ARBITRATION SCHEMES

FOR

GENERALIZED TWO-PERSON GAMES

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Section I

Introduction

1.1 John von Neumann and Oscar I. Morgenstern in (1) analyze the discrete zero-sum two-person game and indicate that to each such game their exists a unique imputation to the players and a pair of convex sets of optimal strategies. The solution of this type of game has a strong stability in the sense that if both players are made aware of a saddle point (in the generalized space of mixed strategies) there is no tendency for either player to act out of conformity with the predicted mathematical solution. However, in the two-person non-zero-sum game, no attempt is made in (1) to establish the solution as a unique imputation, but the solution is defined as a family of imputations which are characterized by a set of requirements. It is felt that this multiplicity of solutions is in harmony with realizations of this mathematical model (eg., the duopoly or the duposony behavior in economics). The actual solution in a given real life situation depends on the psychological behavior of the players in this bargaining problem. In essence, we are faced with the problem of the two players choosing a "best" element from a set of incomparable elements. The utter futility of requiring a unique solution in this case often results empirically in an imputation to the two players which is not an element of the solution set and which is actually dominated by every imputation of the solution set. In situations of this type it behooves the players to offer their case for arbitration. To be objective, the arbiter must seek an imputation from the solution set which satisfies a set of so-called "fair" requirements which is dictated by the sociological, economic or political environment in which the game is embedded. It is only in this extended sense that a unique imputation

will ever be associated, in this paper, to all two-person discrete non-zero-sum games. Each methodology for obtaining a unique imputation of the solution set will be called a convention. Each convention will be characterized mathematically by a transformation. In a given society the convention is usually institutionalized No attempt will be made on the formal level to single out a "best" convention; any preference for a given convention will be purely subjective and rationalizations for the use of a particular convention will be on the metatheoretic level.

1.2 We now cite a rough analysis of a classical economic approach to this problem. Consider a two-person market where each player has initially a given amount of two homogeneous and infinitely divisible goods g_1 and g_2. The vector quantities (x_1, x_2) where x_i denotes the quantity of good g_i are simply ordered, according to preference, by each individual. Each player decomposes the set of (x_1, x_2) into equivalence classes or indifference curves. We now define a set Ω which is characterized by the following properties: 1) for any (x_1, x_2) there exists an $(x_1', x_2') \in \Omega$ which compositely dominates (x_1, x_2), i. e., both players either prefer (x_1', x_2') to (x_1, x_2), or both are indifferent, or one prefers and the other is indifferent; and 2) no element of Ω is compositely dominated by another element of Ω. The subset of Ω, the elements of which compositely dominate the initial market condition of the players, is commonly called the "socially optimum" set and is analogous to von Neumann's solution set (the initial point corresponds to the minimum amounts both players could get regardless of the behavior of their opponents). The economist is called upon to establish a market institution by means of which a social optimum may be reached. One possible approach to this problem is to establish a means of trade by a price structure with the accompanying supply and demand arguments. Essentially, the market price of the goods establishes a common unit of measurement and enables the individuals to arrive at a unique imputation of the socially optimum

set. This analysis serves later as a motivation for plausibility requirements of a solution of the general two-person game.

1.3 As in most models, both in the social and physical sciences, the measurement problem is extremely important. J. von Neumann and O. I. Morgenstern in (1) discuss the nature of the elements of the payoff matrix. Corresponding to a pair of pure strategies each player associates a certain utility value to the outcome. It is assumed that each player can simply order these utility values and a weighted combination of two utility values is comparable to a third. From these requirements the elements of the payoff matrix can be represented by real numbers whose relations have significance up to the group of transformations $a\alpha + b$ where $a > o$. There has been some controversy whether numerical utilities are actually justified in replacing the indifference curves of the classical economic theory. Jacob Marschak in (1), discusses the measurement problem and also arrives at numerical utilities (defined up to the group of positive linear transformations) but the approach used by Marschak is more "transparent and the postulates have a more immediate intuitive meaning by which their appropriateness may be judged directly". The economist, Paul A. Samuelson, asserted that it is desirable to make investigations using numerical utilities and that some attention should be given to non-transferable utilities. Accordingly, in this paper, we shall use numerical utilities which may be transferable or non-transferable.

In a given game, let the payoff be (a_{ij}, b_{ij}) where a_{ij} and b_{ij} are the respective payoffs to player I and II, corresponding to the pair of pure strategies (i, j). The question then arises whether the theory should be invariant up to independent linear transformations on the a_{ij} and the b_{ij}, or must one use the same linear transformation on the a_{ij} and b_{ij}. For example, consider

the "matching pennies" game, viz:

$$\begin{array}{cc} & \text{II} \\ & \begin{array}{cc} \text{H} & \text{T} \end{array} \\ \text{I} \begin{array}{c} \text{H} \\ \text{T} \end{array} & \begin{bmatrix} (1, -1) & (-1, 1) \\ (-1, 1) & (1, -1) \end{bmatrix} \end{array}$$

where, for example, if player I chooses H and player II chooses T then the return to players I and II are - 1 and + 1 respectively. Now multiply player II's scale by the factor of 2 giving

$$\begin{array}{c} \\ \begin{array}{cc} \text{H} & \text{T} \end{array} \\ \begin{array}{c} \text{H} \\ \text{T} \end{array} \begin{bmatrix} (1, -2) & (-1, 2) \\ (-1, 2) & (1, -2) \end{bmatrix} . \end{array}$$

Certainly the structure of this game has been altered if bribery is permissible. In the game with a transferable utility a common unit of measurement must be postulated and then independent linear transformations would destroy that common unit. If bribery were not permitted in the above game the linear transformations would change the game from a zero-sum game into a non-constant-sum game. Accordingly, in the sequel, the concept of linear and non-linear games are introduced and this concept remains invariant under such independent linear transformations.

In this report two distinct cases will be considered: the existence or non-existence of a common unit of measurement. In the case of a transferable utility, a common unit is hypothesized; however, we shall also consider the possibility of the existence of a common unit of measurement where bribery is not permissible. For example, the entries may be in monetary units but physical barriers might require that no exchange take place between the individuals. A common unit might arise if the entries were in psycho-physical units. Or perhaps a common unit might be defined by the introduction of idealized stimuli which

serve to normalize the player's scales. In order not to prejudge the problem the models introduced will incorporate all such possibilities. The theory of the non-zero-sum game as developed by von Neumann and Morgenstern uses a transferable utility and the theory of John Nash,Jr., (2),(3),(4), deals exclusively with the non-existence of a common unit of measurement.

1.4 Let us consider a two person non-zero-sum game in which we allow bribery (transferable utility and common unit of measurement). In this case the imputations of von Neumann and Morgenstern's solution set have the property that the sum of the returns is a constant and equals the largest sum of the payoff entries. Now, say we are successful in developing a dynamic model of the bargaining process, and that eventually the bargaining terminates (this is questionable if there is no negative utility for time consumption) at an imputation of the solution set. Further, let us assume that the "psychologies" of the players are to be given in the dynamic model by a set of parameters, whose domain forms a set P_i for player i. Hence, if we keep the original non-zero sum game fixed we have a mapping (dictated by the dynamic model) from $P_1 \times P_2$ to the solution set. This mapping in turn forms a <u>constant sum</u> game, with perhaps an infinite number of strategies, which may be possible to handle by the techniques of A. Wald, (1). If the players are rational and fully informed, and if the induced game is strictly determined in the sense of Wald, then the value of this game yields a unique imputation of the solution set. If the original non-zero-sum game is symmetric (to be defined later) and if $P_1 \equiv P_2$ then the induced constant-sum game should be symmetric and one would expect the value of the game to each player to be the same. At best the above argument is merely a rationalization for plausibility solutions of symmetric games in the pre-formal stage.

1.5 We will assume that eabh player's primary motivation is to maximize his own return. In the analysis of a given game a contestant might threaten to minimize his opponent's return or to maximize his own relative return, but this is done with the ulterior motive of ultimately maximizing his own absolute return. However, in the case there is a multiplicity of strategies that can be employed, subject to the above maximization principle, we will differentiate three possible alternatives:

a) chance device,

b) minimize opponent's return,

c) maximize opponent's return.

It might be argued that it would be wise for the players to use case c), since in subsequent games this overall philosophy would serve to increase the player's long range expected return which should be his sole objective. However, if viewed in the context that this particular game is embedded in a large random set of possible games, then a good case can be made for both players choosing a strategy which will maximize their total sum in each new game. But on the other hand one can argue that minimizing your opponent's return might aid your position in future games. Wherever possible it will be assumed that such obvious considerations will already be reflected in the payoff entries of the particular game under consideration. These possibilities will be accounted for in the sequel by considering various partial orderings of the strategy spaces.

1.6 In analyzing the general two-person game we will distinguish three types:

A) Collusion without bribery:

i) Use of independent mixed strategies,

ii) Use of dependent mixed strategies.

B) Collusion with bribery.

C) No collusion:

i) Frequency interpretation,

ii) Single game interpretation.

In all these cases both players are aware of all the entries of the payoff matrix.

Case A. In this type of game there is pre-play communication between the players with an ability to make binding agreements. This case is of primary importance when a non-transferable utility is postulated.

In cases where the payoffs are psychological utilities it may be meaningless to allow bribery even though the players can discuss their possible strategies. For example, a husband and wife may have two possibilities for an evening's entertainment, the movies or a hockey game. Say each player attaches a negative utility to the possibility of splitting up. The husband might prefer the hockey game and the wife the movies. If the payoff matrix takes the form:

	movie	hockey
movie	(2, 1)	(-1,-1)
hockey	(-1,-1)	(1, 2)

then a plausible agreement would be to toss a coin with the stipulation that they both will go to the movies if heads and both will go to the hockey game if tails. According to the utility postulates of von Neumann (1) or Marschak (1), this combined event has a utility value of 3/2 to each player. We might expect this value if bribery were meaningful. The mixed strategy of each player is (1/2, 1/2) but their expected values are not obtained by the usual product measure - i. e. in this interpretation the players not only choose their marginal distributions but they can decide on a joint distribution other than the one

defined by the independence of the marginal distributions. This interpretation is a generalization of von Neumann's scheme of mixed strategies but in practice it fulfills the strict interpretation of the game, namely, that the game is resolved by the players giving the umpire a single pair of pure strategies. It will be shown in the sequel that this concept does not serve any important role in the description of generalized two-person games with transferable utilities. In the case of the non-transferable utility it is a useful concept since it serves to convexify certain regions in the Euclidean plane (to be discussed in Section 2).

At this point the question arises whether the concept of dependent mixed strategies alters the theory of the two-person zero-sum games since this also does not depend on a transferable utility. A little reflection will show that the opposite interests of the players quells these fears.

. . Case B. The main features of the case of collusion with bribery are the existence of pre-play communication and a transferable utility. Since there is an extensive discussion of this case in von Neumann and Morgenstern (1), we defer any discussion of this case to the formal development in Sections 2 and 4.

. . Case C. The case of the non-cooperative (no collusion) game is marked by a lack of pre-play communication and thus the transferability of utility values is unimportant.

The theory of the two-person zero-sum game falls under this theory. This theory as developed by von Neumann (2), (3), and later extended by Ville (1) and Wald (1) shows the existence of a value to be associated with a large variety of games which include those in which the strategy spaces are (separable) compact metric spaces and where the payoff function is continuous. In the subsequent theory of generalized two-person games it will be important

in many conventions to find the value of a corresponding zero-sum game. For a computational scheme and a comparatively self-contained theory of this problem cf., Raiffa, Thompson, and Thrall (1).

In the zero-sum game it is never advantageous for a player to inform his opponent of his strategy. However, in the non-zero-sum case there are many examples where each player strives to convince his opponent that he is choosing a particular strategy - i.e. each player wishes to go first. For example, in the game of 1.6.A if the husband were to play first, he would choose the hockey game and the wife would have to tag along; the situation is reversed if the wife plays first.

The solution of the two person zero-sum game has the property that neither player has any tendency to act out of conformity with the predicted results. John F. Nash, Jr. (1), has extended this concept to generalized n-person games in which the primary concept is an equilibrium point. By the use of fixed-point theorems he shows that every finite non-cooperative game has an equilibrium point. In the non zero-sum two-person game, a pair of strategies (s_1, s_2) is an equilibrium point if player I gains no advantage in changing s_1 if s_2 is held fixed, and II gains no advantages in changing s_2 if s_1 is held fixed. While there are several cases where the equilibrium point concept is fruitful there are several drawbacks which were pointed out by Nash. To cite a few: 1) not all equilibrium points yield the same imputation to the players; 2) if the strategy pairs (s_1, s_2) and (s_1', s_2') are equilibrium points then (s_1, s_2') is not necessarily an equilibrium point. (These objections are satisfactorily allayed in the zero-sum case; indeed, there the set of equilibrium points are all possible pairs of optimal strategies); 3) there are examples in which all pairs of strategies are equilibrium points yielding a continuum of

possible imputations. Quite independently of Nash, Gerald Thompson and I developed an analogous concept for the two-person non-zero sum game which was reported to a seminar at the University of Michigan. We classified the equilibrium points into weak and strong stationary points. A particle in the product space of the mixed strategy spaces of the two players was constrained to move in a path given by a pair of simultaneous differential equations. At an equilibrium point the particle remained stationary and these points were classified according to whether the displaced particle moved away or remained in the vicinity of the stationary point.

1.7 The theory of the non-cooperative two-person game is not satisfactory in the sense that so called "rational behavior" can only be defined for a restricted class of games. While most economic games are of the cooperative variety, many military situations can be classified as non-cooperative games. In Section 3 some general concepts of non-cooperative games and their limitations are discussed. When the players are allowed pre-play communication many of the difficulties of the non-cooperative game vanish and in their stead other impediments appear. After all strikes do occur! In certain labor-management disputes the participants can jointly agree to offer their case for arbitration and then they are legally committed to abide by the arbiter's decision. In section 4, various arbitration schemes for generalized two-person games are presented and several simple but provocative games are arbitrated according to the conventions. These games are presented to aid in the subjective evaluation of the arbitration schemes. Since it is usually much easier to arbitrate a complicated system (when the players have only hazy notions) than a clear-cut simple situation (when the players have dogmatic preferences) these games aid in testing the realism of the conventions. For a convention to be realistic it must reflect the threat powers or the bargaining advantages of the players in the non-cooperative game.

Section II

Preliminary Definitions and Concepts

In this section we shall discuss some of the formal properties of two-person games. These properties will mainly be the formulation of concepts to be utilized in discussion of the non-cooperative (Section 3) and cooperative (Section 4 games. Because of the very specialized techniques of Section 4) it will be advantageous for us to defer to that section many obvious topological c[illegible]derations for the non-discrete case. Generalizations of specific techniques from the discrete to the non-discrete case may be effected by imposing very weak topological sufficiency conditions. However, it will appear much too artificial to involve various sets of sufficiency conditions before we motivate these conditions by a discussion of the discrete case.

2.1 Mixed Strategy Spaces.

Definition 1-a. Let $\mathcal{A} = \{\alpha\}$ be the set of pure strategies of player I. If the set $\mathcal{A}$ is finite, denote the pure strategies by $\alpha_1, \alpha_2, \ldots, \alpha_m$.

Definition 1-b. Let $\mathcal{B} = \{\beta\}$ be the set of pure strategies of player II. If the set $\mathcal{B}$ is finite, denote the pure strategies by $\beta_1, \beta_2, \ldots, \beta_n$.

Definition 2-a. Let $A(\alpha, \beta)$ denote the return to player I corresponding to a pair of pure strategies (α, β). In the finite case we abbreviate $A(\alpha_i, \beta_j)$ by a_{ij}.

Definition 2-b. Let $B(\alpha, \beta)$ denote the return to player II corresponding to a pair of pure strategies (α, β). In the finite case we abbreviate $B(\alpha_i, \beta_j)$ by b_{ij}.

Definition 3. Let $R \equiv \{(A(\alpha,\beta), B(\alpha,\beta)) : (\alpha,\beta) \in \mathcal{A} \times \mathcal{B}\}$. That is, R is the range of the mapping, $A \times B : \mathcal{A} \times \mathcal{B} \longrightarrow R_2$ where R_2 represents Euclidean 2-space.

Definitions 4-a and 4-b. Let F_1 $[F_2]$ be a Borel field of subsets of $\mathcal{A}$ $[\mathcal{B}]$ containing all pure strategies.

Definition 5-a. Let $S_I = \{\xi\}$ be the set of all probability measures over $\mathcal{A}$. Thus each ξ is defined on F_1 to the interval $[0, 1]$, and ξ is a completely additive measure function. In the finite case we let $\xi(\alpha_i) \equiv \xi_i$.

Definition 5-b. Let $S_{II} = \{\eta\}$ be the set of all probability measures over $\mathcal{B}$. We abbreviate $\eta(\beta_j) \equiv \eta_j$ in the finite case.

Definitions 6-a and 6-b. Corresponding to the pair (ξ, η) associate a vector $(A(\xi, \eta), B(\xi, \eta))$ where

$$A(\xi, \eta) = \int_{\mathcal{B}}\int_{\mathcal{A}} A(\alpha, \beta)\, d\xi\, d\eta \quad \left(= \sum_{i,j} a_{ij}\, \xi_i\, \eta_j \text{ in the discrete case}\right)$$

$$B(\xi, \eta) = \int_{\mathcal{B}}\int_{\mathcal{A}} B(\alpha, \beta)\, d\xi\, d\eta \quad \left(= \sum_{i,j} b_{ij}\, \xi_i\, \eta_j \text{ in the discrete case}\right).$$

The expressions $A(\xi, \eta)$ and $B(\xi, \eta)$ represent the expected returns to players I and II respectively corresponding to the pair (ξ, η). We explicitly assume:

$A(\alpha, \beta)$, $B(\alpha, \beta)$ are finite and summable with respect to all $(\xi \times \eta) \in S_I \times S_{II}$ and the double integrals are equal to the iterated integrals. Without worrying about the logical niceties we will equate the pure strategy α with the mixed strategy ξ such that $\xi(\alpha) = 1$. In this sense $\mathcal{A} \subset S_I$ and $\mathcal{B} \subset S_{II}$. We take

the further liberty of confusing the expressions $A(\xi, \eta)$, $A(\alpha, \beta)$, $A(\xi, \beta)$ etc. since the domain of the function will be clear by the symbols used for the arguments.

Definition 7. Let $R^* = \{(A(\xi, \eta), B(\xi, \eta)) : (\xi, \eta) \in S_I \times S_{II}\}$. That is, R^* is the range of the mapping, $A \times B$: $S_I \times S_{II} \longrightarrow R_2$.

In the finite case S_I and S_{II} are closed simplexes in R_m (Euclidean m-space) and R_n respectively; thus these spaces are convex, connected and compact (cf. Raiffa, Thompson and Thrall [1] page 6).

If we wish the above assertions to hold for the infinite case (we can get along on less for certain techniques in Section 4) we can impose the following sufficiency requirements: Let $\mathcal{A}$ and $\mathcal{B}$ be compact, metric spaces; let F_1 and F_2 be minimal Borel fields containing the open and closed sets; let $A(\alpha, \beta)$, $B(\alpha, \beta)$ be continuous functions on $\mathcal{A} \times \mathcal{B}$. If we topologize the spaces S_I and S_{II} in the ordinary sense (cf. Wald(1), page 49) then S_I, S_{II} and $S_I \times S_{II}$ become convex, compact, connected Hausdorff spaces (cf. Wald (1) , page 50). Since $A \times B$ is continuous on $S_I \times S_{II}$ the region R^* is also compact and connected. If the above sufficiency conditions are not met, we can metrize $\mathcal{A}$ and $\mathcal{B}$ in the sense of Wald to make A and B continuous. If this topology makes $\mathcal{A}$ and $\mathcal{B}$ compact then again the assertions hold. In the sequel we shall only utilize the assertion that, in the finite case, R^* is connected and compact; hence we will not reproduce the proofs of the above assertions for the infinite case.

For the purpose of giving counter-examples and to develop our intuition we will have frequent recourse to discuss examples in which both players have only two pure strategies. To this end, consider the game

$$\begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} (a_{11}, b_{11}) & (a_{12}, b_{12}) \\ (a_{21}, b_{21}) & (a_{22}, b_{22}) \end{bmatrix} \end{array},$$

and thus if $\xi = \xi_1\alpha_1 + \xi_2\alpha_2$ (to be read: α_1 is played with probability ξ_1 and α_2 with probability ξ_2) and $\eta = \eta_1\beta_1 + \eta_2\beta_2$ we have

$$A(\xi, \eta) = a_{11}\xi_1\eta_1 + a_{12}\xi_1(1-\eta_1) + a_{21}(1-\xi_1)\eta_1 + a_{22}(1-\xi_1)(1-\eta_1)$$
$$= (a_{11} - a_{12} - a_{21} + a_{22})\xi_1\eta_1 + (a_{12} - a_{22})\xi_1 + (a_{21} - a_{22})\eta_1 + a_{22},$$

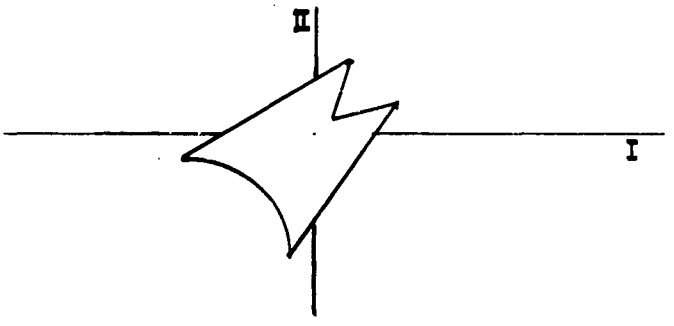
and similarly

$$B(\xi, \eta) = (b_{11} - b_{12} - b_{21} + b_{22})\xi_1\eta_1 + (b_{12} - b_{22})\xi_1 + (b_{21} - b_{22})\eta_1 + b_{22}.$$

From the above equations we note that for a fixed ξ, a straight line is transcribed from $[A(\xi, \beta_1), B(\xi, \beta_1)]$ to $[A(\xi, \beta_2), B(\xi, \beta_2)]$ (cf. remark following definition 6 concerning interpretation of notation). From this observation we <u>cannot</u> conclude that R^* is given by the totality of points bounded by the lines given by $\xi_1 = 0$, $\xi_1 = 1$, $\eta_1 = 0$, $\eta_1 = 1$ or that R^* is convex. Convexity is certainly not invariant of a continuous mapping and thus convexity of R^* cannot be deduced from convexity of $S_I \times S_{II}$. The above has been emphasized, for this difficulty must be overcome in the sequel and in our context cannot be assumed. Since a counter-example is in order we offer

$$g \equiv \begin{bmatrix} (2, 1) & (-1, -4) \\ (-4, -1) & (1, 2) \end{bmatrix}$$

for which the corresponding R^* is:



2.2 Joint Strategy Space.

Definition 8-a. Let $J' = \{\zeta\}$ be the set of all probability measures over $F_1 \times F_2$. In the finite case, let $\zeta(\alpha_i, \beta_j) = \zeta_{ij}$. Thus $\zeta_{ij} \geq 0$ for all i, j and $\sum_{ij} \zeta_{ij} = 1$. The space J' will be restricted to a subspace J in definition 8-b (which follows definition 9-b).

Elements ξ and η will be called mixed strategies and elements ζ will be called joint mixed strategies. To every pair (ξ, η) we can associate a joint mixed strategy which is the product measure (i.e. $\xi \times \eta = \zeta$ where $\zeta_{ij} = \xi_i \cdot \eta_j$) of ξ and η. It is in this sense that we will consider $S_I \times S_{II} \subset J'$. For motivation of J' see again Section 1.6, Case A.

Definition 9-a. If $A(\alpha, \beta)$, $B(\alpha, \beta)$ are summable with respect to any $\zeta \in J'$, then for any $\zeta \in J'$ let

$$(A(\zeta), B(\zeta)) = \int_{\alpha \times \beta} (A(\alpha, \beta), B(\alpha, \beta))\, d\zeta \quad \left(= \sum_{ij} (a_{ij}, b_{ij})\zeta_{ij} \text{ in the discrete case}\right).$$

Now define $\tilde{R}^{**} = \overline{\{(A(\zeta), B(\zeta)) : \zeta \in J'\}}$; that is, $\tilde{R}^{**}$ is the closure of the range of the mapping, $A \times B : J' \longrightarrow R_2$.

Using the product measure, the product space $S_I \times S_{II} \subset J'$. Further, $R^* \subset \tilde{R}^{**}$. In the finite case we note that $\tilde{R}^{**}$ is the minimal convex set containing R. The boundary of $\tilde{R}^{**}$ is composed of one-simplexes whose vertices are elements of R.

Definition 9-b. In the infinite case it is always possible to define R. We now define R^{**} as the minimal closed convex set containing R. This definition is non-vacuous and unique by noting that R_2 is closed and convex and contains R, and also that the intersection of closed convex sets is closed and convex. Thus for all games we can associate a region R^{**}.

We can give an equivalent characterization of R^{**} as follows: let U be the point set which is the union of all line segments joining any two elements of R. Let V be the point set which is the union of all line segments joining any two elements of U. We will now show that V is convex. To this end, let v_1 and v_2 belong to V, and for $0 \leq \alpha \leq 1$ we wish to show $\alpha v_1 + (1-\alpha) v_2 \in V$. Let u_1, u_2, u_3, u_4 be elements of U used in defining v_1 and v_2, and let $e_1, e_2, \ldots e_8$ be the elements of R used in defining u_1, u_2, u_3, u_4. It is easily seen that $\alpha v_1 + (1-\alpha) v_2$ lies in the minimal convex set containing $e_1, e_2, \ldots, e_8$. Thus $\alpha v_1 + (1-\alpha) v_2$ lies on a line segment joining two boundary points of this minimal convex set. Hence we conclude that V is convex. Since the closure of a convex set is convex, we finally have $\overline{V} = R^{**}$.

We now assert that if $\tilde{R}^{**}$ is well defined that $\tilde{R}^{**} = R^{**}$. From the above characterization of V, it is seen that V is contained in the image of J' under the mapping $A \times B$. Taking closures, we get $R^{**} \subset \tilde{R}^{**}$. That $\tilde{R}^{**} \subset R^{**}$ follows from the observation that the image of J' is contained in R^{**}. Again taking closures, we get $\tilde{R}^{**} \subset R^{**}$ and consequently $\tilde{R}^{**} = R^{**}$.

Since R^{**} is always defined we will have no recourse in the sequel to use the notation $\tilde{R}^{**}$. If the set R is compact we will now show that the minimal convex set containing R (in the above notation this is the set V) is also compact. Since R is bounded the set V is also bounded. It remains to show that V is closed. Let $\{v_n\}$ be a sequence of elements of V which have v for a sequential limit point. To show $v \in V$. For each n, we have

$$v_n = \lambda_{n1} e_{n1} + \lambda_{n2} e_{n2} + \lambda_{n3} e_{n3} + \lambda_{n4} e_{n4}$$

where $\sum_{i=1}^{4} \lambda_{ni} = 1$, $\lambda_{ni} \geq 0$, $e_{ni} \in R$. By a diagonalization process there is a subsequence $\{v_{n_m}\}$ such that $\lambda_{n_m i} \longrightarrow \lambda_i$, $e_{n_m i} \longrightarrow e_i$; and further $\sum_{i=1}^{4} \lambda_i = 1$, $\lambda_i \geq 0$, $e_i \in R$.

Consequently

$$V = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4$$

and thus $v \in V$ which proves the assertion. Recalling that V is contained in the image of J' we can conclude that if R is compact then every element of R^{**} has an inverse image in J'. Although in the general case, not every element of R^{**} is the image of an element of J', this will not be a serious handicap because of the context in which R^{**} is utilized.

The primary role of region R^{**} is in the cooperative game. In this case the players must agree on a joint mixed strategy. Although initially we took a rather general viewpoint it should now be clear that we can restrict the space J' to all normalized measures over the ring of sets of $\mathcal{A} \times \mathcal{B}$ which consists of all sets with a finite number of elements. This effectively eliminates any worry over integrability conditions for the expression $\int_{\mathcal{A}\times\mathcal{B}} (A(\alpha,\beta), B(\alpha,\beta)) d\mu$

Definition 8-b. Let $J = \{\mu\}$ be the set of all normalized measures over the ring of sets of $\mathcal{A} \times \mathcal{B}$ which consists of all sets with a finite number of elements.

2.3 Composite Order Relations.

We now shall define two partial orderings of R_2.

Definition 10-a. The element (x_2, y_2) compositely dominates (x_1, y_1),

$$(x_1, y_1) \prec (x_2, y_2) \equiv x_1 \leq x_2,\ y_1 \leq y_2 \text{ and } x_1 + y_1 < x_2 + y_2 .$$

Definition 10-b. The element (x_2, y_2) compositely uniformly dominates (x_1, y_1),

$$(x_1, y_1) \ll (x_2, y_2) \equiv x_1 < x_2 \text{ and } y_1 < y_2 .$$

In the sets $\mathcal{A} \times \mathcal{B}$, $S_I \times S_{II}$, and J we introduce three relationships $\simeq$, $\prec$, and $\ll$.

Definition 11-a. Let e_1 and e_2 be two elements of $\mathcal{A} \times \mathcal{B}$ or $S_I \times S_{II}$ or J. Now

e_1 is equivalent to e_2, $e_1 \simeq e_2$, if and only if they have the same image in R or R^* or R^{**}. This equivalence relation decomposes the spaces $\alpha x \beta$, $S_I \times S_{II}$, and J into equivalence classes.

Definition 11-b. We now introduce the partial orderings $\prec, \ll$ in $\alpha x \beta$, $S_I \times S_{II}$ and J by requiring the mappings $A \times B : \alpha \times \beta \longrightarrow R$, $A \times B : S_I \times S_{II} \longrightarrow R^*$, and $A \times B : J \longrightarrow R^{**}$ be isotone (that is, $e_1 \prec e_2 \equiv (A \times B)(e_1) \prec (A \times B)(e_2)$ and similarly for $\ll$).

Definition 12. Let $M_\prec, M^*_\prec, M^{**}_\prec$ be the maximal elements of R, R^* and R^{**} with respect to the partial order $\prec$. (An element e of a set is maximal if there is no other element e' of the set such that $e' > e$.) The inverse images of $M_\prec$, $M^*_\prec$, $M^{**}_\prec$ will be denoted by $\tilde{M}_\prec$, $\tilde{M}^*_\prec$ and $\tilde{M}^{**}_\prec$ and will be called the admissible elements of $\alpha x \beta$, $S_I \times S_{II}$ and J. (It is quite possible that $M^{**}_\prec$ is non-vacuous while $\tilde{M}^{**}_\prec$ is vacuous). Similarly we can define the maximal and admissible sets with respect to the partial ordering $\ll$.

In case $\alpha x \beta$ is finite it is quite obvious that the maximal and admissible sets are non-vacuous. In the infinite case to insure the above sets are non-vacuous it is quite sufficient to impose a compactness condition on the set R.

Theorem 1-a. If an arbitrary set $U \subset R_2$ is compact then the set of maximal points, W, with respect to the partial ordering $\ll$ is compact.

Proof: It suffices to show W is closed since W is bounded trivially. To this end let p be a limit point of W and $p \notin W$. Note that $p \in U$ since U is compact. Then there exists a $q \gg p$ and $q \in U$; thus there exists a vicinity of p, $V(p)$, such that $V(p) \ll q$. But $V(p) \cap W \neq \emptyset$ which contradicts the nature of W.

Theorem 1-b. If $U \subset R_2$ is compact and convex then the set of maximal points, W, of U, with respect to the partial ordering $\prec$ is compact.

Proof: We proceed as above but now there exists a $q \succ p$. If $q \gg p$ we are through by the argument of theorem 1 a. Hence assume $q \succ p$ and $q \not\gg p$. Choose $V_1(p)$, such that $V_1(p) \cap q = \emptyset$. Let $r \in V_1(p) \cap W$. Clearly $q \not\succ r$. If $r \gg p$ then there exists a $V_2(p)$ such that $V_2(p) \cap r \neq \emptyset$, $V_2(p) \ll r$ and since $V_2(p) \cap W \neq \emptyset$ we have a contradiction. Hence assume r does not satisfy $p \succ r$, $q \succ r$ or $r \gg p$. Then there exists a $V_3(p)$ which does not intersect the segment from q to r and such that every element of $V_3(p)$ is compositely dominated ($\prec$) by some element of the segment q to r. But $V_3(p) \cap W \neq \emptyset$ contradicts the nature of W.

Theorem 1 c. If $U \subset R_2$ is compact and convex then the set of maximal points W, of U, with respect to $\prec$ or $\ll$ is connected.

Proof: Let $W = X \cup Y$ where $X \cap Y = \emptyset$, $X \neq \emptyset$, $Y \neq \emptyset$. Let $a_1 \in X$ and $b_1 \in Y$. Let $c_1 \in W$ and $c_1 \succeq \frac{a_1 + b_1}{2}$ $\left[\text{or } c_1 \gg \frac{a_1 + b_1}{2}\right]$. If $c_1 \in X$, call c_1 by a_2 and b_1 by b_2; if $c_1 \in Y$, call a_1 by a_2 and c_1 by b_2. We continue this process by replacing a_n and b_n by a_{n+1} and b_{n+1}. We thus get a sequence of pairs $\{a_n, b_n\}$ where $a_n \in X$ and $b_n \in Y$. Let the distance between a_n and b_n be $d(a_n, b_n) = d_n$. It is easily verified that $d_n \to 0$. Let c be common limit point of $\{a_n\}$ and $\{b_n\}$. By theorem 1-b, $c \in W$. Hence $(\bar{X} \cap Y) \cup (X \cap \bar{Y}) \neq \emptyset$ and W is connected.

Corollary to Theorems 1-a-b-c. If R is bounded then M_1^{**} and M_4^{**} are compact and connected. (For, if R is bounded then R** is compact and convex.)

We observe that for any $\zeta \in J$, such that $(A \times B)(\zeta)$ belongs to the boundary of R**, there exists a $\zeta^{(0)} \simeq \zeta$ such that $\zeta^{(0)}$ uses positive weights for at most two pairs of pure strategies. Hence for any $\zeta \in J$ there exists a $\zeta^{(0)} \simeq \zeta$ such that $\zeta^{(0)}$ uses positive weights for at most four pairs of pure strategies.

2.4 Bayes Solution and Equilibrium Points.

Definition 13. An equivalence class in $S_I \times S_{II}$ is said to be permutable if

$$(\xi^{(1)}, \eta^{(1)}) \simeq (\xi^{(2)}, \eta^{(2)}) \Longrightarrow (\xi^{(1)}, \eta^{(2)}) \simeq (\xi^{(2)}, \eta^{(1)}) \simeq (\xi^{(1)}, \eta^{(1)}).$$

Definition 14-a. The strategy $\xi^{(0)}$ is said to be a Bayes solution with respect to the strategy $\eta^{(0)}$ (or, $\xi^{(0)}$ is "good against" $\eta^{(0)}$) if $A(\xi^{(0)}, \eta^{(0)}) = \sup_{\xi} A(\xi, \eta^{(0)})$.

Definition 14-B. The strategy $\eta^{(0)}$ is said to be a Bayes solution with respect to the strategy $\xi^{(0)}$ (or $\eta^{(0)}$ is "good against" $\xi^{(0)}$) if $B(\xi^{(0)}, \eta^{(0)}) = \sup_{\eta} B(\xi^{(0)}, \eta)$.

Definition 15. The pair of strategies $(\xi^{(o)}, \eta^{(o)})$ is an equilibrium point if each strategy is good against the other. If besides $\alpha^{(o)}$ and $\beta^{(o)}$ are such that $\xi^{(o)}(\alpha^{(o)}) = 1$ and $\eta^{(o)}(\beta^{(o)}) = 1$ then $(\alpha^{(o)}, \beta^{(o)})$ will be called a saddle point. The totality of saddle and equilibrium points will be denoted by $\widehat{\mathcal{E}}$ and $\widetilde{\mathcal{E}}^*$ respectively.

To illustrate some of these definitions we return to the example on page 14 of this section. The strategy pairs (α_1, β_1) and (α_2, β_2) are both admissible and saddle points. However, these elements are not equivalent. The elements (2, 1) and (1, 2) are the maximal elements of R and R^* under the partial orderings $\prec$ and $\lll$. The region R^{**} is the minimal convex region containing the four vertices (2, 1), (-1, -4), (-4, -1), (1, 2). The elements of M^{**} lie on the segment from (2, 1) to (1, 2). The admissible elements, $\widetilde{M}^{**}$, of J are of the form $\alpha S_{11} + (1 - \alpha) S_{22}$ for $0 \leq \alpha \leq 1$.

2.5 Region Corresponding to a Transferable Utility.

Thus far in Section 2 we have not assumed any common unit of measurement. However, if there exists a common unit of measurement and bribery is permissible then it is possible for the players to achieve imputations outside of R^{**}. Let $c = \max_{(\alpha, \beta)} [A(\alpha, \beta) + B(\alpha, \beta)]$. In the infinite case this is assumed to exist. (For example, it is quite sufficient to assume compactness for $\mathcal{A}$ and $\mathcal{B}$ and continuity for A and B.)

Definition 16. Let $R^{***} = \{(x_1, x_2): x_1 + x_2 \leq c\}$. Also let M^{***} be the maximal points of R^{***}. Indeed, for partial orderings $\prec$ and $\lll$ we have $M^{***} = \{(x_1, x_2): x_1 + x_2 = c\}$.

To summarize, we have defined four sets in R with the inclusion relations $R \subset R^* \subset R^{**} \subset R^{***}$. In the non-cooperative game and the cooperative

game with independent mixed strategies we must consider R and R^*; in the cooperative case with dependent mixed strategies we must consider R^{**}; finally, R^{***} pertains to the cooperative game with bribery.

2.6 Generalization to n-Players.

The possibility of generalizing the concepts of this section to more than two players is quite apparent. Most definitions have a rather trivial extension. For the n-person game the regions R, R^*, R^{**} and R^{***} are subsets of Euclidean n-space and the orderings $\prec$, $\prec\prec$ can be generalized in a natural fashion. We now must define a Bayes solution for a given player with respect to a set of mixed strategies for the remaining players. A set of mixed strategies is an equilibrium point if each mixed strategy is a Bayes solution with respect to the remaining mixed strategies.

Section III

Non-Cooperative Games

3.1 Incomparable, Convex and Linear Games.

We shall now consider an important subclass of two-person games which we shall call incomparable games. These games are characterized by the fact that the elements of R are incomparable in the composite ordering $\prec$. Corresponding to a pair of pure strategies, (α, β) there is a stimulus situation $s(\alpha, \beta)$. If each player simply orders these stimuli according to preference, then in an incomparable game the orderings of both players are exactly reverse. Hence the players have strictly opposing interests with respect to pairs of pure strategies.

Lemma 1: In an incomparable game the set $\widetilde{\mathcal{E}}$ of saddle points(assuming $\widetilde{\mathcal{E}} \neq \emptyset$) forms a permutable equivalence class.

Proof: Let (α_1, β_1) and (α_2, β_2) be a pair of saddle points. Since (α_1, β_1) is a saddle point we have

a) $$A(\alpha_2, \beta_1) \leq A(\alpha_1, \beta_1), \qquad B(\alpha_1, \beta_2) \leq B(\alpha_1, \beta_1).$$

Since the game is incomparable, we have from a)

b) $$B(\alpha_2, \beta_1) \geq B(\alpha_1, \beta_1), \qquad A(\alpha_1, \beta_2) \geq A(\alpha_1, \beta_1).$$

But (α_2, β_2) is a saddle point and thus

c) $$A(\alpha_2, \beta_2) \geq A(\alpha_1, \beta_2), \qquad B(\alpha_2, \beta_2) \geq B(\alpha_2, \beta_1).$$

From b) and c) we have

d) $$A(\alpha_2, \beta_2) \geq A(\alpha_1, \beta_1), \qquad B(\alpha_2, \beta_2) \geq B(\alpha_1, \beta_1).$$

Since the game is incomparable, both equalities must hold in d) and therefore

equalities must hold in c), b) and a). Consequently, we have

$$(\alpha_1, \beta_1) \simeq (\alpha_1, \beta_2) \simeq (\alpha_2, \beta_1) \simeq (\alpha_2, \beta_2) \quad . \qquad \text{Q.E.D.}$$

3.1.1 An incomparable game is said to be <u>convex</u> if $R \subset M_i^{**}$. Thus in a convex incomparable game all pairs of pure strategies are compositely admissible.

3.1.2 An incomparable game is said to be <u>linear</u> if $R^* = R^{**}$ = a segment of a line. Otherwise a game is said to be non-linear. A linear game is also convex, and also $R^* = M_i^{**}$.

The concepts of incomparable, convex incomparable, and linear games are invariant under independent positive linear transformations of the payoff entries. The linear game is a natural generalization of a two-person zero-sum game. In a linear game no assumption need be made on a common unit of measurement between the two players.

3.2 <u>The Rational Solution.</u>

3.2.1 A game is said to have a <u>rational solution</u> if $\tilde{\mathcal{E}}^* \cap \tilde{M}_i^*$ forms a permutable equivalence class. The imputation corresponding to this equivalence class will be called the value of the game. J. von Neumann's solution of the zero-sum two-person game would be a <u>rational solution</u> in our sense.

If both players are aware of the above rational solution of a game (provided it exists) then there is no tendency for either player to act out of conformity with this solution. Furthermore, even if both players were permitted to enter a cooperative game without bribery and without correlated strategies then there still would not be any composite motivation to move from this solution.

3.2.2 In a linear game there is no ambiguity involved in discussing the majorant and minorant games (or the auxiliary games of the two players) and in the finite case one can show the existence of minimax solutions. That is, there exists a $(\xi^{(o)}, \eta^{(o)})$ such that

A) $$\max_{\xi} \min_{\eta} A(\xi, \eta) = \min_{\eta} \max_{\xi} A(\xi, \eta) = A(\xi^{(o)}, \eta^{(o)}) .$$

and

B) $\max_{\eta}\min_{\xi} B(\xi,\eta) = \min_{\xi}\max_{\eta} B(\xi,\eta) = B(\xi^\circ,\eta^\circ)$.

The extended game, in which mixed strategies now become pure strategies, is an incomparable game and thus the saddle points are all compositely admissible and form a permutable equivalence class. Consequently, such a game has a rational solution. The existence of saddle points will follow from a corollary of the principal theorem of section 4. If (ξ°,η°) is a saddle point then (ξ°,η°) satisfies the above equations and is a minimax solution, for: since the extended game is incomparable we have

a) $\min_{\eta}\max_{\xi} A(\xi,\eta) \gtrless A(\xi^\circ,\eta^\circ) \longleftrightarrow \max_{\eta}\min_{\xi} B(\xi,\eta) \lesseqgtr B(\xi^\circ,\eta^\circ)$

b) $\max_{\xi}\min_{\eta} A(\xi,\eta) \gtrless A(\xi^\circ,\eta^\circ) \longleftrightarrow \min_{\xi}\max_{\eta} B(\xi,\eta) \lesseqgtr B(\xi^\circ,\eta^\circ)$

and since (ξ°,η°) is a saddle point

c) $\max_{\xi}\min_{\eta} A(\xi,\eta) \geqslant A(\xi^\circ,\eta^\circ)$, $\max_{\eta}\min_{\xi} B(\xi,\eta) \geqslant B(\xi^\circ,\eta^\circ)$

and, in general, it is easy to show

d) $\max_{\xi}\min_{\eta} A(\xi,\eta) \leqslant \min_{\eta}\max_{\xi} A(\xi,\eta)$,

$\max_{\eta}\min_{\xi} B(\xi,\eta) \leqslant \min_{\xi}\max_{\eta} B(\xi,\eta)$.

Say $\max_{\eta}\min_{\xi} B(\xi,\eta) \neq B(\xi^\circ,\eta^\circ)$; then by c) the $>$ holds. By a) we have $\min_{\eta}\max_{\xi} A(\xi,\eta) < A(\xi^\circ,\eta^\circ)$ and by d) we have $\max_{\xi}\min_{\eta} A(\xi,\eta) < A(\xi^\circ,\eta^\circ)$ which contradicts c). If now $\max_{\xi}\min_{\eta} A(\xi,\eta) \neq A(\xi^\circ,\eta^\circ)$ then repeat the argument using b) instead of a). Thus we have shown the validity of A) and B). Conversely, any pair of minimax strategies is an equilibrium point.

3.2.3 To illustrate the concept of rational solution consider the simple game

$$\begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} (1,\ 1) & (0,\ 0) \\ (0,\ 0) & (2,\ 2) \end{bmatrix} \end{array}.$$

Here (α_1, β_1) and (α_2, β_2) are equilibrium points. Since only (α_2, β_2) is compositely admissible this is the rational solution. Note that if the set $\tilde{\mathfrak{M}}$ has only one element then this element is also an equilibrium point and hence that is the solution. But nothing can be concluded if $\mathfrak{M}$ has only one element. Consider, for example,

$$\begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} (1,\ 1) & (0,\ 0) \\ (0,\ 0) & (1,\ 1) \end{bmatrix} \end{array}.$$

Here $\tilde{\mathcal{E}}^* \wedge \tilde{\mathfrak{M}}^*$ is not a permutable equivalence class. In this case $\left[\left(\frac{\alpha_1}{2} + \frac{\alpha_2}{2}\right), \left(\frac{\beta_1}{2} + \frac{\beta_2}{2}\right)\right]$ is an equilibrium point yielding the imputation $\left(\frac{1}{2}, \frac{1}{2}\right)$ which is a plausible solution if the game is to be played once.

In the game

$$\begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} (1,\ 3) & (2,\ 3) \\ (1,\ 1) & (2,\ 1) \end{bmatrix} \end{array}$$

all pairs of mixed strategies are equilibrium points but only $(\alpha_1, \beta_2) \in \tilde{\mathfrak{M}}_4^*$ and hence this is the rational behavior solution (however, in this degenerate case any individual behavior pattern is as good as any other).

To illustrate that the set $\tilde{\mathcal{E}}^* \wedge \tilde{\mathfrak{M}}_4^*$ may be empty, consider the game

$$\begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} (0,\ 0) & (6,-2) \\ (3,\ 1) & (4,\ 2) \end{bmatrix} \end{array}.$$

This game has only one equilibrium point (ξ, η) where $\xi = \frac{1}{3}\alpha_1 + \frac{2}{3}\alpha_2$ and $\eta = \frac{2}{5}\beta_1 + \frac{3}{5}\beta_2$ and the corresponding imputation is $(\frac{19}{5}, \frac{2}{3})$ which is not a maximal element of R^*.

3.3 The Completely Reduced Game.

The mixed strategy $\sum_{j=1}^{k} \xi_{ij}\alpha_{ij}$ where $\sum_{j=1}^{k} \xi_{ij} = 1$, $\xi_{ij} \geq 0$ for $j = 1, 2, \ldots k$ is said to dominate α_0 relative to the set $\mathcal{B}_0 \subset \mathcal{B}$ if

$$\sum_{j=1}^{k} \xi_{ij} A(\alpha_{ij}, \beta) \geq A(\alpha_0, \beta)$$

for all $\beta \in \mathcal{B}_0$ and if there exists a $\beta \in \mathcal{B}_0$ for which the equality does not hold.

The completely reduced game $\tilde{\mathcal{G}}$ of $\mathcal{G}$ is defined as follows: Let $\mathcal{A}_1 \subset \mathcal{A}$ be the set of undominated pure strategies of player I relative to the set $\mathcal{B}$. Let $\mathcal{B}_1$ be the set of undominated pure strategies of player II relative to the set $\mathcal{A}_1$. Continuing in this fashion we get a sequence

$$\mathcal{A} \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \cdots$$

and

$$\mathcal{B} \supset \mathcal{B}_1 \supset \mathcal{B}_2 \supset \cdots$$

where $\mathcal{A}_{i+1}$ is the set of undominated strategies of $\mathcal{A}_i$ relative to the set $\mathcal{B}_i$ and $\mathcal{B}_i$ is the set of undominated strategies of $\mathcal{B}_{i-1}$ relative to $\mathcal{A}_i$. Let $\tilde{\mathcal{A}} = \bigcap \mathcal{A}_i$ and $\tilde{\mathcal{B}} = \bigcap \mathcal{B}_i$. It is clear that $\tilde{\mathcal{A}} \times \tilde{\mathcal{B}}$ is non-empty; we now show that it does not depend on whether the reduction process starts with player I or II. For if we consider the process where $\mathcal{B}_1^*$ is defined relative to $\mathcal{A}$ and $\mathcal{A}_1^*$ is defined relative to $\mathcal{B}_1^*$ etc., then we have

$$\mathcal{A} \supset \mathcal{A}_1^* \supset \mathcal{A}_2^* \supset \cdots$$

and

$$\mathcal{B} \supset \mathcal{B}_1^* \supset \mathcal{B}_2^* \supset \cdots .$$

Then it follows readily that

$$\mathcal{A} \supset \mathcal{A}_1 \supset \mathcal{A}_1^* \supset \mathcal{A}_2 \supset \mathcal{A}_2^* \supset \cdots$$

and

$$\mathcal{B} \supset \mathcal{B}_1^* \supset \mathcal{B}_1 \supset \mathcal{B}_2^* \supset \mathcal{B}_2 \supset \cdots$$

and therefore

$$\cap \mathcal{A}_i = \cap \mathcal{A}_i^* \quad , \quad \cap \mathcal{B}_i = \cap \mathcal{B}_i^* \quad .$$

The game $\tilde{\tilde{g}}$ is defined over $\tilde{\mathcal{A}} \times \tilde{\mathcal{B}}$ and agrees with g in this restricted set. The game g is said to have a non-cooperative rational solution <u>in the wide sense</u> if $\tilde{\tilde{g}}$ has a rational solution according to 3.2.1. Every equilibrium point of $\tilde{\tilde{g}}$ is also an equilibrium point of g ; however, in general, the maximal points of $\tilde{\tilde{g}}$ are not maximal points of g . In a linear game the value of $\tilde{\tilde{g}}$ is the same as the value of g . Thus in the computational technique to determine the solution of a two-person zero-sum game (cf. Raiffa, Thompson and Thrall (1)) it is more expedient to first make an effort to completely reduce the game before attempting to find the intersections of the column or row hyperplanes.

To illustrate the above concept in the non-linear case let us analyze the following game:

	β_1	β_2	β_3	β_4	β_5
α_1	(-1,3)	(2, 1)	(-4, 5)	(1, 2)	(0, 10)
α_2	(4, 0)	(3, 1)	$(-3, \frac{3}{4})$	(1, 3)	(2, 2)
α_3	(0, 4)	(2, 0)	(1, 9)	(0, 6)	(4, 8)
α_4	(-2, 2)	(1, 3)	(6, 2)	(3, 1)	(8, 1)
α_5	(2,-1)	(0, 0)	(5, 8)	(0, 6)	(9, 3)

With reference to $\mathcal{B}$, α_2 dominates α_1 and $\frac{\alpha_2}{2} + \frac{\alpha_4}{2}$ dominates α_3 . Thus $\mathcal{A}_1 = \{\alpha_2, \alpha_4, \alpha_5\}$. With reference to $\mathcal{A}_1$, β_2 dominates β_1 and

$\frac{1}{4}\beta_2 + \frac{3}{4}\beta_4$ dominates β_5. Hence $\mathcal{B}_1 = \{\beta_2, \beta_3, \beta_4\}$. With reference to $\mathcal{B}_1$, α_4 dominates α_5. Hence $\mathcal{A}_2 = \{\alpha_2, \alpha_4\}$. With reference to $\mathcal{A}_2$, $\frac{\beta_2}{2} + \frac{\beta_4}{2}$ dominates β_3 and thus $\mathcal{B}_2 = \{\beta_2, \beta_4\}$. Here the process terminates. The completely reduced game is of the form

$$\begin{array}{c} \\ \alpha_2 \\ \alpha_4 \end{array} \begin{array}{c} \begin{array}{cc} \beta_2 & \beta_4 \end{array} \\ \begin{bmatrix} (3,\ 1) & (1,\ 3) \\ (1,\ 3) & (3,\ 1) \end{bmatrix} \end{array}$$

which is a linear game and thus has a rational solution. In particular, the non-cooperative rational solution in the wide sense of $\mathcal{G}$ is $(\frac{\alpha_2}{2} + \frac{\alpha_4}{2}, \frac{\beta_2}{2} + \frac{\beta_4}{2})$ which yields the imputation $(2, 2)$.

3.4 The Reduced Game.

The reduced game $\tilde{\mathcal{G}}$ of $\mathcal{G}$ (not completely reduced) is defined on the set $\mathcal{A} \times \tilde{\mathcal{B}}$ where $\mathcal{A}$ is the set of undominated strategies of player I relative to the set $\mathcal{B}$ and $\tilde{\mathcal{B}}$ is the set of undominated strategies of player II relative to the set $\mathcal{A}$. If $\tilde{\mathcal{G}}$ has a rational solution then $\mathcal{G}$ is said to have a non-cooperative rational solution. The pragmatic distinction between the reduced and the completely reduced game will be discussed in 3.4.1 and 3.5.

3.4.1 If in a given two-person game there exists a pure strategy α_0 which dominates all other strategies then α_0 is a "good" strategy in the sense of "rational behavior". Likewise if β_0 dominates all other strategies of player II then β_0 is a "good" strategy and the pair (α_0, β_0) represents a saddle point or an equilibrium point in pure strategies (of course, not every equilibrium point arises from a pair of dominant strategies). In a game in which there exists a pair of dominating strategies (α_0, β_0) then this pair will be the non-cooperative rational solution. If only player I has a dominant

strategy α_0 then the non-cooperative rational solution in the wide sense is $(\alpha_0, \beta_{\alpha_0})$ where β_{α_0} is any Bayes solution for player II with respect to α_0 (i.e. β_{α_0} is a "good" strategy against α_0). Further, if α_0 is a dominant strategy then α_0 is a Bayes solution against any a priori strategy for player II. (cf. Wald [1] for a discussion of Bayes solution in the statistical decision problem).

For any mixed strategy β of player II, there exists a Bayes solution for player I which is defined over the set $\mathcal{A}_1 = \tilde{\mathcal{A}}$. Similarly for any mixed strategy α of player I there exists a Bayes solution for player II which is defined over the set $\mathcal{B}_1^* = \tilde{\mathcal{B}}$. Consequently, each player can confine himself without any disadvantage to the reduced strategy space even if his opponent does not. However, it is not always possible to find a Bayes solution relative to an a priori mixed strategy if the player confines himself to his completely reduced strategy space. For example, if player I suspects that his opponent will not restrict himself to $\tilde{\tilde{\mathcal{B}}}$ then I may be at a disadvantage by confining himself to $\tilde{\tilde{\mathcal{A}}}$.

3.5 An Example (detailed discussion).

We shall now illustrate these concepts by the analysis of a game which has the same structure as a game first suggested by Dr. Dresher at Rand Corporation. The game is the following:

$$\begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} (9,\ 9) & (-10,10) \\ (10,-10) & (-9,-9) \end{bmatrix} \end{array}$$

Again because of symmetry, a plausible solution of the cooperative game is (9, 9). However, the non-cooperative game requires a bit more reflection. If player II plays either pure strategy β_1 or β_2, player I gains an advantage by playing α_2. Consequently, α_2 dominates α_1, and similarly β_2 dominates β_1 and the non-cooperative rational solution is (α_2, β_2) which yields the imputation (-9, -9). We emphasize that the analysis of the above non-cooperative game is to be considered as a game isolated in time without any stigma attached to the players resulting from a given move. Thus we see the obvious advantage to both players in cooperative behavior in which agreements are strictly enforced. The above game is the simplest case where the non-cooperative rational solution is not a maximal element.

When the above game is tested empirically a common behavior pattern is (α_1, β_1) yielding (9,9). The argument yielding (α_1, β_1) is usually as follows: "In general, the cooperative game yields better results than the non-cooperative game. This cooperative game has a unique symmetric solution and thus if both players act accordingly they both will profit". Obviously, a pair of "irrational players" will both get higher returns than a pair of "rational players". Then in what sense is (α_2, β_2) the non-cooperative rational behavior solution? The crux of the situation is that the rational player fares better than the irrational player against any type of opponent! We now proceed to give an analysis of this game where the game instead of being played once is a) repeated indefinitely, b) played a finite number, N, of times. But before expanding on this viewpoint we will generalize this concept to a large variety of games.

Let $\{g_i\}$ be any set of generalized two person games in normal form where the generic pure strategies of player I and II are respectively $\alpha^{(i)}$ and $\beta^{(i)}$

and the payoff entries for each game for an individual player have the same scale. Now let us form a new game

$$\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2 \times \cdots \times \mathcal{G}_n$$

which is in the extensive form. In the terminology of Gerald Thompson (1), this game is a very special case of a simultaneous game in which at the first stage each player makes a choice, giving $(\alpha^{(1)}, \beta^{(1)})$. At the second stage there is complete information and another choice $(\alpha^{(2)}, \beta^{(2)})$ is made etc. A given play of the game $\mathcal{G}$ is a sequence of moves $(\alpha^{(1)}, \beta^{(1)}; \alpha^{(2)}, \beta^{(2)}; \ldots; \alpha^{(n)}, \beta^{(n)})$ and the payoffs to players I and II are respectively $A = \sum_{i=1}^{n} A_i(\alpha^{(i)}, \beta^{(i)})$ and $B = \sum_{i=1}^{n} B_i(\alpha^{(i)}, \beta^{(i)})$ where A_i and B_i are the payoff functions in $\mathcal{G}_i$. If each game $\mathcal{G}_i$ is linear (all with the same slope) then $\mathcal{G}$ is linear and any sequence of moves for player I is an optimal solution if the i^{th} move is a personal mixed move which is an optimal strategy for the game $\mathcal{G}_i$. Thus the game $\mathcal{G}$ in the normalized form admits a saddle point in the pure strategy sense. From the "rational behavior" viewpoint of von Neumann and Morgenstern there is no need to consider strategies which correlate the moves of a given player with the previous moves of his opponent and himself.

In the linear game an optimal strategy is always a Bayes solution against an opposing optimal strategy. But, in general, an optimal strategy is not a Bayes solution against an opposing non-optimal strategy. Thus in the theory of statistical decision functions the minimax estimator quite often gives exceptionally poor results, especially since "nature" is an "irrational" player (cf. Lehman (1)). Now let us consider the game $\mathcal{G}$ which is a direct product of N (large) matching pennies games. An optimal solution in this case is to use a personal chance move at each stage which utilizes heads and tails with proba-

bilities ($\frac{1}{2}$, $\frac{1}{2}$). Now during a given play of this game, say player I keeps a statistical analysis of the past moves and determines a pattern of behavior of his opponent which indicates (up to a certain significance level) that his opponent is not using an optimal strategy. Then player I should utilize this information in his overall strategy - of course, player I should not play a Bayes solution for fear of teaching his opponent. An overall strategy for the game $\mathcal{G}$ should not merely be a repetition of optimal strategies for each component game as this would be tantamount of throwing away information that is available in the extensive form of the game.

Now let us return to the game which is a direct product of component games

$$\begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} (+9,\ +9) & (-10,\ 10) \\ (10,-10) & (-9,\ -9) \end{bmatrix} \end{array} .$$

As a preliminary to the n-fold direct product let us consider that the game is to be played indefinitely where each play is to be considered as a non-cooperative game and the players are informed of the results of a particular play of the game before the next play.

If player I chooses α_2 or player II chooses β_2 in the first play, then the mutual distrust thus engendered would probably result in the pair (α_2 , β_2) in the ensuing play. Consequently, the "intelligent" play should be strategy α_1 or β_1 on the first move. The same reasoning can be applied to subsequent moves. If (α_1 , β_1) is realized on a given play, then there is no tendency in the future to act out of conformity with this mode of behavior for the chaotic influences would result in an overall smaller expected value. This plausibility argument for repeated play of (α_1 , β_1) can be formalized - however, it is this plausibility argument which must be formally reckoned with in

the case for finite n. It should be noted that strategies $(\alpha_2^{(\infty)}, \beta_2^{(\infty)})$ (repeated application of α_2 and β_2) constitute an equilibrium value. The above situation is an example where maximization of a factor in a sequence of trials does not result from maximization of that factor in each individual trial.

We now return to the case where the game is to be played a finite number, n, of times. Recalling that in the single play situation α_2 dominates α_1 and β_2 dominates β_1, we can proceed with the complete reduction of the n-fold direct product as follows: Any strategy of player I which does not utilize α_2 in the last simultaneous move of the extensive form is dominated by the identical strategy which merely utilizes α_2 in the last simultaneous move. Thus all elements of $\mathcal{A}_1$ have α_2 in the last move. Relative to $\mathcal{A}_1$, the undominated strategies $\mathcal{B}_1$ of player II must have β_2 in the last two simultaneous moves. Consequently, all elements of $\mathcal{A}_2$ have α_2 in the last three simultaneous moves. Proceeding in this fashion the completely reduced game reduces to a single strategy pair, namely, (α_2, β_2) in all moves. The non-cooperative rational solution in the wide sense then yields the imputation (-9n, -9n). This strategy pair constitutes an equilibrium point. If a player knows a priori that his opponent will confine himself to his completely reduced strategy space then that player should also restrict himself to his completely reduced strategy space. We shall now investigate the Bayes solutions with respect to various a priori strategies which have some "intuitive appeal".

Let us define a set of pure strategies $\tilde{\alpha}_0, \tilde{\alpha}_1, \ldots \tilde{\alpha}_n$ for player I (similarly, for player II) where $\tilde{\alpha}_i$ is the strategy in which player I uses α_2 in moves n-i+1, n-i+2,..., n, and for $j \leq n - i + 1$ player I uses α_1 provided player II has used β_1 for moves 1, 2, ...j-1 and I uses α_2 in moves $k + 1, k + 2, \ldots, n$ if II uses β_2 in move k. Thus $\tilde{\alpha}_n$ is the strategy where

I uses α_2 in every move, and $\tilde{\alpha}_0$ is the strategy where player I uses α_1 in a given move if player II has always used β_1 in previous moves. It is easily seen that all strategies $\tilde{\alpha}_1$, $\tilde{\alpha}_2$, ... $\tilde{\alpha}_n$ are strategies of the reduced strategy space (not completely reduced). In a similar fashion define $\tilde{\beta}_0$, $\tilde{\beta}_1$, ..., $\tilde{\beta}_n$. In our present terminology ($\tilde{\alpha}_n$, $\tilde{\beta}_n$) is an equilibrium point.

We now wish to consider the Bayes solution of player II corresponding to an a priori mixed strategy where player I utilizes strategies $\tilde{\alpha}_0$, $\tilde{\alpha}_1$, $\tilde{\alpha}_2, \ldots, \tilde{\alpha}_n$ with probabilities p_0, p_1 , ..., p_n. If E_j represents the expected return to player II, if player II uses $\tilde{\beta}_j$ against the above a priori distribution then

$$E_j = \sum_{i=0}^{n} B(\tilde{\alpha}_i , \tilde{\beta}_j)\, p_i$$

$$= [9(n-j) + 10 - 9(j-1)] \sum_{i<j} p_i + [9(n-j) - 9j] p_j + \sum_{i>j} [9(n-i) - 10 - 9(i-1)]\, p_i$$

$$= 9n - 18\left[j \sum_{i<j} p_i + \sum_{i>j} i\, p_i\right] + 19 \sum_{i<j} p_i - \sum_{i>j} p_i \quad .$$

We then have

$$\Delta E_j = E_{j+1} - E_j$$

$$= p_j + p_{j+1} - 18 \sum_{i=0}^{j-1} p_i \qquad \text{if } j \geq 1$$

$$= p_0 + p_1 \qquad \text{if } j = 0$$

and consequently $\tilde{\beta}_{j+1}$ is preferred to $\tilde{\beta}_j$ with respect to the a priori distribution if

3.5.1) $$p_j + p_{j+1} > 18 \sum_{i=0}^{j-1} p_i \qquad \text{for} \qquad j \geq 1$$

and $\tilde{\beta}_0$ is never preferred to $\tilde{\beta}_1$.

From consideration of (3.5.1) it is easily seen that the Bayes solution with respect to the strategy $\tilde{\beta}_i$ is $\tilde{\alpha}_{i+1}$ for i= 1, 2, ... n-1 and $\tilde{\beta}_{i+1}$ is the Bayes solution with respect to the a priori distribution $\tilde{\alpha}_i$. Actually $\tilde{\alpha}_n$ is the worst strategy against $\tilde{\beta}_i$ for $i < n-1$. The secret of this game is to

break away from using the first strategy, one move before your opponent. In the extensive form of the game the present move can depend on the past moves and consequently the present move may affect the future moves. It is true that $\tilde{\alpha}_n$ minimizes the maximum possible loss to player I (which is -9n), but there are other factors which should enter into the determination of an overall strategy. If a player is certain that his opponent will be guided solely by consideration of the completely reduced game then the player must act similarly. But the above example shows that (in this case) if player I confines himself to $\tilde{\alpha}$ then this is "almost always" the worst strategy against a player who is not similarly indoctrinated. If the payoff entries for both players have the same unit of measurement then $\tilde{\alpha}_n$ is a good strategy if one is primarily interested in the relative advantage of the players. For in considering relative advantage we must consider the matrix $(a_{ij} - b_{ij},\ b_{ij} - a_{ij})$ which constitutes a zero-sum game. In the above example, we have

$$\begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} (0,\ 0) & (-20,20) \\ (20,-20) & (0,\ 0) \end{bmatrix} \end{array}$$

and α_2 is an optimal strategy in the minimax sense. Then in the product game, successive use of α_2 at each move is again optimal.

In the product game where n is unknown to the players, but finite, then repeated application of the second choice (α_2 or β_2) constitues a minimax strategy, and is an optimal strategy for the relative game. Further, a pair of such strategies is an equilibrium point. But if one is primarily interested in maximizing his "(subjective) expected return" then one must realize the possibility of his opponent effectively using strategy $\tilde{\beta}_n$ or $\tilde{\alpha}_n$ .From consideration

of (3.5.1), a reasonable behavior pattern is not to use the second choice α_2 or β_2 before your opponent. Of course, in so doing one immediately forfeits the possibility of winning the relative game. Again if n is unknown to both players then the product game can not be reduced. If both players willingly attempt to forfeit the relative game then this mode of behavior is to their mutual advantage.

In peroration, the above game where n = 1 has a non-cooperative rational solution, namely α_2 or β_2. For n finite, but known, the product game has a non-cooperative rational solution in the wide sense but an appraisal of one's opponent keeping in mind 3.5.1 (recall that this inequality depended on metric considerations) is indispensible. For n unknown, a player should not play α_2 or β_2 before his opponent and for n = ∞ the same analysis holds.

If we now were to replace the component game by a game with similar structure, say,

$$\begin{bmatrix} (1, 1) & (-100, 100) \\ (100,-100) & (-1, -1) \end{bmatrix}$$

then the analysis for n = 1 would be similar. However, the metric conditions are such that n must be very large before a player should be tempted not to play his only completely reduced strategy.

3.6 Maximin Value in Wide Sense.

Any mixed strategy ξ° for player I is said to be a maximin strategy relative to the set C x D where $C < S_I$, $D < S_{II}$ if $\xi^\circ \varepsilon C$ and

$$\min_{\eta \varepsilon D} A(\xi^\circ, \eta) = \max_{\xi \varepsilon C} \min_{\eta \varepsilon D} A(\xi, \eta).$$

Any maximin strategy, $\eta^{(\circ)}$, for player II relative to the set C x D must satisfy the relation, $\eta^\circ \varepsilon D$ and

$$\min_{\xi \varepsilon C} B(\xi, \eta^{(\circ)}) = \max_{\eta \varepsilon D} \min_{\xi \varepsilon C} B(\xi, \eta)$$

Let $C_0 = S_I$, $D_0 = S_{II}$ and let $C_1 = \{\xi^0\}$, $D_1 = \{\eta^0\}$ be the maximin strategies of players I and II respectively relative to $C_0 \times D_0$. Now let us consider the extended game G defined over $S_I \times S_{II}$ restricted to the spaces $C_1 \times D_1$. We now can repeat this process and get a sequence $\{C_n \times D_n\}$ where $C_n \times D_n$ is the product space of maximin strategies relative to $C_{n-1} \times D_{n-1}$. Since the maximin operator is continuous, C_n and D_n are closed and convex and thus $C_n \times D_n$ is closed and convex in $S_I \times S_{II}$. Further $(C_n \times D_n) \supset (C_{n+1} \times D_{n+1})$ and thus we know that $\bigcap_n (C_n \times D_n)$ is non-empty closed and convex; denote this set by $\tilde{C} \times \tilde{D}$. We have also

$$\max_{\xi \in C_n} \min_{\eta \in D_n} A(\xi, \eta) = \max_{\xi \in C_{n+1}} \min_{\eta \in D_n} A(\xi, \eta) \leq \max_{\xi \in C_{n+1}} \min_{\eta \in D_{n+1}} A(\xi, \zeta) \leq A(\xi', \zeta')$$

for all $(\xi', \eta') \in (C_{n+2} \times D_{n+2})$,

and similarly,

$$\max_{\zeta \in D_n} \min_{\xi \in C_n} B(\xi, \zeta) \leq \max_{\zeta \in D_{n+1}} \min_{\xi \in C_{n+1}} B(\xi, \eta) \leq B(\xi', \zeta')$$ for all $(\xi', \eta') \in (C_{n+2} \times D_{n+2})$. It now follows that for any $\xi' \in \tilde{C}$, $\eta' \in \tilde{D}$ we have

$$A(\xi', \eta') \geq \lim_n \max_{\xi \in C_n} \min_{\eta \in D_n} A(\xi, \eta) \equiv \gamma_1$$

$$B(\xi', \eta') \geq \lim_n \max_{\eta \in D_n} \min_{\xi \in C_n} B(\xi, \eta) \equiv \gamma_2$$

since if $(\xi', \eta') \in (\tilde{C} \times \tilde{D})$ then $(\xi', \zeta) \in (C_n \times D_n)$ for all n. To show that the equality need not hold in the above equations we exhibit the game

$$\begin{bmatrix} (0, 0) & (10, 0) \\ (0, 100) & (10, 0) \end{bmatrix}$$

in which $\tilde{C} \times \tilde{D} = S_I \times S_{II}$ and $\gamma_1 = \gamma_2 = 0$.

If both players are conservative they can restrict themselves to $\tilde{C} \times \tilde{D}$ and thus guarantee an imputation which is compositely just as good as (γ_1, γ_2). The imputation (γ_1, γ_2) will be termed the <u>composite maximin value in the wide sense.</u>

If $\xi^{(0)}$ is such that $\max_{\xi} \min_{\eta} A(\xi, \eta) = \min_{\eta} A(\xi, \eta) = v_1$ then $\xi^{(0)}$ and v_1 are said to be the maximin strategy and value respectively for player I. Similarly there will be a maximin strategy $\eta^{(0)}$ and maximin value v_2 for player II. The imputation (v_1, v_2) will be termed as the composite maximin value which will be equal or compositely dominated $(\prec)$ by the maximin value in the wide sense. The strategy $\xi^0 [\eta^0]$ enjoys a security of $v_1 [v_2]$ and no other strategy enjoys a security greater than $v_1 [v_2]$.

3.7 <u>Psychological Factors.</u>

3.7.1 An interesting game illustrating some of these points goes as follows:

$$\begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} (2, 1) & (-1, -1) \\ (-1, -1) & (1, 2) \end{bmatrix} \end{array} .$$

The region R^* is not convex and <u>independent</u> mixed strategies $(\frac{\alpha_1}{2} + \frac{\alpha_2}{2}, \frac{\beta_1}{2} + \frac{\beta_2}{2})$ yields the imputation $(\frac{1}{4}, \frac{1}{4})$. However, $(\frac{\alpha_1}{2} + \frac{\alpha_2}{2}, \frac{\beta_1}{2} + \frac{\beta_2}{2})$ is not an equilibrium point. The Bayes solution against $\frac{\alpha_1}{2} + \frac{\alpha_2}{2}$ is β_2 yielding the return $(0, \frac{1}{2})$. If player I plays his maximin strategy $\frac{2}{5}\alpha_1 + \frac{3}{5}\alpha_2$, to guarantee a return to himself of $\frac{1}{5}$ then player II by playing β_2 gets a return of $\frac{4}{5}$. Hence β_2 is a Bayes solution against a priori distributions $\alpha_2, \frac{\alpha_1}{2} + \frac{\alpha_2}{2}$ and $\frac{2}{5}\alpha_1 + \frac{3}{5}\alpha_2$. Similarly α_1 is a Bayes solution against a priori distributions $\beta_1, \frac{\beta_1}{2} + \frac{\beta_2}{2}, \frac{3}{5}\beta_1 + \frac{2}{5}\beta_2$. Even though both players can gain with certainty

an expected return of at least $\frac{1}{5}$ it might be advantageous for a third party to actually be the banker in such a gambling situation; for the strategies $\frac{2}{5}\alpha_1 + \frac{3}{5}\alpha_2$ and $\frac{3}{5}\beta_1 + \frac{2}{5}\beta_2$ are such "sure bets" that there is a strong temptation for each player to play the Bayes solutions α_1 and β_2 giving the banker a gain of two units. But the players realize that the banker's sole motive to play the game is that each player will play a Bayes solution against his opponent's maximin strategy. This is all the more reason to play safe by playing the maximin strategy which is all the more reason to play the Bayes solution against the maximin strategy. From here on the argument is cyclic and where it ends is not of a mathematical nature.

3.7.2 The bulk of two-person games written down at random do not have a non-cooperative rational solution in any sense mentioned in this section. The difficulty in extending the theory realistically to all games is adequately illustrated by the game of 3.7.1. However, one could attempt to extend the domain of analysis by introducing a more subtle partial ordering of equilibrium points. We first observe that if $(\xi^{(i)}, \eta^{(i)})$ is an equilibrium point then

$$(A(\xi^{(i)}, \eta^{(i)}), B(\xi^{(i)}, \eta^{(i)})) \geq (v_1, v_2)$$

where (v_1, v_2) is the composite maximin value. To verify the above assertion, let $\xi^{(o)}$ be a maximin strategy; that is, $\max_{\xi} \min_{\eta} A(\xi, \eta) = \min_{\eta} A(\xi^{(o)}, \eta)$. Now since $\xi^{(i)}$ is good against $\eta^{(i)}$ we have

$$v_1 = \min_{\eta} A(\xi^{(o)}, \eta) \leq A(\xi^{(o)}, \eta^{(i)}) \leq A(\xi^{(i)}, \eta^{(i)}).$$

Similarly; $v_2 \leq B(\xi^{(i)}, \eta^{(i)})$ which proves the assertion. Hence if we can single out an equilibrium point as a solution then, at least, it will compositely

dominate or be equal to the values that the players can absolutely guarantee themselves (the example of 3.5 shows that this may not be a great deal). Now let us consider the game

$$\begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} (1,\ 2) & (-1,\ -4) \\ (-4,\ -1) & (2,\ 1) \end{bmatrix} \end{array} .$$

The maximin strategy and value to player I are $\xi^{(\omega)} = (\frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2)$ and $V_1 = -\frac{1}{4}$; the maximin strategy and value to player II are $\eta^{(\omega)} = (\frac{5}{8}\beta_1 + \frac{3}{8}\beta_2)$ and $V_2 = -\frac{1}{4}$. In this case $(\gamma_1, \gamma_2) = (-\frac{1}{4}, -\frac{1}{4})$ also. But now the Bayes solution against $\eta^{(\omega)}$ and $\xi^{(\omega)}$ are $\xi_{\eta^{(\omega)}} = \alpha_1$ and $\eta_{\xi^{(\omega)}} = \beta_1$ respectively. We then note that (α_1, β_1) is an equilibrium and an admissible point yielding (1, 2). For want of a better name we can say (α_1, β_1) "psychologically dominates" (α_2, β_2).

As another example, consider the game

$$\begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} (4,\ -30) & (10,\ 6) \\ (8,\ 8) & (5,\ 4) \end{bmatrix} \end{array} .$$

Now (α_2, β_1) and (α_1, β_2) are equilibrium points and the maximin strategies are α_2 and β_2. However (α_2, β_2) is not an equilibrium value. But again we may say that (α_1, β_2) "psychologically dominates" (α_2, β_1) since player I takes less of a gamble in playing α_1 than player II has in playing β_1.

Another possibility of extending the domain of analysis for non-cooperative games goes roughly as follows: If the players could communicate, if it is then "obvious" what they should do, and if the cooperative solution

forms a permutable equivalence class of equilibrium points then choose any such point as the non-cooperative solution. The difficulty in formalizing this concept is that the cooperative game can only be solved in general by means of conventions of arbitration.

We can even put a damper on the notion of rational solution by consideration of the following game:

$$\begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} (4, -3000) & (10, 6) \\ (12, 8) & (5, 4) \end{bmatrix} \end{array}$$

Although (α_2, β_1) represents both an equilibrium and admissible point (and this is the rational solution) the players may not act accordingly. Player I may argue that β_2 "psychologically dominates" β_1 and thus player I may decide to play α_1 which is the Bayes solution for β_2. On the other hand, player II can give some rationale for player I using α_1 and thus player II does not dare to play β_1. The argument is cyclic in nature and leads to (α_1, β_2). Hence even if there is a unique maximal element (with respect to partial order $\prec\prec$) in the set of equilibrium points it is still a moot question whether the players would play in accordance with this element.

What I have attempted to demonstrate in this section is that the mathematical axiomatic structure of game theory does not reflect fully the psychological factors of the non-cooperative game. Nevertheless the factors introduced in this section are realistic enough to demand scrutiny before coming to a decision in a non-cooperative game.

3.8 Generalizations to n-Players.

The extensions of the results of this section to n-person games carry over partially. The concepts of incomparable, convex incomparable, and linear games have immediate extensions. It is unfortunate, however, that the saddle points of an incomparable game do not form an equivalence class. The rest of the material does generalize though. We still can define the notion of rational solution and then sections 3.2 to 3.4 carry over almost intact. We can give an interesting interpretation of the game discussed in 3.5 by generalizing to n-players. Let the n participants be wheat farmers, and let each player's strategy be the amount of wheat brought to market. Further restrict each player to two strategies: full or restricted production. If all players use restricted production the price of wheat soars and the players fare well. However, regardless of what the other players do, it behooves each farmer to produce fully as his contribution does not affect the price substantially. Consequently full production dominates restricted production and if all farmers behave accordingly the price drops so low that they fare poorly. As in section 3.5 we can iterate this game in time and the discussion of 3.5 carries over substantially. Thus in the mature market there would be a marked degree of collusion without any pre-play communication between the players. When non-cooperative games are iterated the results are more in accord with the cooperative case. The difficulty in the military and political spheres is that the players usually have a single play orientation.

Section IV

Cooperative Games

41. Existence Theorem.

In the finite cooperative game without bribery the players by using joint strategies $\zeta \in J$ can realize any element of R^{**}. Consequently, if the players decide to enter into an active collusion they should confine themselves to the consideration only of elements of $M^{**}_{\prec\prec}$ (or to make the further restriction to $M^{**}_{\prec}$) since any imputation of R^{**} not in $M^{**}_{\prec\prec}$ is compositely dominated by an element of $M^{**}_{\prec\prec}$. If $M^{**}_{\prec\prec}$ consists of one element the game is trivially resolved. However, if $M^{**}_{\prec\prec}$ consists of more than one element then the elements of this set are compositely incomparable. If $\tilde{E}^{*} \cap \tilde{M}^{**}_{\prec\prec}$ is non-empty and forms an equivalence class then the corresponding imputation could be taken as the value of the game by convention. But as in the non-cooperative game this leaves the large majority of games unresolved. Even if $\tilde{E}^{*} \cap M^{**}_{\prec\prec}$ is a permutable equivalence class this solution may not be realistic in case there is pre-play communication (cf. example on page 57).

We now propose to exhibit a class of "conventions" such that each "convention" will uniquely resolve <u>all</u> finite two-person games. The arbitrated cooperative solution will always belong to $M^{**}_{\prec}$ in case of no bribery and to M^{***} in case bribery is permissible. Further, this imputation will be such that each player will receive at least as much as the maximin value of his non-cooperative game. (For players I and II this is respectively $\max_{\xi} \min_{\eta} A(\xi, \eta)$ and $\max_{\eta} \min_{\xi} B(\xi, \eta)$). In this section it will be understood that all games are finite unless explicitly stated otherwise. For each convention we will find a characterization of the arbitrated solution and in some cases this characterization will be generalized to the infinite case.

Let $\mathcal{G}$ be a given game with region R^{*}. Let $\mathfrak{P}: R^{*} \longrightarrow\!\!\!\!\rightarrow M^{**}_{\prec}$ be a mapping with the following properties:

a) T is continuous.

b) $T(e) \geq e$ for all $e \in R^*$.

c) If $T(e) = T(e')$ then $T(\alpha e + (1-\alpha) e') = T(e) = T(e')$

for $e, e' \in R^*$ and all $0 \leq \alpha \leq 1$, such that

$\alpha e + (1-\alpha) e' \in R^*$.

Now consider the composite mapping:

$$S_I \times S_{II} \xrightarrow{A \times B} R^* \xrightarrow{T} M_i^{**}$$

where

$$(\xi, \eta) \longrightarrow (A(\xi, \eta), B(\xi, \eta)) \longrightarrow T[A(\xi, \eta), B(\xi, \eta)] \equiv$$

$$\equiv (A^{**}(\xi, \eta), B^{**}(\xi, \eta)) .$$

Let us now define a new game g^{**} where players I and II have the spaces of pure strategies $\mathfrak{A}^{**} = \{\xi\}$ and $\mathfrak{B}^{**} = \{\eta\}$ and the payoff is $(A^{**}(\xi, \eta), B^{**}(\xi, \eta))$. The game g^{**} is now a convex incomparable game and consequently if saddle points exist they form a permutable equivalence class. The principal theorem asserts that saddle points always exist and the corresponding imputation will be termed the arbitrated solution relative to the convention T.

If g is a linear game then T is the identity mapping (which trivially satisfy conditions a)-c)) and consequently as a corollary of the principal theorem we have that all finite linear games have a rational solution.

We will utilize the following lemma which was first proved by von Neumann, who utilized the Brouwer Fixed Point Theorem; it was later proved by Kakutani as an immediate consequence of the Kakutani Fixed Point Theorem (cf. Kakutani (1)).

Lemma: Let S_m and S_n be two bounded, closed convex sets in Euclidean Spaces R_m and R_n respectively. If U and V are two closed subsets of $S_m \times S_n$, such

that

$$U_{\xi^{(0)}} \equiv \{ \eta \in S_n : (\xi^{(0)}, \eta) \in U \}$$

and

$$V_{\eta^{(0)}} \equiv \{ \xi \in S_m : (\xi, \eta^{(0)}) \in V \}$$

are both non-empty, closed and convex subsets of S_m and S_n respectively, for all $\xi^{(0)} \in S_m$ and $\eta^{(0)} \in S_n$, then $U \cap V \neq \emptyset$.

Principal Theorem: The game g^{**} has at least one saddle point.

Proof: Let S_m and S_n be the probability spaces of players I and II respectively and let

$$U = \{(\xi, \eta_\xi) : B^{**}(\xi, \eta_\xi) = \max_\eta B^{**}(\xi, \eta)\}$$

and

$$V = \{(\xi_\eta, \eta) : A^{**}(\xi_\eta, \eta) = \max_\xi A^{**}(\xi, \eta)\}.$$

That is, U is the set of elements (ξ, η) for which η is "good against" ξ relative to the game g^{**}; and V is the set of elements (ξ, η) for which ξ is "good against" η relative to the game g^{**}. Since the mapping $T[(A \times B)(\xi, \eta)]$ is continuous on $S_m \times S_n$ it follows that $\max_\eta B^{**}(\xi, \eta)$ is a comtinuous function of ξ and thus U is closed. Similarly V is closed. The set

$$U_{\xi^{(0)}} = \{ \eta_{\xi^{(0)}} : B^{**}(\xi^{(0)}, \eta_{\xi^{(0)}}) = \max_\eta B^{**}(\xi^{(0)}, \eta)\}$$

is non-empty and closed because of the compactness of S_n and from continuity considerations. As regards convexity, let η'_{ξ° and η''_{ξ° both belong to U_{ξ° and let $\max_\eta B^{**}(\xi^{(0)}, \eta) = m_o$. Then $T[(A \times B)(\xi^{(0)}, \eta'_{\xi^{(0)}})] = T[(A \times B)(\xi^{(0)}, \eta''_{\xi^{(0)}})] = m_o$. From property c) of mapping T, and the restricted linearity of $A \times B$ it follows that $T[(A \times B)(\xi^\circ, \alpha\eta'_{\xi^\circ} + (1-\alpha)\eta''_{\xi^\circ})] = m_o$ and thus $\alpha\eta'_{\xi^\circ} + (1-\alpha)\eta''_{\xi^\circ}$ belongs to U_{ξ°. Similarly, the set

$$V_{\eta^{(0)}} = \{ \xi_{\eta^{(0)}} : A^{**}(\xi_{\eta^{(0)}}, \eta^{(0)}) = \max_\xi A^{**}(\xi, \eta^{(0)})\}$$

is non-empty, closed and convex. Consequently, $U \cap V \neq \emptyset$.

Let $(\xi^{(\omega)}, \eta^{(\omega)}) \in U \cap V$. Then $\xi^{(\omega)}$ is "good against" $\eta^{(\omega)}$ and $\eta^{(\omega)}$ is "good against" $\xi^{(\omega)}$ and therefore $(\xi^{(\omega)}, \eta^{(\omega)})$ is a saddle point. Q.E.D.

We will show that the class of T mappings satisfying properties a)-c) is non-empty by considering various specific T conventions. If there exists a common unit of measurement and bribery is permissible then let $T: R^{*} \longrightarrow M^{***}$. The induced game g^{***} is then a constant sum game and by the same argument has a saddle point; again these saddle points form a permutable equivalence class which yields the arbitrated solution relative to the convention T.

John F. Nash, Jr. has shown that all discrete n-person games have equilibrium points. For n = 2 one merely follows the argument of the principal theorem by replacing A^{**} and B^{**} by A and B respectively and T by the identity mapping.

4.2 Stable Conventions.

If a slight perturbation of the payoff entries would result in a drastic change in the arbitrated solution, then the convention would not be stable enough to be practical. After all, the payoff entries are at best only hazy appraisals of the possible terminal outcomes of the game. The desire to capture this continuity notion motivates this section.

Let $g_k = \{(a_{ij}^{(k)}, b_{ij}^{(k)})\}$ for i = 1, 2, ..., m; j = 1, 2, ..., n and for k = 1, 2, ... be a sequence of two person games. The sequence $\{g_k\}$ is said to converge to the game $g = \{(a_{ij}, b_{ij})\}$ if for all i, j it follows that $\lim_k (a_{ij}^{(k)}, b_{ij}^{(k)}) = (a_{ij}, b_{ij})$.

The sequence of sets $\{H_k\}$ in R_2 is said to converge to the set H provided for every $h \in H$ there exists a sequence of elements $h_k \in H_k$ such that $\lim_k h_k = h$ and conversely for every $\tilde{h}$ which belongs to the complement of H there

is a neighborhood of $\tilde{h}$ which intersects at most a finite number of H_k.

If the sequence of games $\{g_k\}$ converges to the game g then the sequences of sets $\{R_k\}$, $\{R_k^*\}$, $\{R_k^{**}\}$, $\{R_k^{***}\}$ converge to R, R^*, R^{**} and R^{***} respectively. However, it does not follow that $\{M_k\}$, $\{M_k^*\}$ or $\{M_k^{**}\}$ converges to M, M^* and M^{**} respectively. But if the linear graph of M^* or M^{**} with respect to the ordering $\ll$ does not contain any horizontal or vertical one simplex then the above convergence follows with respect to the orderings $\prec$ and $\ll$.

Let $(V_1(T, g), V_2(T, g))$ be the arbitrated solution of the game g relative to the convention T. The convention T is said to be <u>stable</u> if for every sequence $\{g_k\}$ converging to g such that $\{M_k^{**}\}$ converges to $\{M^{**}\}$ we have $\lim_k [V_1(T, g_k), V_2(T, g_k)] = [V_1(T, g), V_2(T, g)]$. We will indicate in 4.9 the reason for requiring the convergence of $\{M_k^{**}\}$ to M^{**}.

4.3 <u>Symmetric Conventions.</u>

The game $g^\tau = \{(c_{ji}, d_{ji})\}$ will be said to be the transpose of the game $g = \{(a_{ij}, b_{ij})\}$ provided that $c_{ji} = b_{ij}$, and $d_{ji} = a_{ij}$. Thus the transpose of a game results from an interchange of the roles of the players. A convention T is <u>symmetric</u> if

4.3.1 $$[V_1(T, g^\tau), V_2(T, g^\tau)] = [V_2(T, g), V_1(T, g)] .$$

We shall confine ourselves to the discussion of conventions for which the arbitrated solution is invariant with respect to any permutations of the pure strategies $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ and $(\beta_1, \beta_2, \ldots, \beta_n)$. All conventions will be understood to have this invariance property.

4.4 <u>The Payoff Entries Have a Common Unit of Measurement.</u>

In the case the payoff entries of both players have a common unit of measurement then we require that T satisfy the following property:

4.4.1) $V_i(T, \{\lambda (a_{ij} - c_1), \lambda (b_{ij} - c_2)\}) =$

$= \lambda\{V_i(T, \{(a_{ij}, b_{ij})\}) - c_i\}$ for i = 1, 2,

where $\lambda > 0$, c_1, c_2 are arbitrary real numbers.

A game $\mathcal{G} = \{(a_{ij}, b_{ij})\}$ is said to be symmetric provided: m = n, and there exist permutations $(\alpha_{p_1}, \alpha_{p_2}, \ldots, \alpha_{p_m})$ and $(\beta_{q_1}, \beta_{q_2}, \ldots, \beta_{q_n})$ and a real number c such that $\mathcal{G}_P = \{(a_{p_i q_j}, b_{p_i q_j} - c)\}$ is its own transpose (i.e. $b_{p_i q_j} - c = a_{q_j p_i}$). If T is a symmetric convention and $\mathcal{G}$ is the above symmetric game then it follows from 4.3.1 and 4.4.1 that

$$V_1(T, \mathcal{G}) = V_2(T, \mathcal{G}) + c .$$

For example, the game

$$\begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} (2, 0) & (-10, -11) \\ (-10, -11) & (1, 1) \end{bmatrix} \end{array}$$

is a symmetric game (let c = -1 and permute (α_1, α_2) to (α_2, α_1)) and with respect to any symmetric convention the value of this game is (3/2, 1/2).

If $\mathcal{G}$ is symmetric then there exists a value c such that the regions R, R^* and R^{**} are symmetric about the line through (0, c) with direction numbers (1, 1). However, the converse is not true; namely, the regions may be symmetric about some line with direction numbers (1, 1) without having the game symmetric. An example of such a game is as follows:

$$\begin{bmatrix} (2, 1) & (-4, -1) \\ (-1, -4) & (1, 2) \end{bmatrix} .$$

4.5 Convention T_0.

In this section we shall give example of a stable, symmetric convention T_0 satisfying condition 4.4.1. We shall first consider the cooperative

case in which bribery is allowed and then the case where bribery is not permitted.

The maximal elements of R^{***} corresponding to the game $\{(a_{ij}, b_{ij})\}$ all have the property that the sum of their components is a constant, say c. Corresponding to the pair of mixed strategies (ξ, η) we get $(A(\xi, \eta), B(\xi, \eta))$. Let each of the players then receive an additional amount x such that the total income is c. Thus we have

$$[A(\xi, \eta) + x] + [B(\xi, \eta) + x] = c,$$

and corresponding to the pair (ξ, η) there is an induced payoff

$$A^{***}(\xi, \eta) = \frac{c}{2} + \frac{A(\xi, \eta) - B(\xi, \eta)}{2}$$

and

$$B^{***}(\xi, \eta) = \frac{c}{2} + \frac{B(\xi, \eta) - A(\xi, \eta)}{2}.$$

We now normalize this constant-sum game by a simultaneous change of origin which yields the zero-sum game $\left\{\left(\frac{A(\xi, \eta) - B(\xi, \eta)}{2}, \frac{B(\xi, \eta) - A(\xi, \eta)}{2}\right)\right\}$. By linearity of the operations A and B this continuous game is equivalent to the discrete game with payoff matrix of the form

$$\left\{\left(\frac{a_{ij} - b_{ij}}{2}, \frac{b_{ij} - a_{ij}}{2}\right)\right\}.$$

If $(V_j - V)$ is the value of the induced zero-sum game then the value of the induced constant-sum game is $(V + \frac{c}{2}, -V + \frac{c}{2})$. We note that if a game is its own transpose then the zero-sum game with matrix $\frac{(a_{ij} - b_{ij})}{2}$ is anti-symmetric and thus has value zero. The arbitrated solution of such a game with the above convention is $(\frac{c}{2}, \frac{c}{2})$.

The mapping T_o corresponding to this convention maps each element e of R^* into e' of M^{***} in such a manner that the ray from e to e' has

direction numbers (1, 1). The mapping T_o trivially satisfies a) -c) of page 44 ; that T_o is symmetric follows from the fact that the transposed game yields the induced game,

$$\left\{\left(\frac{b_{ji} - a_{ji}}{2} \quad , \quad \frac{a_{ji} - b_{ji}}{2}\right)\right\}.$$

The value of this zero-sum game is $(-v, v)$ and hence the value of the induced constant sum game, $(B^{***}(\eta, \xi), A^{***}(\eta, \xi))$, is $(-v + \frac{c}{2}, v + \frac{c}{2})$ and therefore

$$[V_1(T_o, g), V_2(T_o, g)] = [V_2(T_o, g^T), V_1(T_o, g^T)] .$$

To show that T_o is stable let $\{g_k\}$ be a sequence of games converging to g. If $c_k = \max_{i,j} \{ a_{ij}^{(k)} + b_{ij}^{(k)} \}$ then $\{c_k\}$ converges to c. Further if $(v^{(k)}, -v^{(k)})$ is the value of the game

$$\left\{\left(\frac{a_{ij}^{(k)} - b_{ij}^{(k)}}{2} \quad , \quad \frac{b_{ij}^{(k)} - a_{ij}^{(k)}}{2}\right)\right\}$$

then $(v^{(k)}, -v^{(k)})$ approaches $(v, -v)$. Consequently $\{[V_1(T_o, g^{(k)}), V_2(T_o, g^{(k)})]\}$ approaches $[V_1(T_o, g), V_2(T_o, g)]$. Finally, T_o satisfies 4.4.1 since in a zero-sum game, we have

$$v[(\mu a_{ij} + \nu, -\mu a_{ij} - \nu)] = \mu v[(a_{ij}, -a_{ij})] + \nu$$

where $v[(a_{ij}, -a_{ij})]$ represents the value of the zero-sum game $\{(a_{ij}, -a_{ij})\}$ as defined by von Neumann.

We now generalize, by considering a finite game $\{(A(\alpha, \beta), B(\alpha, \beta))\}$ in which there is a common unit of measurement and the game is understood in the sense of collusion <u>without</u> bribery.

Let the convention $T_o: R^* \longrightarrow M_4^{**}$ be defined as follows: first define T_o for a subset Q of R^{**}. Let $e \in Q$ if there exists an $e' \in M_4^{**}$ such that the

ray e to e' has direction numbers (1, 1) and then define $T_o(e) = e'$. Now extend the mapping T_o from Q to R^{**} by requiring that for every $e' \in M_{\prec}^{**}$ the set $\{e \in R^{**}: T_o(e) = e'\}$ is connected. This extension is unique and thus T_o is uniquely defined on R^{**}. Now restrict the domain to R^* and note that T_o satisfies conditions a) -c) of page 44. It is suggested that the reader carry out this mapping for the diagram on page 53.

Motivated by our previous argument let us consider the zero-sum game with payoff $K(\alpha, \beta) = A(\alpha, \beta) - B(\alpha, \beta)$ and let us assume that this game has a minimax solution $(v, -v)$ with generic optimal strategies $\xi^{(o)}$ and $\eta^{(o)}$. By the linearity of operators A and B we then have $A(\xi^{(o)}, \eta) - B(\xi^{(o)}, \eta) \geq v$ for all η and $A(\xi, \eta^{(o)}) - B(\xi, \eta^{(o)}) \leq v$ for all ξ, and $A(\xi^{(o)}, \eta^{(o)}) = B(\xi^{(o)}, \eta^{(o)}) + v$. Hence, all elements of the form $(A(\xi^{(o)}, \eta^{(o)}), B(\xi^{(o)}, \eta^{(o)}))$ where $(\xi^{(o)}, \eta^{(o)})$ are pairs of optimal strategies of $K(\alpha, \beta)$, lie on a ray, r_o, with direction numbers (1, 1). By referring to the diagram on page 53, it should be clear that depending on the value of v, the ray r_o may or may not intersect $M_{\prec}^{**}$.

If we define $T^{(1)}$ and $T^{(2)}$ (corresponding to T) by the relation $T(A \times B)(\xi, \eta) = [T^{(1)}(A \times B(\xi, \eta), T^{(2)}(A \times B)(\xi, \eta)]$ and if $r_o \cap M_{\prec}^{**} \neq \emptyset$ then for the convention T_o we have

$$T_o^{(1)}(A \times B)(\xi^{(o)}, \eta) - T_o^{(2)}(A \times B)(\xi^{(o)}, \eta) \geq V, \text{ all } \eta$$

and

$$T_o^{(1)}(A \times B)(\xi, \eta^{(o)}) - T_o^{(2)}(A \times B)(\xi, \eta^{(o)}) \leq V, \text{ all } \xi .$$

Consequently if $r_o \cap M_{\prec}^{**} \neq \emptyset$, then the intersection defines some element, say (m_1, m_2) which is the arbitrated solution of the game g relative to the convention T_o. By choice of a $\xi^{(o)}$, player I gets at least m_1 and by choice of an $\eta^{(o)}$, player II gets at least m_2. By the incomparability of the elements of

$M_{\prec}^{**}$, this is an equilibrium situation. If $r_o \wedge M_{\prec}^{**} = \emptyset$ then the arbitrated solution is one of the end points of the continuum $M_{\prec}^{**}$ which is closest to the ray r_o.

For all finite games the above characterization determines the value $[v_1(T_o, \mathcal{G}), v_2(T_o, \mathcal{G})]$ and in these cases the value of the corresponding zero-sum game can be actually computed (cf. Raiffa Thompson and Thrall (1)).

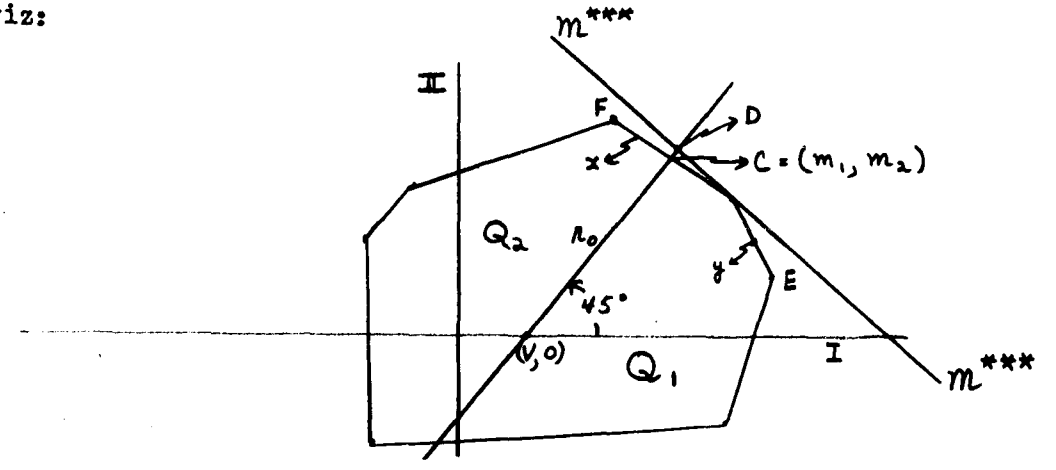
Relative to the convention T_o we can now give a characterization of the arbitrated solution for a wide class of generalized two person games. Let $\mathcal{G} = \{(A(\alpha, \beta), B(\alpha, \beta)\}$ be any generalized game for which $\mathcal{A} = \{\alpha\}$ and $\mathcal{B} = \{\beta\}$ are arbitrary sets with or without a topology. Assume that the zero-sum game $K(\alpha, \beta) = A(\alpha, \beta) - B(\alpha, \beta)$ is strictly determined in the sense of A. Wald (1), i.e.

$$\sup_{\xi} \inf_{\eta} \int_{\mathcal{A}} \int_{\mathcal{B}} K(\alpha, \beta)\, d\eta\, d\xi = \inf_{\eta} \sup_{\xi} \int_{\mathcal{A}} \int_{\mathcal{B}} K(\alpha, \beta)\, d\eta\, d\xi = v$$

where $\xi[\eta]$ is a completely additive probability measure over some suitably chosen Borel field of $\mathcal{A}[\mathcal{B}]$ which contains all pure strategies. (We assume also that $K(\alpha, \beta)$ is bounded and measureable over the product space). Then define r_o as the ray through the point $(v, 0)$ with direction numbers (1, 1). If $r_o \wedge M_{\prec}^{**} \neq \emptyset$, call this the arbitrated solution of the game $\mathcal{G}$ relative to the convention T_o. If $r_o \wedge M_{\prec}^{**} = \emptyset$ then call any maximal element of R which is nearest (assumption made) to the ray r_o the arbitrated solution. Thus, if R is bounded (therefore $M_{\prec}^{**}$ is compact) and $K(\alpha, \beta)$ is strictly determined, then an arbitrated solution exists and this solution can be approximated arbitrarily close by the players using a joint strategy which utilizes at most two pairs of pure strategies.

We now consider a further rationalization (again metatheoretic) for convention T_o besides the ones already mentioned (i.e. agreement with von Neumann's solution in linear games, stability, symmetry and existence of a

computational technique in the finite case). We first note that the ray r_o partitions R^{**} into two regions denoted by Q_1 and Q_2 in the accompanying figure, viz:



Let us consider the actual bargaining situation involved in this game. If both players are aware of the region R^{**} and agree to confine their cooperation behavior to consideration only of elements of M_1^{**}, then player II desires point F and player I desires point E. In the ensuing argument each player intimidates his opponent by threatening to play non-cooperatively unless his opponent will agree to his terms. In jockeying about for bargaining advantage a threat by player I to minimize player II's maximum possible gain is not very convincing if at the same time player I has to suffer more than player II. For example, if player II desires point x of M_1^{**} (cf. figure) then player I can choose a strategy $\xi^{(0)}$ which will guarantee an imputation on r_o or in Q_1. Consequently, relative to the point x player II has "more to lose" than player I in a threat of this variety. If player I wishes point y then by the choice of an $\eta^{(0)}$ player II can confine the result of the non-cooperative game to r_o or Q_2 and thus relatively inflict a heavier loss on player I than upon himself. Thus in the argumentative stage the players threaten to play the non-cooperative relative game - i.e. they consider entries of the form $\{A(\alpha, \beta) - B(\alpha, \beta)\}$ which will be termed the

"bargaining payoff, relative to the convention T_o". At point C neither player has a bargaining advantage and the corresponding imputation is the arbitrated solution relative to the convention T_o. If bribery is possible then the players can realize any element of M^{***}. The same intuitive analysis prevails and this is reflected in the formal theory by noting that convention T_o yields the imputation denoted by D. In the case $r_o \cap M_<^{**} \neq \emptyset$ then the bribery case affords each player with the same additional amount over the non-bribery case.

4.5.1 Examples: (All games are assumed to have a common unit of measurement)

1. Let

$$\mathcal{G}_1 \equiv \begin{bmatrix} (2, 1) & (-1, -1) \\ (-1, -1) & (1, 2) \end{bmatrix} .$$

This game is symmetric and thus all symmetric conventions relative to the cooperative game with or without bribery yield the imputation (3/2, 3/2). For a discussion of this non-cooperative game see Paragraph 3.7.

2. Let

$$\mathcal{G}_2 \equiv \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} (0, 0) & (1, -3) \\ (0, 1) & (-1, -2) \end{bmatrix} \end{array} .$$

We get $[V_1(T_o, \mathcal{G}_2), V_2(T_o, \mathcal{G}_2)] = (1/5, 1/5)$ and $(1/2, 1/2)$ for the cooperative game without and with bribery respectively. The non-cooperative game $\mathcal{G}_2$ can be reduced since α_1 dominates α_2 and β_1 dominates β_2. The non-cooperative rational solution is (0, 0).

3. Let

$$\mathcal{G}_3 \equiv \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} (0, 0) & (10, 0) \\ (0, 100) & (10, 0) \end{bmatrix} \end{array} .$$

This is a particularly frustrating non-cooperative game. In the altruistic partial ordering $\prec$ the game is resolvable. Player II will always use β_1 and thus player I uses α_2. However, if we consider a product of games of variety $\mathcal{G}_3$ then player I should utilize α_1 to force II into a correlated mixed strategy in which II condescends to use β_2 also. The n-fold product game does not have a solution in the wide sense but a modus operandi for both players is obvious. In the cooperative case we have $[V_1(T_0, \mathcal{G}_3), V_2(T_0, \mathcal{G}_3)] = (\frac{100}{11}, \frac{100}{11})$ and (50, 50) for the non-bribery and bribery cases respectively. The fact that in the bribery case the arbitrated return to player I exceeds all his payoff entries is not surprising. Indeed, if we changed the tens to zeros the arbitrated result is still (50, 50). After all, player I has a distinct bargaining advantage!

4. and 5. Let

$$\mathcal{G}_4 = \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} (1, 2) & (-1, -4) \\ (-4, -1) & (2, 1) \end{bmatrix} \end{array} .$$

This game is <u>not</u> symmetric. Player II has a bargaining advantage in the cooperative game and this is reflected in the convention T_0 by noting that $[V_1(T_0, \mathcal{G}_4), V_2(T_0, \mathcal{G}_4)] = (1, 2)$ for both the bribery and non-bribery case. Noting the asymmetry of this game a non-cooperative behavior pattern of (α_1, β_1) is plausible especially since this is an equilibrium point (cf. Paragraph 3.8 for a discussion of this non-cooperative game).

If we alter the entries and consider the game

$$\mathcal{G}_5 = \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} (1, 2) & (-4, -1) \\ (-1, -4) & (2, 1) \end{bmatrix} \end{array}$$

then player I has the bargaining advantage and $[V_1(T_0, \mathcal{G}_5), V_2(T_0, \mathcal{G}_5)] = (2, 1)$.

Note however, that the regions R, R^*, R^{**}, R^{***} are the same for $\mathcal{G}_4$ and $\mathcal{G}_5$. These game illustrate that it is not realistic to arbitrate solely from consideration of the region R^{**}. If the positions of the payoff entries are interchanged, the threat powers and bargaining advantages of the players may be drastically altered but the region R^{**} remains invariant.

6. and 7. Let

$$\mathcal{G}_6 \equiv \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} (2, -2) & (3, 1) \\ (0, 0) & (1, 2) \end{bmatrix} \end{array} .$$

The non-cooperative game has a rational solution, namely (3, 1), and since player I has a bargaining advantage this is again the arbitrated solution relative to the convention T_0. However, if a non-cooperative game has a rational solution then this is not necessarily the arbitrated solution. For example, we can permute the entries of $\mathcal{G}_6$ to get

$$\mathcal{G}_7 \equiv \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} (1, 2) & (3, 1) \\ (0, 0) & (2, -2) \end{bmatrix} \end{array} .$$

If there is no pre-play communication then (1, 2) is the rational solution ((α_1, β_1) is both an equilibrium and admissible point). However, if there is pre-play communication then player I can threaten to use α_2 to force II to use β_2. We then have $[V_1(T_0, \mathcal{G}_7), V_2(T_0, \mathcal{G}_7)] = (5/3, 5/3)$ and (2, 2) in the case of no bribery and bribery respectively. The imputation (5/3,5/3) is realized by the dependent mixed strategy $2/3\,(\alpha_1, \beta_1) + 1/3(\alpha_1, \beta_1)$ or $(\alpha_1, \frac{2}{3}\beta_1 + \frac{1}{3}\beta_2)$. In game $\mathcal{G}_7$ player II is better off playing a non-cooperative game than a game with pre-play communication in which player I can actually make binding threats.

We have indicated that for any game g and convention T satisfying a) -c) page 44, we have

$$[V_1(T, g), V_2(T, g)] \succeq [\max_{\xi} \min_{\eta} A(\xi, \eta), \max_{\eta} \min_{\xi} B(\xi, \eta)] ;$$

however, the arbitrated solution need not compositely dominate the maximin pair in the wide sense, (γ_1, γ_2) (Paragraph 3.6) as the above example illustrates. There the maximin pair in the wide sense is (1, 2) which is incomparable with the arbitrated result (5/3, 5/3). It is possible to discuss classes of conventions such that

$$[V_1(T, g), V_2(T, g)] \succeq [\gamma_1, \gamma_2]$$

by suitably altering the range of the mapping T and replacing conditions a) -c). But from considerations of a game of the following variety,

$$\begin{bmatrix} (1, 2) & (3, 1) \\ (0, -200) & (2, -300) \end{bmatrix}$$

where $(\gamma_1, \gamma_2) = (1, 2)$, this requirement does not suitably reflect the bargaining situation. Certainly a "plausible" solution of this game with pre-play communication is (3, 1).

8. To illustrate the mechanics of obtaining the arbitrated solution we now consider a game with more than two pure strategies for each player.

Let

$$g_8 = \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{array} \begin{array}{c} \begin{array}{ccccc} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \end{array} \\ \begin{bmatrix} (4,3) & (0,-3) & (2,0) & (\frac{12}{5},0) & (8,\frac{14}{3}) \\ (7,5) & (\frac{13}{2},\frac{13}{2}) & (2,-\frac{1}{2}) & (-1,-2) & (\frac{27}{4},\frac{25}{4}) \\ (5,2) & (4,2) & (2,1) & (\frac{8}{5},0) & (1,0) \end{bmatrix} \end{array}.$$

The bargaining matrix is of the form

$$K \equiv \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{array} \begin{array}{c} \begin{array}{ccccc} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \end{array} \\ \begin{bmatrix} 1 & 3 & 2 & \frac{13}{5} & \frac{10}{3} \\ 2 & 0 & \frac{5}{2} & 1 & \frac{1}{2} \\ 3 & 2 & 1 & \frac{8}{5} & 1 \end{bmatrix} \end{array} .$$

(cf. Raiffa, Thompson, and Thrall [1], for an analysis of this two-person zero-sum game). The value of the game K is $(\frac{13}{7}, -\frac{13}{7})$ (and the optimal strategy for player I is $(\frac{3}{7}\alpha_1 + \frac{2}{7}\alpha_2 + \frac{2}{7}\alpha_3)$ and the basic optimal strategies for player II are: $(\frac{7}{21}\beta_1 + \frac{4}{21}\beta_2 + \frac{10}{21}\beta_3)$, $(\frac{3}{7}\beta_1 + \frac{5}{14}\beta_3 + \frac{13}{14}\beta_5)$, $(\frac{21}{63}\beta_1 + \frac{22}{63}\beta_3 + \frac{20}{63}\beta_4)$. The ray r_0 through the point $(\frac{13}{7}, 0)$ with direction numbers (1, 1) divides the region R^{**} into two parts Q_1 and Q_2. The maximal points of R are: $P_1 = (\frac{13}{2}, \frac{13}{2})$, $P_2 = (\frac{27}{4}, \frac{25}{4})$ $P_3 = (7, 5)$, $P_4 = (8, \frac{14}{3})$ where P_1, $P_2 \in Q_2$ and P_3, $P_4 \in Q_1$. The segment joining P_2 and P_4 intersects r_0 at a point of M_1^{**} yielding an imputation of $(7\frac{33}{238}, 5\frac{117}{238})$ which is the arbitrated solution. If bribery is permissible we have:

$$V_1(T_0, \mathcal{G}_8) + V_2(T_0, \mathcal{G}_8) = 13 = \max_{i,j} (a_{ij} + b_{ij})$$

$$V_1(T_0, \mathcal{G}_8) - V_2(T_0, \mathcal{G}_8) = \frac{13}{7} .$$

Hence

$$[V_1(T_0, \mathcal{G}_8), V_2(T_0, \mathcal{G}_8)] = (\frac{104}{14}, \frac{78}{14}).$$

In the non-cooperative game $\mathcal{G}_8$ we have β_3 and β_4 both dominated by β_1. With respect to $\{\beta_1, \beta_2, \beta_5\}$ we have α_3 dominated by α_2. With respect to $\{\alpha_1, \alpha_2\}$ we have β_1 dominated by β_5 and hence we get $\tilde{\mathfrak{A}} = \{\alpha_1, \alpha_2\}$ and $\tilde{\mathfrak{B}} = \{\beta_2, \beta_5\}$. The strategies α_2 and β_5 are the maximin strategies for players I and II respectively for the completely reduced game, $\tilde{\mathcal{G}}_8$. But since α_1 is a Bayes solution against β_5 and β_2 is the Bayes solution against α_2 we have a structure analogous to game $\mathcal{G}_1$.

We previously remarked that the convention T_0 is symmetric. Whether the arbiter should employ a symmetric convention depends on factors not reflected in the payoff entries. For example, if player I represented the combined interests of a large group and player II represented a single individual it would be a moot question whether or not a symmetric convention were desirable.

4.6 Convention T_1. (the payoff entries do not have a common unit of measurement)

In the case of the payoff entries of both players do not have a common unit of measurement then we require that T satisfy the following property:

4.6.1) $V_i(T, \{\lambda_1(a_{ij} - c_1), \lambda_2(b_{ij} - c_2)\}) = \lambda_i V_i(T, \{a_{ij}, b_{ij}\}) - \lambda_i c_i$ for i = 1, 2, $\lambda_i > 0$, c_i arbitrary real numbers.

The game $\mathcal{G} = \{(a_{ij}, b_{ij})\}$ is said to be symmetric provided: m = n, there exists independent positive linear transformations (utility transformations) of each player's entries and a permutation of the pure strategies such that the resulting game is its own transpose.

To show the existence of a stable, symmetric convention, T_1, satisfying condition 4.6.1 we proceed heuristically. For a given finite game (select any fixed but arbitrary utility scale) the mapping $T_1 : R^* \longrightarrow M_\prec^{**}$ defines a decomposition of R^* where elements of R^* are equivalent provided they map by T_1 into the same element of $M_\prec^{**}$. By property c) of page 44, any segment joining any two equivalent elements of R^* also lies in the equivalence class. Let us consider the case where $\mathcal{G}$ is symmetric and the utility scales are such that $\mathcal{G} = \mathcal{G}^t$. Then by symmetry of the convention we would want the elements of R^* lying on the line x = y to be equivalent. If this line does not meet $M_\prec^{**}$ at a point of R then the slope of $M_\prec^{**}$ at this point is -1. We now investigate the

nature of the shearing of R^{**} together with its induced equivalence classes when g undergoes utility transformations. We observe that in the symmetric case that the slope of the line (that was $x = y$) is always the negative of the slope of the segment of M_i^{**} which intersects this line. This gives us our cue. We further note that if m is the slope of a line in R^* and if g undergoes utility transformations $\{\lambda_1(a_{ij} - c)\ ,\ \lambda_2(b_{ij} - c)\}$ then by the linearity of the operator A x B the slope of the transformed line is $\frac{\lambda_2}{\lambda_1} m$. With this heuristic background we now proceed to define T_1 explicitly: For any element $e' \in M_i^{**}$ which does not belong to R, let $T_1(e) = e'$, where $e \in R^{**}$, provided the slope of the ray from e to e' is the negative of the slope of M_i^{**} at e'. This defines T_1 for a subset Q of R^{**}. Now extend T_1 to all of R^{**} such that the mapping is continuous and all equivalence classes of R^{**} are connected. This extension is unique and thus T_1 is uniquely defined on R^{**}. Now restrict the domain of T_1 to R^*. To get a better intuitive notion of this mapping see the diagram on page 63. Since the negative of the slope of any segment of M^{**} is positive, condition b) of page 44 is fulfilled. Conditions a) and c) are obviously satisfied. Hence we can utilize the principal theorem to show the existence of a solution $[V_1(T_1, g), V_2(T_1, g)]$. If $(\xi^{(0)}, \eta^{(0)})$ is a saddle point of the game with payoff $T_1(A \times B)(\xi, \eta)$, then $(\xi^{(0)}, \eta^{(0)})$ is again the saddle point of the game which is subjected to independent utility transformations. Consequently, 4.6.1) is satisfied.

[In section 4.10 another, and perhaps a more natural, motivation for T_1 is given. The convention T_1 was first obtained by John F. Nash, Jr., by an entirely different rationale. However, the above presentation is quite independent of his].

4.6.2) Examples: The games $\mathcal{G}_i$, (i = 1, 2, ... 7) below refer to the games of Section 4.5 where the entries are to be interpreted without a common unit of measurement.

The analysis of the game $\mathcal{G}_1$ with convention T_1 is the same as in Section 4.5 which employed convention T_0. In $\mathcal{G}_2$ we make the utility transformation $4\,a_{ij}$ which yields the game

$$\mathcal{G}_2' \equiv \begin{bmatrix} (0,\ 0) & (4,\ -3) \\ (0,\ 1) & (-4,\ -2) \end{bmatrix},$$

and for $\mathcal{G}_2'$ we have $[V_1(T_1, \mathcal{G}_2'),\ V_2(T_1, \mathcal{G}_2')] = [V_1(T_0, \mathcal{G}_2'),\ V_2(T_0, \mathcal{G}_2')] = (\frac{1}{2},\ \frac{1}{2})$; finally changing back to $\mathcal{G}_2$ we have $[V_1(T_1, \mathcal{G}_2),\ V_2(T_1, \mathcal{G}_2)] = (\frac{1}{8}, \frac{1}{2})$. In $\mathcal{G}_3$ we proceed similarly and get $[V_1(T_1, \mathcal{G}_3),\ V_2(T_1, \mathcal{G}_3)] = (5,\ 50)$. In $\mathcal{G}_4$ and $\mathcal{G}_5$ we have $T_1 \equiv T_0$ and the same analysis holds as in Section 4.5. In $\mathcal{G}_6$ the convention T_1 favors player I more than convention T_0 and hence we still have $[V_1(T_1, \mathcal{G}_6),\ V_2(T_1, \mathcal{G}_6)] = (3,\ 1)$. To analyze $\mathcal{G}_7$, make the utility transformation $\frac{1}{2}(a_{ij} - 1)$ which yields

$$\mathcal{G}_7' \equiv \begin{bmatrix} (0,\ 2) & (1,\ 1) \\ (-\frac{1}{2},\ 0) & (\frac{1}{2},\ -2) \end{bmatrix}$$

and now $[V_1(T_1, \mathcal{G}_7'),\ V_2(T_1, \mathcal{G}_7')] = [V_1(T_0, \mathcal{G}_7'),\ V_2(T_0, \mathcal{G}_7')] = (\frac{3}{4}, \frac{5}{4})$ and therefore $[V_1(T_1, \mathcal{G}_7),\ V_2(T_1, \mathcal{G}_7)] = (\frac{5}{2}, \frac{5}{4})$ by (4.6.1).

4.7 Computational Methodology for Convention T_1.

Let $\mathcal{G}$ be an arbitrary finite generalized two-person game where the entries have no common unit of measurement. (We shall first state the procedure,

and then verify it in the next paragraph). Let P_0, P_1, ... P_r be the vertices of the linear graph of M_1^{**} where $P_i = (c_i, d_i)$ for i = 0, 1, 2, ..., r. Let σ_i be the open one-simplex, (P_{i-1}, P_i). The $\{\sigma_i\}$ can be determined directly from R without much difficulty. Let U_k be a utility transformation of game $\mathcal{G}$ which sends $\mathcal{G}$ into $\mathcal{G}^{(k)} = \{(a_{ij}^{(k)}, b_{ij}^{(k)})\}$ where U_k is chosen such that $U_k(\sigma_k)$ has slope -1. Determine the arbitrated solution of $\mathcal{G}^{(k)}$ relative to the convention T_0 and where bribery is permissible. This solution is characterized by $[V_1(T_0, \mathcal{G}^{(k)}), V_2(T_0, \mathcal{G}^{(k)})]$ where

(4.7.1)

a) $V_1(T_0, \mathcal{G}^{(k)}) + V_2(T_0, \mathcal{G}^{(k)}) = \max_{i,j} [a_{ij}^{(k)} + b_{ij}^{(k)}]$

$= $ sum of coordinates of $U_k(P_k)$

b) $V_1(T_0, \mathcal{G}^{(k)}) - V_2(T_0, \mathcal{G}^{(k)}) = [$ value to player I of the zero-sum game $\{(a_{ij}^{(k)} - b_{ij}^{(k)})\}]$.

There will exist at most one k, say k_0, for which

$$[V_1(T_0, \mathcal{G}^{(k)}), V_2(T_0, \mathcal{G}^{(k)})] \in U_{k_0}(\overline{\sigma_{k_0}}).$$

If such a k_0 exists then

(4.7.2) $$[V_1(T_1, \mathcal{G}), V_2(T_1, \mathcal{G})] = U_{k_0}^{-1}[V_1(T_0, \mathcal{G}^{(k_0)}), V_2(T_0, \mathcal{G}^{(k_0)})].$$

If no such k_0 exists then $[V_1(T_1, \mathcal{G}), V_2(T_1, \mathcal{G})]$ will be one of the vertices $P_0, P_1, \ldots, P_r$. In this case, if there exists an i (for i = 1, 2, ..., r-1) such that

(4.7.3) $V_2(T_0, \mathcal{G}^{(i)}) \leq d_i^{(i)}$ (or, $V_1(T_0, \mathcal{G}^{(i)}) \geq c_i^{(i)}$), where $U_i(P_i) = (c_i^{(i)}, d_i^{(i)})$

and

(4.7.4) $V_2(T_0, \mathcal{G}^{(i+1)}) \geq d_i^{(i+1)}$ (or, $V_1(T_0, \mathcal{G}^{(i+1)}) \leq c_i^{(i+1)}$), where $U_{i+1}(P_i) = (c_i^{(i+1)}, d_i^{(i+1)})$

then $[V_1(T_1, \mathcal{G}), V_2(T_1, \mathcal{G})] = P_i$. If no such i exists then there remain two

possibilities, namely P_o or P_r. If

(4.7.5) $\qquad V_2(T_o, g^{(i)}) \geq d_o^{(i)}, \qquad$ where $(c_o^{(i)}, d_o^{(i)}) \equiv U_i(P_o)$,

then $\quad [V_1(T_1, g), V_2(T_1, g)] = P_o$. If

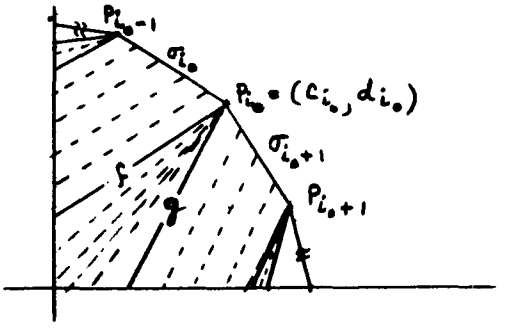
(4.7.6) $\qquad V_2(T_o, g^{(r)}) \leq d_r^{(r)} \qquad$ where, $(c_r^{(r)}, d_r^{(r)}) \equiv U_r(P_r)$

then

$$[V_1(T_1, g), V_2(T_1, g)] = P_r.$$

Verification of above procedure:

Let us suppose $(\xi^{(o)}, \eta^{(o)})$ is a saddle point of the game $T_o\{U_{k_o}[A \times B](\xi, \eta)\}$ whose corresponding solution belongs to $U_{k_o}(\overline{\sigma_{k_o}})$. In the region $U_{k_o}(R^{**})$, the convention T_o yields a ray r_o and two subregions Q_1 and Q_2 (cf. page 53). By the choice of $\xi^{(o)}$ player I confines player II to $Q_1 \cup r_o$ and by choice of $\eta^{(o)}$ player II confines player I to $Q_2 \cup r_o$. But $(T_1(Q_1 \cup r_o) \subset Q_1 \cup r_o$ and $T_1(Q_2 \cup r_o) \subset Q_2 \cup r_o$ and $T_1\{U_{k_o}[A \times B](\xi^{(o)}, \eta^{(o)})\} = T_o\{U_{k_o}[A \times B](\xi^{(o)}, \eta^{(o)})\}$. Consequently, $(\xi^{(o)}, \eta^{(o)})$ is also a saddle point of the game $T_1\{U_{k_o}[A \times B](\xi, \eta)\}$ and the arbitrated solution is characterized by 4.7.1 and 4.7.2. Since, in an incomparable game, all saddle points form a permutable equivalence class there exists at most one k_o satisfying the above requirement. Now suppose there does not exist such a k_o, but there exists an $i_o \neq 0$, r which satisfies 4.7.3 and 4.7.4. From 4.7.3, we conclude that (in the figure below)



player I can confine player II to the line f or the region below it. From 4.7.4

player II can confine player I to the line g or the region above it. Consequently, by convention T_1 player I can get at least c_{i_0} and player II can hold player I to at most c_{i_0} (or get at least d_{i_0}) and thus P_{i_0} is the arbitrated solution relative to the convention T_1. In the case 4.7.5 is valid and $\eta^{(o)}$ is an optimal strategy for $(T_0, g^{(1)})$ then player II can guarantee the arbitrated solution P_0 relative to convention T_1 by employing the same strategy $\eta^{(o)}$. An identical argument holds if 4.7.6 is valid - namely, player I by use of an optimal strategy ξ° for $(T_0, g^{(r)})$ can use $\xi^{(o)}$ to guarantee the solution P_r of (T_1, g).

It remains to show that these possibilities are mutually exhaustive. To this end, we note that $[V_1(T_1, g), V_2(T_1, g)]$ can either belong to $\overline{\sigma_k}$ for k = 1, 2, ..., r, or be a P_i for i = 0, 1, 2, ..., r. In the former case, it is readily seen that a k_0 exists yielding an analysis leading to 4.7.2. Possibilities enumerated in 4.7.3, 4.7.4, and 4.7.5 and 4.7.6 exhaust the latter case.

Although this procedure may be quite involved formally, in practice it is often possible to locate readily the $\overline{\sigma_k}$ which contains the arbitrated solution.

4.8 Convention T_2.

In order to motivate consideration of another symmetric convention satisfying 4.6.1 we investigate the following example.

$$g_9 = \begin{array}{c} \\ \alpha_1 \\ \alpha_2 \end{array} \begin{array}{c} \begin{array}{cc} \beta_1 & \beta_2 \end{array} \\ \begin{bmatrix} (1, 100) & (0, 0) \\ (2, 0) & (0, 0) \end{bmatrix} \end{array} .$$

The non-cooperative rational solution is (2, 0). If there is a common unit of measurement, then $[V_1(T_0, \mathcal{G}_9), V_2(T_0, \mathcal{G}_9)] = (\frac{200}{101}, \frac{200}{101})$.

If there is no common unit of measurement then

$[V_1(T_1, \mathcal{G}_9), V_2(T_1, \mathcal{G}_9)] = (1, 100)$. This last result is seen immediately by noting that use of β_2 results in an imputation of (0, 0) and $T_1(0, 0) = (1, 100)$. This example illustrates that although convention T_1 has some desirable properties it does not always yield a realistic solution.

We shall now exhibit another symmetric, stable convention T_2 which satisfies 4.6.1. For an arbitrary game $\mathcal{G} = \{(a_{ij}, b_{ij})\}$, let

$$\tilde{U}(a_{ij}, b_{ij}) = [\lambda_1(a_{ij} - c_1), \lambda_2(b_{ij} - c_2)]$$

where

$$c_1 = \min_{i,j} a_{ij}, \frac{1}{\lambda_1} = \max_{i,j} (a_{ij} - c_1) \text{ and } c_2 = \min_{i,j} b_{ij}, \frac{1}{\lambda_2} = \max_{i,j} (b_{ij} - c_2).$$

The utility transformation $\tilde{U}$ normalizes the players entries, in the sense that the entries 1 and 0 refer to the maximum and minimum that each player can possibly get from the game. Let $\tilde{U}\mathcal{G}$ be the game with entries $\{\tilde{U}(a_{ij}, b_{ij})\}$. For the game $\tilde{U}\mathcal{G}$, let $T_2(e') = T_0(e')$ for all elements $e' \in R^{**}$ of $\tilde{U}\mathcal{G}$. Now define

4.8.1) $$T_2(e) \equiv \tilde{U}^{-1} \cdot T_0 \cdot \tilde{U}(e)$$

for all $e \in R^{**}$ of $\mathcal{G}$. The convention T_2 obviously satisfies conditions a),b), c) of page 44, and thus $[V_1(T_2, \mathcal{G}), V_2(T_2, \mathcal{G})]$ is well defined. Since

$$\tilde{U}^{-1} T_0 \tilde{U} (A \times B)(\xi^{(o)}, \eta^{(o)}) = T_2(A \times B)(\xi^{(o)}, \eta^{(o)})$$

it is easily seen that

4.8.2) $$\tilde{U}^{-1}[V_1(T_0, \tilde{U}\mathcal{G}), V_2(T_0, \tilde{U}\mathcal{G})] = [V_1(T_2, \mathcal{G}) V_2(T_2, \mathcal{G})]$$

As regards 4.6.1 let U_1 be an arbitrary utility transformation of $\mathcal{G}$ and let

$\tilde{U}$ and $\tilde{U}_1$ be the normalizing utility transformations of g and $U_1 g$ respectively. Noting that $\tilde{U} = \tilde{U}_1 U_1$ and $\tilde{U}_1^{-1} = U_1 \tilde{U}^{-1}$, we then have

$$\begin{aligned}
[V_1(T_2, U_1 g), V_2(T_2, U_1 g)] &= \tilde{U}_1^{-1}[V_1(T_0, \tilde{U}_1 U_1 g), V_2(T_0, \tilde{U}_1 U_1 g)] \\
&= \tilde{U}_1^{-1}[V_1(T_0, \tilde{U} g), V_2(T_0, \tilde{U} g)] \\
&= U_1 \tilde{U}^{-1}[V_1(T_0, \tilde{U} g), V_2(T_0, \tilde{U} g)] \\
&= U_1 [V_1(T_2, g), V_2(T_2, g)],
\end{aligned}$$

which verifies 4.6.1. The stability of T_2 reduces to the stability of T_1. Finally the symmetry of T_2 follows from the symmetry of T_0 and the utility transformation $\tilde{U}$.

For a rationalization of convention T_2 one only needs to rationalize the use of T_0 for game $\tilde{U} g$. The introduction of $\tilde{U}$ symmetrically establishes a common psychological unit. With this common unit one proceeds in $\tilde{U} g$ to forward the same argument as for use of T_0.

Returning to game g_9 (of this section), we get

$$\tilde{U} g_9 \equiv \begin{bmatrix} (\frac{1}{2}, 1) & (0, 0) \\ (1, 0) & (0, 0) \end{bmatrix},$$

and

$[V_1(T_0, \tilde{U} g_9), V_2(T_0, \tilde{U} g_9)] = (2/3, 2/3)$. Hence we get

$[V_1(T_2, g_9), V_2(T_2, g_9)] = (4/3, 2 0/3)$.

By convention T_2 one can also resolve a large class of infinite two person games. We merely require $\max_{(\alpha,\beta)} A(\alpha, \beta)$, $\min_{(\alpha,\beta)} A(\alpha, \beta)$, $\max_{(\alpha,\beta)} B(\alpha, \beta)$, $\min_{(\alpha,\beta)} B(\alpha, \beta)$ to exist and be finite so that the normalizing utility transformation can be defined. Then sufficiency conditions that suffice to yield a solution

for $\tilde{U}g$ with convention T_0 (cf. page 52) carry over to g with convention T_2.

Examples: All the examples refer back to Section 4.5.1 Relative to the convention T_2, the arbitrated solutions are: for g_1, $(\frac{3}{2}, \frac{3}{2})$; for g_2, $(\frac{1}{6}, \frac{1}{3})$; for g_3, (5 , 50); for g_4, (1, 2); for g_5, (2, 1); for g_6, (3, 1); for g_7, $(\frac{15}{11}, \frac{20}{11})$.

4.9 Stability of Arbitrated Solutions.

In each of the conventions discussed it was possible to obtain a characterization of the arbitrated solution based principally on the notion of the value of an induced zero-sum game. Since the value of a zero-sum game is stable (although the optimal strategies of a zero-sum game are not necessarily stable) we were able to deduce the stability of the conventions T_0, T_1, and T_2. However, in defining stability we not only required that $\{g_k\}$ approach g but also that $\{M^{**}_{k,\prec}\}$ approach $M^{**}_{\prec}$. We will now show that some such restricting qualification must be applied. To this end, consider the game,

$$g \equiv \begin{bmatrix} (0, 0) & (0, 0) \\ (0, 1) & (0, 0) \end{bmatrix} .$$

The arbitrated solution of the game without bribery is (0, 1); for, in the altrustic ordering $\prec$ this is the only maximal point. The sequence of games, $\{g_k\}$ where

$$g_k \equiv \begin{vmatrix} (0, 0) & (\frac{1}{k}, 0) \\ (0, 1) & (0, 0) \end{vmatrix} ,$$

approaches g but $\{M^{**}_{k,\prec}\}$ does not approach $M^{**}_{\prec}$. According to conventions

T_1 or T_2, the arbitrated solution of g_k is $(\frac{1}{2k}, \frac{1}{2})$ which approaches $(0, \frac{1}{2})$. We notice, however, that $\{M^{**}_{k,\prec\prec}\}$ does approach $M^{**}_{\prec\prec}$ and thus there is a tendency to use the ordering $\prec\prec$ and take $(0, \frac{1}{2})$ for the solution of game g. But if we consider the sequence $\{g'_k\}$ where

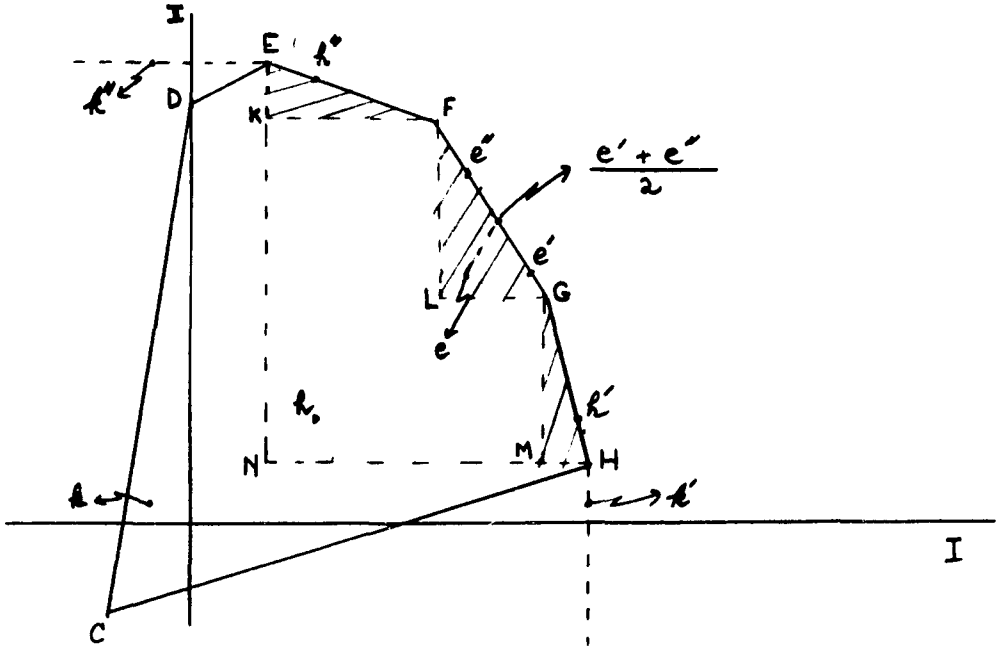
$$g'_k = \begin{bmatrix} (0, 0) & (-\frac{1}{k}, 0) \\ (0, 1) & (0, 0) \end{bmatrix}$$

then $\{M^{**}_{k,\prec}\}$ approaches $M^{**}_{\prec}$ but $\{M^{**}_{k,\prec\prec}\}$ does not approach $M^{**}_{\prec\prec}$. Further, the arbitrated solution of g'_k is $(0, 1)$ which approaches $(0, 1)$.

It should now be clear that in the definition of a stable convention that instead of requiring that $M^{**}_{k,\prec}$ approach $M^{**}_{\prec}$ it would be quite sufficient to require that the arbitrated solution not be an element of a closed one simplex of $M^{**}_{\prec\prec}$ which has a horizontal or vertical slope.

4.10 A Non-Linear Convention.

Let us consider a game g with an R^{**} of the following form (R^{**} is bounded by (C, D, E, F, G, H)):



where the game g is to be interpreted without a common unit of measurement. Consider the point e in the subregion (F,L.G). We wish to map e into some element of the segment e' to e''. If we desire a symmetric convention it is quite natural to map e into $\frac{e' + e''}{2}$. It is also evident that this mapping will be invariant with respect to utility transformations. With this rationale we can now define the mapping from subregions (E,K,F), (F,L,G) and (G,M,H) onto the maximal elements. In convention T_1 we merely extend linearly the linear equivalence classes of the above subregions and then fill in the mapping in a natural fashion. Now instead of considering a linear extension we propose the following: consider the element h which we wish to map into some maximal element between h' and h''. Through h consider a lineal element whose slope is $\frac{\overline{hh''}}{\overline{hh'}}$ where $\overline{hh''}$ and $\overline{hh'}$ are the distances from h to h'' and h' respectively. Thus to each element of (N,E,F,G,H) we can associate a lineal element and stemming from this motivation we can decompose the subregion (N,E,F,G,H) into equivalence classes where each equivalence class is characterized by an arbitrary element of the class and a differential equation. Thus we have defined a mapping from the subregion (N,E,F,G,H) onto M^{**}. The extension of the mapping to all of R^{**} is not quite as natural but is in the same spirit. To the element k we associate a lineal element of slope $\frac{\overline{kk''}}{\overline{kk'}}$ and now proceed to define the equivalence classes by a differential equation. It should be obvious how the three stages of the intuitive definition of the mapping can merge together into a concise formal definition. Although this mapping is perhaps more realistic (subjective opinion) than mappings T_1 and T_2 it has the serious handicap that the equivalence classes of R^{**} may be non-linear. In the existence theorem, this non-linearity is reflected in non-convexity of certain sets which prohibits the use of the Kakutani

Fixed Point Theorem. The existence of an arbitrated solution for this non-linear convention, with the support of a computational technique will be left as an open question in this report.

BIBLIOGRAPHY

Kakutani, S., (1), "A generalization of Brouwer's fixed point theorem," Duke Mathematical Journal 8 (1941), 457-459.

Lehman, E. L., (1), Notes on the Theory of Estimation, Ch. I - V, Berkeley, University of California, 1950.

Marschak, J., (1), "Rational behavior, uncertain prospects, and measurable utility," Econometrica 18 (1950), 111-141.

Morgenstern, O., von Neumann J., (1), Theory of Games and Economic Behavior, Princeton (1947, 2nd ed.). (see von Neumann, J., Morgenstern, O.)

Nash, J. F., (1), "Equilibrium points in n-person games," Proceedings, National Academy of Sciences 36 (1950), 48-49.

----, (2), "The bargaining problem," Econometrica 18 (1950), 155-162.

----, (3), "Two-person cooperative games," Rand Corporation Report, P-172, August 1950.

von Neumann, J., (1), (See Morgenstern, O., (1)).

----, (2), "Zur Theorie der Gesellshaftsspiele," Mathematische Annalen 100 (1928) 295-320.

----, (3), "Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwer'schen Fixpunktsatzes," Ergebnisse eines Math. Kolloquiums 8 (1937), 73-83.

Raiffa, H., Thompson, G., Thrall, R., (1), An Algorithm for the Determination of all Solutions of a Two-Person Zero-Sum Game with a Finite Number of Strategies, Report R-28, Engineering Research Project M720-1, University of Michigan, (Sept. 1950).

Thompson, G., (see Raiffa, H.).

Thrall, R., (see Raiffa, H.).

Ville, J., (1), "Sur la théorie générale des jeux où intervient l'habilité des joueurs," Traité du Calcul des Probabilités et de ses Applications, by E. Borel and collaborators, Paris (1938), Vol. 2, No. 5, 105-113.

Wald, A., (1), Statistical Decision Functions, John Wiley and Sons, N. Y. (1950).