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LOCAL FLAT DUALITY OF ABELIAN VARIETIES

by
Michal Bester

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INTRODUCTION.

Let K be a local field (i.e. a field which is complete for the topology defined by a non-trivial discrete valuation) with a perfect residue field k . Assume that the characteristic of K is $p \neq 0$. Let B be an abelian variety over K with the dual abelian variety $\hat{B}$. If k is finite, there is a duality theorem [10], [19], stating that the cup-product pairings

$$H^1(K, B) \times H^{1-1}(K, \hat{B}) \longrightarrow H^2(K, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}$$

induced by the Poincaré divisorial correspondence give a perfect duality of topological groups. Here $H^i(K, -)$ refers to cohomology with respect to the flat topology on $\text{Spec } K$. The theorem fails when k is not finite. However, for k algebraically closed, and B with good reduction, Šafarevič proved that there exists a perfect duality between $H^1(K, \hat{B})$ and $\pi_1(A)$, except possibly on the p -primary components of the groups, see [15]. Here A is the abelian scheme defined over the ring of integers R in K , such that $A \otimes_R K = B$, and $\pi_1(A)$ is the fundamental group of the proalgebraic group associated to A [18]. Also, Vvedenskiĭ computed both groups for certain classes of elliptic curves and showed by explicit computations that they are dual, including p -primary components, [20], [21], [22].

The aim of this thesis is to show, that for abelian varieties with good reduction defined over a field $K = k((t))$, with k perfect of characteristic $p \neq 0$, one has an analogue to Šafarevič's theorem. To state it, let A be an abelian scheme over the ring of inte-

gers R in K with the dual abelian scheme $\hat{A}$. Let $A_K = A \otimes_R K$ be the generic fibre of A , and let $\bar{K}$ be the maximal unramified extension of K , $\bar{A} = A \otimes_R \bar{R}$. Then there exists a non-degenerate pairing

$$H^1(\bar{K}, \hat{A} \otimes_R \bar{K}) \times \pi_1(\bar{A}) \longrightarrow \mathbb{Q}/\mathbb{Z}$$

which is a perfect duality of topological G -modules, G being the galois group of the algebraic closure of k over k . In order to obtain such a pairing we first prove a duality theorem for finite flat commutative group schemes over R which are killed by a power of p . Let N be such a group scheme. Let $R_1 = R/(t^{1+1})$. There exist exact sequences [14]

$$0 \longrightarrow N \otimes_R R_1 \longrightarrow H_1 \longrightarrow H_1' \longrightarrow 0$$

of group schemes over R (all group schemes in the paper are commutative), with H_1 , H_1' smooth, connected with geometrically connected fibres. We define a proalgebraic group, $\mathcal{F}(N)$, over k to be the pro-group (F_1) where $F_1 = \text{Coker}(\pi_1(H_1) \longrightarrow \pi_1(H_1'))$. The assignment $N \longmapsto \mathcal{F}(N)$ is functorial in N and $\mathcal{F}(N)$ does not depend on the particular "pro-resolution" of N ,

$$0 \longrightarrow (N \otimes_R R_1) \longrightarrow (H_1) \longrightarrow (H_1') \longrightarrow 0.$$

We can also regard $\mathcal{F}(N)$ as a pro-sheaf for the flat topology on $\text{Spec } R$. The Cartier duality pairing $N \times N^D \longrightarrow \mu_{p^\infty}$ yields the pairing $\mathcal{F}(N) \times N^D \longrightarrow \mathcal{F}(\mu_{p^\infty})$, which in turn gives the cup-product pairing

$$(*) \quad H_Z^2(R, N^D) \times \mathcal{F}(N) \longrightarrow H_Z^2(R, \mathcal{F}(\mu_{p^\infty})) = \mathbb{Q}_p / \mathbb{Z}_p$$

where $H_Z^2(R, -)$ denotes relative flat cohomology. The existence and non-degeneracy of the pairing is proved using methods developed by Artin and Milne in [2]. To obtain duality for abelian varieties we apply (*) to the p^n -torsion of A and $\hat{A}$ and pass to the limit in the pairings

$$H_Z^2(R, \hat{A}_{p^n}) \times \mathfrak{F}(A_{p^n}) \longrightarrow \mathbb{Z}/p^n\mathbb{Z}.$$

I. DEFINITION OF THE FUNCTOR $\mathcal{F}$.

Let k be the perfect field of characteristic $p \neq 0$ and let $S = \text{Spec } k$. R will denote the ring of formal power series $k[[t]]$ and $X = \text{Spec } R$. For a scheme T , let T_{fl} denote the flat (fppf) site over T . The finite flat site T_{ffl} is the site whose underlying category consists of all schemes finite and flat over T and the coverings are surjective families of finite flat morphisms. All sheaves are sheaves of abelian groups and all group schemes are commutative.

1). The Greenberg construction.

Let $R_i = R/(t^{i+1})$, $i = 0, 1, 2, \dots$, and let $X_i = \text{Spec } R_i$. The map $\alpha_i : X_i \longrightarrow S$ induced by $a \longmapsto a$, $k \longrightarrow R_i$ gives the morphism of sites

$$\alpha_i : X_{i, fl} \longrightarrow S_{fl} ,$$

for any sheaf F on $X_{i, fl}$ and any map $T \longrightarrow S$ locally of finite type,

$$(\alpha_{i*} F)(T) = F(T \times_S X_i) .$$

If F is representable by a group scheme G over X_i then $\alpha_{i*} G$ is representable by a group scheme over S [3; I, 1, 6.6] . Let now G be a group scheme over R and let $G_i = \alpha_{i*}(G \otimes_R R_i)$. It represents the functor

$$\begin{array}{ccc} \underline{M}_k & \longrightarrow & \underline{Ab} \\ A \longmapsto & G(A_i) & , \end{array}$$

where $\underline{M}_k$ denotes the category of k -algebras, $\underline{Ab}$ the category of

abelian groups, and $A_1 = A[t]/(t^{1+1})$.

We define the functor $\hat{\alpha}_* : \underline{S}(X_{f1}) \longrightarrow \text{Pro } \underline{S}(S_{f1})$ from the category of sheaves on X_{f1} to the category of pro-sheaves on S_{f1} by

$$(\hat{\alpha}_* F) = \varprojlim \alpha_{1*}(F|X_1)$$

where " $\varprojlim$ " denotes the inverse limit in the procategory and $F|X_1$ means F restricted to X_1 . Hence any sheaf on X_{f1} defines a pro-sheaf on S . In particular for any group scheme G over R ,

$\hat{\alpha}_* G$ is a pro-group scheme over S which we denote by $\tilde{G}$.

We shall say that $\tilde{G}$ is connected (respectively smooth, etc.) if it is prorepresented by a projective system of connected (respectively smooth, etc.) group schemes.

Let $F = (F_i)$ be a pro-sheaf on S . Let T be a scheme over S , we define the points of F in T ,

$$F(T) = \varprojlim F_i(T).$$

Lemma 1.1. (1). $G \longmapsto \tilde{G}$ is an exact functor on smooth group schemes.

(11). If G is smooth, connected with geometrically connected fibres, then $\tilde{G}$ is geometrically connected and smooth.

Proof : (1). It is enough to show that $R^1 \hat{\alpha}_* G = 0$ for any smooth group scheme G over X . By definition $R^1 \hat{\alpha}_* G$ is the pro-sheaf $(R^1 \alpha_{1*} G_{R_1})$ and as $R^1 \alpha_{1*} G_{R_1} = 0$ for all i [5;11.7], it follows that $R^1 \hat{\alpha}_* G = 0$.

(11). It follows from the description of α_{1*} as a functor that $\alpha_{1*} G$ is smooth if G is. Indeed let A be a k -algebra and let $J \subset A$ be an ideal of square zero. Then the map

$G_1(A) \longrightarrow G_1(A/J)$ is surjective, since the map
 $G(A_1) \longrightarrow G(A_1/JA_1)$ is, because G is smooth.

If G has geometrically connected fibres, then $\alpha_{0*}G = G_0$ is geometrically connected. Assume now, that G_j is geometrically connected for $j < 1$. There is an exact sequence [3; II, 4, 3.5],

$$0 \longrightarrow U_1 \longrightarrow G_1 \longrightarrow G_{1-1} \longrightarrow 0$$

with $U_1 = \underline{\text{Hom}}(\omega_{G_0}, I)$, where I is the vector group associated

to the ideal $(t^1)/(t^{1+1})$ and ω_{G_0} denotes the cotangent space of

G_0 over S [3; II, 4, 3.4]; hence U_1 is a connected unipotent group scheme over S . By induction, G_{1-1} is geometrically connected since the functor $G \longmapsto G/G^0$ is right exact, where G^0 denotes the connected component of G .

Lemma 1.2. Let N be a finite flat group scheme of height 1 (i.e. killed by Frobenius) over R . Then $\tilde{N}$ is connected.

Proof : By [8; 2.1.2], we can write

$$N = \text{Spec} \frac{R[T_1, \dots, T_r]}{(T_1^p, \dots, T_r^p)}, \text{ and since}$$

$$\frac{R[T_1, \dots, T_r]}{(T_1^p, \dots, T_r^p)} \approx \frac{R[T_1]}{(T_1^p)} \otimes_R \dots \otimes_R \frac{R[T_r]}{(T_r^p)}$$

(as R -algebras) it is enough to show that $\alpha_{1*} \text{Spec}(R_1[T]/(T^p))$ is a connected scheme. But it is easy to check this directly, namely (as schemes)

$$\alpha_{1*} \text{Spec} \frac{R_1[T]}{(T^p)} \approx \mathbb{A}_k^{1-i_0} \times \text{Spec} k^{\text{Spec}} \frac{k[T_1, \dots, T_{1_0}]}{(T_1^p, \dots, T_{1_0}^p)}$$

where $i_0 = [\frac{1}{p}]$, see e.g. the proof of the lemma (2.3).

2). Homotopy groups.

Let $\underline{G}_k$ (respectively $\underline{Fe}_k$) be the category of commutative group schemes over k (repectively its subcategory of finite, étale group schemes over k).

Let $\pi_0 : \underline{G}_k \longrightarrow \underline{Fe}_k$ be the functor $G \longmapsto G/G^0$, i.e.

$\pi_0(G)$ is the "largest" quotient of G in $\underline{Fe}_k$. It is known, that π_0 is left adjoint to the inclusion functor $\underline{Fe}_k \subset \underline{G}_k$, hence it is right exact. In the usual manner π_0 extends to the right exact functor (also written π_0)

$$\pi_0 : \text{Pro}(\underline{G}_k) \longrightarrow \text{Pro}(\underline{Fe}_k)$$

which commutes with projective limits. We denote by

π_j , $j = 0, 1, \dots$, the derived functors of π_0 , $\pi_j = L_j \pi_0$. For more detailed discussion see [3; $\bar{V}$, 3]. Let now G be a flat group scheme over R . For any $j \gg 0$ $\pi_j(\tilde{G})$ is a pro-étale group scheme over S , in particular it is a pro-sheaf $\pi_j(\tilde{G}) = \varprojlim \pi_j(G_1)$. Depending on the context we shall also regard $\pi_j(\tilde{G})$ as a sheaf on X_{ffl} (finite flat site) or on S_{ffl} in the following way.

(i) on X_{ffl} . Let R' be a finite flat R -algebra. Then $R' \otimes_R k$ is a finite k -algebra and $(\text{Spec } R')_{\text{red}} = \coprod \text{spec } l_i$, where l_i are finite extensions of k . Let $\pi_j(\tilde{G})$ be the following presheaf

on X_{ffl} :

$$R' \longrightarrow \pi_j(\widetilde{G}_{R'}) (R' \otimes_R k) = \prod \pi_j(\widetilde{G}_{R'}) (1_i) \quad .$$

Lemma 2.1. Let G be a flat group scheme over R . Then the pre-sheaf $\pi_1(\widetilde{G})$ as defined above is in fact a sheaf.

Proof: It is enough to show the lemma for local R -algebras. So let $X' = \text{Spec } R'$ be a finite local flat R -algebra, and let $Y \longrightarrow X'$ be a covering, i.e. a finite surjective flat morphism. Then $Y = \coprod$ local schemes, hence we may assume, that $Y = \text{Spec } R''$, with R'' a local, faithfully flat R' -algebra. Also, we have a factorization

$$\begin{array}{ccc} R' & \xrightarrow{\quad} & R'' \\ & \searrow & \nearrow \\ & R''_0 & \end{array}$$

with R''_0/R' unramified (so étale) and R''/R''_0 totally ramified. We treat both cases separately.

(a) R''/R' unramified. Then it is proved in [16;2.3, prop.4], that with the above notation,

$$\pi_1(\widetilde{G}_Y)^\Gamma = \pi_1(\widetilde{G}_{X'})$$

where Γ is the galois group of R''/R' , so the lemma follows.

(b) R''/R' totally ramified. Here the assertion follows from the

left exactness of π_1 and $\hat{\alpha}_*$, and from the faithfully flat descent [3; I, 2, 2.7].

(ii) on S_{ffl} . Let $\alpha : X \longrightarrow S$ be the map induced by $k \longrightarrow R$. Define $\pi_j(\tilde{G})$ on S_{ffl} as $\alpha_* \pi_j(\tilde{G})$.

In the following the groups $H^1(X, \pi_j(\tilde{G}))$ should be understood as a cohomology for finite flat site of the sheaf $\pi_j(\tilde{G})$.

Remark 2.2. Let F be a sheaf on X_{ffl} , and let

$u : X_{\text{fl}} \longrightarrow X_{\text{ffl}}$ be a canonical morphism. Then $u_* u^* F \approx F$ and moreover $H^1(X, u^* F) \approx H^1(X, F)$, so we can compute the cohomology of $\pi_j(\tilde{G})$ on the full flat (fppf) site. The same is true of direct images.

Lemma 2.3. $\pi_1(\mu_p^{\sim n}) = 0$.

Proof : It is enough to show that $\pi_1(\mu_p^{\sim}) = 0$. Indeed consider

the exact sequence, $0 \longrightarrow \mu_p \longrightarrow \mu_p^n \longrightarrow \mu_p^{n-1} \longrightarrow 0$

and assume that $\pi_1(\mu_p^{\sim m}) = 0$ for $m < n$. Applying $\pi_1(\sim)$ yields

an exact sequence $0 \longrightarrow \pi_1(\mu_p^{\sim}) \longrightarrow \pi_1(\mu_p^{\sim n}) \longrightarrow \pi_1(\mu_p^{\sim n-1})$,

and since $\pi_1(\mu_p^{\sim}) = \pi_1(\mu_p^{\sim n-1}) = 0$ by assumption, hence

$\pi_1(\mu_p^{\sim n}) = 0$.

Let U_1 be the product of $i-1_0$ copies of $\mathbb{G}_{a,S}$ where $1_0 = [\frac{1}{p}]$.

Let $N = \mu_p$. Define a map $\varphi_1 : U_1 \longrightarrow N_1$ to be a truncated exponential,

$$\varphi_1(a_{i_0+1}, \dots, a_i) = \sum_{j=0}^{p-1} \frac{1}{j!} \left(\sum_{k=i_0+1}^i a_k t^k \right)^j.$$

One checks easily that φ_1 is a homomorphism. Let $M_1 = \text{Coker } \varphi_1$:

$$0 \longrightarrow U_1 \xrightarrow{i} N_1 \longrightarrow M_1 \longrightarrow 0.$$

Consider the commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & U_1 & \longrightarrow & N_1 & \longrightarrow & M_1 \longrightarrow 0 \\ & & \uparrow 0 & & \uparrow \eta_1 & & \uparrow \\ 0 & \longrightarrow & U_{1p} & \longrightarrow & N_{1p} & \longrightarrow & M_{1p} \longrightarrow 0 \end{array}$$

where η_1 is the map induced by $R_{1p} \longrightarrow R_1$. It is clear that $U_{1p} \longrightarrow U_1$ is the zero map. For each $i > 0$, we have an exact sequence (cf. (1.3))

$$0 \longrightarrow \underline{\text{Hom}}(\omega_{N_0}, I) \longrightarrow N_1 \longrightarrow N_{1-1},$$

(see (1.3)). It is easy to see - by induction on i - that

$\dim U_i = \dim N_i = i - i_0$ (if $p \mid i$ then $\dim N_i = \dim N_{i-1}$), hence M_1 is a finite group scheme over k and consequently $\pi_1(M_1) = 0$.

Applying π_1 to the diagram yields therefore,

$$\begin{array}{ccc} \pi_1(U_1) & \xrightarrow{\cong} & \pi_1(N_1) \\ \uparrow & & \uparrow \pi_1(\eta_1) \\ \pi_1(U_{1p}) & \xrightarrow{\cong} & \pi_1(N_{1p}) \end{array}$$

It follows that $\pi_1(\eta_1) = 0$ and $\pi_1(\tilde{\mu}_p) = 0$.

3). The functor $\mathcal{F}$.

Definition. Let D be a finite flat group scheme over the base

scheme T . We call an exact sequence of group schemes over T

$$0 \longrightarrow D \longrightarrow G \longrightarrow H \longrightarrow 0$$

a smooth (respectively smooth, connected) resolution of D if G, H are smooth (respectively smooth connected with geometrically connected fibres) group schemes of finite type over T .

Lemma 3.1. Let N be a finite flat group scheme over R .

(i) There exists a smooth resolution of N .

(ii) There exists a smooth connected formal resolution of N , i.e. there are smooth connected resolutions

$$0 \longrightarrow N_{R_1} \longrightarrow G_1 \longrightarrow H_1 \longrightarrow 0 \quad \text{over } R_1 \text{ such that}$$

$$0 \longrightarrow N_{R_{1-1}} \longrightarrow G_{1-1} \longrightarrow H_{1-1} \longrightarrow 0 = (0 \longrightarrow N_{R_1} \longrightarrow G_1 \longrightarrow H_1 \longrightarrow 0) \otimes_{R_1} R_{1-1},$$

for all i .

(iii) If N is connected then it has a smooth connected resolution.

Proof: (i) See [7].

(ii) See [14] where it is shown that N can be resolved by formal abelian schemes.

(iii) It follows from (i) that there exists an exact sequence $0 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 0$ with G, H smooth, finite type over R . Since N is connected we can replace G by the union of connected components of the fibres containing the image of the unit section $X \longrightarrow N$, call it G' ; it is smooth and connected so we get the required exact sequence

$$0 \longrightarrow N \longrightarrow G' \longrightarrow H' \longrightarrow 0 \quad \text{with } H' = G'/N. \quad G' \text{ and } H' \text{ have then geometrically connected fibres (cf [4, VI}_B, 3.12]).$$

Let N be a finite flat group scheme over R . For each i choose a smooth connected formal resolution of N (e.g. by formal abelian schemes):

$$0 \longrightarrow N \longrightarrow (A_i) \xrightarrow{a=(a_i)} (B_i) \longrightarrow 0.$$

Let $\mathcal{F}_a(N) = (\text{Coker}(\pi_1(\alpha_{1*}A_1) \longrightarrow \pi_1(\alpha_{1*}B_1))) = (F_{1,a})$ be a pro-sheaf as above, where $F_{1,a}$ is the sheaf associated to the pre-sheaf cokernel of

$$\pi_1(\alpha_{1*}A_1) \xrightarrow{a_1} \pi_1(\alpha_{1*}B_1).$$

In particular $\mathcal{F}_a(N)$ is a proalgebraic group over k .

Lemma 3.2. $\mathcal{F}_a(N)$ depends only on N but not on the particular resolution chosen.

Proof : The following is a slight modification of a similar proof from [12]. It is enough to show that each $F_{1,a}$ is independent of the resolution. To simplify notation, let $M = N_{R_1}$, and let

$$F_a = F_{1,a}. \text{ Let } 0 \longrightarrow M \longrightarrow A \xrightarrow{a} B \longrightarrow 0 \text{ and}$$

$0 \longrightarrow M \longrightarrow A' \xrightarrow{b} B' \longrightarrow 0$ be two smooth connected resolutions of M . Exactly as in [12;3.2], one constructs the following exact commutative diagram,

$$\begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \downarrow & & \downarrow & \\ & 0 & \longrightarrow & A' & \longrightarrow & A' & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & M & \longrightarrow & A \oplus A' & \xrightarrow{c} & A'' \longrightarrow 0 \\ & \approx \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & M & \longrightarrow & A & \xrightarrow{a} & B \longrightarrow 0 \\ & & & \downarrow & & \downarrow & \\ & & & 0 & & 0 & \end{array}$$

with A'' smooth, connected. Applying $\pi_1 \circ \alpha_{1*}$ yields,

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 & \pi_1(A_1') & \xrightarrow{\approx} & \pi_1(A_1') & \longrightarrow & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \pi_1(A_1 \oplus A_1') & \longrightarrow & \pi_1(A_1'') & \longrightarrow & F_c & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & \pi_1(A_1) & \longrightarrow & \pi_1(B_1) & \longrightarrow & F_a & \longrightarrow 0 \\
 & \downarrow & & \downarrow & & & \\
 & 0 & & 0 & & &
 \end{array}$$

where $A_1 = \alpha_{1*}A$, etc. The first column is exact since both π_1 and α_{1*} are additive functors, the second is exact, since α_{1*} is exact on smooth group schemes (1.1) and $\pi_0(A_1') = 0$ (1.1). The construction is symmetric in a and b , so

$$F_a \approx F_c \approx F_b, \text{ hence}$$

$$\mathcal{F}_a(N) \approx \mathcal{F}_b(N).$$

Definition 3.3. Let N be a finite flat group scheme over R . We define

$$\mathcal{F}(N) = \mathcal{F}_a(N) = (F_{1,a})$$

where $0 \longrightarrow N \longrightarrow (A_1) \xrightarrow{(a_1)} (B_1) \longrightarrow 0$ is some smooth connected formal resolution of N .

Remark 3.4. If N is connected then we can compute $\mathcal{F}(N)$ from some smooth connected resolution of N over R ,

$$0 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 0, \text{ namely,}$$

$\mathcal{F}(N) = \text{Coker} (\pi_1(\tilde{G}) \longrightarrow \pi_1(\tilde{H}))$, the proof of (3.2) carries over with A, B replaced by G, H and M replaced by N .

In general however, we do not have a good resolution of N so we have to replace it by a formal one.

Lemma 3.5. $\mathcal{F}$ is a functor.

Proof : It is clear that $\mathcal{F}(1_N) = 1_{\mathcal{F}(N)}$. Let $f : N \longrightarrow N'$

be a morphism of finite flat group schemes over R . Let

$$0 \longrightarrow N \longrightarrow (A_1) \longrightarrow (B_1) \longrightarrow 0 \quad \text{and}$$

$$0 \longrightarrow N' \longrightarrow (A'_1) \longrightarrow (B'_1) \longrightarrow 0 \quad \text{be smooth connected formal re-}$$

solutions of N and N' . We shall show that for all i , an in-

duced map $f_{R_1} : N_{R_1} \longrightarrow N'_{R_1}$ gives rise to the map $F_1 \longrightarrow F'_1$,

where $\mathcal{F}(N) = (F_1)$ and $\mathcal{F}(N') = (F'_1)$ (see 3.3).

Let A''_1 be the quotient $(A_1 \oplus A'_1)/N_{R_1}$ where the map $N_{R_1} \longrightarrow A'_1$

is the composition $N_{R_1} \xrightarrow{f} N'_{R_1} \longrightarrow A'_1$. A''_1 exists [3;III,2,3.2]

since N_{R_1} is finite and flat and as A_1 and A'_1 are smooth, so is

A''_1 [3;III,3,2.7] ; hence we obtain a diagram $(B''_1 = A''_1/N'_{R_1})$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & N_{R_1} & \longrightarrow & A_1 & \longrightarrow & B_1 \longrightarrow 0 \\ & & \downarrow f_{R_1} & & \downarrow & & \downarrow \\ 0 & \longrightarrow & N'_{R_1} & \longrightarrow & A''_1 & \longrightarrow & B''_1 \longrightarrow 0 \end{array} .$$

It follows that there is a map $F_1 \longrightarrow F'_1$, hence a map

$$\mathcal{F}(N) \longrightarrow \mathcal{F}(N') \quad .$$

Let now $K^\bullet$ be the complex

$$0 \longrightarrow \pi_1(\tilde{A}) \longrightarrow \pi_1(\tilde{B}) \longrightarrow 0 \quad , \text{ where } \tilde{A} = (\alpha_{1*}A_1) \quad ,$$

$$\tilde{B} = (\alpha_{1*}B_1) \quad \text{and let } L^\bullet \text{ be the complex}$$

$$0 \longrightarrow \pi_1(\tilde{A}') \longrightarrow \pi_1(\tilde{B}') \longrightarrow 0 \quad \text{where } \tilde{A}' \text{ and } \tilde{B}' \text{ are respectively } (\alpha_{1*}A'_1) \text{ and } (\alpha_{1*}B'_1) \quad .$$

By the above there exists another smooth formal resolution $0 \longrightarrow N' \longrightarrow (A''_1) \longrightarrow (B''_1) \longrightarrow 0$

which gives a complex L'' defined in the same way as $K^\bullet$ and $L^\bullet$.

Also, it follows from the above construction that $L^\bullet$ is quasi-isomorphic to L'' . Therefore there are maps

$$H^1(K^\bullet) \longrightarrow H^1(L'') \xleftarrow{\approx} H^1(L^\bullet)$$

and since $H^1(K^\bullet) = \mathcal{F}(N)$, $H^1(L'') = \mathcal{F}(N')$, it follows that $\mathcal{F}$ is a functor.

Lemma 3.6. Let N be a finite flat group scheme over R . Then

$$R^q \hat{\alpha}_{*} N = 0 \quad \text{for } q > 1 \quad .$$

Proof : $R^q \hat{\alpha}_{*} N$ is the pro-sheaf $(R^q \alpha_{1*} N_{R_1})$, and since we have

$$\text{an exact sequence } 0 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 0 \quad \text{with } G, H$$

$$\text{smooth} \quad \text{it follows that } R^q \alpha_{1*} N_{R_1} = 0 \quad \text{for } q > 1, \text{ since}$$

$$R^q \alpha_{1*} G_{R_1} = R^q \alpha_{1*} H_{R_1} = 0 \quad \text{for } q > 0 \quad [5; 11.7] \quad .$$

Lemma 3.7. Let N be a finite flat group scheme over R . There is an exact sequence,

$$0 \longrightarrow \pi_0(\tilde{N}) \longrightarrow \mathcal{F}(N) \longrightarrow \pi_1(R^1 \hat{\alpha}_{*} N) \longrightarrow 0$$

of pro-sheaves on X_{ffl} .

Proof : First observe that $R^1 \hat{\alpha}_* N$ has the natural structure of a pro-algebraic group, since $R^1 \hat{\alpha}_* N$ is the cokernel of the map

$\tilde{G} \longrightarrow \tilde{H}$ induced from some smooth resolution

$0 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 0$ of N . $\pi_1(R^1 \hat{\alpha}_* N)$ is the following sheaf on X_{ffl} . Let R' be a finite flat R -algebra, then $N_{R'} \otimes_{R'} R_1$ is a group scheme over R_1 and $\pi_1(R^1 \hat{\alpha}_* N)$ is the sheaf associated to the presheaf

$$R' \longrightarrow \varprojlim \pi_1(R^1 \alpha_{1*}(N_{R'} \otimes_{R'} R_1))(R' \otimes_{R'} R^k) ,$$

$$\begin{array}{ccc} \text{Spec } R_1 & \longleftarrow & \text{Spec } R' \otimes_{R'} R_1 \\ \downarrow & & \downarrow \\ \text{Spec } R & \longleftarrow & \text{Spec } R' \end{array} .$$

The assertion of the lemma is equivalent to the existence of the projective system of exact sequences

$$(*) \quad 0 \longrightarrow \pi_0(\alpha_{1*} N_{R_1}) \longrightarrow F_1 \longrightarrow \pi_1(R^1 \alpha_{1*} N_{R_1}) \longrightarrow 0 .$$

Choose a smooth, connected, formal resolution of N ,

$$0 \longrightarrow N \longrightarrow (A_1) \longrightarrow (B_1) \longrightarrow 0 . \text{ Let us fix } 1$$

and to simplify notation denote N_{R_1} by M . Applying α_{1*} to

$$0 \longrightarrow M \longrightarrow A_1 \longrightarrow B_1 \longrightarrow 0 , \text{ yields the exact sequence}$$

$$0 \longrightarrow \alpha_{1*} M \longrightarrow \alpha_{1*} A_1 \longrightarrow \alpha_{1*} B_1 \longrightarrow R^1 \alpha_{1*} M \longrightarrow 0 .$$

Upon factoring the cokernel Q of $\alpha_{1*} M \longrightarrow \alpha_{1*} A_1$, we obtain two exact sequences,

$$\pi_1(\alpha_{1*} M) \longrightarrow \pi_1(\alpha_{1*} A_1) \longrightarrow \pi_1(Q) \longrightarrow \pi_0(\alpha_{1*} M) \longrightarrow 0$$

and

$$0 \longrightarrow \pi_1(Q) \longrightarrow \pi_1(\alpha_{1*} B_1) \longrightarrow \pi_1(R^1 \alpha_{1*} M) \longrightarrow 0 .$$

Exactness on the right of the second sequence follows from the fact that $\pi_0(Q) = 0$; Q is connected since $\alpha_{1*}A_1$ maps onto it and by (1.1.) $\alpha_{1*}A_1$ is connected. Now from the Ker-Coker sequence of the triangle

$$\begin{array}{ccc} \pi_1(\alpha_{1*}A_1) & \longrightarrow & \pi_1(\alpha_{1*}B_1) \\ & \searrow & \nearrow \\ & \pi_1(Q) & \end{array}$$

we deduce (*), hence the lemma.

Remark 3.8. It follows from the above, that if N is finite étale, then $\mathcal{F}(N) \approx \pi_0(\tilde{N})$, and if N is infinitesimal, then $\mathcal{F}(N) \approx \pi_1(R^1\hat{\alpha}_*N)$, since by (1.2.), if N is infinitesimal, then $\tilde{N}$ is connected.

Lemma 3.9. $\mathcal{F}$ is a right exact functor.

Proof : Let $0 \longrightarrow N \longrightarrow N' \longrightarrow N'' \longrightarrow 0$ be an exact sequence of finite flat group schemes over R , and let

$$0 \longrightarrow N \longrightarrow (A_1) \longrightarrow (B_1) \longrightarrow 0$$

(respectively $0 \longrightarrow N' \longrightarrow (A'_1) \longrightarrow (B'_1) \longrightarrow 0$) be a smooth connected resolution of N (respectively of N'). Let us fix i .

We construct an exact commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & N_{R_1} & \longrightarrow & A_1 & \longrightarrow & B_1 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & N'_{R_1} & \longrightarrow & C_1 & \longrightarrow & D_1 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & N''_{R_1} & \longrightarrow & A''_1 & \longrightarrow & B''_1 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

with all terms in the second and third columns being smooth connected group schemes of finite type over R with geometrically connected fibres. To simplify notation let $M = N_{R_1}$, $M' = N'_{R_1}$, $M'' = N''_{R_1}$ and let $A = A_1$, $A' = A'_1$ and $B' = B'_1$. It is enough to construct the following diagram

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M & \longrightarrow & A \\
 & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M' & \longrightarrow & C \\
 & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M'' & \longrightarrow & A'' \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
 \end{array}
 \quad (*)$$

with A, C, A'' smooth connected The third column will then exist and will have the required properties (same references as in (3.2)), it will be exact by the serpent lemma.

Let $C = (A \oplus A')/N$. By arguments of (3.2) it exists, is smooth etc. Since C is a push-out $(M \longrightarrow A' = M \longrightarrow M' \longrightarrow A')$,

$$\begin{array}{ccc}
 M & \longrightarrow & A \\
 \downarrow a & & \downarrow b \\
 A' & \longrightarrow & C
 \end{array}
 ,$$

$\text{Coker } a = \text{Coker } b$, and consequently we obtain (*) with

$A'' = A'/M$. Therefore from an exact sequence

$0 \longrightarrow N \longrightarrow N' \longrightarrow N'' \longrightarrow 0$ we constructed a short exact sequence of smooth, connected formal resolutions, hence the exact se-

quence :

$$0 \longrightarrow \pi_1(\tilde{N}) \longrightarrow \pi_1(\tilde{N}') \longrightarrow \pi_1(\tilde{N}'') \longrightarrow \mathcal{F}(N) \longrightarrow \mathcal{F}(N') \longrightarrow \mathcal{F}(N'') \longrightarrow 0$$

which proves the lemma.

Remark 3.10. It will be shown later (II,6) that $\mathcal{F}$ is left exact as well.

II. DUALITY THEOREMS.

1). The pairing.

Let N be a finite flat group scheme over X . The Cartier duality pairing,

$$N \times N^D \longrightarrow \mu_{p^\infty}$$

induces the pairing

$$(1.1) \quad \mathcal{F}(N) \times N^D \longrightarrow \mathcal{F}(\mu_{p^\infty})$$

as follows. Upon writing $N^D = \underline{\text{Hom}}(N, \mu_{p^\infty})$ one sees, in view of the functoriality of $\mathcal{F}$, that any map $N \longrightarrow \mu_{p^\infty}$ induces a map $\mathcal{F}(N) \longrightarrow \mathcal{F}(\mu_{p^\infty})$; whence (1.1). Also interpreting N as N^{DD} yields in the same way the pairing

$$(1.2) \quad \mathcal{F}(N^D) \times N \longrightarrow \mathcal{F}(\mu_{p^\infty}).$$

2). Relative cohomology.

Let T be a scheme, U an open subscheme of T and Z a complementary closed subscheme, $Z = T - U$. Denote by i and j the corresponding immersions

$$Z \xrightarrow{i} T \xleftarrow{j} U,$$

so that T is a disjoint union of $i(Z)$ and $j(U)$. Define a functor $\underline{S}(T_{f1}) \longrightarrow \underline{\text{Ab}}$, $F \longmapsto H_Z^0(T, F) = \text{Ker}(F(T) \longrightarrow F(U))$. $H_Z^0(T, -)$ is left exact, denote its i -th right derived by $H_Z^i(T, -)$.

Lemma 2.1. Let F be any sheaf on T_{f1} . There is an exact sequence;

$$(2.2) \quad 0 \longrightarrow H_Z^0(T, F) \longrightarrow F(T) \longrightarrow F(U) \longrightarrow \dots \longrightarrow H^1(T, F) \longrightarrow \\ H^1(U, F|_U) \longrightarrow H_Z^{1+1}(T, F) \longrightarrow \dots ,$$

where $F|_U$ denotes F restricted to U , and cohomology group $H^1(T, F)$ (respectively $H^1(U, F|_U)$) denotes the flat (fppf) cohomology groups.

Proof : See Appendix. We just mention, that

$H_Z^1(T, F) \approx \text{Ext}^1(\mathbb{Z}_Z, F)$, where $\mathbb{Z}_Z = i_* i^* \mathbb{Z}$, $\mathbb{Z}$ denoting the constant sheaf $\mathbb{Z}$.

We apply the definition to our situation i.e. $T = X$, $U = \text{Spec } K$, $Z = S = \text{Spec } k$ and F a group scheme on X ; here K is the field of fractions of R .

Lemma 2.3. $H_Z^2(X, -)$ is a left exact functor $\underline{N}_R \longrightarrow \underline{\text{Ab}}$ where $\underline{N}_R$ is the category of finite flat group schemes over R .

Proof : This is an easy consequence of (2.1); for any $N \in \text{Ob}(\underline{N}_R)$, the map $N(R) \longrightarrow N(K)$ is an isomorphism, and the map $H^1(X, N) \longrightarrow H^1(U, N|_U)$ is injective since any torsor for N which has a point in K already has a point in R , as R is integrally closed in K . Therefore $H_Z^0(X, N) = H_Z^1(X, N) = 0$.

Lemma 2.4. Suppose that G is a smooth group scheme over X , and assume that k is algebraically closed. Then

$$H_Z^1(X, G) = G(K)/G(R) .$$

Proof: This also follows from (2.1) and [5; 11.7] :

$$H^1(X, G) \approx H^1(k, G) = 0 \quad \text{for } i > 0 .$$

Definition 2.5. Let F be a sheaf on X . We define a sheaf on $S_{\text{ét}}$, $\underline{H}_Z^j(X, F)$ as follows. Let k' be a finite extension of k . Let R' be the unramified extension of R with the residue field k' , $Z' = \text{Spec } k'$, and let $X' = \text{Spec } R'$. Then we define a sheaf $\underline{H}_Z^j(X, F)$ to be a sheaf associated to the presheaf

$$k' \longmapsto H_Z^j(X', F|_{X'}) .$$

Lemma 2.6. (i) Let $\underline{H}_Z^j(X, F)_{\bar{k}}$ denote the stalk of $\underline{H}_Z^j(X, F)$.

Then

$$\underline{H}_Z^j(X, F)_{\bar{k}} = H_{\bar{Z}}^j(\bar{X}, F) ;$$

hence for k algebraically closed, $\underline{H}_Z^j(X, F)(k) = H_Z^j(X, F)$.

(ii) $\underline{H}_Z^2(X, -)$ is left exact on finite flat group schemes over R .

Proof : (i) Since cohomology commutes with projective limits of schemes,

$$\underline{H}_Z^j(X, F)_{\bar{k}} = \varinjlim_{k' \supset k} H_Z^j(X', F) = H_{\bar{Z}}^j(\bar{X}, F) , \text{ where the}$$

limit is taken over all finite algebraic extensions of k contained in $\bar{k}$.

(ii) Follows from (i) and (2.3).

Lemma 2.7. Let G be a vector group over R . Then $H_Z^j(X, G) = 0$ for $j \gg 2$.

Proof : It follows from (2.1) that $H_Z^j(X, G) = H^{j-1}(U, G_U)$, $j \gg 2$, for any smooth group scheme G over X . Since G is

smooth $H^j(U, G_U) \approx H_{\text{ét}}^j(K, G_K)$, [3; III, 5, 6.10], and as G is a vector group,

$$H_{\text{ét}}^j(K, G_K) = 0 \quad \text{for } j \gg 1 .$$

3). Statement of the duality theorem for finite flat group schemes over R .

Theorem 3.1. Let N be a finite flat group scheme over R . Then

$$H_Z^2(X, \mathcal{F}(\mu_{p^\infty})) \approx \mathbb{Q}_p / \mathbb{Z}_p ,$$

and the cup product

$$H_Z^2(X, N^D) \times \mathcal{F}(N) \longrightarrow \mathbb{Q}_p / \mathbb{Z}_p$$

is a perfect duality on S_{f1} , i.e.

$$\mathcal{F}(N) \approx \underline{\text{Hom}}(H_Z^2(X, N^D) , \mathbb{Q}_p / \mathbb{Z}_p) ,$$

where this isomorphism should be understood as the isomorphism between the pro-sheaf $\mathcal{F}(N)$ and the pro-sheaf $\varprojlim \underline{\text{Hom}}(F, \mathbb{Q}_p / \mathbb{Z}_p)$

where F runs through the finite subsheaves of $H_Z^2(X, N^D)$.

Remark 3.2. Another possible interpretation of (3.1).

(a) $H_Z^2(X, N^D)$ is a torsion sheaf, so it is in a natural way an ind-sheaf and $H_Z^2(X, N^D) \approx \underline{\text{Hom}}(\mathcal{F}(N) , \mathbb{Q}_p / \mathbb{Z}_p)$ where both ind-sheaves may be regarded as sheaves on S by passing to the limit.

(b) The $G(\bar{k}/k)$ -modules $\mathcal{F}(N)_{\bar{k}}$ (pro-finite constant group over $\bar{k}$) and $H_Z^2(X, N^D)_{\bar{k}}$ (discrete constant group over $\bar{k}$) are topologically dual, $G(\bar{k}/k)$ being the galois group of $\bar{k}/k$.

4). Basic Lemmas.

Through this section N will denote a finite flat group scheme of height 1 over X , and N_K its generic fibre $N_K = N \otimes_R K$. We quote below, without proofs, several facts from [2]. Let ω_N be the cotangent space of N over X ; it is a locally free $\mathcal{O}_X$ -module of finite type [8; II, 2.1.4], and as such it may be regarded as a vector group over X . As usual $\hat{\omega}_N$ will denote the Lie algebra of N . We shall often write $H_Z^j(F)$ instead of $H_Z^j(X, F)$.

Lemma 4.1. [2; 1.1]. There is an exact sequence of finite type group schemes on X ,

$$(4.2) \quad 0 \longrightarrow N^D \longrightarrow \omega_N \xrightarrow{\varphi} \omega_N^{(p)} \longrightarrow 0$$

The map $N^D \longrightarrow \omega_N$ is a canonical inclusion as explained in [2; §1]; $\varphi = \varphi_0 - \varphi_1$, where φ_0 is induced by the Verschiebung morphism $V_N : N^{(p)} \longrightarrow N$, and φ_1 is the relative Frobenius for ω_N over X , $\varphi_1 : \omega_N \longrightarrow \omega_N^{(p)}$.

To state the following lemmas we need some notation. Let $X' = X^{(1/p)}$, let $F : X' \longrightarrow X$ and $W : X \longrightarrow X'$ be such morphisms that $FW = F_{\text{abs}}$ on X and $WF = F_{\text{abs}}$ on X' .

Lemma 4.3. [2; 2.4]. There is an exact sequence of sheaves on $X_{\text{ét}}$,

$$(4.4) \quad 0 \longrightarrow R^1 f_* N \xrightarrow{\partial} \hat{\omega}_N \otimes_{\mathcal{O}_X} \Omega_X^1 \xrightarrow{\psi} \hat{\omega}_N \otimes_{\mathcal{O}_X} \Omega_X^1 \longrightarrow 0.$$

Here $f : X_{f1} \longrightarrow X_{\text{ét}}$ is the morphism of sites induced by the

identity map on X , ∂ is explained in [2;2.7] and

$\psi = \psi_0 - \psi_1$, $\psi_0 = 1 \otimes C$, C being an $\mathcal{O}_X$ -linear Cartier operator; $\psi_1 = *^{(p)} \otimes W^*$, $*^{(p)}$ denotes the p -th power map on the Lie algebra $\hat{\omega}_N$ of N and W^* is the map induced by W .

Ω_X^1 stands for $\Omega_{X'/S}^1$ and Ω_X^1 for $\Omega_{X/S}^1$.

Lemma 4.5. [2;2.2]. There is an exact sequence of group schemes on X ,

$$(4.6) \quad 0 \longrightarrow N \longrightarrow F_* N_X \longrightarrow \hat{\omega}_N \otimes \Omega_X^1 \xrightarrow{\psi} \hat{\omega}_N \otimes \Omega_X^1 \longrightarrow 0$$

(we identify locally free sheaves with the corresponding vector groups over X).

Lemma 4.7. [2;2.6]. Let $B = \text{Ker}(\psi : \hat{\omega}_N \otimes \Omega_X^1 \longrightarrow \hat{\omega}_N \otimes \Omega_X^1)$ in (4.6). Then B is a smooth group scheme of finite type over X and $f_* B = R^1 f_* N$, where $f : X_{f1} \longrightarrow X'_{\text{ét}}$ is the morphism of sites induced by the identity map on X and $X'_{\text{ét}}$ is the small étale site.

Lemma 4.8. [2;3.4]. Let $V^*(N)$ be the complex

$$(\omega_N \xrightarrow{\varphi} \omega_N^{(p)}) \quad , \text{ and}$$

$$U^*(N) = (\hat{\omega}_N \otimes \Omega_X^1 \xrightarrow{\psi} \hat{\omega}_N \otimes \Omega_X^1) \quad , \text{ so } U^*(\mu_p) = (\Omega_X^1 \xrightarrow{C-W^*} \Omega_X^1) .$$

The system of pairings,

$$\langle a, g \otimes \eta \rangle_{00} = g(a)\eta : \omega_N \times \hat{\omega}_N \otimes \Omega_X^1 \longrightarrow \Omega_X^1 \quad ,$$

$$\langle a, g \otimes \eta \rangle_{01} = g(a)\eta : \omega_N \times \hat{\omega}_N \otimes \Omega_X^1 \longrightarrow \Omega_X^1$$

$$\langle a, g \otimes \eta \rangle_{10} = g(W(a))\eta^W : \omega_N^{(p)} \times \hat{\omega}_N \otimes \Omega_X^1 \longrightarrow \Omega_X^1 \quad ,$$

defines a pairing of complexes :

$$V^*(N) \times U^*(N) \longrightarrow U^*(\mu_p) \quad ,$$

i.e.

$$(4.9) \quad (C-W^*) \langle a, g \otimes \eta \rangle_{00} = \langle \varphi(a), g \otimes \eta \rangle_{10} + \langle a, \psi(g \otimes \eta) \rangle_{01} \quad .$$

To explain the notation observe that there are maps

$$\omega_N \xrightarrow{F} \omega_N^{(p)} \xrightarrow{W} \omega_N \quad , \text{ so if } a \in \omega_N^{(p)}(T) \quad , \quad W(a) \in \omega_N(T) \quad .$$

$$\text{Also } X' \xrightarrow{F} X \xrightarrow{W} X' \quad \text{induce maps} \quad \Omega_X^1 \xrightarrow{W^*} \Omega_X^1 \xrightarrow{F^*} \Omega_X^1,$$

$$\text{which we write also as } F^* = (\eta \longrightarrow \eta^F) \quad , \quad W^* = (\eta \longrightarrow \eta^W) \quad .$$

The maps φ and ψ are the maps of (4.1) and (4.3) respectively.

Lemma 4.10. [9;2.1] For any vector group V of finite type over S there is a canonical isomorphism,

$$r : \underline{\text{Ext}}^1(V, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \underline{\text{Hom}}_{\mathcal{O}_S}(V, \mathcal{O}_S)$$

and

$$\underline{\text{Ext}}^1(V, \mathbb{Z}/p\mathbb{Z}) = 0 \quad \text{for } i \neq 1 \quad .$$

Lemma 4.11. [2;2.4] $R^q f_* N = 0$ for $q \neq 1$, where

$f : X_{fl} \longrightarrow X'_{\text{ét}}$ is the morphism of sites induced by the identity map on X , and $X'_{\text{ét}}$ is the small étale site.

5). Duality for finite flat group schemes of height 1 .

Notation. In this section N will denote a finite flat group scheme over R of height 1 , i.e. $F_N = 0$, N_K its generic fibre, and $U = \text{Spec } K$ an open point of X , $U' = U^{(1/p)}$.

Let $\text{res} = \text{res}_t : \Omega_{U/S}^1 \longrightarrow k$ be the residue map,

$$\text{res} \left(\left(\sum_{j=-\infty}^{\infty} a_j t^j \right) dt \right) = a_{-1}$$

where t is an uniformizing parameter of K .

It follows from [17; II, 11, 5], that the residue map is independent of the choice of the uniformizing parameter of K .

Consider now the pairing,

$$(5.1) \quad \begin{array}{ccc} \text{res} \langle , \rangle_{10} : \omega_{N_K}^{(p)} & \times & \hat{\omega}_{N_K} \otimes \Omega_{K'}^1 \\ \uparrow \varphi & & \downarrow -\psi \\ \text{res} \langle , \rangle_{01} : \omega_{N_K} & \times & \hat{\omega}_{N_K} \otimes \Omega_K^1 \end{array} \quad \begin{array}{c} \nearrow \\ \searrow \end{array} \quad \Omega_K^1 \xrightarrow{\text{res}} k$$

where Ω_K^1 stands for $\Omega_{U/S}^1$ and $\Omega_{K'}^1$ for $\Omega_{U'/S}^1$.

Lemma 5.2. In the pairings (5.1) $\hat{\omega}_N \otimes \Omega_X^1$ is an exact annihilator of $\omega_N^{(p)}$ and $\hat{\omega}_N \otimes \Omega_X^1$ is an exact annihilator of ω_N .

Proof : The fact that the corresponding groups annihilate each other is clear.

a) Suppose that $\text{res} \langle a, g \otimes \eta \rangle_{01} = 0$ for all $g \otimes \eta \in \omega_N \otimes \Omega_X^1$, then $a \in \omega_N$.

b) Suppose that $\text{res} \langle a, g \otimes \eta \rangle_{01} = 0$ for all $a \in \omega_N$.

Let $v_1, \dots, v_n$ be a basis of ω_N over R , and $v'_1, \dots, v'_n$ a dual basis, so $v'_1 \otimes 1, \dots, v'_n \otimes 1$ is the basis of $\hat{\omega}_{N_K}$ over K .

These exist since ω_N is a free R -module.

Assume that $g = \sum_{i=1}^n g_i v_i' \otimes 1$, $\eta = cdt$. Let $v : K \longrightarrow \mathbb{Z}$ be

the standard valuation on K , let $m_1 = v(g_1)$, $c' = v(c)$.

Assume for simplicity that $v(g_1) = m_1 = \min \{v(g_i)\}$. If $c' + m_1 \geq 0$

then $g \otimes \eta \in \hat{\omega}_N \otimes \Omega_X^1$, since $g \otimes \eta = g \otimes cdt = gc \otimes dt$, and

$v(g_1 c) \geq v(g_1 c) \geq 0$. If $c' + m_1 < 0$ choose $a \in \omega_N$,

$a = \sum_{i=1}^n a_i v_i$, with $v(a_1) = -c' - m_1 - 1$ (this is possible since

$-c' - m_1 > 0$) and $v(a_1) \geq v(a_1) + 1$. Then

$$\text{res } g(a) = \text{res} \left(\sum a_i g_i cdt \right) = \text{res}(a_1 g_1 cdt + \sum_{i=2}^n a_i g_i cdt)$$

$$= \text{res } a_1 g_1 cdt \neq 0 \quad \text{since} \quad v(a_1 g_1 c) = v(a_1) + v(g_1) + v(c) = -1.$$

This is a contradiction, so $c' + m_1 \geq 0$ and $g \otimes \eta \in \omega_N \otimes \Omega_X^1$.

c) The proof for the pairing $\text{res} \langle , \rangle_{10}$ is almost identical, one has to only account for the various p -th powers involved.

Proposition 5.3. The pairings (5.1) induce an isomorphism

$$\mathcal{V}^{\ell} : \underline{H}_{\mathbb{Z}}^2(X, N) \xrightarrow{\sim} \underline{\text{Hom}}(\mathcal{F}(N^D), \mathbb{Z}/p\mathbb{Z})$$

of sheaves on S_{fl} in the sense explained on page 23.

Proof : We may assume that k is algebraically closed. The sequence

(4.2) induces an exact sequence of (homotopy) sheaves

$$(5.4) \quad \pi_1(\tilde{\omega}_N) \longrightarrow \pi_1(\tilde{\omega}_N^{(p)}) \longrightarrow \mathcal{F}(N^D) \longrightarrow 0.$$

Taking $\underline{\text{Hom}}$ of the above into $\mathbb{Z}/p\mathbb{Z}$ we obtain

$$(5.5) \quad 0 \longrightarrow \underline{\text{Hom}}(\mathcal{F}(N^D), \mathbb{Z}/p\mathbb{Z}) \longrightarrow \underline{\text{Hom}}(\pi_1(\omega_N^{(p)}), \mathbb{Z}/p\mathbb{Z}) \\ \longrightarrow \underline{\text{Hom}}(\pi_1(\tilde{\omega}_N), \mathbb{Z}/p\mathbb{Z}) \quad .$$

But

$$(5.6) \quad \underline{\text{Hom}}(\pi_1(\tilde{G}), \mathbb{Z}/p\mathbb{Z}) \approx \underline{\text{Ext}}^1(G, \mathbb{Z}/p\mathbb{Z}) \quad [3, V, 3.4.1] \text{ so we}$$

can rewrite (5.5) as

$$(5.7) \quad 0 \longrightarrow \underline{\text{Hom}}(\mathcal{F}(N^D), \mathbb{Z}/p\mathbb{Z}) \longrightarrow \underline{\text{Ext}}^1(\omega_N^{(p)}, \mathbb{Z}/p\mathbb{Z}) \\ \xrightarrow{\varphi} \underline{\text{Ext}}^1(\omega_N, \mathbb{Z}/p\mathbb{Z})$$

where Ext is computed in the category of proalgebraic groups over S .

Furthermore the relative cohomology sequence of (4.4) yields,

$$(5.8) \quad 0 \longrightarrow \underline{H}_Z^1(R^1 f_* N) \longrightarrow \underline{H}_Z^1(\hat{\omega}_N \otimes \Omega_X^1) \xrightarrow{\psi} \underline{H}_Z^1(\hat{\omega}_N \otimes \Omega_X^1) \quad .$$

By (4.11) the spectral sequence

$$\underline{H}_Z^p(R^q f_* N) \Longrightarrow \underline{H}_Z^{p+q}(N)$$

degenerates and hence $\underline{H}_Z^1(R^1 f_* N) \approx \underline{H}_Z^2(N)$, and using (2.4)

we can rewrite (5.8) as

$$(5.9) \quad 0 \longrightarrow \underline{H}_Z^2(N) \longrightarrow \frac{\hat{\omega}_{N_K} \otimes \Omega_U^1}{\hat{\omega}_N \otimes \Omega_X^1} \xrightarrow{\psi} \frac{\hat{\omega}_{N_K} \otimes \Omega_U^1}{\hat{\omega}_N \otimes \Omega_X^1}$$

By (5.2) there are isomorphisms,

$$i_1 : \underline{H}_Z^1(\hat{\omega}_N \otimes \Omega_X^1) \longrightarrow \underline{\text{Hom}}_{O_S}(\omega_N^{(p)}, O_S) \quad ,$$

and

$$i_2 : \underline{H}_Z^1(\hat{\omega}_N \otimes \Omega_X^1) \longrightarrow \underline{\text{Hom}}_{O_S}(\omega_N, O_S)$$

given by

$$i_1(\overline{g \otimes \eta}) = \text{res} \langle -, g \otimes \eta \rangle_{10} \quad , \text{ and}$$

$$i_2(\overline{g \otimes \eta}) = \text{res} \langle -, g \otimes \eta \rangle_{01} \quad .$$

Here $\overline{g \otimes \eta}$ denotes the class of $g \otimes \eta$ in $H_Z^1(\omega_N \otimes \Omega_X^1)$, respectively in $H_Z^1(\hat{\omega}_N \otimes \Omega_X^1)$.

Also, the isomorphism r of (4.10) induces an exact sequence :

$$(5.10) \quad 0 \longrightarrow \underline{\text{Hom}}_{O_S}(\omega_N^{(p)}, O_S)_o \longrightarrow \underline{\text{Hom}}_{O_S}(\omega_N^{(p)}, O_S) \xrightarrow{\varphi_o} \underline{\text{Hom}}_{O_S}(\omega_N, O_S)$$

where φ_o is the map induced by φ of (5.7) and

$\underline{\text{Hom}}_{O_S}(\omega_N^{(p)}, O_S)_o$ is the kernel of φ_o .

It follows from the five lemma, that

$$\underline{\text{Hom}}_{O_S}(\omega_N^{(p)}, O_S)_o \approx \underline{\text{Hom}}(\mathcal{F}(N^D), \mathbb{Z}/p\mathbb{Z}) \quad .$$

We shall call this last isomorphism r_o . Combining all above we obtain a diagram with exact rows :

$$\begin{array}{ccccccc} 0 \longrightarrow & \underline{\text{Hom}}(\mathcal{F}(N^D), \mathbb{Z}/p\mathbb{Z}) & \longrightarrow & \underline{\text{Ext}}^1(\omega_N^{(p)}, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\varphi} & \underline{\text{Ext}}^1(\omega_N, \mathbb{Z}/p\mathbb{Z}) & \\ & \uparrow r_o & & \uparrow r^{(p)} & & \uparrow r & \\ 0 \longrightarrow & \underline{\text{Hom}}_{O_S}(\omega_N^{(p)}, O_S)_o & \longrightarrow & \underline{\text{Hom}}_{O_S}(\omega_N^{(p)}, O_S) & \xrightarrow{\varphi_o} & \underline{\text{Hom}}_{O_S}(\omega_N, O_S) & \\ & \uparrow i_o & & \uparrow i_1 & & \uparrow i_2 & \\ 0 \longrightarrow & H_Z^2(X, N) & \longrightarrow & H_Z^1(X, \hat{\omega}_N \otimes \Omega_X^1) & \xrightarrow{\psi} & H_Z^1(X, \hat{\omega}_N \otimes \Omega_X^1) & \end{array}$$

To prove the existence of i_o we have to show that the right side of the diagram commutes. Then we define $\psi^{\ell} = r_o i_o$. Consider therefore a diagram

$$\begin{array}{ccc}
 \text{Ext}^1(\omega_N^{(p)}, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\varphi} & \text{Ext}^1(\omega_N, \mathbb{Z}/p\mathbb{Z}) \\
 \uparrow \mathcal{J}(p) & & \uparrow \mathcal{J} \\
 \text{Hom}(\omega_N^{(p)}, \mathbb{G}_{a,S}) & \xrightarrow{\varphi} & \text{Hom}(\omega_N, \mathbb{G}_{a,S}) \\
 \uparrow j_1 & & \uparrow j_2 \\
 \text{Hom}_{O_S}(\omega_N^{(p)}, O_S) & \xrightarrow{\varphi \circ} & \text{Hom}_{O_S}(\omega_N, O_S) \\
 \uparrow i_1 & & \uparrow i_2 \\
 H_Z^1(X, \hat{\omega}_N \otimes \Omega_X^1) & \xrightarrow{\psi} & H_Z^1(X, \hat{\omega}_N \otimes \Omega_X^1)
 \end{array} \quad (5.11)$$

where $\mathcal{J}$ and $\mathcal{J}(p)$ are boundary maps in the sequence obtained by taking $\underline{\text{Hom}}$ of ω_N and $\omega_N^{(p)}$ into the exact sequence

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{G}_{a,S} \longrightarrow \mathbb{G}_{a,S} \longrightarrow 0,$$

j_1 and j_2 are natural inclusions.

By (4.10) $\mathcal{J} \circ j_2 = r : \text{Hom}_{O_S}(\omega_N, O_S) \approx \text{Ext}^1(\omega_N, \mathbb{Z}/p\mathbb{Z})$

and

$$\mathcal{J}(p) \circ j_1 = r^{(p)} : \text{Hom}_{O_S}(\omega_N^{(p)}, O_S) \approx \text{Ext}^1(\omega_N^{(p)}, \mathbb{Z}/p\mathbb{Z}).$$

The upper and middle squares of (5.11) commute, so it is enough to show that

$$\mathcal{J}(p)(a \mapsto \text{res}(\varphi j_1 i_1)(g \otimes \eta)(a)) = \mathcal{J}(a \mapsto \text{res}(j_2 i_2(-\psi))(g \otimes \eta)(a)),$$

i.e. that

$$\mathcal{J}(a \mapsto \text{res}(\langle \varphi(a), g \otimes \eta \rangle_{10} + \langle a, \psi(g \otimes \eta) \rangle_{01})) = 0,$$

but the last expression equals, by (4.9), to

$$\mathcal{J}(a \mapsto \text{res}((C-W^*) \langle a, g \otimes \eta \rangle_{00})) = \mathcal{J}(a \mapsto \text{res}(g(a)C\eta - g(a)^p \eta^W)).$$

The only term which contributes to the residue, assuming that

$$\eta = \sum_{j=m}^{\infty} a_j s^j ds, \quad s^p = t, \quad \text{is } a_{-1} \frac{ds}{s} \text{ and}$$

$$\text{res}(C-W^*)(a_{-1} \frac{ds}{s}) = \text{res}(a_{-1} \frac{dt}{t} - a_{-1}^p \frac{dt}{t}) = (1-F)a_{-1}$$

It follows from the proof of [9;2.2], that $\mathcal{J} \circ (1-F) = 0$, hence

$$\mathcal{J}(\text{res}(C-W^*) \langle -, g \otimes \eta \rangle_{00}) = 0$$

which concludes the proof.

Consider the diagram of sites (cf[9;2.8]) :

$$(5.12) \quad \begin{array}{ccc} X_{j, f1} & \xrightarrow{f_j} & (\underline{\text{Pf}} X_j/S)_{\text{ét}} \\ \alpha_j \downarrow & \searrow \beta_j^{f_j} & \downarrow \beta_j \\ S_{f1} & \xrightarrow{f'} & (\underline{\text{Pf}} /S)_{\text{ét}} \end{array}$$

where the sites $(\underline{\text{Pf}} /S)_{\text{ét}}$ and $(\underline{\text{Pf}} X_j/S)_{\text{ét}}$ are explained in [9;§2] :

$(\underline{\text{Pf}} /S)_{\text{ét}}$ is the category of all perfect schemes over S with the étale topology, and $\underline{\text{Pf}} X_j/S$ is the category whose objects are pairs (Y, T) with T in $\underline{\text{Pf}} /S$ and Y an étale $X_j \times_S T$ -scheme.

$(\underline{\text{Pf}} X_j/S)_{\text{ét}}$ denotes this category together with its étale topology.

The map $f_j : X_{j, f1} \longrightarrow (\underline{\text{Pf}} X_j/S)_{\text{ét}}$ is induced by the map $(Y, T) \longmapsto Y$.

Lemma 5.13. Let $B = B(N) = \ker \psi$ in the sequence

$$(*) \quad 0 \longrightarrow N \longrightarrow F_* N \longrightarrow \hat{\omega}_N \otimes \Omega_X^1 \xrightarrow{\psi} \hat{\omega}_N \otimes \Omega_X^1 \longrightarrow 0$$

of (4.5). Then :

$$\pi_1(\tilde{B}) \approx \mathcal{F}(N) \quad .$$

Proof: We shall show that $\pi_1(B_{p1}) = F_{p1}$, where

$$B_{p1} = B(N_{R_{p1}}) \quad , \text{ and } (F_j) = \mathcal{F}(N) \quad , \text{ i.e.}$$

$$F_j = \text{Coker } (\pi_1(\alpha_{j*}A_j) \longrightarrow \pi_1(\alpha_{j*}A'_j))$$

for some smooth formal resolution

$$0 \longrightarrow N \longrightarrow (A_j) \longrightarrow (A'_j) \longrightarrow 0 \quad \text{of } N \quad .$$

From the diagram (5.12) we obtain two spectral sequences with the same abutment :

$$R^p \beta_{j*} R^q f_{j*} N_{R_j} \Longrightarrow R^{p+q}(\beta_{j*} f_{j*})_{*} N_{R_j} \quad ,$$

and

$$R^p f_{*} R^q \alpha_{j*} N_{R_j} \Longrightarrow R^{p+q}(\beta_{j*} f_{j*})_{*} N_{R_j} \quad .$$

and corresponding exact sequences of terms of low degree :

$$\begin{array}{ccccccc} 0 \longrightarrow (R^1 \beta_{j*}) f_{j*} N_{R_j} & \longrightarrow & R^1(\beta_{j*} f_{j*})_{*} N_{R_j} & \longrightarrow & \beta_{j*}(R^1 f_{j*} N_{R_j}) & \longrightarrow & (R^2 \beta_{j*}) f_{j*} N_{R_j} \\ & & \parallel & & & & \\ 0 \longrightarrow (R^1 f'_{*}) \alpha_{j*} N_{R_j} & \longrightarrow & R^1(\beta_{j*} f_{j*})_{*} N_{R_j} & \longrightarrow & f'_{*}(R^1 \alpha_{j*} N_{R_j}) & \longrightarrow & (R^2 f'_{*}) \alpha_{j*} N_{R_j} \end{array}$$

It is clear that $R^i \beta_{j*} N_{R_j} = 0$, $i \geq 1$, since for any smooth

k-algebra A , we can identify the étale sites on Spec A and

Spec(A ⊗_k R_j) . Also, since $k = \bar{k}$, $R^i f'_{*} F = 0$ for $i \geq 1$ and

any group scheme F on S_{f1} . It follows that

$$(R^1 f'_*) \alpha_{j* N_{R_j}} = (R^2 f'_*) \alpha_{j* N_{R_j}} = (R^1 \beta_{j*}) f_{j* N_{R_j}} = (R^2 \beta_{j*}) f_{j* N_{R_j}} = 0 ,$$

$$\text{and } \beta_{j*}(R^1 f_{j* N_{R_j}}) \approx f'_*(R^1 \alpha_{j* N_{R_j}}) .$$

By (I, (3.7)) $\pi_1(R^1 \alpha_{j* N_{R_j}}) \approx F_j$, and $R^1 f_{j* N_{R_j}} \approx B_j$ if $p \nmid j$,

(see (*)), so it is enough to show that f'_* preserves projectives.

But the functor $G \longmapsto G^{\text{pf}}$ from the category $\text{Pro}(\text{group schemes of finite type over } S)$ to the category of perfect groups over S which are perfection of groups of finite type has an exact right adjoint : $H \longmapsto H$, so the assertion follows. Here G^{pf} , in the notation of [9;§2], is the group

$$\varprojlim G(p^{-n}) , \text{ the perfection of } G .$$

Proposition 5.14. The pairings (5.1) induce an isomorphism

$$\mathcal{V}^1 : \underline{H}_Z^2(X, N^D) \longrightarrow \underline{\text{Hom}}(\mathcal{F}(N), \mathbb{Z}/p\mathbb{Z}) \text{ of pro-sheaves on } S_{f1} .$$

Proof : Consider an exact sequence

$$0 \longrightarrow B \longrightarrow \hat{\omega}_N \otimes \Omega_X^1 \xrightarrow{\psi} \hat{\omega}_N \otimes \Omega_X^1 \longrightarrow 0 . \text{ We know, (4.7), that } B \text{ is}$$

smooth and, (5.13), that $\pi_1(\tilde{B}) \approx \mathcal{F}(N)$. Let $U^0 = U^0(N)$

$= \hat{\omega}_N \otimes \Omega_X^1$, and let $U^1 = U^1(N) = \hat{\omega}_N \otimes \Omega_X^1$. Since $\pi_1(\sim)$ is

exact on smooth group schemes (cf(I, 1.1)), it follows that there is an exact sequence

$$(5.15) \quad 0 \longrightarrow \pi_1(\tilde{B}) \longrightarrow \pi_1(\tilde{U}^0) \longrightarrow \pi_1(\tilde{U}^1) \longrightarrow 0$$

of sheaves on S_{f1} . On the other hand, the sequence (4.2) induces an exact sequence

$$(5.16) \quad 0 \longrightarrow \underline{H}_Z^1(\omega_N) \longrightarrow \underline{H}_Z^1(\omega_N^{(p)}) \longrightarrow \underline{H}_Z^2(N^D) \longrightarrow 0 \quad .$$

The sequence is exact on the right since $\underline{H}_Z^2(\omega_N) = 0$ by Lemma 2.7.

Taking $\underline{\text{Hom}}$ of (5.15) into $\mathbb{Z}/p\mathbb{Z}$ yields,

$$\begin{aligned} 0 &\longrightarrow \underline{\text{Hom}}(\pi_1(\tilde{U}^1), \mathbb{Z}/p\mathbb{Z}) \longrightarrow \underline{\text{Hom}}(\pi_1(\tilde{U}^0), \mathbb{Z}/p\mathbb{Z}) \\ &\longrightarrow \underline{\text{Hom}}(\mathcal{F}(N), \mathbb{Z}/p\mathbb{Z}) \longrightarrow 0 \end{aligned}$$

which can be rewritten as

$$\begin{aligned} 0 &\longrightarrow \underline{\text{Ext}}^1(U^1, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \underline{\text{Ext}}^1(U^0, \mathbb{Z}/p\mathbb{Z}) \\ &\longrightarrow \underline{\text{Hom}}(\mathcal{F}(N), \mathbb{Z}/p\mathbb{Z}) \longrightarrow 0 \quad . \end{aligned}$$

We can also rewrite (5.16) as

$$0 \longrightarrow \frac{\omega_{N_K}}{\omega_N} \longrightarrow \frac{\omega_{N_K}^{(p)}}{\omega_N^{(p)}} \longrightarrow \underline{H}_Z^2(N^D) \longrightarrow 0 \quad .$$

Using arguments similar to those of the proof of (5.3) we construct a diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \underline{\text{Ext}}^1(U^1, \mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\sim} & \underline{\text{Ext}}^1(U^0, \mathbb{Z}/p\mathbb{Z}) & \longrightarrow & \underline{\text{Hom}}(\mathcal{F}(N), \mathbb{Z}/p\mathbb{Z}) \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \underline{\text{Hom}}_S(U^1, \mathcal{O}_S) & \longrightarrow & \underline{\text{Hom}}_S(U^0, \mathcal{O}_S) & \longrightarrow & \underline{\text{Hom}}(\mathcal{F}(N), \mathbb{Z}/p\mathbb{Z}) \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \underline{H}_Z^1(\omega_N) & \longrightarrow & \underline{H}_Z^1(\omega_N^{(p)}) & \longrightarrow & \underline{H}_Z^2(N^D) \longrightarrow 0 \end{array}$$

with all vertical arrows being isomorphisms and which commutes. This concludes the proof.

6. Proof of the Theorem 3.1.

Lemma 6.1. Let N be a finite flat group scheme over R . Then N has a composition series

$$N = N_0 \supset N_1 \supset \dots \supset N_m = 0$$

in which N_j is a finite flat group scheme and each quotient N_j/N_{j+1} is a finite flat group scheme killed by F or V .

Proof: The lemma is true for N_K . Indeed

$F : N_K \longrightarrow N_K^{(p)}$ is an isomorphism if and only if N_K is étale,

and $V : N_K^{(p)} \longrightarrow N_K$ is an isomorphism if and only if N_K is infinitesimal. The lemma now follows from the well known statement: given N over R , any exact sequence $0 \longrightarrow M' \longrightarrow N_K \longrightarrow M'' \longrightarrow 0$ of finite flat group schemes over K extends uniquely to an exact sequence of finite flat group schemes over R ,

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0.$$

Proof of the theorem. We may assume that k is algebraically closed.

If N is killed by F (respectively by V) the theorem is proved by (5.14) (respectively by (5.3)) once we check that

(a) $H_Z^2(X, \mathcal{F}(\mu_{p^\infty})) = \mathbb{Q}_p/\mathbb{Z}_p$, and

(b) The pairings of (5.3) and (5.14) coincide with the cup product pairings induced by (1.1) and (1.2). This is done below.

In general (once we have (a) and (b)) we use induction on the length of the composition series (6.1). Assume that the theorem is true for

groups with a composition series of length j , and that N has a composition series (6.1) of length $j+1$,

$$N = N_0 \supset \dots \supset N_{j+1} = 0.$$

From the exact sequence :

$$0 \longrightarrow N_j \longrightarrow N \longrightarrow N/N_j \longrightarrow 0,$$

we obtain two exact sequences :

$$\mathcal{F}(N_j) \longrightarrow \mathcal{F}(N) \longrightarrow \mathcal{F}(N/N_j) \longrightarrow 0$$

and

$$0 \longrightarrow \underline{H}_Z^2(X, (N/N_j)^D) \longrightarrow \underline{H}_Z^2(X, N^D) \longrightarrow \underline{H}_Z^2(X, N_j^D),$$

and therefore a diagram :

$$\begin{array}{ccccccc} \mathcal{F}(N_j) & \longrightarrow & \mathcal{F}(N) & \longrightarrow & \mathcal{F}(N/N_j) & \longrightarrow & 0 \\ \approx \downarrow & & \downarrow & & \downarrow \approx & & \\ 0 \longrightarrow & \underline{H}_Z^2(X, N_j^D)^* & \longrightarrow & \underline{H}_Z^2(X, N^D)^* & \longrightarrow & \underline{H}_Z^2(X, (N/N_j)^D)^* & \end{array}$$

$$\text{where } \underline{H}_Z^2(X, M)^* = \underline{\text{Hom}}(\underline{H}_Z^2(X, M), \mathbb{Q}_p/\mathbb{Z}_p).$$

The first and third arrows are isomorphisms, hence so is the middle one.

It remains to check (a) and (b); this is done in the next two lemmas.

Lemma 6.2. Suppose that k is algebraically closed. Then

$$\underline{H}_Z^2(X, \mathcal{F}(\mu_{p^\infty})) \approx \mathbb{Q}_p/\mathbb{Z}_p.$$

Proof : Since $\underline{H}_Z^2(X, \mathcal{F}(\mu_{p^\infty})) = \varinjlim \underline{H}_Z^2(X, \mathcal{F}(\mu_{p^n}))$ it is enough to

show that $\underline{H}_Z^2(X, \mathcal{F}(\mu_{p^n})) \approx \mathbb{Z}/p^n\mathbb{Z}$. Also since k is algebraically closed,

$$\underline{H}_Z^2(X, \mathcal{F}(\mu_{p^n})) \approx H_Z^2(X, \mathcal{F}(\mu_{p^n})) \quad (\text{cf}(2.6)).$$

Consider the Kummer sequence :

$$0 \longrightarrow \mu_{p^n} \longrightarrow \mathbb{G}_{m,R} \longrightarrow \mathbb{G}_{m,R} \longrightarrow 0$$

and the induced sequence of pro-sheaves on X_{ffl} ,

$$0 \longrightarrow \pi_1(\mathbb{G}_{m,R}^\sim) \longrightarrow \pi_1(\mathbb{G}_{m,R}^\sim) \longrightarrow \mathcal{F}(\mu_{p^n}) \longrightarrow 0 .$$

The sequence is exact since $\pi_1(\mu_{p^n}^\sim) = 0$ by (I,2.3)). Applying relative cohomology yields:

$$0 \longrightarrow \underline{H}_Z^2(\pi_1(\mathbb{G}_{m,R}^\sim)) \xrightarrow{p^n} \underline{H}_Z^2(\pi_1(\mathbb{G}_{m,R}^\sim)) \longrightarrow \underline{H}_Z^2(\mathcal{F}(\mu_{p^n})) \longrightarrow 0 ,$$

where the cohomology is computed on the finite flat site. It follows from the universal covering sequence and the fact that universal covering commutes with cohomology, that

$$\underline{H}_Z^2(\pi_1(\mathbb{G}_{m,R}^\sim)) \approx \underline{H}_Z^1(\mathbb{G}_{m,R})$$

(cf[16;2.5]). Also it is enough to compute $H_Z^1(X, \mathbb{G}_{m,R})$ since k is algebraically closed. But, by (2.4),

$$H_Z^1(X, \mathbb{G}_{m,R}) \approx K^*/R^* \approx \mathbb{Z}$$

$$\text{so } \underline{H}_Z^2(X, \mathcal{F}(\mu_{p^n})) = \text{Coker}(\mathbb{Z} \xrightarrow{p^n} \mathbb{Z}) = \mathbb{Z}/p^n \mathbb{Z} .$$

Lemma 6.3. Let N be a finite flat group scheme of height 1 over R . Then the pairings of (5.3) and (5.14) coincide with the cup-product pairing induced by (1.1) and (1.2).

Proof : (a) The pairing $H_Z^2(X, N) \times \mathcal{F}(N^D) \longrightarrow \mathbb{Z}/p\mathbb{Z}$ of (5.3).

Notation: If $\underline{A}_E$ is a category A with a topology E , then $\underline{S}(\underline{A}_E)$ denotes the category of sheaves, and $\text{Pro-}\underline{S}(\underline{A}_E)$ the corresponding procategory. Let $\underline{A}_1 = \text{Pro-}\underline{S}((\underline{\text{Pf}} X/S)_{\text{ét}})$, $\underline{A}_2 = \text{Pro-}\underline{S}((\underline{\text{Pf}}/S)_{\text{ét}})$, $\underline{A}_3 = (\underline{\text{Pf}} X/S)_{\text{ét}}$, and let $f : X_{f1} \longrightarrow \underline{A}_3$ be the map induced by $(Y, T) \longmapsto Y$. Consider the following commutative diagram (cf[2;4.6]) :

$$\begin{array}{ccc} \text{Rf}_* N & \longrightarrow & \underline{\text{RHom}}(\text{Rf}_* N^D, \text{Rf}_* \mu_p) \\ \downarrow & & \downarrow \\ \text{Rf}_* Z^*(N) & \longrightarrow & \underline{\text{RHom}}(\text{Rf}_* N^D, \text{Rf}_* Z^*(\mu_p)) \\ \uparrow & & \uparrow \\ \text{Rf}_* U^*(N)[1] & \longrightarrow & \underline{\text{RHom}}(V^*(N), U^*(\mu_p)[1]) \end{array},$$

where $Z^*(N) = (F_* N_X \longrightarrow U^*(N))$,

$U^*(N) = (\hat{\omega}_N \otimes \Omega_X^1 \longrightarrow \hat{\omega}_N \otimes \Omega_X^1)$ and $V^*(N) = (\omega_N \longrightarrow \omega_N(p))$,

and $\underline{\text{RHom}}$ is computed in $D(\underline{S}(\underline{A}_3))$, the derived category of $\underline{S}(\underline{A}_3)$.

Let $R\mathcal{F} = R\pi_1^* R\hat{\beta}_*$, where $\hat{\beta}_* : \underline{S}(\underline{A}_3) \longrightarrow \underline{A}_2$ is defined in a similar way as $\hat{\alpha}_*$. It is clear that $R\mathcal{F} = L\pi_0^* R\hat{\beta}_*$. By applying $R\mathcal{F}$ to the right column of the diagram we obtain the commutative diagram:

$$\begin{array}{ccc}
 Rf_* N & \xrightarrow{d} & R\text{Hom}(R\mathcal{F} \circ Rf_* N^D, R\mathcal{F} \circ Rf_* \mu_p) \\
 \downarrow & & \downarrow \\
 Rf_* Z^*(N) & \longrightarrow & R\text{Hom}(R\mathcal{F} \circ Rf_* N^D, R\mathcal{F} \circ Rf_* Z^*(\mu_p)) \\
 \uparrow & & \uparrow \\
 Rf_* U^*(N)[1] & \longrightarrow & R\text{Hom}(R\mathcal{F}(V^*(N)), R\mathcal{F}(U^*(\mu_p))[1])
 \end{array}$$

in which only the map d requires explanation. Its existence follows from the theorem of Breen (unpublished, but see [9;2.2]) - which states that on the perfect site the sheaf Ext agrees with a group Ext in the following way:

Applying $R\hat{\beta}_*$ yields a map

$$R\text{Hom}(Rf_* N^D, Rf_* \mu_p) \longrightarrow R\text{Hom}(R\hat{\beta}_* Rf_* N^D, R\hat{\beta}_* Rf_* \mu_p) \text{ where}$$

the second $R\text{Hom}$ is computed in $D(\underline{A}_2)$, which is unambiguous, since an object of $(\underline{\text{Pf}} X/S)$ is a pair (Y, T) , with $T \in \text{Ob}(\underline{\text{Pf}}/S)$, so we have a well defined object on which we can evaluate $R\text{Hom}$ (namely T). Now using Breen's theorem we take a projective resolution of $R\hat{\beta}_* Rf_* N^D$ and apply $R\pi_1$ to obtain the map d . Next, we apply $R\Gamma_Z$ and take (hyper) cohomology. Then the assertion follows from the reasoning analogous to that of [2;4.6], and the observation that

$$\Gamma(X, \mathcal{F}(N^D)) \approx \underline{\text{Hom}}_X(\mathbb{Z}, \mathcal{F}(N^D)) \text{ , and}$$

$$\Gamma_Z \underline{\text{Hom}}_X(\mathbb{Z}, \mathcal{F}(N^D)) \approx \underline{\text{Hom}}_X(\mathbb{Z}_Z, \mathcal{F}(N^D)) \text{ .}$$

(b) The pairing $H_Z^2(X, N^D) \times \mathcal{F}(N) \longrightarrow \mathbb{Z}/p\mathbb{Z}$ of (5.14).

Consider the diagram of pairings

$$\begin{array}{ccccc}
 \mathcal{F}(N) & \times & N^D & \longrightarrow & \mathcal{F}(\mu_p) \\
 \downarrow & & \parallel & & \downarrow \\
 \pi U^* & \times & N^D & \longrightarrow & \pi U^*(\mu_p) \\
 \parallel & & \downarrow & & \parallel \\
 \pi U^* & \times & V^* & \longrightarrow & \pi U^*(\mu_p)
 \end{array}$$

where $\pi U^* = (\pi_1(\tilde{U}^0) \longrightarrow \pi_1(\tilde{U}^1))$.

The upper pairing is obtained by applying $\pi_1(\sim)$ to the diagram

$$\begin{array}{ccccc}
 R^1 f_{*} N & \times & N^D & \longrightarrow & R^1 f_{*} \mu_p \\
 \downarrow & & \parallel & & \downarrow \\
 U^0 & \times & N^D & \longrightarrow & U^0(\mu_p)
 \end{array} ,$$

which commutes by (4.3). The lower part, when decomposed, leads to two diagrams, which are obtained by applying $\pi_1(\sim)$ to the diagrams

$$\begin{array}{ccccc}
 U^0 & \times & N^D & \longrightarrow & U^0(\mu_p) \\
 \parallel & & \downarrow & & \parallel \\
 U^0 & \times & V^0 & \longrightarrow & U^0(\mu_p)
 \end{array}$$

and

$$\begin{array}{ccccc}
 U^1 & \times & N^D & \longrightarrow & U^1(\mu_p) \\
 \parallel & & \downarrow & & \parallel \\
 U^1 & \times & V^0 & \longrightarrow & U^1(\mu_p)
 \end{array}$$

which commute by [2;4.6]. The rest follows from [loc.cit.].

7). Duality for abelian varieties.

Let A be an abelian scheme over R with the dual abelian scheme $\hat{A}$.

Let $\bar{K}$ denote the maximal unramified extension of K , G will de-

note the galois group of k , i.e. the galois group of the algebraic closure $\bar{k}$ of k over k . Let $\bar{A} = A \otimes_R \bar{R}$ where $\bar{R}$ is the ring of integers in $\bar{k}$. If E is an abelian group, then E_m is the subgroup of elements killed by m and $E(p)$ is the p -primary component of E , also, $E^{(m)}$ will denote the cokernel of multiplication by m , $E^{(m)} = E/mE$. We shall write $\pi_1(A)$ instead of $\pi_1(\tilde{A})$.

Theorem 7.1. There is a non-degenerate pairing of topological G -modules,

$$H^1(\bar{K}, \hat{A} \otimes_R \bar{K}) \times \pi_1(\bar{A}) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$

Proof: In [15;p.96] the assertion of the theorem is proved for all 1-primary components of the groups in question, $1 \neq p$. Hence it is enough to show that there exists a non-degenerate pairing between the p -primary components of the groups which is induced by the e_m -pairing [11;§23].

Consider an exact sequence

$$(*) \quad 0 \longrightarrow A_{p^n} \longrightarrow A \xrightarrow{p^n} A \longrightarrow 0,$$

which we may view as a smooth, connected resolution of A_{p^n} , and

therefore $\mathcal{F}(A_{p^n}) \approx \pi_1(A)^{(p^n)}$. From a similar sequence of multi-

plication by p^n on $\hat{A}$ we obtain an isomorphism

$$(7.2) \quad H_Z^2(X, \hat{A}_{p^n}) \approx H_Z^2(X, \hat{A})_{p^n}.$$

We may assume that k is algebraically closed. Then

$$H^1(K, \hat{A} \otimes_H K)(p) = \varinjlim H_Z^2(R, \hat{A}_{p^n}) \quad , \text{ and}$$

$$\pi_1(A) = \varprojlim \pi_1(A)^{(p^n)} = \varprojlim \mathcal{F}(A_{p^n}) \quad .$$

The first assertion follows immediately from (7.2) and (2.1). The second follows from (*). Hence we only have to show that we can pass to the limit in the pairings

$$H_Z^2(X, \hat{A}_{p^n}) \quad \times \quad \mathcal{F}(A_{p^n}) \longrightarrow \mathbb{Z}/p^n \mathbb{Z} \quad .$$

But we know, that for each $m \gg 0$ we have a non-degenerate cup-product pairing (3.1),

$$H_Z^2(X, \hat{A}_{p^m}) \quad \times \quad \mathcal{F}(A_{p^m}) \longrightarrow \mathbb{Z}/p^m \mathbb{Z} \quad ,$$

induced from the e_{p^m} -pairing

$$\hat{A}_{p^m} \quad \times \quad A_{p^m} \longrightarrow \mu_{p^m}$$

(recall, (6.2), that $H_Z^2(X, \mathcal{F}(\mu_{p^m})) \approx \mathbb{Z}/p^m \mathbb{Z}$).

Consider the diagram :

$$\begin{array}{ccccc} \hat{A}_{p^n} & \times & A_{p^n} & \longrightarrow & \mu_{p^n} \\ f_1 \downarrow & & f_2 \uparrow & & \uparrow \\ \hat{A}_{p^{n'}} & \times & A_{p^{n'}} & \longrightarrow & \mu_{p^{n'}} \end{array}$$

with $n' \gg n$. Here f_1 is the inclusion map and $f_2 = p^{n'-n}$.

It follows from [11, §23], that

$$e_{p^{n'}}(f_1 a, b)^{p^{n'}-n} = e_{p^n}(a, f_2 b) \quad ,$$

so the diagram commutes, which concludes the proof.

III. LOCAL CLASS FIELD THEORY.

Let $K = k((t))$ be a field of formal power series with coefficients in a perfect field k of characteristic $p \neq 0$. Let R be the ring of integers in K , $R = k[[t]]$, and $X = \text{Spec } R$. We shall write $\pi_1(N)$ instead of $\pi_1(\tilde{N})$.

We shall show how to obtain a Serre-Hazewinkel class field theory for K (cf [6] and [16]). To simplify explanations assume for a moment that k is algebraically closed. According to (II,3.1) the pairing

$$\mathcal{F}(\mu_p) \times \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathcal{F}(\mu_p)$$

induces via cup-product an isomorphism

$$H_{\mathbb{Z}}^2(X, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\sim} \text{Hom}(\mathcal{F}(\mu_p), \mathbb{Z}/p\mathbb{Z}) .$$

It is clear from (II,2.1) and from the Artin-Schreier sequence

$$0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{G}_a \longrightarrow \mathbb{G}_a \longrightarrow 0$$

that

$$H_{\mathbb{Z}}^2(X, \mathbb{Z}/p\mathbb{Z}) \approx H^1(K, \mathbb{Z}/p\mathbb{Z}) = \text{Hom}(G, \mathbb{Z}/p\mathbb{Z})$$

where G is the Galois group of K . To compute $\mathcal{F}(\mu_p)$, we may use the Kummer sequence :

$$0 \longrightarrow \mu_p \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow 0 .$$

It follows from the definition of $\mathcal{F}$ that

$$\mathcal{F}(\mu_p) = \pi_1(\mathbb{G}_m) / \pi_1(\mathbb{G}_m)^p .$$

Therefore

$$\text{Hom}(\mathcal{F}(\mu_p), \mathbb{Z}/p\mathbb{Z}) \approx \text{Hom}(\pi_1(\mathbb{G}_m), \mathbb{Z}/p\mathbb{Z}) .$$

Therefore we obtain an isomorphism

$$\text{Hom}(G, \mathbb{Z}/p\mathbb{Z}) \longrightarrow \text{Hom}(\pi_1(\mathbb{G}_m), \mathbb{Z}/p\mathbb{Z})$$

or, dualizing, an isomorphism

$$\mathcal{L}_p : \pi_1(\mathbb{G}_m)/p\pi_1(\mathbb{G}_m) \longrightarrow G^{ab}/pG^{ab}$$

where G^{ab} is the galois group of the maximal abelian extension of K . In the same way, we obtain isomorphisms

$$\mathcal{L}_{p^n} : \pi_1(\mathbb{G}_m)/p^n\pi_1(\mathbb{G}_m) \approx G^{ab}/p^nG^{ab}$$

from the pairings

$$\mathcal{F}(\mu_{p^n}) \times \mathbb{Z}/p^n\mathbb{Z} \longrightarrow \mathcal{F}(\mu_{p^n}) .$$

Passing to the limit yields an isomorphism

$$\mathcal{L}(p) : \pi_1(\mathbb{G}_m)(p) \longrightarrow G^{ab}(p)$$

between the p -primary components of the groups. For $l \neq p$, the isomorphism

$$\mathcal{L}(l) : \pi_1(\mathbb{G}_m)(l) \longrightarrow G^{ab}(l)$$

follows from the fact that $\pi_1(\mathbb{G}_m)(l)$ = set of all l^n -th roots of unity of k , $n = 0, 1, \dots$, and from Hensel's lemma. Thus, providing that our map $\mathcal{L}$ agrees with the map Θ_K of [16], this is the content of Serre's class field theory.

Returning to the general situation, i.e. k perfect, we have an isomorphism [3; V, 3.4.2],

$$\text{Hom}(\pi_1(\mathbb{G}_m), \mathbb{Z}/p\mathbb{Z}) \approx \text{Hom}(\gamma(\mathbb{G}_m), \mathbb{Z}/p\mathbb{Z})$$

where $\gamma(\mathbb{G}_m)$ is the maximal constant quotient of $\pi_1(\mathbb{G}_m)$.

Lemma 1.1. Let α_K be the inertia subgroup of G^{ab} . Then

$$\Gamma(S, H_Z^2(X, \mathbb{Z}/p^n \mathbb{Z})) = \text{Hom}(\alpha_K, \mathbb{Z}/p^n \mathbb{Z}) .$$

Proof : It follows from (II, (2.1)), that

$$H_Z^2(X, \mathbb{Z}/p^n \mathbb{Z}) = \text{Coker } (H^1(R, \mathbb{Z}/p^n \mathbb{Z}) \longrightarrow H^1(K, \mathbb{Z}/p^n \mathbb{Z})) ,$$

and since $G^{ab} \approx \alpha_K \times G(\bar{k}/k)$ the assertion follows. Here $G(\bar{k}/k)$ is the galois group of $\bar{k}/k$.

Hence we obtain an isomorphism (which we also call $\mathcal{B}$)

$$\mathcal{B} : \gamma(\mathbb{G}_m) \longrightarrow \alpha_K .$$

This gives Hazewinkel's theory [6].

Theorem 1.2. The above isomorphism $\mathcal{B}$ agrees with Serre's isomorphism.

Proof : It is enough to check the assertion for k algebraically closed. However, Serre's map comes via cup-product from the natural pairing [16; p.124] :

$$\pi_1(\mathbb{G}_m) \times \mathbb{Z} \longrightarrow \pi_1(\mathbb{G}_m)$$

and the theorem follows, since the diagram

$$\begin{array}{ccc} \pi_1(\mathbb{G}_m) & \times & \mathbb{Z} \longrightarrow \pi_1(\mathbb{G}_m) \\ \downarrow & & \downarrow \\ \mathcal{F}(\mu_p) & \times & \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathcal{F}(\mu_p) \end{array} \quad \text{commutes and } \mathcal{B}$$

is obtained also via cup-product from the lower pairing.

APPENDIX.

We give here the proof of lemma (II,2.1) (see also [1, exp.V]).

Let T be a scheme, U an open subscheme of T and Z a complementary closed subscheme, $Z = T - U$. Denote by i and j the corresponding immersions,

$$Z \xrightarrow{i} T \xleftarrow{j} U,$$

so that T is the disjoint union of $i(Z)$ and $j(U)$. Define a functor $\underline{S}(T_{f1}) \longrightarrow \underline{Ab}$, from the category of sheaves on T_{f1} to the category of abelian groups.

$$F \longmapsto H_Z^0(T, F) = \text{Ker}(F(T) \xrightarrow{F(j)} F(U)).$$

H_Z^0 is a left exact functor, denote by $H_Z^i(T, -)$ its i -th right derived.

Lemma. Let F be a sheaf on T_{f1} . There is a long exact sequence

$$\begin{aligned} 0 \longrightarrow H_Z^0(T, F) \longrightarrow F(T) \longrightarrow F(U) \longrightarrow \dots \longrightarrow H^1(T, F) \longrightarrow H^1(U, F|U) \\ \longrightarrow H_Z^{i+1}(T, F) \longrightarrow \dots, \end{aligned}$$

where $F|U$ denotes F restricted to U .

Proof: Let $\underline{Z}$ be a constant sheaf on T_{f1} defined by $\underline{Z}$. Let $\underline{P}(T)$ denote the category of presheaves on T , and let

$$\underline{P}(T) \xrightleftharpoons[b]{a} \underline{S}(T)$$

be inclusion and sheafification functors respectively.

Let $j^p : \underline{P}(T) \longrightarrow \underline{P}(U)$ be the inverse image functor.

Let $j_! : \underline{P}(U) \longrightarrow \underline{P}(T)$ be the left adjoint of j^P , which in this particular case can be defined explicitly :

For any $V \xrightarrow{e} T$, and $P \in \text{Ob } \underline{P}(U)$,

$$(j_! P)(V) = \bigsqcup_{f \in \text{Hom}_T(V, U)} P(V_f)$$

where V_f denotes the scheme V regarded as a scheme over U via f ,

$$\begin{array}{ccc} & V & \\ f \swarrow & & \searrow e \\ U & \xrightarrow{j} & T \end{array}$$

Denote also by $j_!$ the functor

$$a \circ j_! \circ b : \underline{S}(U) \longrightarrow \underline{S}(T)$$

then $j_!$ is left adjoint to j^* and as such it is right exact. Since it is a composition of exact and left exact functors it is also left exact. Denote by $\mathbb{Z}_U$ the sheaf $j_! j^* \mathbb{Z}$ and let $\mathbb{Z}_Z = i_* i^* \mathbb{Z}$. Then there exist maps g and h such that the sequence

$$(*) \quad 0 \longrightarrow \mathbb{Z}_U \xrightarrow{g} \mathbb{Z} \xrightarrow{h} \mathbb{Z}_Z \longrightarrow 0$$

is exact. Indeed, let g be the map induced by the identity map on $j^* \mathbb{Z}$, and let h be induced by the identity on $i^* \mathbb{Z}$. Let

$V \xrightarrow{e} T$ be a morphism locally of finite type. We may assume that V is connected. By definition

$$\mathbb{Z}_U(V) = \bigsqcup_{f \in \text{Hom}_T(V, U)} \mathbb{Z}(V_f)$$

But if e factors through U it factors uniquely, and j^* simply

takes $\mathbb{Z}$ as a sheaf on T to $\mathbb{Z}$ as a sheaf on U , hence

$\mathbb{Z}_U(V) = \mathbb{Z}$ if there exists $f : V \longrightarrow U$ such that $e = j \circ f$,

$$\begin{array}{ccc} & V & \\ f \swarrow & & \searrow e \\ U & \xrightarrow{j} & T \end{array},$$

and $\mathbb{Z}_U(V) = 0$ otherwise. In the first case $\mathbb{Z}_Z(V) = 0$ since $V_Z = V \times_T Z$ is an empty scheme and $(i_* i^* \mathbb{Z})(V) = \mathbb{Z}(V_Z)$. In the second case, i.e. when e does not factor through U , $\mathbb{Z}_U(V) = 0$ and $\mathbb{Z}_Z(V) = \mathbb{Z}$. Hence the sequence (*) reads

$$0 \longrightarrow \mathbb{Z} \xrightarrow{1_{\mathbb{Z}}} \mathbb{Z} \longrightarrow 0 \longrightarrow 0$$

in the first case, and

$$0 \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{1_{\mathbb{Z}}} \mathbb{Z} \longrightarrow 0$$

in the second case. This proves exactness of (*).

Applying $\text{Hom}_{\underline{S}(T)}(-, F)$ to (*) yields the long exact sequence

$$\dots \longrightarrow \text{Ext}^1(\mathbb{Z}, F) \longrightarrow \text{Ext}^1(\mathbb{Z}_U, F) \longrightarrow \text{Ext}^{1+1}(\mathbb{Z}_Z, F) \longrightarrow \dots$$

The groups $\text{Hom}_{\underline{S}(U)}(\mathbb{Z}, j^* F)$ and $\text{Hom}_{\underline{S}(T)}(\mathbb{Z}_U, F)$ are canonically isomorphic, hence $\text{Ext}^1(\mathbb{Z}_U, F)$ is the 1-th derived of

$F \longmapsto \text{Hom}_{\underline{S}(U)}(\mathbb{Z}, j^* F)$. Since j^* has an exact right adjoint and

therefore preserves injectives, it follows that

$\text{Ext}^1(\mathbb{Z}_U, F) \approx H^1(U, F|_U)$ since j^* is also exact.

By definition $\text{Ext}^1(\mathbb{Z}, F) = H^1(T, F)$. Finally $\text{Ext}^1(\mathbb{Z}_Z, F)$ is the

1-th right derived functor of $F \longmapsto \text{Hom}_{\underline{S}(T)}(\mathbb{Z}_Z, F)$, and it follows

from (*) , that $\text{Hom}_{\underline{S}(T)}(\mathbb{Z}_Z, F) \approx H_Z^0(T, F)$, hence

$$\text{Ext}^1(\mathbb{Z}_Z, F) \approx H_Z^1(T, F) \quad .$$

This concludes the proof of the lemma.

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