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OPTIMAL PRICING BY DOMINANT FIRMS

by

Darius William Gaskins, Jr.

A dissertation submitted in partial fulfillment
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ABSTRACT

OPTIMAL PRICING BY DOMINANT FIRMS

by

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This dissertation is a theoretical study of the optimal pricing strategy for a dominant firm or a group of joint profit-maximizing oligopolies facing potential entry into its product market. The dominant firm is assumed to maximize the present value of its stream of profits, subject to a differential equation constraint relating the rate of rival producer entry to the current product price.

Under this general framework various models are explored utilizing the mathematics of optimal control theory to determine optimal price trajectories. The basic model postulates that the rate of rival entry is proportional to product price and that the dominant firm's relative cost vis à vis rivals remains constant over time. The following general conclusions are shown to hold for this model:

1. The optimal price will always be less than the myopic profit-maximizing price.
2. It is rational to drive out competitors if the dominant firm enjoys a sufficient cost advantage over its rivals.

3. For an infinite time horizon, the optimal price approaches the limit price (that price which induces zero net entry) asymptotically.
4. Optimal pricing by a dominant firm with no cost advantage over rival producers will lead to the loss of the dominant firm's market power.

A model postulating that the rate of rival entry is a quadratic function of the current product price (implying that rivals enter more readily than they exit, and that entry increases more than proportionally as price increases) yields roughly the same results as the basic linear model.

Optimal price trajectories are found for dominant firms whose cost advantage over rival producers is gradually eroded or undergoes discrete jumps (i.e. when a basic patent expires). The optimal price trajectories for infinite time horizons approach the limit price asymptotically in both of these cases.

A model in which the product market is assumed to grow with increasing disposable income is examined. The optimal price trajectories under this growth model differ from the results of the basic model in the long run, the optimal price no longer converges to the limit price and dominant firms with no cost advantage no longer lose all their market power.

A model in which market demand and rival production have some stochastic variation yields an optimal pricing strategy relating the price at any instant to the current level of rival output and market demand.

Finally, we examine the effect of constraints on the dominant firm which prevent it from enlarging its market share even if it enjoys a substantial cost advantage. This is the only case in which the optimal pricing strategy is to set price at the limit price over most of the firm's planning period.

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INTRODUCTION

The purpose of this study is to establish the optimal pricing strategy for a dominant firm or a joint-profit maximizing oligopoly under various models which hold that future market conditions are determined by current and past pricing decisions. The dominant firm is assumed to strive to maximize the present value of its profit stream over some specific time period subject to induced entry into its product market. The models considered postulate that the rate of entry of rival producers is a function of the contemporaneous price level established by the dominant firm. The study proceeds in the following way.

Chapter I establishes the optimal pricing strategy under a basic entry model which postulates that the rate of entry is a linear function of current price. Optimal price trajectories are found for both finite and infinitely long planning periods. Chapter II explores optimal pricing under a quadratic entry model which holds that the rate of entry increases more than proportionally as price increases. Chapter III explores optimal pricing when the dominant firm's cost advantage over its rivals varies over time. The effect of vital patent expiration is specifically analyzed in this chapter. In Chapter IV I analyze the effects of market growth and stochastic variation in the product market. In Chapter V I examine the impact of imposing constraints

on the amount of entry or exit by rival producers. Chapter VI contains a brief discussion of the implications of this study for economic theory and public policy.

CHAPTER I

THE BASIC MODEL

This thesis attempts to determine the optimal pricing strategy for a dominant firm or a group of joint profit maximizing oligopolists faced by potential entry into the product market. The model explored here has broader implications, but we limit our initial discussion to a dominant firm able to deduce a residual demand schedule through knowledge of the output of a well behaved competitive fringe. Contemporary writings on this subject for the most part contend that the dominant firm will maximize its present value by either charging the short-run profit maximizing price and allowing its market share to decline or by setting price at the limit price and precluding all entry.[1] A firm practicing short-run profit maximization would have to ignore continually the reality of induced entry. Conversely, a firm charging the limit price has to be convinced that its prevailing market share is optimal. There has been no analytic justification for this strategic dichotomy, and intuition suggests that the optimal strategy would entail a balancing between current profits and future market share.

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[1]See for example Dale Osborne, "The Role of Entry in Oligopoly Theory," Journal of Political Economy (August 1964) p. 396.

The basic premise of this study is that the rate of entry of rival producers is a function of current price. Mansfield has shown that the variation in rate of firms entering or exiting an industry is positively correlated with the level of industry profits.[2] It follows that a dominant firm with high current price and profit levels is sacrificing some future profits through erosion of its market share. The dependence of future market share on the current price level means that the dominant firm's pricing strategy can only be determined in a dynamic framework.

The Model

In general terms, the optimal pricing strategy will maximize the present value of firm profits, as given by Equation 1.1 below.

$$V[p(t),c(t),q(p(t),t),t] = \int_0^T (p(t)-c(t))q(p(t),t)e^{-rt}dt \qquad (1.1)$$

$p(t)$ = current market price

$c(t)$ = average total cost of production

$q(p(t),t)$ = current sales

r = dominant firm's discount rate

We have deliberately chosen a finite time horizon to include the case of a dominant firm with a patent advantage which will expire at some specified future time. The

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[2] E. Mansfield, "Entry, Gibrat's Law, Innovation and the Growth of Firms," American Economic Review (December 1962) p. 1026.

behavior of a firm with an infinite time horizon can be studied by allowing T to increase without bound. This formulation excludes any consideration of residual or scrap value at the end of the planning period.

The specific functional dependence of sales on time is determined by the nature of the entry phenomenon. We assume initially that the level of the dominant firm's current sales can be decomposed into additive univariate functions of price and time respectively.

$$q(p(t),t) = q_o(p(t)) + x(t) \qquad (1.2)$$

where $q_o(p)$ = initial demand schedule

$x(t)$ = the change in level of rival sales

The residual demand curve $q(p(t),t)$ at any specific instant is found by subtracting the output of the competitive fringe from the total market demand. Equation 1.2 indicates that the net effect of rival entry into the product market is to shift the dominant firm's residual demand curve laterally. This would be the case if rival producers' short-run supply curve is completely inelastic.[3] The rate of entry of rival producers $[\dot{x}(t)]$ is surely determined by their expected rate of return. If potential entrants view current product price as a proxy for future price, the rate of entry will be a monotonically non-decreasing function of current price. We assume that this relationship between rate of entry and

---

[3]The residual demand curve will also shift laterally if a more general rival supply curve shifts by a constant amount at any price level.

current price is linear, as shown by Equation 1.3.[4]

$$\dot{x}(t) = k[p(t)-\bar{p}] \qquad (1.3)$$

$\bar{p}$ = limit price

k = response coefficient ≥ 0

The limit price $\bar{p}$ is defined in this model as that price level at which net entry is equal to zero. The model implies therefore that pricing below $\bar{p}$ will cause negative entry or exit from the product market by rival producers.[5] The difference between the limit price and the dominant firm's average total cost is a measure of the cost advantage enjoyed by the dominant firm. It seems reasonable to restrict the limit price to be greater than or equal to the dominant firm's average total cost. While there may be examples of dominant firms with cost disadvantages, their fate is obvious with or without optimal pricing. We similarly restrict the limit price to be no greater than the static monopoly price, arguing that entry is not blockaded.[6]

To expedite the analysis further, we assume that the initial demand schedule $q_o(p(t))$ is linear over the relevant

---

[4]This can be viewed as a first order approximation of a more complicated functional relationship between $\dot{x}$ and $(p(t)-\bar{p})$. A quadratic entry model which allows asymmetric entry and exit and holds that the rate of entry increases more than proportionally as price increases has been explored in Chapter II.

[5]Mansfield, op. cit., p. 1029, has shown that below normal profits are correlated with net exit of producers.

[6]For a detailed discussion of this term, see Joe S. Bain, Industrial Organization. rev. ed., (New York, John Wiley & Sons, 1968), p. 274.

range and that the limit price $\bar{p}$ and the dominant firm's average total cost do not vary over time.[7] With these simplifying assumptions, it is possible to determine the optimal pricing strategy analytically using the mathematics of optimal control. In the language of modern control theory, we wish to maximize the functional:

$$V = \int_0^T \underbrace{[\bar{p}+\delta(t)-c][a-b(\bar{p}+\delta(t))-x(t)]e^{-rt}}_{= L(\delta(t),x(t),t)} dt \qquad (1.4)$$

subject to

$$\dot{x}(t) = k\delta(t); x(0) = 0 \qquad (1.5)$$

where $\delta(t) = p(t) - \bar{p}$ $\qquad c \leq \bar{p} \leq \frac{a+bc}{2b}$

$q(t) = a-b[\bar{p}+\delta(t)]-x(t)$

a, b, $\bar{p}$, c are positive constants

a = output intercept of original demand curve

b = slope of demand curve

c = dominant firm's average total cost

In the control theory framework $x(t)$ is the state variable and $\delta(t)$ is the control variable. Since the limit price $\bar{p}$ is constant, knowledge of the optimal path for $\delta(t)$ will provide the optimal price trajectory.

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[7] Cases involving continuous variation and discrete jumps in $\bar{p}$ have been considered in Chapter III.

The continuity and concavity of L and $\dot{x}$ with respect to δ and x assure us that at least one optimal path exists.[8] We may derive necessary conditions for the optimal path by using Pontryagin's Maximum Principle. The Hamiltonian for this problem is given by

$$H = L(\delta(t), x(t), t) + z(t)k\delta(t) \qquad (1.6)$$

The adjoint variable z(t) appearing in the Hamiltonian is equal to $\frac{\partial V}{\partial x}(t)$ and can be interpreted as the shadow price of an additional unit of rival entry at any point in time.[9] The first term of Equation 1.6 (the integrand of Equation 1.4) is the change in present value accruing from current sales. The second term which is the product of $\dot{x}(t)$ and z(t) reflects the effect of current entry on future profits. Intuitively, maximizing the Hamiltonian with respect to $\delta(t)$ involves balancing the flows of present value from present and future sales. The maximum principle states that for a maximum V

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[8] An existence theorem relevant to this problem may be found in Lamberto Cesari, "Existence Theorems for Optimal Solutions in Pontryagin and Lagrange Problems," SIAM Journal of Control, Vol. 3, No. 3, 1966, pp. 478-79. It is noted that this existence theorem requires that the control variable and state variable be contained in compact sets. We can satisfy this restriction by imposing the constraint $-\bar{p} \leq \delta(t) \leq 1000\bar{p}$ on the control variable, and the constraint $-a \leq x(t) \leq \frac{k(\bar{p}-c)}{r} + b(2\bar{p}-c)$ on the state variable. As long as trajectories satisfying necessary conditions for optimality do not encounter these constraints we may apply this existence theorem.

[9] Strictly speaking this interpretation requires finding a twice differentiate solution to the Hamilton-Jacobi partial differential equation for this problem. This has been done and may be found in Chapter IV.

to exist, it is necessary that there exist a $z(t)$ such that:

i) $\dot{x}^*(t) = k\delta^*(t)$ $\qquad x^*(0) = 0$

ii) $\dot{z}^*(t) = -\frac{\partial H}{\partial x}(x^*(t),z^*(t),\delta^*(t),t) = (\bar{p}+\delta^*(t)-c)e^{-rt}$ $\qquad z^*(T) = 0$

iii) $H(x^*(t),z^*(t),\delta^*(t),t) = \text{Max } H(x(t),z(t),\delta(t),t)$ w.r.t. $\delta(t)$

In the three conditions stated above, the superscript $(\ )^*$ denotes variables along the optimal trajectory. Since H is differentiable and concave in δ, condition iii is equivalent to $\frac{\partial H}{\partial \delta} = 0$ and implies that:

iii') $\delta^*(t) = \frac{kz^*(t)e^{rt}+a+bc-x^*(t)-2b\bar{p}}{2b}$

Making the substitution $y(t) = z(t)e^{rt}$, the necessary conditions become a simple two point boundary value problem involving ordinary linear differential equations with constant coefficients. Specifically, the optimal price trajectory must be a solution to

$$\dot{y}(t) = \alpha_1 y(t)+\alpha_2 x(t)+\alpha_3 \qquad y(T)e^{-rT} = 0 \qquad (1.7)$$

$$\dot{x}(t) = \beta_1 y(t)+\beta_2 x(t)+\beta_3 \qquad x(0) = 0 \qquad (1.8)$$

where

$$\alpha_1 = r+\frac{k}{2b} \qquad \alpha_2 = \frac{-1}{2b} \qquad \alpha_3 = \frac{a-bc}{2b}$$

$$\beta_1 = \frac{k^2}{2b} \qquad \beta_2 = \frac{-k}{2b} \qquad \beta_3 = \frac{k(a-2b\bar{p}+bc)}{2b}$$

Because the boundary conditions on x and y are not given at the same instant in time, the explicit solution for all time is somewhat cumbersome. The general solution is given by Equations 1.9, 1.10 and 1.11 below:[10]

$$y(t) = d_1 e^{\lambda_1 t} + d_2 e^{\lambda_2 t} - \frac{\bar{p}-c}{r} \tag{1.9}$$

$$x(t) = \phi_1 d_1 e^{\lambda_1 t} + \phi_2 d_2 e^{\lambda_2 t} + \theta \tag{1.10}$$

$$\delta(t) = \frac{\dot{x}(t)}{k} = \frac{\lambda_1 \phi_1 d_1 e^{\lambda_1 t} + \lambda_2 \phi_2 d_2 e^{\lambda_2 t}}{k} \tag{1.11}$$

where

$$\lambda_{1,2} = \frac{r}{2} \pm \frac{\sqrt{r^2 + \frac{2kr}{b}}}{2} \qquad \lambda_1 > r \qquad \lambda_2 < 0$$

$$\phi_1 = \frac{\lambda_1 - \alpha_1}{\alpha_2} \qquad \phi_2 = \frac{\lambda_2 - \alpha_1}{\alpha_2} \qquad 0 < \phi_1 < \phi_2$$

$$\theta = \frac{k}{r}(c-\bar{p}) + b(c-2\bar{p}) + a$$

d_1, d_2 = constants determined by the conditions

$$x(0) = 0, \quad y(T)e^{-rT} = 0$$

The boundary conditions are uniquely satisfied by an appropriately selected initial value of the adjoint variable

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[10] A good reference on the solution of simultaneous linear differential equations is W. Kaplan, Ordinary Differential Equations (Reading, Mass.: Addison Wesley Publishing Co., Inc.), pp. 219-99.

$y(0)$.[11] The resulting trajectory must be optimal as we have previously argued that at least one optimal path exists.

Equation 1.11 shows that the deviation of the optimal price trajectory from $\bar{p}$ is given by the sum of two exponential terms, one of which increases with time. It is apparent that for non-zero d_1, the optimal price trajectory will eventually be dominated by the growing exponential. We shall see that the optimal price trajectory given by Equation 1.11 can take any of several distinct forms which are determined by the length of the time horizon and the sign of θ.

Infinite Time Horizon

To analyze the specific trajectories possible under this model, we consider first a firm with an infinite time horizon.[12]

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[11] Application of the boundary conditions for finite T yields the algebraic equations,

$$e^{\lambda_1 T} d_1 + e^{\lambda_2 T} d_2 = \frac{\bar{p}-c}{r} \qquad \phi_1 d_1 + \phi_2 d_2 = -\theta$$

These equations will have a unique solution since the quantity $e^{\lambda_1 T}\phi_2 - e^{\lambda_2 T}\phi_1$ is strictly positive.

[12] Formally there are problems involved in optimizing a functional when it is an indefinite integral. The major difficulty for our purposes is that most existence theorems have been proved only for optimization over finite time intervals. This difficulty is avoided in our case by making the substitutions; 1) $\gamma(t) = \delta(t)e^{bt}$ $\quad 0 < b < r$
2) $\quad \tau = 1 - e^{-bt}$

The second substitution above maps the time interval $(0,\infty)$ and into the τ interval $(0,1)$ and therefore transforms the original indefinite integral into a definite intergral with respect to τ. The first substitution is necessary to maintain a well behaved differential constraint over the interval $0 \leq \tau \leq 1$. The previously cited existence theorem (see footnote 8) applies to this transformed problem. We have to assume that the control variable $\gamma(t)$ of the transformed problem is also bounded. This assumption is met by the optimal path which is generated by the necessary conditions.

Pontryagin argues that the maximum principle holds for a functional taking the form of an indefinite integral if the integral converges and the boundary condition on the adjoint variable is defined as a limit rather than a strict equality.[13] In our case the boundary condition $\lim y(T)e^{-rT} = 0$ as $T \to \infty$ can only be met if the constant d_1 is zero. Because λ_1 is greater than r the term $y(t)e^{-rt}$ will not approach a finite limit for $d_1 \neq 0$. Application of the additional boundary condition that $x(0) = 0$ to Equation 1.11 generates the unique solution given by Equations 1.9' and 1.10'.

$$y^*(t) = -\frac{\theta}{\emptyset_2} e^{\lambda_2 t} - \frac{\bar{p}-c}{r} \qquad (1.9')$$

$$x^*(t) = \theta(1-e^{\lambda_2 t}) \qquad (1.10')$$

The optimal paths of x(t) and y(t) are both declining exponentials which asymptotically approach θ and $-(\bar{p}-c)/r$ respectively. Differentiating Equation 1.10' with respect to time and noting the definition of the price level, we get the result:

$$p^*(t) = \bar{p} - \frac{\lambda_2 \theta e^{\lambda_2 t}}{k} \qquad (1.12)$$

Figure 1 below shows the possible price trajectories for infinite time horizons. The optimal price trajectory is a

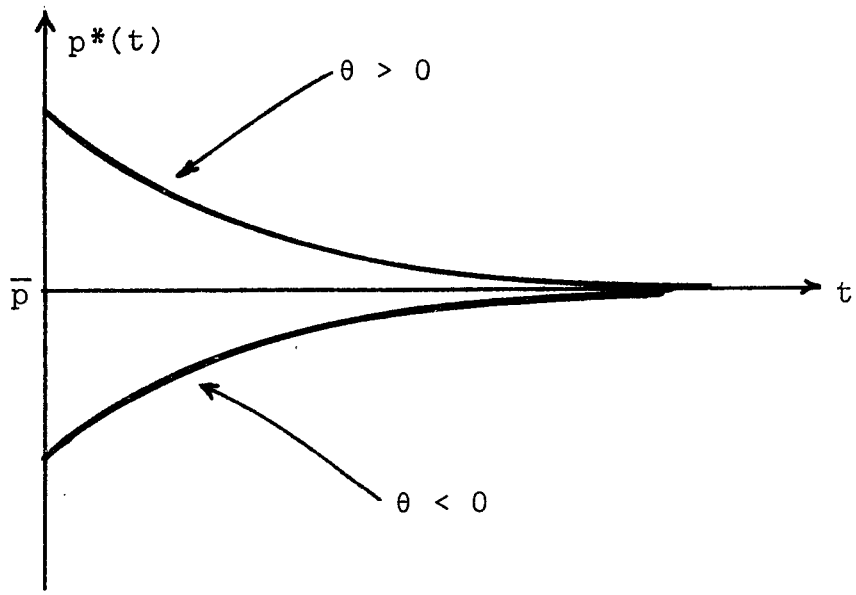
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[13] L.S. Pontryagin, V. Boltyanskii, R. Gamkrelidze, and E. Mishchenko, The Mathematical Theory of Optimal Processes, (New York: Interscience Publishers, Inc., 1962), pp. 189-90.

decaying exponential approaching $\bar{p}$ from above or below as θ is positive or negative. In either case the long-run equilibrium price is $\bar{p}$.

FIGURE 1

Optimal Price Trajectories for an Infinite Horizon



Equation 1.10' indicates that the optimal pricing strategy with an infinite time horizon can be viewed as pricing to achieve some optimal long-run market share. A positive (negative) value of θ justifies pricing above (below) the limit price to decrease (increase) the dominant firm's market by exactly θ units.

We recall that

$$\theta = \frac{k}{r}(c-\bar{p})+b(c-2\bar{p})+a$$

We note that a higher cost advantage ($\bar{p}$-c), a lower discount rate (r) or a more highly responsive entry (k) all tend to decrease θ. Conversely, a larger initial market (a) or a less steeply inclined instantaneous demand schedule (b) both tend to increase θ. A very large cost advantage characterized by large ($\bar{p}$-c) will cause θ to be negative, ceteris paribus, and will justify pricing below the limit price to drive out some portion of the competitive fringe.

An important practical case is the dominant firm with no long-run cost advantage, i.e., $\bar{p} = c$. In this case θ is equal to a-bc, which will be positive if any profitable output is possible. Optimal pricing will result in a leftward shift of the residual demand curve by a-bc units and an attendant decline of the dominant firm's output to zero.[14] We note that our model will cease to apply when the decline in market share of the dominant firm destroys the firm's ability to set prices.

Our model indicates that a dominant firm with little or no cost advantage will deliberately sacrifice its market share in maximizing its present value. The decline of United States Steel market share from 65 percent in 1901 to 24 percent in 1967[15] may have been the outcome of quasi optimal pricing.

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[14]The output of a dominant firm following the optimal pricing strategy will asymptotically approach the equilibrium level: $\hat{q} = (k/r+b)(\bar{p}-c)$.

[15]F.M. Scherer, Industrial Structure and Economic Performance (Manuscript), pp. 217-18.

If there are no long run cost advantages, entry ($k > 0$) requires that rational dominant firms decline under this model.

It is possible by differentiation of Equation 1.12 to verify the following "comparative static" results concerning the effect of various structural parameters on the optimal price at any instant in time.

i) $p^*_{\bar{p}}(t) < 0$[16] iv) $p^*_r(t) > 0$

ii) $p^*_c(t) > 0$ v) $p^*_a(t) > 0$

iii) $p^*_k(t) < 0$

Conditions i) and ii) jointly indicate that as a firm's relative cost advantage increases the optimal price will decline. Increasing $\bar{p}$, ceteris paribus, will increase the current profit level without affecting future market share. A fall in c will similarly raise current profits relative to future profits. Intuitively, in either case a decrease in the current price will tend to restore the optimal balance between present and future returns.

Condition iii) indicates that the optimal price is lowered as rivals respond more quickly to a profit stimulus. A faster entry response will mean a larger volume of future profits sacrificed for the marginal current profit dollar. Intuitively, to restore the optimal balance between current and future profits one must lower current price.

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[16] The subscript indicates the partial derivative of $p^*(t)$ with respect to that particular parameter, e.g.,

$$p^*_k(t) = \frac{\partial p^*}{\partial k}\Big|_t$$

Condition iv) indicates that as the dominant firm's discount rate increases optimal price will rise. Intuitively, the discount rate is a measure of the weight attached to current versus future profits by the firm. A rise in r puts more emphasis on current profits, hence justifying a rise in current price and a sacrifice of some future profits.

Condition v) indicates that an increase in original market size will justify raising the current price.

In addition by differentiating λ_2, the rate of approach of the optimal price trajectory towards the limit price, we establish that increasing k or r increases $|\lambda_2|$. We conclude that an increase in the response of rivals to price signals or an increase in the dominant firm's discount rate will cause the optimal price trajectory to approach $\bar{p}$ more rapidly.

Finite Time Horizons

When we turn to firms with a finite time horizon we get markedly different optimal behavior near the end of the planning period. The dominant firm in this instance is assumed to maximize its present value over some finite period, during which time the firm enjoys a constant cost advantage. An example might be a firm owning a strong patent which provides a distinct cost advantage for seventeen years. Mathematically, the finite horizon implies that $d_1 \neq 0$, and hence the optimal trajectory Equation 1.11 becomes dominated by a diverging positive exponential as t becomes large. Economically, the firm is willing to sacrifice a greater share of its market as the end of its period of cost advantage approaches.

We can verify that the optimal price is always less than or equal to the current short run profit maximizing price. Subtracting the output of the fringe (Equation 1.10') from the initial demand schedule and maximizing static profits imply the static profit-maximizing price at any moment in time $p_s(t)$ given by Equation 1.13.[17]

$$p_s(t) = c + \frac{a-bc-x(t)}{2b} \tag{1.13}$$

As we have shown above, condition iii) of Pontryagin's Maximum Principle applied to this model yields

$$\text{iii)} \quad \delta^*(t) = \frac{kz(t)e^{rt}+a+bc-x(t)-2b\bar{p}}{2b}$$

The optimal dynamic price is therefore

$$p^*(t) = \bar{p}+\delta^*(t) = \frac{kz(t)e^{rt}}{2b} + p_s(t) \tag{1.14}$$

Since $z(t)$, the shadow price of entry, is always non-positive becoming zero only at the end of the finite planning period, the optimal dynamic price will remain below the static profit-maximizing price, approaching it from below as $t \to T$.[18]

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[17] It is noted that the "static" profit maximizing price is not constant over time but varies as the dominant firm's residual demand curve shifts. It is a myopic profit maximizing price.

[18] The non-positivity of $z(t)$ can be rigorously demonstrated by phase plane analysis of the possible solutions of Equations 1.7 and 1.8. In the case of an infinite tome horizon the optimal dynamic price remains always below $p_s(t)$. Equation 1.9' indicates that the adjoint variable $y(t) = z(t)e^{rt}$ approaches a non-zero equilibrium in this case.

For a fairly long planning period (T ≈ 20 years), the optimal price trajectory becomes approximately

$$p^*(t) \approx \bar{p}+\lambda_1\phi_1 \frac{\bar{p}-c}{kr}e^{\lambda_1(t-T)} - \frac{\lambda_2\theta e^{\lambda_2 t}}{k} \qquad (1.15)$$

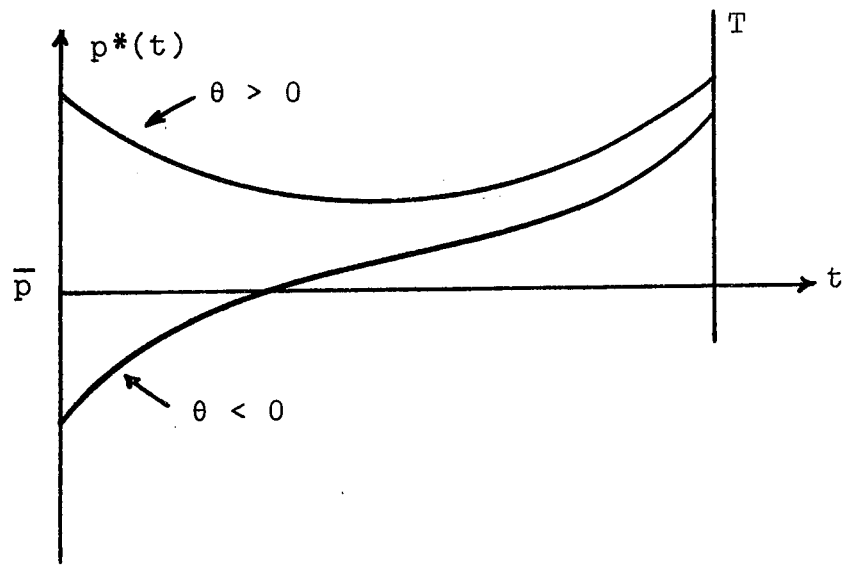
The initial portion of the price trajectory is essentially identical to the trajectory with an infinite time horizon because $e^{\lambda_1 T} >> 0$. We conclude therefore that the comparative effects of structural parameters upon the initial portion of a finite planning period are identical to those derived above for an infinite time horizon.

In general we expect the optimal price trajectories for finite time horizons to be composed of a diverging exponential added on to an initial portion resembling optimal paths for an infinite horizon. Figure 2 presents representative price trajectories for finite T. It is quite plausible that a substantial price advantage will be enjoyed by a dominant firm over some finite period implying a negative value of θ. The optimal strategy with a large price advantage is to drive out some portion of the competitive fringe initially.

Because the optimal price trajectory is a somewhat complicated function of time for a finite time horizon we shall consider some specific cases to demonstrate further the characteristics of the optimal price trajectory. Figures 3 through 8 below are optimal trajectories generated by an analog computer solution of Euqations 1.7 and 1.8 for specific parameter values. The basic model simulated by the computer,

FIGURE 2

Optimal Price Trajectories for a Finite Time Period



except for variations noted in each figure, is given by the following nominal parameter values.[19]

---

[19]The ratio k/b is approximately the elasticity of the shift in the dominant firm's demand curve with respect to product price. A discrete version of our basic entry model holds that the shift in dominant firm's output over time Δq is given by,

$$1) \quad \Delta q = \Delta x = \frac{p(t)k(p(t)-\bar{p})}{p(t)}$$

If the price elasticity along the residual demand curve is approximately unity we know that

$$2) \quad p(t) \tilde{=} \frac{q(t)}{b}$$

We may therefore write Equation 1 as

$$3) \quad \frac{\Delta q}{q(t)} = \frac{k}{b}\frac{\Delta p}{p(t)} \qquad \text{where } \Delta p = p(t)-\bar{p}.$$

We would expect the ratio k/b to typically fall in the interval (0,2) arguing that a price level ten percent above the limit price should not cause the dominant firm to lose more than 20 percent of its market in one year.

$a = 100$ $b = 1$ $c = 10$ $\bar{p} = 15$

$k = 1$ $r = .1$

In each case $\delta^*(t)$ is shown plotted against time. The optimal price level can be found by adding the relevant $\bar{p}$ to any value of $\delta^*(t)$.

Figures 3 and 4 present the optimal price trajectories for the nominal model with horizons of 20 and 10 years respectively. The effect of shortening the time horizon is to raise the optimal price level throughout its length. Intuitively, a shorter time horizon means that induced entry is a less important consideration in determining the optimal current price.

Figure 5 demonstrates somewhat of an exception to the typical finite optimal trajectory. With no cost advantage ($\bar{p} = c = 10$), we do not see the influence of the growing exponential near the end of the time period. We still have the result $p^*(T) = p_s(T)$, but substantial entry of rivals has driven the static profit maximizing price $p_s(t)$ down rapidly enough so that the optimal price level declines monotonically. This result can be shown to hold for any case where $\bar{p} = c$. For the specific case shown, the dominant firm's market share has declined to only 15 percent of its original level by the end of the time horizon. We find that dominant firms with no cost advantage decline continuously even with a finite time horizon.

Figure 6 and 7 demonstrate that either a high cost advantage or a rapid entry response with some cost advantage

FIGURE 3

Optimal Price Trajectory

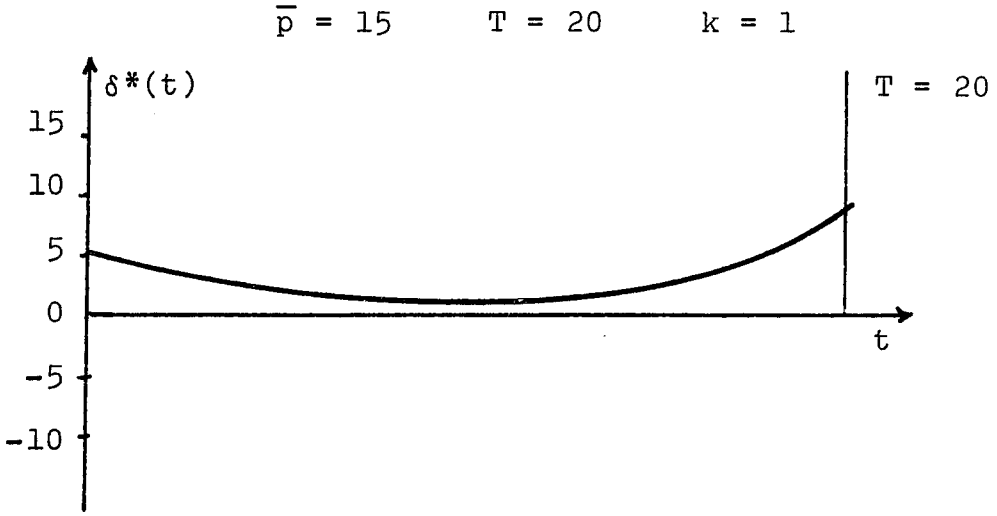


FIGURE 4

Optimal Price Trajectory

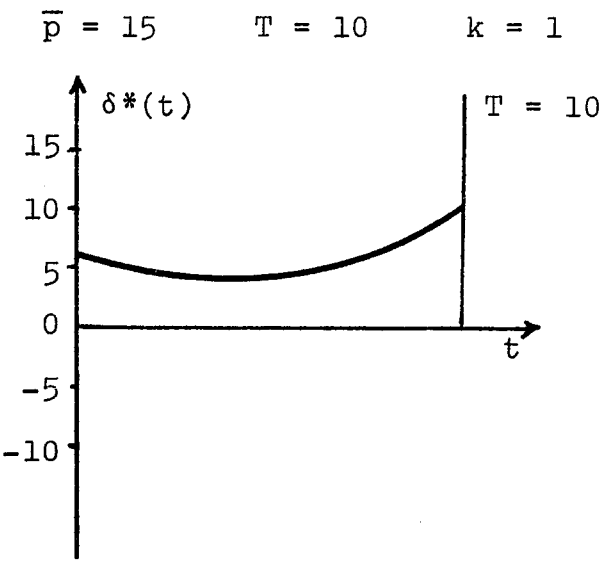


FIGURE 5

Optimal Price Trajectory

$\bar{p} = 10$ T = 15 k = 1

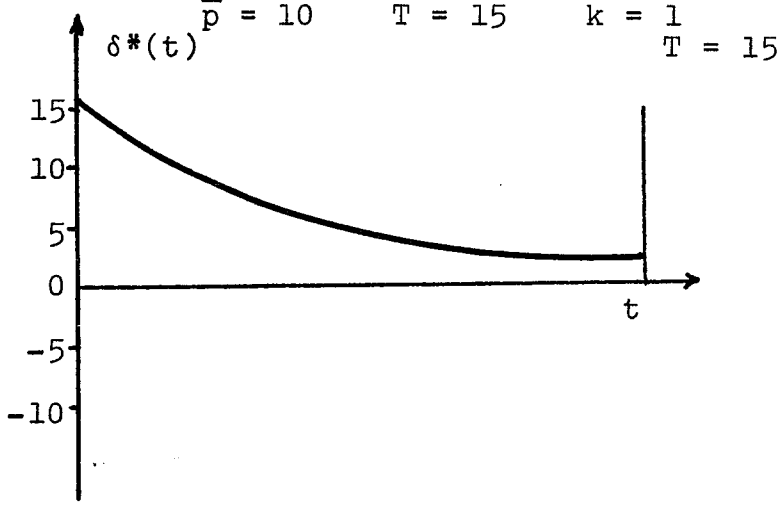


FIGURE 6

Optimal Price Trajectory

$\bar{p} = 20$ T = 20 k = 1

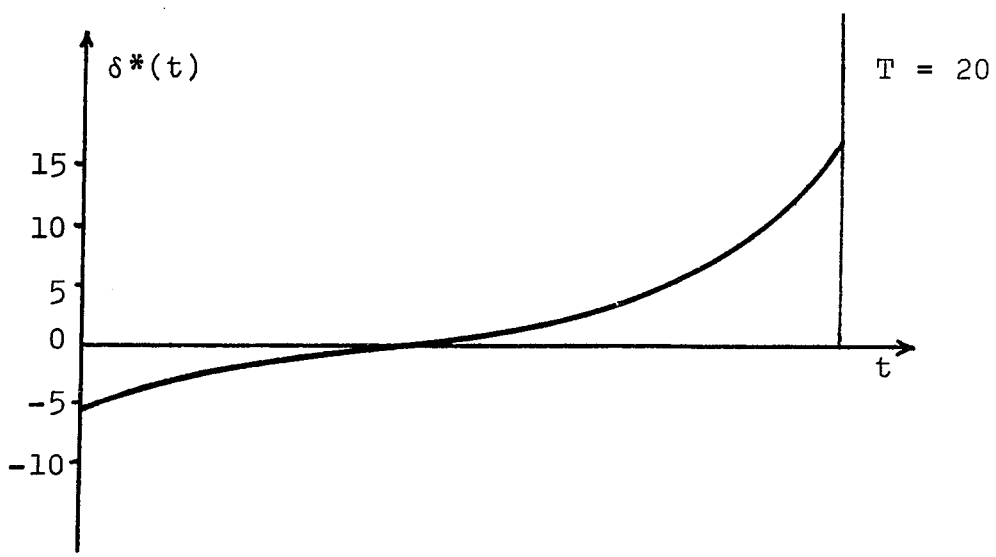


FIGURE 7

Optimal Price Trajectory

$\bar{p} = 15$ $T = 15$ $k = 2$

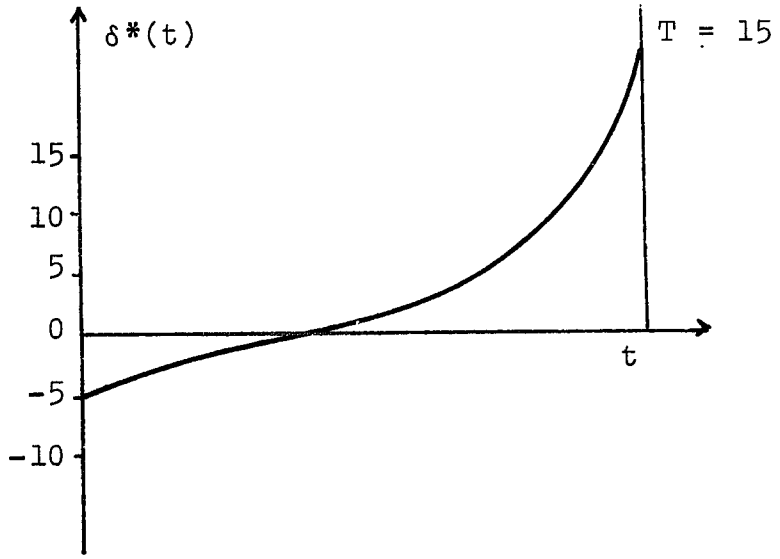
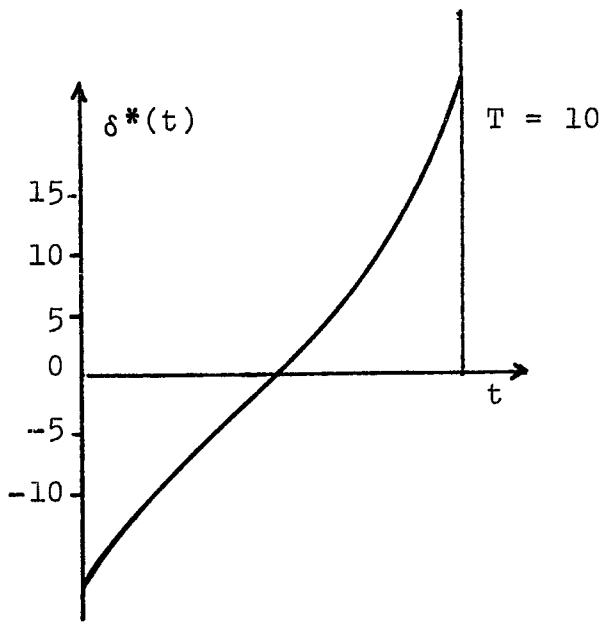


FIGURE 8

Optimal Price Trajectory

$\bar{p} = 20$ $T = 10$ $k = 2$



justifies pricing below the limit price initially to enlarge one's market share. Equation 1.3 reminds us that the area under $\delta^*(t)$ is proportional to the amount of induced entry. In both cases shown, the final market share is less than the original market share, even though rivals are driven out for a time.

Figure 8 presents a case of high cost advantage and rapid entry response. The cumulative effect is sufficient to justify initially pricing below cost to increase market share. The optimal price at $t = 0$ is only 3 while average total cost is 10. Positive profits are earned only after $\delta^*(t)$ rises above -10 in Figure 8. We conclude that it may be rational to incur short-run losses under certain conditions.

In Figures 6 through 8, a portion of the original competitive fringe is driven out of the product market. In the most extreme case shown by Figure 8, the dominant firm market share is increased by 80 percent before positive entry is allowed. This model could only be valid if there was a very large competitive fringe initially.[20]

Computer solution of specific models made it easy to evaluate the sensitivity of firm profits to non-optimal pricing. For each of the optimal trajectories shown above, the value of the firm's present value evaluated at $t = 0$ was calculated and presented as V^* in Table 1 below. In addition

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[20] An alternate model which allows the dominant firm to be constrained by the original size of the competitive fringe is explored in Chapter V.

for each case the maximum present value possible under a constant price strategy was determined by iteration and present as $\overline{V}$ in Table 1. In each case the specific constant price which maximized the firm's present value is noted as a subscript.

TABLE 1

Present Value Under the Best Constant Price Policy

Case shown in:	Optimal Trajectory	Best Constant Price Level
Figure 3	$V^* = 4,164$	$\overline{V}_{\delta=3} = 4,162$
Figure 4	$V^* = 3,690$	$\overline{V}_{\delta=6} = 3,688$
Figure 5	$V^* = 1,268$	$\overline{V}_{\delta=7} = 1,144$
Figure 6	$V^* = 7,020$	$\overline{V}_{\delta=0} = 6,376$
Figure 7	$V^* = 3,472$	$\overline{V}_{\delta=0} = 3,232$
Figure 8	$V^* = 6,210$	$\overline{V}_{\delta=0} = 4,960$

We observe that when the optimal price does not vary much (Figures 3 and 4) a constant price at approximately the average level of the optimal dynamic price will involve virtually no loss in firm's present value. For firms with a moderate cost advantage, a constant price level will be for all practical purposes the optimal strategy if we impute a significant cost to price changes.

Table 1 does indicate that when the optimal price varies substantially over time (Figures 5 through 8) there may be a significant loss of present value even under the best constant price strategy. It is interesting, however, that the best constant price strategy when the optimal strategy is to

initially price below $\bar{p}$ (Figures 6 through 8) is to set price approximately at the limit price. In the worst case, shown in Figure 8, such a limit pricing strategy entails a loss of approximately 20 percent in firm's present value.

Summary

The major conclusions from the analysis of this basic entry model are summarized below. For the case of an infinite time horizon we demonstrated:

1. The optimal price will always be below the static profit maximizing price.
2. It will be rational to drive out competitors if a sufficient cost advantage exists. A polar case of this strategy is to incur initial short term losses to enlarge one's market share.
3. An increase in the dominant firm's cost advantage or an increase in rivals' response to price changes will lower the dominant firm's optimal price level at any point in time, ceteris paribus.
4. The equilibrium optimal price is the limit price, and the rate of approach to this equilibrium increased, the faster rivals respond to price signals and the more the dominant firm's discounts future profits.

We showed that the initial portion of a finite optimal price trajectory was similar to an infinite horizon path. For a finite horizon we further demonstrated:

1. The optimal price trajectory approached the static profit maximizing price from below as $t \to T$.
2. For a moderate cost advantage a specific constant price level will be nearly optimal.
3. Shortening the time horizon raises the optimal price trajectory.

CHAPTER II

QUADRATIC ENTRY

Several major simplifications are embodied in the formulation of the basic model. Perhaps the least defensible assumption is the implied symmetry of entry and exit by rivals. There is good reason to expect entrenched rivals to respond more slowly to negative or below average profits (price below $\bar{p}$) than potential entrants respond to positive profits of the same magnitude. A firm with a large fixed investment will be hesitant to abandon a given market solely on the basis of current losses.

A related model deficiency is the strict linear relationship between current price and entry rate for prices above the limit price. Mansfield found that positive entry rate was nearly a linear function of current profit rate.[1] We would expect, however, that the competitive fringe's expected profit rate will increase more than proportionally as product price increases. Moreover, there may be some expected profit threshold which if exceeded will bring a torrent of entry.

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[1] Mansfield's entry model was of the form $x = k\pi^{\alpha}$...and his empirical estimate of α was 1.15. E. Mansfield, op. cit., pp. 159-167.

In an effort to overcome these two model deficiencies we turn now to a quadratic entry model. In this case we still maximize the functional V of Equation 1.1, but it is now subject to the differential constraint:

$$\dot{x}(t) = k_1\delta(t) + k_2\delta(t)^2 \quad - \frac{k_1}{k_2} \leq \delta(t) \tag{2.1}$$

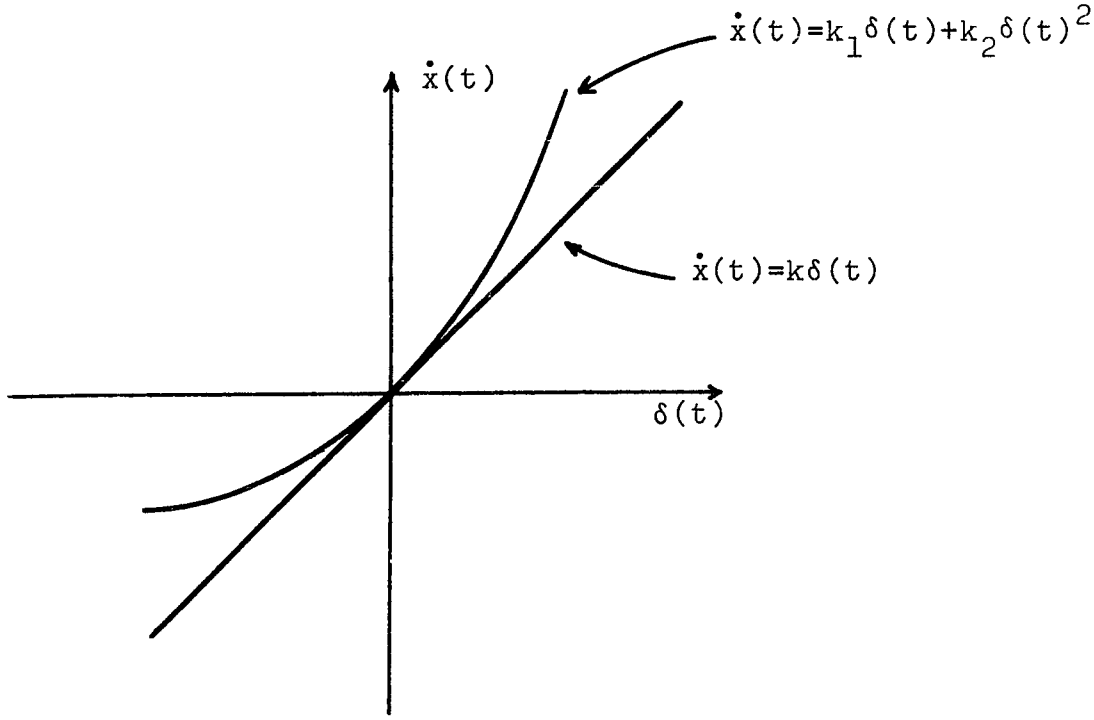
where k_1, k_2 are positive constants

The rate of entry is a parabolic function of $\delta(t)$ rising explosively as price rises above the limit price. We restrict δ to be greater than $-k_1/2k_2$ so that our solution cannot include the unrealistic left half of the parabola, where the rate of entry would rise as price fell. Figure 9 below presents $\dot{x}$ versus δ for $k_1=1$, $k_2=.1$ contrasted to the linear entry model of the previous analysis with k=1. In this case the linear model can be viewed as a linear approximation of the quadratic model near $\delta(t) = 0$. It is apparent from the figure that prices above the limit price ($\delta(t)>0$) will cause a more rapid market share loss than under the comparable linear model. Alternatively, the gain in market share resulting from negative $\delta(t)$ will be less under the quadratic model. Even without resort to Pontryagin it seems clear that the optimal price trajectory will be closer to $\bar{p}$ under the quadratic model than under the comparable linear model.

In this case application of Pontryagin's necessary conditions yields the following non-linear differential equations.

FIGURE 9

Quadratic Entry Model



$$\dot{x}(t) = k_1\delta(t) + k_2\delta(t)^2 \qquad x(0) = 0 \qquad (2.2)$$

$$\dot{y}(t) = ry(t) + \delta(t) + \bar{p}-c \qquad y(T)e^{-rT} = 0 \qquad (2.3)$$

where

$$\delta(t) = \frac{k_1 y(t) + a-b(2\bar{p}-c)-x(t)}{2b-2k_2 y(t)}$$

The functions $\dot{x}(t)$ and $\dot{y}(t)$ and their partial derivatives with respect to x(t) and y(t) are continuous for non-positive values of y(t). We argue as before that since $y(t) = \frac{\partial V}{\partial x}e^{rt}$ and the shadow price of entry is surely non-positive, y(t) will always be non-positive along the optimal path.[2] The

---

[2]Our interpretation of z as equal to $\frac{\partial V}{\partial x}$ depends strictly on finding a solution to the Hamilton-Jacobi equation for this case.

existence theorem cited in Chapter I still applies, and we are sure that at least one optimal path exists. Computer solution of Equation 2.2 and 2.3 for specific parameter values indicates that there is a unique optimal trajectory. Unfortunately, we have not been able to show that there is in general a unique solution to the necessary conditions of this system.

Infinite Time Horizon

The explicit solution to this system is not trivial, but in the case of an infinite time horizon we can find the equilibrium point and establish the asymptotic behavior of the optimal path. The only permissible equilibrium point for this system is identical to the equilibrium point of the linear system, i.e.,[3]

$$\hat{x} = \theta; \ (k = k_1) \tag{2.4}$$

$$\hat{y} = \frac{c-\bar{p}}{r} \tag{2.5}$$

$$\hat{\delta} = 0 \tag{2.6}$$

where (^) designates the equilibrium value of the variable.

When we linearize Equations 2.2 and 2.3 in the region near this equilibrium point, we get the set of linear equations

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[3]It is easily shown that the other possible equilibrium point entails a positive value of y which, we have argued, is not possible.

$$\dot{y}'(t) = \alpha_1' y'(t) + \alpha_2' x'(t) \tag{2.7}$$

$$\dot{x}'(t) = \beta_1' y'(t) + \beta_2' x'(t) \tag{2.8}$$

where

$$\alpha_1' = r + \frac{k_1}{2b + 2k_2 \frac{(\bar{p}-c)}{r}} \qquad \alpha_2' = \frac{-1}{2b + 2k_2 \frac{(\bar{p}-c)}{r}}$$

$$\beta_1' = \frac{k_1^2}{2b + 2k_2 \frac{(\bar{p}-c)}{r}} \qquad \beta_2' = \frac{-k_1}{2b + 2k_2 \frac{(\bar{p}-c)}{r}}$$

$$x'(t) = x(t) - \theta \qquad y'(t) = y(t) - \frac{c-\bar{p}}{r}$$

It is obvious that this set of equations is similar to the linear system given by Equations 1.7 and 1.8. The only difference between the set of coefficients above and α_1, α_2, β_1 and β_2 of the linear model is that in each case b has been replaced by a new constant which we designate $b' = b + k_2\frac{(\bar{p}-c)}{r}$. We note with particular interest that in the case of a dominant firm with no competitive advantage ($\bar{p}=c$), linearized equations in the quadratic case are identical to Equations 1.7 and 1.8 evaluated near their equilibrium point. We can conclude without further analysis that the asymptotic solution for the quadratic entry model for an infinite time horizon is

$$y'^*(t) = d_1 e^{\lambda'_1 t} + d_2 e^{\lambda_2' t} \tag{2.9}$$

$$x'^*(t) = \phi_1' d_1 e^{\lambda_1' t} + \phi_2' d_2 e^{\lambda_2' t} \tag{2.10}$$

where

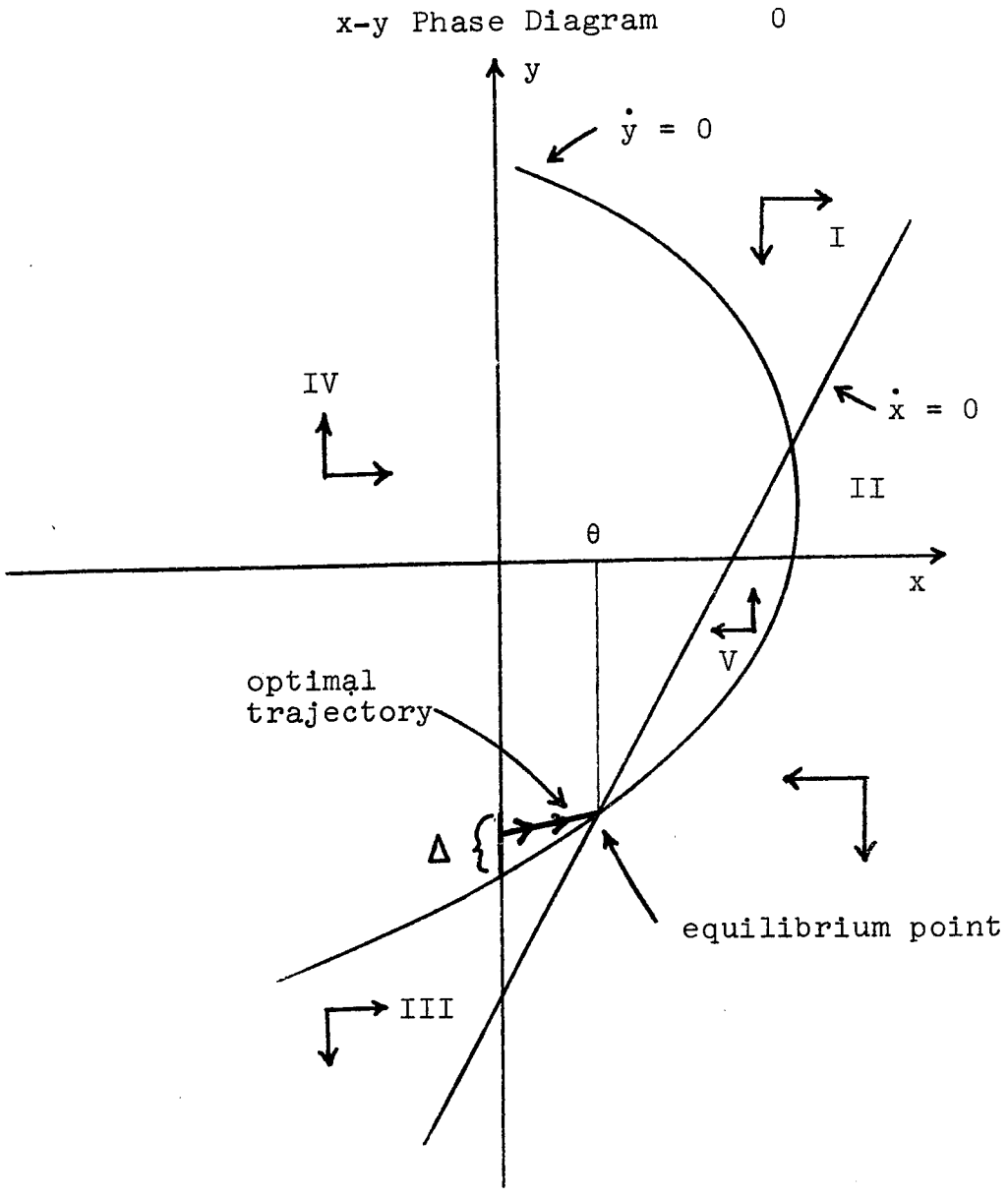
$$\lambda_{1,2}' = \frac{r}{2} \pm \frac{\sqrt{r^2 + \frac{2kr}{b'}}}{2} \qquad \lambda_1' > 0 \qquad \lambda_2' < 0$$

$$\phi_1' = \frac{\lambda_1' - \alpha_1'}{\alpha_2'} \qquad \phi_2' = \frac{\lambda_2' - \alpha_1'}{\alpha_2'}$$

The terminal boundary condition $\lim y^*(T)e^{-rT} = 0$ on the non-linear system 2.2, 2.3 applies equally to the linearized system and therefore implies that $d_1 = 0$ for $T \to \infty$ because $\lambda_1' > r$. Equation 2.10 becomes therefore an exponential decaying at a rate equal to λ_2'. It follows that $x^*(t) = x'^*(t) + \theta$ declines exponentially to θ. We see that, as in the linear case, the optimal pricing strategy involves obtaining some final optimal market share by altering competitive output by θ units.

Unfortunately, we cannot apply the initial condition of the non-linear system $x^*(0) = 0$ to determine the sign of d_2 in the linearized Equation 2.10 because the linear system only approximates the non-linear system near the equilibrium point when $\delta(t) \to 0$. We can however use the phase plane portrait shown in Figure 10 to describe the optimal trajectory qualitatively. In Figure 10 we plot the loci of $\dot{x}^*(t) = 0$ and $\dot{y}^*(t) = 0$ (a straight line and parabola respectively) which divide the phase plane into five distinct regions. Any trajectory governed by Equations 2.2 and 2.3 in one of these regions must have the direction shown by the arrows in the Figure. It is apparent that the feasible equilibrium point $\dot{x}^*(t) = \dot{y}^*(t) = 0$ is attainable from only a small portion of region IV. Because any trajectory in region IV must move in a northeasterly direction, an optimal trajectory starting on the y-axis ($x^*(0)=0$) and necessarily reaching the equilibrium

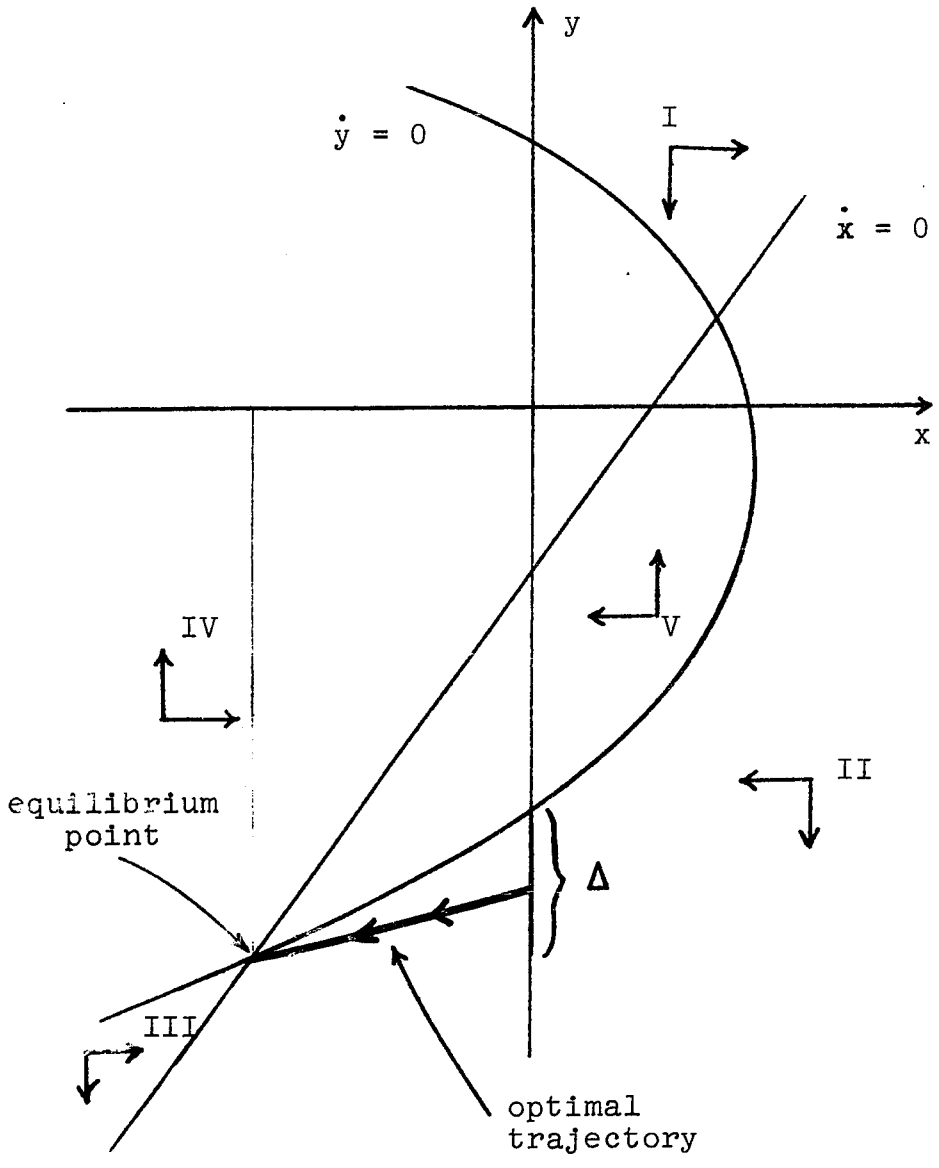
FIGURE 10[4]



[4] We note that the sign of the y-coordinate of the focal point of the parabola depends on the sign of $(k_1+2br-k_2(\bar{p}-c))$. Large values of $(\bar{p}-c)$ consistent with negative values of θ may cause the focal point to move below the x-axis, as shown in Figure 11.

FIGURE 11

x-y Phase Diagram $\theta<0$



point can only originate on the portion of the y-axis marked Δ in Figure 10. Moreover, any trajectory originating on Δ which reaches the equilibrium point must remain in region IV. Because the initial value of $x^*(t)$ is zero, the optimal trajectory must originate on the y-axis and terminate at the equilibrium point with $\dot{x}^*(t)$ always remaining positive along this trajectory. Similarly, Figure 11 shows that for negative θ, $\dot{x}^*(t)$ will remain negative on the optimal trajectory. Our restriction on the allowable values of $\delta(t)$ requires that sign $\delta^*(t)$ = sign $\dot{x}^*(t)$ and we can conclude that positive (negative) values of θ mandate prices above (below) the limit price throughout the infinite planning period.

Finite Time Horizon

The system will not appraoch an equilibrium point and np explicit solution has been found for $\delta^*(t)$. It is possible to verify, as in the linear model, that the optimal dynamic price is equal to the static optimal price at the end of the planning period. For a finite horizon the boundary condition $y^*(T)e^{-rT} = 0$ implies that $y^*(T) = 0$. We can write the optimal price at the end of the planning period as,

$$p^*(t) = \bar{p}+\delta^*(t) = \bar{p} + \frac{a-b(2p-c)-x^*(t)}{2b} = \frac{a+bc-x^*(t)}{2b}$$

This final optimal dynamic price is equal to the static profit maximizing price for a residual demand curve whose output intercept is equal to (a-x). We observe the same end-game behavior under the quadratic model as we saw in the linear case because of our somewhat unrealistic assumption that the

firm ceases production and has no scrap value at time T.

We turn to co puter solutions of Equations 2.2 and 2.3 to explore the characteristics of optimal paths for finite T. The model simulated is specified by the following parameter values:

$$a = 100 \quad b = 1 \quad k_1 = 1 \quad k_2 = .1$$
$$c = 10 \quad \bar{p} = 15 \quad r = .1$$

Figures 12 through 17 present trajectories based on these nominal parameters except as noted on each figure. For each of these optimal trajectories, the linear model counterpart was presented in Figures 3 through 8. The only change in each case has been addition of the $k_2\delta^2$ term to the entry model. In each case we observe the intuitively appealing result that the optimal price trajectory has been compressed toward the limit price throughout the first half of the planning period. The compression toward $\bar{p}$ in the moderate cost advantage case (Figures 12 and 13) causes the optimal price to be nearly constant over the first portion of the planning period. Under conditions which justify enlarging one's market by pricing below the limit price (Figures 15, 16, 17) the quadratic model substantially decreases the portion of the competitive fringe driven out by raising the optimal price substantially toward $\bar{p}$ and by making rival response much slower to a given negative δ. The quadratic model compresses the optimal price toward $\bar{p}$ far enough in Figure 17 to eliminate the initial below-cost pricing we found justified

FIGURE 12

Optimal Price Trajectory

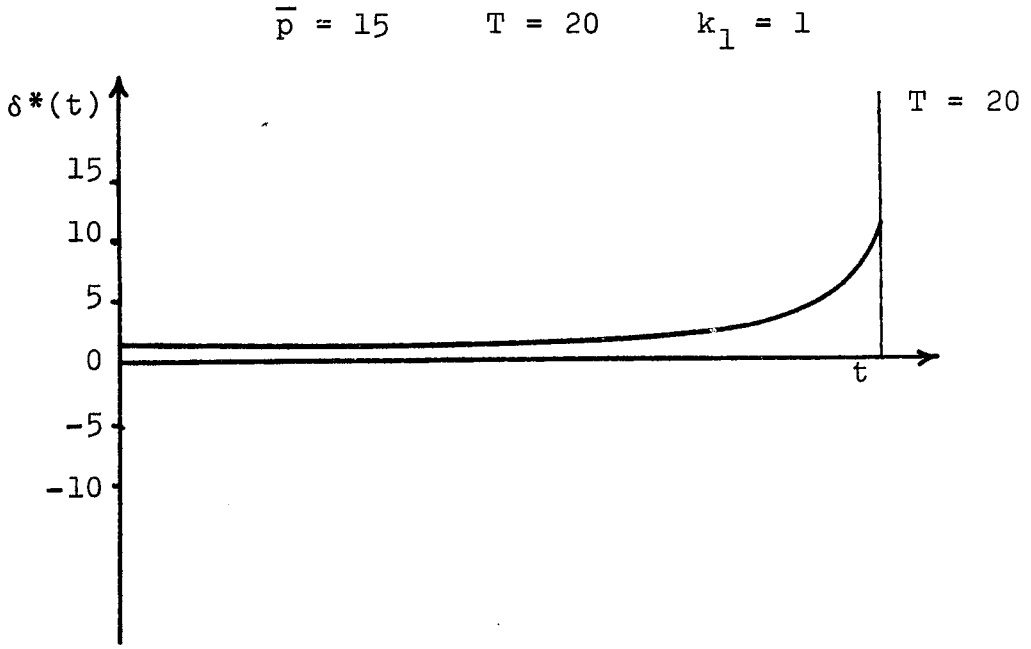


FIGURE 13

Optimal Price Trajectory

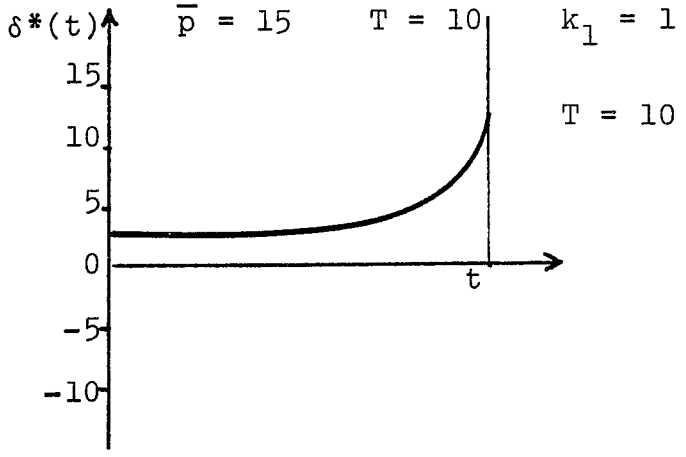


FIGURE 14

Optimal Price Trajectory

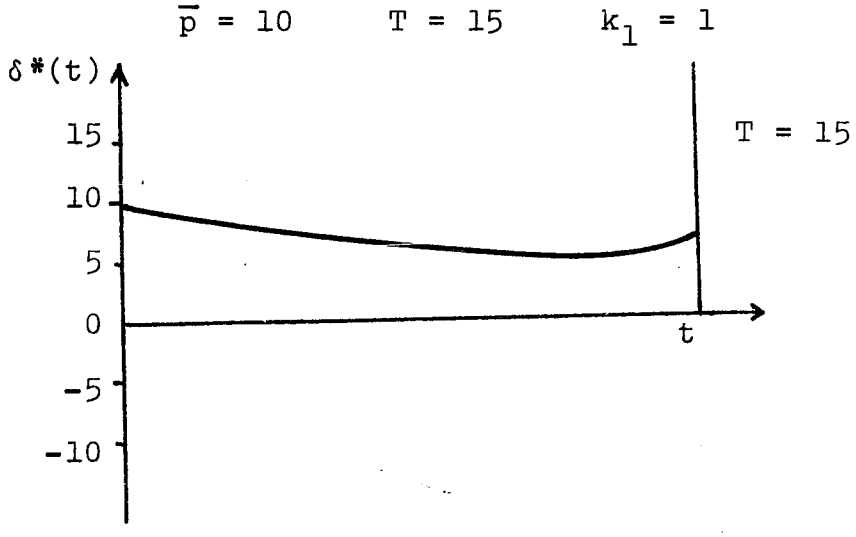


FIGURE 15

Optimal Price Trajectory

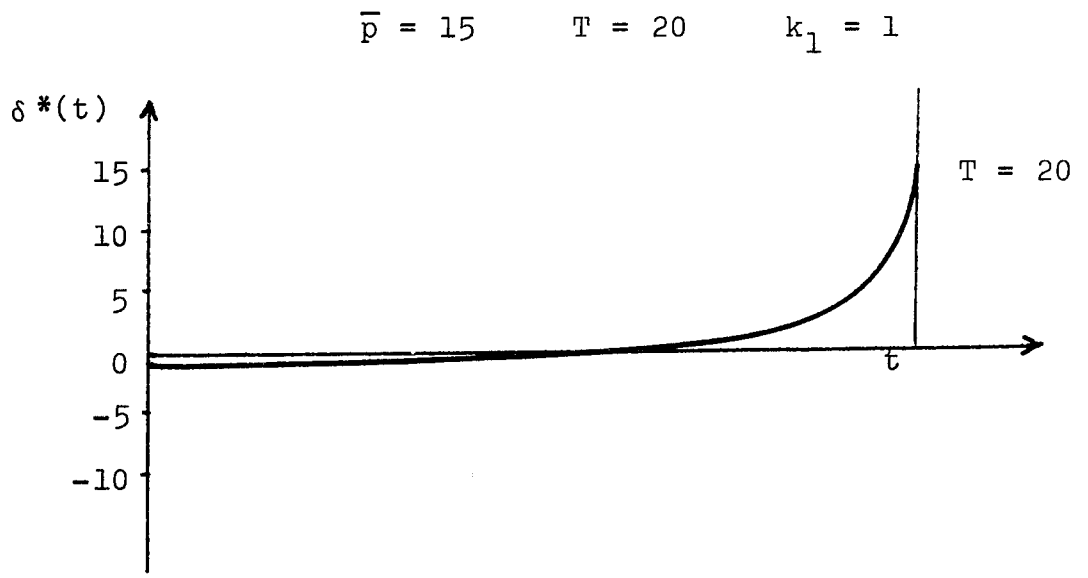


FIGURE 16

Optimal Price Trajectory

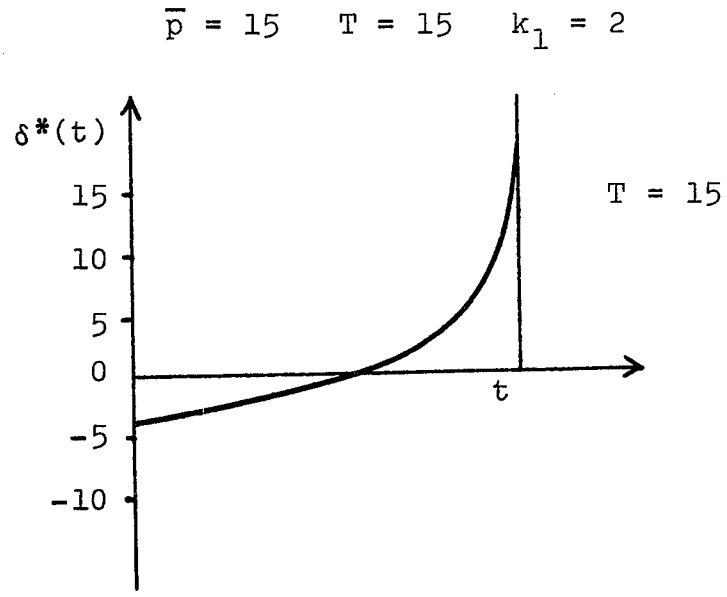
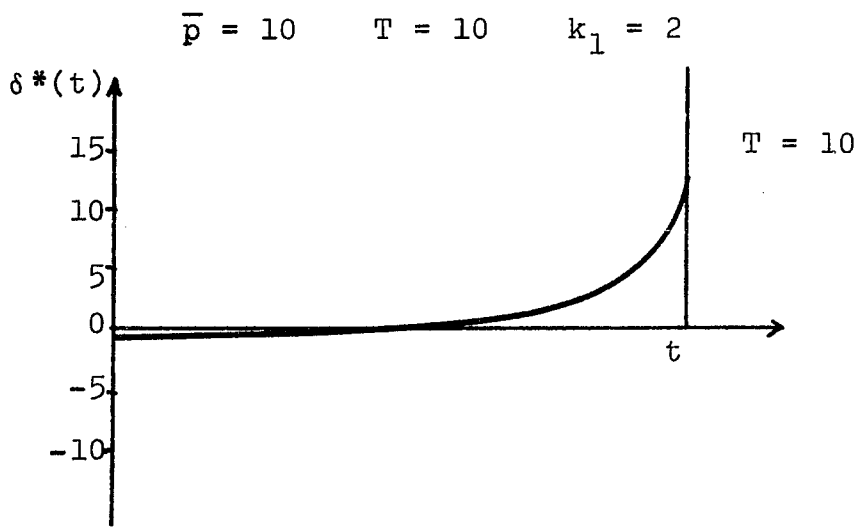


FIGURE 17

Optimal Price Trajectory



under similar conditions in the linear model. We have not shown, however, that this phenomenon will not occur for even larger negative values of θ.

We find another difference between optimal pricing under quadratic and linear models when we examine the final portion of the optimal price trajectories. We have shown above that the optimal dynamic price will approach the static profit-maximizing price as the end of a finite horizon approaches. A firm enjoying no cost advantage over its rivals will retain far more of its original market under the quadratic model. Therefore, the static optimal price approached as the end of the planning period nears is much higher under the quadratic entry model. The difference between linear and quadratic entry models is most clearly seen in the case of a dominant firm with no cost advantage ($\bar{p} = c$), shown by Figures 14 and 15. Under the linear model, the market share has diminished so much that the static optimal price falls to meet p^* as $t \to T$. Under the quadratic model there is the more typical (at least for our finite horizon models) exponential increase in the dynamic optimal price as $t \to T$. We observe essentially the same phenomenon in each of the other cases shown, but the contrast with their linear counterparts is more subdued.

We find therefore that except for a bit more of a price upswing at the end of the finite planning periods, our intuitive notions about the quadratic model are well supported by these specific solutions. While there may be other anomolies not found by specific search, it seems reasonable to accept the hypothesis that the more realistic quadratic

entry model has not altered the qualitative conclusions of the basic linear model.

CHAPTER III

VARIATION IN THE DOMINANT FIRM'S COST ADVANTAGE

The models considered so far have maintained that the dominant firm's cost advantage and hence the limit price are constant throughout the planning period. We could hardly expect the limit price to remain fixed in the real world with its technological break-throughs and changing market conditions. We now turn to consider the important practical cases of a firm with an infinite time horizon enjoying a significant cost advantage for only a finite period and a firm with a continuously declining competitive advantage. In both of these cases it would seem reasonable to postulate an infinitely long planning period for the dominant firm.

Patent Expiration

The first time-varying limit price case we consider is a dominant firm holding a strong patent which expires at a specified moment in time T. The patent gives the dominant firm a cost advantage over its rivals which we assume remains constant over the life of the patent. At the termination of the patent, the dominant firm's cost advantage vanishes (i.e., $\bar{p}(t>T)=c$) and the firm's planning period extends over both periods. In this case we must maximize dominant firm's present value as given by:

$$V = \int_0^T (\bar{p}(t)+\delta(t)-c)q(t)e^{-rt}dt + \int_T^\infty \delta(t)q(t)e^{-rt}dt \quad (3.1)$$

We assume that q(t) is generated by the original linear entry model. The present value is seen to be the sum of discounted profits over the two distinct periods. We find that the optimal trajectory over the second time period (t>T) depends on the level of market penetration by rival producers during the first time period. For a specified value of market penetration by the competitive fringe when the patent expires x(T), the second integral in Equation 3.1 can be maximized by applying our previous results for dominant firms with no cost advantage and an infinite time horizon. Under the simple linear model we find that this latter trajectory for t>T must necessarily satisfy

$$y^*(t) = d_1 e^{\lambda_1 t} + d_2 e^{\lambda_2 t} \qquad y(t)e^{-rt} \to 0, \; t \to \infty \quad (3.2)$$

$$x^*(t) = \phi_1 d_1 e^{\lambda_1 t} + \phi_2 d_2 e^{\lambda_2 t} + \theta' \qquad x(T) = x_T \quad (3.3)$$

where $\theta' = a-bc$ (because $\bar{p} = c$)

$\phi_1, \phi_2, \lambda_1, \lambda_2$ as defined in Chapter I

We argue as before that because $\lambda_1 > r$, the terminal boundary condition on y(t) implies that d_1 must be zero. With the somewhat different boundary condition on x(t), the optimal trajectory for t>T becomes

$$p^*(t) = c+\delta^*(t) = c + \frac{\lambda_2(x_T-\theta')}{k}(e^{\lambda_2(t-T)}) \quad (3.4)$$

This pricing strategy will induce rival production to increase exponentially to an equilibrium level θ' units greater than the level at $t = T$. Formally, the change in the level of output from the competitive fringe is given by Equation 3.5:

$$x(t) = (x_T-\theta')e^{\lambda_2(t-T)}+\theta' \qquad t>T \tag{3.5}$$

We can now solve for the value of the second definite integral of Equation 3.1 which we designate

$$W = \int_T^{\infty} \delta(t)(a-b(\bar{p}+\delta(t))-x(t))e^{-rt}dt \tag{3.6}$$

Substituting the results of Equations 3.4 and 3.5 into Equation 3.6 and performing the indicated integration we see that W is an explicit function of x_T:

$$W(x_T) = \frac{(x_T-\theta')^2}{r-2\lambda_2}\left(\frac{-\lambda_2}{k} - \frac{b\lambda_2^2}{k^2}\right)e^{-rT} \tag{3.6'}$$

It is now possible to reinterpret the overall present value maximization of Equation 3.1 as optimization over a finite time period ($0\leq t\leq T$) with a scrap value at $t=T$ given by $W(x_T)$. We now wish to maximize:

$$V = \int_0^T (p+\delta(t)-c)q(t)e^{-rt}dt+W(x_T) \tag{3.1'}$$

$$\text{subject to } \dot{x} = k\delta(t)$$

The necessary conditions for optimizing the functional V of Equation 3.1' are the same as the basic linear model of Chapter I with the single exception that the boundary condition on the adjoint variable z(T) is modified to be the

partial derivative of $W(x_T)$ with respect to x_T.[1] Intuitively, the shadow price of entry at t=T becomes the change in scrap value for an additional unit of rival output, i.e., $\partial W/\partial x_T$.

Making the substitution $y(t) = z(t)e^{rt}$ to eliminate time explicitly from the differential equations, we get the following results:

$$\dot{y}(t) = \alpha_1 y(t)+\alpha_2 x(t)+\alpha_3 \qquad y(T) = e^{rT}\frac{\partial W}{\partial x_T} \qquad (3.7)$$

$$\dot{x}(t) = \beta_1 y(t)+\beta_2 x(t)+\beta_3 \qquad x(0) = 0 \qquad (3.8)$$

It is apparent that the boundary condition on y specified in Equation 3.9 will be strictly negative because the value of x_T must be less than $\theta' = a-bc$ if profitable operation was possible before patent expiration.

$$y(T) = e^{rT}\frac{\partial W}{\partial x_T} = \frac{2(x_T-\theta')}{r-2\lambda_2}\left(\frac{-\lambda_2}{k} - \frac{b\lambda_2^2}{k^2}\right) \qquad (3.9)$$

We can rewrite Equation 3.9 in the form

$$y(T) = Ax(T)+B$$

$$\text{where } A = \frac{2}{r-2\lambda_2}\left(-\frac{\lambda_2}{k} - \frac{b\lambda_2^2}{k^2}\right) > 0$$

$$\text{and } B = -\theta' A < 0$$

---

[1] A more general formulation of Pontryagin's Maximum Principle holds that the adjoint variables must satisfy certain transversality conditions. In this case the transversality condition is equivalent to the boundary condition $z(T) = \partial W/\partial x_T$. The Appendix contains a brief description of this more general form of the Maximum Principle.

The phase plane portrait shown in Figure 18 contrasts optimal trajectories for the basic linear finite horizon model and the present case. In the basic model a finite optimal trajectory (i in the Figure) originates on the y-axis and ends on the x-axis. In the patent expiration case, the finite portion of an optimal trajectory (ii) again originates on the y-axis but it terminates on the line $y(T) = Ax(t)+B$. It is obvious from the figure that an optimal trajectory for the patent expiration case is coincident with some optimal trajectory for the basic model which pertains to a longer time period. The same result holds for negative values of θ, as shown by Figure 19. We see that in both figures the optimal trajectory with patent expiration terminates in the region to the left of the locus $\dot{x}(t) = 0$. Because $\dot{x}(t)$ will always be strictly greater than zero in this region we can conclude that the optimal trajectory will entail a price above the limit price at the point of patent termination even for the negative θ. (This follows directly from $\delta(t) = \dot{x}(t)/k$.)

We conclude therefore that the optimal trajectory during the finite patent life is similar to the optimal price trajectories seen under the basic linear model for finite horizons. The only difference between the two situations is that the trajectory under the patent expiration case ends along the line $y(T) = Ax(T)$ B while the basic model trajectory terminates on the x-axis. We notice in the figures that the optimal trajectory with a finite patent life is coincident with a portion of an optimal trajectory under the basic model.

FIGURE 18

x-y Phase Diagram $\theta > 0$

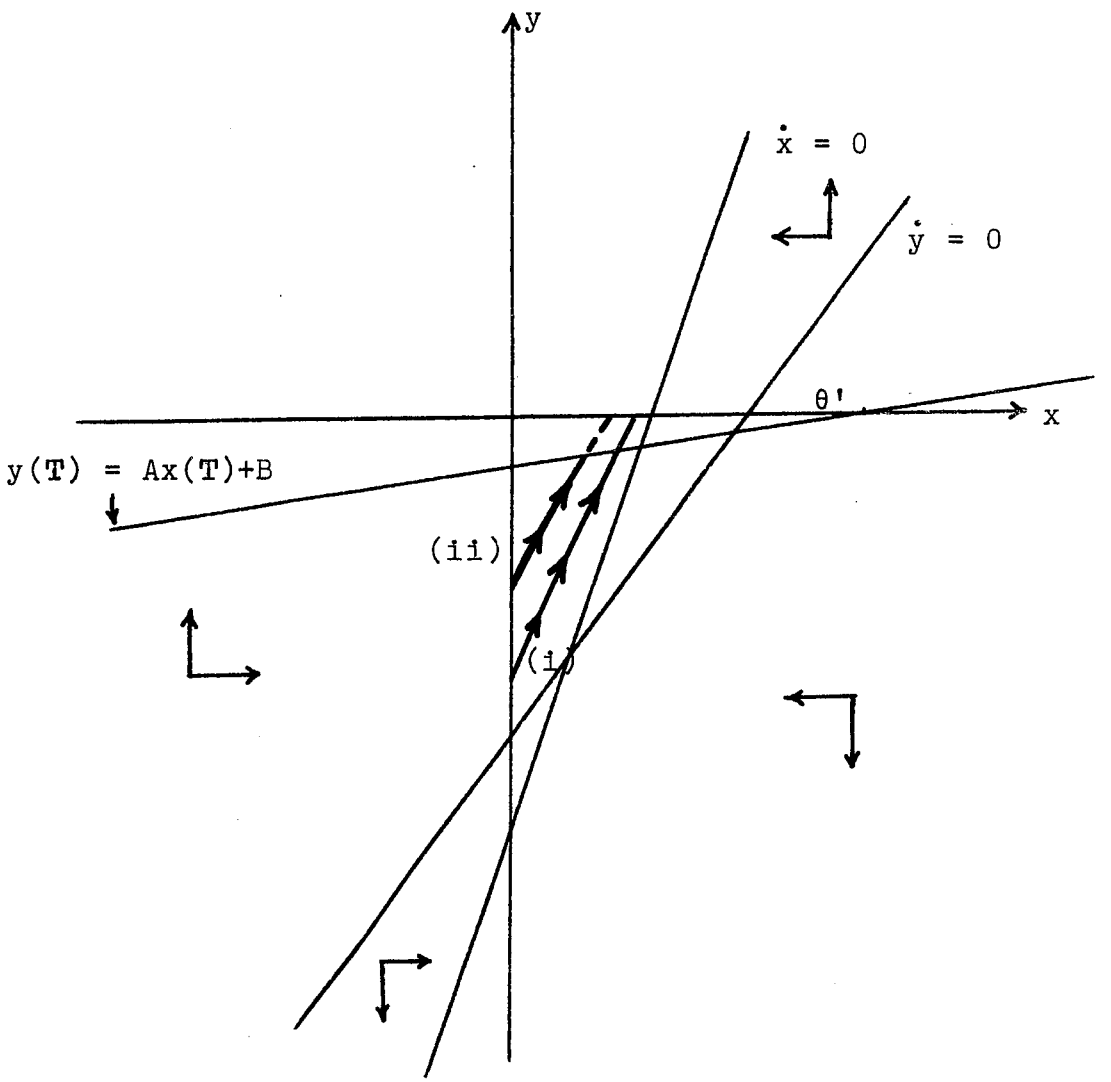
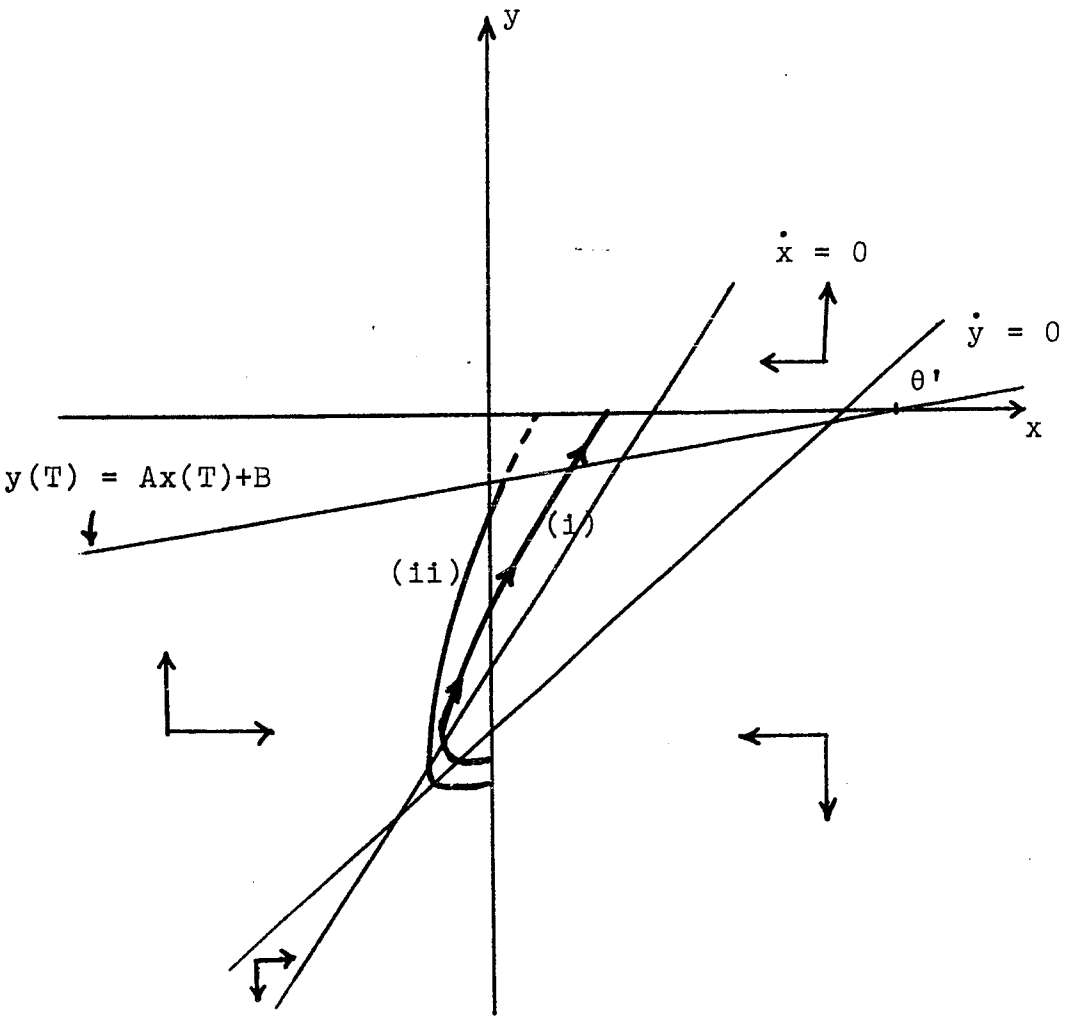


FIGURE 19

x-y Phase Diagram $\theta < 0$



The optimal path of the finite patent case necessarily ends sooner than this coincident basic model trajectory, and hence will not exhibit the full exponential rise found as the planning period ends under the constant cost advantage model. Because the patent expiration trajectory ends before its counterpart under the basic model the optimal price at the point of expiration will always be strictly less than the current static profit maximization price.

We now have a general description of the finite portion of the optimal price trajectory with an expiring patent and an infinite time horizon. It remains only to patch on the remainder of the optimal price trajectory. We previously derived Equation 3.4, which indicates that after the patent terminates, the optimal trajectory assumes the form of a decaying exponential with time constant $1/\lambda_2$ and the initial value:

$$p^*(T^+) = c + \frac{\lambda_2(x_T-\theta')}{k}$$

It can be shown that this decaying portion of the optimal price trajectory begins exactly at the price level at which

the patent expires, i.e., $p^*(T^-) = p^*(T^+)$.[2]

Figures 20 and 21 present representative optimal price trajectories for finite patents which bestow moderate or high cost advantages on the dominant firm. Both trajectories in Figures 20 and 21 exhibit a distinct peak at the instant of patent expiration. In the case of a patent conferring only a moderate cost advantage (Figure 20) we see that the initial portion of the optimal path is relatively flat: even more so than the analagous trajectories under the basic model.

Extending our previous results on the sensitivity of the firm's present value to near-optimal constant pricing, we conclude that a constant-price strategy through the life

---

[2] From Equation 3.10 we have that 1) $p^*(T^+) = \frac{c+\lambda_2(x_T-\theta')}{k}$; we also know that 2) $p^*(T^-) = \bar{p}+\delta^*(T^-)$; and since Pontryagin's third necessary condition indicates that

$$\delta^*(t) = \frac{ky^*(t)-a-x^*(t)+bc-2b\bar{p}}{2b},$$

we find that

$$p^*(T^-) = \frac{ky^*(T^-)+a-x_T+bc}{2b}.$$

We argue that $x(T^+) = x(T^-) = x_T$ for infinitesmal displacements in time. The boundary condition on Equation 3.7 indicates that

$$y(T^-) = \frac{2(x_T-\theta')}{r-2\lambda_2}\left(-\frac{\lambda_2}{k} - \frac{b\lambda_2^2}{k^2}\right)$$

Substituting this expression for $y(T^-)$ in Equation 2 and subtracting the result for Equation 1 we get the following result after some simplification:

$$3)\quad p^*(T^+)-p^*(T^-) = (x_T-\theta')\left(\frac{1}{2b} + \frac{\lambda_2}{k} + \frac{\lambda_2}{b\sqrt{r^2+\frac{2kr}{b}}} + \frac{\lambda_2^2}{k\sqrt{r^2+\frac{2kr}{b}}}\right)$$

The terms inside the last bracket in the above equation sum to zero after some manipulation and hence we see that $p^*(T^+)=p^*(T^-)$.

FIGURE 20

Optimal Price Trajectory

Patent Expiration $\theta > 0$

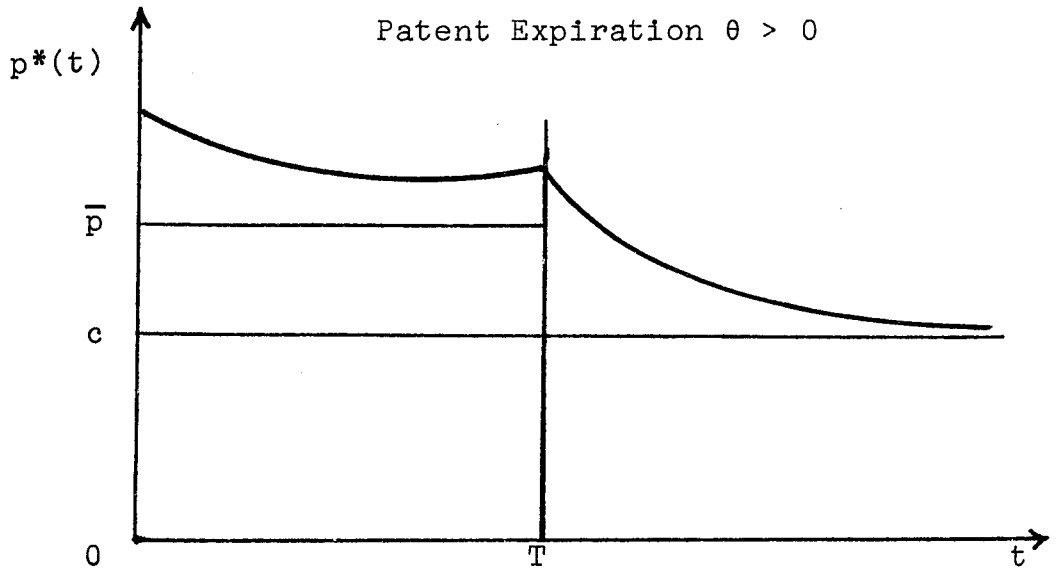
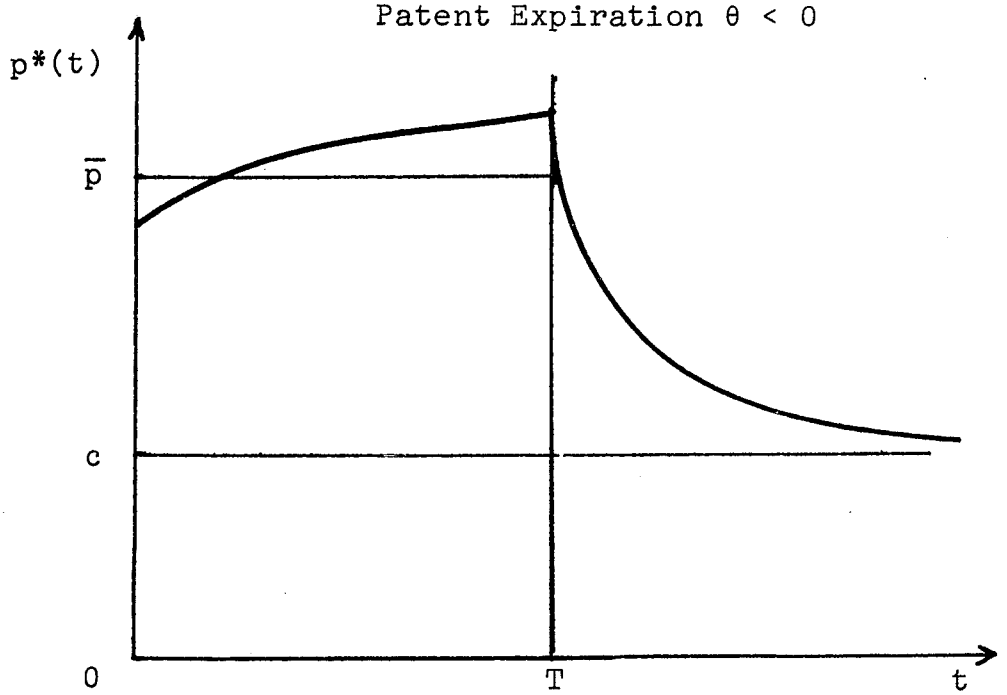


FIGURE 21

Optimal Price Trajectory

Patent Expiration $\theta < 0$



of the patent would entail only a small loss in potential profits. This model strongly indicates that a specific constant price level would be a nearly optimal pricing strategy for a firm enjoying a moderate temporary cost advantage.

Both figures demonstrate that after patent expiration has eliminated all cost advantages the optimal price decays exponentially toward the level of average total cost. This indicates that rational dominant firms will decline if they cannot renew or maintain their cost advantage.

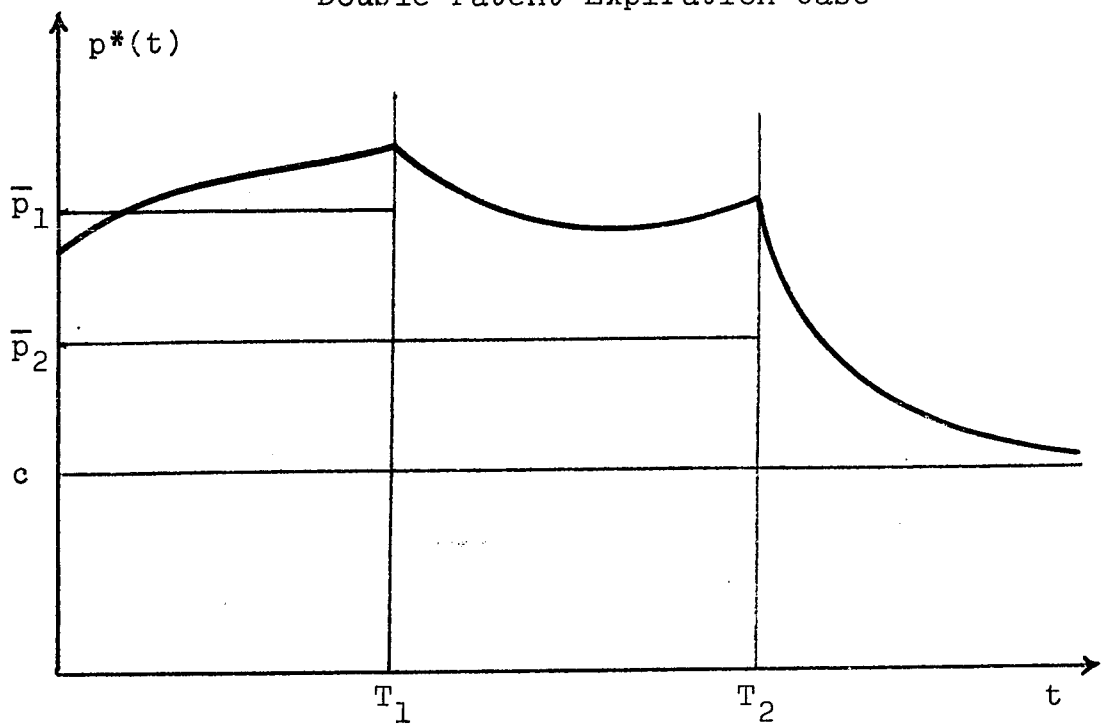
It should now be clear that we could now generate optimal trajectories under any series of discrete changes in either the limit price or dominant firm's average total cost. The analytic technique is to work backwards from the final portion of the optimal path, expressing each separate section of the path in terms of the size of the competitive fringe at the beginning of that section.[3] An example of the optimal price trajectory with the limit price falling in two distinct steps is shown by Figure 22. The optimal trajectory is based on heuristic extension of the preceding results. The figure might represent a trajectory under a patent portfolio consisting of a primary and secondary patent. At the expiration of the primary patent at time T_1 the limit price falls from $\bar{p}_1$ to

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[3]This procedure is quite analagous to Bellman's dynamic programming. A good reference to this alternate method of optimizing under dynamic constraint is Stuart E. Dreyfus, Dynamic Programming and the Calculus of Variations, (New York: Academic Press, 1965).

FIGURE 22

Double Patent Expiration Case



$\bar{p}_2$ and the trajectory changes from a shape justified by a high cost advantage. At time T_2, when the limit price falls to c, the optimal trajectory assumes the declining exponential shape characteristic of a dominant firm with no cost advantage.

Continuously Declining Limit Price

We now turn to the case of a limit price which declines continuously toward some lower limit--specifically the level of long-run average total cost. We cannot justify any specific form of this declining limit price other than to observe that without a new patent or the acquisition of some scarce resource we would expect the limit price to decline monotonically. An economic scenario which might justify a continuously declining limit price involves an established firm which enjoys a

competitive advantage over rival producers via product differentiation. Conceivably, the cost advantage, (as measured by the difference between dominant firm cost and the limit price) narrows as rival products slowly build up brand loyalty. Alternately, a cost advantage established by the aggregation of a talented group of individuals would gradually erode as rivals gained experience and employed better trained personnel.

We assume for the sake of analytic expedience that we can represent the continuously declining limit price as the declining exponential given by Equation 3.11.

$$\bar{p}(t) = (\bar{p}_o - c)e^{-\sigma t} + c \qquad (3.11)$$

where $\bar{p}_o$ = initial level of the limit price

σ = rate of decline of limit price

The necessary conditions of the basic model can now be reformulated with the static $\bar{p}$ replaced by its dynamic equivalent from Equation 3.11. These new necessary conditions become the following linear differential equations with constant coefficients.

$$\dot{y}(t) = \alpha_1 y(t) + \beta_2 x(t) + \beta_3 \qquad y(T)e^{-rT} = 0 \qquad (3.12)$$

$$\dot{x}(t) = \beta_1 y(t) + \beta_2 x(t) + \beta_3(t) \qquad x(0) = 0 \qquad (3.13)$$

where $\beta_3(t) = \dfrac{k(a - bc - 2b\bar{p}(t))}{2b}$

and α_1, α_2, α_3, β_1, β_2 are identical to the constants defined in Equations 1.7 and 1.8.

The fact that Equation 3.11 has a time varying term $\beta_3(t)$ adds slightly to the complexity of the solution. At least one solution to this set of equations must exist because an optimal trajectory must exist. If there is a solution to Equations 3.12 and 3.13, it can be shown to assume the form:

$$y^*(t) = d_1 e^{\lambda_1 t} + d_2 e^{\lambda_2 t} + \frac{k(\bar{p}_o - c)e^{-\sigma t}}{2b(\sigma^2 + \sigma r - kr/2b)} \qquad (3.14)$$

$$x^*(t) = \phi_1 d_1 e^{\lambda_1 t} + d_2 \phi_2 e^{\lambda_2 t} + a - bc + k\psi e^{-\sigma t} \quad \sigma \neq |\lambda_2|^4 \qquad (3.15)$$

$$\text{where } \psi = \frac{(\bar{p}_o - c)(r + k/(2b+\sigma))}{\sigma^2 + \sigma r - kr/2b}, \quad \phi_1 = \frac{\lambda_1 - \alpha_1}{\alpha_2}, \quad \phi_2 = \frac{\lambda_2 - \alpha_1}{\alpha_2}$$

It is easily verified that there is a unique solution to these equations satisfying the boundary conditions $x^*(0) = 0$ $y^*(T)e^{-rT} = 0$.[5]

Equations 3.15 can be differentiated to provide the optimal price trajectory over any planning period. The equations describing a finite optimal path are cumbersome and

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[4] The denominator of this expression vanishes as σ approaches $-\lambda_2$. The solution when $\sigma = -\lambda_2$ takes a more complicated mathematical form, but the nature of the optimal trajectory is the same.

[5] We have previously shown that $\lambda_1 > 0, \lambda_2 < 0, 0 < \phi_1 < \phi_2$. When we apply the boundary conditions to these equations we get a matrix equation of the form

$$[A]\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = B.$$

The conditions cited above are sufficient for non-singularity of A and hence unique solutions for d_1 and d_2.

difficult to interpret, but the finite horizon is not particularly relevant under this model. We will restrict the rest of our analysis of this model to the infinite time horizon case. As T becomes very large, we find as before that d_1 will necessarily approach zero. We find that the optimal price trajectory is composed of two exponential terms which both decay to the equilibrium level c, as shown by Equation 3.16:

$$p^*(t) = c+(\overline{p}_o-c+\sigma\psi)e^{-\sigma t}+(\lambda_2\psi - \frac{(a-bc)}{k})e^{\lambda_2 t} \qquad (3.16)$$

We can rewrite equation 3.16 to show that the optimal price trajectory is the sum of three distinct terms; the declining limit price, a monotonically declining exponential with time constant $1/\lambda_2$ and an exponential term which first increases and then declines. These three distinct terms are shown below in Equation 3.16':

$$p^*(t) = \overline{p}(t) - \frac{\lambda_2(a-bc)}{k} e^{\lambda_2 t} +\psi(\sigma e^{-\sigma t}+\lambda_2 e^{\lambda_2 t}) \qquad (3.16')$$

The third term in Equation 3.16' is always negative at t = 0. For values of $\sigma<|\lambda_2|$, ψ is necessarily positive and hence $\psi(\sigma+\lambda_2)$ is negative. Conversely, when ψ is negative, $(\sigma+\lambda_2)$ becomes positive, requiring their product to be negative. In either case the term $\psi(\sigma e^{-\sigma t}+\lambda_2 e^{\lambda_2 t})$ will increase and eventually become positive as t increases. Asymptotically this term approaches either $\psi\sigma e^{-\sigma t}$ or $\psi\lambda_2 e^{\lambda_2 t}$ as σ is less than or greater than $|\lambda_2|$. In both cases we see that the sign will be such that the asymptotic path declines toward zero from above.

Figure 23 presents the three distinct terms of Equation 3.16' along with their sum p*(t) for the case of a dominant firm with a moderate cost advantage and $\sigma<|\lambda_2|$. The three separate components of the optimal price trajectory combine to generate a trajectory which looks predominantly like a monotonically declining exponential characterized by a time constant equal to $1/\sigma$.

We note that the optimal price trajectory under this model, as given by Equation 3.14 is identical to our results under the basic model when $\sigma = 0$. We further verify that as σ (the rate of decline of the limit price) is increased, the initial optimal limit price will be raised.[6] This increase in the initial optimal price level comes solely from a compression of the term $\psi(\sigma+\lambda_2)$ toward zero from below. We conclude therefore that for a dominant firm with a medium to low cost advantage Figure 23 captures the spirit of the optimal price trajectory. Higher levels of σ will raise the initial level and increase the observed rate of decline of the optimal price trajectory.

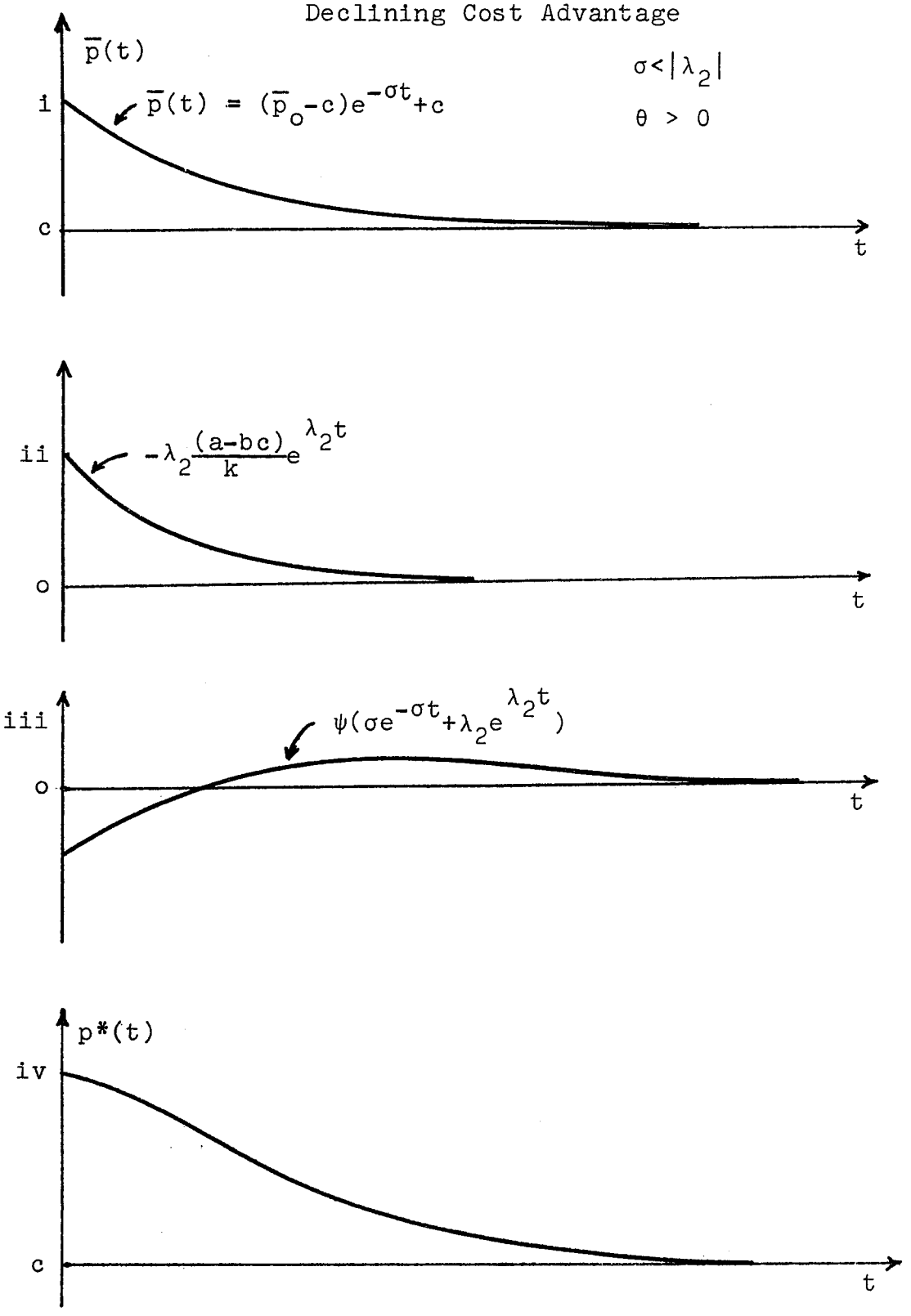
In the case of a dominant firm enjoying a very high initial cost advantage, we find that there is a greater variety among possible optimal paths. Figure 24 presents the three components of Equation 3.16' and their sum p*(t). The

---

[6] It can be shown by extensive algebraic manipulation that $\frac{\partial p^*(0)}{\partial \sigma} > 0$ as long as the quantity kr/b is less than unity. For an practical case this most certainly will be true as we have previously argued that k/b should be less than 2 and the appropriate discount rate for a dominant firm will always be less than 50 percent.

FIGURE 23

Declining Cost Advantage



case shown is based on a relatively low value of σ and a highly responsive competitive fringe (large k). In this case we observe that the optimal price trajectory exhibits what might be termed an overshoot. Initially the strong competitive advantage, high response rate and low σ are sufficient to justify pricing below cost to drive out the fringe. The optimal price rises above cost, reaches a peak and then declines exponentially toward c from above as the dominant firm matures. We verify that this should be so by noting that the limiting case as $\sigma \to 0$ is the basic model previously explored, under which a high cost advantage justified pricing below the limit price. For a dominant firm with less of a cost advantage or a less responsive competitive fringe, the initial limit price would be above the level of average cost, but the price trajectory will still exhibit the hump seen in Figure 24.

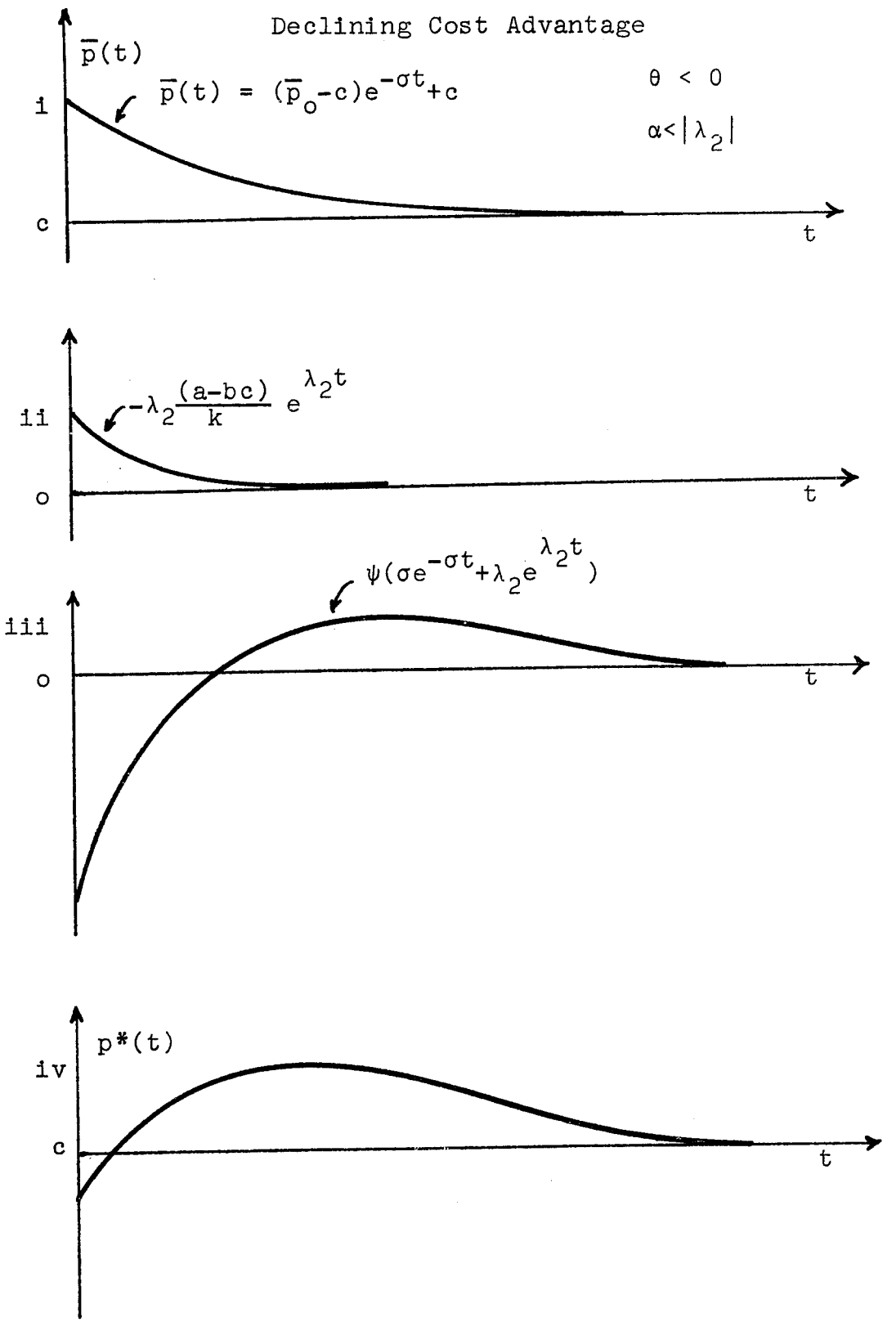
We still have the result that $\frac{\partial p^*(0)}{\partial \sigma} > 0$. The only component of the initial optimal price affected by an increase in σ is the negative term $\psi(\sigma+\lambda_2)$, which is driven monotonically toward $(c-\bar{p}_o)$.[7] It follows that for very large values of σ the initial optimal price may be above the limit price, depending on the relative magnitudes of $(c-\bar{p}_o)$ and $\lambda_2(a-bc)/k$. Equation 3.16' indicates that in the limiting case when $\sigma \to \infty$ the initial optimal price level will be $\emptyset = (-\lambda_2(a-bc)/k-(\bar{p}_o-c))$

---

[7] We can write $\psi(\sigma+\lambda_2)$ as equal to $\frac{(\bar{p}_o-c)(\frac{k}{rb} + r + \sigma)\sigma}{\frac{kr}{2b} - r\sigma - \sigma^2}$

It is obvious that the limit of this expression as $\sigma \to \infty$ is $\frac{(\bar{p}_o-c)\sigma^2}{-\sigma^2} = c-\bar{p}_o$.

FIGURE 24


Declining Cost Advantage
$\bar{p}(t)$
$\bar{p}(t) = (\bar{p}_o - c)e^{-\sigma t} + c$
$\theta < 0$
$\alpha < |\lambda_2|$
i
c
t
ii
$-\lambda_2 \frac{(a-bc)}{k} e^{\lambda_2 t}$
o
t
iii
$\psi(\sigma e^{-\sigma t} + \lambda_2 e^{\lambda_2 t})$
o
t
iv
p*(t)
c
t

units above the contemporary limit price. It is apparent that a very high initial cost advantage ($\bar{p}_o$-c) relative to the initial market size (a) will force $\emptyset$ to be less than zero and we will still observe an initial price below the limit price.

Increasing σ will also tend to shorten the duration of the initial overshoot displayed by the optimal trajectory. For values of $\sigma<|\lambda_2|$, increasing σ speeds the decline of $\sigma e^{-\sigma t}$ the dominant term in the expression $\psi(\sigma e^{-\sigma t}+\lambda_2 e^{\lambda_2 t})$. If σ increases beyond $|\lambda_2|$ the asymptotic behavior of this expression becomes $\psi\lambda_2 e^{\lambda_2 t}$, which will decline more rapidly than the path (iii) shown in Figure 24. As σ increases beyond $|\lambda_2|$ the magnitude of ψ declines continuously and the effect of this whole term on the optimal price becomes imperceptible.

Figures 25 through 30 present several analog computer solutions for this model. Unfortunately, these trajectories are for finite time horizons because of computational limitations. The later portions are not representative of the infinite planning period problem, but the initial portion of each trajectory is relevant to our discussion of the infinite time horizon case. Figures 25, 26, 27 reflect the effect of increasing σ (the rate of decline of the limit price) on a dominant firm with a moderate cost advantage. For these three cases the values of the model parameters are

$a = 100 \qquad b = 1 \qquad r = .1 \qquad c = 10$

$\bar{p}_o = 15 \qquad \sigma =$ variable

Figures 28, 29 and 30 show the effect of increasing σ on a dominant firm possessing a large cost advantage over rival

FIGURE 25

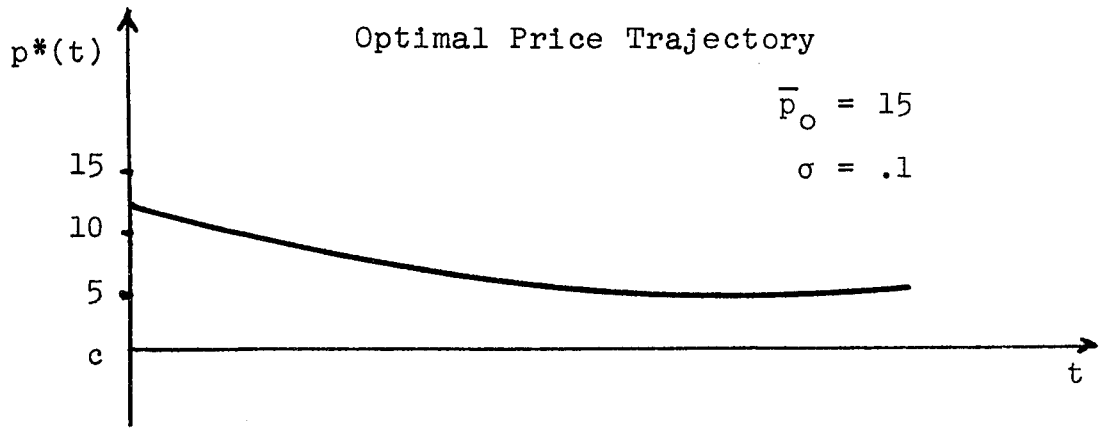


FIGURE 26

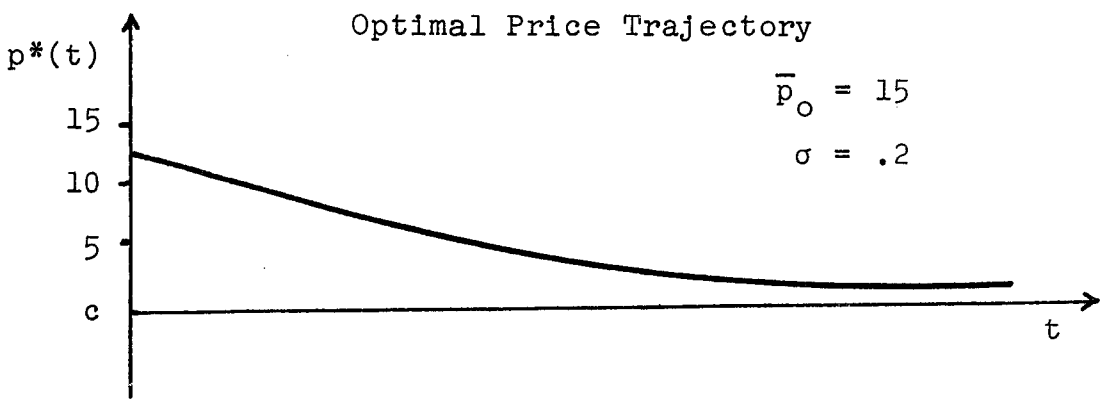


FIGURE 27

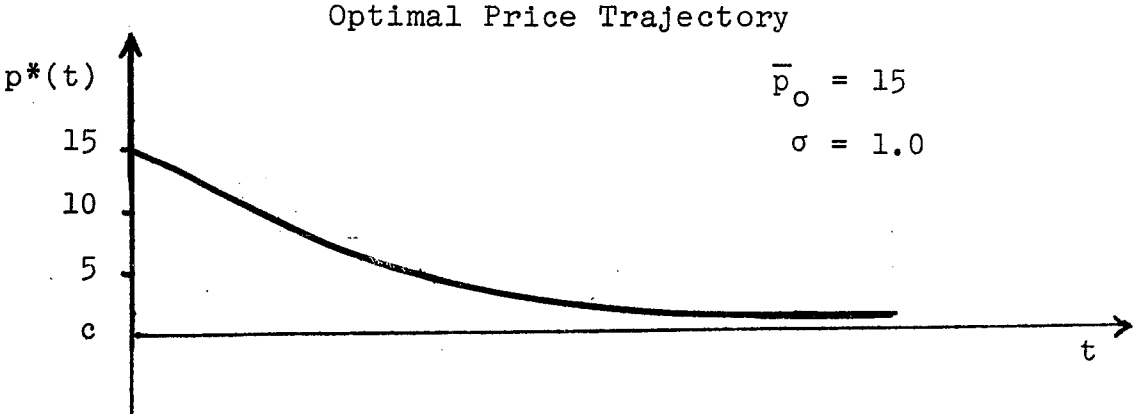


FIGURE 28

Optimal Price Trajectory

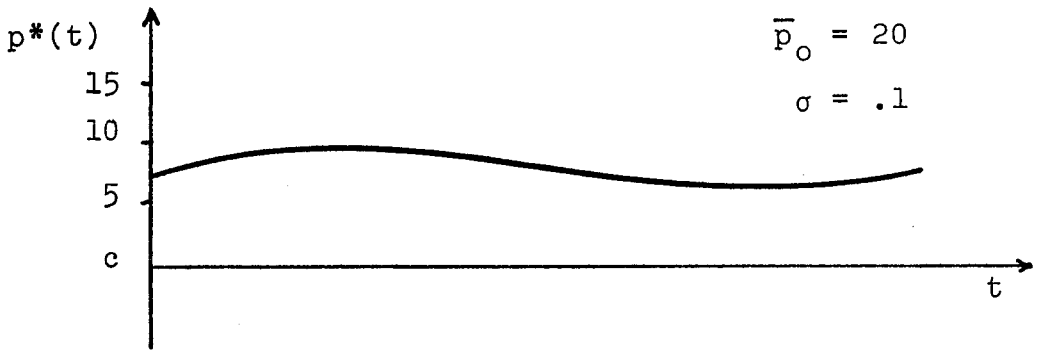


FIGURE 29

Optimal Price Trajectory

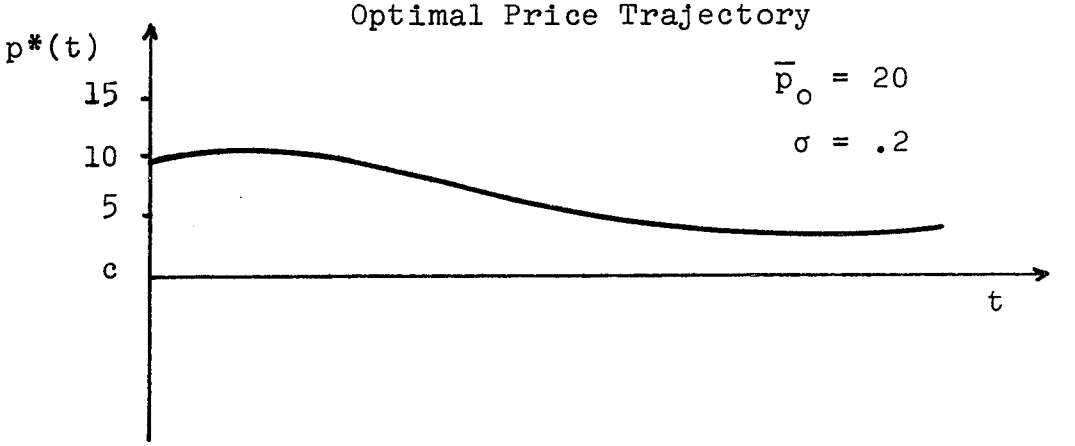
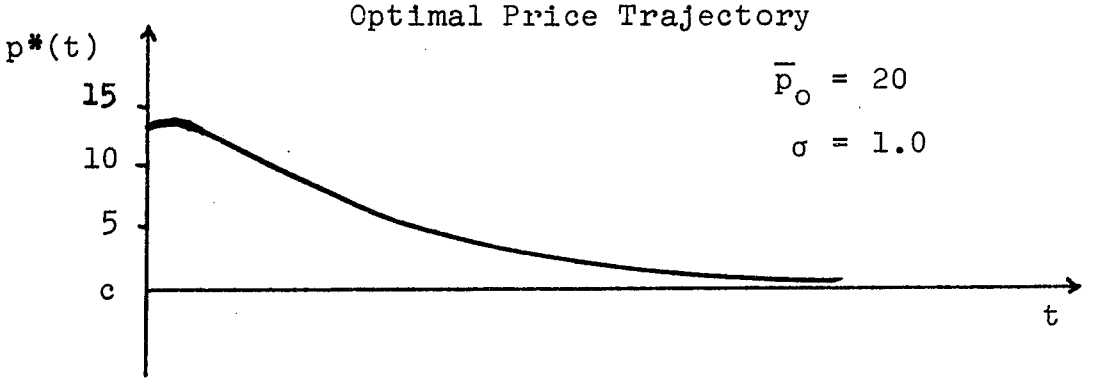


FIGURE 30

Optimal Price Trajectory



producers. This model differs from previous cases only in that $\bar{p}_o$ is equal to 20. We see in this case that a high initial cost advantage causes the optimal price trajectory to exhibit an initial hump. The figures give evidence that the initial overshoot is muted as the limit price declines more rapidly.

The major conclusion to be drawn from this model of a continuously declining limit price is that after a possible initial overshoot, the optimal price will decline exponentially toward the level of average cost. A more rapidly declining limit price, a smaller cost advantage, or a less responsive competitive fringe will all tend to eliminate the initial hump in the optimal price trajectory. Our dominant firm in all cases will deliberately sacrifice market share by pricing above the contemporary limit price in the later stages of its life. If in fact the relevant limit price declines continuously because product differentiation or cost advantages erode gradually, we would expect the universal decline of rational dominant firms.

CHAPTER IV

GROWTH AND STOCHASTIC VARIATION IN THE PRODUCT MARKET

In this chapter we explore three models relaxing the assumption of a static product market. First we consider the case of long-run growth of the product market arising from the steady increase in disposable income. Under this growth model we postulate that the price elasticity at any prescribed price level remains constant over time. The second growth model we explore attempts to approximate the growth of demand for a revolutionary new product. The price elasticity at a given price level continually decreases with market growth under this model. The optimal pricing strategy is significantly modified under both of these growth models.

The third model allows stochastic variation in the dominant firm's residual demand schedule. This random variation could occur either because of unexpected shifts in the output of the competitive fringe or in the market demand. The optimal pricing strategy for this stochastic model is shown to be governed by a control law which relates the optimal price to the contemporary size of the competitive fringe.

Constant Elasticity Growth

The real world, with its growing population and increasing per capita income, is obviously not consistent with our

previous assumption of static market demand. We turn now to a model under which the steady growth in population and per capita income steadily increase the output demanded at any given price level. We assume that the output intercept and the slope of the dominant firm's original demand curve $q_o(p(t))$ are both increasing exponentially over time. With this growth pattern, the demand elasticity will remain constant at any prescribed price level. The economic justification for such an assumption is that an increase in disposable income should cause a proportional increase in output demanded at all price levels. Consonant with growing disposable income, we also assume that the entry response coefficient k is now a growing exponential function of time. A doubling of disposable income should double the quantity of resources potential entrants could invest in a particular product market given some profit incentive. Formally we assume that the static parameters a, b, and k of the basic model become:

$$a(t) = a_o e^{ut}$$

$$b(t) = b_o e^{ut}$$

$$k(t) = k_o e^{ut}$$

where u is the long run growth rate of real disposable income.

We now wish to maximize the functional

$$V = \int_0^T (\bar{p}+\delta(t)-c)(a_o e^{ut}-b_o e^{ut}(\bar{p}+\delta(t))-x(t))e^{-rt}dt \tag{4.1}$$

subject to

$$x(t) = k_o e^{ut}\delta(t) \qquad x(0) = 0 \tag{4.2}$$

We argue that u, the growth rate of disposable income, will be less than r, the dominant firm's discount rate and therefore the integral converges for $T \to \infty$ as long as the dominant firm's profit is bounded.[1] Application of Pontryagin's Maximum Principle yields the following necessary conditions:

$$\text{i)} \quad \dot{x}(t) = k_o e^{ut}\delta(t) \qquad x(0) = 0$$

$$\text{ii)} \quad \dot{z}(t) = (\bar{p}-c+\delta(t))e^{-rt} \qquad z(T) = 0$$

$$\text{iii)} \quad \delta(t) = \frac{ke^{(r+u)t}z(t) + (a_o+b_oc-2b_op)e^{ut}+x(t)}{2b_oe^{ut}}$$

The substitutions $w(t) = x(t)e^{-ut}$, $y(t) = z(t)e^{rt}$ convert these necessary conditions into the simultaneous autonomous differential equations,

$$\dot{y}(t) = \alpha_4 y(t)+\alpha_5 w(t)+\alpha_6 \qquad y(T)e^{-rT} = 0 \qquad (4.3)$$

$$\dot{w}(t) = \beta_4 y(t)+\beta_5 w(t)+\beta_6 \qquad w(0) = 0 \qquad (4.4)$$

where $\alpha_4 = \dfrac{2rb_o+k_o}{2b_o}$ $\qquad \alpha_5 = -\dfrac{1}{2b_o}$

$$\alpha_6 = \frac{a_o-b_oc}{2b_o}$$

$\beta_4 = \dfrac{k_o}{2b_o}$ $\qquad \beta_5 = -\dfrac{k_o+2b_ou}{2b_o}$

$$\beta_6 = \frac{k_o(a_o+b_oc-2b_o\bar{p})}{2b_o}$$

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[1] Unfortunately we were unable to prove the existence of an optimal path for this model for $t \to \infty$. The substitution $\tau = 1-e^{-bt}$ did not enable us to cite Cesari's Existence Theorem for this problem, because the transformed control variable was not bounded along the trajectory satisfying the necessary conditions.

These necessary differential equations have the unique solution given by Equation 4.5 and 4.6:

$$y^*(t) = d_1 e^{\lambda_3 t} + d_2 e^{\lambda_4 t} + \hat{y} \tag{4.5}$$

$$w^*(t) = \phi_3 d_1 e^{\lambda_3 t} + \phi_4 d_2 e^{\lambda_4 t} + \hat{w} \tag{4.6}$$

where $\lambda_{3,4} = \frac{r-u}{2} \pm \frac{\sqrt{(r+u)^2 + \frac{2k(u+r)}{b}}}{2}$ $\quad r < \lambda_3 \quad \lambda_4 < -u$

$$\hat{w} = \frac{k_o r}{k_o(u+r)2b_o ur} \qquad \hat{y} = \frac{k_o(c-\bar{p}) - u(a_o - b_o c)}{k_o(u+r) + 2b_o ur}$$

$$\phi_3 = \frac{\lambda_3 - \alpha_4}{\alpha_5} \qquad \phi_4 = \frac{\lambda_4 - \alpha_4}{\alpha_5}$$

d_1 and d_2 are determined by applying the boundary conditions $w(0) = 0$, $y(T)e^{-rT} = 0$

Because this is a long-run growth model we are primarily interested in the case of an infinite time horizon. When we allow T to increase without bond, the terminal condition $\lim y(T)e^{-rT} = 0$ implies that d_1 is identically zero because λ_3 is greater than r. Application of the boundary condition $w(0) = 0$ yields the results:

$$w^*(t) = \hat{w}(1 - e^{\lambda_4 t}) \tag{4.7}$$

$$x^*(t) = \hat{w}(1 - e^{\lambda_4 t})e^{ut} \tag{4.8}$$

$$\dot{x}(t) = (\hat{w}u - \hat{w}(u+\lambda_4)e^{\lambda_4 t})e^{ut} \tag{4.9}$$

The optimal price trajectory can be written:[2]

$$p^*(t) = \bar{p} + \frac{\dot{x}^*(t)}{k_o e^{ut}} = \bar{p} + \frac{\hat{w}u - \hat{w}(u+\lambda_4)e^{\lambda_4 t}}{k_o} \qquad (4.10)$$

The optimal pricing strategy is again largely determined by the sign of the parameter θ. We note that $\hat{w} = \frac{k_o r\theta}{k_o(u+r)+2b_o ur}$ is a positive multiple of θ. Equation 4.10 indicates that for positive (negative) θ the optimal price trajectory declines (rises) continuously from above (below) to an equilibrium level $\hat{w}u/k_o$ units above (below) the limit price $\bar{p}$. Figure 31 portrays the optimal paths for the two possible cases.

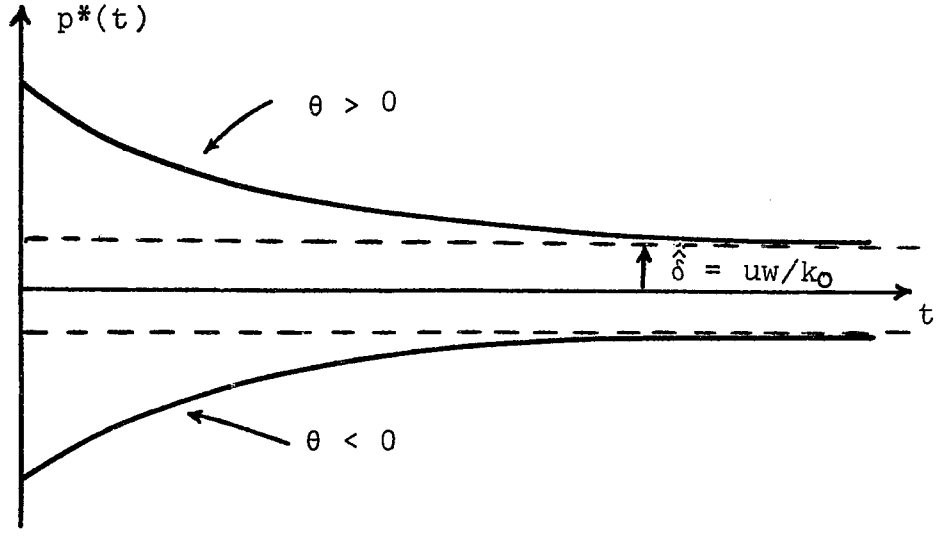
The effect of growth in the product market has been to alter the asymptotic level of the optimal price trajectory. In the long run the dominant firm no longer prices to preclude all entry. Equation 4.8 indicates that the competitive fringe grows asymptotically at the rate u. As under the basic model, we see that the optimal pricing strategy is to seek some optimal long-run market share. For positive θ (the most likely case, since a dominant firm will probably not enjoy a high cost advantage over the long run) the dominant firm yields up a fraction of its market Δ_2 equal to:

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[2]We may now verify that the present value integral converges by substituting the asymptotic results into equation 4.1. Therefore our application of the necessary conditions was valid.

FIGURE 31

Optimal Price Trajectories
with Constant Elasticity Growth



$$\Delta_2 = \frac{\hat{x}(t)}{a_o e^{ut} - b_o e^{ut}\hat{p}} = \frac{\hat{w}}{a_o - b_o(\bar{p} + \frac{u\hat{w}}{k_o})}$$

We can contrast this result with the optimal strategy under the basic model assuming the same original market. A comparable dominant firm facing a static product market would yield a fraction of its market Δ_1 equal to

$$\Delta_1 = \frac{x(t)}{a_o - b_o\hat{p}} = \frac{\theta}{a_o - b_o\bar{p}}$$

Fro small values of u (we might expect disposable income to grow at 4% per year) it is clear that $\Delta_2 < \Delta_1$.[3] Growth of the

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[3] We get this result by noting that for small values of u and r the ur term in Δ_2 may be neglected, yielding:

$$\Delta_2 = \frac{r\theta}{a_o(u+r) - b_o p(u+r)} \Rightarrow \frac{\Delta_2}{\Delta_1} = \frac{r}{u+r}$$

which is no less than unity for any positive growth rate.

product market increases the long-run optimal/market share of the dominant firm, ceteris paribus.

It is easily shown and quite plausible that the comparative statics of the initial optimal price level under growth are identical to the results of the basic model. The major conclusions remain, i.e., that any increase in the dominant firm's cost advantage or the rate of entry response will raise the initial optimal price level.

The comparative statics of the equilibrium optimal price are of some interest, since we no longer find the optimal price trajectory asymptotically approaching the limit price. We have shown that asymptotically the optimal price differs by $\hat{\delta} = \hat{w}u/k_o$ dollars from the limit price. Differentiation of $\hat{\delta}$ with respect to the pertinent model parameters yields these comparative static results.

i) $\text{sign } \hat{\delta}_u = \text{sign } \theta$

ii) $-1 < \hat{\delta}_{\bar{p}} < 0$

iii) $\hat{\delta}_c > 0$

iv) $\hat{\delta}_r > 0$

v) $\hat{\delta}_{k_c} < 0; \ \theta \geq 0$

(The subscript refers to partial differentiation with respect to that parameter.)

The first of these comparative static results indicates that the equilibrium gap between the optimal price and the limit price monotonically increases with the rate of growth of the market. Conditions ii, iii, and iv show that $\hat{\delta}$ declines as $\bar{p}$ increases, c decreases, or r decreases. An increase in the

firm's cost advantage will decrease the gap between $\hat{p}$ and $\bar{p}$ for positive θ and increase it if θ is less than zero. An increase in the firm's discount rate increases (decreases) the gap when θ is positive (negative). Condition v shows that speeding the entry response of rival producers will decrease the gap $\hat{\delta}$ if θ is positive.

The conclusion of this growth model that in the long run the optimal price remains above the limit price ($\theta>0$) while the dominant firm's market share stabilizes differs markedly from the conventional economic wisdom, as well as from our previous results with a static total market. It is of some interest to determine if the anomolies of this model are quantitatively significant. Table 2 presents the equilibrium values δ, $\delta/\bar{p}$, Δ_2 for specific cases. In each case we used the following parameters:

$a_o = 100$ $b_o = 1$ $k_o = 1$ $r = .1$ $c = 10$

The parameters p and u were varied as indicated in the margin of the table. The quantity Δ_2 is the fraction of the market yielded by the dominant firm to the competitive fringe. The dominant firm's long-run market share of course depends on the initial level of output supplied by the competitive fringe. It will equal $1-\Delta_2$ only if the competitive fringe started from a zero base.

TABLE 2

Long-Run Optimal Price
and Market Share under Growth

	u = 0	u = .02	u = .04	u = .08
$\bar{p} = 15$	$\hat{\delta} = 0$ $\hat{\delta}/\bar{p} = 0$ $\Delta_2 = .353$	$\hat{\delta} = .483$ $\hat{\delta}/\bar{p} = .032$ $\Delta_2 = .284$	$\hat{\delta} = .757$ $\hat{\delta}/\bar{p} = .050$ $\Delta_2 = .222$	$\hat{\delta} = 1.22$ $\hat{\delta}/\bar{p} = .082$ $\Delta_2 = .183$
$\bar{p} = 10$ (No Cost Advantage)	$\hat{\delta} = 0$ $\hat{\delta}/\bar{p} = 0$ $\Delta_2 = 1.00$	$\hat{\delta} = 1.45$ $\hat{\delta}/\bar{p} = .145$ $\Delta_2 = .819$	$\hat{\delta} = 2.43$ $\hat{\delta}/\bar{p} = .243$ $\Delta_2 = .694$	$\hat{\delta} = 3.67$ $\hat{\delta}/\bar{p} = .367$ $\Delta_2 = .532$

The first column of Table 2 represents the case of no growth in market demand (u = 0). The quantitative difference between the results shown in that column and the other columns are solely attributable to growth of the product market. Table 2 indicates that growth leads to a long run optimal price substantially above the limit price. Perhaps the most striking conclusion from this table is that a dominant firm with absolutely no cost advantage could price 24% above the limit price and still hold on to 30% of the total market in the long run if the product market is growing at 4% per year. The demise of a dominant firm with no cost advantage is no longer certain.

This model has disturbing policy implications. Entry is no longer sufficient to drive the long-run price to the limit price. Dominant firms with little or no cost advantages

can maintain significant market power while maximizing their present value if there is a rapidly growing product market.

Decreasing Elasticity Growth

The second growth model we explore attempts to approximate the growth of demand for a revolutionary new product, as in the contemporary data processing and copying industries. Both of these industries are presently dominated by single firms possessing substantial market power. The pattern of demand growth in such an industry should differ substantially from the pattern resulting from a steady increase in aggregate disposable income. The demand curve for a new product should shift outward dramatically as it becomes a commercial success. We expect the demand curve's vertical intercept (reflecting the price at which no output will be sold) to increase as the public discovers new uses for the product.

We incorporate this type of market growth into our basic model by allowing the parameter a, which specifies the position of initial residual demand schedule, to grow exponentially (i.e., $a(t) = a_o e^{ut}$).[4] The necessary conditions for this model are given by Equations 4.1 and 4.2 below.

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[4] We note however that it is unreasonable to expect the demand curve for a new product to continue to shift in this manner forever. Eventually the product is no longer "new" and its market demand should grow only with disposable income. It may be argued that the rate of growth should be again incorporated explicitly in the determination of k (the rate of entry) since rivals will obviously be more attracted by a growing product market. For the sake of analytic expediency we have not done this. For any specific case (u and k known) our optimal results are valid, but the comparative static effect of u cannot be found from our results without specifying the functional dependence of k on u.

$$\dot{y}(t) = \alpha_1 y(t)+\alpha_2 x(t)+\frac{a_o e^{ut}}{2b} - \frac{c}{2} \qquad (4.1)$$

$$Y(T)e^{-rT}=0$$

$$\dot{x}(t) = \beta_1 y(t)+\beta_2 x(t)+k(\frac{a_o e^{ut}}{2b} + \frac{c}{2} - \bar{p}) \qquad (4.2)$$

$$x(0) = 0$$

We note that these conditions are predicated upon convergence of the present value integral. For a finite time horizon this convergence is assured, but when we allow the horizon to extend indefinitely we restrict u to be less than r.[5] Equations 4.1 and 4.2 have the unique solution given by Equations 4.3 and 4.4

$$y^*(t) = d_1 e^{\lambda_1 t} + d_2 e^{\lambda_2 t} + \frac{c-\bar{p}}{r} - \frac{a_o e^{ut}}{kr/u+2b(r-u)} \qquad (4.3)$$

$$x^*(t) = d_1\phi_1 e^{\lambda_1 t} + d_2\phi_2 e^{\lambda_2 t} - 2b(p-\frac{c}{2})+k(\frac{c-\bar{p}}{r}) \qquad (4.4)$$
$$+ \frac{a_o e^{ut}}{r/(r-u)+2bu/k} \qquad u < r$$

where d_1 and d_2 are determined by $y^*(T)e^{-T} = 0$ and $x^*(0) = 0$, the boundary conditions.

The parameters λ_1, λ_2, ϕ_1, and ϕ_2 in these equations are identical to those previously defined for the basic model. It is reassuring to note that if u is set equal to zero the optimal paths described in Equations 4.3 and 4.4 are identical

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[5] We note that restricting the value of u doesn't guarantee convergence but makes it plausible. If the dominant firm's profit remains bounded the integral will converge.

to results under the basic model.

Under this model we are primarily interested in the nature of the optimal pricing strategy over a finite time horizon. Unfortunately, the explicit solution to Equations 4.3 and 4.4 is cumbersome and difficult to interpret. We can however approximate the initial portion of a finite optimal path by the initial portion of the optimal price trajectory for a boundless horizon.[6] The infinite horizon case is therefore of some interest even though we should not expect this type of growth to be perpetual. An unbounded planning horizon and the terminal condition of the adjoint variable y(t) imply as before that the constant d_1 must be zero. We can characterize the optimal price trajectory by Equation 4.5, which is derived by applying the relevant boundary conditions, differentiating Equation 4.4, and noting that $p^*(t) = \bar{p} + \frac{\dot{x}^*(t)}{k}$:

$$p^*(t) = \bar{p} + G_1 e^{\lambda_2 t} + G_2 e^{ut} \qquad (4.5)$$

$$\text{where } G_1 = \frac{\lambda_2}{k}\left(b(2\bar{p}-c) - \frac{k(c-\bar{p})}{r} - \frac{kG_2}{u}\right)$$

$$G_2 = \frac{ua_0(u-r)}{2b\left(u^2 - ru - \frac{rk}{2b}\right)} > 0$$

The optimal price trajectory is composed of a constant term (the limit price) and two exponential terms. The first of these exponential terms decays toward zero from below while

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[6] This approximation depends on T being large enough to insure that the term $e^{(u-\lambda_1)T} \cong 0$.

the second term continues to grow at rate u throughout the firm's lifetime. It is easily shown that G_2 is necessarily positive because u is positive and restricted to be less than r.[7] The quantity G_1 can be positive or negative depending upon the relative magnitude of a_o, $\bar{p}$, c, k, r and u. When u is set equal to zero we see that G_2 is zero and G_1 is equal to $-\lambda_2\theta/k$, and hence we have the result found under the basic model.[8]

The steady-state behavior of the optimal price trajectory is to increase at an exponential rate equal to u the growth rate of the product market. Formally, we find that the asymptotic path of the optimal price trajectory is:

$$p^*(t) = \bar{p}+G_2e^{ut} \qquad G_2 > 0 \qquad (4.6)$$

We find therefore that dominant firms facing a growing product market always price above the limit price in the long run. In the long run there will be a growing volume of entry by rival producers. Interestingly enough even though the optimal price is always above the limit price dominant firms do not decline either absolutely or relatively in the long run under this growth model.

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[7]The expression $u^2-ru-rk/2b$ is identical to the characteristic equation of the basic entry model and therefore has roots at λ_1 and λ_2. For positive values of u less than r this expression is necessarily negative since r is less than λ_1. It follows therefore that G_2 is positive, because its numerator is also strictly negative.

[8]We can also verify that the present value integral converges for u<r by substituting the asymptotic values of x(t) and p(t) into the indefinite integral of discounted profits.

The market share of the competitive fringe at any instant $\Delta(t)$ is given by the expression:

$$\Delta(t) = \frac{x(t)}{a(t) - bp^*(t)} \tag{4.7}$$

For sufficiently large t, $\Delta(t)$ approaches the constant ratio $kG_2/u(a_o-bG_2)$. It is easily shown that for relevant rates of growth ($0 \leq u \leq r$), the constant ratio is strictly less than one. Moreover, this asymptotic output of the fringe does not depend on $\bar{p}$ or c, and hence is not a function of the dominant firm's relative cost advantage. We conclude that the competitive fringe will eventually produce a constant proportion of total output, even though the dominant firm maintains a price continuously above the limit price. This result should not be intuitively surprising. Setting price at $\bar{p}$ in a growing market would necessarilly imply a growing market share for the dominant firm, since all additional output will be supplied by the dominant firm. For it to maintain a constant share of the market it must set its price above $\bar{p}$. All of these long run equilibrium results depend, of course, on our assumption of continued outward shifting of the market demand curve.

We are primarily interested in the initial portion of this infinitely long optimal path. Equation 4.5 indicates that in the short run the optimal price level will be below its asymptotic trajectory when $G_1 e^{\lambda_2 t} < 0$. The obvious question is whether the term $G_1 e^{\lambda_2 t}$ is ever sufficiently negative to cause the short-run optimal price ever to be below $\bar{p}$.

Because the second term in Equation 4.5 decays and the positive exponential grows, the lower limit on $p^*(t)$ occurs when $t = 0$. It is apparent that the optimal price can be less than $\bar{p}$ only if the sum of G_1 and G_2 is negative. Noting the allowed range of u we see that if $u = 0$ (the lowest value considered) $G_1 + G_2 = \frac{-\lambda_2\theta}{k}$. We get the familiar results that the optimal price will be below the limit price if $\theta<0$. Looking at the upper range of possible values for u we find that as u approaches r, $G_1 + G_2$ approaches $\frac{\lambda_2}{k}(b(2\bar{p}-c)+\frac{k}{r}(\bar{p}-c))$, which is obviously negative. Relatively high growth rates mandate pricing below the limit price initially.

We find that either a high cost advantage ($\theta<0$) or a high growth rate ($u\to r$) can cause for the optimal price trajectory to start off below the limit price.[9] Low rates of growth and moderate cost advantages warrant initial prices above $\bar{p}$. It can be shown that for plausible model parameters and $0\leq u\leq r$, the level of the initial portion of the optimal price trajectory decreases monotonically as the expected rate of market growth increases. Had we made the rate of entry k depend upon the growth rate in this model, we would certainly strengthen the case for this outcome. Intuitively, a higher market growth rate should generate a higher k, ad we proved in Chapter I that $\frac{\partial p^*(0)}{\partial k} < 0$.

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[9]We note that $\bar{p}$ is restricted to be greater than or equal to c.

We can also demonstrate that the slope of the initial optimal price trajectory will increase as the rate of growth increases. Table 3 illustrates the effect of various rates of market growth on the initial level and the slope of the optimal path. The model parameters for the specific cases shown were:

$a = 100 \qquad k = 1 \qquad b = 1$

$r = .1 \qquad c = 10$

We see from the table that as u increases, $\delta(0)$ (the level of the optimal price above the limit price) declines and $\dot{\delta}(0)$ (the optimal trajectory's slope) increases. It is apparent that an increase in the rate of market growth has the same effect on the initial optimal path as an increase in the firm's cost advantage.

TABLE 3

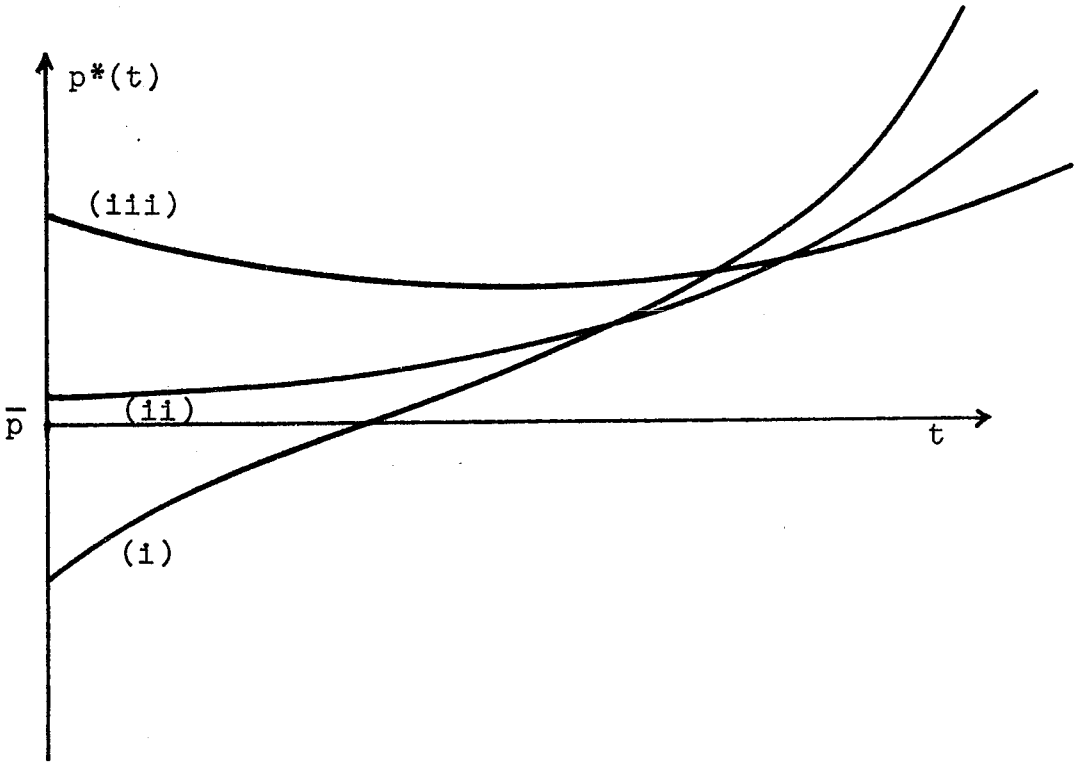
Initial Level and Slope of Optimal Price Trajectories Under Decreasing Elasticity Growth

	u = 0	u = .01	u = .04	u = .08
$\bar{p} = 10$ (No Cost Advantage)	$\delta(0)=16.10$ $\dot{\delta}(0)=-2.88$	$\delta(0)=14.91$ $\dot{\delta}(0)=-2.50$	$\delta(0)=10.75$ $\dot{\delta}(0)=-1.42$	$\delta(0)=3.23$ $\dot{\delta}(0)=-.18$
$\bar{p} = 15$	$\delta(0)=5.37$ $\dot{\delta}(0)=-.96$	$\delta(0)=4.18$ $\dot{\delta}(0)=-.58$	$\delta(0)=1.8$ $\dot{\delta}(0)=.18$	$\delta(0)=-7.50$ $\dot{\delta}(0)=1.74$
$\bar{p} = 20$	$\delta(0)=-5.37$ $\delta(0)=.96$	$\delta(0)=-6.56$ $\dot{\delta}(0)=1.34$	$\delta(0)=10.70$ $\dot{\delta}(0)=2.42$	$\delta(0)=-18.25$ $\dot{\delta}(0)=3.67$

We conclude that three distinct optimal trajectories are possible under this model. Representative examples are shown in Figure 32. In the case of a significant cost advantage or relatively high growth of market (shown by trajectory i in Figure 32), the optimal trajectory starts below $\bar{p}$ and rises monotonically approaching the term $G_2e^{ut}+\bar{p}$ asymptotically. For a lower growth rate and a moderate cost advantage, the optimal trajectory is always above $\bar{p}$ but rises monotonically over time. Trajectory ii in Figure 31 represents such a case. Trajectory iii in the figure illustrates the case of a very low expected rate of growth and little or no cost advantage. This last trajectory still resembles the optimal path under a static market.

FIGURE 32

Optimal Paths for Decreasing Elasticity Growth



We conclude that the principal effect of this kind of market growth is to lower the level and increase the slope of the initial portion of the optimal price trajectory. A relatively high rate of growth warrants a price trajectory which increases monotonically over time. The dominant firm's market share does not continually decline to zero under this model, but rather appraoches a constant fraction of the total market in the long run. The equilibrium market share of the dominant firm is independent of its relative cost advantage.

Stochastic Entry

All our previous results are based upon a deterministic entry model under which future entry by rivals is fully anticipated. If there is unexpected entry at some future time, the price trajectory previously calculated would no longer be optimal. It would be necessary to recompute the optimal path treating the market conditions after this unexpected entry as a new initial state. If we allow the entry model to be stochastic in the sense that there is always a certain amount of random entry or exit in addition to the deterministic entry, our previous results are not particularly operational because we are required to recompute the optimal trajectory continually. We can overcome this difficulty by finding the optimal strategy in terms of a feedback control law. The term feedback refers to the fact that the optimal control at any instant is based on the current level of the state variable x(t). Our previous

results are referred to as open-loop control in the liturgy of control theory, since the optimal trajectory is based only on the original state of the system. The optimal feedback control law takes into consideration not only the initial state of the system but also all successive states up to the current time.[10] The optimal control law we seek is of the form:

$$\delta^*(t) = h(x(t),t) \tag{4.8}$$

Formally, the price trajectory resulting from the application of the optimal control law given by Equation 4.8 will be identical to our previous result for the basic model if in fact there are no disturbances to the system. The Hamilton-Jacobi partial differential equation provides us the means to determine the specific h (x(t),t) which will generate the optimal price path.[11]

We shall first outline briefly the procedures to be followed to establish the desired control law. Our starting point in deriving the optimal control law is again the

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[10]We should note in passing that the control law we seek will be optimal only if we have no prior information about the stochastic disturbances. Given knowledge of the specific form of the disturbance the optimal trajectory can be found by a control law which takes into consideration future expectations at any point in time. A good discussion of this subject is found in Stuart E. Dreyfus, "Some Types of Optimal Control of Stochastic Systems," SIAM Journal of Control, 2: 120-34 (1964).

[11]The Appendix contains a more complete description of the Hamilton-Jacobi approach to finding optimal solutions.

Hamiltonian of the basic model. We first maximize the Hamiltonian with respect to $\delta(t)$, obtaining a maximum value of the Hamiltonian which we designate $H^{o}(x^{*}(t),z^{*}(t),t)$ and a specific optimal $\delta^{*}(t)$ as a function of $x(t)$ and $z(t)$. The Hamilton-Jacobi equation may therefore be written in the form:[12]

$$V_t^{*}+H^{o}[x(t),V_x^{*},t] = 0 \qquad (4.9)$$

where $V^{*}(x(t),t)$ is the optimal discounted stream of profits evaluated at any point in time.[13] In addition the solution $V^{*}(x(t),t)$ must satisfy the appropriate boundary condition which for our case is $V^{*}(x(T),T) = 0$, since the firm faces no scrap value at the end of the planning period. If we are able to find a solution V^{*} to Equation 4.9, we can merely differentiate with respect to x to yield $z(t)$, as we have verified our implicit assumption that $z(t) = V_x^{*}$. We can use this result to establish $\delta^{*}(t)$ as a function of $x(t)$ and t, and this is the optimal control law we seek.

Following the outlined procedure for our problem we find first that

$$\delta^{*}(t) = \frac{z(t)ke^{rt}+a+bc-2b\bar{p}-x(t)}{2b} \qquad (4.10)$$

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[12]The Appendix contains a more complete description of the Hamilton-Jacobi approach to finding optimal solutions.

[13]The subscript $V^{*}_{(\ )}$ indicates as usual the partial derivative with respect to the subscripted variable.

and that

$$H^o = \frac{-e^{-rt}}{4b}[z^2(t)k^2e^{2rt}-(a-bc-x(t))^2] \qquad (4.11)$$

$$+z(t)\frac{k}{2b}[z(t)ke^{rt}+a+bc-2b\bar{p}-x(t)]$$

The Hamilton-Jacobi equation becomes for this case

$$4bV^*_t+k^2e^{rt}(V^*_x)^2+e^{-rt}(a-bc-x(t))^2+ \qquad (4.12)$$

$$2k(a+bc-2b\bar{p}-x(t))V^*_x = 0$$

It is easily shown that this partial differential has a solution of the form

$$V^* = (V_1(t)x^2+V_2(t)x+V_3(t))e^{-rt}; \qquad (4.13)$$

where the time paths of $V_1(t)$ satisfy the ordinary differential equations:

$$\dot{V}_1(t) = (k/b+r)V_1(t)- \frac{k^2}{b}V_1^2(t)-1/4b \qquad (4.14)$$

$$\dot{V}_2(t) = (r+\frac{k}{2b})V_2(t) - \frac{k^2}{b}V_1(t)V_2(t)-\frac{k}{b}(a+bc-2b\bar{p})V_1(t)$$

$$+ \frac{a-bc}{2b} \qquad (4.15)$$

$$\dot{V}_3(t) = rV_3(t)- \frac{k^2}{4b}V_2^2(t)- \frac{k}{2b}(a+bc-2b\bar{p})V_2(t)-\frac{(a-bc)^2}{4b}$$

$$(4.16)$$

The boundary condition that $V^*(x(t),t)$ is zero when $t=T$ implies (for finite T) that $V_1(T) = V_2(T) = V_3(T) = 0$ which enables up to specify unique time paths for the variables in

Equations 4.14, 4.15, and 4.16.[14] The exact solution to this system of simultaneous differential equations is somewhat cumbersome, but if we restrict our attention to $t<<T$ (certainly the case when we look at a dominant firm with a very long time horizon), we get a simple appealing result. We observe that Equation 4.14 is not coupled with the other variables and takes the form of a scalar Riccati equation. Its solution with the appropriate boundary condition applied is[15]

$$V_1(t) = \frac{n_1(1-e^{\sqrt{F}(T-t)})}{1-\frac{n_1}{n_2}e^{\sqrt{F}(T-t)}} \qquad (4.17)$$

where $F = r^2+2rk/b$

and $n_{1,2} = \frac{k+br}{2k^2} \pm \frac{\sqrt{b^2r^2+2brk}}{2k^2}$

We see that for $t<<T$, $V_1(t)$ is essentially equal to n_2, a constant. If we continue to restrict our attention to optimal pricing with a long time horizon and set $V_1(t) = n_2$, Equation 4.15 becomes a simple first order uncoupled

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[14] Solutions to this system of equations exist and are unique because $V_1(t)$ is bounded on the interval $0\leq t\leq T$ as indicated by Equation 4.15. We have a complete set of terminal conditions and the functions $V_i(t)$ are continuous and continuously differentiable with respect to $V_1(t)$, $i = 1,2,3$. Existence and uniqueness for such a system of equations are proved in Pontryagin, Ordinary Differential Equations, p. 196.

[15] This follows because Equation 4.14 is separable in $V_1(t)$ and t and may be factored into the form: $\frac{dV_1}{\lambda(V_1-n_1)(V_1-n_2)}=dt$ and integrated directly.

differential equation.

$$V_2(t) = \lambda_1 V_2(t) + (\bar{p}-c) - \frac{(a+bc-2b\bar{p})}{k}\lambda_2; \qquad (4.15')$$

$$\text{where } \lambda_{1,2} = \frac{r}{2} \pm \frac{\sqrt{r^2+2kr/b}}{2}$$

are identical to the roots of the characteristic equation for the open-loop basic model.

The solution to 4.15' is easily shown to be

$$V_2(t) = \frac{c-\bar{p}}{\lambda_1} + \lambda_2 \frac{(a+bc-2b\bar{p})}{k\lambda_1}\left(1 - \frac{e^{\lambda_1 t}}{e^{\lambda_1 T}}\right) \qquad (4.18)$$

We see that for $t \ll T$, $V_2(t)$ will be the constant

$$V_2(t) = k(c-\bar{p}) + \frac{\lambda_2(a+bc-2b\bar{p})}{k\lambda_1} \qquad (4.19)$$

because λ_1 is strictly positive and the exponential term in Equation 4.16 will be negligible.

There is no need to solve Equation 4.16 because we only need the partial of V* with respect to x to establish the optimal control law. Noting that V^*_x is equal to z(t) because we have established V* as the solution of the Hamilton-Jacobi euqation, we may now write the explicit time path of the adjoint variable along the optimal trajectory:[16]

$$z(t) = (2V_1(t)x(t)+V_2(t))e^{-rt} \qquad (4.20)$$

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[16] Strictly speaking V* must also be twice differentiable in both its arguments for z(t) to be equal to V_x^*. See the Appendix for a brief discussion of this subject.

We now substitute the result of Equation 4.19 and the fact that $V_1(t) = n_2$ into Equation 4.20 and then in turn substitute that result into Equation 4.10 to get the desired control law. After appropriate algebraic manipulation we obtain the appealing result:

$$\delta^*(t) = \frac{\lambda_2}{k}(x(t)-\theta) \tag{4.21}$$

where $\theta = \frac{k}{r}(c-\bar{p})+b(c-2\bar{p})+a$

as before.

It is reassuring to see that this same result could have been derived heuristically from our previous results with the basic model. We demonstrated in Chapter I (see Equation 1.12) that the optimal level of the control variable $\delta^*(t)$ for a sufficiently long time horizon is given by

$$\delta^*(t) = \frac{-\lambda_2\theta e^{\lambda_2 t}}{k} \tag{4.22}$$

Equation 4.20 tells us that for any value of θ the initial value of the control variable is

$$\delta^*(0) = \frac{-\lambda_2\theta}{k} \tag{4.23}$$

We next note by the geometry of our model that a change in $x(t)$ is exactly equivalent to a change in the output intercept of our original demand schedule. We can therefore define the appropriate parameter for calculating the open-loop optimal trajectory at any point in time as

$$\hat{\theta} = \theta-x(t)$$

It follows therefore that if we were to recompute the optimal trajectory at every instant in time, taking into consideration the current level of fringe output, the constantly calculated initial value of the control variable will be given by:

$$\delta^*(0) = \frac{-\lambda_2}{k}(\theta - x(t)) \tag{4.25}$$

If a random fluctuation in x(t) (or an equivalent random shift in the current demand schedule) occurred at every instant in time the only part of the constantly recomputed optimal paths germane to the optimal path would be the initial points, which are given by Equation 4.25. We see (by comparing Equations 4.25 and 4.21) that this feedback law is identical to the result derived by the Hamilton-Jacobi equations as shown. Our ardous labors have not been entirely in vain, because we have demonstrated that the Hamilton-Jacobi equation does have a solution which is a necessary and sufficient condition for our previous interpretation of the adjoint variable z(t) as the shadow price of entry (i.e., $z(t) = V^*_x(x(t),t)$) at any point in time.

Our result in any case is an extremely simple control law which will generate the optimal price trajectory, given knowledge of the original state of the model and information on the position of the contemporary demand schedule. All the comparative statics of the path generated by this optimal control law are identical to the results derived under the

basic open loop model with a sufficiently long time horizon.[17] In addition, Equation 4.21 reveals the obvious comparative static implication that the optimal price level varies inversely with the contemporaneous change in rival output (or directly with shift in demand).

We conclude that adopting an optimal pricing strategy in the form of a control law relating current price to the size of the competitive fringe has not changed any of the basic properties of the optimal trajectory. We still observe, for example, that an increase in the dominant firm's cost advantage will induce a decrease in the optimal price level, ceteris paribus. This implies that a dominant firm will respond to an increase in its relative cost advantage by lowering its price to secure a larger future share of the market.

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[17] See page 13 of the first chapter.

CHAPTER V

OPTIMAL PRICING WHEN PRICES BELOW THE LIMIT PRICE ARE PROHIBITED

In all of the models considered thus far, we found that certain market conditions (i.e., when a dominant firm has a high cost advantage and when rival producer respond rapidly to price stimuli) were sufficient to justify pricing below the limit price over some time interval. This pricing behavior leads to an outward shift in the dominant firm's contemporary demand schedule due to the induced exit of rival producers. The dominant firm's market share increases as long as the optimal price remains below the limit price. Such a pricing strategy implicitly assumes that the competitive fringe was originally large enough so that the dominant firm's output never exceeds the total market demand. But if for example the competitive fringe produced no output originally, it would not be possible for the dominant firm to enlarge its market share, even though it had an enormous cost advantage. There is the further possibility that political and legal constraints prevent a dominant firm from pricing to increase its market share, even if there is a sufficiently large competitive fringe. It seems doubtful, for example, that General Motors regards pricing to gain a substantially larger market share as a politically viable strategy. A

more realistic model should therefore include some lower limit on the value of x(t), representing the maximum amount by which the total production of the competitive fringe can be decreased.

We wish therefore to explore the effect of imposing a lower limit on the value of the state variable x(t). Formally we do this by imposing the constraint $x(t) \geq -\bar{x}$. The constraint will be binding when x(t) is strictly equal to $-\bar{x}$, and we regard $x(t) = -\bar{x}$ as a lower boundary on the allowable values of our state variable x(t). When the constraint is binding, the dominant firm can no longer shift its residual curve outward by pricing below the limit price. When the x(t) trajectory is in contact with this boundary ($x(t) = -\bar{x}$) the necessary conditions previously derived will no longer be valid. The appendix contains a brief exposition of the relevant mathematics. We are seeking necessary conditions of the basic form proposed by Pontryagin.[1]

Finite Time Horizon

We consider first optimal pricing over a finite time horizon. If the minimum allowable value of the state variable $\bar{x}$ is negative, the initial point of an optimal trajectory will not be on the boundary, as we have defined x(0) = 0. The boundary $x(t) = -\bar{x}$ will not be struck until the optimal price has been below the limit price for some finite period. We

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[1] L.W. Pontryagin, V. Boltyanskii, R. Gamkrelidze, and E. Mishenko. The Mathematical Theory of Optimal Processes. (New York: Interscience Publishers, Inc., 1962), Chapter VI.

will strike the boundary at time t if and only if $\int_0^t \delta(t)dt = -\bar{x}$. We have seen before that the optimal price trajectory for a finite planning period with no scrap value is normally characterized by a rapidly rising price near the end of the period. We expect that the optimal price, even if it moves along the boundary ($x(t) = -\bar{x}$) for some period, will eventually move off it as the approaching end of the time horizon encourages higher prices, inducing net entry by rivals. A typical optimal trajectory will therefore remain on the boundary for only an intermediate portion of a finite planning period.

The portions of the optimal price trajectory when the state constraint is not binding can be generated by the Maximum Principle and are identical in form to results previously derived. For our particular problem, the optimal pricing strategy when x(t) is on its boundary is to charge the limit price. A price above the limit price would induce entry and the dominant firm would move off the boundary. A price below $\bar{p}$ would gain no market share as long as x(t) is at its minimum value, and would therefore needlessly sacrifice current profits. Formally once the boundary has been struck the control variable is chosen to maximize the Hamiltonian subject to the binding constraint. We will remain on the boundary as long as this maximization entails setting $\delta^*(t) = 0$. The relevant differential equations describing the system along the boundary are:

$$\dot{x}(t) = k\delta(t) = 0 \qquad (5.1)$$

$$\dot{z}(t) = (\bar{p}-c+\delta(t))e^{-rt} = (\bar{p}-c)e^{-rt} \qquad (5.2)$$

Since the Hamiltonian is strictly concave in δ maximizing H subject to the constraint will obviously imply setting $\delta^*(t)$ equal to zero as long as $\frac{\partial H}{\partial \delta} < 0$.

When the boundary is first struck, the quantity $\frac{\partial H}{\partial \delta}|\delta = 0$ will necessarily be negative because the value of $\delta(t)$ which maximized H at that instant was necessarily less than zero. The optimal path will depart from the boundary at that point in time when $\frac{\partial H}{\partial \delta}|\delta = 0 > 0$, since maximization of the Hamiltonian will then imply setting $\delta(t) > 0$. At any point in time we may write

$$\frac{\partial H}{\partial \delta}|\delta = 0 = (a-x(t)-2b\bar{p}+bc)e^{-rt}+z(t)k \qquad (5.3)$$

Along the boundary this expression must be increasing with time because x(t) is equal to the constant $-\bar{x}$, and z(t) is necessarily increasing, as shown by Equation 5.2. For a given cost advantage $(\bar{p}-c)$, the length of time that the optimal path remains on the boundary is a monotonically decreasing function of $z(t_1^+)$, the value of the adjoint variable after the boundary is first struck. (We use t_1^+ and t_1^- to designate the instant just after and just before the initial contact with the boundary.) We recall that to find a solution to the necessary differential equation for our various models we had to pick that specific value of the adjoint variable at the initial point so as to satisfy the terminal condition $z(T) = y(T)e^{-rT} = 0$. We have always found previously that there was in fact a unique initial

value y(0) which caused the trajectory to meet the terminal condition. It seems intuitive in the present case that if z(t) were continuous throughout the planning period we would again be able to select a unique z(0) which would solve the two point boundary value problem. Unfortunately, Pontryagin's necessary conditions for this problem allow a finite jump in the level of the adjoint variable z(t) at every point that the optimal path strikes or leaves the boundary. In general, the adjoint variable must meet certain jump conditions at every such junction point. In other words, if we consider the junction point t_1 where the optimal path first strikes the constraint, we have the necessary condition

$$z(t_1^+) = z(t_1^-)+M;$$

where M is an arbitrary constant.[2]

The difficulty caused by discrete jumps in the adjoint variable can be illustrated by Figure 33. The figure presents both the state variable x(t) and the adjoint variable z(t) plotted against time. We are considering a dominant firm with a very high cost advantage. The optimal price trajectory without any lower bound on x(t), is a decaying exponential (i). When the state constraint $-x-x(t) \leq 0$ is imposed a possible optimal path is trajectory (ii), which originates below and remains below the unconstrained optimal path until time t_1.

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[2]This is the jump condition for our very simple problem. In general the jump condition is more complicated. The Appendix presents the general form of these conditions.

We recall that as long as x(t) is not on the boundary, maximization of the Hamiltonian with respect to δ requires that

$$\delta(t) = \frac{z(t)ke^{rt}+a-x(t)-2b\bar{p}+bc}{2b} \qquad (5.4)$$

Equation 5.4 shows that at t = 0 the magnitude of z varies directly with the level of the initial optimal price because x(t), the only other variable in the equation, is always zero at t = 0. Equation 5.2 indicates that z(t) will approach zero more slowly along (ii) than along (i) in the interval $0 \leq t \leq t_1$. We conclude therefore that at point t_1^- the value of the adjoint variable $z(t_1^-)$ along (ii) is necessarily more negative than along (i).

At the instant t_1 the adjoint variable is decreased by the amount Δ_1 which is sufficient to hold the trajectory on the boundary until time t_2.[3] When the trajectory (ii) leaves

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[3]We recall that the trajectory will remain along the boundary until $\frac{\partial H}{\partial \delta}|\delta = 0$ becomes positive. It is easily shown that along the boundary constraints the adjoint variable is a decaying exponential given by

$$z(t) = \frac{c-\underline{p}}{r} - z(t_1^+) - \frac{\underline{p}-c}{r} e^{-r(t=t_1^+)} \quad t_1^+ \leq t \leq t_2 \qquad (5.5)$$

By selecting $z(t_1^+)$ we can determine exactly when z(t) will pass through any specified level. Since $\frac{\partial H}{\partial \delta}|\delta = 0$ is a linear function of z(t) we have the ability to determine the specific t_2 at which the optimal trajectory leaves the boundary by the appropriate choice of $z(t_1^+) = z(t_1^-) + \Delta_1$.

the boundary at t_2 the adjoint variable is increased by Δ_2, which is exactly large enough so that the adjoint variable passes through zero at time T. We note that whenever the adjoint variable undergoes a finite jump, the optimal price level makes a similar discrete movement.

The trajectory (ii) shown in Figure 33 is only one of an infinite family of trajectories, all satisfying the necessary conditions. In the unconstrained model we had only to find the unique initial value of the adjoint variable z(0) which allowed the terminal condition z(T) = 0 to be met. Now we can select any one of an infinite number of sets of z(0), Δ_1, and Δ_2 to meet the same final condition.

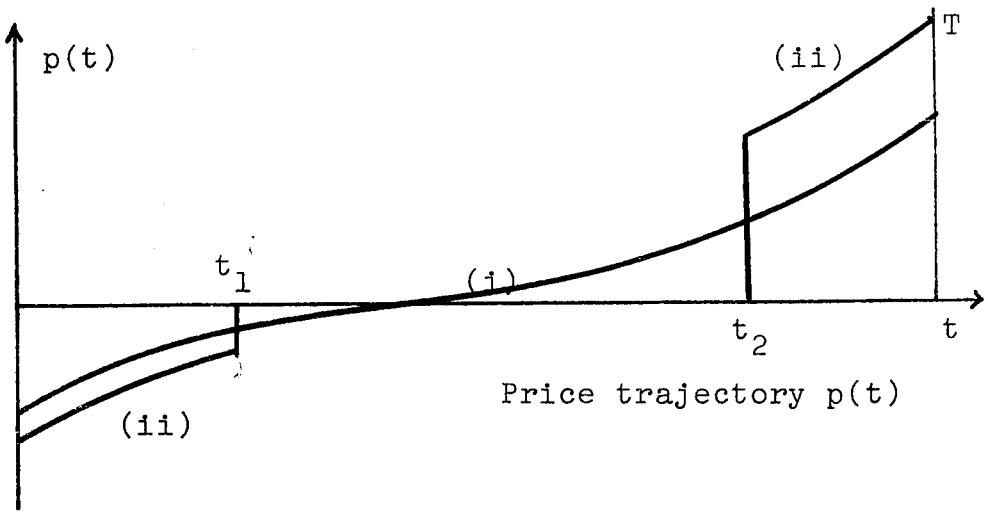
To find the actual optimal path from the infinite family of possible trajectories it is necessary to evaluate the present value integral for each candidate trajectory and to select that path which yields the largest stream of discounted profits. Given specific model parameters an easy way of searching out the optimal trajectory from the whole family of paths satisfying the necessary differential equations is to utilize a hybrid analog-digital computer to solve the necessary differential equations repetitively over the range of possible values of Δ_1, Δ_2, and z(0).[4]

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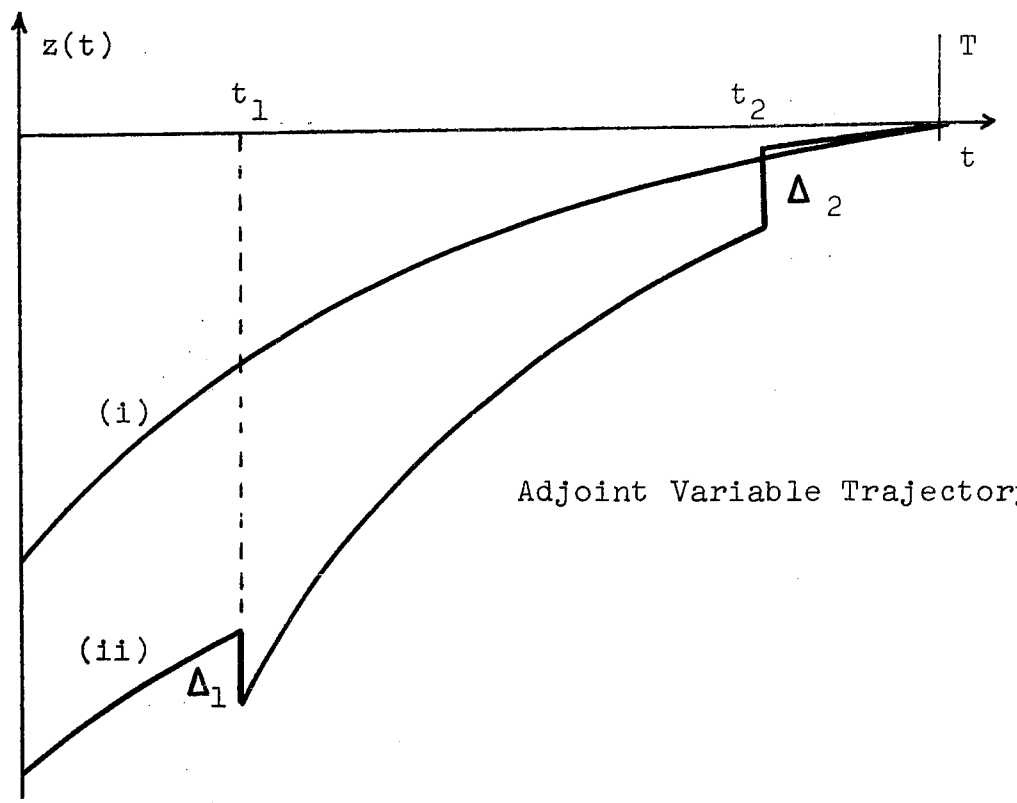
[4] A hybrid computer is very useful in this case because the necessary differential equations can change their form at every junction point. The digital logic section of the computer can be used to select the relevant equations automatically.

FIGURE 33

Price Trajectory When x(t) is Constrained



Price trajectory p(t)



Adjoint Variable Trajectory z(t)

The basic method is to fix arbitrarily two of the elements, say Δ_1 and Δ_2, and to select that value of z(0) which allows the terminal condition z(T) = 0 to be met. It is a simple matter to simultaneously evaluate the value of V for that particular path. The same procedure is repeated for another choice of Δ_1 and Δ_2. Fortunately there is only one value of z(0) for each pair Δ_1 and Δ_2 satisfying the terminal conditions. Furthermore it was soon discovered that in our particular problem $\Delta_2 = 0$ always yielded the maximum stream of discounted profits for any value of Δ_1 and z(0).[5] The results of this search procedure for a specific case are shown below. The case considered is specified by the parameter values:

$a = 100$ $\quad b = 1$ $\quad c = 10$ $\quad r = .1$

$\bar{x} = 10$ $\quad \bar{p} = 20$ $\quad k = 1$

Figure 34 shows the optimal pricing trajectory when there is a binding constraint on the state variable. The optimal path (indicated by the solid line) undergoes a finite jump when the boundary is first struck. The optimal trajectory for the same time horizon without a constraint on the value of x(t) is also shown by a dashed line in the figure. The existence of a binding constraint has the effect (at least in this case) of raising the initial portion of the optimal

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[5]There are necessary conditions of a slightly different form due to J. E. Funk and E. G. Gilbert. "Some Sufficient Conditions for Optimality in Control Problems with State Space Constraints," forthcoming in SIAM Journal of Control. These conditions are substantially more powerful for our particular problem. Application of these conditions verified that $\Delta_2=0$ along the optimal path.

FIGURE 34

Optimal Price Trajectory
for a Finite Time Horizon

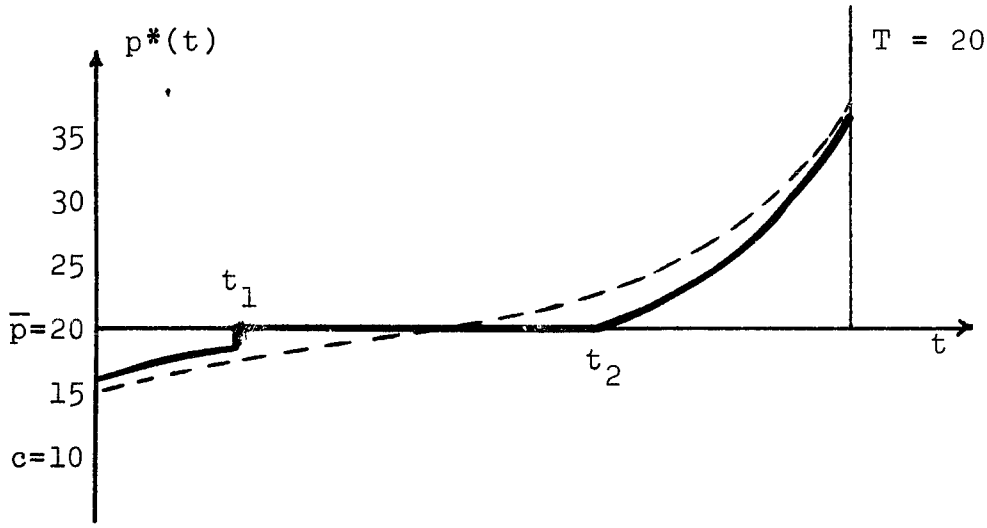
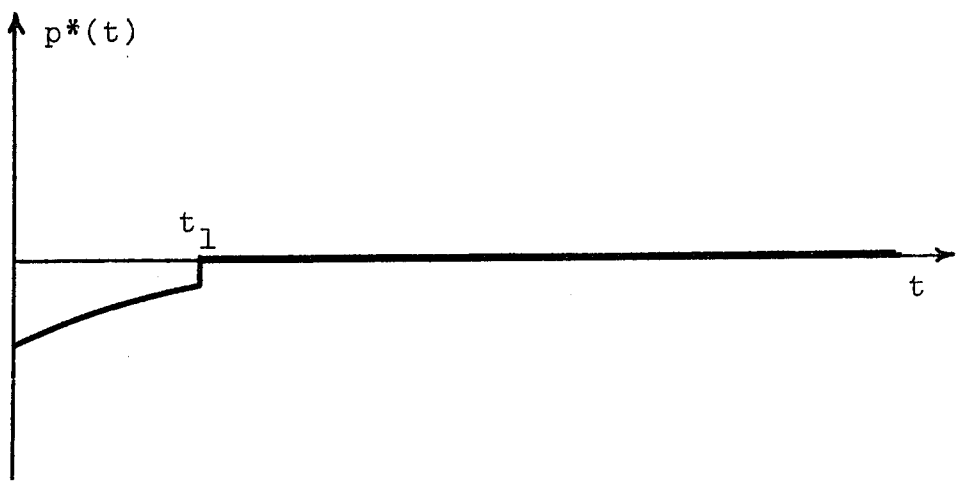


FIGURE 35

Optimal Price Trajectory
for an Infinite Time Horizon



price trajectory toward the limit price. We notice that the presence of a binding constraint has compressed the optimal price trajectory toward the limit price over the whole planning period.

Infinite Time Horizon

Many dominant firms should be correctly modeled as having a very long or essentially infinite planning period. There is no reason to believe that a firm will cease production at some specified time. We argue that the optimal pricing trajectory for a dominant firm with a high cost advantage will strike and remain indefinitely on the binding constraint. Figure 35 contains one member of the family of trajectories which satisfy the necessary conditions. Maximization of the Hamiltonian will cause the trajectory to originate below the limit price. The cost advantage of the dominant firm justifies pricing below $\bar{p}$, which drives $x(t)$ to the boundary $-\bar{x}$ at time t_1. The adjoint variable undergoes a jump at time t_1 just sufficient to hold the trajectory on the boundary forevermore. We already know that after t_1 the optimal trajectory will not go below $\bar{p}$, because to do so would sacrifice current profits with no attendant gain in future market share.

We argue that $p^*(t)$ cannot move above $\bar{p}$ because as we have shown for the basic model, the transversality condition $\lim z(T) = y(T)e^{-rT} = 0$ and maximization of the Hamiltonian without constraint imply that

$$p^*(t) = \bar{p} - \frac{\lambda_2 \theta e^{\lambda_2 t}}{k} \qquad (1.12)$$

If the trajectory were to rise above the limit price, the constraint would immediately cease to bind. The optimal path would then be given by a version of Equation 1.12 with θ replaced by $\theta' = \theta+\bar{x}$. The constant θ' reflects the contemporaneous position of the dominant firm's residual demand schedule. We next observe that θ' must be negative. We have previously shown that the long-run result of optimal pricing without any constraint on $x(t)$ is a rightward shift of θ units of the residual demand schedule. The fact that the constraint $-x(t)-\bar{x}\leq 0$ is binding in this case indicates that the desired shift has not been completed and hence $|\bar{x}|<|\theta|$. Since we know that θ is negative, it follows that θ' is also less than zero. But the fact that $\theta'<0$ indicates that the optimal price $p^*(t)$ is below rather than above $\bar{p}$. It is apparent that any movement above $\bar{p}$ is inconsistent with the transversality condition and maximization of the Hamiltonian.

Unfortunately we still have an infinite family of possible optimal trajectories from which to choose. We have already indicated that along the binding constraint the adjoint variable $z(t)$ is given by:

$$z(t) = \frac{c-\bar{p}}{r} - z(t_1^+) - \frac{\bar{p}-c}{r}e^{-r(t-t_1^+)} \; ; \; t_1^+ \leq t \qquad (5.5)$$

It is readily apparent that if the adjoint variable jumps to the level $z(t_1^+) = \frac{c-\bar{p}}{r}$, the transversality condition $\lim z(T) = 0$ as $T \to \infty$ will necessarily be met. This indicates that for any initial trajectory that encounters the boundary, there is a discrete jump Δ_1 equal to the difference between $\frac{c-\bar{p}}{r}$ and

$z_1(t_1^-)$ which will enable the whole trajectory to meet the necessary conditions.

Unlike the finite time horizon case we can no longer select the unique optimal trajectory by computer search procedures. We can only observe what happens as we increase T in the finite case. For a reasonably long time horizon (20-25 years) we found the only perceptible effect of increasing T was to lengthen the portion of the trajectory remaining on the boundary. We also can specify the optimal path when the dominant firm is prevented from pricing below the limit price initially (i.e., $\bar{x} = 0$). In this case a dominant firm with a substantial cost advantage and an infinite time horizon will always set its product price at $\bar{p}$.

Conclusion

This model yields a major conclusion which differs substantially from our previous results. We found that dominant firms enjoying relatively large cost advantages and constrained from driving out more of the competitive fringe will price at the limit price over much of its time horizon. All of our previous models found that dominant firms maximizing their present value price at the limit price only asymptotically (if at all). Only in this constrained model do we get results identical to the naive limit pricing model under which a dominant firm always prices at the limit price.

A dominant firm in the American economy whose behavior might reasonably be described by this constrained model might be General Motors. It is reputed to be more efficient than

its rivals and has demonstrated considerable market power. It is obviously inhibited from increasing its market share substantially. The fact that General Motors maintains a constant share of the automobile market implies that it has set its price at the limit price over recent years.

CHAPTER VI

IMPLICATIONS FOR THEORY AND PRACTICE

The optimal pricing strategies derived in this thesis challenge several major tenets of the conventional economic wisdom. The results also have important implications for antitrust and patent policies.

From the outset, the models I have presented have questioned the validity of any theoretical model based upon static profit-maximization. It has been shown that a far-sighted dominant firm will always price below the myopic profit-maximizing price. The existence of potential entrants will always justify some exchange of current profits for future market share (and hence future profits). It has been shown further that the optimal price at any moment in time will vary with changes in the dominant firm's discount rate, the rate of rival entry, or the level of the limit price. Under a static profit-maximizing model none of these parameters should affect the current optimal price.

An alternate conventional hypothesis postulates that dominant firms set their price at the limit price to preclude entry by rivals. Maintaining the limit price over most of the planning period was found to be the best strategy under only one of the models considered here. I demonstrated that a dominant firm maximizing its present value will be observed

to maintain the limit price only when prevented from driving out rivals because it already has the whole market or because of threats of antitrust action.

Under several other models, we found that the optimal price trajectory asymptotically approaches the limit price as the dominant firm moves toward its desired long run market share. It is improbable, however, that a dominant firm will be observed to possess its equilibrium market share at any particular moment in time. In a world of shifting consumer demand and technological break-throughs, most dominant firms will not be at the equilibrium defined by their long run optimal market share.

Furthermore, it has been shown that in the empirically important case of a steadily growing product market the optimal price does not even asymptotically approach the limit price. In this case a dominant firm with little or no cost advantage will maximize its discounted profit stream by continually pricing above the limit price.

The dominant firm's relative cost advantage is crucial in determining its optimal pricing strategy. A dominant firm with little or no cost advantage will deliberately surrender its market share, while a firm with a large cost advantage over its rivals will price to enlarge its market (or maintain it, if constrained from driving out more of the fringe). The models considered above indicate that the diverse pricing policies of U.S. Steel, Xerox, General Motors, and Alcoa may all have been optimal.

United States Steel's share of the steel market has almost continuously declined from 65 percent in 1901 to 24 percent in 1967.[1] U.S. Steel had no real cost advantage over rival producers, and it has priced to allow substantial entry over the past seventy years.[2]

In the copying industry, Xerox enjoys a cost advantage which varies directly with the volume of copies produced per time period. Xerox has been observed to price so as to induce entry in the low volume market, where it enjoys its least advantage, and to maintain or increase its market share in the high volume market.[3] This pricing policy coincides exactly with the predictions of my analysis.

General Motors is recognized as the most efficient auto manufacturer, and it possesses considerable market power. Any substantial increase in General Motors share of the United States auto market would undoubtedly bring antitrust action by the Justice Department. The optimal strategy for such a constrained dominant firm is to set price at the limit

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[1]Scherer, Industrial Structure and Economic Performance, pp. 217-18.

[2]It is observed that the U.S. Steel has probably been replaced by a group of American steel producers as the price determining unit in the steel industry. Since American producers have no advantage over foreign producers, we should expect a continual decline in the market share of this price setting group.

[3]Erwin A. Blackstone, The Copying Machine Industry. A Case Study (unpublished Ph.D. dissertation, University of Michigan, 1968), pp. 140-59.

price, and thus to maintain a constant market share. General Motors has in fact maintained a fairly stable fraction of the domestic auto market over the last two decades.[4]

Prior to the end of World War II, Alcoa enjoyed a virtual monopoly in the domestic production of primary aluminum. Alcoa reported only average profits during the prosperous 1920's and 1940's. During the depression years Alcoa's profits were noticeably above the average for all manufacturing industries.[5] This pricing behavior is consistent with my model if we assume that severe depression reduced the willingness or ability of potential rivals to enter the aluminum industry. This is equivalent to a reduction of the parameter k, which causes the optimal price trajectory and current profits to rise.

The implications of these models for normative patent policy are substantial. First, it was shown that a decrease in the lifetime of a patient would raise the product price throughout the life of the patent. Secondly, it was shown that lowering the limit price (through a mandatory licensing policy or a reduction in the protection granted a patent

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[4] General Motors share of the domestic automobile market for the past seventeen years as reported by Automotive Industries has been:

1953	45.1%	1959	42.1%	1965	50.0%
1954	50.7%	1960	43.7%	1966	48.1%
1955	50.8%	1961	46.5%	1967	49.6%
1956	50.8%	1962	51.9%	1968	46.7%
1957	44.8%	1963	51.0%	1969	46.8%
1958	46.3%	1964	49.1%		

[5] L.W. Weiss, Economics and American Industry (New York: John Wiley & Sons, Inc., 1961), pp. 201-2.

holder) will similarly raise the current optimal price level. It is important to note, however, that both of these actions will speed the decline of the dominant firm, and that lowering $\bar{p}$ will reduce the product price in the long run.

This study has mixed implications for antitrust policy. On one hand, it was demonstrated that if dominant firms facing static product markets eventually lose their cost advantages, they will ultimately decline and certainly lose their market power. Unfortunately, this sanguine conclusion is not valid for a constantly growing product market. Growth allows a dominant firm to maintain considerable market power even with little or no cost advantage over rival producers.

A serious philosophical problem concerns the justification of an antitrust policy which would prevent the rationalization of an industry (eliminating inefficient producers) by a dominant firm with a very large cost advantage. If legal restrictions prevent such a firm from driving out relatively inefficient rivals by pricing below the limit price, a higher product price will necessarily ensue. It is ironic that while high prices are normally associated with market power, in this case the price rises precisely because market power is attacked.

A clear policy prescription evident from these results is that any increase in the rate of entry by rivals for a given price stimulus will always lower the product price. Any action which speeds the reaction of rival producers to extra-normal profits will serve to drive down the relevant product price.

In closing it should be re-emphasized that all of these conclusions depend upon the specific models explored. I believe, however, that the models are sufficiently realistic to make the results interesting and plausible.

APPENDIX A

THE MAXIMUM PRINCIPLE[1]

Pontryagin's Maximum Principle provides necessary conditions for the maximization of a functional of the form

$$V(x_o,t_o,u(t)) = {}_{t_0}\int^{T} L(x(t),u(t),t)dt+k(x(T),T) \qquad (A.1)$$

where u(t) is a vector of <u>control</u> variables and x(t) is a vector of <u>state</u> variables whose time path is generated by vector differential equation

$$\dot{x}(t) = f(x(t),u(t),t) \qquad x(t_o) = x_o \qquad (A.2)$$

The functions L and F above are assumed to be continuously differentiable in all their arguments. The function k is assumed to be twice differentiable in its arguments. We are concerned with a version of the above problem called the <u>free end point</u> problem. This term refers to the fact that there is no prescribed final value or target for the state vector x(T).

The necessary conditions for a maximum value of V take the form of a set of differential equations. These necessary

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[1]This particular presentation of the Maximum Principle is found in Athans, M., and Falb, P.L., <u>Optimal Control</u> (New York: McGraw-Hill, 1966), pp. 254-267.

equations are established by first forming the Hamiltonian with the inclusion of a new vector of variables z(t) called the adjoint variables.

$$H(x(t),z(t),u(t),t) = L(x(t),u(t),t) + z'(t)f(x(t),u(t),t)$$

The Hamiltonian is the sum of L, the integrand of the functional and the product of the transposed adjoint vector $z'(t)$ and $\dot{x}(t)$. The necessary conditions can now be written

i) $x^*(t) = \frac{\partial H}{\partial z}(x^*(t),z^*(t),u^*(t),t) \qquad x(t_o) = x_o$

ii) $z^*(t) = -\frac{\partial H}{\partial x}(x^*(t),z^*(t),u^*(t),t)$

iii) $u^*(t)$ maximizes $H(x^*(t),z^*(t),u(t),t)$ at every instant for $t_o \leq t \leq T$

iv) $z^*(T) = \frac{\partial k}{\partial x}(x^*(T),T)$

The first three conditions yield a system of differential equations specified up to some arbitrary constants. Condition iv) called the transversality condition provides additional boundary conditions sufficient to establish a specific trajectory (or trajectories) which satisfy the necessary differential equations. If there is no residual of the function a time T (i.e., $k \equiv 0$) we have the result that $z(T) = 0$ as a terminal boundary condition. From a practical standpoint these terminal values of the adjoint variables generally correspond to a unique set of initial values $z(0)$ of the adjoint variables. Finding a solution to the necessary differential equations therefore consists of finding that

unknown set of initial adjoint variables which will generate trajectories meeting the terminal conditions. Except in very simple cases this two point boundary value problem cannot be solved explicitly. Frequently the asymptotic behavior of optimal paths (if they exist) can be determined without explicit solution of the necessary differential equations.

The problem as posed does not strictly apply to infinite time horizons (i.e., $T \to \infty$). Pontryagin et al. argue that the necessary conditions will still apply for maximizing an indefinite integral as long as the integral converges and the terminal transversality conditions are defined as limits as $T \to \infty$.[2] In most economic problems convergence of the integral is assured by the presence of a discounting factor in the definite integral. All of the models in this study exhibited convergent indefinite integrals. The transversality condition for the problem stated above under an infinite time horizon becomes $\lim z(T) = \frac{\partial k}{\partial x}(x^*(T),T)$ as $T \to \infty$. For the no scrap value case $k = 0$ the terminal transversality condition becomes $\lim z(T) = 0$.

Optimization with State Constraints[3]

Frequently it is required that the functional (Equation

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[2]L.S. Pontryagin, V. Boltyanskii, R. Gamkrelidze and E. Mischenko, The Mathematical Theory of Optimal Processes (New York: Interscience Publishers, Inc., 1962), pp. 189-90.

[3]This discussion is based on Chapter VI of Pontryagin, ibid., pp. 257-316.

A.1) be maximized subject to additional constraints on the state variables. We may generally write this vector of constraints in the form

$$g(x(t)) \leq 0$$

Pontryagin argues that any portion of an optimal path not in contact with the constraint $g(x(t)) = 0$ must satisfy necessary conditions provided by the Maximum Principle.

When the constraint is binding the optimal path must meet the following necessary conditions:

i) $\dot{x}^*(t) = \frac{\partial H}{\partial z}(x^*(t), z^*(t), u^*(t), t) \qquad x(t_o) = x$

ii) $\dot{z}^*(t) = -\frac{\partial H}{\partial x}(x^*(t), z^*(t), u^*(t), t)$

$$+ \lambda(t)\frac{\partial p(x^*(t), u^*(t))}{\partial x}^4$$

iii) $u^*(t)$ maximizes $H(x^*(t), z^*(t), u(t), t)$

with respect to $u(t) \varepsilon w(x(t))$

The function $p(x(t), u(t))$ seen in condition ii) is defined as $p(x(t), u(t)) = \text{grad } g(x(t)) f(x(t), u(t), t)$
The value of the Lagrangian multiplier $\lambda(t)$ in condition ii) above is given by the equation

$$\frac{\partial H(x(t), z(t), u(t), t)}{\partial u} = \lambda(t) \frac{\partial p(x(t), u(t))}{\partial u}$$

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[4] In the case explored in Chapter V of this study this condition is greatly simplified because

$$\frac{\partial p}{\partial x} = 0.$$

and $\frac{d\lambda(t)}{dt} \leq 0$.

Condition iii) states that the Hamiltonian is maximized only over the set of control variables contained in the set W(x(t)). This set W(x(t)) is defined as the set of all values of u(t) with respect to which x(t) is regular. We say that x(t) is regular with respect to u(t) if the following conditions are satisfied:

a) $p(x(t),u(t)) = 0$

b) $\frac{\partial p(x(t),u(t)}{\partial u} \neq 0$

The requirement that $p(x(t),u(t)) = 0$ essentially requires that the trajectory continues to remain on the constraint boundary. We are therefore maximizing the Hamiltonian over only the set of control variables which will keep the trajectory on the boundary. These conditions apply when the trajectory remains along the boundary. The trajectory will move off the boundary if there is a control which increases the Hamiltonian while moving the trajectory into an allowable region.

If the adjoint variables z(t) were continuous the necessary conditions for optimal trajectories would be completely specified by the above conditions and the Maximum Principle and the normal transversality conditions on z(T). Unfortunately at every junction point (a point at which the optimal trajectory strikes or departs from the boundary) the adjoint variables are not necessarily continuous. At each junction point (t_1) the optimal path must satisfy one of two

mutually exclusive conditions called the jump conditions. These conditions are either

$$z(t_1^+) = z(t_1^-)+u \text{ grad } g(x(t_1))$$

or

$$z(t_1^-)+ u \text{ grad } g(x(t_1)) = 0, u \neq 0$$

where u is some arbitrary constant.

In general the task of selecting an optimal path consists of selecting a unique trajectory from one of a family of trajectories all meeting the necessary conditions on and off the boundary and the above jump conditions. This is an almost impossible task without recourse to specific computer solutions. Even if the family of possible trajectories are known explicitly it is necessary to evaluate the value of the functional Equation A.1 along each of them to find the optimal path.

Funk and Gilbert have provided alternate necessary and possibly sufficient conditions which may eliminate some of the arbitrariness in possible optimal trajectories with state constraints.[5] They utilize an adjoint variable which is continuous, while the jump condition is handled by a non-increasing scalar multiplier.

Hamilton-Jacobi Equation

An alternate approach to maximizing the functional $v(x_o,t_o,u(t))$, is to solve the $V^*(x(t),t)$ which satisfies the

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[5]Funk and Gilbert, op. cit.

classical Hamilton-Jacobi partial differential equation. For our problem this equation takes the form[6]

$$\frac{\partial V^*}{\partial t} + H^o(x(t),\frac{\partial V^*}{\partial x},t) = 0$$

where $H^o(x(t),\frac{\partial V^*}{\partial x},t)$ is the absolute maximum value of the Hamiltonian with respect to all possible values of the control variable u(t) at time t. The particular value of u(t) which maximizes the Hamiltonian at any instant in time is designated as u^o and may generally be written

$$u^o = h(x(t),\frac{\partial V^*}{\partial x},t)$$

A solution V*(x(t),t) must further satisfy the terminal boundary conditions, specifically

$$V^*(x(T),T) = \frac{\partial k}{\partial x}(x(T),T)$$

We note that this partial differential equation is based on the assumption that the adjoint vector z(t) is equal to $\frac{\partial V^*}{\partial x}$. Kalman proves that if there is a twice differentiable V* satisfying the Hamilton-Jacobi equations then $z(t) = \frac{\partial V^*}{\partial x}(x(t),t)$.[7] We may now conclude that the adjoint variables z(t) are identical to the marginal contributions to the value of the functional of small changes in

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[6]This discussion is based on R.E. Kalman, "The Theory of Optimal Control and the Calculus of Variations," in R. Bellman, (ed.), Mathematical Optimization Techniques (Berkeley: University of California Press, 1963), pp. 309-311.

[7]Kalman, ibid, pp. 318-20.

the level of the state variables x(t) along the optimal path.

Solution of the Hamilton-Jacobi equation provides an optimal control law of the form:

$$u^{o} = h(x(t), \frac{\partial V^{*}}{\partial x}(x(t),t),t)$$

We recall that this is the value of u(t) which maximizes the Hamiltonian at any point in time. It is apparent therefore that controlling the system according to this law will generate an optimal path of x(t) and u(t). We note that if there are no exogenous disturbances to the system, the trajectory generated by this optimal control law will be identical to the optimal path found by application of Pontryagin's Maximum Principle.

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