\tfrac{\xi}{2})z'(\tfrac{\xi}{2})\right|$$

$$\leq \frac{1}{2} \sup_{|\xi| \leq \frac{1}{2}} \left| 2z(\xi) + z'(\xi) \right| \tag{254}$$

and since the right-hand side of (253) is independent of x,y, we have $T(E_1 \times F_o)$ is simply Bdd.

We turn to the symmetry condition.

$$\left[\frac{\partial T}{\partial E}(x,y)(h)(k)\right](\xi) = h(\tfrac{\xi}{2})\, y(\tfrac{\xi}{2})\, k(\tfrac{\xi}{2}) \tag{255}$$

and

$$\left[\frac{\partial T}{\partial F}(x,y)(T(x,y)(k))(h)\right](\xi) = x(\tfrac{\xi}{2})\, x(\tfrac{\xi}{2})\, y(\tfrac{\xi}{2})\, k(\tfrac{\xi}{2})\, h(\tfrac{\xi}{2}) . \tag{256}$$

Since both (254), (255) are symmetric, in h,k, then equation (190) is satisfied. Thus the conditions of theorem (4.6.2) are satisfied. In addition, let us compute E_o. It is clear from (252) that

$$T(E_1 \times F_o)(\tfrac{2}{3} E_1) \subset F_o . \tag{257}$$

Thus we may take $\tilde{E}_o = \frac{2}{3} E_1$ and $E_o = \frac{1}{6} E_1 = V(\frac{1}{6}, \frac{1}{2})$. Hence we may assert the existence and uniqueness of a function $f: V(\frac{1}{6}, \frac{1}{2}) \longrightarrow V(1, \frac{1}{2})$ such that f(0) is prescribed and

$$[f'(x)(z)](\xi) = x(\tfrac{\xi}{2})\, [f(x)](\tfrac{\xi}{2})\, z(\tfrac{\xi}{2}) . \tag{258}$$

5. Linear Differential Equations

5.1 Introduction

5.1.1

The existence theorems of the previous section do not solve every differential equation which has a solution. This is not surprising since the same situation exists in the classical theory. However, there are some rather simple cases in which our theorems do not apply. Consider the case in which $E = R$ and let us define $T: R \times F \longrightarrow F$ by

$$T(x, y) = y \ . \tag{1}$$

Theorem (4.4.1) does not apply and yet it will be shown below that there is a solution for every initial condition which is, in fact, unique.

5.1.2

The additional assumption of simple boundedness on the range of T is the difficulty which eliminates such problems as (1). In trying to remove this assumption, we turn in this section, to linear differential equations and the exponential function. The classical procedure of replacing a linear differential equation (or system) by a corresponding matrix equation is generalized by considering a matrix as defining a certain linear transformation. The exponential function is then defined for linear transformations by an infinite series. With these generalizations, much of the standard theory of linear differential equations can be developed for normed spaces.

5.1.3

In this section, we first generalize the notion of matrix of solutions (Theorems (5.2.1) and (5.2.2)). This, and all the theorems

of Sections 1, 2 and a symmetry condition, is enough to develop the classical variation of constants formula (Theorem (5.2.7)). The proof is formally the same as the classical one. The exponential function is defined and its properties are developed in lemmas (5.3.2), (5.3.5), (5.3.7), (5.3.8). Finally the uniqueness and existence of solutions of linear differential equations in the cases of "variable coefficients" (Theorem (5.4.3)) and "constant coefficients" (Theorem (5.4.4)) is proven.

5.2 The matrix of solutions

5.2.1

First we define some terms which express certain properties of differential equations in matrix language.

5.2.2

Let E,F be LCHVTs and E_1 a tonneau in E. Let $T: E_1 \times F \to \mathcal{L}(E,F)$ and for each $x \in E_1$ we define $T_x: F \to \mathcal{L}(E,F)$ by

$$T_x(y) = T(x,y) \quad . \tag{2}$$

If $T_x \in \mathcal{L}(F, \mathcal{L}(E,F))$ for all $x \in E_1$, then we shall say that T defines a linear differential equation. If T is independent of x, then we say that the differential equation has constant coefficients. Let $E_o \subset E_1$ be another tonneau in E. If for each Nbd of 0,V in F and for each $x \in E_o$ there exists a Nbd of 0,U in F such that if $c_1, c_2 \in F$ and $c_1 - c_2 \in U$, then there exist, independently of x, functions $y_{c_1}, y_{c_2}: E_o \to F$ satisfying the differential equation and such that

$$y_{c_1}(0) = c_1 , \quad y_{c_2}(0) = c_2$$

and

$$y_{c_1}(x) - y_{c_2}(x) \in V \quad , \tag{3}$$

and then we say that the differential equation has solutions which are continuous w.r.t. initial conditions.

Let $Y:E_o \longrightarrow \mathcal{L}(F,F)$. Then if Y' exists, we have $Y':E_o \longrightarrow \mathcal{L}(E,\mathcal{L}(F,F))$. We say that T possesses a solution matrix if such a function exists, and for all $h \in E$, $k \in F$

$$Y'(x)(h)(k) = [T_x Y(x)](k)(h) = T_x(Y(x)(k))(h) . \quad (4)$$

If in addition, Y(x) is 1-1 for all $x \in E_o$ and Y(0) is onto, we say that T possesses a fundamental matrix. The next two theorems relate these "matrices" to solutions of the differential equation.

5.2.3 Theorem

i) If T possesses a solution matrix, then there exists a subvector space $G \subset F$ such that for each $c \in G$, there is a solution of the differential equation y_c such that

$$y_c(0) = c . \quad (5)$$

ii) If T possesses a fundamental matrix then G = F and if $c_1 \neq c_2$ then $y_{c_1} - y_{c_2}$ never vanishes.

pf. Let Y be a solution matrix. We define G = Y(0)(F). G is a subvector space because Y(0) is linear. Let $c \in G$. Then there exists $d \in F$ such that Y(0)(d) = c. We define $y_c:E_o \longrightarrow F$ by

$$y_c(x) = Y(x)(d) . \quad (6)$$

Then, with the comment that limits in the following considerations are to be taken uniformly w.r.t. h in some Bdd subset of E, we have from (6), (4)

$$y_c'(x)(h) = \lim_{\lambda \to 0} \frac{y_c(x + \lambda h) - y_c(x)}{\lambda}$$

$$= \lim_{\lambda \to 0} \frac{Y(x + \lambda h)(d) - Y(x)(d)}{\lambda} = Y'(x)(h)(d)$$

$$= T_x(Y(x)(d))(h) = T_x(y_c(x))(h) \quad . \tag{7}$$

Further, from (6),

$$y_c(0) = Y(0)(d) = c \quad . \tag{8}$$

This proves the first assertion. Now suppose that Y is a fundamental matrix. Since Y(0) is onto and G = Y(0)(F), then G = F. Now let $c_1, c_2 \in F$ and suppose that there exists $x \in E_o$ such that

$$y_{c_1}(x) = y_{c_2}(x) \quad . \tag{9}$$

Then

$$Y(x)(c_1) = y_{c_1}(x) = y_{c_2}(x) = Y(x)(c_2) \tag{10}$$

and this contradicts the assumption that Y(x) is 1-1. ∎

5.2.4 Theorem

T possesses a fundamental matrix if and only if the differential equation has a solution for each initial condition, any two of which do not meet, and such that these solutions are continuous w.r.t. initial conditions.

pf. If T possesses a fundamental matrix, then everything but the continuity follows from th.1. Let V be a Nbd of 0 in F and let $x \in E_o$. Since $Y(x) \in \mathcal{L}(F,F)$ there is a Nbd of 0, U in F such that

$$Y(x)(U) \subset V \ . \tag{11}$$

Let $c_1, c_2 \in F$, $c_1 - c_2 \in U$. Then from (11),

$$y_{c_1}(x) - y_{c_2}(x) = Y(x)(c_1) - Y(x)(c_2) = Y(x)(c_1 - c_2) \in V \ . \tag{12}$$

Now let us assume that the functions $y_c(x)$ exist. We define the matrix Y by

$$Y(x)(c) = y_c(x) \ . \tag{13}$$

Equation (4) follows from reading equation (7) backwards, so Y is a solution matrix. Since $y_c(0) = c$ and since y_c exists for each c, Y(0) is onto. Similarly, if the discussion leading to equation (9) is read backwards, and since the functions y_c do not intersect each other, we obtain the fact that Y(x) is 1-1. Thus Y is a fundamental matrix. Finally, the argument at the beginning of this proof is also reversible so that Y(x) is continuous. ∎

5.2.5

We observe that one can show in the usual manner that the set of solutions of a linear differential equation is closed under addition and scalar multiplication. Also, if Y is a solution matrix and $c \in \mathcal{L}(F,F)$, then Y(x)c is a solution matrix, although cY(x) may not necessarily be one.

5.2.6

The next theorem derives the classical variation of constants formula. This formula gives the solution of a differential equation in explicit form when T is the sum of a linear term and a term which is constant on E_0 and when a solution matrix is known for the case in which the non-linear part of T is zero.

The only unusual condition is the symmetry condition on T which was needed in Section 4.

5.2.7 Theorem

Let E,F be Montel spaces, E_o a tonneau in E. Let $T:E_o \times F \to \mathcal{L}(E,F)$, $A:E_o \to \mathcal{L}(F,\mathcal{L}(E,F))$, and $B:E_o \to \mathcal{L}(E,F)$ such that

$$T(x,y) = A(x)(y) + B(x) \quad . \tag{14}$$

Let T satisfy equation (4.190) and let A possess a fundamental matrix Y such that Y^{-I} exists and is continuous (see equation 2.51). Let B,B' be continuous.

Then for each $c \in F$ there is a solution y_c of

$$y' = T(x,y) \tag{15}$$

such that

$$y_c(0) = c \quad . \tag{16}$$

Furthermore, we have explicitly,

$$y_c(x) = Y(x)Y^{-I}(0)(c) + Y(x) \int_0^x Y^{-I}(\xi)\, B(\xi)\, d\xi \quad . \tag{17}$$

Furthermore, for each x, $y_c(x)$ is continuous in c.

pf. First we must establish that the expression $\int_0^x Y^{-I}(\xi)\, B(\xi)\, d\xi$ is defined and has a derivative which is equal to $Y^{-I}(x)B(x)$. For this we will use theorem (3.3.8) and some results from Section 2. By th.(2.6.2), (2.6.7), and eq. (4), $(Y^{-I}B)'$ exists and

$$(Y^{-I}B)'(x)(h)(k) = [(Y^{-I})'(x)(h)](B(x)(k)) + Y^{-I}(x)(B'(x)(h)(k))$$

$$= - [Y^{-I}(x)Y'(x)\ Y^{-I}(x)](h)(B(x)(k)) + Y^{-I}(x)(B'(x)(h)(k))$$

$$= Y^{-I}(x)\ [B'(x)(h)(k) - (A(x)\ Y(x)\ Y^{-I}(x)(B(x)(k))(h)]$$

$$= Y^{-I}(x)\ [B'(x)(h)(k) - A(x)(B(x)(k))(h)]\ . \qquad (18)$$

Now using th.(2.4.1), (1.2.5.3) and eq. (4.190), (14) we have

$$A'(x)(h)(y)(k) + B'(x)(h)(k) + A(x)(A(x)(y)(h))(k) + A(x)(B(x)(h))(k) =$$

$$= A'(x)(k)(y)(h) + B'(x)(k)(h) + A(x)(A(x)(y)(k))(h)$$

$$+ A(x)(B(x)(k))(h)\ . \qquad (19)$$

In particular, for $y = 0$ we obtain

$$B'(x)(h)(k) + A(x)(B(x)(h))(k) = B'(x)(k)(h) + A(x)(B(x)(k))(h) \qquad (20)$$

or

$$B'(x)(h)(k) - A(x)(B(x)(k))(h) = B'(x)(k)(h) - A(x)(B(x)(h))(k). \qquad (21)$$

From (18) it then follows that

$$(Y^{-I}B)'(x)(h)(k) = (Y^{-I}B)'(x)(k)(h)\ . \qquad (22)$$

Thus equation (3.22) is satisfied and we may apply th.(3.1) to assert that $\int_0^x Y^{-I}(\xi)B(\xi)d\xi$ is defined and its derivative is $Y^{-I}(x)B(x)$ which, we note, is continuous. Thus the function y_c is well defined. By (17), (4) and th.(2.5),

$$y_c'(x) = Y'(x)Y^{-I}(0)(c) + Y'(x)\int_0^x Y^{-I}(\xi)B(\xi)d\xi + Y(x)Y^{-I}(x)B(x)$$

$$= A(x)(Y(x)(Y^{-I}(0)(c))) + A(x)(Y(x)(\int_0^x Y^{-I}(\xi)B(\xi)d\xi)) + B(x)$$

$$= A(x)\left[Y(x)Y^{-I}(0)(c) + Y(x)\int_0^x Y^{-I}(\xi)B(\xi)d\xi\right] + B(x)$$

$$= A(x)(y_c(x)) + B(x) = T(x,y_c(x)) \quad . \tag{23}$$

Eq. (16) follows immediately from (17). The continuity of $y_c(x)$ in c follows from the fact that the second term of (17) is independent of c and the first term of (17) is continuous in c. ∎

5.3 The exponential function

5.3.1

Now we shall use th.(5.2.3),(5.2.4) to strengthen our existence theory. We assume for the remainder of this section that T defines a linear equation. Let us consider first the constant coefficients case, i.e. T is constant on E; or

$$T \in \mathcal{L}(F,\mathcal{L}(E,F)) \quad . \tag{24}$$

In this case, th.(4.4.1) demands that for some tonneau F_o in F, $T(F_o)$ is a simply Bdd set. Let us see what this condition implies.

5.3.2 Lemma

Let $T \in \mathcal{L}(F,\mathcal{L}(E,F))$, F_o a Nbd of 0 in F such that $T(F_o)$ is a simply Bdd subset of $\mathcal{L}(E,F)$. Then for each $x \in E$, and for each Nbd of 0, V in F there exists a Nbd of 0, W in F and a scalar λ such that

$$W \subset V \tag{25}$$

and

$$T(W)(x) \subset \lambda W \quad . \tag{26}$$

If in addition, E is tonnelé, then x may be replaced by an arbitrary Bdd subset of E.

pf. Let $W = V \cap F_o$ so that (25) is immediately satisfied. Further, since $W \subset F_o$, $T(W)(x)$ is a Bdd subset of F and hence it is contained in some scalar multiple of W.

If E is tonnelé, then by th.(1.2.6.1), T(W) will be uniformly Bdd, i.e. if B is a Bdd subset of E, T(W)(B) is a Bdd subset of F and is again contained in a scalar multiple of W. ∎

5.3.3

The converse of this lemma is false. The function defined by (1) is a counterexample. As was remarked earlier, the assumption of simple boundedness is too strong. It will be shown that a strictly weaker condition, i.e. the conclusion of lemma (5.3.2) suffices in the linear case.

5.3.4

We now proceed to define the exponential function and develop some of its properties. If $A \in \mathcal{L}(F,F)$, we define $e^A \in \mathcal{L}(F,F)$ formally by

$$e^A(u) = \sum_{i=0}^{\infty} A^i(u) \qquad (27)$$

where

$$A^0 = \text{identity}, \qquad A^{i+1} = AA^i . \qquad (28)$$

The next lemma gives an important sufficient condition for the convergence of the sum in (27). This condition is, essentially, the conclusion of lemma (5.3.2).

5.3.5 Lemma

Let F be complete, $A \in \mathcal{L}(F,F)$ such that for each Nbd of 0, V in F there exists a Nbd U of 0 in F and a scalar λ such that

$$U \subset V, \qquad A(U) \subset \lambda U \ . \tag{29}$$

Then e^A is defined.

pf. Let $u \in F$, V a convex Nbd of 0 in F. Then we have a corresponding U, λ satisfying (29). Let $\lambda_0 > 0$ be a scalar such that

$$u \in \lambda_0 U \ . \tag{30}$$

From the convergence of the numerical series for e^λ, we can find an integer N such that

$$\sum_{i=N}^{\infty} \frac{\lambda^i}{i!} < \frac{1}{\lambda_0} \ . \tag{31}$$

Then if $m > n \geq N$ it follows from (29), (30), and (31) that

$$\sum_{i=0}^{m} \frac{1}{i!} A^i(u) - \sum_{i=0}^{n} \frac{1}{i!} A^i(u) = \sum_{i=n+1}^{m} \frac{1}{i!} A^i(u) \in \lambda_0 \sum_{i=n+1}^{m} \frac{\lambda^i}{i!} U \subset$$

$$\subset \left(\lambda_0 \sum_{i=n+1}^{m} \frac{\lambda^i}{i!}\right) U \subset \left(\lambda_0 \sum_{i=N}^{\infty} \frac{\lambda^i}{i!}\right) U \subset$$

$$\subset \lambda_0 \frac{1}{\lambda_0} U = U \subset V \ . \tag{32}$$

Thus the sequence of partial sums is Cauchy and since F is complete, the series converges. The linearity of e^A follows from the linearity of each A^j and the fact that the linear operations, being continuous, will commute with the summation sign. For the continuity we have, if $u \in U$,

$$e^A(u) = \sum_{i=0}^{\infty} \frac{1}{i!} A^i(u) \in \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} U \subset e^\lambda U \ . \tag{33}$$

Thus $e^A \in \mathcal{L}(\Gamma, F)$. ∎

5.3.6 Corollary

Let F be complete, $A \in \mathcal{L}(F,F)$ such that there exists a Nbd of F_o in F such that $A(F_o)$ is a Bdd set.

Then e^A is defined.

pf. This follows from lemma (5.3.5) and an argument almost identical to that used in lemma (5.3.2).

The first property that we wish to establish for the exponential function is the addition formula.

5.3.7 Lemma

Let F be complete, $A_1, A_2 \in \mathcal{L}(F,F)$ such that A_1 and A_2 satisfy the conditions of lemma 2, and

$$A_1 A_2 = A_2 A_1 \quad . \tag{34}$$

Then $e^{A_1+A_2}$ is defined and

$$e^{A_1+A_2} = e^{A_1} e^{A_2} = e^{A_2} e^{A_1} \quad . \tag{35}$$

pf. Let V be a Nbd of 0 in F and let $u \in F$. Then corresponding to A_1 we have U_1, λ_1 such that

$$U_1 \subset V \quad , \qquad A_1(U_1) \subset \lambda_1 U_1 \quad . \tag{36}$$

Then corresponding to A_2 and U_1, we have U_2, λ_2 such that

$$U_2 \subset U_1 \subset V \quad , \qquad A_2(U_2) \subset \lambda_2 U_2 \quad . \tag{37}$$

Let $\mu > 0$ such that

$$u \in \mu U_2 \quad . \tag{38}$$

It follows from the properties of the exponential function for real numbers that

$$\sum_{i=0}^{\infty} \frac{(\lambda_1+\lambda_2)^i}{i!} = e^{\lambda_1} e^{\lambda_2} \quad . \tag{39}$$

Therefore there exists an integer N such that

$$\left| e^{\lambda_1} e^{\lambda_2} - \sum_{i=0}^{N} \frac{(\lambda_1+\lambda_2)^i}{i!} \right| \leq \frac{1}{\mu} \quad . \tag{40}$$

Now, from (34) we may compute

$$e^{A_1} e^{A_2} - \sum_{i=0}^{N} \frac{(A_1 + A_2)^i}{i!} = e^{A_1} \sum_{j=0}^{\infty} \frac{A_2^j}{j!} - \sum_{i=0}^{N} \sum_{j=0}^{i} \frac{A_1^j A_2^{i-j}}{j!(i-j)!}$$

$$= \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{A_i^i A_2^j}{i!\, j!} - \sum_{i=0}^{N} \sum_{j=0}^{i} \frac{A_1^j A_2^{i-j}}{j!(i-j)!}$$

$$= \sum_{j=0}^{N} \sum_{i=N+i-j}^{\infty} \frac{A_1^i A_2^j}{i!\, j!} + \sum_{i=0}^{\infty} \sum_{j=N+1}^{\infty} \frac{A_1^i A_2^j}{i!\, j!} \quad . \tag{41}$$

Therefore, if we set

$$\Delta = e^{A_1} e^{A_2}(u) - \sum_{i=0}^{N} \frac{(A_1 + A_2)^i}{i!}(u) \quad . \tag{42}$$

Then we have

$$\Delta \in \mu \sum_{j=0}^{N} \sum_{i=N+i-j}^{\infty} \frac{\lambda_2^j}{j!\,i!} A_1^i(U_2) + \mu \sum_{j=0}^{\infty} \sum_{j=N+1}^{\infty} \frac{\lambda_2^j}{j!\,i!} A_1^i(U_2)$$

$$\subset \mu \sum_{j=0}^{N} \sum_{i=N+i-j}^{\infty} \frac{\lambda_2^j}{j!\,i!} A_1^i(U_1) + \mu \sum_{i=0}^{\infty} \sum_{j=N+1}^{\infty} \frac{\lambda_2^j}{j!\,i!} A_1^i(U_1)$$

$$\subset \mu \left(\sum_{j=0}^{N} \sum_{i=N+i-j}^{\infty} \frac{\lambda_1^i \lambda_2^j}{i!\, j!} + \sum_{i=0}^{\infty} \sum_{j=N+1}^{\infty} \frac{\lambda_1^i \lambda_2^j}{i!\, j!} \right) U_1$$

$$= \mu \left(e^{\lambda_1} e^{\lambda_2} - \sum_{i=0}^{N} \frac{(\lambda_1+\lambda_2)^i}{i!} \right) U_1 \subset \mu \frac{1}{\mu} U_1 = U_1 \subset V \quad . \tag{43}$$

Thus

$$e^{A_1+A_2} = \sum_{i=0}^{\infty} \frac{(A_1+A_2)^i}{i!} = e^{A_1} e^{A_2} . \tag{44}$$

The other half of (35) is proved in a similar manner. ∎

Now we suppose that A is a function whose range is $\mathcal{L}(F,F)$. The last property of the exponential function which we will need is the derivative of $e^{A(x)}$ in terms of the derivative of A(x).

5.3.8 Lemma

Let E be a Montel space, F complete, and E_o a tonneau in E. Let $A:E_o \longrightarrow \mathcal{L}(F,F)$ such that A' exists and is continuous, and for each $x \in E_o$, A(x) satisfies the conditions of lemma (5.3.5) uniformly for x in any Bdd subset of E_o. Let

$$A(x_1)\,A(x_2) = A(x_2)\,A(x_1) \tag{45}$$

for all $x_1, x_2 \in E_o$.

Then $\left[e^{A(x)}\right]'$ exists and

$$\left[e^{A(x)}\right]' = e^{A(x)}\,A'(x) = A'(x)\,e^{A(x)} . \tag{46}$$

pf. By lemma (5.3.7), where I is the identity map,

$$\begin{aligned} e^{A(x+\lambda h)} - e^{A(x)} &= e^{A(x)}\left[e^{A(x+\lambda h)-A(x)} - I\right] \\ &= e^{A(x)} \sum_{i=1}^{\infty} \frac{1}{i!} \left[A(x+\lambda h) - A(x)\right]^i \\ &= e^{A(x)} \left[A(x+\lambda h)-A(x)\right] + \\ &\quad + e^{A(x)} \left[A(x+\lambda h)-A(x)\right]^2 \sum_{i=2}^{\infty} \frac{1}{i!} \left[A(x+\lambda h)-A(x)\right]^{i-2}. \end{aligned} \tag{47}$$

Thus if we set

$$\Delta = \frac{e^{A(x+\lambda h)} - e^{A(x)}}{\lambda} - e^{A(x)} A'(x)(h) \quad , \tag{48}$$

then from (47), (48) and th.(1.2.5.6)

$$\Delta = e^{A(x)} \left[\frac{A(x+\lambda h)-A(x)}{\lambda} - A'(x)(h) \right] +$$

$$+ \lambda e^{A(x)} \left[\int_0^1 A'(x+\lambda th)(h)dt \right]^2 \sum_{i=0}^{\infty} \frac{[A(x+\lambda h)-A(x)]^i}{(i+2)!} \quad . \tag{49}$$

Now let B be a Bdd equilibrant subset of E, and let $\langle B_1, V \rangle$ be a Nbd of 0 in $\mathscr{L}(F,F)$. That is, B_1 is Bdd in F and V is a Nbd of 0 in F. Since $x + B$ is again a Bdd subset of E, there is a Nbd of 0, U in F and a scalar ρ such that

$$U \subset V \tag{50}$$

and if $\xi \in x + B$, then

$$A(\xi)(U) \subset \rho U \quad . \tag{51}$$

Let α be a scalar such that

$$B_1 \subset \alpha U \quad . \tag{52}$$

Then if $u \in B_1$, it follows from (50), (51), (52) that

$$\sum_{i=0}^{\infty} \frac{[A(x+\lambda h)-A(x)]^i}{(i+2)!}(u) = \alpha \sum_{i=0}^{\infty} \sum_{j=0}^{i} \binom{i}{j} \frac{A^j(x+\lambda h)A^{i-j}(x)}{(i+2)!} \left(\frac{u}{\alpha}\right)$$

$$\subset \alpha \sum_{i=0}^{\infty} \sum_{j=0}^{i} \binom{i}{j} \frac{\rho^i}{(i+2)!} U \subset \frac{\alpha}{2} e^{\rho} U \subset \frac{\alpha}{2} e^{\rho} V \, . \tag{53}$$

Since (53) is valid for any V, (with α, ρ perhaps varying) it follows that

$$B_2 = \sum_{i=0}^{\infty} \frac{[A(x+\lambda h)-A(x)]^i}{(i+2)!} (B_1) \tag{54}$$

is a Bdd subset of F.

Further, since A' is continuous and E is a Montel space, it follows from th(1.2.7.4) that A'(x+B) is a Bdd subset of $\mathcal{L}(E,\mathcal{F}(F,F))$. Hence, if $|\lambda| \leq 1$, and if we set

$$B_3 = \left(\int_0^1 A'(x+\lambda tB)(B)\,dt\right) (B_2) \tag{55}$$

and

$$B_4 = \left(\int_0^1 A'(x+\lambda tB)(B)\,dt\right) (B_3) \tag{56}$$

then B_3, B_4 are Bdd subsets of F. Further, if we set

$$B_5 = e^{A(x)} (B_4) \tag{57}$$

then since $e^{A(x)}$ is linear and continuous, it follows from th(1.2.7.4) that B_5 is also a Bdd subset of F. Thus, if $h \in B$, $|\lambda| \leq 1$, $u \in B_1$ and

$$\Delta_1 = \lambda\, e^{A(x)}\left[\int_0^1 A'(x+\lambda th)(h)dt\right]^2 \sum_{i=0}^{\infty} \frac{[A(x+\lambda h)-A(x)]^i}{(i+2)!} (u)\,, \tag{58}$$

then

$$\Delta_1 \in \lambda(B_4,B_5)(B_3,B_4)(B_2,B_3)(B_1,B_2)\ (B_1) \subset \lambda\, B_5\,. \tag{59}$$

Let $\lambda_0 > 0$ such that if $|\lambda| \leq \lambda_0$, then

$$\lambda B_5 \subset \frac{1}{2} V\,. \tag{60}$$

Now we turn to the first term of Δ. From the continuity of $e^{A(x)}$, there exists a Nbd of 0, V_1 in F such that

$$e^{A(x)}(V_1) \subset \frac{1}{2} V\,. \tag{61}$$

Further, there exists a $\lambda_1 > 0$ such that if $|\lambda| \leq \lambda_1$ $h \in B$, then

$$\frac{A(x+\lambda h)-A(x)}{\lambda} - A'(x)(h) \in \langle B_1, V_1 \rangle \quad . \tag{62}$$

Thus if $|\lambda| \leq \min(1, \lambda_0, \lambda_1)$ and $h \in B$, it follows from (59), (60), (61), (62), (49) that

$$\Delta \in \langle V_1, \tfrac{1}{2} V \rangle \langle B_1, V_1 \rangle + \langle B_1, \tfrac{1}{2} V \rangle \subset \langle B_1, V \rangle \quad . \tag{63}$$

This shows that

$$\left[e^{A(x)}\right]' = e^{A(x)} A'(x) \quad . \tag{64}$$

The other part of (46) is proven in an analogous manner. ∎

5.4 Linear differential equations

5.4.1

We are now ready to apply these results to linear differential equations. We have $T: E_0 \times F \longrightarrow \mathcal{L}(E,F)$ and T is linear and continuous on F so that if $x \in E_0$, $h \in E$, then

$$T(x, \cdot)(h) \in \mathcal{L}(F,F) \quad . \tag{65}$$

We also make the following notational convention,

$$T^2(x, \cdot)(h) = T(x, T(x, \cdot)(h))(h) \quad . \tag{66}$$

5.4.2

In ch.(5.4.6), we will have a special symmetry condition which embodies both the symmetry demanded in the theorems of Section 4 and the commutativity assumed in lemma (5.3.8). In the special case in which the differential equation has constant coefficients, we shall see in theorem (5.4.4) that the old symmetry condition takes care of both.

5.4.3 Theorem

Let E be a Montel space, F complete, and E_0 a tonneau in E. Let $T: E_0 \times F \longrightarrow \mathcal{L}(E,F)$ such that T defines a linear differential equation, T,T' are continuous and

$$\text{i)}\quad \frac{\partial T}{\partial E}(x,y)(h)(k) = \frac{\partial T}{\partial E}(x,y)(k)(h) \tag{67}$$

$$\text{ii)}\quad T(tx_1, T(sx_2,y)(x_2))(x_1) = T(sx_2, T(sx_1,y)(x_1))(x_2)$$

for all $s,t \in [0,1]$, $x_1,x_2 \in E_o$, $y \in F$, $h,k \in E$.

Let the map $T(tx, \cdot)(x)$ satisfy the conditions of lemma (5.3.5) uniformly w.r.t. x in any Bdd subset of E_o and $t \in [0,1]$.

Then the differential equation has a solution for each initial condition, these solutions are continuous w.r.t. initial conditions, and the fundamental matrix is given by

$$Y(x) = e^{\int_0^x T(\xi,\cdot)\,d\xi} . \tag{68}$$

pf. Let $A: E_o \longrightarrow \mathcal{L}(F,F)$ be defined by

$$A(x) = \int_0^x T(\xi,\cdot)\,d\xi . \tag{69}$$

By i) and the fact that T' is continuous, it follows from th.(5.3.1) that A is defined, A' exists and

$$A'(x) = T(x,\cdot) \tag{70}$$

and since T is continuous, A' is continuous. Further, we may write by th.(1.2.5.1)

$$A(x) = \int_0^1 T(tx,\cdot)(x)\,dt \tag{71}$$

so A satisfies the conditions of lemma (5.3.5) uniformly w.r.t. x in any Bdd subset of E_o.

Let $x_1,x_2 \in E_o$. Then from i), and the fact that T is linear and continuous,

$$A(x_1)\,A(x_2) = \int_0^{x_1} T(\xi_1,\cdot)\,d\xi_1 \int_0^{x_2} T(\xi_2,\cdot)\,d\xi_2$$

$$= \int_0^{x_1} T(\xi_1, \int_0^{x_2} T(\xi_2,\cdot)d\xi_2)\,d\xi_1$$

$$= \int_0^1 T(tx_1, \int_0^1 T(sx_2,\cdot)(x_2)ds)\,(x_1)dt \quad .$$

$$= \int_0^1 \int_0^1 T(tx_1, T(sx_2,\cdot)(x_2))\,(x_1)ds\,dt$$

$$= \int_0^1 \int_0^1 T(tx_2, T(sx_1,\cdot)(x_1))\,(x_2)ds\,dt$$

$$= A(x_2)\,A(x_1) \quad . \tag{72}$$

Hence by lemma (5.3.5) and (70), we have

$$e^{A(x)}{}' = A'(x)\,e^{A(x)} = T(x,\cdot)\,e^{A(x)} = T(x, e^{A(x)}) \; . \tag{73}$$

Thus if Y is defined according to (68), then

$$Y'(x) = T(x, Y(x)) \tag{74}$$

and the desired result follows from th.(5.2.4). ∎

Finally, we have, for the constant coefficients case,

5.4.4 Theore

Let E be a Montel space, F complete, and E_0 a tonneau in E. Let $T: E_0 \times F \longrightarrow \mathcal{L}(E,F)$ such that T defines a linear differential equation with constant coefficients (i.e. T is constnat on E_0). Let $T(\cdot)(h)$ satisfy the conditions of lemma (5.3.5) uniformly w.r.t. h in any Bdd subset of E. Let T satisfy the symmetry condition given by equation (4.190). Then the differential equation has a solution on E_0 for each initial condition,

these solutions are continuous w.r.t. initial conditions and the fundamental matrix is given by

$$Y(x) = e^{T(\cdot)(x)} \quad . \tag{75}$$

pf. By th.(1.2.5.3), T' is constant so it is continuous. Also, T is continuous. Since T is constant on E_0, condition i) of th.(5.4.3) is trivially satisfied. Equation (4.190) gives

$$0 + T(T(y)(k))(h) = 0 + T(T(y)(h))(k) \quad . \tag{76}$$

Condition ii) of th.(5.4.3) in this case is

$$T(T(y)(x_2))(x_1) = T(T(y)(x_1))(x_2) \tag{77}$$

so it, too, is satisfied. Finally,

$$T(x,\cdot)(x) = T(\cdot)(x) \tag{78}$$

so all the conditions of th.(5.4.3) are satisfied and its conclusion gives the desired result with the observation that

$$\int_0^x T(\xi,\cdot)\,d\xi = \int_0^x T(\cdot)\,d\xi = T(\cdot)(x) \quad . \tag{79}$$

We remark that if $E = R$, then the only auxiliary condition which must be assumed is ii) of th.(5.4.3).

5.4.5

We close with two examples of linear differential equations which can now be solved, but to which the theorems of Section 4 do not apply.

5.4.6 Example

Let $E = R$ and let F be a complete LCTVT. Let E_0 be an interval containing 0 in the interior and let $\alpha : E_0 \longrightarrow E$ be any continuously differentiable function. We define $T: E_0 \times F \longrightarrow F$ by

$$T(x,y) = \alpha(x)y \quad . \tag{80}$$

T is clearly continuous. We have

$$T'(x,y)(s,z) = \alpha'(x)\,sy + \alpha(x)\,z \quad , \tag{81}$$

so T' is continuous. Also,

$$\frac{\partial T}{\partial E}(x,y)(r)(s) = \alpha'(x)sry = \frac{\partial T}{\partial E}(x,y)(s)(r) \tag{82}$$

so condition (67) i) is satisfied.

$$T(tx_1, T(sx_2,y)(x_2))(x_1) = \alpha(tx_1)\,\alpha(sx_2)\,x_2x_1y$$

$$= T(tx_2, T(sx_1,y)(x_1))(x_2) \tag{83}$$

which verifies (67) ii).

Finally,

$$T(tx,y)(x) = \alpha(tx)y \quad , \tag{84}$$

so if x is in a Bdd set (finite interval) and $t \in [0,1]$, $\alpha(tx)$ is Bdd since α is continuous and this bound is just the scalar demanded by lemma (5.3.5), with U = V.

Thus th.(5.4.3) can be applied to solve this differential equation.

We remark that if $\alpha(x) = 1$ for all x, then this is the equation described by (1) and for which the solution was proposed.

Further, we can write the solution

$$y_x(x) = e^{\int_0^x \alpha(t)dt}(e) \quad . \tag{85}$$

5.4.7 Example

Let $E = F = \mathcal{D}$, E_0 any convex in E. If we define $T: E_0 \times \Gamma \to \mathcal{L}(E,F)$ by

$$T(x,y)(z)(t) = y(t)z(x) \tag{86}$$

then we are in the constant coefficients case and th. (5.4.4) may be used. We have

$$T(y)(h) = hy \quad . \tag{87}$$

Let $V = V(\{\varepsilon_\nu\}, \{\varepsilon_\nu\})$ be a Nbd of 0 in $\mathcal{D}$ and let $B = B(K,\{M_\nu\})$ where $K > 0$ be a Bdd set in $\mathcal{D}$ (i.e. B consists of functions which

are supported by [-K,K] and have the ν^{th} derivative Bdd by M_ν).

Let ν_0 be the smallest integer greater than or equal to K and let

$$M = \operatorname{Max}_{\nu \leq \nu_0} M_\nu \ . \tag{88}$$

Suppose $\nu < \nu_0$ and $|\cdot| \in [\nu,\nu_0]$. Then if $h \in B$, $y \in V$,

$$\left|\frac{d^{m_\nu}}{dt^{m_\nu}}(hy)(\cdot)\right| \quad \left|\sum_{\text{[illegible]}}^{m_\nu} \binom{m_\nu}{\cdot} \left(\frac{d^{m_\nu-\cdot}}{dt^{m_\nu-\cdot}} h(\cdot)\right)\left(\frac{d^{\cdot}}{dt^{\cdot}} y(\cdot)\right)\right|$$

$$\leq M^{\cdot\nu} \sup_{|\cdot|\leq\nu} \left|\frac{d^{\cdot}}{dt^{\cdot}} y(\cdot)\right| \leq 2^{m_{\nu_0}} M c_\nu \ . \tag{89}$$

On the other hand, if $|\cdot| \geq \nu_0 \geq K$, then

$$\frac{d^{\cdot\nu}}{dt^{\cdot\nu}}(hy)(\cdot) = \ . \tag{90}$$

Thus, if $h \in B$, $y \in V$, then

$$\leq 2^{m_{\nu_0}} M V \ . \tag{91}$$

Thus the conditions of Lemma (8.6.7) are satisfied uniformly w.r.t. $h \in B$. The other conditions of theorem (8.6.4) are easy to verify and we consider only the symmetry condition in the form (7.6),

$$T(T(y)(h))(x) = hy = hy = T(T(y)(h))(x) \ . \tag{92}$$

Thus theorem (8.6.4) may be applied and we can assert the existence of solutions. In fact, we may write

$$y(x) = Y(x)(e) = e^{T(\cdot)(x)}(e) \tag{93}$$

or

$$y_e(x) = e^{xx} \tag{94}$$

where e^x, $x \in \mathcal{O}$ can be considered to be defined by (93) or by the series defining the exponential function

$$ce^x = \sum_{i=0}^{\infty} \frac{1}{i!} cx^i \quad . \tag{95}$$

Note that $c \in \mathcal{D}$. If not, e^x itself is not necessarily defined, i.e. the series (95) may not converge.

We remark that example (5.4.7) can be considerably generalized by replacing (87) by

$$T(x,y)(z) = \left(\frac{d^\nu}{dt^\nu}(x)\right)\left(\frac{d^\nu}{dt^\nu}(z)\right) y \quad . \tag{96}$$

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