ON THE STABILITY OF POISEUILLE FLOWS

by

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The present work is a study of the stability of plane and of axially symmetric Poiseuille flows by the small perturbation method. The need for a re-examination of this classical topic arises from the following considerations.

Relatively little has been done on the stability of fully developed axi-symmetric pipe flows. Existing analytical results which are at variance with the evidence of experiments are shown to be either incomplete or incorrect. The inconclusiveness of previous work provides an incentive for a new analysis.

While extensive study has been made of plane Poiseuille flow, disagreement still exists as to the basic results. Some authors have claimed unconditional stability while others have defined regions of stability and regions of instability. It thus seems desirable to analyse the problem anew, to attempt to complete its solution and to survey the areas of agreement and of disagreement between the authors who have treated plane Poiseuille flow. The present work is divided into two main parts.

In the first part, comprising Chapters I and II, axially symmetric and plane Poiseuille flows are treated in similar fashion. The starting point is provided by the linear fourth

order Orr-Sommerfeld equations. Two asymptotic and two inviscid solutions are derived. A detailed study of the Stokes phenomenon is made in conjunction with the asymptotic solutions. Connection formulae valid throughout the complex plane are derived. They allow the use of asymptotic solutions for any value of the characteristic number. Asymptotic relations are given between the eigen values, the wave number and the Reynolds number, for both plane and axi-symmetric Poiseuille flows. Limit formulae for very high values of the Reynolds number are derived. The asymptotic solutions are discussed. They are shown to be poor approximations when the characteristic number becomes vanishingly small.

In the second part, including Chapters III and IV, the works of C. C. Lin and C. L. Pekeris on plane Poiseuille flows are reviewed and compared to the present study. The correctness of Lin's procedure is demonstrated for small values of the characteristic numbers. It is shown that for the limiting case of vanishing wave numbers the results of Lin and of Pekeris are equivalent and can be derived from each other. Lin's methods are then adapted to prove the stability of axially symmetric pipe flows.

In Chapter V are found a review of previous work on the stability of axi-symmetric flow by Sexl and Pekeris as well as the results of a solution of that problem by an electronic differential analyzer. The main conclusions presented in this

investigation are:

1) While plane Poiseuille flow is unstable for large enough Reynolds number, fully developed flow in a circular pipe is stable to axi-symmetric disturbances.

2) The eigen values which characterize the time dependence and the rate of propagation of small disturbances have for both Poiseuille flows the following properties. For large enough Reynolds there are two distinct families of eigen values, one of which tends to zero, the other to unity as the Reynolds number tends to infinity.

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ACKNOWLEDGEMENTS

Research is, at best, a collaboration with past and with contemporary efforts. In that sense, no work should ever bear any single name. In the present case, guidance was provided by previous studies of the problem of fluid stability, in particular by those of Professor C. C. Lin and of C. L. Pekeris.

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I N T R O D U C T I O N

The greatest single source of complexity in the dynamics of a viscous fluid is the physical existence of unsteady velocity fields. Whenever a fluid flows over a surface, the region of flow which is appreciably affected by the viscosity of the fluid, frequently acquires turbulent characteristics.

The transition from the so-called laminar regime to the turbulent regime may occur for the whole flow as in uniform pipes or for part of the flow as in the boundary layer. In both cases it constitutes the boundary between two types of flow which are dissimilar in all important respects. A turbulent layer of fluid will exert different shear than a laminar layer and the phenomena of heat transfer, of diffusion and of mass transfer will also be altered.

It is therefore important to be able to predict the occurence of such a transition and to determine its causes. It would in addition be most desirable to obtain from a study of transition some information about the early stages of the turbulent regime.

So far no theory of transition has contributed significantly to the latter aim. In general, the point of view which has generally been adopted is that transition is caused by a state of instability within the laminar flow region, which tends to amplify infinitesimal disturbances. Interest is then focused not so much on the mode of growth of disturbances as on the criterion for growth or decay.

One can imagine on the other hand other mechanisms than that of the growth of small disturbances, to explain the appearance of a disorderly flow. If no restrictions are placed on the degrees of freedom of the flow, pressure disturbances perpendicularly to the direction of the flow, or local vortices are at least conceivably capable of interaction with the main flow to increase locally its vorticity. The plausibility of any theory of transition will, of course, be measured in terms of the success with which it accounts both quantitatively and qualitatively for the phenomenon.

In the present work only two very simple flows are examined by the small perturbation method. They are the so-called plane and axi-symmetric Poiseuille flows. While from the point of view of applications these flows hold only a modest interest, it is useful to examine them in order to test the theory. The stability of such flows presents in addition an interesting mathematical problem which has claimed the attention of many a researcher for more than half a century. One would find there, if it were needed, sufficient incentive to attempt to complete the analysis of the stability of Poiseuille flows by the small perturbation theory.

It has already been mentioned that the theory postulates that the transformation of the laminar regime into a turbulent one is caused by the amplification of infinitesimal disturbances which are bound to be generated in any flow. If the disturbances are small enough originally, the equations which describe their behaviour can be linearized. Therefore, we are at liberty to consider them as the sum of elemental disturbances of different frequencies and amplitudes. If the disturbances grow steadily,

the equations of motion indicate that their behaviour will depart from linearity. This means in particular that their frequencies and amplitudes will be interdependent; that a multiplicity of motions of the order of magnitude of that of the main flow will in turn be modified by ever growing small disturbances and that the flow will loose all characteristics of steadiness. Since we know of no stable intermediate regime between a purely laminar flow and a turbulent flow the presumption is strong that the growth of small disturbances signifies ultimately the onset of turbulence.

The energy furnished to small disturbances, in order for them to grow, must of necessity be borrowed from the main stream. The small perturbation theory purports to determine when this transfer of energy takes place and when it is ruled out by purely kinematic means.

The purpose of the present analysis is two fold.

(a) It is intended first as an investigation of the stability of fully developed pipe flow when infinitesimal axi-symmetric disturbances are present. This particular problem has received, as will be seen in Chapter Five, surprisingly little attention. It will be treated in Chapters I and IV.

(b) The second aim of this undertaking is to review, evaluate and coordinate the works of other authors concerning the stability of plane Poiseuille flow. The latter problem has raised sufficient controversy to warrant attempts at clarification. The discussion of plane Poiseuille flow will be found in Chapters II and III.

The following plan is adopted.

An asymptotic method of solution is presented successively for the axi-symmetric flow and the plane Poiseuille flow. The same formal development can be used for both cases. An analysis is then made of recent work by two authors on plane Poiseuille flow for which the asymptotic method of solution presented first will serve as common reference. Finally the solution for axi-symmetric flow is completed.

CHAPTER ONE

The stability of axi-symmetric pipe flow

Section I. The Orr-Sommerfeld equations. The steady incompressible flow through a uniform circular pipe has a particularly simple solution. The Navier-Stokes equations of motion yield a parabolic velocity profile which is invariant in the axial direction. If the assumption of a steady motion is lifted, this solution is not unique. We propose to determine by the small perturbation method under what conditions the simple solution describes the flow in a pipe and when the laminar flow tends to depart from a steady state.

We will presently start from the Navier-Stokes equations for incompressible flow. We introduce terms representing small disturbances and linearize the resultant equations to arrive at the well known Orr-Sommerfeld equations.

The Navier-Stokes equation for the axi-symmetric motion of an incompressible fluid is written:

$$\frac{\partial \tilde{u}}{\partial t} + \tilde{u}\frac{\partial \tilde{u}}{\partial r} + \tilde{w}\frac{\partial \tilde{u}}{\partial x} = -\frac{\partial \tilde{p}}{\partial r} + \nu\left[\frac{\partial^2 \tilde{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{u}}{\partial r} - \frac{\tilde{u}}{r^2} + \frac{\partial^2 \tilde{u}}{\partial x^2}\right] \qquad (1.1)$$

$$\frac{\partial \tilde{w}}{\partial t} + \tilde{u}\frac{\partial \tilde{w}}{\partial r} + \tilde{w}\frac{\partial \tilde{w}}{\partial x} = -\frac{\partial \tilde{p}}{\partial x} + \nu\left[\frac{\partial^2 \tilde{w}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{w}}{\partial r} + \frac{\partial^2 \tilde{w}}{\partial x^2}\right] \qquad (1.2)$$

$$\frac{\partial \tilde{u}}{\partial r} + \frac{\tilde{u}}{r} + \frac{\partial \tilde{w}}{\partial x} = 0 \qquad (1.3)$$

where $\tilde{w}$ and $\tilde{u}$ are respectively the axial and radial components of velocity, r the radial distance from the center of the pipe, x the axial distance along the pipe and ν the coefficient of kinematic viscosity.

We now assume the flow is the superposition or sum of a large (steady state) velocity component and of a small disturbance component. We make a similar assumption regarding pressures and define

$$\tilde{u} = u \quad ; \quad \tilde{w} = W + w \quad ; \quad \tilde{p} = P + p$$

from now on our observations will be confined to the case when

$$w \ll W \quad ; \quad p \ll P$$

W was defined as the large steady state component. More explicitly it is given by the solution of (1), (2), (3) when $\frac{\partial \tilde{u}}{\partial t}$ and $\frac{\partial \tilde{w}}{\partial t}$ are 0 W is therefore the velocity profile corresponding to laminar Poiseuille flow in a pipe. P is the pressure in the undisturbed flow. The pressure gradient is constant. Since the main flow satisfies continuity, so does the superimposed disturbance flow and we are lead to define a stream function ψ such that

$$w = -\frac{1}{r}\frac{\partial \psi}{\partial r} \qquad u = +\frac{1}{r}\frac{\partial \psi}{\partial x} \qquad (1.4)$$

We then express (1.1) and (1.2) in terms of the stream function, neglecting products of disturbance components. We are able to obtain a purely kinematic equation which has been linearized.

The non-dimensional form of this equation is:

$$R\left[\frac{\partial}{\partial t} + w\frac{\partial}{\partial x}\right]\left[\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial x^2}\right] + \left[\frac{W'}{r} - W''\right]\frac{\partial \psi}{\partial x} =$$

$$\left[\frac{\partial^4 \psi}{\partial r^4} - \frac{2}{r}\frac{\partial^3 \psi}{\partial r^3} + \frac{3}{r^2}\frac{\partial^2 \psi}{\partial r^2} - \frac{3}{r^3}\frac{\partial \psi}{\partial r} + 2\frac{\partial^4 \psi}{\partial r^2 \partial x^2} + \frac{\partial^4 \psi}{\partial x^4}\right] \qquad (1.5)$$

Where R is the Reynolds number for which we choose as characteristic length, the radius of the pipe and as characteristic velocity the maximum velocity W(o). We now assume that the functional dependence of ψ on r, x and t can be separated in a simple way. We shall require an elemental solution of ψ to satisfy:

$$\psi(r, x, t) = \phi(r)\, e^{i\alpha(x - ct)}$$

and substituting in (1.5)

$$(W - c)\left[\phi'' - \frac{\phi'}{r} - \alpha^2\phi\right] + \phi\left[\frac{W'}{r} - W''\right] = \frac{-i}{\alpha R}\left[\phi^{IV} - \frac{2}{r}\phi''' + \frac{3}{r^2}\phi'' - \frac{3}{r^3}\phi - 2\alpha^2\left(\phi'' - \frac{\phi'}{r}\right) + \alpha^4\phi\right] \qquad (1.6)$$

On the walls of the pipe, both u and w must be zero, while at the center for an axi-symmetric flow w is finite while u is zero. We shall express the boundary conditions as

$$\left.\begin{array}{l} \phi(1) = \phi'(1) = 0 \\ \lim_{r\to 0}\left(\frac{\phi}{r}\right) = 0 \qquad \lim_{r\to 0}\left(\frac{\psi}{r}\right) \text{ is bounded} \end{array}\right\} \quad (1.7)$$

We call equations (1.6) and (1.7) the Orr-Sommerfeld equations for the case of incompressible flow in a pipe.

Equations (1.6) and (1.7) define an eigen value problem. The parameters upon which ϕ depend are c, α, αR. Furthermore, for a given value of αR and α, only a set of discrete values of the complex number c will be found for which (1.6) and (1.7) are satisfied. In that sense (1.6) and (1.7) define c and from a solution of that system we can expect to determine c. If some values of the characteristic number c are found such that its imaginary part is positive we shall conclude that a disturbance characterized by that value of c would grow in time. Since a physical disturbance is likely to be the aggregate of a continuous spectrum of frequencies the small perturbation theory would predict that the flow is unstable for the corresponding Reynolds number, even though only one disturbance frequency might be amplified.

Equation (1.6) can be satisfied by four linearly independent solutions which are entire functions of α, αR and c so that the complex general solution of (1.6) is a linear function of those four solutions

$$\phi = c_1\phi_1 + c_2\phi_2 + c_3\phi_3 + c_4\phi_4$$

On the other hand, our boundary conditions define the value of the constants C_n in the following manner:

$$c_1 \phi_1(1) + c_2 \phi_2(1) + c_3 \phi_3(1) + c_4 \phi_4(1) = 0$$

$$c_1 \phi_1'(1) + c_2 \phi_2'(1) + c_3 \phi_3'(1) + c_4 \phi_4'(1) = 0$$

$$\lim_{r \to 0} \left[c_1 \frac{\phi_1}{r}(r) + c_2 \frac{\phi_2}{r}(r) + c_3 \frac{\phi_3}{r}(r) + c_4 \frac{\phi_4}{r}(r) \right] = 0$$

$$\lim_{r \to 0} \left[c_1 \frac{\phi_1'}{r}(r) + c_2 \frac{\phi_2'}{r}(r) + c_3 \frac{\phi_3'}{r}(r) + c_4 \frac{\phi_4'}{r}(r) \right] = 0$$

The solution of such a system will give a complex relation of the form $c = F(\alpha, \alpha R)$ from which c_i and c_r are determined. According to our terminology, disturbances will be amplified, neutrally excited or damped respectively if

$$c_i > 0 \quad ; \quad c_i = 0 \quad ; \quad c_i < 0$$

For a fully developed flow in a pipe (or Poiseuille flow) the system (1.6; 1.7) can be written in a different way. We give here this formulation which we shall have occasion to utilize later and which was first used by Theodore Sexl (1).

For such a case, the velocity profile of the main stream is written non dimensionally $W = 1 - r^2$

Hence $\frac{W'}{r} - W'' = 0$

This simplification is utilized to separate (1.6) into two differential equations

$$\frac{d^2 f}{dr^2} - \frac{1}{r} \frac{df}{dr} - \alpha^2 f - (1 - r^2 - c) i \alpha R f = 0 \qquad (1.8a)$$

$$\frac{d^2 \varphi}{dr^2} - \frac{1}{r} \frac{d\varphi}{dr} - \alpha^2 \varphi = f(r) \qquad (1.8b)$$

SECTION II

An asymptotic solution of the pipe flow stability problem.

The solution suggested presently is a combination of two asymptotic expansions and of the so-called "perfect fluid solutions" which have been made to satisfy one of the boundary conditions individually. Our starting point is the system (1.6) (1.7) in which the main stream velocity profile is the parabola $W = 1 - r^2$ given by the steady state solution of the Navier-Stokes equations in a pipe. Equations (1.6) and (1.7) are thus written:

$$(W-c)\left[\phi'' - \frac{\phi'}{r} - \alpha^2\phi\right] = -\frac{i}{\alpha R}\left[\phi^{IV} - \frac{2}{r}\phi''' + \frac{3}{r^2}\phi'' - \frac{3}{r^3}\phi' - 2\alpha^2\left(\phi'' - \frac{\phi'}{r}\right) + \alpha^4\phi\right] \quad (1.9)$$

$$\lim_{r\to 0}\left(\frac{\phi}{r}\right) = 0 \qquad \lim_{r\to 0}\left(\frac{\phi'}{r}\right) \text{ is bounded}$$

$$\phi(1) = \phi'(1) = 0$$

The system (1.8) assures us that solutions of

$$\phi'' - \frac{\phi'}{r} - \alpha^2\phi = 0$$

satisfy differential equation (1.9).

We thus obtain two solutions in closed form which we shall call the perfect fluid solutions:

$$\phi_1 = r J_1(i\alpha r) \quad ; \quad \phi_2 = r Y_1(i\alpha r) \qquad (1.10)$$

ϕ_2 does not satisfy our boundary conditions at the origin and must be discarded. ϕ_1 however, has acceptable order properties at the origin and it is continuous and bounded within the interval. We expect two additional linearly independent solutions to exist and we assume them to be of the form $\phi = e^{g(r)}$.

substituting into (1.22) we get

$$g^{IV}+4g'''g'+3g''^2+6g''g'^2+g'^4-\frac{2}{r}\left[g'''+g''g'+g'^3+2g'g''\right]+$$
$$+\left(\frac{3}{r^2}-2\alpha^2\right)\left[g''+g'^2\right)-\left[\left\{\frac{3}{r^3}-\frac{2}{r}\alpha^2\right\}g'+\alpha^4\right]=i\alpha R(W-c)\left[g''+g'^2-\frac{g'}{r}-\alpha^2\right] \quad (1.$$

We now assume that for large enough values of αR the function g can be represented by

$$g(r)=(\alpha R)^{1/2}g_0+g_1+(\alpha R)^{-1/2}g_2+\cdots$$

Substituting and equating powers of (αR) we obtain

$$g_0'^4=i(W-c)\,g_0'^2$$
$$g_0=\pm\int_0^r\sqrt{i(W-c)}\,dr$$

and

$$g_1=\ln\left[\frac{\ln r^{1/2}}{(W-c)^{5/4}}\right]$$

using only the two first terms of the expansions we write for the two remaining solutions:

$$\phi=\frac{Ar^{1/2}}{(W-c)^{5/4}}e^{+\int_{r_0}^r\sqrt{i\alpha R(W-c)}\,dr}+\frac{Br^{1/2}}{(W-c)^{5/4}}e^{-\int_{r_0}^r\sqrt{i\alpha R(W-c)}\,dr} \quad (1.12$$

where the limit r_0 is arbitrary only before A and B are specifi

There are several methods of deriving asymptotic representations (1.12) although the method indicated is probably the clearest one when the differential equation is of higher order. It is worth noting that the derivation gave us an immediate knowledge of the terms in equation (1.9) which retain their importance as αR increases. Specifically the only terms

utilized from (1.11) were:

$$g_1'^4 + 6g''g'^2 - \frac{2}{r}g'^3 = i\alpha R\left[(w-c)g'^2 - \frac{g'}{r} + g''\right]$$

these terms came from

$$\Phi^{IV} - \frac{2}{r}\phi''' = i\alpha R(w-c)\left[\phi'' - \frac{1}{r}\phi'\right]$$

The solution of (1.9) can now be written in general

$$\phi = A\frac{r^{1/2}}{(w-c)^{5/4}}e^{+\int_{r_0}^{r}\sqrt{i\alpha R(w-c)}\,dr} + B\frac{r^{1/2}}{(w-c)^{5/4}}e^{-\int_{r_0}^{r}\sqrt{i\alpha R(w-c)}\,dr} + CJ_1(i\alpha R) \quad (1.13)$$

It is possible to outline as follows the steps which have to be taken to complete the problem:

We must first determine a relationship between A and B in order for the asymptotic solutions to satisfy the boundary conditions at $r = o$. We must then ensure that the asymptotic solution remains continuous and is a valid approximation to Φ throughout the interval except possibly in the immediate neighborhood of the singularity of g_1. Finally we must impose on the three solutions in (1.13) the boundary conditions at the wall. The last step will determine the values of c.

A) Evaluation of the relationship between A and B at the origin:

The procedure consists in equating the asymptotic expressions to asymptotic expansions of a Bessel function solution which is valid around the origin. We first derive the approximate solution valid only within the region $\zeta \ll 1$. In that region equation (1.8 a) can be rewritten:

$$\frac{d^2 f}{dy^2} + \frac{i\alpha R}{4}\left(\frac{1-c}{y}\right) f = 0 \quad ; \qquad y \equiv r^2$$

the solutions of that equation can be written by inspection:

$$\left.\begin{aligned} f_{o1} &= r J_1\left[i\left(i\alpha R(1-c)\right)^{1/2} r\right] = r J_1(i\beta r) \\ f_{o2} &= r Y_1\left[i\left\{i\alpha R(1-c)\right\}^{1/2} r\right] = r Y_1(i\beta r) \end{aligned}\right\} \text{ where } \beta = \left[i\alpha R(1-c)\right]^{1/2}$$

We reject f_{o2} which does not satisfy the boundary condition at $r = o$; therefore, we have

$$\varphi'' - \frac{\varphi'}{r} - \alpha^2 \varphi = r J_1(i\beta r) \qquad (1.14)$$

since

$$\varphi''_1 - \frac{\varphi'_1}{r} - \beta^2 \varphi = o \quad \text{for } \varphi = r J_1(i\beta r)$$

it is clear that the solution of (1.14) is:

$$\varphi = 2 D r J_1(i\beta r) = D\left\{H_1^{(1)}(i\beta r), r H_1^{(2)}(i\beta r)\right\},$$

where D is a constant.

We now expand this solution by means of the asymptotic representations of $H_1^{(1)}$ and $H_1^{(2)}$. This procedure is valid because although we have restricted r to small values, we are at liberty to assume αR large enough for the asymptotic representation of the Hankel functions to hold. We thus obtain

$$\varphi = K_1\left[r^{1/2} e^{-\left[i\alpha R(1-c)\right]^{1/2} r - \frac{3}{4}\pi i} + r^{1/2} e^{+\left[i\alpha R(1-c)\right]^{1/2} r + i\frac{3\pi}{4}}\right] \qquad (1.15a)$$

on the other hand near the origin (1.12) can be written approximately

$$\varphi = \frac{B r^{1/2}}{(1-c)^{5/4}} e^{\sqrt{i\alpha R(1-c)}\, r} + \frac{A r^{1/2}}{(1-c)^{5/4}} e^{-\sqrt{i\alpha R(1-c)}\, r} \qquad (1.15b)$$

by comparing directly (1.15a) and (1.15b) we conclude that

$$\frac{A}{B} = e^{-3/2\pi i} = i \qquad (1.16)$$

B) Stokes' Phenomenon and the crossing substitution

The ratio of B to A has been evaluated in the neighborhood of $r = o$, and it becomes of interest to ask whether this ratio remains the same throughout the interval $o < r < 1$. One condition must be imposed upon our asymptotic representation: It should be single valued within the interval. On the other hand, the existence of roots of $(W - c)$ within the interval implies that the asymptotic expressions in (1.12) are multi-valued. In order to clarify the resulting ambiguity we shall analyze in greater detail the behaviour of these multi-valued functions in the complex r plane. The following development is inspired by W. H. Furry (2). We denote the value of r for which $W = c$ as K, and we define

$$U \stackrel{D}{=} \int_K^r \sqrt{i(W-c)}\, dr$$

In the neighborhood of $r = K$, U can be written as the first term of its Taylor series expansion

$$W - c = 1 - r^2 - c \cong 2(1-c)^{\frac{1}{2}}(1-c-r)$$

and if

$$r - K \stackrel{D}{=} \eta$$

we have in the neighborhood of K

$$U \cong \int_0^{r-K} \sqrt{-i(2K)\eta}\, d\eta$$

$$\cong \frac{2}{3} K^{\frac{1}{2}} \sqrt{-2i}\, \eta^{3/2}$$

We seek the values of the arguments of η for which U is a pure real number. If we define $\arg(i) = +\frac{\pi}{2}$ they are in the neighborhood of K:

$$\arg(\eta) = -\frac{\pi}{2} + \frac{2}{3}N\pi$$

where N is an integer. We call Stokes lines the lines on which U is purely real (Fig. 1). They originate at $r = K$ and in the neighborhood of that point they are at 120° from each other. The sign of U alternates from one line to the other, so that for one given Stokes line it depends on the number of revolutions around the point K.

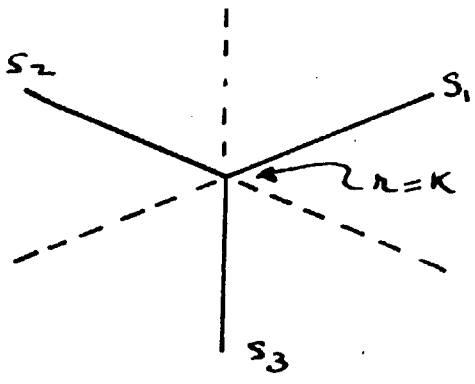


Fig. 1

Corresponding to each Stokes line there exists a line on which the value of U is purely imaginary. Those lines are given in the neighborhood of $\eta = o$ by

$$\arg(\eta) = \frac{\pi}{2} + \frac{2}{3}N\pi$$

On a Stokes line the exponential terms in the asymptotic expansions:

$$e^{+\sqrt{\alpha R}\,u} \quad ; \quad e^{-\sqrt{\alpha R}\,u}$$

become either very large or very small, depending on the sign of U associated with that line. This is a consequence of the assumption that αR is a large number. Thus if we were to traverse the η plane in such a manner as to cross Stokes lines, we would successivel see the coefficient of A in (1.12) become much larger than the coefficient of B, then the coefficient of B overpower the coefficient of A etc..... The small exponential term is in fact on the Stokes lines smaller than the order of approximat

of the asymptotic representation. On the other hand, on the lines of pure imaginary U the coefficients of A and B would have precisely the same magnitude. It is apparent from this observation that if we need to change our coefficients A and B within the interval, the proper place to modify them will be on the Stokes lines. There the coefficient of the small term could be modified without materially altering the value of the sum of the two asymptotic expressions.

We now inquire whether the original ratio $\frac{A}{B}$ need be modified within the interval $0 < r < 1$. We define the Stokes lines and lines of imaginary U as shown in Fig. 2.

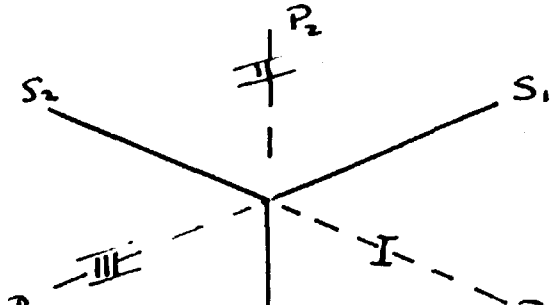


Fig. 2

S_1, S_2, S_3, are the Stokes lines.
P_1, P_2, P_3, are the lines of imaginary U.
I, II, III, are the corresponding regions bounded by the Stokes lines.
Further we choose $U > 0$ on S_1
Hence $U < 0$ on S_2
$U > 0$ on S_3
for the first revolution, and
$U < 0$ on S_1'
after one complete revolution from S_1,

If the asymptotic solution (1.12) is represented in a region I by

$$\varphi = \frac{r^{1/2}}{(w-c)^{5/4}} \left[A_1 e^{r\int_K^z \sqrt{i\alpha R(w-c)}\,dr} + B_1 e^{-\int_K^z \sqrt{i\alpha R(w-c)}\,dr} \right]$$

it can be represented in any other Stokes region by a linear combination of the two terms. Thus

$$A_{\text{II}} = a A_{\text{I}} + b B_{\text{I}}$$

$$B_{\text{III}} = c B_{\text{I}} + d A_{\text{I}}$$

where a, b, c, d, are constants. However, the continuity of our representation requires that the coefficient of the small term only, be modified on the corresponding Stokes line.

Therefore, according to our convention, we have:

$$A_{\text{II}} = A_{\text{I}}$$

$$B_{\text{II}} = B_{\text{I}} + \alpha A_{\text{I}}$$

$$A_{\text{III}} = A_{\text{II}} + \beta B_{\text{II}} = A_{\text{I}} + \beta [B_{\text{I}} + \alpha A_{\text{I}}]$$

$$B_{\text{III}} = B_{\text{II}} = B_{\text{I}} + \alpha A_{\text{I}}$$

and at the initial Stokes line after a full revolution counterclockwise

$$A_{\text{I}}' = A_{\text{III}} = A_{\text{I}} + \beta [B_{\text{I}} + \alpha A_{\text{I}}]$$

$$B_{\text{I}}' = B_{\text{III}} + \gamma A_{\text{III}} = B_{\text{I}} + \alpha A_{\text{I}} + \gamma [A_{\text{I}} + \beta (B_{\text{I}} + \alpha A_{\text{I}})]$$

On the other hand, after one revolution A has become the coefficient of the small term, B, that of the large term and the argument of the expression $(\omega - c)^{5/4}$ has gained an angle $\frac{5\pi}{2}$. In order for the solutions on A_{I} and A_{I}' to be the same we must require

$$A_{\text{I}}' = i B_{\text{I}}$$

$$B_{\text{I}}' = i A_{\text{I}}$$

These values substituted above require, in order for all the above relations to be consistant, that $\alpha = \beta = \gamma = i$

The above relations show that for the representation to be continuous one of the constants A or B must be modified every time η crosses one of the three Stokes lines. This property is known as Stokes phenomenon. We now have a series of connection formulae which permit us to evaluate the discontinuity in A and B on the

Stokes lines. Which connection formula will be used depends on the Stokes regions within which the end points fall and thus on the eigen value c. In particular for the range of eigen values for which the end points $r = o$ and $r = 1$ fall respectively in regions III and I the connection formulae give

$$\left.\begin{aligned} A_I &= -i\, B_{III} + A_{III} \\ B_I &= -i\, A_{III} \end{aligned}\right\} 1.17$$

If A_{III} and B_{III} are known, A_I and B_I are given by (1.17) and hold from the singularity K to the wall $r = 1$. According to equation (1.16) and reversing the order of integration and changing the lower limit of integration, we find that

$$\left.\begin{aligned} A_I &= N\left[e^{\frac{3}{4}\pi i - \int_K^0 \sqrt{i\omega a(w-j)}\,dr} - ie^{-\frac{3}{4}\pi i + \int_0^K \sqrt{i\omega a(w-j)}\,dr}\right] \\ B_I &= N\left[-ie^{+\frac{3}{4}\pi i - \int_K^0 \sqrt{i\omega a(w-j)}\,dr}\right] \end{aligned}\right\} 1.18$$

where N is a constant.

C) Determination of the eigen values c .

The solution of the system (1.6; 1.7) is made up near $r = 1$ of a linear combination of two solutions. The first one, φ_α , is the sum of two asymptotic expansions with coefficients A_I and B_I and the second is the Bessel function solution

$$\varphi_\beta = r J_1(i\omega r).$$

The remaining task before us is to impose on that system the boundary conditions at the wall

$$\phi(1) = \phi'(1) = o$$

which requires

$$\begin{vmatrix} \phi_\alpha(1) & \phi_\beta(1) \\ \phi'_\alpha(1) & \phi'_\beta(1) \end{vmatrix} = o$$

Hence,

$$\begin{vmatrix} \left[\frac{A_1}{B_1}\left(\frac{r^{1/2}}{(w-c)^{5/4}}\right)e^{+\int_\kappa^1\sqrt{ida(w-c)}\,dr} + e^{-\int_\kappa^1\sqrt{ida(w-c)}\,dr}\right]; & r J_1(ida) \\ \frac{d}{dr}\left[\frac{A_1}{B_1}\left(\frac{r^{1/2}}{(w-c)^{5/4}}\right)e^{+\int_\kappa^1\sqrt{ida(w-c)}\,dr} + e^{-\int_\kappa^1\sqrt{ida(w-c)}\,dr}\right]; & \frac{d}{dr}\left[r J_1(ida)\right] \end{vmatrix} = 0 \qquad (1.19)$$

where the determinent is evaluated at $r = 1$ and where A and B are given by (1.18). Equation (1.19) can be rewritten:

$$\frac{A_I}{B_I} e^{2\int_\kappa^1\sqrt{ida(w-c)}\,dr} = \left[\frac{ida\frac{J_1'(ida)}{J_1(ida)}\Big]_1 + 1 + \sqrt{da(-c)} - \frac{1}{2} + \frac{5}{2c}}{-ida\frac{J_1'(ida)}{J_1(ida)}\Big]_1 - 1 + \sqrt{da(-c)} + \frac{1}{2} - \frac{5}{2c}}\right] \qquad (1.20)$$

where $J_1'(ida)$ denotes the derivative with respect to the argument (ida) and where $\frac{A_1}{B_1}$ can be rewritten:

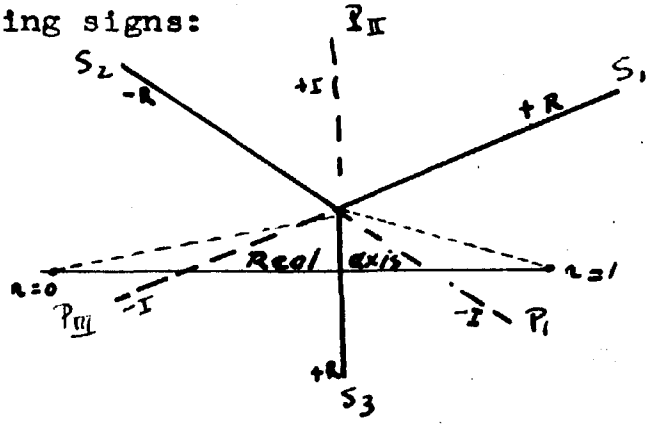
$$\frac{A_1}{B_1} = \left(1 - e^{-3/2\pi i + 2\int_\kappa^0\sqrt{ida(w-c)}\,dr}\right)$$

There remains some ambiguity in expression (1.19) since the arguments of complex quantities are not in themselves single valued. However, the ambiguity disappears if only we consider the convention we have adopted to evaluate the Stokes jumps of A and B. We had chosen for the Stokes lines the following signs:

$U > 0$ on S_I
$U < 0$ on S_{II}
$U > 0$ on S_{III}

As a consequence, on the lines of pure imaginary U :

$U < 0$ on P_I
$U > 0$ on P_{II}
$U < 0$ on P_{III}



Fig

So that we are guided in our choice of the argument π by the sign of the real and the imaginary parts of U in any of the six sextants and by the fact that proceeding around the singularity K we have rotated clockwise. As a consequence

$$\frac{A_1}{B_1} = i\left[1 - e^{Q}\right] \quad \text{where} \quad Q = e^{5/4\pi i}\sqrt{\alpha R}\,\frac{\pi}{2}K^2$$

$$e^{2\int_K^1 \sqrt{i\alpha R(w-c)}\,dr} = e^{\frac{1}{4}\pi i\sqrt{\alpha R}\left[\sqrt{c} - K^2 \log \frac{1+\sqrt{c}}{K}\right]}$$

$$= e^{P}$$

where

$$P = e^{-\frac{1}{4}\pi i\sqrt{\alpha R}}\left[\sqrt{c} - K^2 \log \frac{1+\sqrt{c}}{K}\right]$$

$$\sqrt{i\alpha(-c)} = e^{-\frac{1}{4}\pi i}\sqrt{\alpha R c}$$

Equation (1.20) thus becomes

$$\left(1 - e^{Q}\right)e^{P} = -i\left[\frac{e^{-\frac{\pi i}{4}}\sqrt{\alpha R c} + \frac{\frac{d}{dr}\left[r J_1(ikr)\right]_1}{J_1(ikr)} + \frac{5}{2c} - \frac{1}{2}}{e^{-\frac{\pi i}{4}}\sqrt{\alpha R c} - \frac{\frac{d}{dr}\left[r J_1(ikr)\right]_1}{J_1(ikr)} - \frac{5}{2c} + \frac{1}{2}}\right] \quad (1.21)$$

and for $\alpha = 0$; $\alpha R \neq 0$ and large :

$$\left(1 - e^{Q}\right)e^{P} = -i\left[\frac{e^{-\frac{\pi i}{4}}\sqrt{\alpha R c} + 1.5 + 5/2c}{e^{-\frac{\pi i}{4}}\sqrt{\alpha R c} - 1.5 - 5/2c}\right] \quad (1.22)$$

A solution of the algebraic equation (1.21) yields $c_r(\alpha R, \alpha)$ and $c_i(\alpha R, \alpha)$ where c_r and c_i are respectively the real and the imaginary parts of the eigen value c.

SECTION III

Discussion of the Solution

Equations (1.21) and (1.22) must now be appraised. It is worth emphasizing that they are valid asymptotic determinations of c only if c is such that the ends $r = o$ and $r = 1$ of the interval fall respectively in regions III and I. Furthermore, some question can be raised about their validity or accuracy for special values of c. In general, if the singularity K is not in the immediate proximity of either boundary, the matching at the origin can be made and it is legitimate to represent part of the solution asymptotically at the wall. On the other hand if $c \to o$ as $\alpha R \to \infty$ for a given mode the singularity approaches the wall. This means that since the asymptotic representation in region I will be approximately valid only a finite distance away from the singularity, it may not be valid on the wall where we make explicit use of it. Equations (1.21; 1.22) do give us, as we shall see, one family of eigen values c which has the property $c \to o$ as $\alpha a \to \infty$ at least for $\alpha = o$. While the general trend of eigen values is adequately described by equations (1.21; 1.22) in that case we shall not attempt to decide on the delicate matter of the possibility of positive imaginary parts for c on the strength of these equations alone.

A similar question can be raised for families of eigen values approaching unity for high αR. In the latter case the Stokes point approaches the origin $r = o$. However, as long as $c < 1$ a path can always be found away from the singularity on which the

matching of the asymptotic representation to the Bessel function near the origin would be valid. It will be shown in the following section that for the family of eigen values referred to above, c approaches 1 in such a way that $1-c \sim 1/(\alpha R)^{\frac{1}{2}}$. The Bessel function used in the matching procedure was $J_1(i\beta r)$ where

$$\beta = \left[i\alpha R(1-c)\right]^{1/2}$$

The values of c arrived at would yield for the representation at the origin: $rJ\left(k(\alpha R)^{\frac{1}{2}} r\right)$ where k is a constant. So that the asymptotic representation of this function around the origin is possible for sufficiently large αR. Therefore, we can expect the determination of the eigen values approaching 1 to be accurate within the assumptions inherent in the use of asymptotic expansions.

Keeping the above considerations in mind we derive for purpose of comparison two limiting formulae for c when $\alpha = o$.

Referring ourselves to equation (1.22) and to the definition of P and Q (See Fig. 3) we notice that the terms e^P and e^Q may have for large αR any magnitude depending on the value of $K = \sqrt{1-c}$. For on the value of K depends the path of integration of the quantities e^P and e^Q and in particular whether their real parts are positive or negative. In general we find that as αR increases the eigen values found by (1.21) fall into two categories or families. One family is characterized by $c \to 1$ as $\alpha R \to \infty$ and the second by $c \to o$ as $\alpha R \to \infty$

Limiting values of c for large αR and $\alpha^2 = o$.

a) Assuming $c \to 1$ for large αR . It is seen that the right hand side of (1.22) tends to unity. On the other hand e^P becomes extremely large. Equation (1.22) therefore requires

$$e^Q \to 1 \quad \text{as} \quad \alpha R \to \infty$$

We have then approximately

$$e^{5/4\,\pi i \sqrt{\alpha R}\ \frac{\pi}{2}\cdot\kappa^2} = e^{i(-2N\pi)}$$

or

$$1 - c = -\left(\frac{4N}{\sqrt{\alpha R}}\right) e^{-3/4\,\pi i}$$

$$c = 1 + \left(\frac{4N}{\sqrt{\alpha R}}\right) e^{-\frac{3\pi i}{4}} \qquad (1.23)$$

Expression (1.23) is of course only an approximation to the solution of (1.22). It indicates that the eigen values of a given mode have in the case of large αR an envelop which is a straight line whose slope is 45° . The various modes are given by the various values of N which is the set of positive integers: N 1,2,3, etc..... For negative integers the singularity falls outside of the region for which the particular crossing substitution used applies.

b) Assuming $c \to o$ as $\alpha R \to \infty, \alpha^2 = o$.

We can write

$$P \cong \frac{4}{3} e^{-i\pi/4} \int_o^{1-K} \sqrt{\alpha R (2K)}\ \eta^{3/2}$$

where $\eta = r - k$

and since

$$1-K = 1-\sqrt{1-c} \cong \frac{c}{2}$$

$$P \cong \frac{4}{3}\sqrt{2\alpha R}\ \frac{c^{3/2}}{2\sqrt{2}}\ e^{-\frac{i\pi}{4}}$$

On the other hand e^{Q} tends to e so that $\frac{A_1}{B_1} \longrightarrow 1$

Let us define

$$\delta \equiv (2i\alpha R)^{1/3} c$$

$$\delta^{3/2} = [2i\alpha R]^{1/2} c^{3/2}$$

$$P = \left[\frac{2}{3\sqrt{2}}\, i\delta^{3/2}\right]$$

The right hand of (1.22) side is written approximately

$$\frac{\frac{i\delta^{3/2}}{\sqrt{2}} - \frac{5}{2}}{\frac{i\delta^{3/2}}{\sqrt{2}} + \frac{5}{2}}$$

Hence we have for large values of αR

$$e^{-\left(\frac{2}{3\sqrt{2}}\, i\delta^{3/2}\right)} = -i\left[\frac{\frac{i\delta^{3/2}}{\sqrt{2}} - \frac{5}{2}}{\frac{i\delta^{3/2}}{\sqrt{2}} + \frac{5}{2}}\right]$$

if we define

$$\theta = \tan^{-1}\left[\frac{\sqrt{2}\,\delta^{3/2}}{5}\right]$$

we get

$$e^{-2/3\, i\frac{\delta^{3/2}}{\sqrt{2}}} = e^{i\left[2\theta + \pi - 2N\pi - \pi/2\right]}$$

hence

$$\delta^{3/2} = \frac{3\pi}{2\sqrt{2}}\left[4N - 1 + \frac{4\theta}{\pi}\right] \qquad (1.24)$$

δ has discrete values corresponding to the various modes and as δ increases $\frac{4\theta}{\pi}$ rapidly tends to 2 so that for higher modes formula (1.24) becomes

$$\delta^{3/2} = \frac{3\pi}{2\sqrt{2}}\left[4N+1\right]$$

hence

$$c = \frac{1/2}{(2i\alpha R)^{1/3}}\left[3\pi\left(4N+1\right)\right]^{3/2} \qquad (1.25)$$

Hence for higher modes we obtain another straight line asymptote to small values of c. The accuracy of (1.24) and of (1.25) is, as was suggested earlier, doubtful. On one hand the coefficients A and B have in effect abruptly changed their order of magnitude and their ratio is of the order of unity near the wall. On the other, the argument of the asymptotic functions themselves has been written

$$\int_K^1 \sqrt{i\alpha R(w-c)}\, dr \cong \frac{-2}{3\sqrt{2}}\, i\delta^{3/2}$$

For the lower two modes at least, according to (1.24) this expression is of order unity and the asymptotic expansions do not gain any accuracy as αR increases. We may conclude that for $c \to 0$ the asymptotic representations are in most likelihood in error and that as a consequence of the proximity of the wall to the singularity K, the determinantal equation written at $r = 1$ is probably inaccurate.

Influence of α for the first family of eigen values.

It has been shown that the family of eigen values c which approaches 1 for large αR was characterized by negative values of c_i when α is 0. The same must necessarily be true in the limit for $\alpha \neq 0$. For it is clear from (1.21) that the only influence of α is the dependence on α of the ratio

$$\left[\frac{\frac{d}{dr}\left[r J_1(i\alpha r)\right]}{r J_1(i\alpha r)}\right]_{r=1}$$

Since this ratio is independent of αR its effect vanishes for $\alpha R \longrightarrow \infty$ provided c tends to a non-zero limit. Even for moderate values of αR the ratio above does not modify equation (1.21) notably. About the other family nothing can yet be said with certainty. On the other hand, there are conceivably other ranges of values of c for which the particular crossing substitution that was used does not apply. For such ranges it is natural to wonder whether any set of eigen values will satify the boundary conditions at the wall. We shall briefly consider the other three possible Stokes regions which give rise to different crossing substitutions than the one used to derive (1.11).

a) The end points can be connected by a path included entirely within one Stokes region.

No crossing substitution is required. Expression (1.21) would have been used with A_{III} and B_{III} substituted for A_I and B_I in the determinant at $r = 1$. This would give for c an expression of the kind:

$$-ie^{\int_0^1 \sqrt{i\alpha R(w-c)}\,dr} = \left\{\frac{\sqrt{\alpha R(-c)} + \frac{\frac{d}{dr}\left[r J_1(i\alpha r)\right]_1}{r J_1(i\alpha r)} + \frac{1}{2} + \frac{5}{2c}}{\sqrt{\alpha R(-c)} - \frac{\frac{d}{dr}\left[r J_1(i\alpha r)\right]_1}{r J_1(i\alpha r)} - \frac{1}{2} - \frac{5}{2c}}\right\} \quad (1.26)$$

Since in this case $(W - c)$ never vanishes, the left hand side would be of exponential character. For $c \to$ a non vanishing limit, as $\alpha R \to \infty$, the right hand side would tend to 1, and therefore expression (1.26) could not be satisfied.

For $c \to o$ as $\alpha R \to \infty$,we are again unable to draw any conclusion.

b) The end points are in regions I and II (Fig. 5)

Section (II.B) gives

$$A_{II} = A_I$$
$$B_{II} = B_I + iA_I$$

Hence,

$$A_I = A_{II}$$
$$B_I = B_{II} - iA_{II}$$

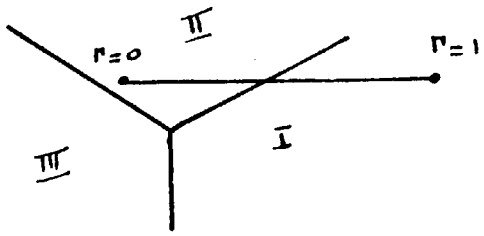


Fig.5

where A_{II} and B_{II} are the values of the coefficients of the asymptotic representations before the Stokes lines jump. Substituting for A_{II} and B_{II} :

$$A_I = e^{3/4\pi i} - \int_K^o \sqrt{i\alpha R(W-c)}\, dr$$

$$B_I = e^{-3/4\pi i} + \int_K^o \sqrt{i\alpha R(W-c)}\, dr \qquad e^{3/4\pi i} - \int_K^o \sqrt{i\alpha R(W-c)}\, dr$$
$$- ie$$

$$\frac{A_I}{B_I} = \frac{1}{e^Q - i}$$

Where the argument of Q is such that

$$Q = e^{3/4\pi i} \sqrt{\alpha R}\, \frac{\pi}{2} K^2$$

the determinant at the wall becomes

$$\frac{1}{e^Q - i} e^P = \left[\frac{e^{-i\pi/4} \sqrt{\alpha R c} + 1.5 + 5/2c}{e^{-i\pi/4} \sqrt{\alpha R c} - 1.5 - 5/2c} \right] \qquad (1.27)$$

where $$P = e^{-\frac{1}{4}\pi i}\sqrt{\alpha R}\left[\sqrt{c} - k^2 \log\frac{1+\sqrt{c}}{k}\right]$$

The region is such that c has a finite limit as $\alpha R \to \infty$. Thus the right hand side must tend to 1. Under these conditions, it is easy to show, by considering the order of magnitude of Q and P, that no value of c will satisfy (1.27).

c) The end points are in region II and III. (Fig. 6)

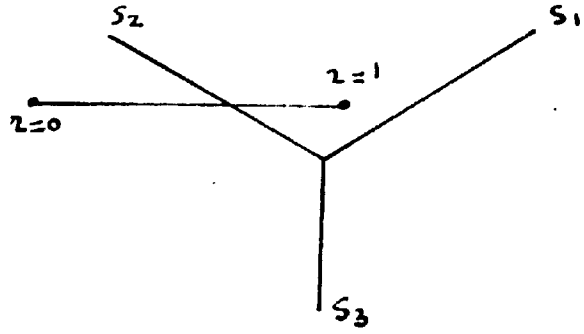
Hence

$$\begin{cases} A_{III} = A_{II} + i\,B_{II} \\ B_{III} = B_{II} \end{cases} \text{ or } \begin{cases} A_{II} = A_{III} - iB_{III} \\ B_{II} = B_{III} \end{cases}$$

$$A_{II} = e^{-\frac{3}{4}\pi i + \int_k^0 \sqrt{ika(w-1)}\,dr} + e^{\frac{3}{4}\pi i - \int_k^0 \sqrt{ika(w-1)}\,dr} - ie$$

$$B_{II} = e^{+\frac{3}{4}\pi i - \int_k^0 \sqrt{ika(w-1)}\,dr}$$

$$\frac{A_{II}}{B_{II}} = \frac{A_{III}}{B_{III}} - i = \left(e^{-Q} - i\right)$$



(Fig. 6)

where $Q = e^{\frac{5}{4}\pi i}\sqrt{\alpha R}\,\frac{\pi}{i}\,k^2$

and the determinant gives

$$\left(e^{-Q} - i\right)c \overset{?}{=} \frac{e^{\frac{1}{4}\pi i}\sqrt{\alpha R c} + 1.5 + 5/2c}{e^{\frac{1}{4}\pi i}\sqrt{\alpha R c} - 1.5 - 5/2c}$$

where

$$P = e^{\frac{1}{4}\pi i}\sqrt{\alpha R}\left[\sqrt{c} - k^2 \log\frac{1+\sqrt{1-k^2}}{k}\right]$$

Here again we postpone considerations on the existence of a family of eigen values c approaching o for increasing αR (although it is apparent that such eigen values would of necessity have a negative imaginary part in that region). On the other hand, we are able to deduce from the same kind of considerations as in b) that no non vanishing eigen values can be found which satisfy (1.18). We, therefore, reach the following tentative conclusions: Of the solutions satisfying the system (1.6; 1.7), one class yielding eigen values c which tend to unity as αR tends to ∞ is characteristically stable. The eigen values are given by relation (1.21) and in the limit by relation (1.23); another class which yields eigen values tending to o as $\alpha R \to \infty$ requires further study. Such a study will be found in chapter IV.

CHAPTER TWO

The stability of plane Poiseuille flow

Plane Poiseuille flow is the fully developed flow between two flat plates and subject to a constant pressure gradient. When this flow is steady, the velocity profile is given nondimentionally as:

$$W = 1 - y^2 \quad ; \quad -1 \leq y \leq 1$$

The stability of this idealized flow to small perturbations has been extensively studied. For a bibliography on the subject, the reader is referred to C. C. Lin (3). The interest for such a flow, as well as for pipe flow, lies in the existence of an exact steady state solution. It is considered here for several reasons. On one hand, the problem of stability of plane Poiseuille flow is so akin to the problem considered in Chapter One, that it is tempting to solve both at once by the same methods. On the other hand, serious disagreement exists between the investigators of this problem on the basic result of their studies. C. C. Lin (3), following Heisenberg (4), has found that the flow was unstable at high enough Reynolds numbers and determined a neutral curve the shape of which is reminiscent of the neutral curve found in laminar boundary layer flow. C. Pekeris (5) on the other hand, arrives at radically different conclusions. According to the latter, plane Poiseuille flow is stable, no matter how large the Reynolds number may be.

While the present investigation was reaching completion, we were informed of a systematic solution of the Orr-Sommerfeld

equations for plane Poiseuille flow on an I. B. M. digital computer, under the direction of L. H. Thomas (6). This computer solution appears to have been so carefully carried out that very little doubt is possible about its veracity. The results are in close agreement with those of C. C. Lin. This development provides an added incentive to analyze the problem, to attempt to complete its solution if possible, and to survey the areas of agreement and disagreement between Lin and Pekeris in an effort to clarify what is, from an analytical standpoint, a rather obscure issue.

We shall first state the problem and indicate a formal asymptotic solution inspired by and very closely parallel to the methods used in Chapter One. We will then be able to discuss the analyses of both Lin and Pekeris.

SECTION I

Statement of the problem

One starts with the Navier-Stokes equations of motion in a plane and follows a procedure identical to that of Chapter I, Section I. After linearization and separation the following non dimentional linear differential equation for ϕ (y) is obtained:

$$(W-c)(\phi''-\alpha^2\phi) - W''\phi = \frac{-i}{\alpha R}(\phi^{IV} - 2\alpha^2\phi'' + \alpha^4\phi) \qquad (2.1)$$

where the stream function ψ is assumed of the form $\psi(x,y,t) = \phi(y)e^{i\alpha(x-ct)}$ and where y is the distance perpendicular to the plates and its interval of definition is: $-1 \leq y \leq 1$

Equation (2.1) has the same general characteristics as equation (1.6). It has a basic system of four solutions which are analytic functions of y wherever the differential equation is analytic, (therefore everywhere except possibly for W = c) and which are entire functions of c .

At this point it is useful to consider any disturbances as a combination of two families of disturbances one of which comprises symmetric functions of y, the other anti-symmetric functions of y. These two types of disturbances are studied separately and require different boundary conditions at y = o. Interest centers around the minumum stability of the least stable type.

We shall concern ourselves with the family of symmetrical disturbance stream functions (anti-symmetrical disturbances). For this case the boundary conditions are imposed at the center of the channel, y = o and on one of the walls, say y = 1.

They are:

$$\phi(1) = \phi'(1) = 0$$

$$\phi'(0) = \phi'''(0) = 0$$

SECTION II

The asymptotic solutions

We intend to represent ϕ by three solutions of (2.1). Two of those solutions are asymptotic expressions of the W.K.B. type and the remaining is a series solution of the inviscid equation. The assumption that $\phi = e^{g}$ where $g = (\alpha R)^{\frac{1}{2}} g_0 + g_1$ leads to two solutions:

$$\left.\begin{aligned} \phi_3(y) &= (w-c)^{-5/4} e^{+\int_{y_0}^{y} \sqrt{i\alpha R(w-c)}\, dy} \\ \phi_4(y) &= (w-c)^{5/4} e^{-\int_{y_0}^{y} \sqrt{i\alpha R(w-c)}\, dy} \end{aligned}\right\} \quad (2.2)$$

We apply the condition that a linear combination of ϕ_3 and ϕ_4 must be symmetric at the origin $y = o$. This requires that around the origin the coefficients of ϕ_3 and ϕ_4 be equal if the lower limit of integration is o. Therefore, between the origin $y = o$ and the singularity $(W - c) = o$ the solution of (2.1) will be the sum of two solutions, one of which is:

$$\phi = E\left[(w-c)^{-5/4} e^{+\int_0^{y}\sqrt{i\alpha R(w-c)}\,dy} + (w-c)^{-5/4} e^{-\int_0^{y}\sqrt{i\alpha R(w-c)}\,dy}\right]$$

where E is a constant and $k^2 = 1 - c$.

It is immediately recognized that in so far as the Stokes phenomenon has bearing on the asymptotic representations used in Chapter I for the case of axi-symmetric flow, it also has bearing on the representations (2.2). Indeed it is clear that the considerations permitting the determination of the discontinuities of the

constants A and B across the Stokes lines are identical and that the connection formulae must be the same. We define the Stokes lines and the region which they determine in the same manner. In particular when $y = o$ is in region III and $y = 1$ in region I we have:

$$A_I = -i B_{III} + A_{III}$$
$$B_I = -i A_{III}$$

The part of our solution ϕ which is represented asymptotically by using (2.2) takes for $k < y \leq 1$ the following form:

$$\varphi_A = \left[A_1 e^{+\int_k^y \sqrt{i\alpha R(w-c)}\, dy} + B_1 e^{-\int_k^y \sqrt{i\alpha R(w-c)}\, dy} \right] \frac{1}{(w-c)^{5/4}}$$

where $A_I = e^{-\int_k^0 \sqrt{i\alpha R(w-c)}\, dy} - ie^{\int_k^0 \sqrt{i\alpha R(w-c)}\, dy}$

$$B_1 = e^{-\int_k^0 \sqrt{i\alpha R(w-c)}\, dy}$$

The other solutions

In addition two approximate solutions can be obtained by using the first approximation of an asymptotic expansion in terms of powers of $(\alpha R)^{-1}$. In this we use two of Lin's solutions. We assume

$$\phi(y) = \phi^{(0)}(y) + (\alpha R)^{-1} \phi^{(1)}(y) + (\alpha R)^{-2} \phi^{(2)}(y) + \cdots$$

if we substitute this expression in (2.1) we obtain by a successive approximation process:

$$(w-c)\left(\phi^{(0)\,II} - \alpha^2 \phi^{(0)}\right) - w'' \phi^{(0)} = 0 \qquad (2.3a)$$

$$(w-c)\left(\phi^{(k)\,II} - \alpha^2 \phi^{(k)}\right) - w'' \phi^{(k)} = -i\left[\phi^{(k-1)\,IV} - 2\alpha^2 \phi^{(k-1)\,II} + \alpha^4 \phi^{(k-1)}\right] \qquad (2.3b)$$

We intend to use only the initial approximation which can be solved by expanding two solutions in terms of increasing powers of α^2:

$$\phi_1^{(o)} = (W-c)\left[h_o(y) + \alpha^2 h_2(y) + \alpha^4 h_4(y) + \cdots\right] \qquad (2.4)$$

$$\phi_2^{(o)} = (W-c)\left[K_1(y) + \alpha^2 K_3(y) + \alpha^4 K_5(y) + \cdots\right] \qquad (2.5)$$

where:

$$h_o(y) = 1 \; ; \quad h_{2n+2}(y) = \int_{y_1}^{y}\left[(W-c)^{-2}\int_{y_1}^{y} d\xi\,[W(\xi)-c]^2\, h_{2n}(\xi)\right] dy \qquad (n \geq 0)$$

$$K_1(y) = \int_{y_1}^{y} (W-c)^{-2}\, dy$$

$$K_{2n+3}(y) = \int_{y_1}^{y} (W-c)^2\left[\int_{y_1}^{y} (W(\xi-c)^{-2}\, K_{2n+1}(\xi)\right] dy$$

Solutions (2.4) and (2.5) are entire functions of α^2 and the only point at which these series might not be convergent is the singular point (W = c) of the inviscid equation (2.3a). Since $\phi_1^{(o)}$ is a symmetric function of y while $\phi_2^{(o)}$ is not, we choose $\phi_1^{(o)}$. We notice that $\phi_I^{(o)}$ reduces to (W - c) when $\alpha^2 = o$.

SECTION III

The determination of the eigen values

We impose the second set of boundary conditions at the wall

$$\begin{vmatrix} \varphi_A(1) & \varphi_B(1) \\ \varphi'_A(1) & \varphi'_B(1) \end{vmatrix} = 0$$

where $\varphi_B = \phi_1(0)$

For values of c such that y = o is in Stokes region III and y = 1 is in Stokes region I the determinant can be written:

$$\left(i + e^{Q}\right)e^{P} = \frac{\left[-\sqrt{\alpha R c}\, e^{-\frac{\pi i}{4}} - \frac{5}{2c} - \frac{\phi'_B}{\phi_B}\right]}{\left[-\sqrt{\alpha R c}\, e^{-\frac{\pi i}{4}} + \frac{5}{2c} + \frac{\phi'_B}{\phi_B}\right]} \qquad (2.6)$$

and if $\alpha^2 = o$

$$\left(i + e^{Q}\right)e^{P} = \frac{\left[-\sqrt{\alpha R c}\, e^{-\frac{\pi i}{4}} - 9/2c\right]}{\left[-\sqrt{\alpha R c}\, e^{-\frac{\pi i}{4}} + 9/2c\right]} \qquad (2.7)$$

where

$$Q = e^{5/4\pi i}\sqrt{\alpha R}\,\frac{\pi}{2}k^2$$

$$P = e^{-1/4\pi i}\sqrt{\alpha R}\left[\sqrt{c} - k^2 \log\frac{1+\sqrt{1-k^2}}{k}\right]$$

Discussion of the results

The eigen values c are given by an expression very similar to expressions (1.21) and (1.22). Expression (2.6) is not in a form which permits easy determination of c, primarily because the representation of ϕ'_B/ϕ_B is a complicated algebraic equation. However, we are able to draw some interesting conclusions out of it.

The limit of c for large αR and $\alpha^2 = 0$

It is possible just as in the case of a pipe to determine for plane Poiseuille flow the limiting values of c for very large αR

For values of c around 1 and very large αR, e^P becomes very large. The right hand side again tends to 1. Hence for the determinantal equation to be satisfied e^Q must tend to $-i$. We write:

$$e^{5/4\pi i}(1-c) = -\frac{i(1+4N)}{\sqrt{\alpha R}}$$

$$c = 1 - \frac{(1+4N)}{\sqrt{\alpha R}} e^{\pi i/4} \qquad (2.8)$$

For values of c around o the same approximations are used as in pipe flow. They give

$$P = \frac{4}{3}\sqrt{2\alpha R}\,\frac{c^{3/2}}{2\sqrt{2}}\,e^{-i\pi/4}$$

$$= -\frac{2}{3}\sqrt{\alpha R}\,c^{3/2}e^{-i\pi/4}$$

We again define $\delta = (2i\alpha R)^{1/3}c$

hence $$P = -\left[\frac{2}{3\sqrt{2}}\,i\delta^{3/2}\right]$$

also since the real part of Q is very large and negative $\frac{A_1}{B_1} \cong i$, the right hand side of (2.7) is written:

$$\left(\frac{+\frac{i\delta^{3/2}}{\sqrt{2}} - \frac{q}{2}}{+\frac{i\delta^{3/2}}{\sqrt{2}} + \frac{q}{2}}\right),$$

and we define

$$\theta = \tan^{-1}\left(\frac{\sqrt{2}\,\delta^{3/2}}{q}\right)$$

we obtain

$$e^{-2/3\, i\, \delta^{3/2}/\sqrt{2}} = -i\left[\frac{Ae^{i(\pi-\theta)}}{Ae^{i\theta}}\right] = e^{i\left[\frac{\pi}{2} - 2\theta - 2N\pi\right]}$$

from which

$$\delta^{3/2} = \frac{3\pi}{2\sqrt{2}}\left[-1 + 4N + \frac{4\theta}{\pi}\right]$$

or

$$\delta = \frac{1}{2}\left[\frac{3}{1}\pi\left(-1 + 4N + \frac{4\theta}{\pi}\right)\right]^{2/3}$$

as δ increases $\tan^{-1}\left[\frac{\sqrt{2}\,\delta^{3/2}}{9}\right]$

rapidly approaches $\frac{\pi}{2}$ and the formula becomes

$$\delta = \frac{1}{2}\left[3\pi\left(1 + 4N\right)\right]^{2/3}$$

Again we have some well founded misgivings about the accuracy of the asymptotic solutions when

$$c \longrightarrow 0 \quad \text{as} \quad \alpha R \rightarrow \infty$$

for reasons identical to those exposed in Chapter I. We will, just as in that case, temporarily refrain from drawing any quantitative conclusion for that family of eigen values. However, we have no reason to disbelieve the results of the asymptotic study for the case:

$$c \longrightarrow 1 \quad \text{as} \quad \alpha R \longrightarrow \infty$$

and we will, for the time being, single out these results (equations 2.6 - 2.8) as a partial answer to our problem.

Concerning the possibility of positive imaginary parts for c and the existence of other eigen values tending to a finite limit as $\alpha R \rightarrow \infty$ it is obvious that the same considerations can be used as were called upon to bear on the pipe case.

The fact that $\frac{A_{III}}{B_{III}}$ in the plane case is different by an argument of $\frac{\pi}{2}$ from the ratio $\frac{A_{III}}{B_{III}}$ in the axi-symmetric case does not alter order of magnitude analyses on Q and P.

Our purely asymptotic study of plane Poiseuille flow leads us therefore to the following tentative conclusions:

The solution of the Orr-Sommerfeld equation by asymptotic expansions reveals the existence of two families of modes, one of which characterized by $c \to 1$ as $\alpha R \to \infty$ is stable and given by relations (2.6) or (2.7) and (2.8). The other family is characterized by $c \to o$ as $\alpha R \to \infty$ is not accurately enough described by asymptotic solutions to ascertain the sign of c_i in all cases. However, for that latter family, we should expect a difference between plane Poiseuille flow and axi-symmetric Poiseuille flow. The difference stems from the fact that in axi-symmetric flow as in Couette flow, what we have called the non-viscous solutions are analytic throughout the interval no matter what value the wave number α assumes. On the contrary, in the case of plane Poiseuille flow, for any $\alpha \neq o$ the non-viscous solutions have a logarithmic singularity at $W = c$. As $c \to o$ this singularity moves towards the wall $y = 1$ and in most likelihood has an influence on the determinental equation written at the wall and which defines c.

CHAPTERS III

An analysis of the works of C. C. Lin and C. Pekeris

C. C. Lin in a series of three papers (3) has given a very detailed treatment of the stability of two dimensional parallel flow. A review of all phases of this author's investigation will not be found here. We are primarily concerned with Lin's study of plane Poiseuille flow, in particular with those f cets of the problem for which disagreement exists.

Lin's solution consists of two solutions of the inviscid equations (those are the solutions which we borrowed from him) and two other solutions which we shall have occasion to discuss in detail and the derivation of which we give presently.

The Orr-Sommerfeld equation

$$(W-c)(\phi''-\alpha^2\phi)-W''\phi=\frac{-i}{\alpha R}\left(\phi^{IV}-2\alpha^2\phi''+\alpha^4\phi\right) \qquad (3.1)$$

is written in terms of a new variable η such that y being measured from the wall

$$(y-K) = \varepsilon\eta$$

$$\varepsilon = (\alpha R)^{-1/3}$$

$$\phi(y) = \chi(\eta)$$

and (3.1) becomes

$$(W-c)\left(\chi''-\alpha^2\varepsilon^2\chi\right)-\varepsilon^2W''\chi=\frac{-i}{\alpha R\varepsilon^2}\left(\chi^{IV}-2\alpha^2\varepsilon^2\chi''+\alpha^4\varepsilon^4\chi\right)$$

the velocity profile is expanded in a power series around the singularity K

$$(W-c)=(W_K-c)+W'_K(\varepsilon\eta)+\frac{W''_K}{2!}(\varepsilon\eta)^2+\cdots\cdots \text{ etc.}$$

where W_k is the velocity profile at $y = k$ and solutions are sought as the sum of successive approximations:

$$\phi(y) = \chi(\eta) = \chi^{(0)}(\eta) + \varepsilon\chi^{(1)}(\eta) + \varepsilon^2\chi^{(2)}(\eta) + \cdots \text{ etc.}$$

The differential equations giving the successive approximations are

$$W'_k \eta \chi^{(0)''} + i\chi^{(0)}_{IV} = 0 \qquad (3.2)$$

$$W'_k \eta \chi^{(n)''} + i\chi^{(n)}_{IV} = L_{n-1}(\chi) \qquad (n \geq 1)$$

where L_{n-1} is a linear combination of $\chi^{(0)}(\eta), \chi^{(1)}(\eta) \cdots \chi^{(n-1)}(\eta)$ for instance

$$L_0(\chi) = W''_k\left[\left(\chi^{(0)}\right) - \tfrac{1}{2}\eta^2\left(\chi^{(0)''}\right)\right]$$

Lin notices that two solutions of equation (3-2) (Stokes' equation for χ'') are:

$$\chi_3^{(0)} = \int_{+\infty}^{\eta} d\xi \int_{+\infty}^{\xi} d\mu\, \mu^{1/2} H^{(1)}_{1/3}\left[\tfrac{2}{3}(i\alpha_0\mu)^{3/2}\right] \qquad (3.3a)$$

$$\chi_4^{(0)} = \int_{-\infty}^{\eta} d\xi \int_{-\infty}^{\xi} d\mu\, \mu^{1/2} H^{(2)}_{1/3}\left[\tfrac{2}{3}(i\alpha_0\mu)^{3/2}\right] \qquad (3.3b)$$

where $\alpha_0 = (W_k)^{1/3}$

The two boundary conditions are:

at $y = 0$

$$\phi'(0) = \phi'''(0) = 0$$

and at $y = 1$ $\qquad \phi'(1) = \phi(1) = 0$

which leads to the following determinant:

$$\begin{vmatrix} \phi_1(1) & \phi_2(1) & \phi_3(1) & \phi_4(1) \\ \phi_1'(1) & \phi_2'(1) & \phi_3'(1) & \phi_4'(1) \\ \phi_1'(0) & \phi_2'(0) & \phi_3'(0) & \phi_4'(0) \\ \phi_1'''(0) & \phi_2'''(0) & \phi_3'''(0) & \phi_4'''(0) \end{vmatrix} = 0$$

where ϕ_3 and ϕ_4 are $\chi_3(0)$ and $\chi_4(0)$ and ϕ_1 and ϕ_2 are the two inviscid solutions. In order to simplify this determinant Lin considers the asymptotic behaviour of ϕ_3 and ϕ_4 by using known asymptotic expansions of Hankel functions of order one third. It will be remembered that this type of Hankel functions has two different asymptotic expansions each of which is valid within a different limit of the argument. For instance:

$$H_{1/3}^{(1)}\left(\tfrac{2}{3}z^{3/2}\right) \sim z^{-\frac{3}{4}} e^{\frac{2}{3}iz^{3/2} - \frac{5\pi i}{12}}$$

$$\text{for } -\tfrac{2\pi}{3} < \arg z < \tfrac{4\pi}{3} \qquad |z| \gg 1$$

This expansion will be referred to as an expansion of the first type. Or:

$$H_{1/3}^{(1)}\left(\tfrac{2}{3}z^{3/2}\right) \sim z^{-\frac{3}{4}} e^{\frac{2}{3}iz^{3/2} - \frac{5\pi i}{12}} + z^{-\frac{3}{4}} e^{-\frac{2}{3}iz^{3/2} - \frac{11\pi i}{12}}$$

$$-\tfrac{4\pi}{3} < \arg z < 0 \qquad |z| \gg 1$$

This expansion will be referred to as an expansion of the second type.

Similarly there are two types of asymptotic expansions for $H^{(2)}_{1/3}\left(\frac{2}{3}z^{3/2}\right)$, valid within two different limits of the argument z. Lin selects the first type which involves only positive exponential powers of the variable ($i\alpha_0\eta$) for $H^{(1)}_{1/3}$ and negative exponential powers only for $H^{(2)}_{1/3}$. Integrating term by term he obtains approximately:

$$\chi_3 \sim (y-k)^{-5/4} \exp\left\{-\int_{y_1}^{y}\sqrt{i\alpha_0^2(y-k)}\,dy\right\} \quad (3.5a)$$

$$\chi_4 \sim (y-k)^{-5/4} \exp\left\{+\int_{y_1}^{y}\sqrt{i\alpha_0^2(y-k)}\,dy\right\} \quad (3.5b)$$

for which the interval of definition is the common interval of definition of the two asymptotic expansions:

$$-\pi/6 < \arg \alpha_0\eta < \pi/6$$

Lin then compares these expansions with the asymptotic solutions obtained by the W.K.B. method and which were used in Chapter II. These expansions are shown to agree formally around the singularity K. However, there is a certain amount of arbitrariness in establishing a one-to-one correspondence between χ_3 and Φ_3 of (2.2); χ_4 and ϕ_4 of (2.2). For as we have mentioned above, Lin could have instead and with more generality written:

$$\chi_3 \sim D\phi_3 + E\phi_4$$

$$\chi_4 \sim F\phi_3 + G\phi_4$$

where D, E, F and G are constants. We shall consider later whether the restriction implied by the choice of asymptotic expansions for $H^{(1)}_{1/3}$ and $H^{(2)}_{1/3}$ has serious consequences. In

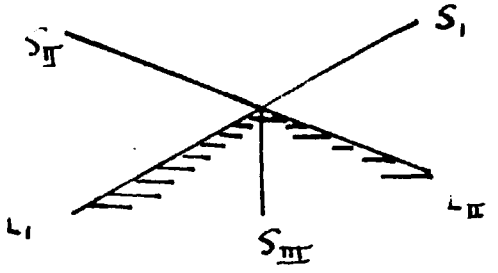
any event the type of asymptotic expansions used by Lin, holds continuously on a connected path between $y = 1$ and $y = 0$ provided

$$-\frac{7\pi}{6} < \arg(\alpha_0 n) < \frac{\pi}{6}$$

everywhere along the path.

As a consequence, since, according to Lin, such a connected path can always be found, this author argues that in order to find the eigen values c no crossing substitution is required.

Let us note that since the origin of y in Lin's notation is the wall $y=1$ while it is for us the center of the channel, on our diagram of branch cuts (Fig. 3) the interval of definition of the asymptotic expansions used by Lin would be represented as shown in Fig. 6.



(Fig. 6)

The cross hatched region is the region in which χ_3 and χ_4 are not given by (3.5). It is, in our notation the region

$$-\frac{7\pi}{6} \leq \arg(y-K) \leq \frac{11\pi}{6}$$

Lin uses the asymptotic form (3.5) to consider the order of magnitude of each term of the determinant (3.4). He writes:

$$\phi_3 = A(y)\, e^{-Y}$$

$$\phi_4 = B(y)\, e^{Y}$$

where A and B are of the order of unity and Y is defined by

$$Y = \int_{y_1}^{y} \sqrt{i\alpha R(w-c)}\, dy$$

He thus expresses the ratios occuring in the determinantal equation (3.4) as a function of A, B, the integrand of Y and e^P where

$$P = \int_{y_1}^{y} \sqrt{i\alpha R(w-c)}\, dy$$

Thus for instance

$$\frac{\phi_3'(1)}{\phi_3(1)} = -\sqrt{i\alpha R(w-c)} + \frac{A_1'}{A_1}$$

$$\frac{\phi_3(0)}{\phi_3(1)} = \frac{A_2}{A_1} e^{-I}$$

Lin notices that the real part of P is, according to his notation, positive throughout the interval if $c_i > 0$ and he neglects quantities of order e^{-P} and $(\alpha R)^{-1}$. This permits him to arrive at a reduced form of the determinant

$$\frac{f_2(\alpha, c)}{f_4(\alpha, c)} = \frac{\phi_3(1)}{\phi_3'(1)} \qquad (3.6a)$$

where

$$f_2(\alpha, c) = \begin{vmatrix} \phi_1(1) & \phi_1'(0) \\ \phi_2(1) & \phi_2'(0) \end{vmatrix} \qquad f_4(\alpha, c) = \begin{vmatrix} \phi_1'(1) & \phi_1'(0) \\ \phi_2'(1) & \phi_2'(0) \end{vmatrix}$$

In order to evaluate (3.6) Lin makes no direct use of the asymptotic expansions of ϕ_3 and ϕ_4. For anti-symmetric disturbances (aymmetricalϕfunctions) it is possible to use only one of the two inviscid solutions(2.3) since one of them (2.3a) contains only symmetric terms. Under these conditions equation (3.5) could have been written:

$$\frac{\phi_3'(1)}{\phi_3(1)} = \frac{\phi_2'(1)}{\phi_1(1)} \qquad (3.6b)$$

For ϕ_3 and ϕ_3' Lin uses the original solutions in terms of a Hankel function of order of one third (relation 3.3a). If equation (3.6) is correct we should be able to identify it with the help of our asymptotic solutions. We shall presently show that this is the case. More precisely we shall show that if the assumption is made that the eigen values are small compared to unity, the correct linear combination of asymptotic expressions near the wall is the asymptotic expansion of the Hankel function denoted by Lin as ϕ_3. So that expression (3.6) is valid provided it is understood that ϕ_3 is defined by (3.3a) not by (3.5a).

Let us write the part ϕ_A of the asymptotic solution found in Chapter II, to be valid between the singularity and the wall:

$$\Phi_A = \frac{1}{(w-c)^{5/4}} \left[A_1 e^{\int_K^4 \sqrt{i\alpha R(w-c)}\, dy} + B_1 e^{-\int_K^4 \sqrt{i\alpha R(w-c)}\, dy} \right]$$

where

$$A_1 = e^{-\int_K^0 \sqrt{i\alpha R(w-c)}\, dy} - ie^{+\int_K^0 \sqrt{i\alpha R(w-c)}\, dy}$$

$$B_1 = -ie^{-\int_K^0 \sqrt{i\alpha R(w-c)}\, dy}$$

so that

$$\frac{B_1}{A_1} = \frac{-ie^{-\int_K^0 \sqrt{i\alpha R(w-c)}\, dy}}{e^{-\int_K^0 \sqrt{i\alpha R(w-c)}\, dy} - ie^{+\int_K^0 \sqrt{i\alpha R(w-c)}\, dy}}$$

$$= \frac{-i}{1 - ie^{+2\int_K^0 \sqrt{i\alpha R(w-c)}\, dy}}$$

Let us assume that both c_r and c_i are small compared to unity, and that c is such that $y = 0$ and $y = 1$ belong to Stokes region III and I respectively. Then the expression

$$e^{+i\int_K^0 \sqrt{i\alpha R(w-c)}\,dy} + e^{5/4\pi i}\sqrt{\frac{\pi}{2\alpha R}}\,K^2$$

is very small since K^2 is the order of unity and very nearly real and positive. We can, therefore, neglect it and write:

$$\phi_A = \frac{1}{(w-c)^{5/4}}\left[e^{+\int_K^y \sqrt{i\alpha R(w-c)}\,dy} - ie^{-\int_K^y \sqrt{i\alpha R(w-c)}\,dy}\right] \quad (3.7)$$

we now simplify (3.7) in a manner equivalent to Lin's first approximation by writing:

$$\int_K^y \sqrt{i\alpha R(w-c)}\,dy \cong \frac{2}{3}\sqrt{2K\alpha R}\,\mu^{3/2}$$

where $\mu = (y-K)$

This approximation consists essentially in taking the velocity profile to be a straight line whose slope is the correct slope at the singularity K. It is in Lin's case and in ours a good approximation only because c is assumed to be small so that the slope at the singularity is very close to the slope on the wall where use is made of the representation. One must emphasize that the reason for the accuracy of Lin's results by a method utilizing first order terms of the form (3.3) is that the solution he developed gave only one class of eigen values, that for which the singularity is close to the wall.

Coming back to equation (3.7) we differentiate ϕ_A twice keeping

only terms of higher order in αR

$$\varphi_A'' \simeq \frac{1}{\mu^{1/4}} \left[e^{2/3 e^{-\pi i/4} \sqrt{2k\alpha R}\, \mu^{3/2}} - i e^{-2/3 e^{-\pi i/4} \sqrt{2k\alpha R}\, \mu^{3/2}} \right] \quad (3.8)$$

where $\mu = y - k$

Now for arguments of μ such that

$$-\frac{2\pi}{3} < \arg (2k\, \alpha R\, i) < \frac{2\pi}{3}$$

a valid asymptotic expansion of

$$\chi'' = A(\mu)^{1/2} H_{1/3}^{(1)} \left[\tfrac{1}{3} (2k\alpha R i)^{1/2} \mu^{3/2} \right] + B \left[\mu^{1/2} H_{1/3}^{(2)} \tfrac{2}{3} (2k\alpha R i)^{1/2} \mu^{3/2} \right]$$

is :

$$\chi'' \sim \frac{1}{\mu^{1/4}} \left[A e^{-2/3 (2k\alpha R)^{1/2} \mu^{3/2} e^{-\pi i/4} - \frac{5}{12}\pi i} + B e^{2/3 (2k\alpha R)^{1/2} \mu^{3/2} e^{-\frac{\pi i}{4}} + 5/12 \pi i} \right] \quad (3.9)$$

A comparison between (3.8) and (3.9) reveals that

$A e^{-5/12 \pi i} = -i$ $\qquad$ or $\qquad$ $A = e^{-1/12 \pi i}$

$B e^{+5/12 \pi i} = 1$ $\qquad$ $\qquad$ $B = e^{-5/12 \pi i}$

we can therefore write

$$\chi'' \sim e^{-\pi i/12} \delta^{1/2} H_{1/3}^{(1)} \left(\tfrac{2}{3} \delta^{3/2} \right) + e^{-5/12 \pi i} \delta^{1/2} H_{1/3}^{(2)} \left(\tfrac{2}{3} \delta^{3/2} \right)$$

or

$$\chi'' \sim e^{i\pi/3} \delta^{1/2} H_{1/3}^{(1)} \left[\tfrac{2}{3} \delta^{3/2} \right] + \delta^{1/2} H_{1/3}^{(2)} \left[\tfrac{2}{3} \delta^{3/2} \right]$$

where in Lin's notation:

$$J = -\left(\omega' K i\right)^{1/3} \eta = -\alpha_0 (i)^{1/3} \eta$$

$$\eta = (y - K)\, \alpha e^{1/3}$$

and y is measured from the wall.

If we define

$$h_1(J) = \left(\tfrac{2}{3} J^{3/2}\right) H^{(1)}_{1/3}\left(\tfrac{2}{3} J^{3/2}\right)$$

$$h_2(J) = \left(\tfrac{2}{3} J^{3/2}\right) H^{(2)}_{1/3}\left(\tfrac{2}{3} J^{3/2}\right)$$

it can be shown (7) that

$$h_1\left(e^{-\frac{2\pi i}{3}} J\right) = e^{\pi i/3}\left[e^{\pi i/3} h_1(J) + h_2(J)\right]$$

so that

$$\chi'' \sim h_1\left(e^{-\frac{2\pi i}{3}} J\right) \sim h_1\left(-e^{-2/3\pi i} \alpha_0 \eta i^{1/3}\right) = h_1(i\alpha_0 \eta)$$

or

$$\chi'' \sim \sqrt{\eta}\, H^{(1)}_{1/3}\left[\tfrac{2}{3}(i\alpha_0 \eta)^{3/2}\right] \qquad (3.10)$$

This is precisely the expression which Lin gets for his ϕ_3" so that we must conclude that his equality

$$\left(\frac{\phi_3'}{\phi_3}\right)_1 = \left(\frac{\phi_1'}{\phi_1}\right)_1 \qquad (3.6)$$

corresponds to the determinantal condition which must be imposed at the wall

$$\left(\frac{\phi_A'}{\phi_A}\right) = \left(\frac{\phi_B'}{\phi_B}\right)$$

where ϕ_B is the symmetric inviscid solution of Lin and ϕ_A is the correct combination of asymptotic expansions in Stokes region I.

However, one must have clearly in mind the limitations inherent in that agreement. It would be impossible to associate our asymptotic solutions near the wall to the first kind of asymptotic expansions of the Hankel function:

$$\phi_3 = \sqrt{\eta}\, H^{(1)}_{1/3}\left[\tfrac{2}{3}(i\alpha_0\eta)^{3/2}\right]$$

outside of the interval given above for the argument. Yet ϕ_3 is a single valued function of η . The conclusions must therefore be reached that:

a) If the relation

$$\left(\frac{\phi_3'}{\phi_3}\right) = \left(\frac{\phi_1'}{\phi_1}\right)$$

holds for a particular value of the argument (c being small) that relation holds for any value of the argument. This relation can therefore be verified a posteriori when values of c are obtained. In Lin's case the values of c associated with the neutral curve all satisfy the restriction imposed on η and therefore the relation (3.6) above is satisfied.

b) The type of expansion chosen for ϕ_3 however, depends on the value of the argument.

The remaining part of Lin's development consists primarily of an algebraic solution of (3.6). Lin defines:

$$\frac{\phi_{31}}{\phi_{31}} = (y-k)\,F(z)$$

where

$$F(z) = \frac{\int_{+\infty}^{-z} d\mu \int_{+\infty}^{\mu} \xi^{1/2} d\xi \, H_{1/3}^{(1)} \left[\frac{2}{3}(i\xi)^{3/2}\right]}{-z \int_{+\infty}^{-z} \xi^{1/2} d\xi \, H_{1/3}^{(1)} \left[\frac{2}{3}(i\xi)^{3/2}\right]}$$

$$z = -\alpha_0 (\eta(1) - \eta_\kappa)$$

The function $F(z)$ was calculated both by Tietjens and by Lin for real values of z, $(c_i = 0)$. The inviscid solution which is regular on the wall is also evaluated for real values of c and equation (3.6) finally gives the locus of points in the α vs $\mathcal{R}$ plane for which c_i vanishes. This is the so-called neutral curve which divides a region of stability (negative values of c_i) from a region of instability (positive values of c_i). The transformations which are required to evaluate this curve from equation (3.6) are numerous and laborious. Accounting for them here would serve no purpose and would only lengthen the present analysis. However, it must be understood that Lin's method which we outlined above was meant to yield a determination of the neutral curve, not to investigate in general the valuesof the characteristic numbers c. An extension of this author's methods for the determination of c_r and c_i would be hampered by the following difficulties:

a) The first approximations to the solution of (3.1) which were designated as $\chi_3^{(0)}$ and $\chi_4^{(0)}$ would be poor for eigen values such that c approaches unity. The reason for that fact has

already been given.

b) Those eigen values which are characterized by:

$$c \to 1 \quad \text{as} \quad \alpha R \longrightarrow \infty$$

have been shown by equation (2.8) to lie in a region such that:

$$\arg(\eta) \cong \frac{\pi}{4}$$

For such an argument, asymptotic expansions of the first type cannot be used. Hence, an order of magnitude analysis such as the one employed by Lin would not be permissible and the particular simplification which resulted could not be made. Both $H^{(1)}_{1/3}$ and $H^{(2)}_{1/3}$ would have to be considered at the wall.

c) For the family $c \to o$ as $\alpha R \to \infty$ the asymptotic representation chosen by Lin for ϕ_3 which has been referred to as an expansion of the first type, is not valid if:

$$\arg(\eta) = \arg(y-k) = \frac{7\pi}{6} + 2N\pi$$

where N is an integer. In addition, that representation can be expected to yield poor accuracy for arg. (η) in the neighborhood of this axis, except for extremely large values of αR . For instance, Lin gives the asymptotic form of $\phi_3(1)/\phi_3'(1)$ as

$$\frac{\phi_3(1)}{\phi_3'(1)} = -\frac{e^{\pi i/4}}{\sqrt{\alpha R c}}$$

For $\alpha = o$ this is certainly in error since for that case

$$\frac{\phi_1'(1)}{\phi_1(1)} = -2c$$

and the value of c_1 thus derived would be positive. This is in contradiction with the rest of Lin's work which yields negative values of c_1 when $\alpha = o$. It is also in contradiction with Pekeris's work and with our own. For that case ($\alpha = o$)

the other type of asymptotic expansion should be used in conjunction with (3.6). Not only will it give asymptotically correct eigen values for the first mode, but also a series of eigen values for each αR corresponding to the higher modes of c. This latter point will be brought up in connection with the analysis of C.L. Pekeris.

SECTION II

C. L. Pekeris, following earlier studies of the problem of plane Poiseuille flow for low values of the Reynolds number has given a solution of that problem for high values of R . The methods used by this author are markedly different from those of Lin and the results are as we have mentioned earlier in contradiction with the ones Lin has advanced. We intend presently to survey the area of agreement and disagreement between these two authors.

Pekeris, unlike Lin, notices the existence of two families of eigen values. They correspond to the two families which we have already discussed in Chapters I and II . We shall examine first the class characterized by the property $c \to 1$ as $\alpha R \to \infty$.

The discussion of Chapter II showed that for that case the influence of α decreases steadily as αR increases. Pekeris, lead by the same kind of consideration first examines the problem for $\alpha^2 = 0$.

For that case the Orr-Sommerfeld equations are written:

$$(W-c)\,\phi'' - W''\phi = -\frac{i}{\alpha R}\,\phi^{IV} \qquad (3.11)$$

equation (3.11) can be integrated once

$$(W-c)\,\phi' - W\phi + k = -\frac{i}{\alpha R}\,\phi'''$$

For symmetrical ϕ functions and since the velocity profile is symmetrical we have at the origin $y = 0$

$$\phi'''(0) = \phi'(0) = W'(0) = 0$$

hence $k = 0$

and since $\phi(1) = \phi'(1) = 0$.

we also have $\phi'''(1) = 0$ (3.12)

by making successively the changes of variable

$z = y^2$ and $x = (i\alpha R)^{\frac{1}{2}} z$ and by assuming for simplicity

$$a = \left[\frac{5 - (1 - C_0) i^{1/2} (\alpha a)^{-1/2}}{4} \right]$$

where C_0 represents the eigen values for $\alpha^2 = 0$ Pekeris is able to identify a solution for the second derivative:

$$\frac{d^2\phi}{dx^2} = e^{-x/2} F\left(a, \frac{5}{12}, x\right) \qquad (3.13)$$

By using the condition (3.12) in conjunction with (3.13) Pekeris concludes that

$$F(a, 3/2, \lambda) + F(a-1, 3/2, \lambda) = 0 \qquad 3.14$$

where $\lambda = (i\alpha a)^{1/2}$ and F is the confluent hypergeometric function.

Pekeris then expands these two hypergeometric functions asymptotically

$$F(\alpha, \gamma, \lambda) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} e^{\lambda} \lambda^{\alpha-\gamma} \left[1 + \frac{(1-\alpha)(\gamma-\alpha)}{\lambda} + \cdots \right] + \cdots$$

In order to satisfy (3.14) both $\Gamma(a)$ and $\Gamma(a-1)$ must become infinite. $\Gamma(a) = \infty$ for $a = 1-m$; $m = 1, 2, 3 \ldots$

$\Gamma(a-1) = \infty$ for all integers for which $\Gamma(a)$ is ∞

so that Pekeris obtains for $\alpha^2 = 0$

$$C_0^m = 1 - \frac{(1+4m)}{\alpha a^{1/2}} i^{1/2} \qquad m = 1, 2, 3 \ldots \text{ etc.} \qquad (3.15)$$

Expression (3.15) is identical to expression (2.8) which was obtained in Chapter II as the limiting formulation of the asymptotic solution for one family of modes. One disagreement between our results and equation (3.15) is the existence according to (2.8) of one lower mode ($N=0$) than the equation (3.15) indicates. The reason for this discrepancy has not been made clear. An analytic expression for the influence of α^2 is not derived by Pekeris in the case of plane Poiseuille flow. It is given by analogy with the pipe case as $C_1 = +\frac{5i}{3\alpha R}$ where $C = C_0 + \alpha^2 C_1$

so that the destabilizing effect of α is seen to decrease very rapidly and to be small for all but very low values of αR

This is certainly in qualitative agreement with equation (2.6). For the family of eigen values such that

$$c \to 0 \quad \text{as} \quad \alpha R \to \infty$$

Pekeris expands the solution in a power series in α^2

$$\phi = \phi_0 + \alpha^2 \phi_2 + \cdots$$

The solution ϕ_0 for $\alpha^2 = 0$ is given. It is made of the sum Lin's inviscid solution for symmetrical ϕ functions and of a function $\varphi(z)$ such that

$$\varphi(z) = \varphi_0(z) + \left(\frac{1}{z}\right)\varphi_1(z) +$$

$$z = e^{i\pi/2}(2\alpha R)^{1/3}$$

$$y = \left(\frac{1-Y_1}{2} + \frac{z}{2}\right)$$

The first approximation for $\varphi(z)$ is given as follows:

$$F(z) = \frac{d^2\varphi_0}{dz^2} = z^{1/2} H^{(2)}_{1/3}\left[\left(\tfrac{2}{3} z^{3/2}\right)\right]$$

It satisfies the condition

$$F'(z_1) = 0$$

which determines $z_1 = \gamma_1$

$$\gamma_1 = e^{i 2\pi/3} \tau_m$$

$$\tau_m = \left(\frac{3\delta_m}{2}\right)^{2/3}$$

and δ_m satisfies

$$\left[J_{2/3}(\delta_m) - J_{-2/3}(\delta_m)\right] = 0$$

The roots of this equation are the roots of Airy's function $A_i(-x)$. So that (7) the first two roots are:

$$\tau_1 = 1.01879297$$
$$\tau_2 = 3.24819758$$

As has been remarked earlier for $\alpha = 0$

$$\phi'''(1) = 0$$

also for the inviscid part of the solution ϕ''' is automatically 0. So that Lin's determinantal equation (3.6) reduces to

$$\phi'''_3 = 0$$

if we differentiate (3.10) once we get

$$\Phi'''_3 = h_1'(i\delta_0 \eta)$$
$$= h_2'(e^{2\pi/3} J)$$

of which the roots are the τ_m found by Pekeris. It is thus seen that for $\alpha = 0$ the first approximation to the solution for ϕ is given by Pekeris and Lin in a completely equivalent form.

In addition it is of interest to note that our formula for δ (in Chapter II)

$$\delta = \frac{1}{2}\left[2\pi\left(1+4N\right)\right]^{2/3}$$

which was approximately derived for high modes agrees completely with Pekeris' approximation for $\alpha = 0$

$$c_0 = \frac{2\tau_m e^{i\pi/6}}{(2\alpha R)^{1/3}}$$

since Pekeris' τ_m is exactly one half of our $\delta = (2i\alpha R)^{1/3} c$

We have so far established the following areas of agreement.

For the family of modes characterized by

$$c \to 1 \quad \text{as} \quad \alpha R \to \infty$$

which Lin does not mention, Pekeris' results and ours agree for $\alpha = 0$ and qualitatively at least for $\alpha \neq 0$. This is all that seems to be required physically since these modes are stable.

For the family of modes characterized by

$$c \to 0 \quad \text{for} \quad \alpha R \to \infty$$

Lin's result for $\alpha = 0$ is the first approximation to Pekeris' result for that case. Our asymptotic solutions also give this result for large αR and higher modes.

Inasmuch as we have verified the correctness of Lin's formulation (3.6) and since the computer solutions of L. H. Thomas confirm closely Lin's numerical answers, it is suggested that Pekeris misjudged the influence of the wave number α. In particular the series

$$c = c_0 + \alpha^2 c_2 + \alpha^4 c_4 + \cdots$$

does not appear to converge fast for moderate values of α.

According to Pekeris' own result for $\alpha = 1$ and $\alpha R = 5430$, C_2 is in magnitude close to half of C_0. If it is realized that the imaginary part of c according to Pekeris is not 5% of the total magnitude and that C_i is positive, it seems legitimate to doubt that c is adequately represented by the first two terms of the above expansion. On the other hand a direct verification of the size of C_4 seems to be a task of some magnitude. At any rate an order of magnitude analysis on this point should certainly not consider the relative size of C_1 and C_2 only, but more particularly the relative size of the imaginary part of each quantity.

An added argument can be made against the accuracy of Pekeris' account of the influence of α :

According to that author, the coefficient C_2 of α^2 involves a term b which is independent of αR. If that term were present or uncancelled by other terms, there should be no reason to suspect our own equation (2.6) of which the right hand side would then tend to unity; e^Q would be negligible and e^P would increase without bounds. Equation (2.6) thus indicates that the existence of a term $\alpha^2 b$ in the representation of c for low valuesof that parameter is impossible.

SECTION III

The higher modes

Lin's work covered only eigen values of the first or lowest mode. Both Pekeris' results and ours indicate that higher modes yield values of c for which the solution for the perturbation velocity components are more stable than for the first (higher negative values of c_i). The physical interest of such modes is therefore not very great. Nevertheless several properties of the solution are worth noting.

The asymptotic formulae derived by Pekeris (3.15) and our limiting formulae (2.8) become worse approximations to the true eigen values for a given αR as the number of the modes increases. We have seen that these formulae show that the eigen values lie along two straight lines in the complex plane for very large αR. In order to keep the same accuracy for higher and higher modes one is forced to select increasing values of αR.

Furthermore only a limited number of modes exist for a given value of αR. The lower αR is, the fewer modes exist. This can be seen from equation (2.6) or from a combination of equations (2.8) and (2.9). For higher modes the values of c are such that the singularity K moves away from either boundary and the two families of eigen values c move toward each other. Unlike Pekeris' formulae for c, equation (2.8) gives values of c such that one family merges smoothly into the other, and we need exclude as inaccurate only those eigen values for which the singularity is close to the wall. These are the eigen values best computed by Lin's method. The phenomenon of the birth of new modes as αR increases has been noticed and discussed by Pekeris.

CHAPTER IV

Stability of pipe flow (conclusion)

It has been said earlier that our asymptotic solutions were probably not accurate enough for the correct determination of eigen values such that $c \to 0$ as $\alpha R \to \infty$

We present here a method of settling the question of stability for axi-symmetric flow, which is inspired by Lin's treatment although it is simplified by the analytic properties of the inviscid solution in the pipe case.

We consider, as we did in Chapter III for plane Poiseuille flow the linear combination of asymptotic solutions between the Stokes point K and the wall $r = 1$. If we assume c to be small compared to unity it is possible to expand the solution of equation (1.6) in a convergent power series around the singularity K. In the immediate neighborhood of that point which includes the real point $r = 1$ the first term of the expansion for ϕ will be a sufficiently accurate representation of the solution. That first term is easily found to be such that

$$\varphi_A'' = \sqrt{\eta}\; H_{1/3}^{(1)}\left[\tfrac{2}{3}(i\alpha\eta)^{3/2}\right]$$

On the other hand for the case of small c, K^2 is very nearly real and positive and e^{Q} is very small. under these conditions:

$$\frac{A_1}{B_1} \simeq 1$$

and the asymptotic solutions are written

$$\varphi_A = \left[e^{+\int_K^r \sqrt{i\alpha R(w-c)}\,dr} - ie^{-\int_K^r \sqrt{i\alpha R(w-c)}\,dr}\right](w-c)^{1/2 s/q - s/q}$$

Since the singularity K is close to the wall, $r^{\frac{1}{2}}$ can be replaced by unity or considered constant throughout the interval $K < r < 1$. It is clear that under these conditions (K^2 close to 1, first mode) expression (4.3) is identical to expression (3.7).. It follows therefore that we can identify the asymptotic part of the solution by:

$$\varphi_A'' = \sqrt{\eta}\, H^{(1)}_{1/3}\left(\tfrac{2}{3}(i\alpha\eta)^{3/2}\right)$$

and again we can write on the boundary $r = 1$

$$\frac{\varphi'_A}{\varphi_A} = \frac{\varphi'_B}{\varphi_B} \qquad (4.1)$$

where φ_B is the inviscid solution:

$$\varphi_B = r J_1(i\alpha r)$$

Hence the ratio $\frac{\varphi'_B}{\varphi_B}$ can be evaluated simply.

Following Lin we write:

$$\frac{\varphi_A}{\varphi'_A} = (1-K)\ F\ (z)$$

where

$$F\ (z) = \frac{\int_{+\infty}^{-z} d\mu \int_{+\infty}^{\mu} d\xi\ H^{(1)}_{1/3}\left[\frac{2}{3}(i\xi)^{3/2}\right]}{-z\left[\int_{+\infty}^{-z} \xi^{1/2}\, d\xi\ H^{(1)}_{1/3}\left[\frac{2}{3}(i\xi)^{3/2}\right]\right.} \qquad (4.2)$$

and where

$$z = (1-K)\,\alpha R^{1/3}\,(2k)^{1/3}$$

On the other hand

$$\frac{\varphi_B}{\varphi'_B} = \left[\frac{r J_1(i\alpha r)}{\frac{d}{dr} r J_1(i\alpha r)}\right]_{r=1}$$

is a pure real quantity since α is real. Therefore, the value of z for which equation (4.1) can be satisfied for $c_i = 0$ are those which make F (z) a purely real function.

From Lin's computation of the Tietjens curve for real z it is apparent that the only value of z which will satisfy this condition is

$z = 2.67$ for which $F(z) = .5194$

Hence we have two relations which would permit the determination of c, and αR corresponding to any α for $c_i = 0$. Those relations however have been derived with the following restrictions:

c is small compared to unity.

α^2 is small compared to αR.

It is apparent that no α can be found which satisfies these conditions. The function

$$\frac{J_1(i\alpha)}{\frac{d}{dr}\left[r J_1(i\alpha r)\right]_{r=1}} = \frac{1}{1 + \frac{1 + \frac{3}{8}\alpha^2 + \frac{5}{192}\alpha^4 + \cdots}{1 + \frac{\alpha^2}{8} + \frac{\alpha^4}{192} + \cdots}}$$

is a decreasing function of α^2 which yields small values of c only for values of α such that α^2 is of the order of αR. None of the work presented here is valid for that case. The conclusion one must draw is that for moderate values of α there is no eigen value c such that its imaginary part vanishes. Hence no possibility of instability.

A strong argument for the stability of pipe flow to axi-symmetric disturbances can also be made by comparison with Couette flow. Such a flow is characterized by a velocity profile $W = y$. It has been shown in particular by Hopf (8) that such a flow is stable and general agreement seems to exist on that point. Lin analyses this flow in the same way as plane Poiseuille flow and it is found that the determinantal equation obtained if c is assumed small:

$$\frac{f_1(\alpha, c)}{f_3(\alpha, c)} = \frac{\phi_3(1)}{\phi'_3(1)}$$

is correct according to a detailed study of the crossing substitution. It gives an expression which is remarkably similar to the one we arrived at for axi-symmetric flow. The inviscid solutions used in this case are $\phi_{B_1} = \sinh(\alpha y)$ $\phi_{B2} = \cosh(\alpha y)$ and the other two solutions are exactly of the same form as for plane Poiseuille or pipe flow so that we have

$$(1-k)\, F(z) = \frac{\varphi_B}{\varphi'_B} = \frac{\sinh(\alpha)}{\alpha \cosh(\alpha)} \qquad (4.3)$$

where $F(z)$ is given by equation (4.2). The expression φ_B / φ'_B takes on all possible values for $\alpha > 0$. Therefore for any value of $J_1(i\alpha r) / \frac{d}{dr}[r J_1(i\alpha r)]_1$ one can find, at different α a value of φ_B / φ'_B for which it is already known that c_1 is negative. Since the left hand side of equation (4.3) is not dependant on α, the stability of Couette flow seems to ensure the stability of axi-symmetric flow in pipes.

It thus appears that while two dimensional Poiseuille flow is unstable, axi-symmetric Poiseuille flow is stable if excited by small disturbances and that the difference between these two flows is entirely due to the character of the solutions of the so-called inviscid equations in both cases. It is worth noting for instance, that axi-symmetric pipe flow, Couette flow and plane Poiseuille flow for $\alpha = 0$, all of which have regular inviscid solutions throughout the interval, are characteristically stable.

CHAPTER V.

Independent analyses of the stability of pipe flows.

While considerable efforts have been made to solve the stability problem in the boundary layer and in plane parallel flows, relatively little has been published until recently on axi-symmetric flows in pipes. As the present work was undertaken we were aware of only two relevant analytical studies on the stability of axially symmetric pipe flows. They were those of Pretsch (9) and Sexl (1). Recently, Pekeris (10) has treated the same subject. We shall show later that while we concurr with the partial results of his analysis, these results can hardly be interpreted as a proof of stability of pipe flow. In addition Tatsumi (11) has examined the stability of entrance flow in a circular pipe.

J. R. Sellars (12) has demonstrated that Pretsch's work is fundamentally erroneous on at least two grounds. There seem to be no need to develop this criticism here. Sexl's approach to the problem is interesting and resourceful and it is of interest to summarize it.

SECTION I The Analysis of Theodore Sexl.

As we have mentioned earlier, Sexl writes equation (1.6) as:

$$\frac{d^2f}{dr^2} - \frac{1}{r}\frac{df}{dr} - \alpha^2 f - (1 - r^2 - c)\, i\alpha R f = 0 \qquad (1.8a)$$

$$\frac{d^2g}{dr^2} - \frac{1}{r}\frac{dg}{dr} - \alpha^2 g = f \qquad (1.8b)$$

The boundary conditions (1.7) are to be applied to this system. The change of variable $y=r^2$ transforms(1.9)into:

$$\frac{d^2f}{dy^2} + \frac{id\alpha}{4}\left(1 - \frac{(1-c)}{y}\right) f - \frac{d^2 f}{4y} = 0 \qquad (5.1)$$

Equation (5.1) has two linearly independent solutions which can be found by power series expansions. Since the origin is a regular singular point and the only singularity in the finite plane, the expansions will converge. This type of equation has one solution of the form

$$f_1 = y - \frac{k_1^2}{2!} y^2 + \frac{k_2^2}{3!2!} y^3 + \cdots$$

while the other solution is:

$$f_2 = B_1 f_1 \log y + B_2 y + B_3 y^2 + \cdots$$

for large values of $d\mathfrak{R}$ those solutions converge extremely slowly and are therefore of no direct use. On the other hand the homogeneous part of (1.8b)

$$\frac{d^2\varphi}{dr^2} - \frac{1}{r}\frac{d\varphi}{dr} - d^2\varphi = 0 \qquad (5.2)$$

has two well known solutions in closed form

$$\varphi_1 = C r J_1(idr)$$
$$\varphi_2 = D r Y_1(idr) \qquad (5.3)$$

where C and D are arbitrary constants. It is therefore possible to express formally the solution of the system (1.8) by using the method of variation of parameters to determine four linearly independent solutions:

$$\phi(y) = A_1\left[-\varphi_1\int_0^y f_1\frac{g_2}{Wr}\,dy + g_2\int_0^y f_1\frac{g_1}{Wr}\,dy\right] + A_2\left[-\varphi_1\int_0^y f_1\frac{g_2}{Wr}\,dy + g_2\int_0^y f_2\frac{g_1}{Wr}\,dy\right] + A_3 r J_1(iar) + A_4 r Y_1(iar) \qquad (5.4)$$

where φ_1 and g_2 are given by (5.3)

f_1 and f_2 are the solutions of (1.8,a)

and Wr is the Wronskian of (5.3)

Expression (1.14) can be rewritten:

$$\varphi/r = C_1 i\frac{\pi}{2}\left\{J_1(iar)\int_0^y Y_1(iar) f_1(ig)\,dr - Y_1(iar)\int_0^r J_1(iar) f_1(y)\,dr\right\}$$
$$+ C_2 i\frac{\pi}{2}\left\{J_1(iar)\int_0^y Y_1(iar) f_2(y)\,dr - Y_1(iar)\int_0^r J_1(iar) f_2(y)\,dr\right\}$$
$$+ C_3 J_1(iar) + C_4 Y_1(iar)$$

Sexl then imposes on this system the boundary conditions (1.7)

The conditions at the origin:

$$\lim_{r\to 0}\left(\frac{\phi}{r}\right) = 0 \;;\; \lim_{r\to 0}\left(\frac{\phi'}{r}\right) \text{ bounded}$$

require that C_2 and C_4 be 0. In effect we are requiring φ to vary in the neighborhood of the origin as r^n where $n \geq 2$.

The remaining boundary conditions at the wall $\phi(1) = \phi'(1) = 0$ are imposed in order to determine the value of C_1/C_3. The determinant of the coefficients of C_1 and C_3 is set equal to 0 at the wall

$$\begin{vmatrix} \phi_A(1) & \phi_B(1) \\ \phi'_A(1) & \phi_B(1) \end{vmatrix} = 0$$

where ϕ_A is the coefficient of C_1 and ϕ_B that of C_3 and differentiations are made with respect to r.

This yields simply

$$\int_0^r J_1(i\alpha r)\, f_1(r^2)\, dr = 0$$

The result is particularly elegant: the second order differential equation and the boundary conditions are all expressed in terms of the new dependent variable f.

The system

$$\begin{cases} f'' + \frac{i\alpha R}{4}\left(1 - \frac{1-c}{y}\right) f - \frac{\alpha^2 f}{4y} = 0 & (a) \\ \varphi'' - \frac{\varphi'}{r} - \alpha^2 \varphi = f(r) & (b) \\ f(0) = 0 & (c) \\ \int_0^1 J_1(i\alpha r)\, f_1(y)\, dr = 0 & (d) \\ y = r^2 & (e) \end{cases} \qquad (5.4)$$

is entirely equivalent to (1.6); (1.7) where $f_1(y)$ is the solution of (5.1) for which f (o) = o

We may add parenthetically that one could easily overestimate what has been gained by formulating the problem in the form (5.4). We may suspect for instance that an integral boundary condition will impose serious restrictions upon the type of representation that can be chosen for Φ . It has been seen earlier

that while good use can be made of (5.4) an analytic solution of the flow stability problem seems to follow much more easily from (1.6; 1.7) than from (5.4).

Sexl's argument does not give the solution of our problem. The conclusion that pipe flow is stable for all Reynolds numbers is attained without a determination of eigen values c or eigen functions f. It follows rather from considerations analogous to those exposed earlier in connection with the stability of a non viscous flow.

Equation (1.8a)

$$\frac{d^2f}{dr^2} - \frac{1}{r}\frac{df}{dr} - \alpha^2 f - (1-r^2-c)i\alpha R f = 0$$

is written

$$\left(\frac{f'}{r}\right)' - \frac{\alpha^2 f}{r} = i\alpha R(1-r^2-c)\frac{f}{r}$$

and split into real and imaginary parts:

$$\frac{1}{\alpha R}\left(\frac{f_r'}{r}\right)' = \frac{f_r}{r}\left(\frac{\alpha^2}{\alpha R} + c_i\right) - \frac{f_i}{r}(1-r^2-c_r)$$

$$\frac{1}{\alpha R}\left(\frac{f_i'}{r}\right)' = \frac{f_r}{r}(1-r^2-c_r) - \frac{f_i}{r}\left(\frac{\alpha^2}{\alpha R} + c_i\right)$$

Sexl then multiplies the top one by $\frac{f_r}{r}$ the bottom one by $\frac{f_i}{r}$ adds the two equations and integrates them across the cross section of the pipe. He thus gets

$$\frac{1}{\alpha R}\int_0^1\int_0^{2\pi}\left[\frac{f_r}{r}\left(\frac{f_r'}{r}\right)' + \frac{f_i}{r}\left(\frac{f_i'}{r}\right)\right]r\,dr\,d\theta = \left[\frac{\alpha^2}{\alpha R} + c_i\right]\int_0^1\int_0^{2\pi}\left\{\left(\frac{f_r}{r}\right)^2 + \left(\frac{f_i}{r}\right)^2\right\}r\,dr\,d\theta$$

The left hand side is transformed by the use of one of Green's theorems and becomes

$$\frac{1}{2\alpha}\int_0^{2\pi}\left[\frac{f_r}{r^2}\frac{\partial f_r}{\partial r}\,d\theta + \frac{f_i}{r^2}\frac{\partial f_i}{\partial r}\,d\theta\right] - \frac{1}{\alpha R}\int_0^1\int_0^{2\pi}\left(\frac{f_r'}{r}\right)^2 + \left(\frac{f_i'}{r}\right)^2 r\,dr\,d\theta$$

where the integrand at the left is evaluated at $r=1$

The left hand side of this expression would vanish only if f/r or $\frac{1}{r}\frac{df}{dr}$ vanished on the wall. In that case we would be left with

$$-\frac{1}{2\alpha}\int_0^1\int_0^{2\pi}\left[\left(\frac{f_r}{r}'\right)^2 + \left(\frac{f_i}{r}'\right)^2\right]r\,dr\,d\theta = \left(c_i + \frac{\alpha^2}{\alpha R}\right)\int_0^1\int_0^{2\pi}\left(\frac{f_r}{r}\right)^2 + \left(\frac{f_i}{r}\right)^2 r\,dr\,d\theta$$

and it would be clear that the integrands being both positive, Ci would have to be negative and $\leq -\frac{\alpha^2}{\alpha R}$

However we have no a priori knowledge of the value of $\frac{f(r)}{r}$ or $\frac{1}{r}\frac{df(r)}{dr}$ on the wall and, unable to determine the sign of Ci at this point, Sexl is forced to probe deeper into the character of the solution f.

The stability of axi-symmetric incompressible flow in a uniform pipe has been made to depend on the value of $\frac{f(r)}{r}$, and $\frac{f'(r)}{r}$, and Sexl then attempts to show that either term vanishes at the wall. Since stability of pipe flow is not questioned at very low Reynolds number, Sexl formulates asymptotic solutions of the W.K.B. type. The function f is thus represented by a linear combination

$$f = \frac{r^{1/2}}{(w-c)^{1/4}}\, e^{+\int_{r_0}^{r}\sqrt{i\alpha R(w-c)}\,dr} + C\frac{r^{1/2}}{(w-c)^{1/4}}\, e^{-\int_{r_0}^{r}\sqrt{i\alpha R(w-c)}\,dr} \qquad (5.5)$$

where the lower limit r_0 is arbitrary and α is a constant.

Sexl then attempts to show that the value of $c(\alpha_R, \alpha)$ which is required to satisfy the boundary conditions for moderate values of α namely

$$\int_0^1 r J_1(i\alpha r) f_1(r) dr = 0 \qquad (5.4d)$$

is precisely one which makes f' disappear on the wall. Differentiating (5.5) and calling $\bar{f}_1$ and $\bar{f}_2$ the two solutions given by that equation it is readily seen that the condition for $f' = \bar{f}_1' + c\bar{f}_2'$ to be zero at the wall is

$$+ C = e^{2\int_{r_0}^1 \sqrt{-i\alpha R(w-c)}\, dr}$$

on the other hand Sexl attempts to integrate (5.4d) by noting that

$$\frac{d}{dr}\left[\frac{J_1(i\alpha r) r^{1/2}}{(w-c)^{1/4}} \frac{e^{\int_0^r \sqrt{i\alpha R(w-c)}\, dr}}{\sqrt{i\alpha R(w-c)}} \right] \sim$$

$$\frac{J_1(i\alpha r) r^{1/2}}{(w-c)^{1/4}} e^{\int_0^r \sqrt{i\alpha R(w-c)}\, dr} + \frac{e^{\int_0^r \sqrt{i\alpha R(w-c)}\, dr}}{\sqrt{i\alpha R(w-c)}} \frac{d}{dr}\left[\frac{J_1(i\alpha r) r^{1/2}}{(w-c)^{1/4}} \right]$$

and that the second term is of higher order than the first for values of α of the order of unity. He thuswrites (5.4d) as

$$\frac{J_1(i\alpha)}{(-c)^{1/4}} \left\{ \frac{e^{\int_{r_0}^1 \sqrt{i\alpha R(w-c)}\, dr}}{\sqrt{i\alpha R c}} + C \frac{e^{-\int_{r_0}^1 \sqrt{i\alpha R(w-c)}\, dr}}{-\sqrt{i\alpha R c}} \right\} = 0 \qquad 5.6$$

which imposes

$$e^{2\int_{r_0}^1 \sqrt{i\alpha R(w-c)}\, dr} = + C$$

Thus Sexl concludes that f' is 0 at the wall.

By his previous argument, therefore, the flow is stable for values of α of the order of unity. Unfortunately this elegant method of proof, while it by-passes several difficulties, is the victim of two fundamental errors. The first and possibly the most obvious one is that Sexl's method of integration of functions of the form:

$$f(r)e^{\int q(r)dr}$$

is incorrect. Expression (5.6) is simply not a close approximation to the integral of $J_1(i\alpha r)$ multiplied by expression (5.5) evaluated at the wall, no matter how large αr may be. For instance there is no reason to suspect that

$$\int_0^r J_1(i\alpha r)\left[\bar{f}_1 + C\,\bar{f}_2\right]dr$$

is unbounded at $W = c$ as Sexl's representation of this expression would indicate.

The second error made in Sexl's proof is a disregard for the property which we have referred to as the Stokes phenomenon. Expressions $\bar{f}_1$ and $\bar{f}_2$ are multi-valued and they have three branch-cuts originating at the singularity. As the argument of these functions crosses one of the branch-cuts the coefficients of either $\bar{f}_1$ or $\bar{f}_2$ must be altered in general if the representation is to be continuous and if it is to converge to the value of f on either side of the point for which $W = c$. Sexl does not indicate he was ever preoccupied by this difficulty and as a consequence his evaluation of C is neither meaningful nor correct.

One might add that suspicions about the correctness of Sexl's method were raised when an attempt to adapt it to the case of Couette flow between two flat plates in relative motion revealed a contradiction.

SECTION II The Investigation of C. L. Pekeris.

Pekeris (10) makes use of Sexl's formulation of the problem (see Chapter V Section I)

$$\left.\begin{aligned} &\frac{d^2f}{dy^2} + \frac{i\alpha}{4}\left(1 - \frac{1-c}{y}\right) f - \frac{d^2f}{4y} = 0 \\ &f(0) = 0 \\ &\int_0^1 J_1(i\alpha r)\, f(y)\, dy = 0 \\ &y = r^2 \\ &f = \varphi'' - \frac{\varphi'}{r} - \alpha^2 \varphi \end{aligned}\right\} \quad 5.4$$

a solution is recognized as:

$$f = e^{-\tau y}\, y\, F(a, 2, 2\tau y)$$

where $\tau = \frac{1}{2}\sqrt{i\alpha}\, e^{-i\pi/4}$

$$a = 1 - \frac{(1-c)\sqrt{i\alpha}}{4} - \frac{\alpha^2}{4\sqrt{i\alpha}}$$

The integral boundary conditions gives:

$$\int_0^1 r\, J_1(i\alpha r)\, e^{-\tau r^2}\, r^2\, F(a, 2, 2\tau r^2)\, dr = 0 \qquad (5.7)$$

For large αR the real part of τ is positive and very large so that

$$e^{\tau y} y F(a,2,2\tau y) \longrightarrow \frac{y e^{\tau y}(2\tau y)^{a-2}}{\Gamma(a)} \left\{ 1 + \frac{(1-a)(2-a)}{2\tau y} + \cdots \right.$$

Provided a is such that the last bracket on the right approaches unity as τ approaches ∞ .

This asymptotic representation satisfies the boundary conditions if $\Gamma(a) = \infty$ a condition which is fulfilled for

$$a = n \qquad n = 0, 1, 2, 3 \cdots \text{etc}$$

so that in the limit of large αR the characteristic value of c has the asymptotic form

$$c = 1 - \frac{4(1+n)}{\sqrt{\alpha R}} e^{\pi i/4} - \frac{i\alpha^2}{\alpha R} \qquad (5.8)$$

$n = 0, 1, 2$ etc...and are seen to insure the convergence of the asymptotic series above.

The first part of c which is independent of α^2 is identically the limiting expression of Chapter I. Again while the second term of equation (5.8) cannot be exactly identified with our results, it seems to show just as equation (1.11) does that the effect of the wavenumber decreases for higher αR . However Pekeris' method is limited in that it fails for eigen values not approaching unity for high αR . Pekeris thus omits the second family of eigen values which was found in Chapter I and which is common to plane Poiseuille and to axi-symmetric flow.

Since this is the class of eigen values which is responsible for the instability of plane Poiseuille flow, the question of the stability of pipe flow was left by Pekeris very far from settled.

SECTION III A Differential Analyzer Solution.

While the reduction of the system (1.6; 1.7) to the system (5.4) does not simplify the task of seeking an analytic solution, good use can be made of formulation (5.4) in connection with a solution by means of an electronic differential analyzer. For that purpose, equation (1.8a) is split into real an imaginary parts:

$$\left.\begin{aligned} f_r'' - \frac{\alpha R}{4} f_i + \frac{\alpha R}{4}\left(\frac{1-c_r}{y}\right) f_i - \frac{\alpha R}{4y} c_i f_r = 0 \\ f_i'' - \frac{\alpha R}{4} f_r - \frac{\alpha R}{4}\left(\frac{1-c_r}{y}\right) f_r - \frac{\alpha R}{4y} c_i f_i = 0 \end{aligned}\right\} (5.9)$$

with the following boundary conditions

$$f_r(0) = f_i(0) = 0 \qquad (5.10)$$

$$\left.\begin{aligned} \int_0^1 J_1(i\alpha r) f_r(y)\, dy = 0 \\ \int_0^1 J_1(i\alpha r) f_i(y)\, dy = 0 \end{aligned}\right\} 5.11$$

The solution is essentially a trial and error process. With (5.10) as an initial condition successive guesses are made for

the correct values for c_i and c_r corresponding to a given R and a given α .*

The eigen values obtained are given below.

Table of the Eigen Values c obtained with a differential analyzer

α^2	αR	c_i	c_r
0	576	.1750	.666
"	829	.1625	.500
"	894	.1565	.588
"	1055	.143	.556
"	1125	.1395	.549
.5	828	.1025	.600
"	872	.161	.592
"	932	.152	.570
"	1032	.1455	.559
"	1135	.1455	.554
"	1255	.142	.550
1.0	811	.160	.592
" "	866	.156	.584
"	908	.154	.578
"	990	.150	.569
"	1080	.152	.565

*For a detailed discussion of electronic differential analyzers and their application to eigen value problems see (13) and (14).

Table of the Eigen Values c obtained with a differential analyzer (Cont)

α	αR	c_i	c_r
2	802	.1685	.588
"	853	.1645	.583
"	895	.1595	.571
"	946	.1560	.559
"	1030	.1545	.555
"	1126	.1545	.552

It was not our intention to check specific values of characteristics numbers, but rather to determine whether pipe flow was stable according to the small disturbance theory in the neighborhood of the value of R for which Reynolds found it unstable in his early experiments. The flow was found stable and substantially damped up to a value of αR of 1255 (where R is based on the radius of the pipe). The imaginary part of C decreased steadily for increasing αR but it could not be ascertained whether in the limit of very high αR, c_i approached zero exponentially or reversed signs. Regions of high αR correspond on the analyzer to low input resistances and result in a marginal operation of the machine for which no accuracy can be guaranteed.

The accuracy of the analyzer results is hard to ascertain in so far as the equipment was experimental and unproven. However the consistency of the results and the accuracy of the components

of the analyzer lead us to believe the characteristic numbers to be accurate to within 5% in general.

It would be interesting to verify systematically the analytical predictions of Chapter I either on a differential analyzer or preferably on a digital computor. The solutions by L. H. Thomas (6) on an I.B.M. machine of the plane Poiseuille case, a comparatively harder problem augurs well for the success of such an enterprise.

C O N C L U S I O N S

The validity of the small perturbation theory, applied as it has been to parallel flows has certainly not been fully established. On the one hand the experimental results of Schubauer and Skramstad on the stability of the boundary layer were encouraging. They agreed so closely with C. C. Lin's analytical results that one was tempted to conclude that the postulates of the small disturbance theory did not violate any significant physical facts. Nevertheless many authors have had reservations about the criteria given by the theory. On the one hand its usefulness is restricted by the fact that the growth of a small disturbance to a size which distorts the main flow appreciably is a function of the original amplitude of that disturbance. No information about this can of course be gathered from the analysis. It is clear therefore that the problem is only determined when outside factors (roughness in the walls of the pipe, turbulence in the flow ahead of a boundary layer and so called contamination) have been adequately described.

Yet one might still maintain that if the flow is predicted unstable it will ultimately become so, provided the disturbances are allowed to grow a sufficient amount of time. The lack of precision of concepts such as a "small" disturbance necessarily restricts the usefulness of the theory in so far as it precludes a a real explanation of the mechanism of the onset of turbulence.

The analytical results bearing on the stability of axi-symmetric pipe flow restrict it even further. For it is a well known fact that pipe flow is unstable at sufficiently high Reynolds numbers. Further, Reynolds himself, commenting on his early experiments with pipes remarked that the Reynolds number of transition seemed to be a function of the size of the initial disturbances. This fact was confirmed late and experiments have shown that the Reynolds number at which transition occurs has a maximum, but no definite value. This seems to suggest that the Reynolds number of the main stream in pipes may not necessarily be the only relevant parameter. Yet it is difficult to define another one without presuming a particular mechanism of instability.

On the other hand, a study recently completed by Tatsumi (11) has attempted to lend analytical strength to the argument originally advanced by Prandtl (15) that instability in pipes is probably caused by the instability in the entrance flow which is more akin to boundary layer regime than to a fully developed parabolic profile.

There is no good reason of course for assuming axi symmetric disturbances in a pipe except that the assumption permits an analytic solution. It is possible that in some instances instability results from lateral disturbances such as small transient pressure gradients or the occurrence of small vortices at the boundary. Th. Theodorsen has recently advanced the theory that the onset of

turbulence is due to the increase of vorticity due to the elongation of small transverse vortex filaments carried away by the main stream. While such hypotheses are still very tentative, it seems reasonable to expect only a limited amount of information to be derived from the study of small perturbations alone. In any case more experimental results are needed to increase our knowledge of the birth of unsteadiness in laminar flows.

SUMMARY

The stability of plane Poiseuille flow and of axi-symmetric Poiseuille flow has been analyzed by the small perturbation method.

For axi-symmetric flow in uniform pipes asymptotic solutions were derived by the W.K.B. method. They give explicit algebraic relations between the eigen values c, the wave number α and the Reynolds number R. For large values of αR these relations give approximations to the eigen-values which should be accurate except for very small values of c. The asymptotic solutions indicate that pipe flow is stable for all Reynolds numbers. Use was made of Bessel and related functions to obtain exact relations permitting the conclusion that pipe flow is stable even for small values of the characteristic number c.

Similar asymptotic solutions were derived for plane Poiseuille flow. It was shown that for large values of c, this flow is also stable. Asymptotic expressions for $c(\alpha, R)$ were given. They are similar to those found in axi-symmetric flow. For small values of c, the solution of C. C. Lin is appraised and justified. The areas of agreement and of disagreement between C. L. Pekeris, C. C. Lin and the work presented here are defined and discussed.

It was found that the solutions of the stability problems for plane Poiseuille and pipe flows indicate the existence in both cases of two families of eigen values characterized respect-

ively by:

$$c \to 1 \quad \text{as} \quad \alpha R \to \infty$$

and

$$c \to 0 \quad \text{as} \quad \alpha a \to \infty$$

The number of higher modes was shown to be a function of αR for both families. However while plane Poiseuille flow is unstable for high enough Reynolds numbers, fully developed pipe flow is found stable to axi-symmetric disturbances. This essential difference is found to be due to the dissimilarity of the non-viscous solutions in these two cases. The non-viscous solutions are regular everywhere in pipe flow, while they have a logarithmic singularity in plane Poiseuille flow.

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