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ORIGINAL ARTICLE

# Arbitrage theory in a market of stochastic dimension

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**Abstract**

This paper studies an equity market of stochastic dimension, where the number of assets fluctuates over time. In such a market, we develop the fundamental theorem of asset pricing, which provides the equivalence of the following statements: (i) there exists a supermartingale numéraire portfolio; (ii) each dissected market, which is of a fixed dimension between dimensional jumps, has locally finite growth; (iii) there is no arbitrage of the first kind; (iv) there exists a local martingale deflator; (v) the market is viable. We also present the optional decomposition theorem, which characterizes a given nonnegative process as the wealth process of some investment-consumption strategy. Furthermore, similar results still hold in an open market embedded in the entire market of stochastic dimension, where investors can only invest in a fixed number of large capitalization stocks. These results are developed in an equity market model where the price process is given by a piecewise continuous semimartingale of stochastic dimension. Without the continuity assumption on the price process, we present similar results but without explicit characterization of the numéraire portfolio.

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**KEYWORDS**
fundamental theorem of asset pricing, local martingale deflator, market viability, numéraire portfolio, open market, optional decomposition theorem, piecewise semimartingale, superhedging

## 1 | INTRODUCTION

Equity markets are usually modeled as *closed*, that is, the total number of assets is assumed to be constant (say $n \in \mathbb{N}$) over time such that entrance to or exit from the market of an individual asset is prohibited. The evolution of individual assets is then modeled by an $n$-dimensional stochastic process, for example, geometric Brownian motion, Itô process, or more general semimartingale. However, this equity market model compels investors to put their money into the immutable space of investable assets, even though the number and composition of companies in actual markets fluctuate over time. In fact, the U.S. stock exchanges experienced dimensional changes quite frequently over the recent decades due to IPOs, bankruptcies, privatizations, and so forth; New York Stock Exchange (NYSE) and American Stock Exchange (AMEX) underwent dimensional changes every 1.68 and 2.57 trading days, respectively, on average, over the last 40 years from 1982 to 2021 (see Section 4.3 of Bayraktar et al., 2023 for more details).

In order to better simulate investors' behavior in real equity markets than in the closed market model, the concept of *open* markets is introduced in Fernholz (2018), and widely studied in Karatzas and Kim (2021). An open market consists of a fixed number (say $m < n$) of higher capitalization stocks within a wider equity universe. Though the dimension of the open market is also assumed to be fixed, it is open in the sense that the constituents change over time according to their capitalizations; it is similar to high-capitalization indexes, in which a stock is replaced when its capitalization falls too low.

However, both the closed and open market models fail to capture the intrinsic property of the real stock market, where new stocks enter as a result of IPOs or spin-offs, and old stocks exit following bankruptcies, privatization, or mergers and acquisitions. To this end, this paper and our companion paper, Bayraktar et al. (2023), study a stock market model of changing dimensions, using the concept of piecewise semimartingales introduced in Strong (2014). Whereas the companion paper focuses on how dimensional changes of the market impact performances of self-financing stock portfolios with empirical results, this paper develops more theoretical aspects of arbitrage theory in the setting of equity market with changing number of assets (thus we refer to Bayraktar et al., 2023 for those who are more interested in practical aspects).

Strong (2014) defines the piecewise semimartingale of stochastic dimension and its stochastic integration, by patching several pieces (or *dissections*) of finite-dimensional semimartingales of different dimensions. Two fundamental theorems of asset pricing (FTAPs) are then developed in the market of stochastic dimension; the equivalence of no free lunch with vanishing risk (NFLVR) to the existence of an equivalent sigma-martingale measure (E$\sigma$MM) for the price process (Delbaen & Schachermayer, 1998), and the equivalence of no arbitrage of the first kind ($\text{NA}_1$) to the existence of a local martingale deflator (ELMD) for the set of nonnegative wealth processes (Kardaras, 2012).

Our paper uses the same notion of piecewise semimartingale to model an equity market, but further extends the latter FTAP ($NA_1 \iff$ ELMD) by establishing most of the results presented in the recent monograph by Karatzas and Kardaras (2021) under two different settings.

First, we assume that each component of the price process is continuous and strictly positive between each dimensional jump. Under this assumption, we present the so-called fundamental theorem (see Theorem 3.14), which is a complete version of the aforementioned FTAP. The central part of this result is to show the existence of the numéraire portfolio (roughly speaking, a portfolio which cannot be outperformed) under $NA_1$ or ELMD. If it exists, the numéraire portfolio can be characterized in terms of local drift and covariation rates of each *dissected market*, that is, each "piece" of the market with a fixed dimension between the dimensional changes. From this characterization, which we call the *structural condition* of the dissected market, we derive that the numéraire portfolio is optimal, in the sense that it attains maximal growth rate and has relative log-optimal property. In particular, the maximal growth rate of the numéraire portfolio implies that each dissected market must have finite local growth if such a portfolio exists. This growth condition on every dissected market is then connected with the market viability, another notion of no-arbitrage; funding a nontrivial cumulative capital withdrawal stream should not be possible from an arbitrarily small amount of initial wealth.

In addition to the fundamental theorem, we also show the optional decomposition theorem (ODT) under the same assumption on the price process. It provides a local martingale (supermartingale) formulation for a nonnegative process, which can be expressed as a stochastic integral (minus a nondecreasing component, respectively) of some investment strategy with respect to the price process of stochastic dimension, under the equivalent conditions of the fundamental theorem.

Moreover, we study an open market embedded in the entire market of stochastic dimension. When investors are only allowed to invest in a fixed number of large capitalization stocks, while the total number of assets in the entire market fluctuates, both the fundamental theorem and the ODT are presented. In order to handle the embedded open market, we combine the idea of *dissecting* the market with the idea of *censoring* the return process from Karatzas and Kim (2021).

The assumption of continuity in the price process is useful to have a complete and explicit treatment of arbitrage theory, including an open market within a larger market of stochastic dimension. It allows us to carry out the arbitrage theory by means of portfolios and leads to explicit characterizations of the numéraire portfolio, the set of local martingale deflators, and the investment strategy, which appears in the ODT, in a pedagogically illustrative way.

Subsequently, in the second main part of the paper, we consider a more general market of stochastic dimension, relaxing the continuity condition on the price process such that each component is only assumed to be a right-continuous with left limits (RCLL) semimartingale between the dimensional changes. When jumps are present in the dynamics of asset prices, two notions of numéraire (supermartingale numéraire and local martingale numéraire) portfolios may not coincide (see Kabanov et al., 2016 and Takaoka & Schweizer, 2014), hence one is not able to exploit the explicit characterization of the numéraire portfolio anymore in developing the arbitrage theory. Thus, the results obtained in the second part, while being more general, are less explicit than the ones in the first part of the paper.

Moreover, Karatzas and Kardaras (2015) noted that the same proof technique does not extend in a nontrivial way to prove the ODT if the price process contains jumps. Hence, we adopt a different method to prove the ODT first, relying on the classical version of ODT from Stricker and Yan (1998). We then apply the ODT for establishing the fundamental theorem; the essential part is again to prove the existence of a wealth process having the supermartingale numéraire property,

under the other equivalent conditions (e.g., the ELMD). Finally, as further applications of the ODT, we present in full generality the superhedging duality and the second FTAP, which provides an equivalent condition for the completeness of the market.

*Preview:* This paper is organized as follows. Section 2 reviews the concept of piecewise semimartingale and its stochastic integration from Strong (2014). Section 3 studies a market of stochastic dimension, where the asset price is assumed to be continuous and strictly positive piecewise semimartingale; once we introduce relevant definitions and preliminary results in succession, the fundamental theorem is presented. We then provide the ODT and study the open market embedded in the entire market. In Section 4, under a more general equity market model with a right-continuous price process, we develop similar results of the ODT and the fundamental theorem, but with different arguments. We also present the superhedging duality and the second FTAP under the general model.

## 2 | PIECEWISE SEMIMARTINGALES

This section reviews the notion of piecewise semimartingales of stochastic dimension, introduced by Strong (2014). We provide the definitions, notations, and summary of the results from Section 2 of Strong (2014), which are useful to develop a full-fledged arbitrage theory for a market with stochastic dimension in the later sections.

We consider a state space $\mathbb{U} := \cup_{n=1}^{\infty} \mathbb{R}^n$, equipped with the topology generated by the union of the standard topologies of $\mathbb{R}^n$. Besides the $n$-dimensional zeros $0^{(n)} \in \mathbb{R}^n$ for each $n \in \mathbb{N}$, we define an additive identity element $\odot$, a topologically isolated point in $\hat{\mathbb{U}} := \mathbb{U} \cup \{\odot\}$ satisfying $\odot + x = x + \odot = x$ and $\odot x = x\odot = \odot$ for each $x \in \hat{\mathbb{U}}$. We define the modified indicator

$$\hat{\mathbb{1}}_A(t,\omega) := \begin{cases} 1 \in \mathbb{R} & \text{for } (t,\omega) \in A \subset [0,\infty) \times \Omega, \\ \odot & \text{otherwise,} \end{cases} \tag{1}$$

which will be useful for dissecting stochastic processes. In order for expressions involving $\hat{\mathbb{1}}$ to have the correct dimension in $\mathbb{U}$, we shall add a zero vector $0^{(n)}$ of an appropriate dimension $n$. Moreover, $1^{(n)}$ denotes the $n$-dimensional vector of ones, and $\mathbb{1}_A$ is the usual indicator function for set $A$.

We use the notations $\mathbb{R}_+ := [0,\infty)$ and $B^\top$ the transpose of a matrix $B$. For any stochastic processes $Y$ and $Z$, the identity $Y = Z$ means that $Y$ and $Z$ are indistinguishable, $Y^\alpha(\cdot) := Y(\alpha \wedge \cdot)$ denotes the process stopped at a random time $\alpha$. All relationships among random variables are understood to hold almost surely. We shall denote the set $[n] := \{1, \dots, n\}$ for every $n \in \mathbb{N}$.

On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ satisfying the usual conditions, let $X$ be a $\mathbb{U}$-valued progressive process having paths with left and right limits at all times. We denote $N := \dim X$ the dimension process of $X$ also having paths with left and right limits at all times. The following definition characterizes time instants of dimensional jumps for a given $\mathbb{U}$-valued process $X$, as a sequence of stopping times.

**Definition 2.1** (Reset sequence). A sequence of stopping times $(\tau_k)_{k\geq 0}$ is called a *reset sequence* for a progressive $\mathbb{U}$-valued process $X$, if the following hold for $\mathbb{P}$-a.e. $\omega$:

(i) $\tau_0(\omega) = 0$, $\tau_{k-1}(\omega) \leq \tau_k(\omega)$ for all $k \in \mathbb{N}$, and $\lim_{k\to\infty} \tau_k(\omega) = \infty$;

(ii) $N(t,\omega) = N(\tau_{k-1}+,\omega)$ for every $t \in (\tau_{k-1}(\omega), \tau_k(\omega)]$ and $k \in \mathbb{N}$;
(iii) $t \mapsto X(t,\omega)$ is right-continuous on $(\tau_{k-1}(\omega), \tau_k(\omega))$ for every $k \in \mathbb{N}$.

When $X$ has a reset sequence $(\tau_k)_{k\geq 0}$, we shall always consider the minimal one $(\hat{\tau}_k)_{k\geq 0}$ in the sense of the fewest resets by a given time:

$$\hat{\tau}_0 := 0, \qquad \hat{\tau}_k := \inf\{t > \hat{\tau}_{k-1} \ : \ X(t+) \neq X(t)\}, \quad k \in \mathbb{N},$$

and assume that the initial dimension is deterministic, that is, $\dim(X(0)) = N_0 \in \mathbb{N}$.

We also emphasize here that there is no restriction on the size of each dimensional change, in the sense that the quantity $|N(\tau_k+) - N(\tau_{k-1}+)|$ can be bigger than or equal to one for each $k \in \mathbb{N}$. This allows us to include several equity market models with a changing number of assets in our setting. For example, the diverse market model of Karatzas and Sarantsev (2016) considers a particular form of splits and mergers between companies; the largest company is split into two companies (modeling a regulatory breakup) and any two of the existent companies merge into one at random times. In their model, the size of each dimensional jump is always one.

In what follows, we fix such $\mathbb{U}$-valued process $X$ with the reset sequence $(\tau_k)_{k\geq 0}$, and define the *dissections* of $\Omega$ and $X$ for every pair $(k,n) \in \mathbb{N}^2$

$$\Omega^{k,n} := \{\tau_{k-1} < \infty, \ N(\tau_{k-1}+) = n\} \subset \Omega, \tag{2}$$

$$X^{k,n} := (X^{\tau_k} - X(\tau_{k-1}+))\hat{\mathbb{1}}_{\rrbracket \tau_{k-1},\infty \rrbracket \cap (\mathbb{R}_+ \times \Omega^{k,n})} + 0^{(n)}. \tag{3}$$

We now introduce piecewise semimartingales as integrators of stochastic integrals, with an appropriate class of integrands. The $(k,n)$-dissection $X^{k,n}$ of $X$, defined in Equation (3), is appropriate when $X$ plays the role of integrator. For integrands, a different definition of dissection is necessary.

**Definition 2.2** (Piecewise semimartingale). A *piecewise semimartingale* $X$ is a $\mathbb{U}$-valued progressive process having paths with left and right limits at all times, and possessing a reset sequence $(\tau_k)_{k\geq 0}$ such that $X^{k,n}$ is an $\mathbb{R}^n$-valued semimartingale for every $(k,n) \in \mathbb{N}^2$.

A piecewise semimartingale $X$ is called *piecewise continuous (RCLL) semimartingale*, if each dissection $X^{k,n}$ is an $n$-dimensional continuous (RCLL, respectively) semimartingale for every $(k,n) \in \mathbb{N}^2$.

We note that even though the piecewise semimartingale $X$ is continuous, an existing component $X_i$ can experience a right-discontinuous jump *at* a reset time $\tau_k$, that is, $X_i(\tau_k) \neq X_i(\tau_k+)$, but every component of $X$ should be continuous *between* two consecutive reset times. We also note that any right-discontinuities of $X$ do not affect the value of the stochastic integral, defined in Equation (6) below.

**Definition 2.3** (Stochastic integral). For a piecewise semimartingale $X$ and its reset sequence $(\tau_k)_{k\geq 0}$, let $H$ be a $\mathbb{U}$-valued predictable process satisfying $\dim H = N = \dim X$. We dissect $H$ in the following manner:

$$H^{(k,n)} := H\hat{\mathbb{1}}_{\llbracket \tau_{k-1},\tau_k \rrbracket \cap (\mathbb{R}_+ \times \Omega^{k,n})} + 0^{(n)}, \qquad \forall (k,n) \in \mathbb{N}^2, \tag{4}$$

and define

$$\mathcal{L}(X) := \{H \text{ predictable } : \dim H = N \text{ and } H^{(k,n)} \text{ is } X^{k,n}\text{-integrable, } \forall (k,n) \in \mathbb{N}^2\}, \tag{5}$$
$$\mathcal{L}_0(X) := \{H \in \mathcal{L}(X) \ : \ H_0 = 0^{(N_0)}\}.$$

For $H \in \mathcal{L}(X)$, the stochastic integral $H \cdot X$ is defined as

$$H \cdot X := H_0^\top X_0 + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (H^{(k,n)} \cdot X^{k,n}). \tag{6}$$

Note that each dissection $H^{(k,n)}$ of Equation (4) is predictable, since the process $H$ and the $(k,n)$-dissection set $[\![\tau_{k-1}, \tau_k]\!] \cap (\mathbb{R}_+ \times \Omega^{k,n})$ are predictable. We also note that Strong (2014) uses the same notation for two dissections $X^{k,n}$ in Equation (3) and $H^{(k,n)}$ in Equation (4), whereas we shall differentiate the two notations throughout this paper. The stochastic integral of Equation (6) generalizes the usual $\mathbb{R}^n$-valued semimartingale stochastic integration, since any increasing sequence $\tau_k \uparrow \infty$ of stopping times is a reset sequence for any semimartingales of fixed dimension. We use interchangeably the notations

$$H \cdot X = \int H dX, \qquad H^{(k,n)} \cdot X^{k,n} = \int H^{(k,n)} dX^{k,n} = \int \sum_{i=1}^{n} H_i^{(k,n)} dX_i^{k,n},$$

for the stochastic integrals of Equation (6). Although we shall not make direct use of the following result, we summarize some of the properties of the stochastic integral from Strong (2014).

**Proposition 2.4.** *Let $X$ be a piecewise semimartingale with reset sequence $(\tau_k)_{k\geq 0}$ and $H, G \in \mathcal{L}(X)$.*

*(i) If $X$ has another reset sequence $(\tilde{\tau}_k)_{k\geq 0}$, then $\tilde{X}^{k,n}$ defined via Equation (3) is an $\mathbb{R}^n$-valued semimartingale for every $(k,n) \in \mathbb{N}^2$. The class $\mathcal{L}(X)$ and the process $H \cdot X$ do not depend on the choice of reset sequence of $X$.*
*(ii) The stochastic integral $H \cdot X$ is an $\mathbb{R}$-valued semimartingale.*
*(iii) $\mathcal{L}(X)$ is a vector space; $H \cdot X + G \cdot X = (H + G) \cdot X$.*
*(iv) For any stopping time $\alpha$, the stopped process $X^\alpha$ is a piecewise semimartingale and the identity $(X^\alpha)^{k,n} = (X^{k,n})^\alpha$ holds for every $(k,n) \in \mathbb{N}^2$.*

## 3 | MARKET WITH CONTINUOUS, POSITIVE PRICE PROCESS

In this section, we develop a full-fledged arbitrage theory in a market of stochastic dimension, when a stock price is modeled as a $\mathbb{U}$-valued piecewise *continuous* semimartingale. This means that between two consecutive dimensional jumps $\tau_{k-1}$ and $\tau_k$, each $n$-dimensional piece (or dissection) of the price process remains *continuous* for every $(k,n) \in \mathbb{N}^2$. Under this setting, we ultimately present the fundamental theorem (Section 3.5) and the ODT (Section 3.7).

### 3.1 | Price process, return process, and portfolio

Using the concept of piecewise semimartingale from Section 2, we describe in this subsection an equity market with a stochastic number of investable assets.

**Definition 3.1** (Price process). A $\mathbb{U}$-valued piecewise continuous semimartingale $S$ is called a *price process* (of stochastic dimension), if every component of $S$ is strictly positive on each dissection set $[\![\tau_{k-1},\tau_k[\![ \cap (\mathbb{R}_+ \times \Omega^{k,n})$ for every $(k,n) \in \mathbb{N}^2$.

The dimension process $N = \dim(S)$ of $S$ represents the number of companies present in the market, and the $n$ components of $S$ on the set $[\![\tau_{k-1},\tau_k[\![ \cap (\mathbb{R}_+ \times \Omega^{k,n})$ represent the discounted stock prices (or the capitalizations) of the $n$ existent companies, for every $(k,n) \in \mathbb{N}^2$. In order to simplify the model, we assume that every stock has a single outstanding share, so that the price of a stock is equal to its capitalization. By definition, every $\cup_{n=1}^{\infty}(0,\infty)^n$-valued piecewise continuous semimartingale can be a price process. We note that the strict positivity of the price process is essential to define the return process in the following.

**Definition 3.2** (Return process). For a given price process $S$, we call $R$ a *return process*, if $R$ is a $\mathbb{U}$-valued piecewise continuous semimartingale with $\dim(R) = N = \dim(S)$, and its dissection is given by

$$R_i^{k,n}(t) = \int_0^t \left( \frac{1}{S_i(u)} \hat{\mathbb{1}}_{[\![\tau_{k-1},\tau_k[\![ \cap (\mathbb{R}_+ \times \Omega^{k,n})} + 0^{(1)} \right) dS_i^{k,n}(u), \quad t \geq 0, \tag{7}$$

for every $i \in [n]$ and $(k,n) \in \mathbb{N}^2$.

Since $S^{k,n}$ is constant out of the interval $[\![\tau_{k-1},\tau_k[\![$, and the increments $dS_i^{k,n}$ and $dS_i$ coincide on the dissection set $[\![\tau_{k-1},\tau_k[\![ \cap (\mathbb{R}_+ \times \Omega^{k,n})$ from the definition (3), the identity of Equation (7) can be loosely rewritten as

$$R_i^{k,n}(t) = \int_0^t \frac{1}{S_i(u)}\, dS_i^{k,n}(u) = \int_0^t \left( \frac{1}{S_i(u)} \hat{\mathbb{1}}_{[\![\tau_{k-1},\tau_k[\![ \cap (\mathbb{R}_+ \times \Omega^{k,n})} + 0^{(1)} \right) dS_i(u),$$

which is reminiscent of the fact that the return of a stock is defined as a stochastic logarithm of its price process.

The quantity $R_i^{k,n}(t)$ can be interpreted as the cumulative return of the $i$th stock until time $t$ when there are $n$ stocks extant between $(k-1)$th and $k$th dimensional changes of the market. We note that each value of $R^{k,n}$ is accumulated only on the interval $[\![\tau_{k-1},\tau_k[\![$ by definition, thus $R_i^{k,n} \equiv 0$ on $[\![0,\tau_{k-1}]\!]$ and $R_i^{k,n} \equiv R_i^{k,n}(\tau_k)$ on $[\![\tau_k,\infty]\!]$. Moreover, $R_i^{k,n}$ and $R_i^{k+1,m}$ may indicate cumulative returns of different companies for $i = 1,\dots,\min(n,m)$ because the indexing of the stocks in the market can be inconsistent between each dimensional change; for example, if the $i$th company exits the market, the $(i+1)$st company inherits the index $i$ from the next epoch, and so on.

We consider the canonical decomposition of each $n$-dimensional semimartingale

$$R_i^{k,n} := A_i^{k,n} + M_i^{k,n} \tag{8}$$

such that $A_i^{k,n}$ is of finite variation, $M_i^{k,n}$ is a local martingale for every $i \in [n]$ and $(k,n) \in \mathbb{N}^2$. Recalling the notation (5), we define portfolios in this market.

**Definition 3.3** (Portfolio). For a given return process $R$, we call $\pi$ a *portfolio*, if $\pi \in \mathcal{L}(R)$. The (cumulative) return process of $\pi$ is defined by $R_\pi := \pi \cdot R$, and the wealth process of $\pi$ is given by

$$X_\pi := \mathcal{E}(R_\pi), \tag{9}$$

where $\mathcal{E}(Z) := \exp(Z - \frac{1}{2}[Z,Z])$ denotes the stochastic exponential of a scalar continuous semimartingale $Z$ with $Z(0) = 0$. A portfolio $\pi$ is called *null portfolio* if its return process is a zero process, that is, $R_\pi \equiv 0$.

In Definition 3.2, there was no condition on the initial vector $R_0$ for the return process, but from now on, we shall assume $R_0 = 0^{(N_0)}$, in order to have the identity $X_\pi(0) = 1$, thus the wealth of any portfolio $\pi$ is always normalized at the initial time. For each $(k,n) \in \mathbb{N}^2$, the component $\pi_i^{(k,n)}$ of dissection of $\pi$ represents the proportion of wealth invested in the $i$th company among $n$ companies, and

$$\pi_0^{(k,n)} = 1 - \sum_{j=1}^{n} \pi_j^{(k,n)}, \tag{10}$$

is the proportion of capital invested in the money market. Here, we assume that the money market earns no interest. Moreover, when the dimension of the market changes from $n$ (in the $k$th epoch) to $m$ (in the $(k+1)$st epoch), redistribution of wealth from $\pi^{(k,n)}$ to $\pi^{(k+1,m)}$ is financed by the money market such that the corresponding portion of wealth changes from $\pi_0^{(k,n)}$ to $\pi_0^{(k+1,m)}$.

Recalling Equation (6), the return process of $\pi$ can be dissected as

$$R_\pi = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \pi^{(k,n)} \cdot R^{k,n} = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} R_\pi^{k,n}, \qquad \text{where} \quad R_\pi^{k,n} := \pi^{(k,n)} \cdot R^{k,n}. \tag{11}$$

We note that the double sum in Equation (11) is a finite sum for each time point and the same is true for the sums of this type that follow.

Given two portfolios $\pi, \rho \in \mathcal{L}(R)$, we denote $C_{\pi\rho}$ the covariation process between the cumulative returns $R_\pi, R_\rho$

$$C_{\pi\rho} \equiv [R_\pi, R_\rho] := \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} C_{\pi\rho}^{k,n}, \tag{12}$$

where

$$C_{\pi\rho}^{k,n} := [R_\pi^{k,n}, R_\rho^{k,n}] = \int_0^{\cdot} \sum_{i=1}^{n}\sum_{j=1}^{n} \pi_i^{(k,n)}(s)\rho_j^{(k,n)}(s)\, d[R_i^{k,n}, R_j^{k,n}](s). \tag{13}$$

Moreover, we write for every $i \in [n]$ and $(k,n) \in \mathbb{N}^2$

$$C_{i\rho}^{k,n} := [R_i^{k,n}, R_\rho^{k,n}]. \tag{14}$$

This notation is justified as $C^{k,n}_{i\rho} \equiv C^{k,n}_{\nu\rho}$ when a portfolio $\nu$ invests all its wealth in the $i$th stock, if at least $i$ stocks are present in the market; that is, $\nu$ is defined via its dissection for a fixed $i \in \mathbb{N}$

$$\nu := 0^{(N_0)} + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \hat{\mathbb{1}}_{\rrbracket \tau_{k-1},\tau_k \rrbracket \cap (\mathbb{R}_+ \times \Omega^{k,n})} \nu^{(k,n)}, \qquad \text{where} \tag{15}$$

$$\nu^{(k,n)} := e^i \hat{\mathbb{1}}_{\rrbracket \tau_{k-1},\tau_k \rrbracket \cap (\mathbb{R}_+ \times \Omega^{k,n}) \cap \{i \le n\}} + 0^{(n)}$$

where $e^i$ denotes the $n$-dimensional unit vector with the $i$th entry equal to one. It is straightforward to check $R^{k,n}_\nu = R^{k,n}_i$ and $C^{k,n}_{\nu\rho} = [R^{k,n}_i, R^{k,n}_\rho]$ whenever $i \le n$.

## 3.2 | Numéraire portfolios

For any given portfolios $\pi, \rho \in \mathcal{L}(R)$, we compare the relative performance of $\pi$ with respect to the other "baseline" portfolio $\rho$ by computing the ratio

$$X^\rho_\pi := \frac{X_\pi}{X_\rho}. \tag{16}$$

The following result gives a representation of $X^\rho_\pi$ as the stochastic exponential of $R^\rho_\pi$, which we call *relative (cumulative) return of* $\pi$ with respect to $\rho$.

**Lemma 3.4.** *For every* $(k,n) \in \mathbb{N}^2$, *recalling the notations (11), (13), and (14), we define*

$$R^{\rho,k,n}_0 := C^{k,n}_{\rho\rho} - R^{k,n}_\rho, \tag{17}$$

$$R^{\rho,k,n}_i := R^{\rho,k,n}_0 + (R^{k,n}_i - C^{k,n}_{i\rho}), \qquad i \in [n], \tag{18}$$

*and*

$$R^\rho_\pi := \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \int_0^\cdot \sum_{i=0}^{n} \pi^{(k,n)}_i(s)\, dR^{\rho,k,n}_i(s). \tag{19}$$

*Then, we have the representation of* $X^\rho_\pi$ *in Equation (16)*

$$X^\rho_\pi = \mathcal{E}(R^\rho_\pi). \tag{20}$$

*Proof.* Using the definitions (17)–(20), (14), and (13), the integral on the right-hand side of Equation (19) can be written as

$$\begin{aligned}\int_0^\cdot \sum_{i=0}^{n} \pi^{(k,n)}_i(s)\, dR^{\rho,k,n}_i(s) &= R^{\rho,k,n}_0(\cdot) + \int_0^\cdot \sum_{i=1}^{n} \pi^{(k,n)}_i(s)\, dR^{k,n}_i(s) - \int_0^\cdot \sum_{i=1}^{n} \pi^{(k,n)}_i(s)\, dC^{k,n}_{i\rho}(s)\\ &= R^{\rho,k,n}_0 + R^{k,n}_\pi - C^{k,n}_{\pi\rho} = C^{k,n}_{\rho\rho} - R^{k,n}_\rho + R^{k,n}_\pi - C^{k,n}_{\pi\rho}\\ &= C^{k,n}_{(\rho-\pi)\rho} - R^{k,n}_{\rho-\pi}.\end{aligned} \tag{21}$$

Thus, we derive

$$\log\left(\mathcal{E}(R_\pi^\rho)\right) = R_\pi^\rho - \frac{1}{2}[R_\pi^\rho, R_\pi^\rho] = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left(C^{k,n}_{(\rho-\pi)\rho} - R^{k,n}_{\rho-\pi} - \frac{1}{2}[R^{k,n}_{\rho-\pi}, R^{k,n}_{\rho-\pi}]\right)$$
$$= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left(R^{k,n}_{\pi-\rho} - \frac{1}{2}C^{k,n}_{\pi\pi} + \frac{1}{2}C^{k,n}_{\rho\rho}\right) = R_{\pi-\rho} - \frac{1}{2}C_{\pi\pi} + \frac{1}{2}C_{\rho\rho}.$$

Here, the second identity uses the fact $[R_\rho^{k,n}, R_\rho^{\ell,m}] = [R_\pi^{k,n}, R_\pi^{\ell,m}] = [R_\rho^{k,n}, R_\pi^{\ell,m}] \equiv 0$ whenever $(k,n) \neq (\ell,m)$, from the property of dissection defined in Equation (11).

On the other hand, we have

$$\log(X_\pi^\rho) = \log\left(\mathcal{E}(R_\pi)\right) - \log\left(\mathcal{E}(R_\rho)\right) = \left(R_\pi - \frac{1}{2}C_{\pi\pi}\right) - \left(R_\rho - \frac{1}{2}C_{\rho\rho}\right),$$

thus the result follows. □

**Definition 3.5** (Numéraire portfolio). A portfolio $\rho \in \mathcal{L}(R)$ is called *supermartingale (local martingale) numéraire portfolio*, if the relative wealth process $X_\pi^\rho$ is a supermartingale (local martingale, respectively) for every portfolio $\pi \in \mathcal{L}(R)$ in the market.

If a supermartingale (or local martingale) numéraire portfolio $\rho$ exists, then it is unique modulo null portfolios (Lemma 3.3 of Karatzas & Kim, 2021). In this case, the wealth process $X_\rho$ is called a supermartingale (or local martingale) numéraire. The next result further shows that two numéraire portfolios are actually the same, whenever they exist, and gives an equivalent characterization.

**Proposition 3.6.** *For a portfolio $\rho \in \mathcal{L}(R)$, the following statements are equivalent:*

*(i)* *$\rho$ is a supermartingale numéraire portfolio.*
*(ii)* *$\rho$ is a local martingale numéraire portfolio.*
*(iii)* *$A_i^{k,n} = C_{i\rho}^{k,n}$ holds for every $i \in [n]$ and $(k,n) \in \mathbb{N}^2$.*

*Proof.* We first show that *(iii)* implies *(ii)*. Thanks to the decomposition (8), $R_i^{k,n} - C_{i\rho}^{k,n} = M_i^{k,n}$ is a local martingale for every $i \in [n]$ and $(k,n) \in \mathbb{N}^2$. Moreover, we deduce from Equation (17) that

$$R_0^{\rho,k,n} = C_{\rho\rho}^{k,n} - R_\rho^{k,n} = [R_\rho^{k,n}, R_\rho^{k,n}] - R_\rho^{k,n} = \left[\int_0^\cdot \sum_{i=1}^n \rho_i^{(k,n)}(s)\, dR_i^{k,n}(s), R_\rho^{k,n}\right] - R_\rho^{k,n} \tag{22}$$
$$= \int_0^\cdot \sum_{i=1}^n \rho_i^{(k,n)}(s)\, dC_{i\rho}^{k,n}(s) - \int_0^\cdot \sum_{i=1}^n \rho_i^{(k,n)}(s)\, dR_i^{k,n}(s) = -\int_0^\cdot \sum_{i=1}^n \rho_i^{(k,n)}(s)\, dM_i^{k,n}(s).$$

Thus, $R_0^{\rho,k,n}$ and every $R_i^{\rho,k,n}$ of Equation (18) are local martingales. Every integral of Equation (19) is then a local martingale, and we apply Lemma 2.12 of Strong (2014) (as in the proof of Theorem 2.15, Corollary 2.16 of Strong, 2014) to conclude that $R_\pi^\rho$, and also $X_\pi^\rho$, are local martingales.

The implication $(ii) \Longrightarrow (i)$ is trivial, since every nonnegative local martingale is a supermartingale by Fatou's lemma.

We now assume $(i)$. Let us fix $(i,j) \in \mathbb{N}^2$ satisfying $i \leq j$, and construct a portfolio $\nu_{(i,j)}$ via dissection

$$\nu_{(i,j)} := 0^{(N_0)} + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \hat{\mathbb{1}}_{\rrbracket \tau_{k-1},\tau_k \rrbracket \cap (\mathbb{R}_+ \times \Omega^{k,n})} \nu_{(i,j)}^{(k,n)}, \qquad \text{where}$$

$$\nu_{(i,j)}^{(k,n)} := e^i \hat{\mathbb{1}}_{\rrbracket \tau_{k-1},\tau_k \rrbracket \cap (\mathbb{R}_+ \times \Omega^{k,n}) \cap \{j \leq n\}} + 0^{(n)}. \tag{23}$$

This portfolio $\nu_{(i,j)}$ is a generalization of the portfolio $\nu$ depending on a single index $i \in \mathbb{N}$, defined in Equation (15). It allocates all wealth to the $i$th stock, if there exist more than or equal to $j$ $(\geq i)$ stocks in the market; otherwise, it invests all wealth into the money market.

From the assumption $(i)$, $X^{\rho}_{\rho+\nu_{(i,j)}}$ and $X^{\rho}_{\rho-\nu_{(i,j)}}$ are supermartingales and their stochastic logarithms $R^{\rho}_{\rho+\nu_{(i,j)}}$ and $R^{\rho}_{\rho-\nu_{(i,j)}}$ in Lemma 3.4 are then local supermartingales. We compute

$$R^{\rho}_{\rho+\nu_{(i,j)}} = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \int_0^{\cdot} \sum_{\ell=0}^{n} \left( \rho_{\ell}^{(k,n)}(s) + \nu_{(i,j),\ell}^{(k,n)}(s) \right) dR_{\ell}^{\rho,k,n}(s)$$

via Equation (19). Using the definitions (17)–(19), it is easy to show that

$$\int_0^{\cdot} \sum_{\ell=0}^{n} \rho_{\ell}^{(k,n)}(s)\, dR_{\ell}^{\rho,k,n}(s) = 0$$

holds, thus we have $R^{\rho}_{\rho-\nu_{(i,j)}} = -R^{\rho}_{\rho+\nu_{(i,j)}}$, and both quantities are local martingales. Moreover, we obtain

$$\begin{aligned} R^{\rho}_{\rho+\nu_{(i,j)}} = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \int_0^{\cdot} \sum_{\ell=0}^{n} \nu_{(i,j),\ell}^{(k,n)}(s)\, dR_{\ell}^{\rho,k,n}(s) &= \sum_{k=1}^{\infty}\sum_{n=1}^{j-1} R_0^{\rho,k,n} + \sum_{k=1}^{\infty}\sum_{n=j}^{\infty} R_i^{\rho,k,n} \\ &= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} R_0^{\rho,k,n} + \sum_{k=1}^{\infty}\sum_{n=j}^{\infty} \left( R_i^{k,n} - C_{i\rho}^{k,n} \right). \end{aligned}$$

The stopped local martingale $(R^{\rho}_{\rho+\nu_{(i,j)}})^{\tau_k}$ is again a local martingale for every $k \in \mathbb{N}$, therefore the difference

$$(R^{\rho}_{\rho+\nu_{(i,j)}})^{\tau_k} - (R^{\rho}_{\rho+\nu_{(i,j)}})^{\tau_{k-1}} = \sum_{n=1}^{\infty} R_0^{\rho,k,n} + \sum_{n=j}^{\infty} \left( R_i^{k,n} - C_{i\rho}^{k,n} \right) =: R(i,j,k)$$

is also a local martingale for every $k \in \mathbb{N}$. Since we fixed $(i,j) \in \mathbb{N}^2$ satisfying $i \leq j$ arbitrarily, we now choose any $n \in \mathbb{N}$, $i \in [n]$, and set $j = n$ and $j = n+1$ to conclude that

$$R(i,n,k) - R(i,n+1,k) = R_i^{k,n} - C_{i\rho}^{k,n}$$

is a local martingale and the condition (iii) holds. □

### 3.3 | Structural condition of each dissected market

The condition (*iii*) of Proposition 3.6 provides an important characterization of the numéraire portfolio $\rho$; for every dissection $(R^{k,n})_{(k,n)\in\mathbb{N}^2}$ of the return process, the identity $A_i^{k,n} = C_{i\rho}^{k,n}$, derived by means of Equations (8) and (14), must hold for each component $i = 1, \dots, n$. This condition is called *structural condition*, and it is known to be connected to the maximal growth rate of the market, in the spirit of Sections 2.1.2 and 2.1.3 of Karatzas and Kardaras (2021). This subsection demonstrates such connection for every *dissected market*. First, we can reformulate the structural condition in terms of *local rates* as in the following.

Let us fix $(k, n) \in \mathbb{N}^2$, recall the decomposition (8), and denote $C^{k,n}$ the $(n \times n)$ matrix-valued process with entries

$$C_{i,j}^{k,n} := [M_i^{k,n}, M_j^{k,n}]. \tag{24}$$

We define the *dissected operational clock* by a real-valued process

$$O^{k,n} := \sum_{i=1}^{n} \int_0^{\cdot} \Big( |dA_i^{k,n}(t)| + dC_{i,i}^{k,n}(t) \Big), \tag{25}$$

where $\int |dA_i^{k,n}(t)|$ denotes the total variation of $A_i^{k,n}$. With respect to this nondecreasing process, we consider the Radon–Nikodým derivatives $\alpha^{k,n} = (\alpha_i^{k,n})_{1\le i\le n}$ and $c^{k,n} = (c_{i,j}^{k,n})_{1\le i,j\le n}$ of $A^{k,n} = (A_i^{k,n})_{1\le i\le n}$ and $C^{k,n}$, respectively,

$$A^{k,n} = \int_0^{\cdot} \alpha^{k,n}(t)\, dO^{k,n}(t), \qquad C^{k,n} = \int_0^{\cdot} c^{k,n}(t)\, dO^{k,n}(t). \tag{26}$$

These predictable processes $\alpha^{k,n}$ and $c^{k,n}$ are called *local return rate* and *local covariation rate* of the dissected market, since they are derived from the *dissected return* $R^{k,n}$. Here, we note that an $n$-dimensional process $\nu$ is $R^{k,n}$-integrable, if and only if,

$$\int_0^T \big( |\nu^\top \alpha^{k,n}| + \nu^\top c^{k,n} \nu \big)(t)\, dO^{k,n}(t) < \infty \qquad \text{holds for any } T \ge 0. \tag{27}$$

We then obtain the representation

$$\begin{aligned} C_{i\rho}^{k,n} = \left[ R_i^{k,n}, \int_0^{\cdot} \sum_{j=1}^{n} \rho_j^{(k,n)}(t)\, dR_j^{k,n}(t) \right] &= \int_0^{\cdot} \sum_{j=1}^{n} \rho_j^{(k,n)}(t)\, dC_{i,j}^{k,n}(t) \\ &= \int_0^{\cdot} \sum_{j=1}^{n} \rho_j^{(k,n)}(t) c_{i,j}^{k,n}(t)\, dO^{k,n}(t), \end{aligned}$$

thus the condition (*iii*) of Proposition 3.6 is equivalent to

$$\alpha^{k,n} = c^{k,n} \rho^{(k,n)}, \qquad (\mathbb{P} \otimes O^{k,n}) - \text{a.e.} \tag{28}$$

for every $(k, n) \in \mathbb{N}^2$. In other words, the existence of numéraire portfolio $\rho$ is equivalent to the structural condition (28) of the $(k, n)$-*dissected market* for every $(k, n) \in \mathbb{N}^2$.

In order to show that the numéraire portfolio $\rho$ has the maximal growth rate, we first need to define the growth (both cumulative and local rate) of portfolios in the market. For fixed $(k,n) \in \mathbb{N}^2$ and a portfolio $\pi \in \mathcal{L}(R)$, recalling the decomposition (8), we consider the decomposition of $R_\pi^{k,n}$ in Equation (11)

$$A_\pi^{k,n} := \pi^{(k,n)} \cdot A^{k,n} = \int_0^\cdot \sum_{i=1}^n \pi_i^{(k,n)}(t)\, dA_i^{k,n}(t), \qquad M_\pi^{k,n} := \pi^{(k,n)} \cdot M^{k,n},$$

and define

$$\Gamma_\pi^{k,n} := A_\pi^{k,n} - \frac{1}{2} C_{\pi\pi}^{k,n}. \tag{29}$$

It is then easy to derive

$$\log\left(\mathcal{E}(R_\pi^{k,n})\right) = R_\pi^{k,n} - \frac{1}{2} C_{\pi\pi}^{k,n} = \Gamma_\pi^{k,n} + M_\pi^{k,n},$$

and moreover, using the property that $R_\pi^{k,n}$ and $R_\pi^{\ell,m}$ are orthogonal, that is, $[R_\pi^{k,n}, R_\pi^{\ell,m}] \equiv 0$, whenever $(k,n) \neq (\ell,m)$, we obtain

$$X_\pi = \mathcal{E}\left(\sum_{k=1}^\infty \sum_{n=1}^\infty R_\pi^{k,n}\right) = \exp\left(\sum_{k=1}^\infty \sum_{n=1}^\infty R_\pi^{k,n} - \frac{1}{2}\sum_{k=1}^\infty \sum_{n=1}^\infty [R_\pi^{k,n}, R_\pi^{k,n}]\right) = \prod_{k=1}^\infty \prod_{n=1}^\infty \mathcal{E}\left(R_\pi^{k,n}\right).$$

Combining the last identities, we have the following representation of the log-wealth process:

$$\log(X_\pi) = \sum_{k=1}^\infty \sum_{n=1}^\infty \left(\Gamma_\pi^{k,n} + M_\pi^{k,n}\right). \tag{30}$$

We call the finite variation process $\Gamma_\pi := \sum_{k=1}^\infty \sum_{n=1}^\infty \Gamma_\pi^{k,n}$ the *cumulative growth* of $\pi$ with dissection $\Gamma_\pi^{k,n}$ for each $(k,n) \in \mathbb{N}^2$. Furthermore, if we denote $\gamma_\pi^{k,n}$ the Radon–Nikodým derivative of $\Gamma_\pi^{k,n}$ with respect to $O^{k,n}$, it is easy to verify the following relationship from Equation (29):

$$\gamma_\pi^{k,n} = (\alpha^{k,n})^\top \pi^{(k,n)} - \frac{1}{2}(\pi^{(k,n)})^\top c^{k,n} \pi^{(k,n)}, \qquad (\mathbb{P} \otimes O^{k,n}) - \text{a.e.} \tag{31}$$

We call $\gamma_\pi^{k,n}$ the *local growth rate* of the portfolio $\pi$ in the $(k,n)$-dissected market.

We now define the *maximal growth rate*

$$g^{k,n} := \sup_{p \in \mathbb{R}^n} \left((\alpha^{k,n})^\top p - \frac{1}{2} p^\top c^{k,n} p\right), \tag{32}$$

achievable in the $(k,n)$-dissected market. Note that $g^{k,n}$ is predictable, since the supremum can be taken over a countable dense subset of $\mathbb{R}^n$. Before presenting the following result, we define *pseudo-inverse* of an $(n \times n)$ matrix-valued process $c$ by

$$c^\dagger := \lim_{m \to \infty} \left(\left(c + \frac{\mathbf{id}_n}{m}\right)^{-2} c\right), \tag{33}$$

where $\mathbf{id}_n$ denotes the $n \times n$ identity matrix.

**Proposition 3.7.** *For a fixed, arbitrary pair $(k,n) \in \mathbb{N}^2$, the following statements are equivalent:*

*(i) There exists an $n$-dimensional $R^{k,n}$-integrable process $\rho^{(k,n)}$ with $\rho^{(k,n)}(0) = 0^{(n)}$ such that $\alpha^{k,n} = c^{k,n}\rho^{(k,n)}$ holds $(\mathbb{P} \otimes O^{k,n})$-a.e.*

*(ii) $\alpha^{k,n} \in \mathbf{range}(c^{k,n})$ for $(\mathbb{P} \otimes O^{k,n})$-a.e., and $\int_0^T (\alpha^{k,n}(t))^\top (c^{k,n}(t))^\dagger \alpha^{k,n}(t)\, dO^{k,n}(t) < \infty$ for every $T \geq 0$.*

*(iii) The $(k,n)$-dissected market has locally finite growth, that means, for every $T \geq 0$, we have $G^{k,n}(T) := \int_0^T g^{k,n}(t)\, dO^{k,n}(t) < \infty$.*

We refer to Sections 2.1.2 and 2.1.3 of Karatzas and Kardaras (2021) for the detailed proof of Proposition 3.7 because the same argument can be applied, if the symbols $\alpha, c, \rho, O, g, G$ there, are replaced with the ones with the superscript $k,n$ here. To provide the idea of the proof, the process

$$\rho^{(k,n)} := (c^{k,n})^\dagger \alpha^{k,n}, \qquad \text{(modulo null portfolio)} \tag{34}$$

satisfies the conditions $(i) - (iii)$ of Proposition 3.7. In this case, $\rho^{(k,n)}$ attains the maximal growth rate $g^{k,n}$ of Equation (32), which is equal to $(1/2)(\alpha^{k,n})^\top (c^{k,n})^\dagger \alpha^{k,n}$. Generally, $g^{k,n}$ of Equation (32) can be expressed as

$$g^{k,n} = \frac{1}{2}\left((\alpha^{k,n})^\top (c^{k,n})^\dagger \alpha^{k,n}\right) \mathbb{1}_{\{\alpha^{k,n} \in \mathbf{range}(c^{k,n})\}} + \infty\, \mathbb{1}_{\{\alpha^{k,n} \notin \mathbf{range}(c^{k,n})\}} \tag{35}$$

and the finiteness of $(iii)$ guarantees the $R^{k,n}$-integrability of the process $\rho^{(k,n)}$ above.

Thanks to Proposition 3.6, the numéraire portfolio exists if and only if the structural conditions of Proposition 3.7 hold for every $(k,n)$-dissected market for $(k,n) \in \mathbb{N}^2$. Then, the numéraire portfolio $\rho$ can be constructed as

$$\rho := 0^{(N_0)} + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \hat{\mathbb{1}}_{]\!]\tau_{k-1}, \tau_k]\!] \cap (\mathbb{R}_+ \times \Omega^{k,n})} \rho^{(k,n)}, \tag{36}$$

where every dissection $\rho^{(k,n)}$ satisfies the condition $(i)$ of Proposition 3.7.

## 3.4 | Arbitrage of the first kind and local martingale deflators

According to the fundamental result (Theorem 2.31 of Karatzas & Kardaras, 2021) of arbitrage theory in equity markets (with fixed number of stocks), there are more concepts related to the equivalent conditions of Propositions 3.6 and 3.7, namely the market viability (or lack of arbitrage of the first kind), and the existence of local martingale deflators. The equivalence between the last two notions is actually proven in Strong (2014) in the setting of market with a stochastic number of assets. In this subsection, we state their result, supplement it with related concepts, and provide connections with the other results from the previous subsections.

We first define investment strategies, which play the role of integrand for the price process, as opposed to portfolios acting as an integrand for the return process.

**Definition 3.8** (Investment strategy). For a given price process $S$ of Definition 3.1, we call $\vartheta$ an *investment strategy*, if $\vartheta \in \mathcal{L}_0(S)$. The wealth process of $\vartheta$ with initial capital $x$ is defined by

$$X(\cdot; x, \vartheta) := x + \vartheta \cdot S = x + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\int_0^{\cdot}\sum_{i=1}^{n}\vartheta_i^{(k,n)}(u)\,dS_i^{k,n}(u). \tag{37}$$

The investment strategy $\vartheta$ with initial capital $x > 0$ is said to be *admissible (strictly admissible)*, if its wealth process is nonnegative (strictly positive) at all times, that is,

$$X(\cdot; x, \vartheta) \geq 0 \qquad (X(\cdot; x, \vartheta) > 0, \text{ respectively}).$$

For a given strictly admissible investment strategy $\vartheta$ with initial capital $x = 1$, we can construct corresponding portfolio

$$\pi := 0^{(N_0)} + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\hat{\mathbb{1}}_{\rrbracket\tau_{k-1},\tau_k\rrbracket\cap(\mathbb{R}_+\times\Omega^{k,n})}\pi^{(k,n)}, \tag{38}$$

where for every $(k, n) \in \mathbb{N}^2$

$$\pi_i^{(k,n)} := \frac{S_i\vartheta_i^{(k,n)}}{X(\cdot; 1, \vartheta)}\hat{\mathbb{1}}_{\rrbracket\tau_{k-1},\tau_k\rrbracket\cap(\mathbb{R}_+\times\Omega^{k,n})} + 0^{(1)}, \qquad i \in [n]. \tag{39}$$

Here, the strict admissibility of $\vartheta$ is necessary as the wealth process appears in the denominator. The wealth $X_\pi$ of this portfolio $\pi$ is then equal to the wealth $X(\cdot; 1, \vartheta)$ of $\vartheta$:

$$\begin{aligned} X_\pi(t) = \mathcal{E}(R_\pi(t)) &= \mathcal{E}\left(\int_0^t \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{S_i(u)\vartheta_i^{(k,n)}(u)}{X(u; 1, \vartheta)}\hat{\mathbb{1}}_{\rrbracket\tau_{k-1},\tau_k\rrbracket\cap(\mathbb{R}_+\times\Omega^{k,n})}\,dR_i^{k,n}(u)\right) \\ &= \mathcal{E}\left(\int_0^t \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{\vartheta_i^{(k,n)}(u)}{X(u; 1, \vartheta)}\hat{\mathbb{1}}_{\rrbracket\tau_{k-1},\tau_k\rrbracket\cap(\mathbb{R}_+\times\Omega^{k,n})}\,dS_i^{k,n}(u)\right) \\ &= \mathcal{E}\left(\int_0^t \frac{dX(u; 1, \vartheta)}{X(u; 1, \vartheta)}\right) = X(t; 1, \vartheta), \qquad \forall\, t \geq 0, \end{aligned} \tag{40}$$

after plugging in Equations (39), (7), (37), and using the property of stochastic exponential.

Conversely, for a given portfolio $\pi \in \mathcal{L}(R)$ (note that its wealth $X_\pi$ is always positive as a stochastic exponential), we can define corresponding investment strategy

$$\vartheta := 0^{(N_0)} + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\hat{\mathbb{1}}_{\rrbracket\tau_{k-1},\tau_k\rrbracket\cap(\mathbb{R}_+\times\Omega^{k,n})}\vartheta^{(k,n)},$$

where for every $(k, n) \in \mathbb{N}^2$

$$\vartheta_i^{(k,n)} := \frac{(X_\pi)\pi_i^{(k,n)}}{S_i}\hat{\mathbb{1}}_{\rrbracket\tau_{k-1},\tau_k\rrbracket\cap(\mathbb{R}_+\times\Omega^{k,n})} + 0^{(1)}, \qquad i \in [n]. \tag{41}$$

It is also easy to check that the wealth $X(t; 1, \vartheta)$ of $\vartheta$ is equal to $X_\pi(t)$ for every $t \geq 0$, by plugging in Equation (41) into Equation (37), and using the identity $dS_i^{k,n}(u) = dS_i(u)$ on the set $\rrbracket\tau_{k-1}, \tau_k\rrbracket \cap (\mathbb{R}_+ \times \Omega^{k,n})$, together with the property of stochastic exponential.

**Definition 3.9** (Arbitrage of the first kind). An *arbitrage of the first kind* for horizon $T > 0$ is an $\mathcal{F}_T$-measurable random variable $h$ satisfying $\mathbb{P}(h \geq 0) = 1$, $\mathbb{P}(h > 0) > 0$, such that for every $x > 0$, there exists an admissible investment strategy $\vartheta$ satisfying $X(T; x, \vartheta) \geq h$. If there are no arbitrages of the first kind, we say that $NA_1$ holds.

Since this weak notion of arbitrage was introduced by Ingersoll (1987), it has appeared under different names; *no asymptotic arbitrage with first kind* in Kabanov and Kramkov (1994), *bounded in probability (or BK)* in Kabanov (1997), *cheap thrills* in Loewenstein and Willard (2000), and *no unbounded profit with bounded risk (NUPBR)* in Karatzas and Kardaras (2007). It is also known to be equivalent to other conditions, which we provide in the following.

**Definition 3.10** (Local martingale deflator). An adapted, strictly positive process $Y$ is called *local martingale deflator*, if $Y(0) = 1$ and $YX_\pi$ is a local martingale for every portfolio $\pi \in \mathcal{L}(R)$.

**Definition 3.11** (Market viability). A nondecreasing, adapted, and right-continuous process $K$ with $K(0) = 0$ is called a *cumulative withdrawal stream*. For any given cumulative withdrawal stream $K$, we define the *superhedging capital associated with* $K$ by

$$x(K) := \inf\{x \geq 0 \mid \exists\, \vartheta \in \mathcal{L}_0(S) \text{ such that } X(\cdot; x, \vartheta) \geq K\}, \tag{42}$$

representing the smallest initial capital starting from which the process $K$ can be financed or hedged in the market. We say that market is *viable*, if $x(K) = 0$ implies $K \equiv 0$.

**Proposition 3.12** Strong (2014) and Karatzas and Kardaras (2021). *The following statements are equivalent:*

*(i)* $NA_1$ *holds.*
*(ii) There exists a local martingale deflator.*
*(iii) The market is viable.*
*(iv) The collection of portfolio wealth processes is bounded in probability, that is,*

$$\lim_{m\to\infty} \sup_{\pi\in\mathcal{L}(R)} \mathbb{P}[X_\pi(T) > m] = 0 \text{ holds for any } T \geq 0. \tag{43}$$

*Proof.* Theorem 3.5 of Strong (2014) proves that the existence of local martingale deflator is equivalent to $NA_1$ in the market of stochastic dimension. The equivalences (i) $\iff$ (iii) and (iii) $\iff$ (iv) follow from Exercise 2.21 and Proposition 2.22 of Karatzas and Kardaras (2021); the argument can be applied to any market (of either fixed or stochastic dimension). □

We also refer to Section 1.4 of Karatzas and Shreve (1998) for some ramifications of this concept of arbitrage, in a classical model of the equity market.

We conclude this subsection with the following explicit expression of local martingale deflators in terms of a local martingale numéraire portfolio. This characterization of local martingale deflators is originally studied by Schweizer (1995), also appears in Propositions 2.3, 3.2 of Larsen and Žitković (2007), and Exercise 2.27 of Karatzas and Kardaras (2021).

Recalling the decomposition (8) of each dissected return, let us denote $\mathcal{Y}$ the collection of local martingale deflators in Definition 3.10, and $\mathcal{M}^{\perp}_{loc}(M)$ the collection of scalar local martingales

$L$ with RCLL paths, satisfying $L(0) = 0$ and the strong orthogonality condition $[L, M_i^{k,n}] = 0$ for every $(k, n) \in \mathbb{N}^2$ and $i \in [n]$.

**Proposition 3.13.** *If there exists a local martingale numéraire portfolio $\rho$, then we have*

$$\mathcal{Y} = \left\{ \frac{\mathcal{E}(L)}{X_\rho} \,\middle|\, L \in \mathcal{M}_{loc}^{\perp}(M), \quad \Delta L > -1 \right\}. \tag{44}$$

*Proof.* Let $Z$ be a local martingale deflator. Recalling the notation (16), $ZX_\rho$ and $ZX_\pi = ZX_\rho X_\pi^\rho$ are strictly positive local martingales for every $\pi \in \mathcal{L}(R)$. Then, it follows that $Q := ZX_\rho$ is strongly orthogonal to $X_\pi^\rho$. Since we can express $Q = \mathcal{E}(L)$, where

$$L = \int_0^\cdot \frac{dQ(t)}{Q(t-)}, \qquad \text{with} \qquad \Delta L(t) = \frac{Q(t) - Q(t-)}{Q(t-)} > -1,$$

and $X_\pi^\rho = \mathcal{E}(R_\pi^\rho)$ from Equation (20), we conclude from the strict positivity of $Q$ and $X_\pi^\rho$, that $L$ is strongly orthogonal to $R_\pi^\rho$.

Now that $\rho$ satisfies the condition $(iii)$ of Proposition 3.6, the notations (17), (18) of Lemma 3.4 have alternative representations from Equation (22)

$$\begin{aligned} R_0^{\rho,k,n} &= -\int_0^\cdot \sum_{i=1}^n \rho_i^{(k,n)}\, dM_i^{k,n}, \\ R_i^{\rho,k,n} &= R_0^{\rho,k,n} + M_i^{k,n}, \qquad\qquad i \in [n], \end{aligned}$$

and a straightforward computation gives for every portfolio $\pi$

$$R_\pi^\rho = \sum_{k=1}^\infty \sum_{n=1}^\infty \int_0^\cdot \sum_{i=1}^n (\pi_i^{(k,n)} - \rho_i^{(k,n)})\, dM_i^{k,n}. \tag{45}$$

For an arbitrary pair $(k, n) \in \mathbb{N}^2$ and number $i \in [n]$, we define a portfolio $\pi$ via the following recipe:

$$\begin{aligned} \pi^{(\ell,p)} &= \rho^{(\ell,p)}, \qquad \text{if } (\ell, p) \neq (k, n), \\ \pi_j^{(k,n)} &= \rho_j^{(k,n)} \qquad \text{if } j \neq i, \\ \pi_i^{(k,n)} &= \rho_i^{(k,n)} + 1, \end{aligned}$$

such that $R_\pi^\rho = M_i^{k,n}$. In other words, $L$ is strongly orthogonal to every $M_i^{k,n}$, thus it belongs to $\mathcal{M}_{loc}^{\perp}(M)$, which establishes the representation $Z = \mathcal{E}(L)/X_\rho$.

Conversely, for any $L \in \mathcal{M}_{loc}^{\perp}(M)$ satisfying $\Delta L > -1$, it is enough to show that $\mathcal{E}(L)X_\pi^\rho$ is a local martingale for every portfolio $\pi$. The identities (20), (45), the strong orthogonality condition on $L$, and Yor's formula prove that

$$\mathcal{E}(L)X_\pi^\rho = \mathcal{E}(L)\mathcal{E}(R_\pi^\rho) = \mathcal{E}(L + R_\pi^\rho)$$

is a local martingale. □

## 3.5 | The fundamental theorem

We are now ready to connect all results (Propositions 3.6, 3.7, and 3.12) in the previous subsections, and state the following cornerstone theorem of arbitrage theory in the market of stochastic dimension.

**Theorem 3.14.** *The following statements are equivalent:*

*(i) There exists a supermartingale (also a local martingale) numéraire portfolio.*
*(ii) There exists a portfolio $\rho$ satisfying $A_i^{k,n} = C_{i\rho}^{k,n}$ for every $i \in [n]$ and $(k,n) \in \mathbb{N}^2$.*
*(iii) Each $(k,n)$-dissected market has locally finite growth, that is, $G^{k,n}(T) < \infty$ for every $T \geq 0$ and $(k,n) \in \mathbb{N}^2$.*
*(iv) $NA_1$ holds.*
*(v) There exists a local martingale deflator.*
*(vi) The market is viable.*

*Proof.* The equivalences between (i) and (iii) are explained in Sections 3.2, 3.3, and the equivalences between (iv) and (vi) are from Proposition 3.12. If a local martingale numéraire portfolio $\rho$ exists, then the reciprocal of its wealth $1/X_\rho$ is a local martingale deflator by definition, thus $(i)$ implies $(v)$. We show in the following the implication $(vi) \Longrightarrow (iii)$, which shows the result.

We assume that there exists a pair of natural numbers $(\ell, p)$ such that the $(\ell, p)$-dissected market fails to have locally finite growth, or the condition $(ii)$ of Proposition 3.7 is violated. Following the proof of Theorem 2.31 of Karatzas and Kardaras (2021), we shall treat two cases: either

(A) the set $\{\alpha^{\ell,p} \notin \mathbf{range}(c^{\ell,p})\}$ fails to be $(\mathbb{P} \otimes O^{\ell,p})$-null, or
(B) the set $\{\alpha^{\ell,p} \notin \mathbf{range}(c^{\ell,p})\}$ is $(\mathbb{P} \otimes O^{\ell,p})$-null, but there exists $T \geq 0$ such that

$$\mathbb{P}\left[\int_0^T \left((\alpha^{\ell,p})^\top (c^{\ell,p})^\dagger \alpha^{\ell,p}\right)(t)\, dO^{\ell,p}(t) = \infty\right] > 0, \qquad \text{holds.} \tag{46}$$

For case (A), we define a $p$-dimensional predictable process

$$\varphi^{(\ell,p)} := \frac{1}{\|\alpha^{\ell,p} - c^{\ell,p}(c^{\ell,p})^\dagger \alpha^{\ell,p}\|^2}\left(\alpha^{\ell,p} - c^{\ell,p}(c^{\ell,p})^\dagger \alpha^{\ell,p}\right)\mathbb{1}_{\{\alpha^{\ell,p} \notin \mathbf{range}(c^{\ell,p})\}}. \tag{47}$$

Recalling the fact that $c^{\ell,p}(c^{\ell,p})^\dagger$ is the projection operator on $\mathbf{range}(c^{\ell,p})$, we can deduce that $\varphi^{(\ell,p)}$ is well-defined, that is, the denominator is nonzero on the set $\{\alpha^{\ell,p} \notin \mathbf{range}(c^{\ell,p})\}$, and the identities $c^{\ell,p}\varphi^{(\ell,p)} \equiv 0$ and $(\varphi^{(\ell,p)})^\top \alpha^{\ell,p} = \mathbb{1}_{\{\alpha^{\ell,p} \notin \mathbf{range}(c^{\ell,p})\}}$ hold. From Equation (27), $\varphi^{(\ell,p)}$ is $R^{\ell,p}$-integrable, however, $c^{\ell,p}\varphi^{(\ell,p)} \equiv 0$ yields $[\int_0^\cdot (\varphi^{(\ell,p)})^\top dM^{\ell,p}] \equiv 0$, thus

$$\int_0^\cdot (\varphi^{(\ell,p)})^\top dM^{\ell,p} \equiv 0. \tag{48}$$

Therefore, we set nondecreasing process

$$K := \int_0^\cdot (\varphi^{(\ell,p)})^\top dR^{\ell,p} = \int_0^\cdot (\varphi^{(\ell,p)})^\top dA^{\ell,p} = \int_0^\cdot \mathbb{1}_{\{\alpha^{\ell,p} \notin \mathbf{range}(c^{\ell,p})\}} dO^{\ell,p}, \tag{49}$$

and the assumption (A) implies $\mathbb{P}[K(\infty) > 0] > 0$. We now define an investment strategy $\vartheta$ via dissection

$$\vartheta_i^{(\ell,p)} := \frac{\varphi_i^{(\ell,p)}}{S_i} \hat{\mathbb{1}}_{\rrbracket \tau_{\ell-1}, \tau_\ell \rrbracket \cap (\mathbb{R}_+ \times \Omega^{\ell,p})} + 0^{(1)}, \qquad i = 1, \dots, p,$$
$$\vartheta^{(k,n)} := 0^{(n)}, \qquad \text{if } (k,n) \neq (\ell, p),$$

then its wealth process is equal to $K$:

$$X(\cdot; 0, \vartheta) = 0 + \vartheta \cdot S = \int_0^{\cdot} (\vartheta^{(\ell,p)})^\top dS^{\ell,p} = \int_0^{\cdot} (\varphi^{(\ell,p)})^\top dR^{\ell,p} = K.$$

This violates the market viability, since we can "finance" nonzero process $K$ from zero initial capital.

For case (B), we assume Equation (46) and we shall deduce that the market is not viable, by showing that the condition (43) is violated. We define a portfolio $\rho$ via dissection

$$\rho^{(\ell,p)} := (c^{\ell,p})^\dagger \alpha^{\ell,p} \hat{\mathbb{1}}_{\rrbracket \tau_{\ell-1}, \tau_\ell \rrbracket \cap (\mathbb{R}_+ \times \Omega^{\ell,p})} + 0^{(p)},$$
$$\rho^{(k,n)} := 0^{(n)}, \qquad \text{if } (k,n) \neq (\ell, p),$$

as well as a sequence of portfolios $(\rho_\eta)_{\eta \in \mathbb{N}}$ such that

$$\rho_\eta^{(k,n)} := \rho^{(k,n)} \mathbb{1}_{\{\|\rho^{(k,n)}\| < \eta\}}, \qquad \text{for every } (k,n) \in \mathbb{N}^2. \tag{50}$$

Recalling Equations (30) and (31), it is straightforward to deduce for every $\eta \in \mathbb{N}$

$$\log(X_{\rho_\eta}) = \frac{1}{2} \int_0^{\cdot} \mathbb{1}_{\{\|\rho^{(\ell,p)}\| < \eta\}} (\rho^{(\ell,p)})^\top c^{\ell,p} \rho^{(\ell,p)} \, dO^{\ell,p} + \int_0^{\cdot} \mathbb{1}_{\{\|\rho^{(\ell,p)}\| < \eta\}} (\rho^{(\ell,p)})^\top \, dM^{\ell,p}.$$

Since the first integral

$$2G^\eta := \int_0^{\cdot} \mathbb{1}_{\{\|\rho^{(\ell,p)}\| < \eta\}} (\rho^{(\ell,p)})^\top c^{\ell,p} \rho^{(\ell,p)} \, dO^{\ell,p}$$

is the quadratic variation of the last integral $\int_0^{\cdot} \mathbb{1}_{\{\|\rho^{(\ell,p)}\| < \eta\}} (\rho^{(\ell,p)})^\top \, dM^{\ell,p}$, which is a local martingale, the Dambis–Dubins–Schwarz representation (Theorem 3.4.6 of Karatzas & Shreve, 1991) with the scaling property of Brownian motion shows that there exists a Brownian motion $W^\eta$, possibly on an enlarged probability space, satisfying

$$\log(X_{\rho_\eta}) = G^\eta + \sqrt{2} W^\eta(G^\eta), \qquad \text{for every } \eta \in \mathbb{N}.$$

Now that $(G^\eta(T))_{\eta \in \mathbb{N}}$ is nondecreasing and converges as $\eta \to \infty$ to

$$\frac{1}{2} \int_0^T (\rho^{(\ell,p)})^\top c^{\ell,p} \rho^{(\ell,p)} \, dO^{\ell,p} = \frac{1}{2} \int_0^T (\alpha^{\ell,p})^\top (c^{\ell,p})^\dagger \alpha^{\ell,p} \, dO^{\ell,p} =: G(T),$$

the condition (46) and the strong law of large numbers imply

$$\lim_{\eta\to\infty}\mathbb{P}\left[\frac{W^\eta(G^\eta(T))}{G^\eta(T)}\le -\frac{1}{2\sqrt{2}},\quad G(T)=\infty\right]=0,$$

thus

$$\lim_{\eta\to\infty}\mathbb{P}\left[\frac{\log\left(X_{\rho_\eta}(T)\right)}{G^\eta(T)}\le \frac{1}{2},\quad G(T)=\infty\right]=0.$$

Therefore, the sequence of random variables $(X_{\rho_\eta}(T))_{\eta\in\mathbb{N}}$ fails to be bounded in probability under the assumption (46), which violates the condition (43). □

Example 2.11 of Karatzas and Kardaras (2021) provides a simple example of a stock market (of a fixed dimension) that is not viable. We can easily generalize it in our setting to give the following example of a market in which any condition of Theorem 3.14 fails to hold.

**Example 3.15.** For a fixed $n\in\mathbb{N}$, consider $n$ independent copies of Brownian motions $W_1,\dots,W_n$ and the $n$-dimensional vector $B:=(|W_1|,\dots,|W_n|)$. Note from Tanaka's formula that the component $B_i$ of $B$ can be decomposed as $B_i=A_i+M_i$, where $A_i$ is the local time process of $W_i$ at the origin, and $M_i=\int_0^{\cdot}\operatorname{sign}(W_i(s))dW_i(s)$ is a Brownian motion. Moreover, $A_i$ is singular with respect to the Lebesgue measure, and $[M_i,M_j](t)\equiv\delta_{i,j}t$ holds for every $i,j\in[n]$. Suppose that the return process of the $(1,n)$-dissected market follows the dynamics of $B$, that is,

$$R_i^{1,n}(t)=B_i^{\tau_1}(t),\qquad \text{for every } t\ge 0,\quad i\in[n]. \tag{51}$$

Then, we have $A_i^{1,n}=A_i^{\tau_1}$, $M_i^{1,n}=M_i^{\tau_1}$, and

$$C_{i\rho}^{1,n}(t)=\int_0^{t\wedge\tau_1}\sum_{j=1}^n\rho_j^{(1,n)}(s)d[M_i^{1,n},M_j^{1,n}](s)=\int_0^{t\wedge\tau_1}\rho_i^{(1,n)}(s)ds.$$

However, the singularity of $A_i$ with respect to the Lebesgue measure concludes that a portfolio $\rho$ satisfying $A_i^{1,n}=C_{i\rho}^{1,n}$ does not exist. Since condition (ii) of Theorem 3.14 is violated, this market is not viable.

For any $k\in\mathbb{N}$, we can also make the return process of $(k,n)$-dissected market follow $B$ by setting

$$R_i^{k,n}(t+\tau_{k-1})=B_i^{\tau_k}(t),\qquad \text{for every } t\ge 0,\quad i\in[n],$$

instead of Equation (51), then the same argument concludes that this market violates condition (ii) of Theorem 3.14 for the pair $(k,n)\in\mathbb{N}^2$.

## 3.6 | Optimal properties of the numéraire portfolio

In Section 3.3, we showed that the (supermartingale) numéraire portfolio attains the maximal growth rate of Equation (32) for every dissected market. This "optimality" in the growth can be formally defined as follows.

**Definition 3.16** (Growth optimal portfolio). A portfolio $\rho \in \mathcal{L}(R)$ is called *growth optimal portfolio*, if

$$\gamma_{\rho}^{k,n} \geq \gamma_{\pi}^{k,n} \tag{52}$$

holds for every $(k,n) \in \mathbb{N}^2$ and every portfolio $\pi \in \mathcal{L}(R)$.

Recalling the cumulative growth $\Gamma_{\pi}^{k,n}$ of $\pi$ in Equation (29) and the fact that $\gamma_{\pi}^{k,n}$ is a Radon–Nikodým derivative of $\Gamma_{\pi}^{k,n}$ with respect to $O^{k,n}$, we note that the inequality of Equation (52) is equivalent to the condition that the process

$$(\Gamma_{\pi}^{\rho})^{k,n} := \Gamma_{\pi}^{k,n} - \Gamma_{\rho}^{k,n}$$

is nonincreasing.

Furthermore, as described in Section 2.3.3 of Karatzas and Kardaras (2021), the (supermartingale) numéraire portfolio (in a market of fixed dimension) has the relative log-optimality, which is related to the maximization of expected logarithmic utility of the wealth process. We generalize this result in our setting of market with a changing number of assets.

**Definition 3.17** (Relative log-optimal portfolio). A portfolio $\rho \in \mathcal{L}(R)$ is called *relatively log-optimal portfolio*, if

$$\mathbb{E}\left[\left(\log X_{\pi}^{\rho}(\tau)\right)^{+}\right] < \infty \qquad \text{and} \qquad \mathbb{E}\left[\log X_{\pi}^{\rho}(\tau)\right] \leq 0 \tag{53}$$

hold for every stopping time $\tau$ and every portfolio $\pi \in \mathcal{L}(R)$.

**Proposition 3.18.** *For a portfolio $\rho \in \mathcal{L}(R)$, the following statements are equivalent:*

*(i)* *$\rho$ is a supermartingale numéraire portfolio.*
*(ii)* *$\rho$ is a growth optimal portfolio.*
*(iii)* *$\rho$ is a relatively log-optimal portfolio.*

The conditions (*ii*) and (*iii*) of Proposition 3.18 can be appended to Proposition 3.6, hence the existence of the growth optimal portfolio or the relative log-optimal portfolio can also be added as an equivalent statement in the fundamental theorem (Theorem 3.14).

*Proof.* The equivalence (i) $\iff$ (ii) can be proven by applying the same argument in the proof of Proposition 2.41 of Karatzas and Kardaras (2021) to each dissected market (with the notations $\pi^{k,n}, \rho^{k,n}, c^{k,n}, \alpha^{k,n}$, etc.) for every $(k,n) \in \mathbb{N}^2$.

For the equivalence (i) $\iff$ (iii), suppose that $\rho$ is a supermartingale numéraire portfolio. From a trivial inequality $(\log x)^{+} < x$, together with the Optional Sampling Theorem and Fatou's lemma

applied to the nonnegative supermartingale $X^{\rho}_{\pi}$, we obtain

$$\mathbb{E}\left[\left(\log X^{\rho}_{\pi}(\tau)\right)^{+}\right] < \mathbb{E}\left[X^{\rho}_{\pi}(\tau)\right] \leq 1$$

for any stopping time $\tau$ and $\pi \in \mathcal{L}(R)$. Jensen's inequality applied to the last inequality also yields the second condition $\mathbb{E}[\log X^{\rho}_{\pi}(\tau)] \leq 0$ of Equation (53).

In order to show the reverse implication, suppose that $\rho$ is a relative log-optimal portfolio. Recalling the proof of Theorem 3.14, we first claim that the set $\{\alpha^{\ell,p} \notin \mathbf{range}(c^{\ell,p})\}$ is $(\mathbb{P} \otimes O^{\ell,p})$-null for every $(\ell, p) \in \mathbb{N}^2$.

For each pair $(\ell, p) \in \mathbb{N}^2$, let us bring back the $p$-dimensional process $\varphi^{(\ell,p)}$, defined in Equation (47), which satisfies the identities

$$c^{\ell,p}\varphi^{(\ell,p)} \equiv 0, \qquad (\varphi^{(\ell,p)})^{\top}\alpha^{\ell,p} = \mathbb{1}_{\{\alpha^{\ell,p} \notin \mathbf{range}(c^{\ell,p})\}}. \tag{54}$$

We construct a portfolio $\varphi$ from these dissections

$$\varphi := 0^{(N_0)} + \sum_{\ell=1}^{\infty}\sum_{p=1}^{\infty} \hat{\mathbb{1}}_{]\!]\tau_{\ell-1},\tau_{\ell}]\!]\cap(\mathbb{R}_+\times\Omega^{\ell,p})}\varphi^{(\ell,p)}.$$

From the representation (30) of the log-wealth process, we obtain

$$\begin{aligned}
\log X^{\rho}_{\rho+\varphi} &= \sum_{\ell=1}^{\infty}\sum_{p=1}^{\infty}\left(\Gamma^{\ell,p}_{\rho+\varphi} - \Gamma^{\ell,p}_{\rho} + M^{\ell,p}_{\rho+\varphi} - M^{\ell,p}_{\rho}\right)\\
&= \sum_{\ell=1}^{\infty}\sum_{p=1}^{\infty}\left(\int_0^{\cdot}(\gamma^{\ell,p}_{\rho+\varphi} - \gamma^{\ell,p}_{\rho})^{\top}\, dO^{\ell,p} + \int_0^{\cdot}(\varphi^{(\ell,p)})^{\top} dM^{\ell,p}\right)\\
&= \sum_{\ell=1}^{\infty}\sum_{p=1}^{\infty}\int_0^{\cdot}\mathbb{1}_{\{\alpha^{\ell,p}\notin\mathbf{range}(c^{\ell,p})\}} dO^{\ell,p}.
\end{aligned}$$

Here, the last equality follows from the definition (31) of the local growth rate, the identities (54) and (48). The log-optimal property $\mathbb{E}[\log X^{\rho}_{\rho+\varphi}] \leq 0$ now yields that each set $\{\alpha^{\ell,p} \notin \mathbf{range}(c^{\ell,p})\}$ should be $(\mathbb{P} \otimes O^{\ell,p})$-null for every $(\ell, p) \in \mathbb{N}^2$.

Next, we recall from Equation (34) that the process $\nu^{(k,n)} := (c^{k,n})^{\dagger}\alpha^{k,n}$ is a candidate for (a $(k,n)$-dissection of) the numéraire portfolio. Let us fix an arbitrary $m \in \mathbb{N}$ and define for every $(k,n) \in \mathbb{N}^2$

$$\begin{aligned}
\rho^{(k,n),m} := \rho^{(k,n)}\mathbb{1}_{\{c^{k,n}\rho^{(k,n)}=\alpha^{k,n}\}} &+ \rho^{(k,n)}\mathbb{1}_{\{c^{k,n}\rho^{(k,n)}\neq\alpha^{k,n},\, \|\nu^{(k,n)}\|>m\}}\\
&+ \nu^{(k,n)}\mathbb{1}_{\{c^{k,n}\rho^{(k,n)}\neq\alpha^{k,n},\, \|\nu^{(k,n)}\|\leq m\}}.
\end{aligned}$$

Since the last term is bounded by $m$ and $\rho^{(k,n)}$ is $R^{k,n}$-integrable, the process $\rho^{(k,n),m}$ is also $R^{k,n}$-integrable. We also define a portfolio $\rho^m$ by collecting the dissections

$$\rho^m := 0^{(N_0)} + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\hat{\mathbb{1}}_{]\!]\tau_{k-1},\tau_k]\!]\cap(\mathbb{R}_+\times\Omega^{k,n})}\rho^{(k,n),m}.$$

In what follows, we shall show that $X_\rho^{\rho^m} = \mathcal{E}(R_\rho^{\rho^m})$ is a local martingale. From Equations (19) and (21), we have the representation

$$R_\rho^{\rho^m} = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\int_0^{\cdot}\sum_{i=0}^{n}\pi_i^{(k,n)}(s)\,dR_i^{\rho^m,k,n}(s) = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left(R^{k,n}_{\rho-\rho^m} - C^{k,n}_{(\rho-\rho^m)\rho^m}\right). \tag{55}$$

Let us fix a pair $(k,n)\in\mathbb{N}^2$. On the set $\Xi^{k,n,m} := \{c^{k,n}\rho^{(k,n)} \neq \alpha^{k,n},\ \|\nu^{(k,n)}\| \leq m\}$, we have $\rho^{(k,n),m} = \nu^{(k,n)}$, which implies $c^{k,n}\rho^{(k,n),m} = \alpha^{k,n}$, hence also $A_i^{k,n} = C_{i\rho^m}^{k,n}$ for every $i\in[n]$ from Equation (28). Then, the expression $R_i^{k,n} - C_{i\rho^m}^{k,n} = M_i^{k,n}$ is a local martingale for each $i\in[n]$. Moreover, from Equations (17), (18), and (22), all integrators $R_0^{\rho^m,k,n},\dots,R_n^{\rho^m,k,n}$ are local martingales, hence the integral $\int_0^{\cdot}\sum_{i=0}^{\infty}\pi_i^{(k,n)}dR_i^{\rho^m}$ is a local martingale. On the complement set $(\Xi^{k,n,m})^c$, we have $\rho^{(k,n),m} = \rho^{(k,n)}$, thus each summand $R^{k,n}_{\rho-\rho^m} - C^{k,n}_{(\rho-\rho^m)\rho^m}$ of Equation (55) vanishes. Therefore, every summand is a local martingale, and Lemma 2.12 of Strong (2014) concludes that $R_\rho^{\rho^m}$ is also a local martingale. This proves the claim that $X_\rho^{\rho^m}$ is a (nonnegative) local martingale, hence also a supermartingale.

If we have $\mathbb{P}[X_\rho(T)\neq X_{\rho^m}(T)]>0$ for some $T>0$, Jensen's inequality and the Optional Sampling Theorem give

$$\mathbb{E}\left[\log X_\rho^{\rho^m}(T)\right] < \log\mathbb{E}\left[X_\rho^{\rho^m}(T)\right] \leq 0$$

which contradicts the relative log-optimality of $\rho$. Thanks to the continuity of $X_\rho$ and $X_{\rho^m}$, we conclude that $X_\rho \equiv X_{\rho^m}$, thus $\rho-\rho^m$ is a null portfolio for every $m\in\mathbb{N}$. Then, we have $c^{k,n}\nu^{(k,n)} = \alpha^{k,n} = c^{k,n}\rho^{(k,n),m} = c^{k,n}\rho^{(k,n)}$ on the set $\Xi^{k,n,m}$, which implies that $\Xi^{k,n,m}$ should be a $(\mathbb{P}\times O^{k,n})$-null set for every $m\in\mathbb{N}$. Therefore,

$$\bigcup_{m=1}^{\infty}\Xi^{k,n,m} = \{c^{k,n}\rho^{(k,n)}\neq\alpha^{k,n}\}$$

is also a $(\mathbb{P}\times O^{k,n})$-null set, and $c^{k,n}\rho^{(k,n)} = \alpha^{k,n}$ holds $(\mathbb{P}\times O^{k,n})$-a.e. for each $(k,n)\in\mathbb{N}^2$. The supermartingale numéraire property of $\rho$ now follows from the structural condition (28). □

## 3.7 | The optional decomposition theorem

When the numéraire (either supermartingale or local martingale) exists in the market, the wealth of any portfolio (or corresponding investment strategy), divided by the numéraire, is a local martingale. The following result shows that this local martingale can be represented as a stochastic integral with respect to the $M^{k,n}$ of Equation (8), on each dissection set.

**Lemma 3.19.** *Let $\rho$ be a numéraire portfolio. For any given initial capital $x\in\mathbb{R}$ and an investment strategy $\vartheta\in\mathcal{L}_0(S)$ with a notation $X(\cdot)\equiv X(\cdot;x,\vartheta) = x+\vartheta\cdot S$, there exists a $\mathbb{U}$-valued process $\eta$ such that each $\eta^{(k,n)}$ is $M^{k,n}$-integrable for every $(k,n)\in\mathbb{N}^2$, and the ratio of its wealth process $X$ to the numéraire $X_\rho$ can be represented as*

$$\frac{X}{X_\rho} = x + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\int_0^{\cdot}\sum_{i=1}^{n}\eta_i^{(k,n)}(u)\,dM_i^{k,n}(u). \tag{56}$$

*Proof.* Using Equations (4), (7), and (8), we derive that

$$\begin{aligned} dX(t) &= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\sum_{i=1}^{n} \vartheta_i^{(k,n)}(t)\hat{\mathbb{1}}_{]\!]\tau_{k-1},\tau_k]\!]\cap(\mathbb{R}_+\times\Omega^{k,n})}\, dS_i^{k,n}(t) \\ &= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\sum_{i=1}^{n} S_i(t)\vartheta_i^{(k,n)}(t)\hat{\mathbb{1}}_{]\!]\tau_{k-1},\tau_k]\!]\cap(\mathbb{R}_+\times\Omega^{k,n})}\, dR_i^{k,n}(t) \\ &= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\sum_{i=1}^{n} S_i(t)\vartheta_i^{(k,n)}(t)\Big(dA_i^{k,n}(t) + dM_i^{k,n}(t)\Big). \end{aligned}$$

On the other hand, let us consider a portfolio $\pi$ investing all its wealth in the money market at all times, that is, $\pi^{(k,n)} \equiv 0^{(n)}$ for every $(k,n) \in \mathbb{N}^2$, such that $X_\pi \equiv 1$. Lemma 3.4 and the identity (22) show that

$$\frac{1}{X_\rho} = \frac{X_\pi}{X_\rho} = \mathcal{E}(R_\pi^\rho) = \mathcal{E}\left(\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} R_0^{\rho,k,n}\right) = \mathcal{E}\left(-\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\int_0^{\cdot}\sum_{i=1}^{n}\rho_i^{(k,n)}(u)\, dM_i^{k,n}(u)\right),$$

thus,

$$d\left(\frac{1}{X_\rho}\right)(t) = -\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{\rho_i^{(k,n)}(t)}{X_\rho(t)}\, dM_i^{k,n}(t).$$

Applying the product rule, we have

$$\begin{aligned} d\left(\frac{X}{X_\rho}\right)(t) &= \frac{1}{X_\rho(t)}\, dX(t) + X(t)\, d\left(\frac{1}{X_\rho}\right)(t) + d\left[X, \frac{1}{X_\rho}\right](t) \\ &= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left[\sum_{i=1}^{n}\frac{S_i(t)\vartheta_i^{(k,n)}(t)}{X_\rho(t)}\Big(dA_i^{k,n}(t) + dM_i^{k,n}(t)\Big) - \sum_{i=1}^{n}\frac{X(t)\rho_i^{(k,n)}(t)}{X_\rho(t)}\, dM_i^{k,n}(t)\right. \\ &\qquad \left. - \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{S_i(t)\vartheta_i^{(k,n)}(t)\rho_j^{(k,n)}(t)}{X_\rho(t)}\, d[M_i^{k,n}, M_j^{k,n}](t)\right] \\ &= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{S_i(t)\vartheta_i^{(k,n)}(t) - X(t)\rho_i^{(k,n)}(t)}{X_\rho(t)}\, dM_i^{k,n}(t). \end{aligned}$$

Here, the finite variation terms vanish in the last equality, thanks to the property of the numéraire portfolio $\rho$ from Proposition 3.6 (*iii*). Setting

$$\eta_i^{(k,n)}(t) := \frac{S_i(t)\vartheta_i^{(k,n)}(t) - X(t)\rho_i^{(k,n)}(t)}{X_\rho(t)}, \qquad \forall\,(k,n) \in \mathbb{N}^2,$$

yields the result (56). □

The converse of Lemma 3.19 also holds, that is, for a given $x \in \mathbb{R}$ and a $\mathbb{U}$-valued process $\eta$ such that each $\eta^{(k,n)}$ is $M^{k,n}$-integrable for every $(k,n) \in \mathbb{N}^2$, we can construct an investment strategy $\vartheta \in \mathcal{L}_0(S)$ satisfying Equation (56), by reversing the proof.

The left-hand side of Equation (56) can be written as $X(1/X_\rho)$, and the expression $1/X_\rho$ is an example of local martingale deflator. Since we know the general form of deflators from Proposition 3.13, we can expect a generalization of the result by replacing $1/X_\rho$ in Equation (56) by any local martingale deflator. This gives rise to the following so-called ODT.

**Theorem 3.20.** *Suppose that the market is viable. For a nonnegative, adapted RCLL process $X$ with $X(0) = x \geq 0$, the following statements are equivalent:*

*(i) For every local martingale deflator $Y \in \mathcal{Y}$, the process $YX$ is a supermartingale.*
*(ii) There exist an investment strategy $\vartheta \in \mathcal{L}_0(S)$ and an adapted, nondecreasing process $K$ satisfying $K(0) = 0$ with right-continuous paths, such that*

$$X = x + \vartheta \cdot S - K = x + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\int_0^{\cdot}\sum_{i=1}^{n}\vartheta_i^{(k,n)}(t)\,dS_i^{k,n}(t) - K. \tag{57}$$

*Proof.* We first prove the implication $(ii) \Longrightarrow (i)$. From Proposition 3.13, any local martingale deflator $Y$ admits the representation $\mathcal{E}(L)/X_\rho$, where $L \in \mathcal{M}_{loc}^{\perp}(M)$, $\Delta L > -1$, and $\rho$ is the local martingale numéraire portfolio. Using the representation from Lemma 3.19, we obtain

$$Y(X+K) = YX(\cdot;x,\vartheta) = \mathcal{E}(L)\frac{X(\cdot;x,\vartheta)}{X_\rho} = \mathcal{E}(L)\left(x + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\int_0^{\cdot}\sum_{i=1}^{n}\eta_i^{(k,n)}(u)\,dM_i^{k,n}(u)\right).$$

As a product of two orthogonal local martingales, the last expression is also a local martingale. Thus, $YX$, being a nonnegative local supermartingale, is a supermartingale.

The proof of the reverse implication $(i) \Longrightarrow (ii)$ is divided into four parts. Following the argument of Sections 3.1.2, 3.1.3 of Karatzas and Kardaras (2021), we first impose Assumption 3.21 in the first three parts A–C, and then give the proof for the general case in the last part D.

**Assumption 3.21.** All local martingales on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ have continuous paths.

**Part A:** Since the market is viable, there exists a local martingale numéraire portfolio $\rho$ by Theorem 3.14. From Proposition 3.13, the process $\mathcal{E}(L)X/X_\rho$ is a supermartingale for any element $L \in \mathcal{M}_{loc}^{\perp}(M)$ satisfying $\Delta L > -1$. The ratio $X/X_\rho$ is then also a supermartingale, thus the Doob–Meyer decomposition yields

$$\frac{X}{X_\rho} = x + V - B, \tag{58}$$

where $V$ is a local martingale (which is continuous from Assumption 3.21) and $B$ is an adapted, nondecreasing process with right-continuous paths, satisfying $V(0) = B(0) = 0$.

**Part B:** We shall prove two claims. We first show that there exist a collection of processes $\eta_i^{(k,n)}$ for every $(k,n) \in \mathbb{N}^2$, $i = [n]$ and a process $L \in \mathcal{M}_{loc}^{\perp}(M)$ such that each $\eta_i^{(k,n)}$ is $M_i^{k,n}$-integrable and the local martingale $V$ admits a decomposition

$$V = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\int_0^{\cdot}\sum_{i=1}^{n}\eta_i^{(k,n)}(t)\,dM_i^{k,n}(t) + L = \eta \cdot M + L. \tag{59}$$

In order to prove this claim, we shall use the Kunita–Watanabe decomposition (KWD) repeatedly in a manner of "double induction." First, we take a localizing sequence of stopping times $(\tau_k)_{k\in\mathbb{N}}$ for the local martingale $V$, and apply KWD to the stopped local martingale $V^{\tau_1}$ such that there exist a $M_1^{1,1}$-integrable process $\eta_1^{(1,1)}$ and a local martingale $L^{\{1,1\}}$ with $L^{\{1,1\}}(0)=0$ satisfying the identities

$$V^{\tau_1} = \int_0^{\cdot} \eta_1^{(1,1)}(t)\, dM_1^{1,1}(t) + L^{\{1,1\}}, \qquad [M_1^{1,1}, L^{\{1,1\}}] \equiv 0.$$

We next apply KWD to the local martingale $L^{\{1,1\}}$ to obtain the identities

$$L^{\{1,1\}} = \int_0^{\cdot} \sum_{i=1}^{2} \eta_i^{(1,2)}(t)\, dM_i^{1,2}(t) + L^{\{1,2\}} = \eta^{(1,2)} \cdot M^{1,2} + L^{\{1,2\}}, \quad [M_i^{1,2}, L^{\{1,2\}}] \equiv 0 \text{ for } i = 1, 2,$$

where $L^{\{1,2\}}$ is a local martingale and $\eta^{(1,2)}$ is a two-dimensional $M^{1,2}$-integrable process. Note that the components of $M^{1,n}$ are flat off the dissection set $\text{diss}^{1,n} := [\![\tau_0, \tau_1]\!] \cap (\mathbb{R}_+ \times \Omega^{1,n})$, and each element of $(\text{diss}^{1,n})_{n\in\mathbb{N}}$ is disjoint with each other, we have $[M_1^{1,1}, M_1^{1,2}] = [M_1^{1,1}, M_2^{1,2}] \equiv 0$, thus $[M_1^{1,1}, L^{\{1,2\}}] \equiv 0$ holds. Continuing this procedure, we obtain a sequence of processes $(\eta^{(1,n)})_{n\in\mathbb{N}}$ such that each $\eta^{(1,n)}$ is $n$-dimensional and $M^{1,n}$-integrable, along with a sequence of local martingales $(L^{\{1,n\}})_{n\in\mathbb{N}}$, satisfying for every $p \in \mathbb{N}$

$$V^{\tau_1} = \sum_{n=1}^{p} \left(\eta^{(1,n)} \cdot M^{1,n}\right) + L^{\{1,p\}}, \qquad [M_i^{1,n}, L^{\{1,p\}}] \equiv 0, \qquad \text{for each } n \in [p],\ i \in [n].$$

We now define

$$L^{\{1\}} := V^{\tau_1} - \sum_{n=1}^{\infty} \left(\eta^{(1,n)} \cdot M^{1,n}\right),$$

then it is easy to show that $L^{\{1\}}$ is a well-defined local martingale with the property

$$[M_i^{1,n}, L^{\{1\}}] \equiv 0 \text{ for each } n \in \mathbb{N},\ i \in [n],$$

from the disjoint property of the set $(\text{diss}^{1,n})_{n\in\mathbb{N}}$.

For the second stopping time $\tau_2$, we consider the local martingale $V^{\tau_2} - V^{\tau_1}$ and apply the same argument to obtain the representation

$$V^{\tau_2} - V^{\tau_1} = \sum_{n=1}^{\infty} \left(\eta^{(2,n)} \cdot M^{2,n}\right) + L^{\{2\}},$$

where $L^{\{2\}}$ is a local martingale satisfying $[M_i^{2,n}, L^{\{2\}}] \equiv 0$ for every $n \in \mathbb{N}$, $i \in [n]$, and each $\eta^{(2,n)}$ is an $n$-dimensional $M^{2,n}$-integrable process. Inductively, we obtain for every $k \in \mathbb{N}$

$$V^{\tau_k} - V^{\tau_{k-1}} = \sum_{n=1}^{\infty} \left(\eta^{(k,n)} \cdot M^{k,n}\right) + L^{\{k\}},$$

where each $L^{\{k\}}$ is a local martingale satisfying $[M_i^{k,n}, L^{\{k\}}] \equiv 0$ for every $n \in \mathbb{N}$, $i \in [n]$, and each $\eta^{(k,n)}$ is an $n$-dimensional $M^{k,n}$-integrable process. Therefore, we arrive at the desired expression

(59)

$$V = \sum_{k=1}^{\infty} (V^{\tau_k} - V^{\tau_{k-1}}) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left(\eta^{(k,n)} \cdot M^{k,n}\right) + L,$$

by setting $L := \sum_{k=1}^{\infty} L^{\{k\}}$. Since $L^{\{k\}}$ is flat off the interval $[\![\tau_{k-1}, \tau_k]\!]$ for each $k \in \mathbb{N}$, we have for each fixed $(\ell, n) \in \mathbb{N}^2$ and $i \in [n]$

$$[M_i^{\ell,n}, L] = [M_i^{\ell,n}, L^{\{\ell\}}] \equiv 0,$$

thus $L$ belongs to $\mathcal{M}_{loc}^{\perp}(M)$. This proves the first claim.

We next claim that the process $L$ is actually a zero process, that is, $L \equiv 0$, so that the identities (58), (59) reduce to

$$\frac{X}{X_\rho} = x + \eta \cdot M - B. \tag{60}$$

For a fixed arbitrary $m \in \mathbb{N}$, the process $mL$ belongs to $\mathcal{M}_{loc}^{\perp}(M)$, thus Proposition 3.13 together with Assumption 3.21 implies that $\mathcal{E}(mL)/X_\rho$ is a local martingale deflator and $X\mathcal{E}(mL)/X_\rho$ is a supermartingale by condition (i). Moreover, since $\mathcal{E}(mL)$ also belongs to $\mathcal{M}_{loc}^{\perp}(M)$, we obtain $[\mathcal{E}(mL), \eta \cdot M] \equiv 0$, and $\mathcal{E}(mL)(\eta \cdot M)$ is a local martingale.

Combining Equation (58) with Equation (59) and multiplying $\mathcal{E}(mL)$, we derive

$$\mathcal{E}(mL)(L - B) = \frac{X\mathcal{E}(mL)}{X_\rho} - \mathcal{E}(mL)(x + \eta \cdot M)$$

and the left-hand side is a local supermartingale from the above observations. By the product rule, the left-hand side can be expressed as

$$\int_0^{\cdot} (L - B)(t-)\, d\mathcal{E}(mL)(t) + \int_0^{\cdot} \mathcal{E}(mL)(t)\, dL(t) + \int_0^{\cdot} \mathcal{E}(mL)(t)\, d([mL, L] - B)(t).$$

The first two integrals are local martingales, so the last integrator $m[L, L] - B$ should be a local supermartingale for every $m \in N$. We conclude that $[L, L] \equiv 0$, thus $L \equiv 0$.

**Part C:** The product rule applied to $X = X_\rho(X/X_\rho)$, along with Equations (9) and (60), yields

$$X = x + \int_0^{\cdot} X(t-)\, dR_\rho(t) - \int_0^{\cdot} X_\rho(t)\, dB(t) + \int_0^{\cdot} X_\rho(t)\, d(\eta \cdot M)(t) + [X_\rho, \eta \cdot M]. \tag{61}$$

Here, recalling the notations (24) and (14), the last term of the right-hand side is equal to

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left[ \int_0^{\cdot} X_\rho(t)\, dR_\rho(t), \int_0^{\cdot} \sum_{i=1}^{n} \eta_i^{(k,n)}(t)\, dM_i^{k,n}(t) \right]$$

$$= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \int_0^{\cdot} X_\rho(t) \sum_{i=1}^{n} \sum_{j=1}^{n} \eta_i^{(k,n)}(t) \rho_j^{(k,n)}(t)\, dC_{ij}^{k,n}(t) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \int_0^{\cdot} X_\rho(t) \sum_{i=1}^{n} \eta_i^{(k,n)}(t)\, dC_{i\rho}^{k,n}(t),$$

whereas the second-last term is expressed as

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\int_0^{\cdot} X_\rho(t)\sum_{i=1}^{n}\eta_i^{(k,n)}(t)\,dM_i^{k,n}(t).$$

From the structural condition (Proposition 3.6 (iii)) of the local martingale numéraire portfolio $\rho$, the sum of the last two expressions is

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\int_0^{\cdot} X_\rho(t)\sum_{i=1}^{n}\eta_i^{(k,n)}(t)\,dR_i^{k,n}(t),$$

thus the identity (61) becomes

$$\begin{aligned} X+\int_0^{\cdot} X_\rho(t)\,dB(t) &= x+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\int_0^{\cdot}\sum_{i=1}^{n}\left(X(t-)\rho_i^{(k,n)}(t)+X_\rho(t)\eta_i^{(k,n)}(t)\right)dR_i^{k,n}(t)\\ &= x+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\int_0^{\cdot}\sum_{i=1}^{n}\left(\frac{X(t-)\rho_i^{(k,n)}(t)+X_\rho(t)\eta_i^{(k,n)}(t)}{S_i(t)}\hat{\mathbb{1}}_{\rrbracket\tau_{k-1},\tau_k\rrbracket\cap(\mathbb{R}_+\times\Omega^{k,n})}+0^{(1)}\right)dS_i^{k,n}(t)\end{aligned}$$

with the help of the definition (7). We now construct the investment strategy $\vartheta$ via dissection

$$\vartheta_i^{(k,n)} := \frac{X(t-)\rho_i^{(k,n)}(t)+X_\rho(t)\eta_i^{(k,n)}(t)}{S_i(t)}\hat{\mathbb{1}}_{\rrbracket\tau_{k-1},\tau_k\rrbracket\cap(\mathbb{R}_+\times\Omega^{k,n})}+0^{(1)} \tag{62}$$

and setting $K := \int_0^{\cdot} X_\rho(t)\,dB(t)$, we arrive at the identity (57) of the condition $(ii)$.

**Part D:** Finally, we provide the proof of $(i)\Longrightarrow(ii)$ without Assumption 3.21. Since any local martingale can be decomposed (uniquely up to indistinguishability) as the sum of the continuous part and the purely discontinuous part (Theorem I.4.18 of Jacod & Shiryaev, 2003), the joint decomposition (58), (59) is transformed into

$$\frac{X}{X_\rho} = x+\eta\cdot M+L^c+L^d-B. \tag{63}$$

Here, the processes $\eta, B$ have the same properties as before, $L^c$ is a continuous local martingale in $\mathcal{M}_{loc}^{\perp}(M)$, and $L^d$ is a purely discontinuous local martingale, which is orthogonal to every continuous local martingale. We now define $\widetilde{B} := B-L^d$, then $\widetilde{B}$ is again orthogonal to every continuous local martingale. Using the argument of Section 3.1.3 of Karatzas and Kardaras (2021), we can show that $\widetilde{B}$ is a nondecreasing process. From the argument in part B, we have $L^c\equiv 0$, thus the identities (60) and (63) become

$$\frac{X}{X_\rho} = x+\eta\cdot M-\widetilde{B}.$$

As in part C, defining $\vartheta$ in (62) and $K := \int_0^{\cdot} X_\rho(t)\,d\widetilde{B}(t)$ proves the condition (ii). □

The name of this decomposition theorem is given from the fact that the process $K$ is optional (being an adapted, right-continuous process). This version of theorem is close to the formulation of Theorem 3.1 of Karatzas and Kardaras (2021) (or originally from Karatzas &

Kardaras, 2015), in a closed market with fixed number of stocks modeled by continuous semimartingales under a general right-continuous filtration. We refer to El Karoui and Quenez (1995), Kramkov (1996), Föllmer and Kramkov (1997), and Stricker and Yan (1998) for earlier studies of the optional decomposition.

The following corollary presents the local martingale version of the ODT.

**Corollary 3.22.** *Suppose that the market is viable. For a nonnegative, adapted RCLL process $X$ with $X(0) = x \geq 0$, the following statements are equivalent:*

*(i) For every local martingale deflator $Y \in \mathcal{Y}$, the process $YX$ is a local martingale.*
*(ii) There exists an investment strategy $\vartheta \in \mathcal{L}_0(S)$ satisfying $X \equiv X(\cdot; x, \vartheta)$, that is,*

$$X = x + \vartheta \cdot S = x + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \int_0^{\cdot} \sum_{i=1}^{n} \vartheta_i^{(k,n)}(t)\, dS_i^{k,n}(t).$$

*Proof.* The implication $(ii) \Longrightarrow (i)$ is easy; as in the proof of Theorem 3.20, $YX$ can be represented as the product of two orthogonal local martingales by Proposition 3.13 and Lemma 3.19.

We now assume $(i)$, then $YX$ is also a supermartingale (being a nonnegative local martingale). From Theorem 3.20, there exist $\vartheta \in \mathcal{L}_0(S)$ and an adapted, nondecreasing process $K$ with right-continuous paths satisfying Equation (57). In what follows, we prove $K \equiv 0$. Let us choose $Y = 1/X_\rho$, the reciprocal of the local martingale numéraire, then

$$YX = \frac{X}{X_\rho} = \frac{X(\cdot; x, \vartheta)}{X_\rho} - \frac{K}{X_\rho}$$

is a local martingale. Moreover, we know from Lemma 3.19 that $X(\cdot; x, \vartheta)/X_\rho$ is also a local martingale. Thus, $K/X_\rho$ is a local martingale and the product rule gives

$$\frac{K}{X_\rho} = \int_0^{\cdot} K(t-)\, d\left(\frac{1}{X_\rho}\right)(t) + \int_0^{\cdot} \frac{1}{X_\rho(t)}\, dK(t).$$

Since the integrator of the first integral on the right-hand side is a local martingale, the last integral is also a local martingale. Being a nonnegative local martingale, the last term is also a supermartingale, and nondecreasing. This shows $K \equiv 0$, since the integrand $1/X_\rho$ is strictly positive. □

## 3.8 | Open market embedded in a market of stochastic dimension

The concept of *open market* was introduced in Fernholz (2018). When the equity market consists of a fixed number $M \in \mathbb{N}$ of stocks, the investors are allowed to invest only in the top $m$ capitalization stocks at all times for $m \leq M$. The fundamental result of arbitrage theory, in the sense of Theorem 3.14, is then proven in Karatzas and Kim (2021) for this top $m$ open market. More recently, some empirical examples of numerically optimized portfolios under open market constraints are given in Campbell and Wong (2022), and the work of Itkin and Larsson (2021) presents a parametric family of market weight models in a slightly generalized setting of an open market. We explore in this subsection the notion of open markets in the setting of investing universe with a stochastic dimension.

Throughout this subsection, we shall fix a positive integer $m \in \mathbb{N}$ for the size of the top open market. One may impose an assumption that the number of assets in the market is always bigger than or equal to $m$, that is, $N = \dim(S) \geq m$. In this case, only the $(k,n)$-dissections for dimension $n \geq m$ of $\mathbb{U}$-valued processes would be necessary. Without this assumption, when the number of stocks in the market is less than $m$, the top open market boils down to the usual *closed* market of the previous subsections, and the same theory can be applied. Therefore, we shall not impose any assumption in the following on $m$, the size of the open market.

For any $n$-dimensional vector $v = (v_1, \dots, v_n)$ for every $n \in \mathbb{N}$, we define the $k$th ranked component $v_{(k)}$ by

$$v_{(k)} := \max_{1 \leq i_1 \cdots \leq i_k \leq n} \min\{v_{i_1}, \dots, v_{i_k}\}, \tag{64}$$

satisfying

$$\max_{i=1,\dots,n} v_i = v_{(1)} \geq v_{(2)} \geq \cdots \geq v_{(n)} = \min_{i=1,\dots,n} v_i.$$

We shall use a lexicographic rule for breaking ties that always assigns a higher rank (smaller $(k)$) to a smaller index $i$.

Moreover, for any $(k,n) \in \mathbb{N}^2$, we recall the notation $[n] := \{1, \dots, n\}$, and define a process $[n] \times [0,\infty) \ni (i,t) \mapsto u_i^{k,n}(t) \in [n]$ such that each $u_i^{k,n}(\cdot)$ is predictable and satisfies

$$\begin{aligned} S_i(t) &= S_{\left(u_i^{k,n}(t)\right)}(t), && \text{on the dissection set } [\![\tau_{k-1}, \tau_k]\!] \cap (\mathbb{R}_+ \times \Omega^{k,n}), \\ u_i^{k,n}(t) &= i, && \text{otherwise,} \end{aligned} \tag{65}$$

for every $i \in [n]$. In other words, on the dissection set, $u_i^{k,n}(t)$ represents the rank of the $i$th stock in terms of capitalization among $n$ stocks at time $t$. Since $u_i^{k,n}$ shall act only on the dissection set, the choice $u_i^{k,n} \equiv i$ on the complement set, is not important. Here, both index and rank of the stocks may shift due to the dimensional change (at each reset sequence $\tau_k$), and we assume that appropriate relabeling of index (and corresponding rank) is performed whenever necessary to inherit each company's dynamics after every dimensional change.

**Definition 3.23** (Censored return process). For a given return process $R$ of Definition 3.2, we define *censored return process* $\widetilde{R}$ via dissection

$$\widetilde{R}_i^{k,n}(t) := \int_0^t \mathbb{1}_{\{u_i^{k,n}(s) \leq m\}} dR_i^{k,n}(s), \qquad i \in [n], \quad k,n \in \mathbb{N}, \quad t \geq 0.$$

The process $\widetilde{R}_i^{k,n}(t)$ represents the cumulative return of the $i$th stock of the $(k,n)$-dissected market, accumulated over $[0,t]$ only when this stock ranks among the top $m$ by capitalization, out of $n$ companies present in the market. When $n < m$, that is, the number of stocks existent in the market is less than the size $m$ of the open market, we note the identity $\widetilde{R}^{k,n} \equiv R^{k,n}$.

Imposing an additional condition that restricts investing in the $i$th stock, whenever the rank of the stock is bigger than $m$, we have the following definitions of portfolio and investment strategy in the top $m$ open market.

**Definition 3.24** (Portfolio and investment strategy in the top $m$ open market). A portfolio $\pi \in \mathcal{L}(R)$ is called a *portfolio among the top m stocks*, if it satisfies for each $(k,n) \in \mathbb{N}^2$ and $i \in [n]$

$$\pi_i^{(k,n)}(t)\mathbb{1}_{\{u_i^{k,n}(t)>m\}} = 0. \tag{66}$$

We denote $\mathcal{L}(R) \cap \mathcal{T}(m)$ the collection of portfolios among the top $m$ stocks. Similarly, an *investment strategy among the top m stocks* is an investment strategy $\vartheta \in \mathcal{L}_0(S)$ satisfying

$$\vartheta_i^{(k,n)}(t)\mathbb{1}_{\{u_i^{k,n}(t)>m\}} = 0,$$

and we denote $\mathcal{L}_0(S) \cap \mathcal{T}(m)$ the collection of investment strategies among the top $m$ stocks.

We note that the condition (66) is equivalent to $\pi_i^{(k,n)}(t)\mathbb{1}_{\{u_i^{k,n}(t)\le m\}} = \pi_i^{(k,n)}(t)$. From Equations (9) and (11), we have the similar representation of the wealth $X_\pi$ of $\pi \in \mathcal{L}(R) \cap \mathcal{T}(m)$

$$X_\pi = \mathcal{E}(R_\pi) = \mathcal{E}\left(\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} R_\pi^{k,n}\right),$$

$$\begin{aligned} R_\pi^{k,n} = \sum_{i=1}^{n}\int_0^{\cdot} \pi_i^{(k,n)}(s)\, dR_i^{k,n}(s) &= \sum_{i=1}^{n}\int_0^{\cdot} \pi_i^{(k,n)}(s)\mathbb{1}_{\{u_i^{k,n}(s)\le m\}}\, dR_i^{k,n}(s) \\ &= \sum_{i=1}^{n}\int_0^{\cdot} \pi_i^{(k,n)}(s)\, d\widetilde{R}_i^{k,n}(s) = \pi^{(k,n)} \cdot \widetilde{R}^{k,n} \end{aligned} \tag{67}$$

where $R^{k,n}$ is replaced by $\widetilde{R}^{k,n}$.

In what follows, we define censored version of the other processes from Sections 3.1–3.3, which are relevant for the open market. We consider the semimartingale decomposition $\widetilde{R}_i^{k,n} := \widetilde{A}_i^{k,n} + \widetilde{M}_i^{k,n}$ and define the cumulative covariation and the local rates

$$\widetilde{C}_{i,j}^{k,n} := [\widetilde{M}_i^{k,n}, \widetilde{M}_j^{k,n}],$$

$$\widetilde{\alpha}_i^{k,n} := \mathbb{1}_{\{u_i^{k,n}\le m\}}\alpha_i^{k,n}, \qquad \widetilde{c}_{i,j}^{k,n} := \mathbb{1}_{\{u_i^{k,n}\le m\}}\mathbb{1}_{\{u_j^{k,n}\le m\}}c_{i,j}^{k,n}$$

for each $(i,j) \in [n]^2$ and $(k,n) \in \mathbb{N}^2$ (cf. Equations 24 and 26). Introducing the diagonal-matrix-valued predictable process $D^{k,n} \equiv (D_{i,j}^{k,n})_{i,j\in[n]}$ with entries

$$D_{i,j}^{k,n}(t) := \begin{cases} \mathbb{1}_{\{u_i^{k,n}(t)\le m\}}, & i \ne j, \\ 0, & i = j, \end{cases} \qquad t \ge 0,$$

for each pair $(k,n) \in \mathbb{N}^2$, we have the alternative matrix form

$$d\widetilde{C}^{k,n}(t) = D^{k,n}(t)\, dC^{k,n}(t)\, D^{k,n}(t), \qquad \widetilde{\alpha}^{k,n} = D^{k,n}\alpha^{k,n}, \qquad \widetilde{c}^{k,n} = D^{k,n}c^{k,n}D^{k,n}.$$

As in Equation (67), these definitions give rise to the following representations:

$$\widetilde{A}_i^{k,n} = \int_0^{\cdot} \mathbb{1}_{\{u_i^{k,n}(t)\le m\}} dA_i^{k,n}(t) = \int_0^{\cdot} \widetilde{\alpha}_i^{k,n}(t)\, dO^{k,n}(t), \qquad \widetilde{C}_{i,j}^{k,n} = \int_0^{\cdot} \widetilde{c}_{i,j}^{k,n}(t)\, dO^{k,n}(t),$$

$$dC_{\pi\rho}^{k,n}(t) = d[R_\pi^{k,n}, R_\rho^{k,n}](t) = \pi^{(k,n)}(t)\, d\widetilde{C}^{k,n}(t)\, \rho^{(k,n)}(t),$$

for every $(k,n) \in \mathbb{N}^2$ and $\pi, \rho \in \mathcal{L}(R) \cap \mathcal{T}(m)$ (cf. Equations 26 and 13). Moreover, we denote

$$\widetilde{C}^{k,n}_{i\rho} := [\widetilde{R}^{k,n}_i, R^{k,n}_\rho], \tag{68}$$

for any $\rho \in \mathcal{L}(R) \cap \mathcal{T}(m)$ (cf. Equation 14).

Combining the argument of Karatzas and Kim (2021) and of the previous subsections, we can derive analogous results, ultimately the fundamental theorem for the top $m$ open market without much effort, by replacing the regular symbols with the corresponding tilde (censored) symbols. We illustrate such results in the following under the framework of the open market.

**Definition 3.25** (Numéraire portfolio in the top $m$ open market). A portfolio $\rho \in \mathcal{L}(R) \cap \mathcal{T}(m)$ is called *supermartingale (local martingale) numéraire portfolio among the top m stocks*, if the relative wealth process $X^\rho_\pi$ is a supermartingale (local martingale, respectively) for every portfolio $\pi \in \mathcal{L}(R) \cap \mathcal{T}(m)$ among the top $m$ stocks.

**Proposition 3.26.** *For a portfolio $\rho \in \mathcal{L}(R) \cap \mathcal{T}(m)$ among the top m stocks, the following statements are equivalent:*

*(i)* *$\rho$ is a supermartingale numéraire portfolio among the top m stocks.*
*(ii)* *$\rho$ is a local martingale numéraire portfolio among the top m stocks.*
*(iii)* *$\widetilde{A}^{k,n}_i = \widetilde{C}^{k,n}_{i\rho}$ holds for every $i \in [n]$ and $k, n \in \mathbb{N}$.*

The statement (20) of Lemma 3.4 remains true for $\rho, \pi \in \mathcal{L}(R) \cap \mathcal{T}(m)$, if we replace the symbols in the parenthesis of Equation (18) with the censored dissected return $\widetilde{R}^{k,n}$ and the expression $\widetilde{C}^{k,n}_{i\rho}$ of Equation (68). For the proof of Proposition 3.26, we define a new portfolio $\widetilde{\nu}_{(i,j)}$ among the top $m$ stocks for every fixed $i \le j$, defined via dissection from Equation (23)

$$\widetilde{\nu}^{(k,n)}_{(i,j)}(\cdot) := \nu^{(k,n)}_{(i,j)}(\cdot)\mathbb{1}_{\{u_i(\cdot) \le m\}}.$$

Using the aforementioned tilde symbols instead of the regular ones in the proof of Proposition 3.6 establishes Proposition 3.26.

Since the condition (iii) of Proposition 3.26 can be reformulated in terms of local rates (cf. Equation 28)

$$\widetilde{\alpha}^{k,n} = \widetilde{c}^{k,n}\rho^{(k,n)}, \qquad (\mathbb{P} \otimes O^{k,n}) - \text{a.e.}, \tag{69}$$

we expect the similar result of Proposition 3.7 for the top $m$ open market. We first define the *maximal growth rate*

$$\widetilde{g}^{k,n} := \sup_{p \in \mathbb{R}^n \cap \mathcal{T}(m)} \left( (\widetilde{\alpha}^{k,n})^\top p - \frac{1}{2} p^\top \widetilde{c}^{k,n} p \right), \tag{70}$$

achievable in the $(k,n)$-dissected, top $m$ open market (cf. Equation 32). Here, $\mathbb{R}^n \cap \mathcal{T}(m)$ denotes the subset of elements in $\mathbb{R}^n$ such that at most $m$ components are nonzero. Applying the same computational technique used in Section 3.3 of Karatzas and Kim (2021) to the tilde symbols of each dissected market, we can derive the following version of Proposition 3.7.

**Proposition 3.27.** *For a fixed, arbitrary pair* $(k,n) \in \mathbb{N}^2$*, the following statements are equivalent:*

*(i)* *There exists an n-dimensional* $\widetilde{R}^{k,n}$*-integrable process* $\rho^{(k,n)}$ *having values in* $\mathbb{R}^n \cap \mathcal{T}(m)$ *with* $\rho^{(k,n)}(0) = 0^{(n)}$ *such that* $\widetilde{\alpha}^{k,n} = \widetilde{c}^{k,n}\rho^{(k,n)}$ *holds* $(\mathbb{P} \otimes O^{k,n})$*-a.e.*

*(ii)* $\widetilde{\alpha}^{k,n} \in \mathbf{range}(\widetilde{c}^{k,n})$ *for* $(\mathbb{P} \otimes O^{k,n})$*-a.e., and* $\int_0^T (\widetilde{\alpha}^{k,n}(t))^\top (\widetilde{c}^{k,n}(t))^\dagger \widetilde{\alpha}^{k,n}(t)\, dO^{k,n}(t) < \infty$ *for every* $T \geq 0$*.*

*(iii)* *The* $(k,n)$*-dissected top m open market has locally finite growth, that means, for every* $T \geq 0$*, we have* $\widetilde{G}^{k,n}(T) := \int_0^T \widetilde{g}^{k,n}(t)\, dO^{k,n}(t) < \infty$*.*

Again, the process given by

$$\rho^{(k,n)} := (\widetilde{c}^{k,n})^\dagger \widetilde{\alpha}^{k,n}, \qquad \text{(modulo null portfolio)} \tag{71}$$

satisfies the conditions $(i) - (iii)$ of Proposition 3.27, and $\rho^{(k,n)}$ attains the maximal growth rate $\widetilde{g}^{k,n}$ of Equation (70), which is equal to $(1/2)(\widetilde{\alpha}^{k,n})^\top (\widetilde{c}^{k,n})^\dagger \widetilde{\alpha}^{k,n}$. Then, we have the similar representation of $\widetilde{g}^{k,n}$ as in Equation (35) and the supermartingale numéraire portfolio $\rho$ among the top $m$ stocks, can be constructed from dissection $\rho^{(k,n)}$ of Equation (71).

We finally present the associated definitions and results of Section 3.4 in the context of top $m$ open markets.

**Definition 3.28** (Arbitrage of the first kind in the top $m$ open market). An *arbitrage of the first kind* for horizon $T > 0$ in the top $m$ open market is an $\mathcal{F}_T$-measurable random variable $h$ satisfying $\mathbb{P}(h \geq 0) = 1$, $\mathbb{P}(h > 0) > 0$, such that for every $x > 0$, there exists an admissible investment strategy $\vartheta$ among the top $m$ stocks, satisfying $X(\cdot; x, \vartheta) \geq 0$ and $X(T; x, \vartheta) \geq h$. If there are no arbitrages of the first kind, we say that $NA_1$ holds in the top $m$ open market.

**Definition 3.29** (Local martingale deflator in the top $m$ open market). An adapted, positive process $Y$ is called *local martingale deflator among the top m stocks*, if $Y(0) = 1$ and $YX_\pi$ is a local martingale for every portfolio $\pi \in \mathcal{L}(R) \cap \mathcal{T}(m)$ among the top $m$ stocks.

**Definition 3.30** (Market viability of the top $m$ open market). For a given nondecreasing, adapted, and right-continuous process $K$ with $K(0) = 0$, we define *financing capital* associated with $K$ in the top $m$ open market by

$$x^m(K) := \inf_{x \geq 0}\{\exists\, \vartheta \in \mathcal{L}_0(S) \cap \mathcal{T}(m) \text{ such that } X(\cdot; x, \vartheta) \geq K\}. \tag{72}$$

We say that the top $m$ open market is *viable*, if $x^m(K) = 0$ implies $K \equiv 0$.

Proposition 3.12 can be immediately extended to the top $m$ open market using the above Definitions 3.28–3.30, when the condition (43) is substituted for

$$\lim_{m \to \infty} \sup_{\pi \in \mathcal{L}(R) \cap \mathcal{T}(m)} \mathbb{P}[X_\pi(T) > m] = 0 \text{ holds for any } T \geq 0.$$

Therefore, we present the fundamental theorem for the top $m$ open market.

**Theorem 3.31.** *The following statements are equivalent:*

*(i)* *There exists a supermartingale (also a local martingale) numéraire portfolio among the top m stocks.*
*(ii)* *There exists a portfolio $\rho$ satisfying $\widetilde{A}_i^{k,n} = \widetilde{C}_{i\rho}^{k,n}$ for every $i \in [n]$ and $(k,n) \in \mathbb{N}^2$.*
*(iii)* *Each $(k,n)$-dissected top m open market has locally finite growth, that is, $\widetilde{G}^{k,n}(T) < \infty$ for every $T \geq 0$ and $(k,n) \in \mathbb{N}^2$.*
*(iv)* *$NA_1$ holds in the top m open market.*
*(v)* *There exists a local martingale deflator among the top m stocks.*
*(vi)* *The top m open market is viable.*

It is sufficient to prove the implication $(vi) \Longrightarrow (iii)$, but the same proof of Theorem 3.14 can be used by replacing the symbols $\alpha^{\ell,p}, c^{\ell,p}, R^{\ell,p}, A^{\ell,p}$, and $M^{\ell,p}$ with the corresponding tilde symbols.

We finally present the ODT in the framework of open market. First, let us denote $\mathcal{Y}^m$ the collection of local martingale deflators among the top $m$ stocks, and $\mathcal{M}_{loc}^{\perp}(\widetilde{M})$ the collection of scalar local martingales $L$ with RCLL paths, satisfying $L(0) = 0$ and the strong orthogonality condition $[L, \widetilde{M}_i^{k,n}] = 0$ for every $(k,n) \in \mathbb{N}^2$ and $i \in [n]$. It is then easy to modify the proof of Proposition 3.13 to derive the characterization

$$\mathcal{Y}^m = \left\{ \frac{\mathcal{E}(L)}{X_\rho} \,\middle|\, L \in \mathcal{M}_{loc}^{\perp}(\widetilde{M}), \quad \Delta L > -1 \right\}. \tag{73}$$

Moreover, we have the following version of Lemma 3.19. Denoting $\rho$ the numéraire portfolio among the top $m$ stocks and $X(\cdot) \equiv X(\cdot; x, \vartheta)$ for any given $x \in \mathbb{R}$ and investment strategy $\vartheta \in \mathcal{L}_0(S) \cap \mathcal{T}(m)$ among the top $m$ stocks, there exists a $\mathbb{U}$-valued process $\eta$ such that each $\eta^{(k,n)}$ is $\widetilde{M}^{k,n}$-integrable, $\eta_i^{(k,n)}(\cdot)\mathbb{1}_{\{u_i^{k,n}(\cdot) \leq m\}} = \eta_i^{(k,n)}(\cdot)$ holds for every $(k,n) \in \mathbb{N}^2$, and satisfies

$$\frac{X}{X_\rho} = x + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \int_0^{\cdot} \sum_{i=1}^{n} \eta_i^{(k,n)}(u)\, d\widetilde{M}_i^{k,n}(u). \tag{74}$$

Indeed, defining

$$\eta_i^{(k,n)}(t) := \frac{S_i(t)\vartheta_i^{(k,n)}(t) - X(t)\rho_i^{(k,n)}(t)}{X_\rho(t)}, \qquad \forall\, (k,n) \in \mathbb{N}^2,$$

as in the proof of Lemma 3.19 shows the existence of such process $\eta$.

Now that we have the representations (73) and (74) in hand, the following optional decomposition results can be obtained by the same arguments in Section 3.7 with the tilde symbols replacing the regular ones.

**Theorem 3.32.** *Assume that the top m open market is viable. For a nonnegative, adapted process $X$ with $X(0) = x \geq 0$, the following statements are equivalent:*

*(i)* *For every local martingale deflator $Y \in \mathcal{Y}^m$ among the top m stocks, the process $YX$ is a supermartingale.*

*(ii) There exist an investment strategy $\vartheta \in \mathcal{L}_0(S) \cap \mathcal{T}(m)$ among the top $m$ stocks and an adapted, nondecreasing process $K$ satisfying $K(0) = 0$ with right-continuous paths, such that $X = x + \vartheta \cdot S - K$ holds.*

*Furthermore, the following statements are also equivalent:*

*(iii) For every local martingale deflator $Y \in \mathcal{Y}^m$ among the top $m$ stocks, the process $YX$ is a local martingale.*
*(iv) There exists an investment strategy $\vartheta \in \mathcal{L}_0(S) \cap \mathcal{T}(m)$ among the top $m$ stocks satisfying $X \equiv X(\cdot; x, \vartheta)$, that is, $X = x + \vartheta \cdot S$.*

We conclude this subsection with the following remark.

*Remark* 3.33. In this subsection (and in the work of Karatzas & Kim, 2021), the essential idea of handling the open market is to censor the return process by the event $\{u_i^{k,n}(s) \le m\}$ in Definition 3.23 to keep the dimension of the open market equal to $m$. However, this censoring technique also works for any $\mathcal{F}_s$-measurable event $E_i(s)$, in other words, we can consider instead

$$\widetilde{R}_i^{k,n}(t) := \int_0^t \mathbb{1}_{E_i(s)} dR_i^{k,n}(s), \qquad i \in [n], \quad k, n \in \mathbb{N}, \quad t \ge 0,$$

in Definition 3.23. This process $\widetilde{R}_i^{k,n}$ now represents the cumulative return of the $i$th stock of the $(k, n)$-dissected market, censored according to the (collection of) events $\{E_i(s)\}_{s\ge 0}$. Then, the corresponding portfolio censored by the events $\{E_i(s)\}_{s\ge 0}$ should satisfy

$$\pi_i^{(k,n)}(t)\mathbb{1}_{(E_i(s))^c} = 0,$$

instead of Equation (66) in Definition 3.24. If we modify all the other definitions accordingly (replacing $\{u_i^{k,n}(s) \le m\}$ with $E_i(s)$), all results in this subsection (Theorems 3.31 and 3.32) remain to hold.

Now that the dimension of the entire market changes over time, we can also consider the open market of stochastic dimension. For example, instead of choosing $m$ largest stocks by their rank at all times, we can select large stocks whose capitalization exceeds a certain threshold $\epsilon > 0$ by setting $E_i(s) = \{S_i(s) \ge \epsilon\}$ to construct another open market composed of large capitalization stocks. Then, this open market has a dimension process equal to $M(t) := |\{i : S_i(t) \ge \epsilon\}|$, and the portfolios in this open market can only invest in the stocks with capitalization greater than or equal to $\epsilon$ at all times.

## 4 | MARKET WITH GENERAL PRICE PROCESS

We now relax the assumption on the continuity and strict positivity of the price process between the dimensional changes. Precisely, we shall consider a piecewise RCLL semimartingale $S$ to represent the price process of assets in the market. This general setting requires different proof techniques for the results presented in the previous section.

## 4.1 | Preliminaries

First, we give a more general definition of the price process than the one given in Definition 3.1. Some preliminary results, which will be frequently used in the proofs in the following subsections, are then presented.

**Definition 4.1** (Price process). A $\mathbb{U}$-valued process $S$ is called a *price process*, if it is a piecewise RCLL semimartingale with a reset sequence $(\tau_k)_{k\geq 0}$.

In Section 3, we considered a concept of investment strategy $\vartheta$ and its corresponding portfolio $\pi$ such that they are connected via Equations (39) and (41). Their wealth process $X_\pi(\cdot) \equiv X(\cdot;1,\vartheta)$ in Equation (40) is a strictly positive process as a stochastic exponential and $\vartheta$ has to be strictly admissible in order to define the portfolio $\pi$. However, since the concept of the portfolio is no longer needed in this section, we can allow the wealth process $X(\cdot;x,\vartheta)$ of an investment strategy $\vartheta$ to hit zero at some time. From this perspective, we do not assume strict positivity, or even nonnegativity, of the price process $S$ in Definition 4.1. Since we shall only consider admissible investment strategies as in the following definition, only the nonnegative condition on the wealth process is important, not on the price process. We refer to Karatzas and Kardaras (2007, Section 4.8) for a more detailed remark regarding the positivity condition of the price process.

**Definition 4.2** (Investment strategy). For a given price process $S$ of Definition 4.1, an investment strategy and its wealth process are defined as in Definition 3.8. The collection of all wealth processes of admissible investment strategies $\vartheta$ with initial capital $x > 0$ is denoted by

$$\mathcal{W} := \{X \equiv X(\cdot;x,\vartheta) \mid \vartheta \in \mathcal{L}_0(S),\ x > 0,\ X \geq 0\}.$$

We denote the subset $\mathcal{X} \subset \mathcal{W}$ of all wealth processes of strictly admissible investment strategies

$$\mathcal{X} := \{X \in \mathcal{W} : X > 0\}.$$

We also need to modify the definition of local martingale deflators accordingly.

**Definition 4.3** (Local martingale deflator). An adapted strictly positive process $Y$ satisfying $Y(0) = 1$ is called *local martingale deflator*, if $YX$ is a local martingale for every $X \in \mathcal{W}$. We denote $\mathcal{Y}$ the set of all local martingale deflators. Moreover, for a given price process $S$ and any $(k,n) \in \mathbb{N}^2$, an adapted, positive process $Y$ satisfying $Y(0) = 1$ is called $(k,n)$-*local martingale deflator*, if the product $Y(x + \vartheta \cdot S^{k,n})$ is a local martingale for every $n$-dimensional $S^{k,n}$-integrable process $\vartheta$ and $x > 0$, satisfying $x + \vartheta \cdot S^{k,n} \geq 0$. We denote $\mathcal{Y}^{k,n}$ the set of all $(k,n)$-local martingale deflators for every $(k,n) \in \mathbb{N}^2$.

Observe that $\mathcal{Y} \subset \mathcal{Y}^{k,n}$ for every $(k,n) \in \mathbb{N}^2$, since for an arbitrary $n$-dimensional $S^{k,n}$-integrable process $\vartheta$, we can construct a $\mathbb{U}$-valued process $\vartheta'$ by setting $\vartheta' := \vartheta$ on the $(k,n)$-dissection set and $\vartheta' := 0^{(m)}$ on any other $(\ell,m)$-dissection sets for $(\ell,m) \neq (k,n)$ such that $\vartheta' \cdot S = \vartheta \cdot S^{k,n}$ holds in view of the definition of stochastic integral in Equation (6). We also note that any local martingale deflator $Y$ and any $(k,n)$-local martingale deflator $Y^{k,n}$ are themselves local martingales by taking $\vartheta(\cdot) \equiv 0^{(N_\cdot)}$ (a $\mathbb{U}$-valued zero vector process with the same dimension process as $S$).

Furthermore, for any $(k,n) \in \mathbb{N}^2$, we note from Proposition 2.5 of Karatzas and Ruf (2017) that there is an equivalent, but seemingly weaker characterization

$$\mathcal{Y}^{k,n} = \mathcal{Z}(S^{k,n}) \tag{75}$$

of the set $\mathcal{Y}^{k,n}$ in Definition 4.3, where $\mathcal{Z}(S^{k,n})$ is the collection of all positive local martingales $Z$ with $Z(0) = 1$ such that $ZS^{k,n}$ is a local martingale.

We end this subsection by providing some lemmas.

**Lemma 4.4.** *Let $Z$ be a local supermartingale (resp. local martingale) and $\tau$ a stopping time such that $Z = 0$ on $[\![0,\tau]\!]$. If $X$ is an adapted process, then $X^\tau Z$ is a local supermartingale (resp. local martingale).*

*Proof.* By the Doob–Meyer decomposition, we have $Z = M - A$ for a local martingale $M$ and a predictable, nondecreasing process $A$. Since $Z = 0$ on $[\![0,\tau]\!]$, we also have $A = M = 0$ on $[\![0,\tau]\!]$. Observe that we can write

$$X^\tau Z = X^\tau M - X^\tau A = \int (X(\tau)\mathbb{1}_{]\!]\tau,\infty[\![}) \, dM - X(\tau)A. \tag{76}$$

Since $X(\tau)\mathbb{1}_{]\!]\tau,\infty[\![}$ is locally bounded and predictable, $\int (X(\tau)\mathbb{1}_{]\!]T,\infty[\![}) \, dM$ is a local martingale and thus $X^\tau Z$ is a local supermartingale. □

In addition to Lemma 4.4, we shall need the following easy extension of Lemma 2.12 of Strong (2014).

**Lemma 4.5.** *Suppose that $\eta$ is a stopping time, $(C_j)_{j\in\mathbb{N}}$ is an $\mathcal{F}_\eta$-measurable partition of $\Omega$, and $Y$ is an $\mathbb{R}$-valued semimartingale equal to 0 on the interval $[\![0,\eta]\!]$.*

*(i) If $Y\mathbb{1}_{C_j}$ is a local martingale for all $j \in \mathbb{N}$, then $Y$ is a local martingale.*
*(ii) If $Y$ is nonnegative and $Y\mathbb{1}_{C_j}$ is a supermartingale for all $j \in \mathbb{N}$, then $Y$ is a supermartingale.*

*Proof.* The assertion (i) is just Lemma 2.12 (2) of Strong (2014). For (ii), Fatou's lemma yields for $0 \le s \le t$

$$\mathbb{E}[Y_t|\mathcal{F}_s] = \mathbb{E}\left[\sum_{j=1}^{\infty} Y_t \mathbb{1}_{C_j} \middle| \mathcal{F}_s\right] \le \sum_{j=1}^{\infty} \mathbb{E}[Y_t \mathbb{1}_{C_j}|\mathcal{F}_s] \le \sum_{j=1}^{\infty} Y_s \mathbb{1}_{C_j} = Y_s.$$

□

## 4.2 | The optional decomposition theorem

In this subsection, we state and prove the ODT for a general piecewise semimartingale price process $S$ with RCLL paths. This will be the cornerstone for proving the following results, including the existence of an investment strategy with the supermartingale numéraire property and the superhedging duality.

Let us mention that the proof of the ODT in Section 3.7 was more direct and illustrative leading to a rather explicit representation of the investment strategy (see Equation 62) in the optional decomposition. However, as we mentioned in the introduction, the same proof technique does not extend to the case when jumps are present in the dynamics of asset prices (see Karatzas & Kardaras, 2015). In contrast, the proof that we present in this subsection relies on the classical ODT for RCLL semimartingales of fixed dimension from Stricker and Yan (1998).

**Theorem 4.6.** *Suppose that $\mathcal{Y} \neq \emptyset$, that is, a local martingale deflator exists. For a nonnegative, adapted process $X$ with $X(0) = x \geq 0$, the following statements are equivalent:*

*(i) For every local martingale deflator $Y \in \mathcal{Y}$, the process $YX$ is a supermartingale.*
*(ii) There exist an investment strategy $\vartheta \in \mathcal{L}_0(S)$ and an adapted, nondecreasing process $K$ satisfying $K(0) = 0$ with right-continuous paths, such that*

$$X = x + \vartheta \cdot S - K = x + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\int_0^{\cdot}\sum_{i=1}^{n}\vartheta_i^{k,n}(t)\,dS_i^{k,n}(t) - K. \tag{77}$$

*Proof.* For the implication, $(ii) \Longrightarrow (i)$, the integration by parts gives $YK = K_- \cdot Y + Y \cdot K$ for $Y \in \mathcal{Y}$. Since $K_- \cdot Y$ is a local martingale and $Y \cdot K$ is a nondecreasing process, $YK$ is a local submartingale. Moreover, $Y(x + \vartheta \cdot S)$ is a local martingale, which implies that $YX$ is a nonnegative local supermartingale and hence a true supermartingale.

For the proof of the implication $(i) \Longrightarrow (ii)$, let us denote $X^k := X^{\tau_k} - X^{\tau_{k-1}}$ and $X^{k,n} := \mathbb{1}_{\Omega^{k,n}}X^k$ to obtain the representation $X = X(0) + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}X^{k,n}$. Our goal is to show that each $X^{k,n}$ admits the decomposition

$$X^{k,n} = \vartheta^{k,n} \cdot S^{k,n} - K^{k,n} \tag{78}$$

for some $n$-dimensional $S^{k,n}$-integrable process $\vartheta^{k,n}$ and an adapted nondecreasing process $K^{k,n}$ with $K^{k,n}(0) = 0$.

Thanks to the assumption $\mathcal{Y} \neq \emptyset$ and the observation $\mathcal{Y} \subset \mathcal{Y}^{k,n}$ following Definition 4.3, we choose an arbitrary $(k,n)$-local martingale deflator $Y^{k,n}$ in $\mathcal{Y}^{k,n}$ for every $(k,n) \in \mathbb{N}^2$, to construct the set $\{Y^{k,n}\}_{(k,n)\in\mathbb{N}^2}$.

We now define

$$Z := \prod_{k=1}^{\infty} Z^k, \qquad \text{where} \quad Z^k := \mathbb{1}_{\{\tau_{k-1}=\infty\}} + \sum_{n=1}^{\infty}\mathbb{1}_{\Omega^{k,n}}\left(\frac{(Y^{k,n})^{\tau_k}}{(Y^{k,n})^{\tau_{k-1}}}\right),$$

then we have $Z \in \mathcal{Y}$ from the proof of Theorem 3.5 of Strong (2014). By the hypothesis $(i)$, $ZX$ is a supermartingale. The Doob–Meyer decomposition yields $ZX = X(0) + M - A$ for a local martingale $M$ and a nondecreasing predictable process $A$ with $M(0) = A(0) = 0$. This implies that $(ZX)^{\tau_k} - (ZX)^{\tau_{k-1}} = M^{\tau_k} - M^{\tau_{k-1}} - (A^{\tau_k} - A^{\tau_{k-1}})$ is a local supermartingale. Now let us write

$$Z^{\tau_k}(X^{\tau_k} - X^{\tau_{k-1}}) = (ZX)^{\tau_k} - (ZX)^{\tau_{k-1}} - (Z^{\tau_k} - Z^{\tau_{k-1}})X^{\tau_{k-1}}.$$

Lemma 4.4 implies that $(Z^{\tau_k} - Z^{\tau_{k-1}})X^{\tau_{k-1}}$ is a local martingale, so $Z^{\tau_k}(X^{\tau_k} - X^{\tau_{k-1}})$ is a local supermartingale. We derive that

$$\mathbb{1}_{\Omega^{k,n}}\mathbb{1}_{\llbracket\tau_{k-1},\infty\rrbracket}\frac{Z^{\tau_k}}{Z^{\tau_{k-1}}}(X^{\tau_k} - X^{\tau_{k-1}}) = \mathbb{1}_{\Omega^{k,n}}Z^k(X^{\tau_k} - X^{\tau_{k-1}}) = \frac{(Y^{k,n})^{\tau_k}}{(Y^{k,n})^{\tau_{k-1}}}X^{k,n},$$

and the last expression is a local supermartingale, again on the strength of Lemma 4.4. Another application of Lemma 4.4 implies that $(Y^{k,n})^{\tau_k}X^{k,n}$ is a local supermartingale. We write $Y^{k,n}X^{k,n} = (Y^{k,n} - (Y^{k,n})^{\tau_k})X^{k,n} + (Y^{k,n})^{\tau_k}X^{k,n}$, and yet another application of Lemma 4.4 with the identity $X^{k,n} \equiv (X^{k,n})^{\tau_k}$ yields that $(Y^{k,n} - (Y^{k,n})^{\tau_k})X^{k,n}$ is a local martingale, thus, $Y^{k,n}X^{k,n}$ is a local supermartingale. Keeping in mind that $X^{k,n} + X(\tau_{k-1})\mathbb{1}_{\Omega^{k,n}}\mathbb{1}_{[\![\tau_{k-1},\infty[\![}$ is nonnegative, we consider an $\mathcal{F}(\tau_{k-1})$-measurable partition of $\Omega$ given by $\cup_{j\in\mathbb{N}}A_j \cup \{\tau_{k-1} = \infty\}$ where $A_j = \{\tau_{k-1} < \infty,\ j-1 \le X(\tau_{k-1}) < j\}$. Hence, $\mathbb{1}_{A_j}\mathbb{1}_{[\![\tau_{k-1},\infty[\![}Y^{k,n}X^{k,n}$ is an adapted process bounded from below for each $j \in \mathbb{N}$.

On the other hand, the Doob–Meyer decomposition implies that $Y^{k,n}X^{k,n} = N^{k,n} - B^{k,n}$ for a local martingale $N^{k,n}$ and a nondecreasing predictable process $B^{k,n}$, both of which satisfy $N^{k,n} = B^{k,n} = 0$ on $[\![0,\tau_{k-1}]\!]$, as $X^{k,n} = 0$ on $[\![0,\tau_{k-1}]\!]$. Since $\mathbb{1}_{A_j}\mathbb{1}_{[\![\tau_{k-1},\infty[\![}$ is locally bounded and simple predictable, we can write

$$\begin{aligned}\mathbb{1}_{A_j}\mathbb{1}_{[\![\tau_{k-1},\infty[\![}Y^{k,n}X^{k,n} &= \mathbb{1}_{A_j}\mathbb{1}_{[\![\tau_{k-1},\infty[\![}N^{k,n} - \mathbb{1}_{A_j}\mathbb{1}_{[\![\tau_{k-1},\infty[\![}B^{k,n}\\ &= \int \mathbb{1}_{A_j}\mathbb{1}_{[\![\tau_{k-1},\infty[\![}dN^{k,n} - \mathbb{1}_{A_j}\mathbb{1}_{[\![\tau_{k-1},\infty[\![}B^{k,n},\end{aligned}$$

where $\int \mathbb{1}_{A_j}\mathbb{1}_{[\![\tau_{k-1},\infty[\![}dN^{k,n}$ is a local martingale and $\mathbb{1}_{A_j}\mathbb{1}_{[\![\tau_{k-1},\infty[\![}B^{k,n}$ is a predictable nondecreasing process. Consequently, we obtain that $\mathbb{1}_{A_j}\mathbb{1}_{[\![\tau_{k-1},\infty[\![}Y^{k,n}X^{k,n} = \mathbb{1}_{A_j}Y^{k,n}X^{k,n}$ is a local supermartingale bounded from below, hence a true supermartingale for each $j \in \mathbb{N}$.

Let us recall the equivalent formulation (75) of $\mathcal{Y}^{k,n}$. Since the process $\mathbb{1}_{A_j}X^{k,n}Y^{k,n}$ is a supermartingale for any $Y^{k,n} \in \mathcal{Y}^{k,n} = \mathcal{Z}(S^{k,n})$, we apply a version of ODT in Stricker and Yan (1998), to obtain the decomposition for each $j \in \mathbb{N}$

$$\mathbb{1}_{A_j}X^{k,n} = \vartheta^{k,n,j} \cdot S^{k,n} - K^{k,n,j} \tag{79}$$

for some $S^{k,n}$-integrable process $\vartheta^{k,n,j}$ and an adapted nondecreasing process $K^{k,n,j}$ satisfying $K^{k,n,j}(0) = 0$. By setting $\vartheta^{k,n} := \sum_{j=1}^{\infty}\mathbb{1}_{A_j}\vartheta^{k,n,j}$ and $K^{k,n} := \sum_{j=1}^{\infty}\mathbb{1}_{A_j}K^{k,n,j}$, we obtain the desired decomposition (78).

Finally, taking $\vartheta := 0^{(N_0)} + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\vartheta^{k,n}\hat{\mathbb{1}}_{[\![\tau_{k-1},\tau_k]\!]\cap(\mathbb{R}_+\times\Omega^{k,n})}$ and $K := \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}K^{k,n}$ yields the result (77) in view of the definition of stochastic integral in Equation (6). □

The following result is the local martingale version of the ODT as in Corollary 3.22. It can be proved in the same manner as Corollary 3.22, except that here we can choose any $Y \in \mathcal{Y}$, instead of the reciprocal $1/X_\rho$ of the local martingale numéraire there.

**Corollary 4.7.** *Suppose that $\mathcal{Y} \neq \emptyset$. For a nonnegative, adapted process $X$ with $X(0) = x \ge 0$, the following statements are equivalent:*

*(i) For every local martingale deflator $Y \in \mathcal{Y}$, the process $YX$ is a local martingale.*
*(ii) There exists an investment strategy $\vartheta \in \mathcal{L}_0(S)$ satisfying $X \equiv X(\cdot; x, \vartheta)$.*

## 4.3 | The fundamental theorem

As the first application of the ODT, we prove that the ELMD implies the existence of an investment strategy, in the general piecewise semimartingale market, such that the corresponding

wealth process has the supermartingale numéraire property. As it was previously mentioned in the introduction, Kabanov et al. (2016) noted that in the case of a finite dimensional RCLL semimartingale market (of a fixed dimension), the supermartingale numéraire portfolio and the local martingale numéraire portfolio may not coincide. In the presence of jumps, the local martingale numéraire portfolio may not even exist under the stronger assumption of the existence of an equivalent martingale measure (see Takaoka & Schweizer, 2014).

Once we have related the existences of a local martingale deflator to that of the supermartingale numéraire investment strategy, we provide the fundamental theorem by connecting them to the other concepts such as $NA_1$ and market viability.

In Definition 4.2, for any given $X \equiv X(\cdot; x, \vartheta)$ in $\mathcal{W}$, a simple normalization leads to define another admissible investment strategy $\tilde{\vartheta} := \vartheta/x$ satisfying $X(\cdot; 1, \tilde{\vartheta}) = X(\cdot; x, \vartheta)/x$, which also belongs to $\mathcal{W}$. Therefore, in what follows, we shall assume without loss of generality that every wealth process in $\mathcal{W}$ has the initial wealth equal to $x = 1$.

**Definition 4.8** (Investment strategy with the supermartingale numéraire property). A strictly admissible investment strategy $\vartheta_*$ with the wealth process $X_* \equiv X(\cdot; 1, \vartheta_*) \in \mathcal{X}$ is said to have the *supermartingale numéraire property*, if the ratio $X/X_*$ is a supermartingale for every $X \in \mathcal{X}$.

**Theorem 4.9.** *Suppose that $\mathcal{Y} \neq \emptyset$. Then there exists an admissible investment strategy $\vartheta_*$ having the supermartingale numéraire property.*

*Proof.* We begin the proof by first noting that $\mathcal{Y}^{k,n} \neq \emptyset$ for every $(k, n) \in \mathbb{N}^2$ from the argument below of Definition 4.3. This implies from the result of Karatzas and Kardaras (2007) that for every $(k, n) \in \mathbb{N}^2$ there exists a wealth process, denoted by $X_*^{k,n}$, which has the supermartingale numéraire property for each $n$-dimensional market where the prices process is given by $S^{k,n}$. Namely, for each $(k, n) \in \mathbb{N}^2$, the ratio $X^{k,n}/X_*^{k,n}$ is a supermartingale for every $X^{k,n}$ of the form $X^{k,n} = 1 + \vartheta^{(k,n)} \cdot S^{k,n} > 0$, for some $S^{k,n}$-integrable process $\vartheta^{(k,n)}$.

We construct the process $X_*$ by the following recipe:

$$X_*^k := \mathbb{1}_{\{\tau_{k-1}=\infty\}} + \sum_{n=1}^{\infty} \mathbb{1}_{\Omega^{k,n}} (X_*^{k,n})^{\tau_k}, \qquad X_* := \prod_{k=1}^{\infty} X_*^k. \tag{80}$$

Let us first prove that $YX_*$ is a local martingale for every $Y \in \mathcal{Y}$. For that purpose, we shall show that the product $(YX_*)^{\tau_m}$ is a local martingale for each $m \in \mathbb{N}$.

For a fixed arbitrary $Y \in \mathcal{Y}$, we also have $Y \in \mathcal{Y}^{k,n}$, thus $YX_*^{k,n}$ is a local martingale for every $(k, n) \in \mathbb{N}^2$. By denoting $\nu_*^{(k,n)}$ the investment strategy corresponding to the numéraire wealth process $X_*^{k,n}$ in each $(k, n)$-dissected market (of fixed dimension), we can write $X_*^{k,n} = 1 + \nu_*^{(k,n)} \cdot S^{k,n}$. Then, $Y(X_*^{k,n} - 1) = Y(\nu_*^{(k,n)} \cdot S^{k,n})$ is also a local martingale for each $(k, n) \in \mathbb{N}^2$. Moreover, we have $X_*^k = 1 + \sum_{n=1}^{\infty} \nu_*^{(k,n)} \cdot S^{k,n}$ and consider an $\mathcal{F}_{\tau_{k-1}}$-measurable partition

$$\bigcup_{n=1}^{\infty} \{\Omega^{k,n}\} \cup \{\tau_{k-1} = \infty\} \tag{81}$$

of $\Omega$. From Lemma 4.5, we obtain that $Y(X_*^k - 1)$, and consequently $YX_*^k$, are local martingales for every $k \in \mathbb{N}$. Furthermore, $(YX_*^k)^{\tau_k} - Y^{\tau_{k-1}}$ is a local martingale as well, and an application of Lemma 4.4 with the identity $(X_*^k)^{\tau_{k-1}} = 1$ yields the local martingale property of $(YX_*^k)^{\tau_k}/Y^{\tau_{k-1}} - 1$, and hence of $(YX_*^k)^{\tau_k}/Y^{\tau_{k-1}}$.

In what follows, we denote $Y^k := Y^{\tau_k}/Y^{\tau_{k-1}}$ for $k \in \mathbb{N}$. For a fixed $m \in \mathbb{N}$, we have $Y^{\tau_m} = \prod_{i=1}^m Y^i$ and $Y^i X^i_*$ is a local martingale for every $i \in [m]$. Let us take a common localizing sequence $(\sigma_\ell)_{\ell\in\mathbb{N}}$ for $\{Y^i X^i_*\}_{i\in[m]}$. We now prove that $(YX_*)^{\tau_m\wedge\sigma_\ell}$ is a martingale for every $\ell \in \mathbb{N}$. Since we have $Y^i X^i_*(s \vee \tau_{i-1}) = Y^i X^i_*(s)$ for each $i \in \mathbb{N}$, we obtain for any $s < t$

$$\begin{aligned}\mathbb{E}\left[(YX_*)^{\tau_m\wedge\sigma_\ell}(t)\middle|\mathcal{F}_s\right] &= \mathbb{E}\left[\prod_{i=1}^m (Y^iX^i_*)^{\sigma_\ell}(t)\,\middle|\,\mathcal{F}_s\right]\\ &= \mathbb{E}\left[\prod_{i=1}^{m-1}(Y^iX^i_*)^{\sigma_\ell}(t)\,\mathbb{E}\left[(Y^mX^m_*)^{\sigma_\ell}(t)\middle|\mathcal{F}_{s\vee\tau_{m-1}}\right]\middle|\mathcal{F}_s\right]\\ &= (Y^mX^m_*)^{\sigma_\ell}(s)\,\mathbb{E}\left[\prod_{i=1}^{m-1}(Y^iV^i_*)^{\sigma_\ell}(t)\,\middle|\,\mathcal{F}_s\right].\end{aligned}$$

By successively conditioning on $\mathcal{F}_{s\vee\tau_{m-2}}, \mathcal{F}_{s\vee\tau_{m-3}}, \dots, \mathcal{F}_{s\vee\tau_1}$, we derive that the last expression is equal to

$$\prod_{i=1}^m (Y^iX^i_*)^{\sigma_\ell}(s) = (YX_*)^{\tau_m\wedge\sigma_\ell}(s),$$

proving the martingale property of $(YX_*)^{\tau_m\wedge\sigma_\ell}$. Consequently, $(YX_*)^{\tau_m}$ for all $m \in \mathbb{N}$, hence also $YX_*$, are local martingales for any $Y \in \mathcal{Y}$.

Thanks to the ODT (Corollary 4.7), we obtain that the existence of an admissible investment strategy $\vartheta_*$ such that $X_* = 1 + \vartheta_* \cdot S$ holds. What is now left to prove is the supermartingale numéraire property of the process $X_*$. For a fixed $X \equiv X(\cdot; 1, \vartheta) \in \mathcal{X}$, we shall show by induction that $(X/X_*)^{\tau_m}$ is a supermartingale for every $m \in \mathbb{N}$; then $X/X_*$ is also a supermartingale as every nonnegative local supermartingale is a supermartingale.

Let us recall the representation $X = 1 + \sum_{k=1}^\infty \sum_{n=1}^\infty \vartheta^{(k,n)} \cdot S^{k,n}$, then the expression

$$\left(\frac{X^{\tau_1}}{X^{\tau_1}_*} - 1\right)\mathbb{1}_{\Omega^{1,n}} = \frac{1 + \vartheta^{(1,n)} \cdot S^{1,n}}{(X^{1,n}_*)^{\tau_1}} - 1 = \frac{1 + \vartheta^{(1,n)} \cdot S^{1,n}}{(X^{1,n}_*)} - 1$$

is a supermartingale for every $n \in \mathbb{N}$ from the first paragraph of this proof. Thanks to Lemma 4.5, $(X/X_*)^{\tau_1} - 1$, and thus $(X/X_*)^{\tau_1}$, are supermartingales.

Assuming that $(X/X_*)^{\tau_m}$ is a supermartingale, we shall show that $(X/X_*)^{\tau_{m+1}}$ is a supermartingale. Since $X^{\tau_{m+1}}_* = X^{\tau_m}_* X^{m+1}_*$ and $X^{\tau_{m+1}} = X^{\tau_m} + \sum_{n=1}^\infty \vartheta^{(m+1,n)} \cdot S^{m+1,n} > 0$, we have for $0 \le s \le t$

$$\mathbb{E}\left[\left(\frac{X}{X_*}\right)^{\tau_{m+1}}(t)\middle|\mathcal{F}_s\right] = \mathbb{E}\left[\left(\frac{X}{X_*}\right)^{\tau_m}(t)\,\mathbb{E}\left[\frac{1 + \frac{\sum_{n=1}^\infty \int_0^t \vartheta^{(m+1,n)}(u)dS^{m+1,n}(u)}{X^{\tau_m}(t)}}{X^{m+1}_*(t)}\,\middle|\,\mathcal{F}_{s\vee\tau_m}\right]\middle|\mathcal{F}_s\right]$$

$$= \mathbb{E}\left[\left(\frac{X}{X_*}\right)^{\tau_m}(t)\,\mathbb{E}\left[\frac{1 + \frac{\sum_{n=1}^\infty \int_0^t \vartheta^{(m+1,n)}(u)dS^{m+1,n}(u)}{X(\tau_m)}\mathbb{1}_{[\![\tau_m,\infty[\![}(t)}{X^{m+1}_*(t)}\,\middle|\,\mathcal{F}_{s\vee\tau_m}\right]\middle|\mathcal{F}_s\right]$$

$$= \mathbb{E}\left[\left(\frac{X}{X_*}\right)^{\tau_m}(t)\,\mathbb{E}\left[\frac{1+\sum_{n=1}^{\infty}\int_0^t \frac{\vartheta^{(m+1,n)}(u)}{X(\tau_m)}dS^{m+1,n}(u)}{X_*^{m+1}(t)}\,\middle|\,\mathcal{F}_{s\vee\tau_m}\right]\middle|\,\mathcal{F}_s\right].$$

Here, the last two identities use the fact that the integrals $\int_0^t \vartheta^{(m+1,n)}(u)dS^{m+1,n}(u)$ take nonzero values only when $t > \tau_m$. We denote $\zeta^{m+1}(t)$ the expression in the inner expectation, that is,

$$\zeta^{m+1} := \frac{1+\sum_{n=1}^{\infty}\left(\vartheta^{(m+1,n)}/X(\tau_m)\right)\cdot S^{m+1,n}}{X_*^{m+1}} = \frac{X^{\tau_{m+1}}/X^{\tau_m}}{X_*^{m+1}} > 0.$$

We note that $\zeta^{m+1}-1$ is equal to 0 on $[\![0,\tau_m]\!]$ and for every $n\in\mathbb{N}$

$$\mathbb{1}_{\Omega^{m+1,n}}(\zeta^{m+1}-1) = \frac{1+\left(\vartheta^{(m+1,n)}/X(\tau_m)\right)\cdot S^{m+1,n}}{X_*^{m+1,n}} - 1$$

is a supermartingale from the first paragraph. Lemma 4.5 again implies that $\zeta^{m+1}-1$, hence also $\zeta^{m+1}$, are supermartingales. From the identity $\zeta^{m+1}(s\vee\tau_m)=\zeta^{m+1}(s)$ and the supermartingale property of $(X/X_*)^{\tau_m}$, we obtain for $0\le s\le t$

$$\mathbb{E}\left[\left(\frac{X}{X_*}\right)^{\tau_{m+1}}(t)\middle|\mathcal{F}_s\right] = \mathbb{E}\left[\left(\frac{X}{X_*}\right)^{\tau_m}(t)\,\mathbb{E}\left[\zeta^{m+1}(t)\middle|\mathcal{F}_{s\vee\tau_m}\right]\right] \le \mathbb{E}\left[\left(\frac{X}{X_*}\right)^{\tau_m}(t)\,\zeta^{m+1}(s)\middle|\mathcal{F}_s\right]$$
$$\le \left(\frac{X}{X_*}\right)^{\tau_m}(s)\,\zeta^{m+1}(s) = \left(\frac{X}{X_*}\right)^{\tau_{m+1}}(s).$$

This proves the supermartingale numéraire property of $X_*$ and completes the proof. □

We are now ready to restate the fundamental theorem in the general case of piecewise semimartingale market with RCLL paths. As we pointed out in the first paragraph of this subsection, there is no structural condition in this general market, which nicely characterizes the supermartingale numéraire strategy in terms of the local rates of the market. Thus, the conditions (*ii*) and (*iii*) in Theorem 3.14, which were derived from the structural condition, do not appear in the following theorem. The definitions of market viability and no arbitrage of the first kind remain the same as in Definitions 3.9 and 3.11 in the present context.

**Theorem 4.10.** *The following statements are equivalent:*

*(i) The market is viable.*
*(ii) $NA_1$ holds.*
*(iii) There exists a local martingale deflator.*
*(iv) An investment strategy $\vartheta_*$ having the supermartingale numéraire property exists, and its wealth process $X_*(T)=X(T;1,\vartheta_*)$ is finite almost everywhere for any $T\ge 0$.*
*(v) The collection $\mathcal{X}$ of wealth processes of strictly positive investment strategies is bounded in probability, that is,*

$$\lim_{m\to\infty}\sup_{X\in\mathcal{X}}\mathbb{P}[X(T)>m]=0 \text{ holds for any } T\ge 0.$$

*Proof.* As in the proof of Proposition 3.12, we refer to Exercise 2.21, Proposition 2.22 of Karatzas and Kardaras (2021) and Theorem 3.5 of Strong (2014) for the equivalence between $(i)$, $(ii)$, $(iii)$, and $(v)$. For the implication $(iii) \Longrightarrow (iv)$, the existence of $\vartheta_*$ with the supermartingale numéraire property follows from Theorem 4.9. Moreover, if $\mathbb{P}[X_*(T) = \infty] > 0$ holds for some $T \geq 0$, then an $\mathcal{F}_T$-measurable random variable $h = xX_*(T)$ for every initial capital $x > 0$ is an arbitrage of the first kind for horizon $T$, such that $\varphi := x\vartheta_*$ is an admissible strategy satisfying $X(T; x, \varphi) = xX(T; 1, \vartheta_*) = h$. This violates the condition $(ii)$, hence $X_*(T)$ should be finite almost everywhere for any $T \geq 0$.

Assuming $(iv)$, we now prove $(v)$. Let us denote $\vartheta^*$ the strategy having the supermartingale numéraire property and $X_* \equiv X(\cdot; 1, \vartheta^*)$ its wealth process. For any $T \geq 0$, $X \in \mathcal{X}$, and $m, \ell \in \mathbb{N}$, Markov's inequality and the supermartingale property of $X/X_*$ yield

$$\begin{aligned}
\mathbb{P}[X(T) > m] = \mathbb{P}\left[\frac{X(T)}{X_*(T)} > \frac{m}{X_*(T)}\right] &\leq \mathbb{P}[X_*(T) > \ell] + \mathbb{P}\left[\frac{X(T)}{X_*(T)} > \frac{m}{X_*(T)},\, X_*(T) \leq \ell\right] \\
&\leq \mathbb{P}[X_*(T) > \ell] + \mathbb{P}\left[\frac{X(T)}{X_*(T)} > \frac{m}{\ell}\right] \\
&\leq \mathbb{P}[X_*(T) > \ell] + \frac{\ell}{m}\mathbb{E}\left[\frac{X(T)}{X_*(T)}\right] \leq \mathbb{P}[X_*(T) > \ell] + \frac{\ell}{m}.
\end{aligned}$$

Therefore, by taking the supremum over all $X \in \mathcal{X}$, we obtain for every $m, \ell \in \mathbb{N}$

$$\sup_{X \in \mathcal{X}} \mathbb{P}[X(T) > m] \leq \mathbb{P}[X_*(T) > \ell] + \frac{\ell}{m}.$$

For any $\epsilon > 0$, we can choose large enough $\ell \in \mathbb{N}$ such that $\mathbb{P}[X_*(T) > \ell] < \epsilon/2$. Then, we take large enough $m$ satisfying $\ell/m < \epsilon/2$ to deduce $\sup_{X \in \mathcal{X}} \mathbb{P}[X(T) > m] < \epsilon$. This establishes $(v)$ and completes the proof. □

## 4.4 | Superhedging duality and the market completeness

As another application of the ODT, we first present in this subsection the superhedging duality. We recall the definitions of cumulative withdrawal stream and its superhedging capital from Definition 3.11. For the results in this subsection, we shall make the assumption $\mathcal{F}_0 = \{\emptyset, \Omega\}$ throughout.

**Theorem 4.11.** *Suppose that $\mathcal{Y} \neq \emptyset$. Let $K$ be a cumulative withdrawal stream and*

$$x(K) := \inf\{x \geq 0 \,|\, \exists\, X(\cdot; x, \vartheta) \in \mathcal{W} \text{ such that } X(t; x, \vartheta) \geq K(t) \text{ for all } t \geq 0\},$$

*the superhedging capital associated with $K$. Then, the following superhedging duality holds:*

$$x(K) = \sup_{Y \in \mathcal{Y}} \mathbb{E}\left[\int_0^\infty Y(s)\, dK(s)\right].$$

*Proof.* Let us denote the right-hand side by $w_K := \sup_{Y \in \mathcal{Y}} \mathbb{E}[\int_0^\infty Y(s) dK(s)]$. We first show the inequality $x(K) \leq w_K$. If $w_K = \infty$, there is nothing to prove. Thus, we assume $w_K < \infty$ and

consider the process

$$X(t) := \operatorname*{ess\,sup}_{Y\in\mathcal{Y}} \mathbb{E}\left[\int_t^\infty \frac{Y(u)}{Y(t)}\, dK(u)\Big|\mathcal{F}(t)\right] \tag{82}$$

with $X(0) = w_K$ (as $\mathcal{F}_0$ is trivial). We shall show that for any local martingale deflator $Z \in \mathcal{Y}$, the process

$$\zeta_Z(t) := Z(t)X(t) + \int_0^t Z(u)\, dK(u), \qquad t \geq 0,$$

is a supermartingale. To this purpose, let us define

$$\mathcal{Y}_t^Z := \{Y \in \mathcal{Y} \mid Y(u) = Z(u) \text{ for all } u \in [0,t]\}$$

and denote

$$\widetilde{\zeta}_Z(t) := \operatorname*{ess\,sup}_{Y\in\mathcal{Y}_t^Z} \mathbb{E}\left[\int_0^\infty Y(u)\, dK(u)\Big|\mathcal{F}(t)\right].$$

We now argue $\zeta_Z(\cdot) \equiv \widetilde{\zeta}_Z(\cdot)$. For a fixed $t \geq 0$, we clearly have $\widetilde{\zeta}_Z(t) \leq \zeta_Z(t)$. For the reverse inequality, for any $Y \in \mathcal{Y}$, we construct another local martingale deflator $Y_Z \in \mathcal{Y}_t^Z$ by concatenation:

$$Y_Z(u) := \begin{cases} Z(u) & \text{for } 0 \leq u \leq t, \\ \frac{Z(t)}{Y(t)} Y(u) & \text{for } u > t. \end{cases}$$

The process $Y_Z$ is a local martingale deflator (see Excercise 2.26 of Karatzas & Kardaras, 2021) with the property that $Y_Z(u)/Y_Z(t) = Y(u)/Y(t)$ holds for every $u \geq t$. Denoting the collection of all such processes $Y_Z$ by $\mathcal{Z}_t^Z$, then $\mathcal{Z}_t^Z \subset \mathcal{Y}_t^Z$, and we can write

$$X(t) = \operatorname*{ess\,sup}_{Y_Z\in\mathcal{Z}_t^Z} \mathbb{E}\left[\int_t^\infty \frac{Y_Z(u)}{Y_Z(t)}\, dK(u)\Big|\mathcal{F}(t)\right].$$

This shows the reverse inequality $\widetilde{\zeta}_Z(t) \geq \zeta_Z(t)$ for an arbitrary $t \geq 0$, hence $\zeta_Z(\cdot) \equiv \widetilde{\zeta}_Z(\cdot)$.

For any two elements $Y_1, Y_2 \in \mathcal{Y}_t^Z$, we construct another element $Y_3$ in $\mathcal{Y}_t^Z$, again from Exercise 2.26 of Karatzas and Kardaras (2021), as follows:

$$Y_3(u) = \begin{cases} Z(u) & \text{for } 0 \leq u \leq t, \\ Y_1(u) & \text{for } u > t, \text{ on the set } \{\mathbb{E}[\int_0^\infty Y_1(s)dK(s)|\mathcal{F}(t)] \geq \mathbb{E}[\int_0^\infty Y_2(s)dK(s)|\mathcal{F}(t)]\}, \\ Y_2(u) & \text{for } u > t, \text{ on the set } \{\mathbb{E}[\int_0^\infty Y_1(s)dK(s)|\mathcal{F}(t)] < \mathbb{E}[\int_0^\infty Y_2(s)dK(s)|\mathcal{F}(t)]\}. \end{cases}$$

Then, we have $\mathbb{E}[\int_0^\infty Y_3(s)dK(s)|\mathcal{F}(t)] \geq \mathbb{E}[\int_0^\infty Y_i(s)dK(s)|\mathcal{F}(t)]$ for $i = 1, 2$, which implies that the collection $\{\mathbb{E}[\int_0^\infty Y(s)\, dK(s)|\mathcal{F}(t)] : Y \in \mathcal{Y}_t^Z\}$ has the so-called "directed upwards" property (see Theorem A.32 of Föllmer & Schied, 2008), and we can find a sequence $(Y_m)_{m\in\mathbb{N}} \subset \mathcal{Y}_t^Z$ such

that the sequence $(\mathbb{E}[\int_0^\infty Y_m(s)\,dK(s)|\mathcal{F}(t)])_{m\in\mathbb{N}}$ is increasing and

$$\zeta_Z(t) = \lim_{m\to\infty} \mathbb{E}[\int_0^\infty Y_m(s)\,dK(s)|\mathcal{F}(t)].$$

By the Monotone convergence theorem, we have

$$\mathbb{E}[\zeta_Z(t)|\mathcal{F}(s)] = \lim_{m\to\infty} \mathbb{E}\left[\int_0^\infty Y_m(u)\,dK(u)\Big|\mathcal{F}(s)\right].$$

Now for $s \le t$, due to the inclusion $\mathcal{Y}_t^Z \subset \mathcal{Y}_s^Z$, we have

$$\begin{aligned}\zeta_Z(s) = \widetilde{\zeta}_Z(s) &= \operatorname*{ess\,sup}_{Y\in\mathcal{Y}_s^Z} \mathbb{E}\left[\int_0^\infty Y(u)\,dK(u)\Big|\mathcal{F}(s)\right] \\ &\ge \operatorname*{ess\,sup}_{Y\in\mathcal{Y}_t^Z} \mathbb{E}\left[\int_0^\infty Y(u)\,dK(u)\Big|\mathcal{F}(s)\right] \ge \mathbb{E}[\zeta_Z(t)|\mathcal{F}(s)],\end{aligned}$$

proving the supermartingale property of $\zeta_Z$.

From integration by parts, we have

$$Z(X+K) = \zeta_Z + \int_0^\cdot K(u-)\,dZ(u).$$

The process $Z(X+K)$ is then a local supermartingale, hence a supermartingale, as it is nonnegative, for all $Z \in \mathcal{Y}$. Thanks to Theorem 4.6, there exist an admissible strategy $\vartheta \in \mathcal{L}_0(S)$ and a cumulative withdrawal stream $F$ such that $X + K = w_K + \vartheta \cdot S - F$ holds. This proves $x(K) \le w_K$.

For the reverse inequality, consider a wealth process $X \in \mathcal{W}$ satisfying $X(\cdot) \ge K(\cdot)$. Since $YX$ is a local martingale for every $Y \in \mathcal{Y}$, the process

$$YX - \int_0^\cdot K(u-)\,dY(u) = Y(X-K) + \int_0^\cdot Y(u)\,dK(u)$$

is a nonnegative local martingale, hence a supermartingale. Therefore, we have the inequality

$$X(0) \ge \mathbb{E}\left[\int_0^\infty Y(u)\,dK(u)\right].$$

Taking supremum over all $Y \in \mathcal{Y}$ and infimum over all such initial capitals $X(0)$ financing $K$, yields the reverse inequality $x(K) \ge w_K$. □

The process $X + K$ in the proof of Theorem 4.11 where $X$ is of Equation (82), which finances the given cumulative withdrawal stream $K$, is actually the *minimal* one, as we present in the following result.

**Theorem 4.12.** *Suppose that $\mathcal{Y} \neq \emptyset$. Let $K$ be a cumulative withdrawal stream and define*

$$\widetilde{X}(t) := K(t) + \operatorname*{ess\,sup}_{Y\in\mathcal{Y}} \mathbb{E}\left[\int_t^\infty \frac{Y(u)}{Y(t)}\,dK(u)\Big|\mathcal{F}(t)\right], \qquad t \ge 0.$$

*Then, $\widetilde{X}$ minimally finances $K$; for any adapted nonnegative process $X$ such that $YX$ is a supermartingale for every $Y \in \mathcal{Y}$ and $X(\cdot) \geq K(\cdot)$, we have $X(\cdot) \geq \widetilde{X}(\cdot)$.*

*Proof.* Since $YX$ is a supermartingale for every $Y \in \mathcal{Y}$, the process

$$YX - \int_0^{\cdot} K(u-)\, dY(u) = Y(X-K) + \int_0^{\cdot} Y(u)\, dK(u)$$

is a nonnegative local supermartingale, thus a supermartingale. Hence, we obtain for $t \geq 0$

$$Y(t)(X(t) - K(t)) + \int_0^t Y(u)\, dK(u) \geq \mathbb{E}\left[\int_0^{\infty} Y(u)\, dK(u) \Big| \mathcal{F}(t)\right].$$

Rearranging the inequality yields for any $Y \in \mathcal{Y}$

$$X(t) \geq K(t) + \mathbb{E}\left[\int_t^{\infty} \frac{Y(u)}{Y(t)}\, dK(u) \Big| \mathcal{F}(t)\right],$$

which proves the result. □

As a corollary to Theorem 4.11, we derive the following superhedging duality for European contingent claims.

**Definition 4.13** (European contingent claim). A pair $(T, \xi_T)$ is called a *European contingent claim*, if $T$ is a finite stopping time and $\xi_T$ is a nonnegative $\mathcal{F}_T$-measurable random variable. Here, $T$ is the maturity of the claim, and $\xi_T$ is the payoff at the maturity.

**Example 4.14.** In addition to the classical examples of European contingent claims, whose payoffs depend on individual asset price at $T$, such as call and put ($\xi_T = (S_i(T) - K)^+$ and $(K - S_i(T))^+$, respectively, for some $K > 0$), we can consider a new type of European contingent claim depending on the number of assets in the market at time $T$, that is, $\xi_T = \mathbb{1}_{\{N(T)>M\}}$ for some $M \in \mathbb{N}$.

Let us remark that any European contingent claim $(T, \xi_T)$ can be expressed as a cumulative withdrawal stream $K^{(T,\xi_T)} = \xi_T \mathbb{1}_{[\![T,\infty[\![}$ and the following result immediately follows.

**Corollary 4.15.** *Suppose that $\mathcal{Y} \neq \emptyset$. Let $(T, \xi_T)$ be a European contingent claim and denote*

$$x^{(T,\xi_T)} := \inf\{x > 0 : \exists\, X(\cdot; x, \vartheta) \in \mathcal{W} \text{ such that } X(T) \geq \xi_T\}$$

*the smallest initial capital starting from which the claim $(T, \xi_T)$ can be financed. Then, the following superhedging duality holds:*

$$x^{(T,\xi_T)} = \sup_{Y \in \mathcal{Y}} \mathbb{E}[Y(T)\, \xi_T]. \tag{83}$$

*If the quantity of Equation (83) is finite, the minimal hedging process for the European contingent claim* $(T, \xi_T)$ *is given by*

$$X^{(T,\xi_T)}(t) = \operatorname*{ess\,sup}_{Y\in\mathcal{Y}} \mathbb{E}\left[\frac{Y(T)\xi_T}{Y(t)}\middle|\mathcal{F}(t)\right] = x^{(T,\xi_T)} + \vartheta \cdot S - K \tag{84}$$

*for some* $\vartheta \in \mathcal{L}_0(S)$ *and a cumulative withdrawal stream* $K$.

In what follows, we discuss the notions of replicability of European claims and market completeness in the present context.

**Definition 4.16** (Replicability). Suppose that $\mathcal{Y} \neq \emptyset$. A European contingent claim $(T, \xi_T)$ such that the superhedging capital of Equation (83) is finite is said to be *replicable*, if the minimal hedging process $X^{(T,\xi_T)}$ of Equation (84) does not have any capital withdrawals, that is, $K \equiv 0$.

**Definition 4.17** (Market completeness). Suppose that $\mathcal{Y} \neq \emptyset$. The market is said to be complete, if every *European contingent claim* $(T, \xi_T)$ with the finite superhedging capital of Equation (83) is replicable.

In view of the ODT (Corollary 4.7), a European contingent claim $(T, \xi_T)$ is replicable if and only if $YX^{(T,\xi_T)}$ is a local martingale for every $Y \in \mathcal{Y}$; the market is complete if and only if $YX^{(T,\xi_T)}$ is a local martingale for every $Y \in \mathcal{Y}$ and for a minimal hedging process $X^{(T,\xi_T)}$ of every European contingent claim with a finite superhedging capital.

In the following result, known as the second FTAP, we prove that the completeness of the market is equivalent to the uniqueness of the local martingale deflator in the market.

**Theorem 4.18.** *Suppose that* $\mathcal{Y} \neq \emptyset$. *The market is complete if and only if* $\mathcal{Y}$ *is a singleton.*

*Proof.* Let us assume that $\mathcal{Y} = \{Y\}$ is a singleton. For any European contingent claim $(T, \xi_T)$ such that the quantity of Equation (83) is finite, we have

$$X^{(T,\xi_T)}(t) = \mathbb{E}\left[\frac{Y(T)\xi_T}{Y(t)}\middle|\mathcal{F}(t)\right].$$

Then $YX^{(T,\xi_T)}$ is a uniformly integrable martingale, thus a direct application of the ODT (Corollary 4.7) yields the replicability of $(T, \xi_T)$ and the completeness of the market.

For the reverse implication, we assume that the market is complete. Let $Y, Y' \in \mathcal{Y}$ and take a common localizing sequence $(\sigma_n)_{n\in\mathbb{N}}$ such that $Y^{\sigma_n}$ and $(Y')^{\sigma_n}$ are uniformly integrable martingales for $n \in \mathbb{N}$. We define two probability measures $Q_n$ and $Q'_n$ on $\mathcal{F}_{\sigma_n}$ by setting

$$\frac{dQ_n}{d\mathbb{P}} = Y(\sigma_n) \qquad \text{and} \qquad \frac{dQ'_n}{d\mathbb{P}} = Y'(\sigma_n) \qquad \text{for each } n \in \mathbb{N}. \tag{85}$$

For any $A \in \mathcal{F}_{\sigma_n}$, a pair $(\sigma_n, \mathbb{1}_A)$ is a European claim such that its superhedging capital of Equation (83) is finite. Since $YX^{(\sigma_n,\mathbb{1}_A)}$ and $Y'X^{(\sigma_n,\mathbb{1}_A)}$ are local martingales, $(X^{(\sigma_n,\mathbb{1}_A)})^{\sigma_n}$ is a local martingale under $Q_n$ and $Q'_n$ (see Lemma 11, Chapter 2 of Jarrow, 2018). Moreover, since $X^{(\sigma_n,\mathbb{1}_A)} \leq 1$, $(X^{(\sigma_n,\mathbb{1}_A)})^{\sigma_n}$ is a martingale under $Q_n$ and $Q'_n$, hence we have $x^{(\sigma_n,\mathbb{1}_A)} = \mathbb{E}^{Q_n}[X^{(\sigma_n,\mathbb{1}_A)}(\sigma_n)] = \mathbb{E}^{Q'_n}[X^{(\sigma_n,\mathbb{1}_A)}(\sigma_n)] = Q_n(A) = Q'_n(A)$ for all $A \in \mathcal{F}_{\sigma_n}$. This implies that $Y^{\sigma_n} = (Y')^{\sigma_n}$ for all $n \in \mathbb{N}$, thus $Y \equiv Y'$. This completes the proof. □

Recalling Definition 4.3 and Theorem 4.18, the completeness of $(k,n)$-dissected market is equivalent to the uniqueness of $(k,n)$-local martingale deflator on the $(k,n)$-dissection set, that is, $Y_1 = Y_2$ on $[\![\tau_{k-1},\tau_k]\!] \cap (\mathbb{R}_+ \times \Omega^{k,n})$ for any $Y_1, Y_2 \in \mathcal{Y}^{k,n}$. Moreover, for given $(k,n)$-local martingale deflators $Y^{k,n} \in \mathcal{Y}^{k,n}$ for all pairs $(k,n) \in \mathbb{N}^2$, we can construct a local martingale deflator $Z \in \mathcal{Y}$ via recipe in the proof of Theorem 3.5 of Strong (2014):

$$Z := \prod_{k=1}^{\infty} Z^k, \qquad \text{where} \quad Z^k := \mathbb{1}_{\{\tau_{k-1}=\infty\}} + \sum_{n=1}^{\infty} \mathbb{1}_{\Omega^{k,n}} \left( \frac{(Y^{k,n})^{\tau_k}}{(Y^{k,n})^{\tau_{k-1}}} \right). \tag{86}$$

We now have the following corollary stating the completeness of the entire market in terms of the completeness of every dissected market.

**Corollary 4.19.** *Suppose that $\mathcal{Y} \neq \emptyset$. The entire market is complete if and only if each $(k,n)$-dissected market is complete for every $(k,n) \in \mathbb{N}^2$.*

*Proof.* Suppose that the entire market is complete but there exists a pair $(k',n')$ such that $(k',n')$-dissected market is not complete. This implies that there exist $Y_1, Y_2 \in \mathcal{Y}^{k',n'}$ such that $Y_1 \neq Y_2$ on the $(k',n')$-dissection set. From the completeness of the entire market, we have $\mathcal{Y} = \{Y\}$ and $Y \in \mathcal{Y}^{k,n}$ for every $(k,n) \in \mathbb{N}^2$. Thus, using the recipe (86), we can construct two different local martingale deflators $Z_1, Z_2 \in \mathcal{Y}$, where we set $Y^{k,n} = Y$ for all other pairs $(k,n) \neq (k',n')$ but we choose $Y_1$ and $Y_2$, respectively, for $Y^{k',n'}$. This contradicts the completeness of the entire market.

For the other implication, assume that there exist $Y_1, Y_2 \in \mathcal{Y}$ such that $Y_1 \neq Y_2$. Then, there exists a $(k,n)$-dissection set such that $Y_1 \neq Y_2$ on that set. Since $\mathcal{Y} \subset \mathcal{Y}^{k,n}$ for all $(k,n) \in \mathbb{N}^2$, we have $Y_1, Y_2 \in \mathcal{Y}^{k,n}$, which contradicts the completeness of $(k,n)$-dissected market. □

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