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Framing, responsiveness, serviceability, and normativity: Categories of perception teachers use to relate to students' mathematical contributions in problem-based lessons

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Abstract

We contribute to the understanding of teacher noticing by focusing on what a teacher may notice in students' mathematical contributions in the context of problem-based lessons. Complementing approaches to research on noticing that focus on individual teachers' perceptual, cognitive, or situated skills, this conceptual paper offers four categories of perception as examples of affordances available in the practice of teaching mathematics through problems. These include (1) the familiar instructional situations available to frame the problem, and the possibility to see student's work as (2) responsive to the problem, (3) serviceable to the knowledge at stake, and (4) normative (with respect to the instructional situation used to frame the problem). The paper shows examples of how teachers recognize responsiveness, serviceability, and normativity of student contributions and calls for research that can further uncover how such recognition may matter in the practice of teaching.

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Introduction

We propose four categories of perception that describe what is available for teachers to notice in students' mathematical contributions to problem-based lessons. In doing so, we suggest that the field of research on mathematics teacher noticing should aim to understand not only what teachers notice and how they do so, but also what the practice of mathematics instruction makes available for the teacher to notice. Indeed, a mathematics teacher, just like any observer of students' mathematical contributions, stands a chance of being surprised, stimulated to think, and even learn from observing what students do. But as they observe students' mathematical activity, teachers are different from other observers (including researchers on students' thinking) in how they are expected to engage with students' contributions.

Cohen (2011) describes teaching as a profession of human improvement, as teachers' success depends on their students' success. Consequently, teachers can be expected to be concerned with students' actions in ways that go beyond intellectual or aesthetic interest. Because a teacher observes students' work in the context of instruction and because, as scholars (e.g., Jacobs et al., 2010; Rowland et al., 2005) have noted, their interpretations of such observations may inform their instructional decisions, the professional vision of a teacher has much more in common with that of other professionals of human improvement (e.g., clinicians, therapists) than with that of researchers interested in understanding a reality in which they did not participate (e.g., the archaeologist in Goodwin, 1994). We argue that theoretical considerations of what teachers notice should be anchored in a theory of the work teachers do in instruction and what instruction makes available to be noticed. This conceptual paper connects teacher noticing with a theoretical framework for analyzing the work of the teacher in mathematics instruction. It does so by defining four categories of perception that scholars may use to anticipate and describe what kinds of things a mathematics

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teacher may notice in students' contributions to a problem-based lesson. We illustrate how such noticing is evinced in secondary school geometry teachers' discussions of a problem-based lesson.

Review of Literature on Noticing

The literature on teachers' professional noticing has grown quite vast in recent years (see König et al, 2020; Santagata et al., 2021; for recent reviews). Much of this work on teacher noticing has built crucially on Goodwin's (1994) notion of *professional vision*—defined as "socially organized ways of seeing and understanding events that are answerable to the distinctive interests of a particular social group" (p. 606). Reflecting on some of the early work in the field, Erickson (2011) describes how research on teacher noticing emerged "under the influence of the child study movement" when "beginning teachers were encouraged to watch closely the children they taught" (p. 17-18) and attend to their thinking processes. Beyond the growing realization about the importance for teachers to pay attention to students' cognition (e.g., Carpenter et al., 1989), Erickson (2011) notes that there was also a growing realization about the need to attend to teacher thinking. From that realization, education researchers have spent time inquiring whether teachers recognize students' conceptions documented in research (Even, 1993) and associated it with conceptions of mathematical knowledge for teaching (Ball et al., 2008; Simon, 2006). According to Erickson (2011), the focus on teacher noticing grew out of this realization about the importance of teachers attending to students' cognition (p. 19).

Since that time, a field of research on teacher noticing has expanded to include a variety of teacher populations (including pre-service, in-service, and teacher educators) and use a variety of methodologies (see Amador et al., 2021; Sherin & Star; 2011; Stahnke et al., 2016). While the research on teacher noticing has expanded since its inception, we argue that, in many ways, it continues to retain its cognitive heritage. On the one hand, recent literature reviews on teacher noticing (Amador et al., 2021; König et al., 2020; Santagata et al., 2021) reveal that a dominant strand

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documents what teachers notice of students' cognition, including both students' cognitive processes and students' conceptions (e.g., Barnhart & van Es, 2015; Fernández et al., 2013; Jacobs et al., 2010; Leatham et al., 2015; Roth McDuffie et al., 2014; Stockero et al., 2017a; van Es, 2011; Van Zoest et al., 2017). On the other hand, there is also a tendency for education scholars to conceive of the teacher's noticing process through a cognitive lens. While important variations in the conceptualizations of teacher noticing exist within the literature (Blömeke & Star 2016; Stahnke et al., 2016), the prevalent conceptualizations account for teacher noticing by borrowing from processes identified within cognitive science—with some scholars defining noticing as *perception* (e.g., Star & Strickland, 2008), while others include other processes such as *interpretation* (e.g., van Es & Sherin, 2002) or *decision making* (e.g., Jacobs et al., 2010; van Es and Sherin 2008; Schack et al., 2013). As a result, a second dominant strand in the teacher noticing literature focuses on teachers' cognition of students' thinking—aiming to understand how teachers notice (or fail to notice), categorize, and distinguish between the various cognitive processes used by students. These two strands of research have been helpful for teacher educators to design interventions to develop individual capacity for noticing (e.g., Fernández et al., 2012; Ivars et al., 2020; Stockero et al., 2017b). For example, Rowland et al.'s (2005) Knowledge Quartet framework, designed specifically to support preservice teacher education in practicum settings, includes within its contingency dimension the ability for teachers to "think on their feet" and their "readiness to respond to children's ideas and a consequent preparedness, when appropriate, to deviate from an agenda set out when the lesson was prepared" (p. 263). Furthermore, such work has often emerged against the background of the categories of student thinking uncovered by researchers on student cognition (e.g., Amador et al., 2022; Jacobs et al., 2010; Sánchez-Matamoros et al., 2015; Schack et al., 2013; Walkoe, 2015).

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Against this backdrop, there have been a growing number of concerns about the field's primary ways of conceiving teacher noticing—with scholars suggesting the phenomenon of teacher noticing has been treated in overly intuitive ways (Sherin & Star, 2011), too often regarded as unidimensional (Rotem & Ayalon, 2022), and too narrowly defined (Scheiner, 2016, 2021). For example, in a commentary on the teacher noticing literature, Scheiner (2016) suggests a problem with the current "theoretical formulations and usage of the construct of teacher noticing" (p. 227), namely that these formulations have served to obscure "critical aspects in the field of teacher competence" (p. 227-228). Specifically, he problematizes ways that the literature has progressed by borrowing terms from cognitive psychology, such as *attending*, *interpreting*, and *decision-making*. In a more recent contribution, Scheiner (2021) expands on this critique—suggesting problems with the ways the literature has too narrowly "construe[d] noticing as a disembodied, purely mental form of seeing … position[ing] the teacher as separated or separable from the observing environment" (p. 92). Drawing on the research of others (e.g., Gibson, 1979; Goffman, 1974), Scheiner (2021) reminds the field that "what teachers notice in classrooms depends in part on how they frame what they are doing" (p. 90). Relatedly, Blömeke et al. (2015) have proposed a model of competence that combines individual dispositions and situated skills to account for observable behavior, in which the processes ordinarily included among teacher noticing are described as situation-dependent.

Our contribution builds on Scheiner's (2021) critique of the field of research on teacher noticing and relates to the situated nature of noticing as described by Blömeke et al. (2015). Along the lines of considering the teacher as an active participant in culturally- and historically-constituted realities, we highlight the institutional position of mathematics teachers and the elements that structure the teacher's work environment that may influence what a teacher could perceive in students' contributions. As an institutional agent charged with teaching and helping students learn, rather than as a detached observer of students' development or cognition, a teacher encounters

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students' contributions in the context of their work environment. Noticing students' contributions thus happens as part of a teacher's way of being in the world of the mathematics classroom. In this paper, we propose some theoretical distinctions that can help researchers describe noticing as a relationship between the teacher's practice in mathematics instruction and students' mathematical contributions. The proposed distinctions are anchored in a theory of the context of teachers' work, which we introduce first.

Practical Rationality: A Descriptive Theory of Teaching Practice

In calling for the field of research on teacher noticing to investigate this phenomenon in its situatedness, we point specifically to the institutional context in which teachers work, which instantiates what Scheiner (2021) refers to as the interpersonal interactions (e.g., with students and colleagues), the cultural-historical frame (e.g., as shaped by mathematical tasks, programs of study, and academic trajectories), or the embodiments and ecologies within which the teacher is involved (e.g., as shaped by the physical and technological conditions of work). Practical Rationality theory (Herbst & Chazan, 2012; Chazan, et al., 2016) provides the means to describe and explain how such a multilayered context makes available noticings and actions a teacher can make while at the same time allowing for individual variability in response to knowledge, motivation, and other personal resources. Practical rationality seeks to provide a framework for observers to interpret teachers' decisions and actions as reasonable or sensible (Leatham, 2006): It does so, in part, by articulating what categories are available for teachers to perceive aspects of what happens in classroom environments so as to anticipate what actions are viable for them to take. We sketch this theory briefly and then point to how it can help understand the situatedness of teacher noticing.

Practical rationality builds on Bourdieu's (1990) logic of practice and, as such, it tries to capture the perspective of the practitioner from the perspective of an observer (Herbst & Chazan, 2023; Simon and Tzur, 1999). Practical rationality builds on the model of the instructional triangle

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proposed by Cohen et al. (2003) whereby instruction is defined as the interactions among teacher, students, and content in environments (Herbst & Chazan, 2012). Further, practical rationality recognizes schools and colleges as institutions that provide environments for instruction to take place and that the institutional work of teaching includes mathematics instruction as one among several activities teachers are expected to do–schoolteachers also mentor individual students, promote social values, and monitor compliance with institutional rules. This distinction between the position of teacher in an institution (responsible to different stakeholders and involved in several activities) and the role of teacher in mathematics instruction is a key resource for the theory to articulate how a mathematics teacher might be tacitly disposed to perceive in particular ways while avowedly committed to principles that might justify alternative perceptions (Chazan et al., 2016).

Mathematics Instruction

Educational institutions make room for mathematics instruction by associating instructional goals with courses of study, student groups, and teacher selection. Thus, when a teacher and some students are related to each other in instruction, the content at stake (each of the institutionally-sanctioned instructional goals, the knowledge and skills that students are expected to acquire in a course of study) pre-exists them: Students and teacher gather in a class to lay claim on that content through the work they do together. In particular, teachers construct work opportunities for students to come to know that content (e.g., teachers design or choose problems and tasks that put those instructional goals at stake) and students do things within the context of those work opportunities that provide evidence of progress toward, or accomplishment of, the instructional goals at stake in those tasks. In these general terms, mathematics instruction can be described as an exchange of meanings: Students enact those meanings as they do their work and the teacher interprets those meanings in terms of the instructional goals that those tasks are meant to put at stake. We elaborate

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the instructional triangle proposed by Cohen et al (Figure 1), to highlight the exchange of meanings involved in the content vertex in Figure 2.

The notion of *instructional exchange*, whereby the teacher needs to observe students' work on tasks and interpret it in terms of the content at stake, is where the relationship between practical rationality and teacher noticing is anchored. An initial interpretation of instructional exchanges–in which a teacher needs to attest to whether students can use what they are being taught when solving problems might cast the research on teachers' noticing of students' thinking as inspecting (1) what mathematics teachers notice in students' contributions and (2) what cognitive processes teachers engage in order to do such noticing. That interpretation casts the teacher as consumer of research on students' cognition or perhaps as a builder of models of students' thinking (Cobb & Steffe, 1983). Such interpretive work could inform some decision making, for example, about what tasks a teacher could assign a student next (Jacobs et al., 2010). But the institutional situatedness of the teacher requires them to attend to more than questions about individual students' cognition.

INSERT FIGURES 1 and 2 SIDE BY SIDE HERE

Practical rationality describes a teacher's work with content as including not only what the student is doing on problems but also the knowledge at stake in those problems. Consider, as an example, learning to solve equations in one variable in Algebra I. Facing the problem of solving an equation such as the one in The Great Divide[1] (Bieda, et al., 2015), in which a high school algebra class is tasked with solving the equation $20x + 5 = 5x + 65$, some students may think of the need to gather like terms first and write $15x = 60$. Other students might think of the convenience of working with smaller numbers, deciding to first divide out common factors and write instead $4x+1=x+13$. While the second case might not reproduce the steps shown in a worked-out example from a typical textbook (Buchbinder et al., 2019), a teacher might notice the reasonableness and the

[1] https://youtu.be/LTIZ-4HwFJI

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algebraic nature of the second student's work. But it bears consideration that students might also think about things quite different from solving equations with symbolic manipulation. Some might think of how to solve the problem posed in the most convenient way, others of how to show they can do what they were taught, and others might act in ways that obtain a value for x without doing algebra (e.g., by guessing and checking) or without engaging mathematically with the problem (e.g., by arguing that those two things are not equal). A teacher might be able to hypothesize what students are thinking in many of those circumstances, but such hypotheses would have different value for the teacher in the context of their work: Some hypotheses might help explain individual students' performance, other hypotheses might help decide how to move instruction forward. Indeed, noticing students' thinking is not valuable in and of itself, but in relation to the demands on the work of the teacher. To identify those demands we use Brousseau's (1997) notion of *didactical contract*.

The Didactical Contract

Practical rationality describes various demands on the teacher and students during instruction in terms of norms of the various structures that frame their work and obligations to the various stakeholders that hold practitioners accountable (Chazan et al., 2016). The practice of teaching mathematics through demonstrating how to solve each of the problem types students will have to solve, which is common in the United States (Stigler & Hiebert, 1999), provides an important context to define those terms. *Teaching through demonstrating problem types* (DPT) is an example of what Brousseau (1997) calls a didactical contract. However, as we see below, other didactical contracts are also possible. The point of identifying contracts is not to promote classifying classes or teachers in terms of their contracts, but to marshal means of understanding the interactions that happen during instructional exchanges: Instructional exchanges respond to the fact that a contract exists that may need to be further negotiated for the case of the work at hand. Every

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class (perhaps excepting kindergarten) has an initial didactical contract for mathematics based on shared expectations of what it means to study mathematics in school. For example, all didactical contracts are likely to rely on norms such as: (1) the teacher has the right to ask students questions to which the teacher knows the answer and (2) the student has the right to assume the teacher's questions have an instructional purpose. Those norms are not explicit but are accepted as appropriate by habit and application of maxims (e.g., respect of elders). The DPT contract relies on additional norms comparably established through habituation. In this contract, the content of studies can be described as a set of problem types that students should become fluent in solving and students may expect that anything they are supposed to learn will be demonstrated to them before they are asked to solve such problems independently. A consequence is that students may assume that they have already been taught what they need to know in order to solve the problems they are posed.

Each problem type that forms part of the DPT contract provides the opportunity to elaborate more specific norms. Hence, in Algebra I, one problem type is to solve equations in one variable. The contractual norm whereby the teacher has the right to assign problems gets specialized into a norm that the teacher has the right to pose equations for the students to solve. And the students' expectation that they have been taught what they need to know to solve the problems posed specializes into the expectation that any equation they are asked to solve gets solved by algebraically manipulating both sides of the equal sign. By *instructional situation*, we refer to each of the problem types and the specific norms that regulate teacher and students work on them. Accordingly, "solving equations" is an example of an instructional situation. Other examples of instructional situations in algebra include graphing functions and doing word problems (Chazan et al., 2012), and examples in high school geometry include doing proofs, calculating measures, and constructing figures (Herbst, 2010). It is important to note that an instructional situation is not only a type of task

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but a type of task in the context of a course of study, along with the norms that specify how the didactical contract applies to the posing and solving of those tasks. Thus, an instructional situation is a local version of the didactical contract; we refer to the norms of an instructional situation as *situational norms*.

As a didactical contract, DPT contrasts with another possible contract that might be called *teaching through problems* (TTP, hereafter; Lampert, 1990, 2001). In this contract, problems are posed with the expectation that students may not immediately know what they need to do to solve them, and that as they use a variety of available resources (e.g., their prior knowledge), they may discover something they don't yet know. The knowledge at stake may be a piece of declarative knowledge (e.g., the statement of a theorem), connections between different representations of a concept, a new procedure, etc. The reform movement in the US has had an interest in TTP since the early days of the NCTM Standards–advocating for teachers to choose worthwhile tasks and engage classes in discourse about those tasks that could lead to students acquiring new knowledge (NCTM, 1991). The Launch-Explore-Summarize sequence used in prominent curriculum materials (e.g., CMP; see Edson et al., 2019) has been one way in which TTP has become institutionalized in some schools. While other contracts might exist and these two might admit further differentiation, DPT and TTP are sufficiently distinct for the purposes of this paper, which are to identify important categories of perception that support teacher noticing. The two contracts play an important role in describing instruction in regular classrooms. In many classrooms where the default contract is DPT, there are occasional problems posed that require students to engage in productive struggle. Additionally, in classrooms used to TTP, teachers may provide additional opportunities for students to hone skills that were developed first through problems but where the practice opportunities rely on sets of norms that resemble instructional situations (see Chazan & Schnepp, 2002; Manouchehri & Goodman, 1998).

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Categories of Perception

Bourdieu (1998) names categories of perception as a salient constituent of practical reason– these categories account for the dispositions of practitioners as syntheses of (1) what their position in a social structure permits them to perceive, and (2) the perceptions that they individually experience or are disposed to have. Accordingly, we use the expression *categories of perception* to name the kinds of things that a teacher could be disposed to notice, given the context in which they work– with this context modeled with the notions of instruction, contract, instructional exchange, and instructional situation. Categories of perception describe (rather than prescribe) what a teacher may notice in ways that tie this noticing to the nature of instruction as described above. By identifying categories of perception rather than the specific things to be perceived, the theory makes room for observing individual differences in what practitioners notice. In this paper, we define some categories of perception as theoretical consequences of instructional exchanges and instructional situations and illustrate them in teacher conversations. Empirical claims about frequency, and group or individual differences are left for another occasion.

This paper focuses on categories of perception that are usable descriptors of teacher noticings of students' contributions to problem-based lessons. The categories of perception for teacher noticing introduced here are motivated by the need to understand instruction in the transition from DTP to TTP. The following section introduces the notion of *framing* to help provide context for the first category of perception.

Framing

The concept of framing was introduced in social research by Bateson (1955/2000) and made popular by Goffman's (1974/1986) analyses and Tversky and Kahneman's (1985) experiments. Framing alludes to the organization of events and individual experiences according to certain principles (e.g., "this is play" "this is fighting;" Bateson, 1956). Scheiner (2021) has described how

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the concept of framing can assist the study of teacher noticing, while Herbst (2006) has argued that the notion of instructional situation is useful to describe how a teacher frames students' work on problems (e.g., "this is construction", "this is proof"; Schoenfeld, 1988). More generally, in the context of working on problems, prior instructional situations that the class has engaged in before might be handy for students to use as ways to frame their work on a problem (Bing & Redish, 2009; Stein et al., 1996). Moreover, teachers might find it useful to propose such framings themselves when using novel problems to teach new ideas. Thus, the set of instructional situations (and larger didactical contracts) that a teacher of a certain course of study believes their students will be familiar with are one important category of perception for a teacher who is managing students' work on a problem. Each of those available instructional situations or framings presents the teacher with an opportunity to see how the students are, or could be, taking on the problem, and hence an opportunity for the teacher to scaffold students' work on the problem.

By *framing work on a problem with an instructional situation* we mean seeing a given problem, which is not necessarily a case of a given instructional situation, as if it was a case of that instructional situation. Figure 3 offers two examples for the case of the tangent circle problem, which is stated in Figure 3a. This problem is also the basis of a lesson that we designed to teach the tangent segments theorem.[2] Figure 3b shows how a teacher might frame the work on the problem as a construction and Figure 3c shows how the teacher might frame the work on the problem as a proof. Note that neither Figure 3b nor Figure 3c present canonical instances of a situation of construction or of proof as these could be seen in US high school geometry classes. Only some elements are present to usher the students into adopting that framing.

INSERT FIGURE 3a, 3b, 3c HERE

[2] The tangent segments theorem says that a circle is tangent to two intersecting lines if and only if the points of tangency are equidistant from the point of intersection. This is the content at stake in the lesson discussed.

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In Figure 3b, the use of the word *construct* and the provision of a diagram with some initial geometric objects (lines, points) to students who have done geometric constructions before frame the task as one in which they are expected to use tools and the given geometric objects to make new geometric objects. The names of the other geometric objects (e.g., circle) and the properties ascribed to them (viz., tangent at *A*) serve as controls for the construction work (e.g., what needs to be constructed, what needs to be checked for). Yet, the framed problem in Figure 3b departs from canonical instances of construction in that more initial objects are given than needed to produce a circle with the desired properties (see Herbst, 2010). However, this is precisely what makes this problem a good candidate for students to learn something new.

Similarly, in Figure 3c, the presence of the word *prove* and of a diagram with labeled points and marked known properties typically activates what they know about doing proofs–for example, the need to interact distally with the diagram, making claims about the figure based on its known properties. As can be noted, the framed problem in Figure 3c also departs from canonical instances of doing proofs in that the givens and the statement to prove have not been identified for the students (see Herbst, 2010). Rather, the students are asked to see what they can prove.

In a classroom where instructional situations such as construction and proof are perceivable as potential framings of a novel task, three other categories of perception are available for a teacher to notice student contributions. We define these and illustrate them using examples of what teachers have said when considering ways that the Tangent Circle lesson may play out.

Data Context

The three categories of perception that we present below have been derived theoretically from the concepts of instructional exchange and didactical contract. Those categories are being used to code conversations among teachers, though this paper does not report on the incidence of these categories in a dataset; rather, here we pick and choose quotes that illustrate how the categories

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appear in teacher discussions. The quotes are from teachers' reactions to students' contributions during two types of lesson-based discussions among practicing teachers. First, data was collected from focus group sessions targeted at discussing problem-based high school geometry lessons with practicing geometry teachers from around the country. These online focus groups investigated (1) the lesson objective and how the lesson could be launched, (2) anticipations of student contributions, and (3) how teachers would respond to students' work. To illustrate the categories of perception we identify below, we draw upon the reactions of 4 teachers (whom we call Labronna, Steve, Valentino, and Victor) to students' work on the Tangent Circle Problem. Second, data was collected from a 6-week online Story*Circle* (Herbst & Milewski, 2018), in which 18 practicing geometry teachers collectively participated in activities to script, visualize, and argue about how they imagine the Tangent Circle lesson unfolding. We draw upon an asynchronous activity in which teachers were asked to annotate ten different samples of student work from the lesson and the contributions from two participants (Quill and Zoey).

Responsiveness, Serviceability, and Normativity

Two categories of perception, *responsiveness* and *serviceability*, are particularly important for a teacher to distinguish if they want to support a TTP contract. A problem posed to promote work that leads to a new item of knowledge is not likely to be solved correctly as soon as it is posed. Rather, it is expected that students will produce a variety of attempts that the teacher then might select and sequence (Stein et al., 2008) for discussion by the whole class to arrive at the knowledge at stake. Additionally, as students are tied to the teacher by the didactical contract, they know that doing something in response to the problem would indicate good will if not also knowledge and, in exchange, they could expect the teacher to look more favorably upon any contribution than on no contribution. Thus, while a teacher might be able to notice correctness or completeness of students' work in these circumstances, these categories of perception may not be enough for the teacher to

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inform their discernment of which students' contributions to use to guide the class's progress toward the knowledge at stake. Thus, our question is, how might we describe the categories of perception a teacher might appeal to in order to perceive and organize the variety of students' contributions that can be observed in response to a problem when teaching through problems?

The identification of different ways in which a teacher may perceive students' contributions serves not only researchers who want to investigate teacher noticing as situated in the practice of instruction but also to model a first-person analysis (Ball, 2000) of how teachers might distinguish among the diverse contributions they might expect (when they anticipate possible approaches; Stein et al., 2008). This, in turn, may help observers understand when and how a teacher may have the opportunity to, and eventually how they do, incorporate different student contributions to the flow of instruction. Indeed, what a teacher makes of students' contributions may likely influence what they decide to do with them: Some contributions may be acknowledged privately while others may be selected to be shown publicly, some may be supported while others confronted (Milewski & Strickland, 2020), some may be given more versus less time in whole class discussion, some may be brought up to the whole class before others, and so on. In other words, through words and deeds a teacher will come across as treating students' contributions differently after taking stock of them. We surmise that this taking stock of student contributions relies neither only on individual perception nor only on individual cognition but also on how the situation the teacher is in activates categories of perception.

The situatedness of noticing (e.g., Blömeke et al., 2015; Kaiser et al., 2017) could be noted at various scales: The practice of mathematics instruction situates the teacher in one of their institutional functions, specific contracts such as DPT or TTP in a course of study (e.g., geometry) further specify such situatedness. But for us the potential exchange between students' work on a problem and the knowledge at stake in the problem, and the instructional situations available to

frame the work on the problem provide even more specificity to the situatedness of teacher noticing. The categories of perception we propose articulate how a teacher might perceive students' contributions at that level of specificity.

Responsiveness

Responsiveness to the problem posed (or *responsiveness*, for short) is one of the categories of perception that describe what teachers may notice in students' contributions. Responsiveness alludes to the possibility that a student's work may be more or less responsive to the question asked in the problem. This category is relevant insofar as the problem posed is likely to spell out for students what kind of product they need to produce (Doyle, 1988). Even if students don't know the knowledge that might allow them to complete the problem correctly, they might use their understanding of what the problem asks them to do to produce a contribution that relates to what has been asked. We propose the word *responsiveness* to describe this category of perception–the extent to which the student's contribution responds to what the problem statement asks them to do. We note that responsiveness does not mean correctness–for example, a student may attempt to produce a circle tangent in response to the problem as posed in Figure 3b and yet fail to do so (or succeed only partially; see Figure 5b, where the student produced a circle, albeit not a tangent one). Also, as with the other categories of perception we present below, we do not claim these are necessarily binaries (i.e., we do not suggest that a teacher classifies contributions into responsive and nonresponsive) but suspect instead that a teacher could perceive student contributions in terms of more or less responsiveness to the problem posed. The distinctions teachers might make within responsiveness are accountable not only to individual differences among teachers but also to the affordances of the task chosen to promote an instructional exchange, particularly to what the task statement identifies as its product (Doyle, 1988; Herbst, 2006). For the sake of introducing responsiveness, we present examples in which responsiveness is distinguished in some contributions

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from lack of, or less responsiveness in others. If we consider the Tangent Circle problem presented in Figure 3a, Figures 4a and 4b present examples of work that we would describe as more responsive, while Fig 4c presents an example of work that we would describe as less responsive.

INSERT FIGURE 4a, 4b, 4c HERE

Consider Figure 4a. The image shows something that looks like a precisely drawn circle that appears to touch the lines at marked points. We see this work as responsive to the task posed. Figure 4b shows a different contribution, where the student also drew something that purports to be a circle and that appears to touch the lines at marked points. Figures 4a and 4b are clearly different in how they respond to the question–with the student who produced Figure 4a obtaining a shape that is more defensibly a circle than the student who produced Figure 4b. But they have in common that they accept starting from what the problem said was given (two intersecting lines and two points) and aim at what the problem requested to produce. They contrast with the work in Figure 4c in which nothing resembling a circle is visible. Note that this lack of responsiveness does not entail a holistic judgment of worthlessness, as a nonresponsive contribution to a problem might nevertheless have other qualities that make it noticeable and usable in instruction, as we see later.

Responsiveness can be recognized by teachers. For example, upon seeing the contribution shown in Figure 5b, Victor, a participant in our focus groups, said:

> My thought would be that they're-they're trying to find some point in the middle of those two lines, saying, "Are those two lines?" And so they're finding a spot that should be equidistant in there–is what they're thinking because it's the middle of those two points. So they're trying to find the circle that fits within would be my guess.

In contrast, upon being asked to comment on the contribution shown in figure 4c in the context of Story*Circles*, Quill said he would ask the student:

> What conclusion do you come to as [a] result of this proof? What needs to change/be added

to our original condition? Where is the circle in the picture?

And Zoey observed that "Circle O is missing."

However, *responsiveness to the problem posed* at the beginning of the lesson is not the only category of perception that might describe what a teacher could notice. Indeed, a second category of perception is exemplified in the differences between Figures 4a and 4b, both of which we have so far described as more responsive.

Serviceability

Serviceability to the instructional goal (or *serviceability* for short) is a second category of perception that describes what teachers may notice in students' contributions. By serviceability of a student's contribution, we mean the extent to which the contribution contains elements that support the class's pursuit of the instructional goal. A wide range of students' productions might be serviceable in this sense: If students' engagement with the problem attested to one or more of the mathematical observations that would need to be made to claim the knowledge at stake, teachers might perceive this student work as serviceable for their goals. Similarly, if student contributions included errors that might prevent progress toward the knowledge at stake, that would make the contribution salient for its lack of serviceability. And this observation highlights why serviceability is a category of perception rather than a particular value in that category. Further, the possibility that a student contribution might both include elements that support progress toward the knowledge at stake and elements that prevent it suggests that attention to the serviceability of a contribution might be associated with a variety of attitudes toward it.

Serviceability may, but also may not, align with responsiveness. Indeed, in general, the instructional goal of a lesson is different from the specific product required by the task. Thus, a student's contribution might have elements serviceable to the instructional goal while not being responsive to the problem posed (e.g., Figure 4c fails to be responsive, but contains serviceable

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aspects in that it offers a proof of the target theorem). Also, a contribution might be responsive to the problem but be less serviceable to the knowledge at stake. The contrast between Figures 4a and 4b highlight this. Some differences between Figure 4a and Figure 4b can be interpreted in terms of the instructional goal of the Tangent Circle lesson. While the task pointedly requires students to find a circle tangent to two given lines at two given points, the instructional goal of the lesson is to reveal the conditions that make the solving of that problem possible–as those conditions are articulated by the Tangent Segments theorem. Figure 4a shows the work of a student who switched the location of point B once they constructed a circle tangent at A: Tacitly, this action suggests that the choice of the points of tangency matters to whether the problem could be solved. In contrast, Figure 4b shows the work of a student who stayed with the two points originally given, applied a conception of circle as a round thing that can be drawn free hand, and did not use the visual feedback (i.e., that the circle looks odd) to reason about the appropriateness either of the approach taken or of the givens of the problem. While there is reason to see both approaches as attempting to address the problem as asked (hence, responsive) and both as having potential to build upon, the work shown in Figure 4a seems more serviceable to making the claim that the solvability of the problem depends on the choice of the points of tangency.

Figure 4b demonstrates that serviceability is not necessarily a dichotomy. The feedback one gets from the diagram (the circle is odd) could be serviceable for, for example, refining a method to solve the problem (e.g., the student could look for an alternative way to make a circle that looks better and as a result get to think about the points given). But the consideration that in Figure 4b's case there are other issues that are likely to arise before a conjecture about the choice of points is warranted might matter at the time of comparing Figure 4a and Figure 4b regarding their serviceability. And yet, the fact that the contribution in Figure 4a may be seen as more serviceable than the one in Figure 4b does not require a specific appreciation or action. Their differential

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serviceability does not make one necessarily better for everything–rather, discerning their differences is useful for a teacher to manage the class's pursuit of the knowledge at stake. A teacher might decide to publicly share Figure 4b before Figure 4a precisely because it is less serviceable and might provide an opportunity for more students to discuss it. This teacher might bring in Figure 4a closer to the time when they would need the conjecture to emerge. Figure 5a shows yet another example of serviceable work, but some of its differences with Figure 4a suggest a third category of perception. Figure 5b shows an example of a less serviceable contribution that, however, displays characteristics a teacher would likely also perceive and that motivates the third category.

Serviceability can be recognized by teachers. For example, upon seeing the contribution on Figure 5a, Valentino noted:

> I feel that that one is… is very much so connected to the theorem that you're … trying to get the students to realize about the points of tangency being equidistant from the point of intersection.

But in reaction to the contribution shown in Figure 5b, Steve indicated that this contribution would need more work to be serviceable, thus:

> So I would prompt them to … see if they can use the tangent to a circle theorem …[about six minutes later] then they would eventually stumble upon that it has to be perpendicular and then I'd ask, is this perpendicular and then like, you know, this would be opened up to the class and they'd recognize that it's not and so then, this doesn't work.

However, there is another difference between the contributions in Figures 5a and 5b that might allow teachers to see something else.

Normativity

Normativity of a student's contribution within an instructional situation (or *normativity*, for short) is a third category of perception that describes what teachers may notice in students' contributions. As

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noted above, instructional situations such as construction or proof could be useful to frame students' work on a novel problem. For the example of the Tangent Circle lesson, while Figure 3a shows the problem in all its novelty, Figure 3b shows how the problem could be framed as a construction problem. In Figure 3a the facts that no diagram is given and that the word *find* is used leave ambiguous how students might proceed. For example, students might choose to construct but they could also approach it in other ways–as Doyle (1988) described, all that openness can invite multiple solution approaches but also can fail to stimulate any work until some ambiguity is removed. Students could frame a novel problem as a case of an instructional situation in which they know how to work and diverse mathematical strategies may be noticed as a result of different framings (e.g., some students may find the circle by figuring out how to construct it while others may do so by proving the properties that such a circle should have). Framing a novel task as a case of an instructional situation is a move the teacher can also make to scaffold students' work. Figure 3b shows the teacher using the word *construct* instead of *find* and providing an actual diagram with givens; these choices would likely trigger in students the expectation that they should use construction tools and actually produce the circle desired. Along those lines, the contribution in Figure 5a shows circles sketched without any evidence that the student has constructed their centers. One may surmise that even if they have been made to look like circles, they have not been constructed as circles.

INSERT FIGURE 5a, 5b, 5c HERE

By *normativity*, or by saying that a student's contribution is more normative, we mean that if the work on the problem is seen as an instance of an instructional situation (e.g., construction), the contribution of the student can be recognized as complying with or breaching (some of) the norms of that instructional situation. We do not use the word *normative* as synonymous of correct in the sense that the problem has been solved in a mathematically correct way; rather we use *normative* to

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allude to the teacher's perception that (some of) what the student has done to solve the problem is appropriate within the frame of an instructional situation. Again, while understanding the meaning of normativity requires at least to distinguish normative from nonnormative work, it is more likely that a teacher would see different degrees of normativity in students' work. Figure 5a is less normative within a construction situation in the sense that the circles have not been constructed. But the lack of tools is only one piece of evidence of its lack of normativity. The student could have also traced round objects (e.g., bottle caps) with care to fit the circles to look tangent or used a compass centered at a point chosen without any clear reason, and, in such cases, one might still note the work as less normative for a construction (because the center of the circle would have not been obtained through stipulable construction steps). More normative construction work would generate desired geometric objects by performing stipulable steps on given geometric objects. One can see in the contributions in Figures 4a and 5b that, regardless of their serviceability, both demonstrate stipulable procedures for creating a circle. In each case, the circle's existence is warranted on a precisely identified center and radius. In Figure 4a, the student seems to have found the center as the intersection of a perpendicular through a point of tangency and the angle bisector; and in Figure 5b, the student seems to have found the center as the midpoint of the segment between the two given points of tangency. The fact that both contributions are more normative in the situation of construction and yet very different in their degree of serviceability highlight the importance of distinguishing these categories of perception.

Normativity can be recognized by teachers, as can be seen in the comments below. Upon offered to consider the contribution shown in Figure 5b, Labronna said:

> This was representative of a lot of the student work, I would push them on why they picked the perpendicular bisector.

And a few seconds later, Labronna reflected:

> the prompt … used the word construction. So I think the student was tempted to do some kind of construction, but whether they knew that this one was appropriate or not, and why I would kind of want to tease that out in discussion.

In contrast, when shown work that we would describe as less normative according to the situation of construction, such as the contribution shown in Figure 4b, Steve indicated:

> this would probably be something that I would share early on to scaffold them towards forcing them to construct if they're not doing it

And:

> this is a common picture and I don't want them to go down this avenue. How do you engender them or push them towards actually constructing to make it so that it does look appropriate?

We offer these three categories of perception–*responsiveness*, *serviceability*, and *normativity*–in addition to *instructional situations available for framing*, as useful ones for researchers to describe what teachers notice when confronted with students' contributions in problem-based lessons. They are categories of perception that can be applied simultaneously to a contribution–that is, a student contribution might be simultaneously perceived as having a degree of responsiveness, a degree of serviceability, and a degree of normativity for a particular framing. One can see, for example, that the contribution shown in Figure 5b was deemed to be more responsive by Victor and more normative by Labronna, while Steve saw it as not so serviceable. Not only different individuals might perceive the same contribution using different categories, but it is also possible that the same individual teacher might be compelled to see the contribution using more than one of those categories.

Discussion

We describe framing, responsiveness, serviceability, and normativity as categories of perception because they name qualities that teachers can use to *see* the problems in which students work (for the

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first case) and the work students do in those problems (for the other three). Clearly *seeing* is a metonymy here as visual input is only one component of the perception involved and unsighted teachers could still perceive these categories with alternative sensory input. Our use of seeing connects to Coliva's (2012) notion of "seeing-as," inasmuch as they present alternative ways of casting the problem (as framed by various instructional situations) or the student's contribution (in terms of its responsiveness, serviceability, or normativity).

The role of categories of perception in describing the work of mathematics teachers

In their conceptualization of competence, Blömeke et al (2015, p. 7) claim that individual processes (such as cognition and affect-motivation) are mediated by situation-specific skills (including perception, interpretation, and decision-making) in the production of observable behavior. Kaiser et al. (2017) have articulated how this model supports understanding the work of mathematics teaching. In that context, it is important to note that the same processes studied by noticing scholars are the ones that Blömeke et al. (2015) describe as situation-specific skills. The work we present here complements Blömeke et al.'s (2015) model by proposing how it is that the situation calls for those situation-specific skills. Indeed, we are not psychologists but mathematics education scholars and this orientation shapes our interests in ways complementary to those of psychologists. A psychologist might be interested in what happens in individuals when, in specific situations, they tend to see, think, feel, decide, and eventually act in ways possibly different than in other situations. In our case, as researchers of the practice of mathematics teaching, we are interested in the situations themselves and on what the affordances are of situations that permit, prevent, or perhaps require practitioners to see, think, feel, decide, and eventually act in particular ways.

Our present contribution adds to earlier work in which we describe the instructional

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situations[3] in which teachers get to perceive, interpret, and make decisions (e.g., Chazan & Lueke, 2009; Herbst, 2010). As noted above, in prior work we have used the constructs of didactical contract and instructional situation as ways to describe more precisely some of the situations that compose the practice of mathematics teaching. The present article contributes by proposing the categories of perception as affordances of teaching practice that can be found in didactical contracts that make room for problem-based instruction while having some instructional situations available to frame students' work on those problems. In contrast with the individual-centered noticing work that would examine how individuals notice or what they notice, these categories of perception are described as properties of the work of instruction, or affordances of practice.

These affordances of practice are four. First, the existence of instructional situations in the class's prior history (e.g., construction, proof, calculation) provides resources for the teacher to frame students' work on a problem as a case of an instructional situation (even if the way of being a case may not be canonical[4]). These affordances are also available for students to frame the problem: Students may look at novel problems using the lenses of their prior work on instructional situations and make contributions that the teacher can notice as attesting to such framing. Second, as students work on a problem that they do not immediately know how to solve, it is possible to see their contributions (regardless of their correctness) in terms of their *responsiveness* to the problem posed. Third, as the problem has been chosen as an instrument to lead the class to the knowledge at stake, students' contributions can be seen in terms of their *serviceability* for foregrounding the knowledge at stake, regardless of their correctness or responsiveness. Fourth, as students' work on the problem may abide by or flout norms that regulate instructional situations used to frame this work, students'

[3] NB: We take Blomeke et al's (2015) use of the word *situation* as much more general than our use of *instructional situation*.

[4] For example, this framing can be done by distancing the problem from a canonical case of a situation. As when a teacher tells their students that they don't have to construct it with straightedge and compass but just describe how they would do it if they had to.

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contributions can be seen in terms of their *normativity* according to an instructional situation (regardless of their correctness, responsiveness, or serviceability).

These categories of perception are made available by the design of a lesson in which a problem has been chosen as the starting point for work that pursues an item of knowledge at stake. While the categories themselves might be found in all problem-based lessons, it is clear that what counts as responsive, serviceable, or normative depends on the problem, the knowledge at stake, and the available instructional situations. Thus, while the skills of perceiving, interpreting, and deciding are implemented by individuals-in-situation, those affordances of practice are available to anticipate how practice might modulate the application of such skills. These categories of perception permit the formulation of conjectures about how teachers' perception of students' work might impact the decisions they make.

Situational affordances for interpretation and decision making

The categories of perception offered in this paper only propose situational affordances for what is there to be perceived by teachers (and likely a partial list at that). More work on these affordances still needs to document how the situation provides resources for teachers to apply skills of interpretation and decision-making (Blömeke et al., 2015). Bourdieu's (1998) description of practical reason alludes not only to categories of perception but also to categories of appreciation. We take the latter to mean the affordances a situation has for practitioners to exercise their situated interpretation skills in the form of attitudes toward what they notice. Future work needs to articulate what the predicates are that a situation makes available for a practitioner to appreciate, judge, or feel about what they perceive and how these categories of appreciation are modulated. What are the public, practical attitudes that a teacher could have toward different student contributions in a problem-based lesson? Delight, distrust, enlightenment, frustration, preference, and puzzlement are some conceivable attitudes toward students' work that might be expressed when outwardly

addressing students' work (with a student or the whole class). But what are those attitudes which teachers deem reasonable or sensible to have towards student contributions? Do they vary according to the instructional situations available to frame the exchanges?

These possible categories of appreciation for students' contributions might also vary according to the task of teaching in which they may be involved. What are those tasks of teaching in which categories of perception and appreciation might afford resources for decision making? Responding to students' contributions individually and privately might contrast with responding to multiple student contributions publicly: These two tasks of teaching may rely on the same categories of perception described above but make available different categories of appreciation. Furthermore, the pursuit of these tasks of teaching happens in a constrained environment (e.g., students may hear what the teacher says to other students, time is limited), how do those perceptions and appreciations of student contributions complement or compete in that constrained environment is a matter of research. Decisions of what contribution to bring to the attention of the class and at what time, and how to respond to those contributions are among the important decisions a teacher needs to make. Thus, categories of perception and appreciation may help better understand the basis upon which teachers engage in selecting and sequencing students' contributions for classroom discussions and in making connections across them (Stein, et al., 2008).

Indeed, researchers examining teachers' selection of students' contributions (e.g., Dunning, 2022) have paid attention to the different mathematical strategies that students employ when solving problems, often relying on distinctions of degree of mathematical sophistication of students' strategies (Stein et al., 2008). Without denying that teachers can make such distinctions, the present contribution suggests that the characteristics of the work of teaching may affect how teachers see those different strategies in the context of the work that the teachers need to do. Mathematical sophistication (e.g., along a learning trajectory) might be usable by teachers to give individual

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feedback (Hanby, 2018). But as it concerns orchestrating the progress of a lesson from a novel problem to a particular item of knowledge at stake, distinctions in mathematical sophistication may not be the only, or even the most salient, ones available for a teacher to make. Some strategies, such as constructing or proving may be seen as different choices to frame the problem as a case of something students know how to do. Within one of those frames (e.g., within construction), different strategies (e.g., to determine a center using an intersection, to trace a round object) may be seen as more versus less normative. Similarly, different mathematical strategies may be further differentiated in terms of their responsiveness to the problem posed or their serviceability toward the instructional goal of the lesson. Researchers on students' mathematical cognition may be attuned to the differences in the mathematical sophistication of students' ideas (Stein et al., 2008) but we have argued that the demands of work of teaching through problems makes available other categories of perception for teachers with which they might make further or different distinctions than the cognitive distinctions researchers make.

Conclusion

The present paper contributes to our field's understanding of teacher noticing of student contributions in the context of problem-based lessons. Complementing individual-centered approaches to research on noticing in which perceptual, cognitive, or situated individual skills are described, this conceptual paper offers four categories of perception as examples of affordances available for the practice of teaching mathematics through problems. These include (1) the instructional situations available to frame the problem, and the possibility to see students' work as (2) responsive to the problem, (3) serviceable to the knowledge at stake, and (4) normative for an instructional situation. The paper shows examples of how teachers recognize responsiveness, serviceability, and normativity and calls for research that can further flesh out the affordances the practice of teaching offers for teachers to do their work.

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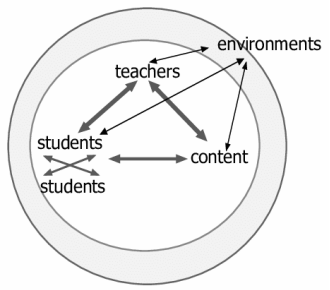


Figure 1. The instructional triangle (from Cohen et al., 2003, p. 124)

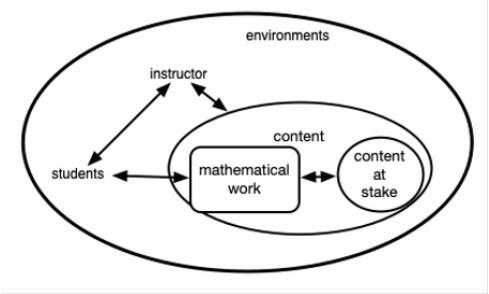


Figure 2. An elaboration of the instructional triangle for the study of mathematics teaching

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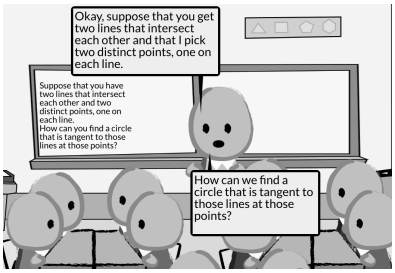


Figure 3a. A teacher starting a lesson centered on the Tangent Circle problem

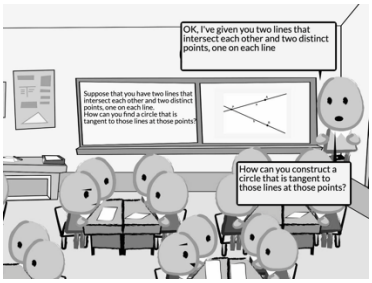


Figure 3b. A teacher framing the Tangent Circle problem as a construction

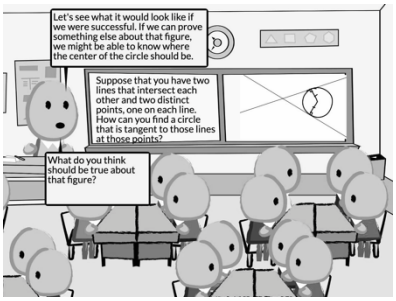


Figure 3c. A teacher framing the Tangent Circle problem as a proof

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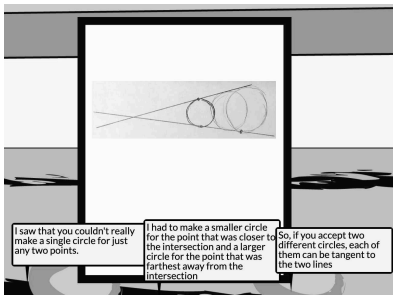


Figure 5a. Work that is more serviceable and responsive, but less normative (construction)

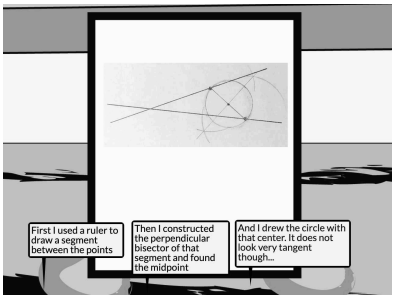


Figure 5b. Work that is more normative (construction) and responsive, but less serviceable

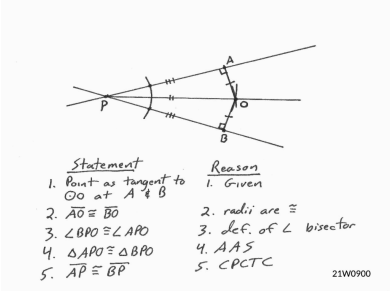


Figure 5c. Work that is more serviceable and normative (proof), but less responsive

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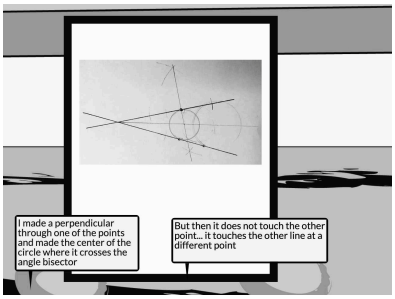


Figure 4a. Example of more responsive work

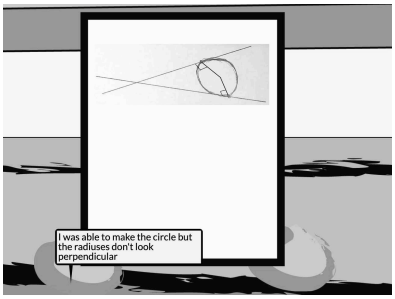


Figure 4b. Another example of more responsive work

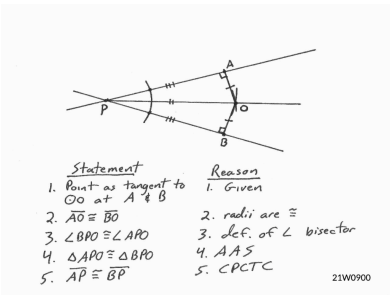


Figure 4c. An example of less responsive work

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