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THE DYNAMICS OF GAS BUBBLES MOVING IN LIQUIDS WITH PRESSURE GRADIENT

by

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ABSTRACT

THE DYNAMICS OF GAS BUBBLES MOVING IN LIQUIDS WITH PRESSURE GRADIENT

by Hsu-Chieh Yeh

The time history of the velocity, size and deformation of a gas bubble moving in the flow field around a point source or sink is studied analytically. Also included in the analysis is the dynamic behavior of a bubble moving in a quiescent liquid subject to a step change in the liquid pressure. The viscous effects of fluids are neglected, the flow is considered irrotational and the velocity potential is assumed to exist. The governing equations for the translatory motion, size, and deformation of the bubble are derived by perturbation theory. These equations were integrated numerically by the Runge-Kutta method using an IBM 7090 digital computer.

When a gas bubble is in translatory motion the bubble may be accelerated or decelerated as the result of the relative velocity between the bubble and the liquid and the rate of bubble collapse or growth. In addition, the relative velocity between the bubble and the liquid may accelerate the rate of growth. This is true for both cases: the case where the liquid is in a source or sink flow and the case where the bubble travels in a quiescent liquid subject to a step change in the liquid pressure. It is conjectured that the conclusion may also be applied to the case where the fluid is in any other type of flow.

The time history of the bubble's translating velocity in a sink flow is monotonically increasing while in a source flow it varies following two typical patterns depending on the initial velocity. When the initial velocity is smaller than a critical value, the translating motion of the bubble is decreased as it collapses. After the bubble comes to complete stop it starts to travel upstream for a short distance then moves downstream again, regrowing. When the initial velocity is greater than the critical value the bubble's translating motion is first decreased as it collapses. The velocity is then increased over a short distance followed by a deceleration and a regrowth of the bubble.

In a sink flow, a bubble can maintain a nearly spherical shape over a rather long distance as it grows. The bubble is then flattened on the downstream side. In a source flow, the bubble shape varies with time in various ways depending

upon the initial velocity. When the initial translating velocity is low, the initially spherical bubble becomes oblate in shape. It is then flattened at the downstream side. As the initial velocity increases, the indentation shifts upstream along the bubble surface. When the initial velocity is further increased the indentation occurs on the upstream side. During the collapse the bubble shape may become unstable when the rate change in the translatory velocity is large.

When an initially spherical bubble moving in a quiescent liquid is subject to a step change in the liquid pressure, the indentation starts at the rear stagnation point.

The characteristic features in the dynamic behavior of a translating gas bubble as described above were also disclosed in a previous photographic study by Ivany et al. of a cavitation bubble in a rectangular venturi tube. The validity of the present analysis is borne out in its agreement with experimental observations.

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NOMENCLATURE

A	area of bubble surface
$\vec{a}$	acceleration of a bubble
a_n	coefficients of P_n as defined by equation (9)
F	equation of bubble surface as defined by (10)
$\vec{n}$	unit normal vector on bubble surface
P_n	Legendre polynomials
p	pressure; p_{out}, for liquid at the bubble surface; p_0, for liquid at $r_1 = r_0$ in the absence of bubble; p_∞, for liquid at infinite; p_{in}, for gas inside bubble $p_{in}(0)$, gas inside bubble at $t = 0$.
R	equivalent bubble radius; $R^{(0)}$, at $t = 0$.
r, θ, φ	moving coordinates with the origin fixed at the center of an unperturbed bubble
r', θ', φ'	stationary coordinates with the origin fixed at a source or sink
r_s	radial distance from the center to the surface of the bubble
r_1	distance from a source or sink to the center of a bubble
r_0	the value of r_1 at $t = 0$
t	time
U	translatory velocity of a bubble moving in a quiescent liquid; the value of U at $t = 0$.
V_1	translatory velocity of a bubble moving in a source flow or sink flow
V_0	the value of V_1 at $t = 0$
V_{0cr}	critical value of V_0
$\vec{V}$	velocity vector
x	displacement of a bubble moving in a quiescent liquid

Greek letters:

β	angle between the normal of bubble surface and radial vector
μ	strength of a source or sink
ρ	density of liquid
σ	surface tension
$\phi, \phi_1, \phi_2, \phi_3$	velocity potential

CHAPTER I

INTRODUCTION

The problems on the translatory motion and deformation of a bubble have been studied by a number of investigators. However due to the difficulty and complexity of the problem, those investigations are limited to some special cases: Plesset and Mitchell (1) and Naude and Ellis (2) considered the dynamics of a stationary bubble in quiescent liquids with uniform system pressure. Since there exists no un-symmetrical force acting upon the bubble from the surrounding liquid, that is, there is no forcing function in the bubble dynamics equations, the instability problem may arise only when the bubble starts to grow or collapse from an initially nonspherical form. Furthermore, since there is no bulk flow of the liquid and the bubble is not in translating motion, the governing equations are not coupled and thus may be solved independently. Hartunian and Sears (3) studied the instability of a translating bubble in a liquid with uniform pressure. They obtained a set of simultaneous coupled equations whose solution may predict the onset of the instability in the bubble's translatory motion. Since the bubble being considered remains the same in size during the course of translating motion the equations determining instability are independent of time. Walters and Davidson (4) investigated the deformation of bubble

shape caused by the translating motion and the pressure gradient in the surrounding liquid. However, since the problem dealt with the case where during the transient the volume of the bubble and the pressure gradient in the liquid are both constant the acceleration of the bubble in the liquid is constant and equal to 2g.

Recently, Ivany, Hammitt and Mitchell (5) have experimentally studied the dynamics of cavitation bubbles in a liquid flowing through a venturi. They observed "rebound" phenomena in the translating motion and deformation of the bubble during the course of its collapse. These phenomena are quite complicated due to the interaction among the liquid flow, the bubble's translating motion, the collapse or growth of the bubble, and the non-uniformity in the pressure gradient.

In this thesis analysis is presented for the dynamics of a gas bubble in translating motion in an inviscid liquid with pressure gradient. Consideration is given to the cases where the surrounding liquid is in a source flow, a sink flow or quiescent. The problem is simplified through the assumptions of constant liquid density, negligible effects of heat and mass transfer, adiabatic processes for ideal gas inside the bubble and ideal liquid flow. Analytical results are obtained for the time history of the bubble's size, shape and translating velocity. The characteristic behavior of a translating gas bubble disclosed in the present analysis was observed in the photographic study of a cavitation bubble in a rectangular venturi (5).

CHAPTER II

LITERATURE SURVEY

The problem of bubble collapse was first solved by Besant (6), and then later resolved by Rayleigh (7) using different approach, i.e., an energy method. Lamb (8) summarized the formulation and solution to the problems of bubble collapse with negligible internal pressure and of isentropic bubble expansion with negligible external pressure. Plesset (9) modified the Rayleigh equation to account for surface tension effects, and Poritsky (104) included viscous effects.

There has been a considereable amount of experiments performed to study the behavior of a rising bubble (11-22). For describing bubble motion in liquids at constant velocity U_∞, Rosenberg (15) suggested the use of three dimensionless parameters: drag coefficient $C_D(=\frac{8Rg}{3U_\infty^2})$, Reynolds number $Re(=2\rho U_\infty/\mu)R$ and a third parameter $M(=g\mu^4/\rho\sigma^3)$, in which R is the bubble radius, μ the liquid viscosity, ρ the liquid density, g the gravitational acceleration, and σ the surface tension. Haberman and Morton (23) conducted the most extensive study on the bubble dynamics including the effects of variation of liquid properties and the wall effect. Rosenberg (15), Davices and Taylor (16), Haberman and Morton (23), and Walters and Davidson (4) disclosed experimentally that the bubble with a spherical-cap shape experienced the same drag coefficient during its rising

through a liquid irrespective of its size. This was confirmed by Walters and Davidson (4), who found that a bubble in a spherical-cap shape was less affected by viscous and surface tension forces. Hickling and Plesset (24) presented some numerical solutions which described the flow in the vicinity of a collapsing spherical bubble considering the effect of liquid compressibility. Ivany and Hammitt made somewhat similar numerical calculations in which liquid compressibility, viscosity and surface tension were included (105). Miyagi (25) observed that there is always an upward current of water which moves after the bubble. Thus, he introduced the concept of "added mass" which enabled him to predict the terminal velocity of the rising bubble in the water.

The problem of mass transfer to or from a single bubble rising in a liquid has also been investigated (26-34). The mass transfer coefficient was generally found to increase with increased bubble velocity and to decrease with the addition of surface active agents to the liquid.

On the basis of the penetration theory, Higbie (26) treated the problem of gas absorption from a bubble rising through a nonvolatile liquid. The theory assumed that the dissolved component penetrates only a short distance from the interface into the liquid. As the bubble rises, the gas in the bubble undergoes a toroidal motion such that there is essentially no slip between the gas flow and the liquid at the interface, and the top of the bubble always

encounters fresh liquid. Under these assumptions, the transfer of gas to the liquid is treated as transient diffusion into a falling film; the exposure time is taken to be the bubble diameter divided by the bubble rise velocity. Pattle (27,28) studied the aeration of water experimentally by measuring the absorption of oxygen from air injected in various bubble sizes. Coppock and Maiklejohn (31) studied bubble formation from horizontal circular orifices and measured mass transfer coefficients for the absorption of oxygen in water for rising bubbles 0.15 to 0.3 cm in diameter. Hammerton and Garner (30) measured mass transfer rates of five gases from single bubbles (0.3 to 0.5 cm in diameter), rising in water and glycerol. They found that the presence of surface-active agents lessened transfer rates considerably. Datta, Napier, and Newitt (14) investigated the absorption of carbon dioxide in air-saturated water from rising bubbles formed from circular capillary orifices submerged in mercury to minimize absorption during bubble formation. Deindoerfer and Humphrey (31) measured instantaneous values of the mass transfer coefficient for bubbles with diameters of 0.1 to 0.5 cm, and found that the mass transfer coefficient decreased with increased age of the bubble; this suggests that internal circulation inside the bubbles decreased with time. Baird and Davidson (32) determined the mass transfer rate from large carbon dioxide bubbles (0.8 to 4.2 cm in diameter) rising in water. The mass transfer coefficient

was also found to decrease with an increase in bubble age, because of the saturation of the bound wake formed in such bubble flow regimes. Leonard and Houghton (33) also observed a decrease in the mass transfer coefficient with bubbles in nitrous oxide-water and ethylene-water systems, but attributed it to the formation of a thin, stagnant layer of liquid around the bubble during its ascension.

The growth and decay of gas bubbles in liquids due to concentration difference was investigated by Keller (34).

Circulation within the bubble contributed not only to increase the mass transfer coefficient above that of an equivalent solid sphere but also to reduce the drag coefficient to several times less than that of a solid sphere (35-40). Circulation patterns were experimentally observed by Garner and Hammerton (41) in bubbles and by Garner and Haycock (40) and Spells (38) in liquid droplets. Hadamard (35) and Rybczynski (36) studied the drag coefficient for circulating bubbles. Hadamard predicted $C_D = 16/Re$ for a circulating bubble, 1.5 times lower than an equivalent nincirculating bubble with $C_D = 24/Re$ Groothuis and Kramers (41) predicted mass transfer rates during drop formation on a capillary tip. By using the Eotvos number, the ratio of the maximum hydrostatic head inside a bubble or droplet to the surface tension forces, Harmathy (42) succeeded in correlating the drag coefficient for bubbles as well as droplets for values of $Re = 500$. Other work has treated to the drag coefficient of a liquid

droplet traveling in another fluid includes References 43-45. Kronig et al., (46,47) analyzed transfer processes from a liquid droplet and a bubble in the presence of internal circulation.

The surface effect has drawn considerable attention. Haberman and Morton (23) Hartunian and Sears, (3), Levich (20) Garner and Hammerton, (39) and Gibon, Houghton, and Coull (49) showed that dirt, impurity, or surface-active agents in liquids strongly affect bubble motion in the spherical and ellipsoidal regimes by impeding circulation.

The time-history of the growth of a vapor bubble in a superheated liquid may be divided into three stages, the initial, intermediate, and asymptotic stages. Stage is determined by the bubble size. In the initial stage, in which the bubble is very small, its growth is described by a differential equation in which the inertia of liquid, the surface tension, and the vapor pressure have significant influence. After the bubble has begun to grow its size increases rapidly and its growth rate reaches a maximum. After the maximum, the asymptotic stage begins: the growth rate decreases continuously, the bubble grows larger, and the effects of viscosity, inertia force, and surface tension become vanishingly small.

Plesset and Zwick (50-52) formulated a coupled thermal and dynamic problem for a uniformly superheated liquid. An approximate solution was obtained for the prediction of the bubble growth in all three stages by the "thin-thermal-

boundary-layer" concept. Another approximation arrived at by the source theory, has been applied to the bubble growth in superheated liquid by Forster and Zuber (53-56). By means of the similarity transformations, Chambre (57) analyzed the coupled dynamic problem of fluid flow and heat transfer in a viscous liquid surrounding a growing solid that is flat-surfaced, cylindrical, or spherical. Griffith (58) studied hemispherical bubbles growing on a heated surface immersed in a liquid subjected to a linearly decreasing superheat. Scriven, (59) using the similarity transformation, obtained an exact solution for bubble growth in a superheated liquid. Bankoff and Mikesell (60) applied the thin-thermal-boundary-layer concept to the growth of spherical bubbles in a liquid whose temperature distribution, is initially linear or exponential. Bubble growth rates in various superheated liquids have been studied experimentally by Dergarabedian (61). Ellion (62) measured radii and radial velocities of bubbles growing on a heated surface. Skinner and Bankoff, neglecting inertial and surface-tension effects, analyzed the growth of a spherical vapor bubble in a superheated liquid with nonuniform temperature field (63) and in binary liquids with spherically symmetrical temperature and concentration fields (64) during an asymptotic stage in which bubble pressure was approximately constant.

For bubbles growing in subcooled nucleate boiling, theoretical models have been based on heat-transfer-

controlled processes (65-67) and on an inertia-controlled process (68). For bubble collapse in highly subcooled boiling, liquid inertia appears to be the dominant mechanism (65,68). Plesset and Zwick (52) found that the effect of heat transfer was insignificant in retarding the collapse rate compared to the liquid inertia effects. Florshuetz and Chao (69,70) studied vapor bubble collapse based on a liquid-inertia-controlled process, a heat-transfer-controlled process, and the intermediate case in which both effects are of importance. The effects of translatory motion on the bubble collapse were found to be important by both spherical and ellispoidal bubbles (72-74).

By determining the rate of heat transfer from droplets of various sizes in various liquid-liquid situations, and comparing their results with several analytical models, McDowell and Myers (75) and Elzinga and Banchero (76) showed what heat-transfer mechanisms predominated in a given droplet-flow regime. Their studies also disclosed that circulation inside the droplet and oscillation of the droplet increased the heat-transfer process.

The bubble dynamics in non-Newtonian fluids was investigated in References 77 and 78.

Scriven (59) formulated the asymptotic stage of growth for a single spherical vapor bubble in a superheated binary mixture. An exact solution for bubble growth controlled by heat and mass transfer in the liquid phase was obtained by means of the similarity transformation.

The corresponding asymptotic solution was found by the source theory (79), which was in good agreement with the exact solution. In analyzing a problem of the diffusion of a gas into an expanding bubble, it was disclosed that interface temperature and composition remained constant throughout the process of growth. (80,81).

Bubble-growth rates in binary mixtures of water and ethylene glycol were experimentally studied by Benjamin and Westwater (82). In boiling binary mixtures, bubbles grow more slowly, and therefore are smaller, than when any of the pure components constituting the mixture is boiled; this was observed in the study of boiling and burn-out in binary mixtures by Van Stralen et al. (83-86) by Grigorev and Usmanov, (87), and by Yang (88).

Injecting a soluble or insoluble gas into a liquid acts to remove heat from the liquid, partly because the liquid evaporates into the gas bubbles until saturation is reached. One application of this principle is to the cooling of cryogenic systems, especially when local and short-duration subcooling is desired. A lumped system analysis of the injection-cooling process was performed by Larsen et al, (90); it was based on the instantaneous attainment of the equilibrium composition in the gas phase. The source theory of Forster (56) was applied to predict the rates of growth and collapse of a non-condensing, insoluble gas bubble in liquids (79). Larson (91) investigated the same problem both analytically and

experimentally, for both soluble and insoluble gas bubbles. The source theory and the momentum integral are applied to obtain approximate solutions. During the collapse of a soluble gas bubble, the instability or shattering of the bubble was experimentally observed.

The cause of instability of bubble motion in the ellipsoidal regine has been studied by several researchers. Haberman and Morton (23) believed that the oscillatory translating motion is induced by the periodic shedding of vortices behind the bubbles. The work or Magarvey and Bishop (45) seems to support this view.

Strasberg (103) studied experimentally the pulsation frequency of gas bubbles formed on a capillary tip and rising through a liquid. The zeroth-order-pulsation frequency was found in accord with the standard frequency of an oscillating gas bubble in calculation using the acoustic approximation. This frequency was not significantly affected by the lower-frequency oscillations (second and higher order) the bubble displays while departing and rising. By using a quasi-static model neglecting liquid motion, Epstein and Plesset (92) obtained approximate solutions for the growth or collapse of a stationary gas bubble due to mass diffusion in a nonvolatile liquid both when it is oversaturated with the soluble gas and when it is undersaturated. Barlow and Langolis (10) solved the same isothermal problem of bubble growth for a more general case, taking into con-

sideration the inertia of the viscous liquid, surface tension, and variation in the concentration of gas. A numerical solution was obtained by means of a digital computer. Approximate analytical solutions were obtained in closed form for the initial and asymptotic stages.

The problem of rectified diffusion of mass into spherical gas bubble in an oscillating pressure field was studied by Hsieh and Plesset (93). The problem is identical to that in which a spherical gas bubble undergoing an undamped harmonic oscillation. Rectified diffusion is the second-order effect of a net mass flow to the gas phase by diffusion from the liquid phase. It arises because the average transfer area available is greater during the expansion half-cycle, when the mass flow is inward, than during the compression cycle, when the mass flow is outward. Further, an investigation of the Mathieu equation of stability indicated that the second mode of harmonic oscillation is most probable. The analytical expression for the rate of mass transfer was given, and sample calculations confirm that the effect is secondary.

In a theoretical study of inception of cavitation and flashing of liquids in hydraulic equipment, Lienhard (94) treats thermodynamic stability in a superheated liquid of vapor bubbles with and without the presence of a permanent gas. The dynamics of a bubble growing because of evaporation are formulated; the effects of

inertia, surface effects, heat transfer, and liquid viscous dissipation are all included. The case of incipient cavitation is solved numerically by the direction field method; heat transfer is neglected. Also treated is the case of isothermal growth resulting from mass diffusion of dissolved gas in the liquid into the bubble. A solution is obtained for a simplified quasistatic model in the absence of liquid motion.

It is quoted from reference 91: Plesset (95) compares boiling and cavitation bubbles and gives an account of the thermodynamics of oscillating gas bubbles, including the first-order damping effect of heat diffusion in the liquid. Criteria in terms of diffusion lengths, thermal properties, and oscillation frequency are given for the thermodynamic behavior of an oscillating gas bubble in a liquid. The second-order rectified-diffusion phenomenon is treated. In any analysis the stability of a growing or collapsing spherical gas bubble, it is found that bubble growth is stable but bubble collapse is unstable. The instability is unaffected by surface tension; it becomes evident when the bubble radius has been reduced by a factor of ten from its initial value. This agrees with experimental observations of the fragmentation of collapsing bubbles. For bubbles with an appreciable amount of permanent gas, the size may not decrease enough to cause instability.

In certain conditions, a stationary vapor bubble in a superheated binary mixture of water and ethyleneglycol displays oscillatory motion, as observed by Benjamin and Westwater (82). Larsen (91) reported the shattering of moving ammonia gas bubble in water and the transition from rectilinear into oscillatory motion during the growth of a nitrogen bubble in heated water. He found that the effect of translatory motion as an increase in the growth rate does not appear significant until the eventual transition of bubble motion. This is an indication of the importance of the investigation of the instability.

Davies and Taylor (96) studied the mechanics of large bubbles rising through extended liquids. The bubble is in the form of spherical cap. They showed that the flow field around the stagnation point essentially followed the potential flow around a solid sphere. They used the fact that pressure on surface is constant since the pressure inside the bubble is constant, and derived a simple equation for the velocity of the bubble $U = \frac{2}{3}\sqrt{gR}$ where g is the gravitational acceleration and R is the radius of the spherical portion of the bubble. Although their analysis took care only the flow at the vicinity of the stagnation point, while completely ignored the drag caused by the wake behind the bubble, the results agreed very well with experiment results. Saffman (97) used the same ideal to obtain the relation between oblate shape and zig-zag motion of the bubble.

Plesset (98) derived the stability equations for a stationary bubble in a quiescent liquid using a linearized, small perturbation theory. Later Plesset and Mitchell (99) solved these equations under the conditions of constant pressure difference and purely inertia controlled motion. It was disclosed that a growing bubble remained stable, while a collapsing bubble became unstable when the ratio of the radius of the bubble to the initial radius was less than 0.2. Florschuetz and Chao (69,70) investigated the collapse of a vapor bubble under zero g condition. It was found that the instability occurred much earlier than those calculated by Plesset and Mitchell. Naude and Ellis (2) studied theoretically and experimentally the collapse of nonhemispherical cavities in contact with a solid boundary. Hartunian and Sears (3) solved the stability problem of a small bubble moving with a uniform velocity in a quiescent liquid. The critical Weber number was found to be 1.23. Walters and Davidson (4) investigated theoretically and experimentally the initial motion of a gas bubble formed in an inviscid liquid. It was disclosed that the acceleration of a rising bubble is 2g in the initial short period of time. The shape of the bubble was also analyzed. Ivany Hammitt and Mitchell (5) studied experimentally the collapse of a cavitation bubble in a venturi. It was observed that bubbles deformed during collapsing and became a torus-shaped body with a jet of liquid pierced through the center of the body.

CHAPTER III

ANALYSIS

III-1 Dynamics of a bubble moving in a source or sink flow

III-1-a Velocity Potential

For incompressible fluid, the equation of continuity can be written as

$$\nabla \cdot \vec{v} = 0 , \tag{1}$$

where $\vec{v}$ is the velocity vector of the fluid. If the flow is irrotational, then the velocity potential exists, that is

$$\vec{v} = -\nabla \phi$$

and equation (1) becomes

$$\nabla^2 \phi = 0. \tag{2}$$

For a flow around a point source (or sink), the solution of equation (2) is well known: In spherical coordinates with the origin 0 located at the source (or sink), as shown in Figure 1, the velocity potential at an arbitrary point $P(r', \theta', \varphi')$ in the flow field may be expressed as

$$\phi = \frac{\mu}{r'} \tag{3}$$

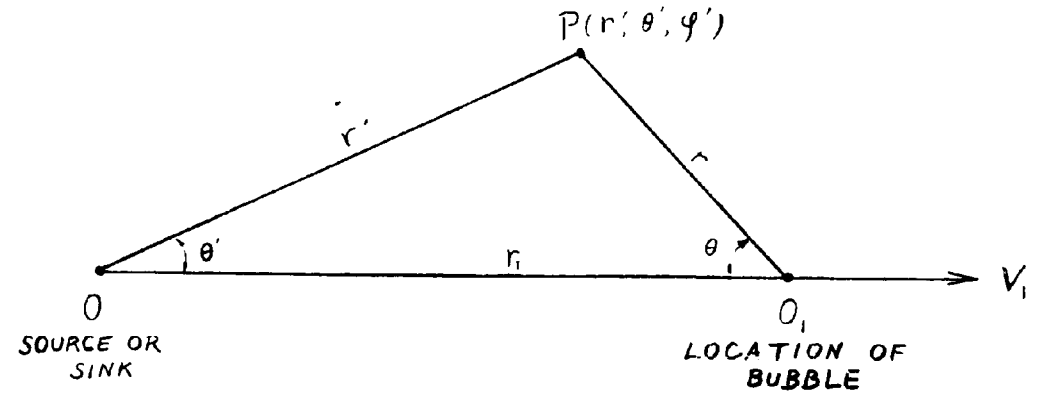


Figure 1. Coordinate systems for analyzing the dynamics of a bubble moving in a source or sink flow

where μ is the strength of the source if positive (the strength of the sink if negative) and r' is the radial distance from the source (or sink) to P. Now, let the coordinates be transformed into a new coordinate system with the origin at O_1 located at $r'=r_1$, as shown in Figure 1, and new independent variables r and θ. Then equation (3) may be rewritten as

$$\phi_1 = \frac{\mu}{[r^2 + r_1^2 - 2 r_1 r \cos\theta]^{1/2}}$$

$$= \begin{cases} \mu \sum_{n=0}^{\infty} \frac{r^n}{r_1^{n+1}} P_n(\cos\theta) & \text{for } r < r_1 \\ \mu \sum_{n=0}^{\infty} \frac{r_1^n}{r^{n+1}} P_n(\cos\theta) & \text{for } r > r_1 \end{cases}$$

or

$$= \mu \sum_{n=0}^{\infty} \frac{r_<^n}{r_>^{n+1}} P_n(\cos\theta), \qquad (4)$$

where $P_n(\cos\theta)$ are Legendre polynomials and $r_<$ and $r_>$ denote the smaller and the larger quantities of r and r_1, respectively. In case the new origin O_1 moves with the velocity V_1, as shown in Figure 1, an additional term, $-V_1 r\cos\theta$ must be added to the right side of equation (4), that is, the velocity potential in the moving coordinate system is

$$\phi_2 = \mu \sum_{n=0}^{\infty} \frac{r_<^n}{r_>^{n+1}} P_n(\cos\theta) - V_1 r \cos\theta. \qquad (5)$$

If there exists a spherical bubble in the flow field and the center of the bubble coincides the origin O_1 moving at the velocity V_1, then the velocity potential in the flow field can be obtained by superposing the velocity potential $\sum_{n=0}^{\infty} b_n \frac{1}{r^{n+1}} P_n$ induced by the bubble on

the right side of equation (5). The method of superposition is possible because equation (2) is linear. Now one has

$$\phi_3 = \mu \sum_{n=0}^{\infty} \frac{r_<^n}{r_>^{n+1}} P_n(\cos\theta) - V_1 r\cos\theta + \sum_{n=0}^{\infty} b_n \frac{1}{r^{n+1}} P_n \qquad (6)$$

as the velocity potential in the flow field, where b_n are the constant coefficients to be determined in the following. Let the instantaneous radius of the bubble be $R(t)$ which varies with time t. Then it is required that the condition

$$\dot{R} = -\frac{\partial \phi}{\partial r}\Big)_{r=R} \qquad (7)$$

be satisfied on the bubble surface, where the dot "." denotes the time derivative. Equation (7) may be employed to determine the unknown coefficients b_n. The substitution of equation (7) into equation (6) yields

$$\phi_3 = \frac{R^2\dot{R}}{r} + \frac{R^3}{2r}\left(\frac{\mu}{r_1^2} - V_1\right)\cos\theta - V_1 r\cos\theta + \sum_{n=2}^{\infty} \frac{n\mu R^{2n+1}}{(n+1) r_1^{n+1} r^{n+1}} P_n + \mu \sum_{n=0}^{\infty} \frac{r_<^n}{r_>^{n+1}} P_n . \qquad (8)$$

For lack of the spherical symmetry in the flow field, an initially spherical bubble cannot, in general, retain its spherical shape as it travels along the flow. Therefore the shape of the bubble is assumed to take the form of

$$r_s = R(t) + \sum_{n=1}^{\infty} a_n(t) P_n(\cos\theta) \qquad (9)$$

where r_s is the distance from the center to the surface of the bubble and a_n are the time-dependent but still unknown coefficients whose square are to be neglected. The terms $\sum_{n=1}^{\infty} a_n(t) P_n(\cos\theta)$ in equation (9) express the amount by which the bubble deviates from its spherical shape.

Now, the surface of the perturbed bubble may also be expressed by the equation

$$F(r_s, \theta, t) = r_s - R - \sum_{n=1}^{\infty} a_n P_n = 0 \tag{10}$$

Then, the condition

$$\frac{DF}{Dt} = 0 \tag{11}$$

has to be satisfied on the bubble surface, where D/Dt is the total derivative. It can be shown that a special case of equation (11) may be reduced to equation (7).

If the bubble travelling with the flow at O_1 in Figure 1 is expressed by equation (9), then the velocity potential in the flow field becomes

$$\phi = \phi_3 + \sum_{n=1}^{\infty} c_n \frac{1}{r^{n+1}} P_n , \tag{12}$$

where c_n is the coefficients which can be expressed in terms of a_n as will be shown in the following.

Following the definition of the total derivative, one can rewrite equation (9) as

$$\begin{aligned} \frac{DF}{Dt} &= \frac{\partial F}{\partial t} + (V_\theta)_{r=r_s} \frac{1}{r_s} \frac{\partial F}{\partial \theta} + (V_r)_{r=r_s} \frac{\partial F}{\partial r_s} \\ &= \dot{R} + \sum_{n=1}^{\infty} \dot{a}_n P_n - (V_\theta)_{r=r_s} \frac{\sin\theta}{r_s} \sum_{n=1}^{\infty} a_n P_n' - (V_r)_{r=r_s} \end{aligned} \tag{13}$$

where the prime denotes the derivative with respective to the argument of P_n, namely $\cos\theta$.

The velocity components V_θ and V_r in equation (13) can be found from the velocity potential (12) as

$$V_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

$$= -\frac{1}{r}\Big[-\mu \sum_{n=0}^{\infty} \frac{r_s^n}{r_s^{n+1}} \sin\theta P_n' + V_1 r \sin\theta - \frac{R^3}{2r^2}\left(\frac{\mu}{r_1^2} - V_1\right)\sin\theta - \mu \sum_{n=0}^{\infty} \frac{n}{n+1} \frac{R^{2n+1}}{r_1^{n+1} r^{n+1}} \sin\theta P_n' - \sum_{n=1}^{\infty} C_n \frac{1}{r^{n+1}} \sin\theta P_n'\Big] \tag{14}$$

$$V_r = -\frac{\partial \phi}{\partial r}$$

$$= V_1 \cos\theta + \frac{R^2 \dot{R}}{r^2} + \frac{R^3}{r^3}\left(\frac{\mu}{r_1^2} - V_1\right)\cos\theta \begin{cases} +\mu \sum_{n=0}^{\infty} (n+1) \frac{r_1^n}{r^{n+2}} P_n & \text{for } r > r_1 \\ -\mu \sum_{n=1}^{\infty} n \frac{r^{n-1}}{r_1^{n+1}} P_n & \text{for } r < r_1 \end{cases}$$

$$+ \sum_{n=2}^{\infty} n \frac{\mu R^{2n+1}}{r_1^{n+1} r^{n+1}} P_n + \sum_{n=0}^{\infty} (n+1) C_n \frac{1}{r^{n+2}} P_n \tag{15}$$

To evaluate V_θ and V_r at the bubble surface $r = r_s$, one substitutes equation (9) into equations (14) and (15), followed by neglecting square and higher order terms of a_n There results

$$(V_\theta)_{r=r_s} = \frac{3}{2}\left(1 - \sum_{n=1}^{\infty} \frac{a_n}{R} P_n\right)\left(\frac{\mu}{r_1^2} - V_1\right)\sin\theta + \mu \sum_{n=2}^{\infty} \frac{2n+1}{n+1}\left(1 - \sum_{n=1}^{\infty} \frac{a_n}{R} P_n\right)\frac{R^{n-1}}{r_1^{n+1}} \sin\theta P_n' + \sum_{n=1}^{\infty} \frac{C_n}{R^{n+2}} \sin\theta P_n' \tag{16}$$

$$(V_r)_{r=r_s} = V_1 \cos\theta + \dot{R}\left(1 - 2\sum_{n=1}^{\infty} \frac{a_n}{R} P_n\right) + \left(\frac{\mu}{r_1^2} - V_1\right)\left(1 - 3\sum_{n=1}^{\infty} \frac{a_n}{R} P_n\right)\cos\theta + 3\mu \sum_{n=1}^{\infty} n \frac{R^{n-1}}{r_1^{n+1}}\left(\sum_{m=1}^{\infty} \frac{a_m}{R} P_m\right) P_n + \sum_{n=0}^{\infty} (n+1) \frac{C_n}{R^{n+2}} P_n \tag{17}$$

where the recurrence formulas for $\cos\theta P_n$ in Appendix II has been used. To avoid undue mathematical complication,

the second- and higher-order terms of R/r_1 are neglected. This is justified so long as the equivalent radius R of the bubble is small in comparison with the distance r_1 between the bubble and the source (or sink). It follows that the second term from the last on the right side of the equation (17) can be reduced to

$$\sum_{n=2}^{\infty} n(2n+1)\frac{\mu R^{n-1}}{r_1^{n+1}} P_n \sum_{m=1}^{\infty} \frac{a_m}{R} P_m$$

$$\approx 15\frac{\mu R}{r_1^3}\sum_{n=1}^{\infty}\frac{a_n}{R}\left\{\frac{(n+1)(n+2)}{(2n+1)(2n+3)}P_{n+2} + \left[\frac{(n+1)^2}{(2n+1)(2n+3)} - \frac{1}{3}\right] + \frac{n(n-1)}{(2n-1)(2n+1)}P_{n+2}\right\} \qquad (18)$$

where the recurrence relations for $P_2 P_n$ in Appendix II has been used. With the substitutions of equations (16) (17) and (18) into equation (13) followed by neglecting the terms involving a_n^2 and using the recurrence formulas for $\sin^2\theta P_n'$ in Appendix II, one obtains

$$\frac{DF}{Dt} = \sum_{n=1}^{\infty}\dot{a}_n P_n + \frac{3}{2}\left(\frac{\mu}{r_1^2} - V_1\right)\sum_{n=1}^{\infty}\frac{a_n}{R}\frac{n(n+1)}{2n+1}\left(P_{n+1} - P_{n-1}\right)$$

$$+ 5\frac{\mu}{r_1^3}\sum_{n=1}^{\infty} a_n\left[\frac{n(n+1)(n+2)}{(2n+1)(2n+3)}P_{n+2} - \frac{n(n+1)}{(2n-1)(2n+3)}P_n - \frac{n(n+1)(n-1)}{(2n+1)(2n-1)}P_{n-2}\right]$$

$$+ 2\dot{R}\sum_{n=1}^{\infty}\frac{a_n}{R}P_n + 3\left(\frac{\mu}{r_1^2} - V_1\right)\sum_{n=1}^{\infty}\frac{a_n}{R}\left(\frac{n+1}{2n+1}P_{n+1} + \frac{n}{2n+1}P_{n-1}\right) - \sum_{n=0}^{\infty}(n+1)\frac{c_n}{R^{n+2}}P_n$$

$$+ 2\frac{\mu R}{r_1^3}\sum_{n=1}^{\infty}\frac{a_n}{R}\left[\frac{3}{2}\frac{(n+1)(n+2)}{(2n+1)(2n+3)}P_{n+2} + \frac{n(n+1)}{(2n-1)(2n+3)}P_n + \frac{3}{2}\frac{n(n-1)}{(2n-1)(2n+1)}P_{n-2}\right]$$

$$+ 8\frac{\mu R}{r_1^3}\sum_{n=1}^{\infty}\frac{a_n}{R}\left[\frac{3}{2}\frac{(n+1)(n+2)}{(2n+1)(2n+3)}P_{n+2} + \frac{n(n+1)}{(2n-1)(2n-3)}P_n + \frac{3}{2}\frac{n(n-1)}{(2n-1)(2n+1)}P_{n-2}\right] \qquad (19)$$

For convenience, let the $\sum_{n=1}^{\infty} c_n \frac{1}{r^{n+1}} P_n$ terms in equation (12) be rewritten as

$$\sum_{n=1}^{\infty}\frac{c_n}{r^{n+1}}P_n = \sum_{n=3}^{\infty}\frac{d_{n-2}}{r^{n-1}}P_{n-2} + \sum_{n=2}^{\infty}\frac{e_{n-1}}{r^n}P_{n-1} + \sum_{n=1}^{\infty}\frac{f_n}{r^{n+1}}P_n$$

$$+ \sum_{n=0}^{\infty}\frac{g_{n+1}}{r^{n+2}}P_{n+1} + \sum_{n=0}^{\infty}\frac{h_{n+2}}{r^{n+3}}P_{n+2}, \qquad (20)$$

where d_{n-2}, e_{n-1}, f_n, g_{n+1}, and h_{n+2} are the coefficients to be determined as follows. Now the $\sum_{n=1}^{\infty}(n+1)\frac{c_n}{R^{n+2}}P_n$ terms in equation (19) become

$$\sum_{n=1}^{\infty}(n+1)\frac{c_n}{R^{n+2}}P_n = \sum_{n=3}^{\infty}(n-1)\frac{d_{n-2}}{R^n}P_{n-2} + \sum_{n=2}^{\infty} n\frac{e_{n-1}}{R^{n+1}}P_{n-1}$$

$$+ \sum_{n=0}^{\infty}(n+1)\frac{f_n}{R^{n+2}}P_n + \sum_{n=0}^{\infty}(n+2)\frac{g_{n+1}}{R^{n+3}}P_{n+1} + \sum_{n=0}^{\infty}(n+3)\frac{h_{n+2}}{R^{n+4}}P_{n+2} \qquad (21)$$

Substituting equation (21) into equation (19) followed by comparing terms one obtains

$$d_{n-2} = -\frac{5(n-2)n}{(2n-1)(2n+1)}\frac{\mu R^n}{r_1^3}a_n$$

$$e_{n-1} = -\frac{1}{2}\frac{n(n-1)}{2n+1}R^n\left(\frac{\mu}{r_1^2} - V_1\right)a_n$$

$$f_n = \frac{R^{n+2}}{n+1}\left(\dot{a}_n + 2\dot{R}\frac{a_n}{R}\right) + \frac{5}{n+1}\frac{\mu R^{n+2}}{r_1^3}a_n\left[\frac{n(n+1)}{2n+1}\left(\frac{n+1}{2n+3} - \frac{n}{2n-1}\right)\right.$$

$$\left. + 3\left\{\frac{(n+1)^2}{(2n+1)(2n+3)} + \frac{n^2}{(2n-1)(2n+1)} - \frac{1}{3}\right\}\right]$$

$$g_{n+1} = \frac{3}{2}\frac{n+1}{2n+1}R^{n+2}\left(\frac{\mu}{r_1^2} - V_1\right)a_n$$

$$h_{n+2} = \frac{15(n+1)(n+2)}{(n+3)(2n+1)(2n+3)}\frac{\mu R^{n+4}}{r_1^3}a_n$$

By substituting these expressions into equation (20), one can write equation (12) as

$$\phi = -V_1 r\cos\theta + \frac{R^2\dot{R}}{r} + \frac{R^3}{2r^2}\left(\frac{\mu}{r_1^2} - V_1\right)\cos\theta + \sum_{n=0}^{\infty}\frac{\mu r_2^n}{r_2^{n+1}}P_n$$

$$+ \sum_{n=2}^{\infty}\frac{n}{n+1}\frac{\mu R^{2n+1}}{r_1^{n+1}r^{n+1}}P_n - \sum_{n=3}^{\infty}\frac{5n(n-2)}{(2n-1)(2n+1)}\frac{\mu R^n}{r_1^3 r^{n-1}}P_{n-2}$$

$$- \sum_{n=2}^{\infty}\frac{1}{2}\frac{n(n-1)}{2n+1}\frac{R^n}{r^n}\left(\frac{\mu}{r_1^2} - V_1\right)a_n P_{n-1} + \sum_{n=1}^{\infty}\frac{5n}{(2n-1)(2n+3)}\frac{\mu R^{n+2}}{r_1^3 r^{n+1}}a_n P_n$$

$$+\sum_{n=1}^{\infty}\frac{1}{n+1}\frac{R^{n+2}}{r^{n+1}}\left(\dot{a}_n+2\dot{R}\frac{a_n}{R}\right)P_n+\frac{3}{2}\sum_{n=1}^{\infty}\frac{n+1}{2n+1}\frac{R^{n+2}}{r^{n+2}}\left(\frac{\mu}{r_1^2}-V_1\right)a_n P_{n+1}$$

$$+15\sum_{n=1}^{\infty}\frac{(n+1)(n+2)}{(n+3)(2n+1)(2n+3)}\frac{\mu R^{n+4}}{r_1^3 r^{n+3}}a_n P_{n+2} \qquad (22)$$

III-1-b Pressure distribution on a bubble surface

The equation of motion in an accelerating coordinate system may be written as

$$\frac{\partial\vec{V}}{\partial t}+(\vec{V}\cdot\nabla)\vec{V}=-\frac{1}{\rho}\nabla p-\vec{a}, \qquad (23)$$

where $\vec{V}$ is the velocity vector, ρ is the density of the fluid, p is the pressure and $\vec{a}$ is the acceleration of the coordinate system. It must be noted that the gravitational acceleration has been neglected in equation (23). According to vector analysis, one can write

$$(\vec{V}\cdot\nabla)\vec{V}=\nabla\left(\frac{V^2}{2}\right)-\vec{V}\times(\nabla\times\vec{V}). \qquad (24)$$

For an irrotational flow, the second term on the right hand side of equation (24) becomes identically zero. Thus, equation (23) becomes

$$\nabla\left(-\frac{\partial\phi}{\partial t}+\frac{V^2}{2}+\frac{p}{\rho}\right)=-\vec{a}. \qquad (25)$$

The r-component of (25) is

$$\frac{\partial}{\partial r}\left(-\frac{\partial\phi}{\partial t}+\frac{V^2}{2}+\frac{p}{\rho}\right)=-a_r=\frac{dV_1}{dt}\cos\theta, \qquad (26)$$

where a_r is equal to $-\frac{dV_1}{dt}\cos\theta$ in the accelerating coordinate system shown in Figure 1. Equation (26) is then integrated from $r=r_1$ to $r=\infty$ to obtain the pressure

distribution on the bubble surface. It yields

$$p(r_s) = p(\infty) + \rho\left[-\frac{\partial\phi}{\partial t} + \frac{v^2}{2}\right]_{r=r_s}^{r=\infty} - \rho\,\frac{dV_1}{dt}\,r\cos\theta\Big|_{r=r_s}^{r=\infty} \qquad (27)$$

The inertia terms in equation (27) can be evaluated from equation (22): By differentiating equation (22) with respect to time, there results

$$\frac{\partial\phi}{\partial t} = -\frac{dV_1}{dt}\,r\cos\theta + \frac{R}{r}\left(2\dot{R}^2 + R\ddot{R}\right) + \frac{3}{2}\,R^2\dot{R}\left(\frac{\mu}{r_1^2} - V_1\right)\frac{1}{r^2}\cos\theta$$

$$+ \begin{cases} -\sum_{n=0}^{\infty}(n+1)\,\dfrac{\mu\, r^n}{r_1^{n+2}}\,V_1 P_n & \text{for } r < r_1 \\ \sum_{n=1}^{\infty} n\,\dfrac{\mu\, r_1^{n-1}}{r^{n+1}}\,V_1 P_n & \text{for } r > r_1 \end{cases}$$

$$- \frac{R^3}{2r^2}\left(2\frac{\mu}{r_1^3}V_1 - \frac{dV_1}{dt}\right)\cos\theta - \sum_{n=2}^{\infty} n\,\frac{\mu R^{2n+1}}{r_1^{n+2}\,r^{n+1}}\,V_1 P_n$$

$$- \sum_{n=3}^{\infty}\frac{n(2n+1)}{n+1}\,\frac{\mu R^{2n}\dot{R}}{r_1^{n+1}\,r^{n+1}}P_n - \sum_{n=3}^{\infty}\frac{5(n-2)n}{(2n-1)(2n+1)}\left(n\,\frac{\mu R^{n-1}\dot{R}}{r_1^3}\,a_n - 3\,\frac{\mu R^n}{r_1^4}V_1 a_n\right.$$

$$\left. + \frac{\mu R^n}{r_1^3}\dot{a}_n\right)\frac{1}{r^{n+1}}P_{n-2} - \frac{1}{2}\sum_{n=2}^{\infty}\frac{n(n-1)}{2n+1}\left[nR^{n-1}\dot{R}\left(\frac{\mu}{r^2} - V_1\right)a_n\right.$$

$$\left. - R^n\left(2\frac{\mu}{r_1^3}V_1 - \frac{dV_1}{dt}\right)a_n + R^n\left(\frac{\mu}{r_1^2} - V_1\right)\dot{a}_n\right]\frac{1}{r^n}P_{n-1}$$

$$+ 5\sum_{n=1}^{\infty}\frac{n}{(2n-1)(2n+3)}\left[(n+2)R^{n+1}\dot{R}\,\frac{\mu}{r_1^3}\,a_n - 3R^{n+2}\,\frac{\mu}{r_1^4}\,V_1\,a_n\right.$$

$$\left. + R^{n+2}\,\frac{\mu}{r_1^3}\,\dot{a}_n\right]\frac{1}{r^{n+1}}P_n + \sum_{n=1}^{\infty}\left[\frac{n+2}{n+1}\,R^{n+1}\,\dot{R}\left(\dot{a}_n + 2\dot{R}\,\frac{a_n}{R}\right)\right.$$

$$\left. + \frac{R^{n+2}}{n+1}\left(\ddot{a}_n + 2\ddot{R}\,\frac{a_n}{R} + 2\dot{R}\,\frac{\dot{a}_n}{R} - 2\dot{R}^2\,\frac{a_n}{R^2}\right)\right]\frac{1}{r^{n+1}}P_n$$

$$+ \sum_{n=1}^{\infty}\frac{3}{2}\,\frac{n+1}{2n+1}\left[(n+2)R^{n+1}\dot{R}\left(\frac{\mu}{r_1^2} - V_1\right)a_n - R^{n+2}\left(\frac{\mu}{r_1^3}V_1 - \frac{dV_1}{dt}\right)a_n\right.$$

$$\left. + R^{n+1}\left(\frac{\mu}{r_1^2} - V_1\right)\dot{a}_n\right]\frac{1}{r^{n+2}}\,P_{n+1} \qquad (28)$$

By differentiating equation (22) with respect to space variable r and θ, respectively there results

$$\frac{\partial \phi}{\partial r} = -V_1 \cos\theta - \frac{R^2 \dot{R}}{r^2} + \frac{R^3}{r^3}\left(\frac{\mu}{r_1^2} - V_1\right)\cos\theta$$

$$+ \begin{cases} -\mu \sum_{n=0}^{\infty} (n+1) \frac{r_1^n}{r^{n+2}} P_n & \text{for } r > r_1 \\ \mu \sum_{n=1}^{\infty} n \frac{r^{n-1}}{r_1^{n+1}} P_n & \text{for } r < r_1 \end{cases}$$

$$- \sum_{n=2}^{\infty} n \frac{\mu R^{2n+1}}{r_1^{n+1} r^{n+2}} P_n + \sum_{n=3}^{\infty} \frac{5(n-2)n(n-1)}{(2n-1)(2n+1)} \frac{\mu R^n}{r_1^3 r^n} a_n P_{n-2}$$

$$+ \frac{1}{2} \sum_{n=2}^{\infty} \frac{n^2(n-1)}{2n+1} \frac{R^n}{r^{n+1}} \left(\frac{\mu}{r_1^2} - V_1\right) a_n P_{n-1}$$

$$- 5 \sum_{n=1}^{\infty} \frac{n(n+1)}{(2n-1)(2n+3)} \frac{\mu R^{n+2}}{r_1^3 r^{n+2}} a_n P_n - \sum_{n=1}^{\infty} \frac{R^{n+2}}{r^{n+2}} \left(\dot{a}_n + 2\dot{R} \frac{a_n}{R}\right) P_n$$

$$- \frac{3}{2} \sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{2n+1} \frac{R^{n+2}}{r^{n+3}} \left(\frac{\mu}{r_1^2} - V_1\right) a_n P_{n+1}$$

$$- 15 \sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{(2n+1)(2n+3)} \frac{\mu R^{n+4}}{r_1^3 r^{n+4}} a_n P_{n+2} \qquad (29)$$

$$\frac{1}{r}\frac{\partial \phi}{\partial \theta} = V_1 \sin\theta - \frac{R^3}{2r^3}\left(\frac{\mu}{r_1^2} - V_1\right)\sin\theta - \sum_{n=0}^{\infty} \frac{\mu r_2^{n-1}}{r_>^{n+1}} \sin\theta P_n'$$

$$+ 5 \sum_{n=} \frac{n}{n+1} \frac{\mu R^{2n+1}}{r_1^{n+1} r^{n+2}} \sin\theta P_n' + 5 \sum_{n=3}^{\infty} \frac{(n-2)n}{(2n-1)(2n+1)} \frac{\mu R^n}{r_1^3 r^n} a_n \sin\theta P_{n-2}'$$

$$+ \frac{1}{2} \sum_{n=2}^{\infty} \frac{n(n-1)}{2n+1} \frac{R^n}{r^{n+1}} \left(\frac{\mu}{r^2} - V_1\right) a_n \sin\theta P_{n-1}'$$

$$- 5 \sum_{n=1}^{\infty} \frac{n}{(2n-1)(2n+3)} \frac{\mu R^{n+2}}{r_1^3 r^{n+2}} a_n \sin\theta P_n'$$

$$- \sum_{n=1}^{\infty} \frac{1}{n+1} \frac{R^{n+2}}{r^{n+2}} \left(\dot{a}_n + 2\dot{R} \frac{a_n}{R}\right) \sin\theta P_n'$$

$$- \frac{3}{2} \sum_{n=1}^{\infty} \frac{n+1}{2n+1} \frac{R^{n+2}}{r^{n+3}} \left(\frac{\mu}{r_1^2} - V_1\right) a_n \sin\theta P_{n+1}'$$

$$- 15 \sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{(n+3)(2n+1)(2n+3)} \frac{\mu R^{n+4}}{r_1^3 r^{n+4}} a_n \sin\theta P_{n+2}' \qquad (30)$$

The kinetic energy per unit mass is

$$\frac{V^2}{2} = \frac{1}{2}(-\nabla\phi)^2 = \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)^2\right] \qquad (31)$$

At $r=\infty$ since all terms except the first one become identically zero, equation (28) gives

$$\left.\frac{\partial\phi}{\partial t}\right)_{r=\infty} = -\left.\frac{dV_1}{dt} r\cos\theta\right|_{r=\infty} \tag{32}$$

Therefore, the $-\rho\left(\frac{\partial\phi}{\partial t}\right)_{r=\infty}$ and $-\rho\left(\frac{dV_1}{dt} r\cos\theta\right)_{r=\infty}$ terms on the right side of equation (27) may cancel each other. At $r=\infty$ equation (31) reduces to

$$\frac{V^2}{2} = \frac{1}{2}\left[(V_1\cos\theta)^2 + (V_1\sin\theta)^2\right] = \frac{1}{2}V_1^2 \tag{33}$$

The expression for the inertia terms at $r=r_s$ can be obtained by substituting equation (9) into equations (28), (29) and (30) followed by neglecting the second- and higher-order terms of a_n and R/r_1. Equation (28) yields

$$\begin{aligned}
\left.\frac{\partial\phi}{\partial t}\right|_{r=r_s} &= 2\dot{R}^2 + R\ddot{R} - \frac{\mu}{r_1^2}V_1 + \left[-\frac{3}{2}\frac{dV_1}{dt}R + \frac{3}{2}\dot{R}\left(\frac{\mu}{r_1^2} - V_1\right)\right.\\
&\left. - 3\frac{\mu R}{r_1^3}V_1\right]\cos\theta - \sum_{n=2}^{\infty}(2n+1)\frac{\mu R^n}{r_1^{n+2}}V_1 P_n\\
&- (2\dot{R}^2 + R\ddot{R})\sum_{n=1}^{\infty}\frac{a_n}{R}P_n - 3\dot{R}\left(\frac{\mu}{r_1^2} - V_1\right)\sum_{n=1}^{\infty}\frac{a_n}{R}\left(\frac{n+1}{2n+1}P_{n+1}\right.\\
&\left. + \frac{n}{2n+1}P_{n-1}\right) - 10\frac{\mu R\dot{R}}{r_1^3}\sum_{n=1}^{\infty}\frac{a_n}{R}\left[\frac{3}{2}\frac{(n+1)(n+2)}{(2n+1)(2n+3)}P_{n+2}\right.\\
&\left. + \frac{n(n+1)}{(2n-1)(2n+3)}P_n + \frac{3}{2}\frac{n(n-1)}{(2n-1)(2n+1)}P_{n-2}\right]\\
&- 5\sum_{n=3}^{\infty}\frac{n(n-2)}{(2n-1)(2n+1)}\left(n\frac{\mu R^{n-1}\dot{R}}{r_1^3}a_n - 3\frac{\mu R^n}{r_1^4}V_1 a_n\right.\\
&\left. + \frac{\mu R^n}{r_1^3}\dot{a}_n\right)\frac{1}{R^{n-1}}P_{n-2} - \frac{1}{2}\sum_{n=2}^{\infty}\frac{n(n-1)}{2n+1}\left[nR^{n-1}\dot{R}\left(\frac{\mu}{r_1^2} - V_1\right)a_n\right.\\
&\left. - R^n\left(2\frac{\mu}{r_1^3}V_1 - \frac{dV_1}{dt}\right)a_n + R^n\left(\frac{\mu}{r_1^2} - V_1\right)\dot{a}_n\right]\frac{1}{R^n}P_n\\
&+ 5\sum_{n=1}^{\infty}\frac{n}{(2n-1)(2n+3)}\left[(n+2)\dot{R}\frac{\mu}{r_1^3}a_n - 3\frac{\mu}{r_1^4}RV_1 a_n + R\frac{\mu}{r_1^3}\dot{a}_n\right]P_n
\end{aligned}$$

$$+\sum_{n=1}^{\infty}\left[\frac{n+2}{n+1}\dot{R}\left(\dot{a}_n+2\dot{R}\frac{a_n}{R}\right)+\frac{R}{n+1}\left(\ddot{a}_n+2\ddot{R}\frac{a_n}{R}+2\dot{R}\frac{a_n}{R}\right.\right.$$

$$\left.\left.-2\dot{R}^2\frac{a_n}{R}\right)\right]P_n+\frac{3}{2}\sum_{n=1}^{\infty}\frac{n+1}{2n+1}\left[(n+2)\frac{\dot{R}}{R}\left(\frac{\mu}{r_1^2}-V_1\right)a_n-\left(\frac{\mu}{r_1^3}V_1-\frac{dV_1}{dt}\right)a_n\right.$$

$$\left.+\left(\frac{\mu}{r_1^2}-V_1\right)a_n\right]P_{n+1}+15\sum_{n=1}^{\infty}\frac{(n+1)(n+2)}{(n+3)(2n+1)(2n+3)}\left[(n+4)\frac{\mu\dot{R}}{r_1^3}a_n\right.$$

$$\left.-\frac{\mu R}{r_1^4}V_1a_n+\frac{\mu R}{r_1^3}\dot{a}_n\right]P_{n+2} \qquad (34)$$

Similarly equations (29) and (30) yield

$$\left(\frac{\partial\phi}{\partial r}\right)_{r=r_s}=-\dot{R}+2\dot{R}\sum_{n=1}^{\infty}\frac{a_n}{R}P_n+3\left(\frac{\mu}{r_1^2}-V_1\right)\sum_{n=1}^{\infty}\frac{a_n}{R}\left(\frac{n+1}{2n+1}P_{n+1}+\frac{n}{2n+1}P_{n-1}\right)$$

$$+10\frac{\mu}{r_1^3}\sum_{n=1}^{\infty}a_n\left(\frac{3}{2}\frac{(n+1)(n+2)}{(2n+1)(2n+3)}P_{n+2}+\frac{n(n+1)}{(2n-1)(2n+3)}P_n\right.$$

$$\left.+\frac{3}{2}\frac{n(n-1)}{(2n-1)(2n+1)}P_{n-2}\right)+5\sum_{n=3}^{\infty}\frac{(n-2)(n-1)n}{(2n-1)(2n+1)}\frac{\mu}{r_1^3}a_nP_{n-2}$$

$$+\frac{1}{2}\sum_{n=2}^{\infty}\frac{n^2(n-1)}{2n+1}\left(\frac{\mu}{r_1^2}-V_1\right)\frac{a_n}{R}P_{n-1}-5\sum_{n=1}^{\infty}\frac{n(n+1)}{(2n-1)(2n+3)}\frac{\mu}{r_1^3}a_nP_n$$

$$-\sum_{n=1}^{\infty}\left(\dot{a}_n+2\dot{R}\frac{a_n}{R}\right)P_n-15\sum_{n=1}^{\infty}\frac{(n+2)(n+1)}{(2n+1)(2n+3)}\frac{\mu}{r_1^3}a_nP_{n+2}$$

$$-\frac{3}{2}\sum_{n=1}^{\infty}\frac{(n+1)(n+2)}{2n+1}\left(\frac{\mu}{r_1^2}-V_1\right)\frac{a_n}{R}P_{n+1} \qquad (35)$$

and

$$\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)_{r=r_s}=-\frac{3}{2}\left(\frac{\mu}{r_1^2}-V_1\right)\sin\theta-\sum_{n=2}^{\infty}\frac{2n+1}{n+1}\frac{\mu R^{n-1}}{r_1^{n+1}}\sin\theta P_n'$$

$$+\frac{3}{2}\left(\frac{\mu}{r_1^2}-V_1\right)\sum_{n=1}^{\infty}\frac{a_n}{R}\sin\theta P_n'+\frac{5}{3}\frac{\mu}{r_1^3}\sum_{n=1}^{\infty}a_n\sin\theta P_nP_2'$$

$$+5\sum_{n=3}^{\infty}\frac{(n-2)n}{(2n-1)(2n+1)}\frac{\mu}{r_1^3}a_n\sin\theta P_{n+2}'$$

$$+\frac{1}{2}\sum_{n=2}^{\infty}\frac{n(n-1)}{2n+1}\left(\frac{\mu}{r_1^2}-V_1\right)\frac{a_n}{R}\sin\theta P_{n-1}'$$

$$-5\sum_{n=1}^{\infty}\frac{n}{(2n-1)(2n+3)}\frac{\mu}{r_1^3}a_n\sin\theta P_n'$$

$$- \sum_{n=1}^{\infty} \frac{1}{n+1} \left(\dot{a}_n + 2\dot{R} \frac{a_n}{R} \right) \sin\theta P_n'$$

$$- \frac{3}{2} \sum_{n=1}^{\infty} \frac{n+1}{2n+1} \left(\frac{\mu}{r_1^2} - V_1 \right) \frac{a_n}{R} \sin\theta P_{n+1}'$$

$$- 15 \sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{(n+3)(2n+1)(2n+3)} \frac{\mu}{r_1^3} a_n \sin\theta P_{n+2}' \qquad (36)$$

respectively. Therefore with the aid of equations (35) and (36) and the recurrence relations for $\sin^2\theta P_n'$, $P_2 P_n$, $\sin^2\theta P_2' P_n'$, $\sin^2\theta P_n$ and $\sin^2\theta P_2'^2 P_n$ from Appendix II, the kinetic energy per unit mass at $r = r_s$ may be written as

$$\frac{1}{2}\left[\left(\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2 \right]_{r=r_s} = -\frac{1}{2}\dot{R}^2 + 2\dot{R}^2 \sum_{n=1}^{\infty} \frac{a_n}{R} P_n + 3\dot{R}\left(\frac{\mu}{r_1^2} - V_1\right) \sum_{n=1}^{\infty} \frac{a_n}{R} \left(\frac{n+1}{2n+1} P_{n+1} \right.$$

$$\left. + \frac{n}{2n+1} P_{n-1} \right) + 10\dot{R} \frac{\mu}{r_1^3} \sum_{n=1}^{\infty} a_n \left[\frac{3}{2} \frac{(n+1)(n+2)}{(2n+1)(2n+3)} P_{n+2} + \frac{n(n+1)}{(2n-1)(2n+3)} P_n \right.$$

$$\left. + \frac{3}{2} \frac{n(n-1)}{(2n-1)(2n+1)} P_{n-2} \right] + 5\dot{R} \sum_{n=3}^{\infty} \frac{n(n-2)(n-1)}{(2n-1)(2n+1)} \frac{\mu}{r_1^3} a_n P_{n-2}$$

$$+ \frac{1}{2}\dot{R} \sum_{n=2}^{\infty} \frac{n^2(n-1)}{2n+1} \left(\frac{\mu}{r_1^2} - V_1 \right) \frac{a_n}{R} P_{n-1} - 5\dot{R} \sum_{n=1}^{\infty} \frac{n(n+1)}{(2n-1)(2n+3)} \frac{\mu}{r_1^3} a_n P_n$$

$$- \dot{R} \sum_{n=1}^{\infty} \left(\dot{a}_n + 2\dot{R} \frac{a_n}{R} \right) P_n - \frac{3}{2}\dot{R} \sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{2n+1} \left(\frac{\mu}{r_1^2} - V_1 \right) \frac{a_n}{R} P_{n+1}$$

$$- 15\dot{R} \sum_{n=1}^{\infty} \frac{(n+1)(n+2)}{(2n+1)(2n+3)} \frac{\mu}{r_1^3} a_n P_{n+2} + \frac{3}{4}\left(\frac{\mu}{r_1^2} - V_1 \right)^2 (P_2 - 1)$$

$$- \frac{3}{2}\left(\frac{\mu}{r_1^2} - V_1 \right) \sum_{n=2}^{\infty} n \frac{\mu R^{n-1}}{r_1^{n+1}} \left(-P_{n+1} + P_{n-1} \right)$$

$$- \frac{5}{2} \frac{\mu R}{r_1^3} \sum_{n=2}^{\infty} n \frac{\mu R^{n-1}}{r_1^{n+1}} \left[-\frac{n+2}{2n+3} P_{n+2} + \frac{2n+1}{(2n-1)(2n+3)} P_n + \frac{n-1}{2n-1} P_{n-2} \right]$$

$$+ \frac{9}{4}\left(\frac{\mu}{r_1^2} - V_1 \right)^2 \sum_{n=1}^{\infty} \frac{n(n+1)}{2n+1} \left(-P_{n+1} + P_{n-1} \right)$$

$$+ \frac{15}{2}\left(\frac{\mu}{r_1^2} - V_1 \right) \frac{\mu}{r_1^3} \sum a_n \left[-\frac{(n+1)(n+2)(n+3)}{(2n+1)(2n+3)(2n+5)} P_{n+3} \right.$$

$$\left. + \frac{2}{5}\frac{n+1}{2n+1} \left(1 - \frac{n(n+2)}{(2n-1)(2n+5)} \right) P_{n+1} + \frac{2}{5}\frac{n}{2n+1} \left(1 - \frac{3}{2}\frac{(n-1)(n+1)}{(2n-3)(2n+3)} \right) P_{n-1} \right.$$

$$+\frac{2}{5}\frac{n}{2n+1}\left(1-\frac{3}{2}\frac{(n-1)(n+1)}{(2n-3)(2n+3)}\right)P_{n-1}-\frac{n(n-1)(n-2)}{(2n-1)(2n+1)(2n-3)}P_{n-3}\Big]$$

$$+\frac{15}{2}\left(\frac{\mu}{r_1^2}-V_1\right)\sum_{n=3}^{\infty}\frac{(n-1)^2n(n-1)}{(2n-1)(2n+1)(2n-3)}\frac{\mu}{r^3}a_n\left(-P_{n-1}+P_{n-3}\right)$$

$$+\frac{3}{4}\left(\frac{\mu}{r_1^2}-V_1\right)^2\sum_{n=2}^{\infty}\frac{n^2(n-1)^2}{(2n+1)(2n-1)}\frac{a_n}{R}\left(-P_n+P_{n-2}\right)$$

$$-\frac{15}{2}\left(\frac{\mu}{r_1^2}-V_1\right)\sum_{n=1}^{\infty}\frac{n^2(n+1)}{(2n-1)(2n+3)(2n+1)}\frac{\mu}{r_1^3}a_n\left(-P_{n+1}+P_{n-1}\right)$$

$$-\frac{3}{2}\left(\frac{\mu}{r_1^2}-V_1\right)\sum_{n=1}^{\infty}\frac{n}{2n+1}\left(\dot{a}_n+2\dot{R}\frac{a_n}{R}\right)\left(-P_{n+1}+P_{n-1}\right)$$

$$-\frac{9}{4}\left(\frac{\mu}{r_1^2}-V_1\right)^2\sum_{n=1}^{\infty}\frac{(n+1)^2(n+2)}{(2n+1)(2n+3)}\frac{a_n}{R}\left(-P_{n+2}+P_n\right)$$

$$-\frac{45}{2}\left(\frac{\mu}{r_1^2}-V_1\right)\sum_{n=1}^{\infty}\frac{(n+1)(n+2)^2}{(2n+1)(2n+3)(2n+5)}\frac{\mu}{r_1^3}a_n\left(-P_{n+3}+P_{n+1}\right)$$

$$+\frac{15}{2}\frac{\mu}{r_1^3}\left(\frac{\mu}{r_1^2}-V_1\right)\sum_{n=1}^{\infty}\frac{n(n+1)}{2n+1}a_n\Big[-\frac{n+2}{2n+3}P_{n+2}+\frac{2n+1}{(2n-1)(2n+3)}P_n$$

$$+\frac{n-1}{2n-1}P_{n-2}\Big]+25\frac{\mu^2R^2}{r_1^6}\sum_{n=1}^{\infty}a_n\Big[-\frac{(n+1)(n+2)(n+3)(n+4)}{(2n+1)(2n+3)(2n+5)(2n+7)}P_{n+4}$$

$$+\frac{(n+1)(n+2)}{(2n+1)(2n+3)}\left(\frac{2}{5}-\frac{3}{5}\frac{n(n+2)}{(2n-1)(2n+5)}-\frac{(n+3)^2}{(2n+5)(2n+7)}\right)P_{n+2}$$

$$+\frac{1}{2n+1}\Big\{2\left(\frac{(n+1)^2}{2n+3}+\frac{n^2}{2n-1}\right)-3\frac{n(n+1)}{(2n-1)(2n+3)}\Big(\frac{(n+1)(n+2)}{2n+5}$$

$$+\frac{(n-1)n}{2n-3}\Big)\Big\}P_n+\frac{(n-1)n}{(2n-1)(2n+1)}\left(\frac{2}{5}-\frac{3}{5}\frac{(n-1)(n+1)}{(2n-3)(2n+3)}\right.$$

$$\left.-\frac{(n-2)^2}{(2n-5)(2n-3)}\right)P_{n-2}-\frac{n(n-1)(n-2)(n-3)}{(2n-5)(2n-3)(2n-1)(2n+1)}P_{n-4}\Big]$$

$$-25\frac{\mu^2R}{r_1^6}\sum_{n=3}^{\infty}\frac{n(n-1)(n-2)^2}{(2n-1)(2n+1)(2n-3)}a_n\left(-\frac{n}{2n-1}P_n\right.$$

$$\left.+\frac{2n-1}{(2n-3)(2n+1)}P_{n-1}+\frac{n-2}{2n-3}P_{n-3}\right)$$

$$-25\frac{\mu^2R}{r_1^6}\sum_{n=1}^{\infty}\frac{n^2(n+1)}{(2n-1)(2n+1)(2n+3)}a_n\left(-\frac{n+2}{2n+3}P_{n+2}\right.$$

$$\left.+\frac{2n+1}{(2n-1)(2n+3)}P_n+\frac{n-1}{2n-1}P_{n-2}\right)$$

$$-5\frac{\mu R}{r_1^3}\sum_{n=1}^{\infty}\left(\dot{a}_n+2\dot{R}\frac{a_n}{R}\right)\frac{n}{2n+1}\left(-\frac{n+2}{2n+3}P_{n-2}\right.$$

$$+\frac{2n+1}{(2n-1)(2n+3)}P_n+\frac{n-1}{2n-1}P_{n-2}\Big)$$

$$-\frac{15}{2}\frac{\mu}{r_1^3}\left(\frac{\mu}{r_1^2}-V_1\right)\sum_{n=1}^{\infty}\frac{(n+1)^2(n+2)}{(2n+1)(2n+3)}a_n\Big(-\frac{n+3}{2n+5}P_{n+3}$$

$$+\frac{2n+3}{(2n+1)(2n+5)}P_{n+1}+\frac{n}{2n+1}P_{n-1}\Big)$$

$$-75\frac{\mu^2R}{r_1^6}\sum_{n=1}^{\infty}\frac{(n+1)(n+2)^2(n+3)}{(n+3)(2n+1)(2n+3)(2n+5)}a_n\Big(-\frac{n+4}{2n+7}P_{n+4}$$

$$+\frac{2n+5}{(2n+3)(2n+7)}P_{n+2}+\frac{n+1}{2n+3}P_n\Big) \qquad (37)$$

Now, one writes the $\frac{dV_1}{dt}r\cos\theta\Big|_{r=r_s}$ term in equation (27) as

$$\frac{dV_1}{dt}r\cos\theta\Big|_{r=r_s}=\frac{dV_1}{dt}\cos\theta\left(R+\sum_{n=1}^{\infty}a_nP_n\right). \qquad (38)$$

Since the pressure at infinity $p(\infty)$ is not affected by the presence of the bubble in the flow field, the application of the Bernoulli's equation yields

$$p(\infty)=p_0+\frac{1}{2}\left(\frac{\mu}{r_0^2}\right)^2, \qquad (39)$$

where r_0 is the initial distance of the bubble from the source or sink, and p_0 and μ/r_0^2 are the pressure and velocity of the liquid at $r=r_0$, respectively.

Substituting equations (32),(33),(34),(37),(38) and (39) into equation (27), one obtains the expression for

the pressure distribution over the bubble surface as

$$\frac{p(r_s)}{\rho} = R\ddot{R} + \frac{3}{2}\dot{R}^2 + \frac{p_0}{\rho} + \frac{1}{2}\left[\left(\frac{M}{r_0^2}\right)^2 - \left(\frac{M}{r_1^2}\right)^2\right] - \frac{1}{4}\left(\frac{M}{r_1^2} - V_1\right)^2 + \left[-\frac{1}{2}R\frac{dV_1}{dt}\right.$$

$$\left. + \frac{3}{2}\dot{R}\left(\frac{M}{r_1^2} - V_1\right) - 3\frac{MR}{r_1^3}V_1\right]\cos\theta + \frac{3}{4}\left(\frac{M}{r_1^2} - V_1\right)^2 P_2$$

$$- \frac{5}{2}\sum_{m=0}^{\infty}\frac{(m+1)(m+2)}{2m+3}\left(\frac{M}{r_1^2}\right)^2\left(\frac{R}{r_1}\right)^{m+2}P_m - \frac{3}{2}\sum_{m=1}^{\infty}(m+1)\left(\frac{M}{r_1^2} - V_1\right)\frac{M}{r_1^2}\left(\frac{R}{r_1}\right)^m \cdot$$

$$\cdot P_m + \sum_{m=2}^{\infty}(2m+1)\frac{M}{r_1^2}\left[-\left(\frac{R}{r_1}\right)^m V_1 + \frac{m}{m+1}\left(\frac{R}{r_1}\right)^{m-1}\dot{R} - \frac{5}{2}\frac{m}{(2m-1)(2m+3)}\cdot\right.$$

$$\left.\left(\frac{R}{r_1}\right)^m\frac{M}{r_1^2}\right]P_m + \frac{3}{2}\sum_{m=3}^{\infty}(2m-1)\left(\frac{M}{r_1^2} - V_1\right)\frac{M}{r_1^2}\left(\frac{R}{r_1}\right)^{m-2}P_m$$

$$+ \frac{5}{2}\sum_{m=4}^{\infty}\frac{(m-2)m}{2m-1}\left(\frac{M}{r_1^2}\right)^2\left(\frac{R}{r_1}\right)^{m-2}P_m + \sum_{m=1}^{\infty}\frac{R}{m+1}\dot{a}_m P_m$$

$$+ \sum_{m=1}^{\infty}\left[\frac{3}{m+1}\dot{R} - \frac{5m}{(2m-1)(2m+3)}\frac{M}{r_1^2}\frac{R}{r_1}\right]a_m P_m - \sum_{m=1}^{\infty}\frac{m}{m+1}\dot{R}\,a_m P_m$$

$$- \sum_{m=1}^{\infty}\frac{5m}{(2m-1)(2m+3)}\left\{(m+2)\frac{M\dot{R}}{r_1^3}a_m - 3R\frac{M}{r_1^4}V_1 a_m + \frac{R}{r_1}\frac{M}{r_1^3}a_m\right\}P_m$$

$$+ \sum_{m=1}^{\infty}\left\{-\left[\frac{5m(m+3)}{(2m-1)(2m+3)}\frac{M\dot{R}}{r_1^3} - \frac{9}{4}\frac{(m+1)^2(m+2)}{(2m+1)(2m+3)}\frac{1}{R}\left(\frac{M}{r_1^2} - V_1\right)^2\right.\right.$$

$$\left.\left. + \frac{15}{2}\frac{m(m+1)}{(2m-1)(2m+3)}\frac{M}{r_1^3}\left(\frac{M}{r_1^2} - V_1\right)\right]a_m P_m + \sum_{m=1}^{\infty}\frac{25}{2m+1}\left\{\frac{(m+1)^2}{2m+3}\cdot\right.\right.$$

$$\left.\left.\cdot\left[\frac{2}{5} - \frac{3}{5}\frac{m(m+2)}{(2m-1)(2m+5)} - 3\frac{(m+2)^2}{(2m+3)(2m+5)}\right]\right.\right.$$

$$\left.\left. + \frac{m^2}{2m-1}\left[\frac{2}{5} - \frac{3}{5}\frac{(m-1)(m+1)}{(2m-3)(2m+3)} - \frac{(m+1)(2m+1)}{(2m+1)(2m+3)^2}\right]\right\}\left(\frac{M}{r_1^2}\right)^2\frac{R}{r_1^2}a_m P_m\right.$$

$$- \frac{3}{4}\sum_{m=2}^{\infty}\frac{m^2(m-1)^2}{(2m-1)(2m+1)}\left(\frac{M}{r_1^2} - V_1\right)^2\frac{a_m}{R}P_m$$

$$- 25\sum_{m=3}^{\infty}\frac{(m-2)^2(m-1)m}{(2m-3)(2m-1)^2(2m+1)}\left(\frac{M}{r_1^2}\right)^2\frac{R}{r_1^2}a_m P_m$$

$$+ \sum_{m=0}^{\infty}\frac{m+1}{2m+3}\left\{-\frac{3}{2}\left(\frac{M}{r_1^2} - V_1\right)\dot{a}_{m+1} + \left[\frac{dV_1}{dt} - 3\frac{\dot{R}}{R}\left(\frac{M}{r_1^2} - V_1\right)\right.\right.$$

$$\left.\left. + \frac{9}{4}(m+2)\frac{1}{R}\left(\frac{M}{r_1^2} - V_1\right)^2 + \left(\frac{2}{5}\frac{m+1}{2m+3} - \frac{15}{2}\frac{(m+1)(m+2)}{(2m+3)(2m+5)}\left(\frac{3}{5}\frac{m}{2m-1}\right.\right.\right.\right.$$

$$\left.\left.\left.\left. + \frac{m+1}{2m+1} + \frac{(m+2)(m+3)}{2m+3}\right)\right)\left(\frac{M}{r_1^2} - V_1\right)\frac{M}{r_1^3}\right]a_{m+1}\right\}P_m$$

$$+\sum_{m=1}^{\infty}\frac{m(m+1)}{2m+3}\Big\{-\frac{1}{2}\Big(\frac{\mu}{r_1^2}-V_1\Big)\dot{a}_{m+1}+\Big[\frac{\mu}{r_1^3}V_1+\frac{1}{2}\frac{dV_1}{dt}$$

$$+\frac{5}{2}\frac{m(m+1)}{(2m-1)(2m+3)}\frac{\mu}{r_1^3}\Big(\frac{\mu}{r_1^2}-V_1\Big)\Big]a_{m+1}\Big\}P_m$$

$$-\sum_{m=2}^{\infty}\frac{15}{2}\frac{(m-1)^2m(m+1)}{(2m-1)(2m+1)(2m+3)}\frac{\mu}{r_1^3}\Big(\frac{\mu}{r_1^2}-V_1\Big)a_{m+1}P_m$$

$$+\frac{3}{2}\sum_{m=2}^{\infty}\Big(\frac{\mu}{r_1^2}-V_1\Big)a_{m-1}P_m-\frac{5}{2}\sum_{m=3}^{\infty}\frac{(m-1)^2(m+2)^2}{(2m-1)^2(2m-3)}\frac{\mu}{r_1^3}\Big(\frac{\mu}{r_1^2}-V_1\Big)\cdot$$

$$\cdot a_{m-1}P_m+\sum_{m=0}^{\infty}\frac{(m+1)(m+2)}{(2m+3)(2m+5)}\Big\{-5\frac{\mu R}{r_1^3}\dot{a}_{m+1}+\Big[-10\frac{\mu\dot{R}}{r_1^3}$$

$$+\frac{3}{4}(m+1)(m+2)\frac{1}{R}\Big(\frac{\mu}{r_1^2}-V_1\Big)^2+\frac{15}{2}(m+3)\frac{\mu}{r_1^3}\Big(\frac{\mu}{r_1^2}-V_1\Big)$$

$$+25\Big(\frac{2}{5}-\frac{3}{5}\frac{(m+1)(m+3)}{(2m+1)(2m+7)}-\frac{m^2}{(2m-1)(2m+1)}-\frac{(m+2)(m+3)}{(2m+3)(2m+7)}\Big)\cdot$$

$$\cdot\Big(\frac{\mu}{r_1^3}\Big)^2\frac{R}{r_1^2}\Big]a_{m+2}\Big\}P_m+\sum_{m=1}^{\infty}\frac{5m(m+2)}{(2m+3)(2m+5)}\Big\{-\frac{\mu R}{r_1^3}a_{m+2}$$

$$+\Big[\frac{\mu R}{r_1^4}V_1-\frac{\mu\dot{R}}{r_1^3}+\frac{5(2m+1)}{(2m-1)(2m+3)}\Big(\frac{\mu}{r_1^2}\Big)^2\frac{R}{r_1^2}\Big]a_{m+2}\Big\}P_m$$

$$+\sum_{m=3}^{\infty}\frac{m}{(2m-3)(2m-1)}\Big\{\Big(15\frac{m-1}{m+1}+5(m-2)\Big)\frac{\mu R}{r_1^3}\dot{a}_{m-2}$$

$$+\Big[\Big(15\frac{m-1}{m+1}+10(m-2)\Big)\frac{\mu\dot{R}}{r_1^3}-15\frac{m-1}{m+1}\frac{\mu R}{r_1^4}V_1$$

$$+\frac{9}{4}(m-1)^2\frac{1}{R}\Big(\frac{\mu}{r_1^2}-V_1\Big)^2-\frac{15}{2}(m-1)(m-2)\Big(\frac{\mu}{r_1^2}-V_1\Big)\frac{\mu}{r_1^3}$$

$$+25(m-1)\Big(\frac{2}{5}-\frac{3}{5}\frac{(m+2)m}{(2m-5)(2m+1)}-\frac{(m+1)^2}{(2m+1)(2m+3)}$$

$$+\frac{(m-2)^2}{(2m-5)(2m-1)}-3\frac{m}{(2m-1)(2m+3)}\Big)\Big(\frac{\mu}{r_1^2}\Big)^2\frac{R}{r_1^2}\Big]a_{m-2}P_m$$

$$+\sum_{m=4}^{\infty}\frac{15}{2}\frac{(m-2)(m-1)(m^2-3)}{(2m-1)(2m-3)(2m-5)}\frac{\mu}{r_1^3}\Big(\frac{\mu}{r_1^2}-V_1\Big)a_{m-3}P_m+$$

$$+\sum_{m=0}^{\infty} 25\frac{(m+1)^2(m+2)(m+3)(m+4)}{(2m+3)(2m+5)(2m+7)(2m+9)}\left(\frac{M}{\lambda_1^2}\right)^2\frac{R}{\lambda_1^2}a_{m+4}P_m$$

$$+\sum_{m=5}^{\infty} 25\frac{(m-3)(m-2)m(4m-7)}{(2m-1)(2m-3)(2m-5)(2m-7)}\left(\frac{M}{\lambda_1^2}\right)^2\frac{R}{\lambda_1^2}a_{m-4}P_m \qquad (40)$$

III-1-c The time history of bubble's translatory motion

Let us now define the surface of a translating bubble as the control surface as shown in Figure 2. Since there is no mass transfer across the control surface, the Newton's second law of motion can be written as

$$f = -m_1\frac{dV_1}{dt} \qquad (41)$$

where m_1 is the mass of the gas inside the bubble and f is the z-direction force induced by the pressure exerting on the control surface.

Consider the unit normal vector $\vec{n}$ on the bubble surface as shown in Figure 3-a. Its components in the

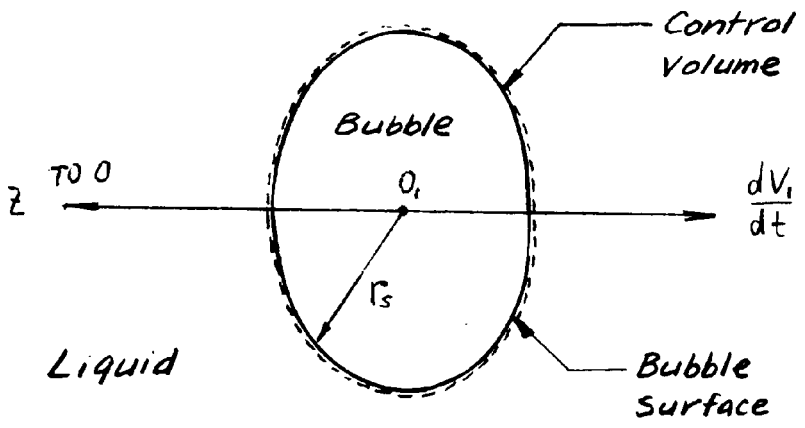


Figure 2. Definition of control volume.

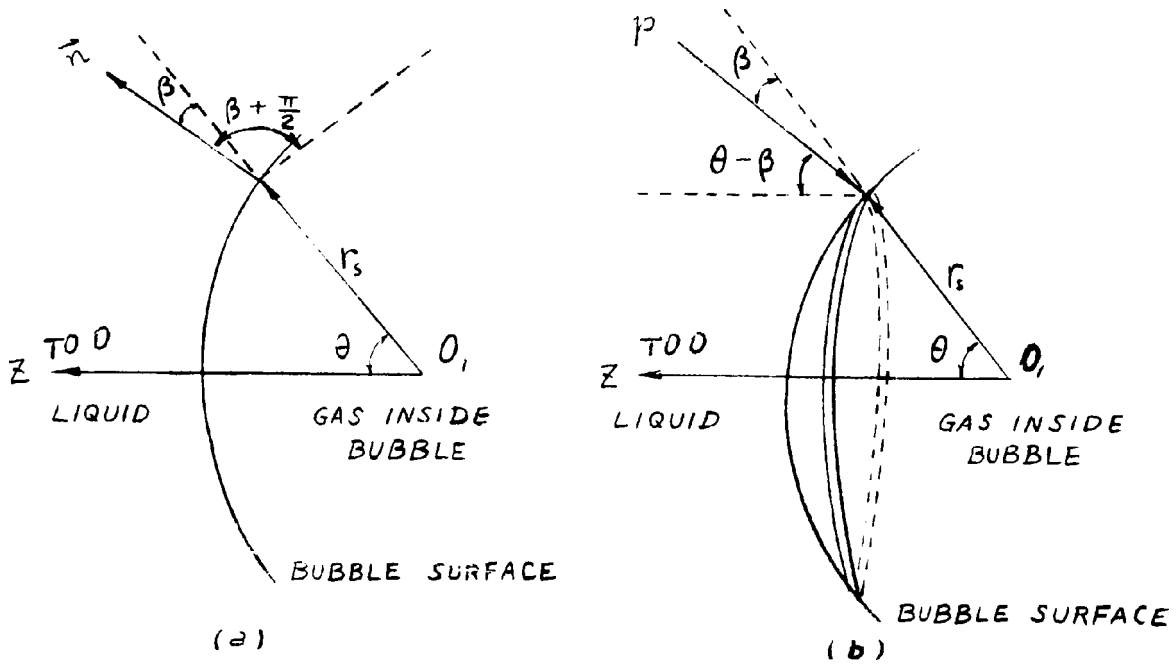


Figure 3. Unit normal and pressure on a bubble surface.

directions of r_s, θ and φ are

$$\frac{F_{r_s}}{\sqrt{F_{r_s}^2 + (\frac{1}{r_s}F_\theta)^2}}, \quad \frac{\frac{1}{r_s}F_\theta}{\sqrt{F_{r_s}^2 + (\frac{1}{r_s}F_\theta)^2}} \quad \text{and} \quad 0 \tag{42}$$

respectively, where the subscripts r_s, and θ denote the derivatives with respect to r_s and θ, respectively.

If β is defined as the angle between the vectors $\vec{n}$ and $\vec{r}_s$ the components of $\vec{n}$ may then be expressed as

$$\frac{F_{r_s}}{\sqrt{F_{r_s}^2 + (\frac{1}{r_s}F_\theta)^2}} = \cos(\vec{n}, \vec{r}_s) = \cos\beta \tag{43}$$

and

$$\frac{\frac{1}{r_s}F_\theta}{\sqrt{F_{r_s}^2 + (\frac{1}{r_s}F_\theta)^2}} = \cos(\vec{n}, \vec{\theta}) = \cos(\beta + \tfrac{\pi}{2}) = -\sin\beta. \tag{44}$$

Since F is defined by equation (10), its derivatives may be obtained from equation (10) as

$$F_{r_s} = 1$$

and

$$\frac{1}{r_s}F_\theta = \frac{1}{r_s}\sum_{n=1}^{\infty} a_n P_n' \sin\theta \approx \frac{1}{R}\left(1 - \sum_{n=1}^{\infty}\frac{a_m}{R}P_m\right)\sum_{n=1}^{\infty} a_n P_n' \sin\theta$$
$$\approx \frac{1}{R}\sum_{n=1}^{\infty} a_n P_n' \sin\theta.$$

Therefore, with the neglect of the terms involving a_n^2 one obtains

$$\sqrt{F_{r_s}^2 + (\frac{1}{r_s}F_\theta)^2} \approx 1.$$

The combination of these expressions with equations (43) and (44) gives

$$\cos\beta \approx 1 \tag{45}$$

$$\sin\beta \approx -\frac{1}{R}\sum_{n=1}^{\infty} a_n P_n' \sin\theta. \tag{46}$$

Now, consider the pressure p exerted upon the bubble surface as shown in Figure 3b. The force induced by the pressure can be written as

$$f = \iint (-p)\cos(\theta-\beta)\,dA. \tag{47}$$

With the aid of equations (45) and (46), the quantity $\cos(\theta-\beta)$ in the integrand of equation (47) can be expressed as

$$\begin{aligned}\cos(\theta-\beta) &= \cos\beta\cos\theta - \sin\beta\sin\theta \\ &\approx \cos\theta + \frac{1}{R}\sum_{n=1}^{\infty} a_n P_n' \sin^2\theta.\end{aligned} \tag{48}$$

Through the combination of equation (A-7) in Appendix III and the relationship

$$\frac{dr_s}{d\theta} = -\sum_{n=1}^{\infty} a_n P_n' \sin\theta.$$

followed by neglecting the terms involving $(a_n)^2$, the differential area dA may be written as

$$\begin{aligned}dA &= 2\pi r_s \sin\theta \sqrt{(r_s\,d\theta)^2 + (dr_s)^2} \\ &= 2\pi r_s \sin\theta \sqrt{r_s^2 + \left(\frac{dr_s}{d\theta}\right)^2}\; d\theta \\ &\approx 2\pi r_s^2 \sin\theta\, d\theta \\ &\approx 2\pi R^2 \left(1 + 2\sum_{n=1}^{\infty} \frac{a_n}{R} P_n\right) \sin\theta\, d\theta.\end{aligned} \tag{49}$$

By substituting equation (48) and (49) into equation (47) one obtains

$$
\begin{aligned}
f &= 2\pi R^2 \int_0^{\pi} p\left(\cos\theta + \frac{1}{R}\sum_{n=1}^{\infty} a_n P_n' \sin^2\theta\right)\left(1 + 2\sum_{n=1}^{\infty}\frac{a_n}{R}P_n\right) d(\cos\theta) \\
&= 2\pi R^2 \int_0^{\pi} p\left(\cos\theta + 2\cos\theta \sum_{n=1}^{\infty}\frac{a_n}{R}P_n + \sum_{n=1}^{\infty}\frac{a_n}{R}P_n' \sin^2\theta\right) d(\cos\theta) \\
&= 2\pi R^2 \int_0^{\pi} p\left[\cos\theta + \sum_{n=1}^{\infty}\frac{1}{2n+1}\frac{a_n}{R}\left\{\left[-(n-2)(n+1)P_{n+1}\right.\right.\right. \\
&\quad \left.\left.\left. + n(n+3)P_{n-1}\right\}\right] d(\cos\theta) \\
&= 2\pi R^2 \int_0^{\pi} p\left[\cos\theta - \sum_{m=2}^{\infty}\frac{(m-3)m}{2m-1}\frac{a_{m-1}}{R}P_m\right. \\
&\quad \left. + \sum_{m=0}^{\infty}\frac{(m+1)(m+4)}{2m+3}\frac{a_{m+1}}{R}P_m\right] d(\cos\theta)
\end{aligned}
\tag{50}
$$

where the recurrence formulas for the $\cos\theta P_n$ and $\sin^2\theta P_n'$ in Appendix II are used. Now, equation (40) for $p(r_s)$ is substituted into equation (50) followed by performing integration utilizing the orthogonality property of Legendre polynomials:

$$
\begin{aligned}
\int_{-1}^{1} P_n(u) P_m(u)\, du &= \frac{2}{2m+1} \qquad \text{if} \quad n = m \\
&= 0 \qquad \text{if} \quad n \neq m
\end{aligned}
\tag{51}
$$

The resulting equation for f is equated to zero, because m_1 is so small that the term $m_1 \frac{dV_1}{dt}$ on the right side of equation (41) may be considered negligible. It must be noted that the neglect of the $m_1 \frac{dV_1}{dt}$ for small value of m_1 does not necessarily imply an infinitely large value for the acceleration $\frac{dV_1}{dt}$, since f itself is a function of $\frac{dV_1}{dt}$. Hence, there results

$$\frac{1}{3}\frac{dV_1}{dt}R-\dot{R}\left(\frac{M}{r_1^2}-V_1\right)+2\frac{M^2}{r_1^4}\left(\frac{R}{r_1}\right)+2\frac{M^2}{r_1^4}\left(\frac{R}{r_1}\right)^3-\frac{1}{3}R\ddot{a}_1$$

$$-\left(\dot{R}-\frac{2}{3}\frac{M}{r_1^2}\frac{R}{r_1}\right)\dot{a}_1+\frac{1}{3}\ddot{R}a_1+\frac{2}{3}\left(3\frac{M\dot{R}}{r_1^3}a_1-3\frac{MR}{r_1^4}V_1a_1\right.$$

$$\left.+\frac{R}{r_1}\frac{M}{r_1^2}\dot{a}_1\right)-\left[-\frac{8}{3}\frac{M\dot{R}}{r_1^3}-\frac{6}{5}\frac{1}{R}\left(\frac{M}{r_1^2}-V_1\right)^2+2\frac{M}{r_1^3}\left(\frac{M}{r_1^2}-V_1\right)\right]a_1$$

$$+\frac{56}{3}\left(\frac{M}{r_1^2}\right)^2\frac{R}{r_1^2}a_1+\frac{2}{5}\left(\frac{M}{r_1^2}-V_1\right)\dot{a}_2-\frac{4}{15}\left[\frac{M}{r_1^3}V_1+\frac{1}{2}\frac{dV_1}{dt}\right.$$

$$\left.+\frac{M}{r_1^3}\left(\frac{M}{r_1^2}-V_1\right)\right]a_2+\frac{4}{7}\frac{MR}{r_1^3}\dot{a}_2-\frac{4}{35}\left[-10\frac{M\dot{R}}{r_1^3}\right.$$

$$\left.+\frac{9}{2}\frac{1}{R}\left(\frac{M}{r_1^2}-V_1\right)^2+30\frac{M}{r_1^3}\left(\frac{M}{r_1^2}-V_1\right)+\frac{85}{9}\left(\frac{M}{r_1^2}\right)^2\frac{R}{r_1^2}\right]a_3$$

$$+\frac{2}{7}\frac{MR}{r_1^3}\dot{a}_3-\frac{2}{7}\left[\frac{MR}{r_1^4}V_1-\frac{M\dot{R}}{r_1^3}+3\left(\frac{M}{r_1^2}\right)^2\frac{R}{r_1^2}\right]a_3$$

$$-\frac{40}{21}\frac{M}{r_1^3}\left(\frac{M}{r_1^2}-V_1\right)a_4-\frac{800}{693}\left(\frac{M}{r_1^2}\right)^2\frac{R}{r_1^2}a_5$$

$$+\frac{1}{5}\left(\frac{M}{r_1^2}-V_1\right)^2\frac{a_1}{R}-5\sum_{m=2}^{\infty}\frac{(m+1)(m+2)(m-3)m}{(2m+3)(2m-1)(2m+1)}\left(\frac{M}{r_1^2}\right)^2\cdot$$

$$\cdot\left(\frac{R}{r_1}\right)^{m+2}\frac{a_{m-1}}{R}-3\sum_{m=2}^{\infty}\frac{(m+1)(m-3)m}{(2m-1)(2m+1)}\left(\frac{M}{r_1^2}-V_1\right)\frac{M}{r_1^2}\left(\frac{R}{r_1}\right)^m\cdot$$

$$\cdot\frac{a_{m-1}}{R}+2\sum_{m=2}^{\infty}\frac{(m-3)m}{2m-1}\frac{M}{r_1^2}\left[-\left(\frac{R}{r_1}\right)^m V_1-\frac{m}{m+1}\left(\frac{R}{r_1}\right)^{m-1}\dot{R}\right.$$

$$\left.-\frac{5}{2}\frac{m}{(2m-1)(2m+3)}\left(\frac{R}{r_1}\right)^m\frac{M}{r_1^2}\right]\frac{a_{m-1}}{R}$$

$$+3\sum_{m=3}^{\infty}\frac{(m-1)(m-3)m}{(2m-1)(2m+1)}\left(\frac{M}{r_1^2}-V_1\right)\frac{M}{r_1^2}\left(\frac{R}{r_1}\right)^{m-2}\frac{a_{m-1}}{R}$$

$$+5\sum_{m=4}^{\infty}\frac{(m-2)m^2(m-3)}{(2m-1)^2(2m+1)}\left(\frac{M}{r_1^2}\right)^2\left(\frac{R}{r_1}\right)^{m-2}\frac{a_{m-1}}{R}$$

$$-\frac{8}{3}\left[R\ddot{R}+\frac{3}{2}\dot{R}^2+\frac{P_0}{\rho}+\frac{1}{2}\left(\left(\frac{M}{r_0^2}\right)^2-\left(\frac{M}{r_1^2}\right)^2\right)-\frac{1}{4}\left(\frac{M}{r_1^2}-V_1\right)^2\right]\cdot$$

$$\cdot\frac{a_1}{R}-\frac{4}{3}\left[-\frac{1}{2}R\frac{dV_1}{dt}+\frac{3}{2}\dot{R}\left(\frac{M}{r_1^2}-V_1\right)-3\frac{MR}{r_1^3}V_1\right]\frac{a_2}{R}$$

$$+5\sum_{m=0}^{\infty}\frac{(m+1)^2(m+2)(m+4)}{(2m+3)^2(2m+1)}\left(\frac{M}{r_1^2}\right)^2\left(\frac{R}{r_1}\right)^{m+2}\frac{a_{m+1}}{R}$$

$$+3\sum_{m=1}^{\infty}\frac{(m+1)^2(m+4)}{(2m+3)(2m+1)}\left(\frac{M}{r_1^2}-V_1\right)\frac{M}{r_1^3}\left(\frac{R}{r_1}\right)^m\frac{a_{m+1}}{R}$$

$$-\frac{4}{15}\left[\frac{dV_1}{dt}-3\frac{\dot{R}}{R}\left(\frac{\mu}{\Lambda_1^2}-V_1\right)+\frac{27}{4}\frac{1}{R}\left(\frac{\mu}{\Lambda_1^2}-V_1\right)^2\right.$$

$$\left.-4.56\left(\frac{\mu}{\Lambda_1^2}-V_1\right)\frac{\mu}{\Lambda_1^3}\right]a_2+\frac{2}{15}\left(\frac{\mu}{\Lambda_1^2}-V_1\right)\dot{a}_2$$

$$-2\sum_{m=2}^{\infty}\frac{(m+1)(m+4)}{2m+3}\frac{\mu}{\Lambda_1^2}\left[-\left(\frac{R}{\Lambda_1}\right)^m V_1+\frac{m}{m+1}\left(\frac{R}{\Lambda_1}\right)^{m-1}\dot{R}\right.$$

$$\left.-\frac{5}{2}\frac{m}{(2m-1)(2m+3)}\left(\frac{R}{\Lambda_1}\right)^m\frac{\mu}{\Lambda_1^2}\right]\frac{a_{m+1}}{R}$$

$$-3\sum_{m=3}^{\infty}\frac{(m-1)(m+1)(m+4)}{(2m+3)(2m+1)}\left(\frac{\mu}{\Lambda_1^2}-V_1\right)\frac{\mu}{\Lambda_1^2}\left(\frac{R}{\Lambda_1}\right)^{m-2}\cdot$$

$$\cdot\frac{a_{m+1}}{R}-5\sum_{m=4}^{\infty}\frac{(m-2)m(m+1)(m+4)}{(2m-1)(2m+3)(2m+1)}\left(\frac{\mu}{\Lambda_1^2}\right)^2\cdot$$

$$\cdot\left(\frac{R}{\Lambda_1}\right)^{m-2}\frac{a_{m+1}}{R}=0 \qquad (52)$$

Equation (52) is the expression for translating motion of bubble, which predicts Λ_1 and $V_1 \left(= d\Lambda_1/dt\right)$

III-1-d The time history of bubble size and shape

The force balance on the differential element of the bubble surface leads to the expression

$$\sigma\left(\frac{1}{R_1}+\frac{1}{R_2}\right)=P_{in}-P_{out}(\Lambda_s), \qquad (53)$$

where σ is the surface tension, R_1 and R_2 are the principal radii for the curvature of bubble surface and P_{in} and $P_{out}(\Lambda_s)$ are the pressures exerted upon the inside and outside surfaces of the bubble, respectively. Since the gas density is low compared with the liquid density, the pressure inside the bubble may be assumed uniform. Assuming that the gas inside the bubble is ideal and undergoes an adiabatic process during the growth

or collapse of the bubble one gets the expression

$$p_{in} = \left[\frac{R(0)}{R(t)}\right]^{3\gamma} p_{in}(0), \tag{54}$$

where $p_{in}(0)$ is the initial pressure of the gas inside the bubble and γ is the specific heat ratio c_p/c_v of the gas. For a perturbed sphere, the sum of the curvatures up to the first-order correction may be written as

$$\frac{1}{R_1} + \frac{1}{R_2} = \sigma\left(\frac{2}{R} + \sum_{n=1}^{\infty} \frac{(n-1)(n+2)}{R^2} a_n P_n\right). \tag{55}$$

The details of the derivation of equation (55) are given in Appendix III. By substituting equations (40), (54) and (55) into equation (53), one obtains

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 + \frac{p_0 - p_{in}}{\rho} + \frac{2\sigma}{R^2\rho} + \frac{1}{2}\left[\left(\frac{\mu}{\lambda_0^2}\right)^2 - \left(\frac{\mu}{\lambda_1^2}\right)^2\right] - \frac{1}{4}\left(\frac{\mu}{\lambda_1^2} - V_1\right)^2 + \left[-\frac{1}{2}R\frac{dV_1}{dt}\right.$$
$$\left. + \frac{3}{2}\dot{R}\left(\frac{\mu}{\lambda_1^2} - V_1\right) - 3\frac{\mu R}{\lambda_1^3}V_1\right]\cos\theta + \frac{3}{4}\left(\frac{\mu}{\lambda_1^2} - V_1\right)^2 P_2 - \frac{5}{2}\sum_{m=0}^{\infty}\frac{(m+1)(m+2)}{2m+3}\left(\frac{\mu}{\lambda_1^2}\right)^2 \cdot$$
$$\cdot\left(\frac{R}{\lambda_1}\right)^{m+2} P_m - \frac{3}{2}\sum_{m=1}^{\infty}(m+1)\left(\frac{\mu}{\lambda_1^2} - V_1\right)\frac{\mu}{\lambda_1^2}\left(\frac{R}{\lambda_1}\right)^m P_m + \sum_{m=2}^{\infty}(2m+1)\frac{\mu}{\lambda_1^2}\left[-\left(\frac{R}{\lambda_1}\right)^m V_1\right.$$
$$\left. + \frac{m}{m+1}\left(\frac{R}{\lambda_1}\right)^{m-1}\dot{R} - \frac{5}{2}\frac{m}{(2m-1)(2m+3)}\left(\frac{R}{\lambda_1}\right)^m\frac{\mu}{\lambda_1^2}\right] P_m + \frac{3}{2}\sum_{m=3}^{\infty}(m-1)\left(\frac{\mu}{\lambda_1^2} - V_1\right) \cdot$$
$$\cdot\frac{\mu}{\lambda_1^2}\left(\frac{R}{\lambda_1}\right)^{m-2} P_m + \frac{5}{2}\sum_{m=4}^{\infty}\frac{(m-2)m}{(2m-1)}\left(\frac{\mu}{\lambda_1^2}\right)^2\left(\frac{R}{\lambda_1}\right)^{m-2} P_m + \sum_{m=1}^{\infty}\frac{R}{m+1}\ddot{a}_m P_m +$$
$$+ \sum_{m=1}^{\infty}\left[\frac{3}{m+1}\dot{R} - \frac{5m}{(2m-1)(2m+3)}\frac{\mu}{\lambda_1^2}\frac{R}{\lambda_1}\right]\dot{a}_m P_m - \sum_{m=1}^{\infty}\frac{m}{m+1}\ddot{R}\,a_m P_m -$$
$$+ \sum_{m=1}^{\infty}\frac{5m}{(2m-1)(2m+3)}\left\{(m+2)\frac{\mu\dot{R}}{\lambda_1^3}a_m - 3R\frac{\mu}{\lambda_1^4}V_1 a_m + \frac{R}{\lambda_1}\cdot\frac{\mu}{\lambda_1^2}\dot{a}_m\right\} P_m$$
$$+ \sum_{m=1}^{\infty}\left\{-\frac{5m(m+3)}{(2m-1)(2m+3)}\frac{\mu\dot{R}}{\lambda_1^3} - \frac{9}{4}\frac{(m+1)^2(m+2)}{(2m+1)(2m+3)}\frac{1}{R}\left(\frac{\mu}{\lambda_1^2} - V_1\right)^2 + \frac{15}{2}\frac{m(m+1)}{(2m-1)(2m+3)}\cdot\right.$$
$$\left.\cdot\frac{\mu}{\lambda_1^3}\left(\frac{\mu}{\lambda_1^2} - V_1\right)\right\} a_m P_m + \sum_{m=1}^{\infty}\frac{25}{2m+1}\left\{\frac{(m+1)^2}{2m+3}\left[\frac{2}{5} - \frac{3}{5}\frac{m(m+2)}{(2m-1)(2m+5)}\right.\right. -$$

$$-3\frac{(m+2)^2}{(2m+3)(2m+5)}\Big]+\frac{m^2}{2m-1}\Big[\frac{2}{5}-\frac{3}{5}\frac{(m-1)(m+1)}{(2m-3)(2m+3)}-\frac{(m+1)(2m+1)}{(2m-1)(2m+3)^2}\Big]\Big\}\cdot$$

$$\cdot\left(\frac{\mu}{\Lambda_1^2}\right)^2\frac{R}{\Lambda_1^2}a_m P_m+\sum_{m=2}^{\infty}\Big[(m-1)(m+2)\frac{\sigma}{R^2}-\frac{3}{4}\frac{m^2(m-1)^2}{(2m-1)(2m+1)}\frac{1}{R}\left(\frac{\mu}{\Lambda_1^2}-V_1\right)^2\Big]\cdot$$

$$\cdot a_m P_m-25\sum_{m=3}^{\infty}\frac{(m-2)^2(m-1)m^2}{(2m-3)(2m-1)^2(2m+1)}\left(\frac{\mu}{\Lambda_1^2}\right)^2\frac{R}{\Lambda_1^2}a_m P_m+\sum_{m=0}^{\infty}\frac{m+1}{2m+3}\cdot$$

$$\cdot\Big\{-\frac{3}{2}\left(\frac{\mu}{\Lambda_1^2}-V_1\right)\dot{a}_{m+1}+\Big[\frac{dV_1}{dt}-3\frac{\dot{R}}{R}\left(\frac{\mu}{\Lambda_1^2}-V_1\right)+\frac{9}{4}(m+2)\frac{1}{R}\left(\frac{\mu}{\Lambda_1^2}-V_1\right)^2$$

$$+\Big(\frac{2}{5}\frac{m+1}{2m+3}-\frac{15}{2}\frac{(m+1)(m+2)}{(2m+3)(2m+5)}\Big(\frac{3}{5}\frac{m}{2m-1}+\frac{m+1}{2m+1}+\frac{(m+2)(m+3)}{2m+3}\Big)\Big)\left(\frac{\mu}{\Lambda_1^2}-V_1\right)\cdot$$

$$\cdot\frac{\mu}{\Lambda_1^3}\Big]a_{m+1}\Big\}P_m+\sum_{m=1}^{\infty}\frac{m(m+1)}{2m+3}\Big\{-\frac{1}{2}\left(\frac{\mu}{\Lambda_1^2}-V_1\right)\dot{a}_{m+1}+\Big[\frac{\mu}{\Lambda_1^3}V_1+\frac{1}{2}\frac{dV_1}{dt}$$

$$+\frac{5}{2}\frac{m(m+1)}{(2m-1)(2m+3)}\frac{\mu}{\Lambda_1^3}\left(\frac{\mu}{\Lambda_1^2}-V_1\right)\Big]a_{m+1}\Big\}P_m-\sum_{m=2}^{\infty}\frac{15}{2}\frac{(m-1)^2m(m+1)}{(2m-1)(2m+1)(2m+3)}\cdot$$

$$\cdot\frac{\mu}{\Lambda_1^3}\left(\frac{\mu}{\Lambda_1^2}-V_1\right)a_{m+1}P_m+\frac{3}{2}\sum_{m=2}^{\infty}\left(\frac{\mu}{\Lambda_1^2}-V_1\right)\dot{a}_{m-1}P_m-\frac{5}{2}\sum_{m=3}^{\infty}\frac{(m-1)^2(m-2)^2}{(2m-1)^2(2m-3)}\cdot$$

$$\cdot\frac{\mu}{\Lambda_1^3}\left(\frac{\mu}{\Lambda_1^2}-V_1\right)a_{m-1}P_m+\sum_{m=0}^{\infty}\frac{(m+1)(m+2)}{(2m+3)(2m+5)}\Big\{-5\frac{\mu R}{\Lambda_1^3}\dot{a}_{m+2}+\Big[-10\frac{\mu\dot{R}}{\Lambda_1^3}$$

$$+\frac{3}{4}(m+1)(m+2)\frac{1}{R}\left(\frac{\mu}{\Lambda_1^2}-V_1\right)^2+\frac{15}{2}(m+3)\frac{\mu}{\Lambda_1^3}\left(\frac{\mu}{\Lambda_1^2}-V_1\right)+25\Big(\frac{2}{5}-$$

$$-\frac{3}{5}\frac{(m+1)(m+3)}{(2m+1)(2m+7)}-\frac{m^2}{(2m-1)(2m+1)}-\frac{(m+2)(m+3)}{(2m+3)(2m+7)}\Big)\left(\frac{\mu}{\Lambda_1^2}\right)^2\frac{R}{\Lambda_1^2}\Big]a_{m+2}\Big\}\cdot$$

$$\cdot P_m+\sum_{m=1}^{\infty}\frac{5m(m+2)}{(2m+3)(2m+5)}\Big\{-\frac{\mu R}{\Lambda_1^3}\dot{a}_{m+2}+\Big[\frac{\mu R}{\Lambda_1^4}V_1-\frac{\mu\dot{R}}{\Lambda_1^3}+\frac{5(2m+1)}{(2m-1)(2m+3)}\cdot$$

$$\cdot\left(\frac{\mu}{\Lambda_1^2}\right)^2\frac{R}{\Lambda_1^2}\Big]a_{m+2}\Big\}P_m+\sum_{m=3}^{\infty}\frac{m}{(2m-3)(2m-1)}\Big\{\Big(15\frac{m-1}{m+1}+5(m-2)\Big)\frac{\mu R}{\Lambda_1^3}\dot{a}_{m-2}$$

$$+\Big[\Big(15\frac{m-1}{m+1}+10(m-2)\Big)\frac{\mu\dot{R}}{\Lambda_1^3}-15\frac{m-1}{m+1}\frac{\mu R}{\Lambda_1^4}V_1+\frac{9}{4}(m-1)^2\frac{1}{R}\left(\frac{\mu}{\Lambda_1^2}-V_1\right)^2$$

$$-\frac{15}{2}(m-1)(m-2)\left(\frac{\mu}{\Lambda_1^2}-V_1\right)\frac{\mu}{\Lambda_1^3}+25(m-1)\Big(\frac{2}{5}-\frac{3}{5}\frac{(m-2)m}{(2m-5)(2m+1)}-$$

$$-\frac{(m+1)^2}{(2m+1)(2m+3)}+\frac{(m-2)^2}{(2m-5)(2m-1)}-3\frac{m}{(2m-1)(2m+3)}\Big)\left(\frac{\mu}{\Lambda_1^2}\right)^2\frac{R}{\Lambda_1^2}\Big]a_{m-2}\Big\}P_m$$

$$+\sum_{m=0}^{\infty}\frac{5(m+1)(m+2)(m+3)}{2(2m+3)(2m+5)(2m+7)}\Big[3m+(m+2)(m+3)\Big]\frac{\mu}{\Lambda_1^3}\left(\frac{\mu}{\Lambda_1^2}-V_1\right)a_{m+3}P_m$$

$$+\sum_{m=4}^{\infty}\frac{15(m-2)(m-1)(m^2-3)}{2(2m-1)(2m-3)(2m-5)}\frac{\mu}{\Lambda_1^3}\left(\frac{\mu}{\Lambda_1^2}-V_1\right)a_{m-3}P_m+$$

$$+\sum_{m=5}^{\infty} 25\,\frac{(m-3)(m-2)m(4m-7)}{(2m-1)(2m-3)(2m-5)(2m-7)}\left(\frac{\mu}{\Lambda_1^2}\right)^2\frac{R}{\Lambda_1^2}\,a_{m-4}\,P_m$$

$$+\sum_{m=0}^{\infty} 25\,\frac{(m+1)^2(m+2)(m+3)(m+4)}{(2m+3)(2m+5)(2m+7)(2m+9)}\left(\frac{\mu}{\Lambda_1^2}\right)^2\frac{R}{\Lambda_1^2}\,a_{m+4}\,P_m$$

$$+\sum_{m=2}^{\infty}\frac{m}{2m-1}\left\{-\frac{1}{2}\frac{dV_1}{dt}+3\,\frac{m-1}{m}\,\frac{\dot R}{R}\left(\frac{\mu}{\Lambda_1^2}-V_1\right)-\frac{3}{2}\,\frac{\mu}{\Lambda_1^3}V_1\right.$$

$$-\frac{9}{4}(m-1)\frac{1}{R}\left(\frac{\mu}{\Lambda_1^2}-V_1\right)^2+\frac{15}{2}\left[\frac{2}{5}+\frac{(m-1)^2}{(2m+1)(2m+3)}\right.$$

$$\left.\left.-\frac{m+1}{2m+3}\left(3\,\frac{m+1}{2m+1}+\frac{3}{5}\,\frac{m-1}{2m-3}+\frac{m}{2m-1}\right)\right]\frac{\mu}{\Lambda_1^3}\left(\frac{\mu}{\Lambda_1^2}-V_1\right)\right\}a_{m-1}$$

$$=0 \qquad (56)$$

By equating all coefficients of the P_n terms in equation (56) to zero, there results a set of second-order differential equations for R and a_n's.

Coefficient of P_0 :

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 + \frac{p_0 - p_\infty}{\rho} + \frac{2\sigma}{R^2\rho} + \frac{1}{2}\left[\left(\frac{\mu}{r_0}\right)^2 - \left(\frac{\mu}{r_1^2}\right)^2\right] - \frac{1}{4}\left(\frac{\mu}{r_1^2} - V_1\right)^2$$
$$- \frac{5}{3}\left(\frac{\mu}{r_1^2}\right)^2\left(\frac{R}{r_1}\right)^2 + \frac{1}{3}\left\{-\frac{3}{2}\left(\frac{\mu}{r_1^2} - V_1\right)\dot{a}_1 + \left[\frac{dV_1}{dt} - 3\frac{\dot{R}}{R}\left(\frac{\mu}{r_1^2} - V_1\right)\right.\right.$$
$$\left.\left. + \frac{9}{2}\frac{1}{R}\left(\frac{\mu}{r_1^2} - V_1\right)^2 - \frac{4}{15}\left(\frac{\mu}{r_1^2} - V_1\right)\frac{\mu}{r_1^3}\right] a_1\right\} - \frac{2}{3}\frac{\mu R}{r_1^3}\dot{a}_2$$
$$+ \left[-\frac{4}{3}\frac{\mu\dot{R}}{r_1^3} + \frac{1}{5R}\left(\frac{\mu}{r_1^2} - V_1\right)^2 + 3\frac{\mu}{r_1^3}\left(\frac{\mu}{r_1^2} - V_1\right) - \frac{10}{21}\frac{\mu^2 R}{r_1^6}\right] a_2$$
$$+ \frac{6}{7}\left(\frac{\mu}{r_1^2} - V_1\right)\frac{\mu}{r_1^3}a_3 + \frac{40}{63}\frac{\mu^2 R}{r_1^6}a_4$$
$$= 0 \qquad \text{(57a)}$$

Coefficient of P_1 :

$$\frac{R}{2}\dot{a}_1 - \frac{1}{2}R\frac{dV_1}{dt} + \frac{3}{2}\dot{R}\left(\frac{\mu}{r_1^2} - V_1\right) - 3\frac{\mu R}{r_1^3}V_1 - 3\left(\frac{\mu}{r_1^2}\right)^2\left(\frac{R}{r_1}\right)^3$$
$$- 3\left(\frac{\mu}{r_1^2} - V_1\right)\frac{\mu}{r_1^2}\left(\frac{R}{r_1}\right) + \left(\frac{3}{2}\dot{R} - \frac{\mu}{r_1^2}\frac{R}{r_1}\right)\dot{a}_1 - \frac{1}{2}\dot{R}a_1$$
$$+ \left\{3\frac{\mu\dot{R}}{r_1^3}a_1 - 3R\frac{\mu}{r_1^4}V_1 a_1 + \frac{R}{r_1}\frac{\mu}{r_1^2}\dot{a}_1\right\} + \left\{-4\frac{\mu\dot{R}}{r_1^3}\right.$$
$$\left. - \frac{9}{5}\frac{1}{R}\left(\frac{\mu}{r_1^2} - V_1\right)^2 + 3\frac{\mu}{r_1^3}\left(\frac{\mu}{r_1^2} - V_1\right)\right\} a_1 - \frac{20}{7}\left(\frac{\mu}{r_1^2}\right)^2\frac{R}{r_1^2}a_1$$
$$+ \frac{2}{5}\left\{-\frac{3}{2}\left(\frac{\mu}{r_1^2} - V_1\right)\dot{a}_2 + \left[\frac{dV_1}{dt} - 3\frac{\dot{R}}{R}\left(\frac{\mu}{r_1^2} - V_1\right)\right.\right.$$
$$\left.\left. + \frac{27}{4R}\left(\frac{\mu}{r_1^2} - V_1\right)^2 + 4.56\left(\frac{\mu}{r_1^2} - V_1\right)\frac{\mu}{r_1^3}\right] a_2\right\}$$
$$+ \frac{2}{5}\left\{-\frac{1}{2}\left(\frac{\mu}{r_1^2} - V_1\right)\dot{a}_2 + \left[\frac{\mu}{r_1^3}V_1 + \frac{1}{2}\frac{dV_1}{dt} + \frac{\mu}{r_1^3}\left(\frac{\mu}{r_1^2} - V_1\right)\right] a_2\right\}$$
$$+ \frac{6}{35}\left\{-5\frac{\mu R}{r_1^3}\dot{a}_3 + \left[-10\frac{\mu\dot{R}}{r_1^3} + \frac{9}{2}\frac{1}{R}\left(\frac{\mu}{r_1^2} - V_1\right)^2 + 30\frac{\mu}{r_1^3}\left(\frac{\mu}{r_1^2} - V_1\right)\right.\right.$$

$$- 9.42\left(\frac{\mu}{r_1^2}\right)^2 \frac{R}{r_1^2}\Big] a_3 \Big\} + \frac{3}{7}\Big\{ - \frac{\mu R}{r_1^3}\dot{a}_3 + \Big[\frac{\mu R}{r_1^4} V_1$$

$$- \frac{\mu \dot{R}}{r_1^3} + 3\left(\frac{\mu}{r_1^2}\right)^2 \frac{R}{r_1^2}\Big] a_3 \Big\} + \frac{20}{7}\frac{\mu}{r_1^3}\left(\frac{\mu}{r_1^2} - V_1\right) a_4 + 1.73\left(\frac{\mu}{r_1^2}\right)^2 \frac{R}{r_1^2} a_5$$

$$= 0 \qquad (57b)$$

Coefficient of P_2 :

$$\frac{R}{3}\ddot{a}_2 + \cdots$$

These equations are coupled not only among themselves but also with equation (52). Therefore it is necessary to solve equations (57) together with equation (52) simultaneously for r, R and a_n's.

III-2 Dynamics of a bubble moving in a quiescent liquid following a step change in liquid pressure

Consider a bubble travelling with the velocity in a quiescent liquid following a step change in the liquid pressure.

If the coordinate systems as shown in Figure 4 is fixed on a stationary reference frame, then the velocity potential ϕ, which is the solution of the Laplace equation (2), can be expanded into a series of spherical harmonics as

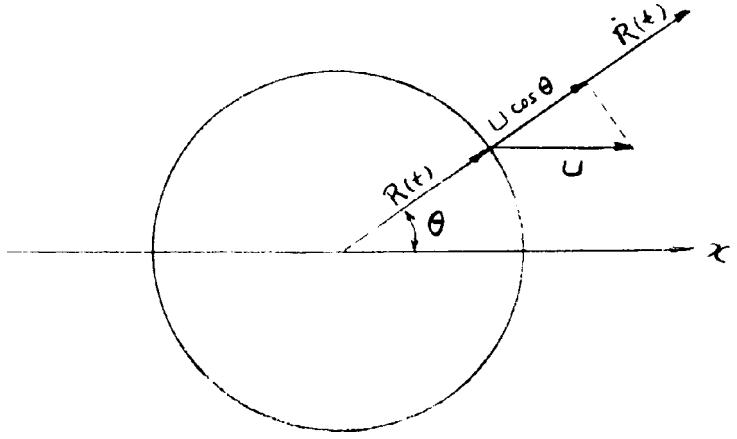


Figure 4. Coordinate systems for analyzing the dynamics of a bubble moving in a quiescent liquid.

$$\phi_2 = \sum_{n=1}^{\infty} A_n \frac{1}{r^{n+1}} P_n(\cos\theta), \tag{58}$$

where are the coefficients to be determined, r is the radial distance from the center of the bubble and θ is the angle between radial vector and the translating velocity U of the bubble. For a spherical bubble, the condition to be satisfied on the surface is

$$-\left.\frac{\partial \phi_2}{\partial r}\right|_{r=R} = U\cos\theta + \dot{R}. \tag{59}$$

By substituting equation (58) into equation (59) followed by equating all the coefficients of the P_n terms to zero, one obtains

$$A_0 = R^2\dot{R}, \qquad A_1 = \frac{1}{2}UR^3$$

$$A_n = 0 \qquad \text{for } n \geq 2.$$

Equation (58) may now be written as

$$\phi_2 = \frac{R^2\dot{R}}{r} + \frac{1}{2}\frac{UR^3}{r^2}\cos\theta \tag{60}$$

Now, if the coordinate systems are fixed at the center of the moving bubble, the velocity potential becomes

$$\phi_3 = \frac{R^2\dot{R}}{r} + \left(r + \frac{1}{2}\frac{R^3}{r^2}\right)U\cos\theta \tag{61}$$

In an analogous manner as the previous case, the shape of the bubble and the velocity potential are expressed in the form of equations (9) and (12), respectively, subject to the surface condition (11). With the neglect of the terms

involving a_n^2 one obtains

$$\phi = \frac{R^2\dot{R}}{r} + \sum_{n=1}^{\infty} \frac{1}{n+1}\left(\dot{a}_n + 2\dot{R}\frac{a_n}{R}\right)\frac{R^{n+2}}{r^{n+1}} P_n$$

$$+ U\left[\left(r + \frac{1}{2}\frac{R^3}{r^2}\right)\cos\theta\right.$$

$$\left. + \frac{3}{2}\sum_{n=1}^{\infty} a_n\left(\frac{n+1}{2n+1}\frac{R^{n+2}}{r^{n+2}} P_{n+1} - \frac{n-1}{2n+1}\frac{R^n}{r^n} P_{n-1}\right)\right] \qquad (62)$$

Equation (27) may also predict the pressure distribution on the bubble surface for the present case. However, the inertia terms in equation (27) must be evaluated from equation (62) with the aid of equation (9). The result may be expressed as

$$\frac{p(r)}{\rho} = R\ddot{R} + \frac{3}{2}\dot{R}^2 + \frac{p_\infty}{\rho} - \frac{1}{4}U^2 - \frac{1}{2}U a_1 - U\frac{\dot{R}}{R}a_1$$

$$- \frac{1}{3}\frac{dU}{dt}a_1 + \left\{\frac{1}{2}R\ddot{a}_1 + \frac{3}{2}\dot{R}\dot{a}_1 - \frac{9}{10}\frac{U^2}{R}a_1 - \frac{3}{10}\left[3U\dot{a}_2\right.\right.$$

$$\left.\left. + \left(\frac{dU}{dt} + 4U\frac{\dot{R}}{R}\right)a_2\right] + \frac{27}{70}\frac{U^2}{R}a_3 + \frac{3}{2}\dot{R}U + \frac{1}{2}R\frac{dU}{dt}\right\} P_1$$

$$+ \left\{\frac{3}{2}U\dot{a}_1 + \left(\frac{1}{3}\frac{dU}{dt} + U\frac{\dot{R}}{R}\right)a_1 + \frac{1}{3}R\ddot{a}_2 + \dot{R}\dot{a}_2\right.$$

$$+ \left(-\frac{1}{3}\ddot{R} - \frac{54}{35}\frac{U^2}{R}\right)a_2 - \frac{3}{14}\left[5U\dot{a}_3 + \left(4\frac{dU}{dt}\right.\right.$$

$$\left.\left. + 6U\frac{\dot{R}}{R}\right)a_3\right] + \frac{U^2}{R}a_4 + \frac{3}{4}U^2\Bigg\} P_2$$

$$+ \sum_{n=2}^{\infty}\left\{\frac{9}{4}\frac{n(n+1)(n+2)}{(2n+3)(2n+5)}\frac{U^2}{R}a_{n+2} - \frac{3}{2}\left[\frac{n+\frac{2}{3}(n+1)}{2n+3}\frac{dU}{dt}a_{n+1}\right.\right.$$

$$+ \frac{(n+1)(n+4)}{2n+3}U\frac{\dot{R}}{R}a_{n+1} + \frac{2n+1}{2n+3}\dot{R}a_n$$

$$+ \frac{9}{4}\frac{U^2}{R}a_n\left[1 - \frac{(n+1)^2}{(2n+1)(2n+3)} + \frac{n^2}{(2n-1)(2n+1)} - \frac{(n+1)^2(n+2)}{(2n+1)(2n+3)}\right.$$

$$\left. - \frac{(n-1)^2 n}{(2n+1)(2n-1)}\right] + \frac{3}{2}\left\{2\frac{n-1}{2n-1}U\frac{\dot{R}}{R}a_{n-1} + U\dot{a}_{n-1}\right.$$

$$+\frac{1}{3}\frac{n}{2n-1}\frac{dU}{dt}a_{n-1}\Big] + \frac{9}{4}\frac{U^2}{R}\frac{(n-2)(n-1)n}{(2n-3)(2n-1)}a_{n-2}\Big\}P_n \qquad (63)$$

When the last expression for P is substituted into equation (50) followed by performing integration, one gets the equation for the translatory motion of the bubble as

$$\begin{aligned}
&\frac{1}{3}R\frac{dU}{dt} + \dot{R}U + \frac{1}{3}R\ddot{a}_1 + \dot{R}\dot{a}_1 - \frac{3}{5}\frac{U^2}{R}a_1 \\
&\quad - \frac{1}{5}\left[3U\dot{a}_2 + \left(\frac{dU}{dt} + 4U\frac{\dot{R}}{R}\right)a_2\right] \\
&\quad + \frac{9}{35}\frac{U^2}{R}a_3 + \frac{1}{5}U^2\frac{a_1}{R} + \frac{8}{3}\left(R\ddot{R} + \frac{3}{2}\dot{R}^2\right. \\
&\quad \left. + \frac{p_\infty}{\rho} - \frac{1}{4}U^2\right)\frac{a_1}{R} \\
&\quad + \frac{3}{5}\left(3U\dot{R} + R\frac{dU}{dt}\right)\frac{a_2}{R} \\
&\quad = 0
\end{aligned} \qquad (64)$$

The equations for R and a_n's can be obtained by substituting equation (63) for $p(r_s)$ into equation (53) followed by equating all the coefficients of the P_n terms to zero. Thus, there results

Coefficient of P_0 :

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 + \frac{p_\infty - p_{in}}{\rho} - \frac{1}{4}U^2 + \frac{2\sigma}{R} - \frac{1}{2}U\dot{a}_1 - U\frac{\dot{R}}{R}a_1 - \frac{1}{3}\frac{dU}{dt}a_1 = 0 \qquad (65a)$$

Coefficient of P_1:

$$\frac{1}{2}R\ddot{a}_1 + \frac{3}{2}\dot{R}\dot{a}_1 - \frac{9}{10}\frac{U^2}{R}a_1 - \frac{3}{10}\left[3U\dot{a}_2 + \left(\frac{dU}{dt} + 4U\frac{\dot{R}}{R}\right)a_2\right] + \frac{27}{70}\frac{U^2}{R}a_3 + \frac{3}{2}\dot{R}U + \frac{1}{2}R\frac{dU}{dt} = 0 \qquad (65b)$$

Coefficient of P_2:

$$\frac{1}{3}R\ddot{a}_2 + \dot{R}\dot{a}_2 + \left(-\frac{1}{3}\ddot{R} - \frac{54}{35}\frac{U^2}{R} + \frac{4\sigma}{\rho R^2}\right)a_2 + \frac{3}{2}U\dot{a}_1 + \left(\frac{1}{3}\frac{dU}{dt} + U\frac{\dot{R}}{R}\right)a_1 - \frac{3}{14}\left[5U\dot{a}_3 + \left(4\frac{dU}{dt} + 6U\frac{\dot{R}}{R}\right)a_3\right] + \frac{U^2}{R}a_4 + \frac{3}{4}U^2 = 0 \qquad (65c)$$

Coefficient of P_n where $n \geq 3$:

$$\frac{R}{n+1}\ddot{a}_n + \frac{3}{n+1}\dot{R}\dot{a}_n - \frac{n-1}{n+1}\ddot{R}a_n + \frac{9}{4}\frac{U^2}{R}a_n\left[1 - \frac{(n+1)^2}{(2n+1)(2n+3)} + \frac{n^2}{(2n-1)(2n+1)} - \frac{(n+1)^2(n+2)}{(2n+1)(2n+3)} - \frac{(n-1)^2 n}{(2n+1)(2n-1)}\right] + \frac{(n-1)(n+2)\sigma}{\rho R^2}a_n + \frac{9}{4}\frac{n(n+1)(n+2)}{(2n+3)(2n+5)}\frac{U^2}{R}a_{n+2} - \frac{3}{2}\left[\frac{n+\frac{2}{3}(n+1)}{2n+3}\frac{dU}{dt}a_{n+1} - \frac{(n+1)(n+4)}{2n+3}U\frac{\dot{R}}{R}a_{n+1} + \frac{2n+1}{2n+3}U\dot{a}_{n+1} - \frac{(n+1)(n-2)}{2n+3}U\frac{\dot{R}}{R}a_{n+1}\right] + \frac{3}{2}\left[2\frac{n-1}{2n-1}U\frac{\dot{R}}{R}a_{n-1} + U\dot{a}_{n-1} + \frac{1}{3}\frac{n}{2n-1}\frac{dU}{dt}a_{n-1}\right] + \frac{9}{4}\frac{U^2}{R}\frac{(n-2)(n-1)n}{(2n-3)(2n-1)}a_{n-2} = 0 \qquad (65d)$$

CHAPTER IV

RESULTS AND DISCUSSION

When both the liquid and bubble are stationary, the kinetic energy term $\frac{1}{2} v_\theta^2|_{r=r_s}$ in equation (27) becomes negligible. The resulting equations for the a_n may then become uncoupled and thus can be solved independently. This is the case studied by Plesset (98) and Plesset and Mitchell (99). However, once the liquid and/or bubble are set in motion, as in the present analysis, the contribution of the kinetic energy $\frac{1}{2} v_\theta^2|_{r=r_s}$ becomes important. As the result the other modes such as a_{n+1} etc. are introduced into the governing equations, which are mutually coupled and have to be solved simultaneously. In case of a source or sink flow, the problem is further complicated by the appearance of a series of spherical harmonic terms in equation (5) for the velocity potential in the main stream. Under such circumstances it becomes impossible to obtain the analytical solution. The only alternative is to try to get the approximate solution by means of a numerical method.

All terms of the equations derived in the previous Chapter may be nondimensionalized through the following definitions.

$$\text{(dimensionless length)} = \text{(length)}/R(0)$$

$$\text{(dimensionless velocity)} = \text{(velocity)}/\left(\frac{\sigma}{\rho R(0)}\right)^{1/2}$$

$$\text{(dimensionless acceleration)} = \text{(acceleration)}/\frac{\sigma}{\rho[R(0)]^2}$$

$$\text{(dimensionless time)} = \text{(time)}/\left(\frac{\rho[R(0)]^3}{\sigma}\right)^{1/2} \qquad (66)$$

$$\text{(dimensionless pressure)} = \text{(pressure)}/\left(\frac{\sigma}{R(0)}\right)$$

$$\text{(dimensionless strength of a point source or sink)} = \text{(strength of a point source or sink)}/\left(\frac{\sigma[R(0)]^3}{\rho}\right)^{1/2},$$

where $R(0)$ is the initial radius of the bubble. The introduction of these definitions does not cause a change in the form of these equations except that ρ and σ now disappear from the expressions. The dimensionless velocity as defined in equation (66) is exactly the definition for the Weber number. In the following discussion we shall always refer to the equations in dimensionless form unless otherwise indicated.

IV-1 Dynamics of a bubble moving in a source or sink flow

Equations (52) and (57) were numerically integrated by the Runge-Kutta method using an IBM 7090 digital computer. Since it is impossible to solve infinitely many equations simultaneously, only the first twelve equations for the a_n were integrated. All other equations for $n>12$ were considered higher order of magnitude and thus were ignored. From the experimental study it is disclosed that sufficiently accurate results may be obtained by retaining the first twelve a_n in the computer program. Therefore, together with equation (52), the total of fourteen simultaneous equations were programmed for τ, R and a_1 to a_{12}. To begin with numerical computation, it is assumed that

the bubble at the start of the transient is in a spherical form. This is equivalent to imposing the conditions that all a_n and $\dot{a}_n$ are zero and R is equal to unity at $t=o$ With these initial conditions, equations (52) and (57b) become identical at $t=0$ and one more condition is required for computation. This condition is $\ddot{a}_1$ being zero at $t=o$ from the reason that the bubble is initially spherical and hence the $-\frac{1}{3}R\ddot{a}_1$ term must be absent from equation (52). Therefore, in computer program $\ddot{a}_1$ is neglected in equation (52) at initial period. The calculations were performed until the values of the a_n become no longer appropriate for the method of perturbation. Separate programs were written for the time history of the equivalent bubble size R and the translating velocity V_1 of the bubble at downstream from the source or upstream from the sink. For such cases equations (52) and (57a) were solved simultaneously for a longer period of time with the neglect of the a_n. This is equivalent to treat the bubble as a spherical one. In the initial stage of bubble growth or collapse where the perturbation method applies, the results of the computations show that this simplification results in a slight deviation in the values of r_1, V_1 and R from those computed with the a_n retained. This is not surprising since those terms involving a_n in equations (52) and (57a) are small compared with the other terms.

IV-1-a The time history of bubble size and velocity

For the sake of convenience in discussion, the relationship between the equivalent bubble radius R and the

translating velocity $V_1 (= dr_1/dt)$ may be obtained by solving equations (52) and (57a) simultaneously with the neglect of the terms involving Q_n. These two equations may be rewritten as

$$\frac{d^2 r_1}{dt^2} = -6\frac{\mu^2}{r_1^5} - 3\frac{\dot{R}}{R}\left(V_1 - \frac{\mu}{r_1^2}\right) - 6\frac{\mu^2 R}{r_1^4}\left(\frac{R}{r_1}\right)^3 \tag{67}$$

and

$$\ddot{R} = \frac{1}{2R}\left[\left(\frac{\mu}{r_1^2}\right)^2 - \left(\frac{\mu}{r_0^2}\right)^2\right] + \frac{1}{4R}\left(V_1 - \frac{\mu}{r_1^2}\right)^2 - \frac{3}{2}\frac{\dot{R}^2}{R} + \frac{P_{in} - P_0}{R} - \frac{2}{R^2} + \frac{5}{3R}\frac{\mu^2}{r_1^4}\left(\frac{R}{r_1}\right)^2, \tag{68}$$

respectively. Equation (67) indicates that the acceleration of the bubble's translating motion consists of three components induced by three different causes. The first one $-6\frac{\mu^2}{r_1^5}$ is due to the acceleration of the liquid in the flow field. This component is independent of the bubble radius and would cause the bubble to accelerate toward the source in case of a source flow or toward the sink in case of a sink flow. It is interesting to compare the magnitude of this component with the acceleration of the bubble's translating motion induced purely by the action of gravitation which is neglected in the present study. Walters and Davidson (5) have disclosed that the acceleration of a bubble with constant size in the liquid under the gravitational acceleration g is equal to $2g$ during the initial stage of bubble motion after transient. In the present analysis the acceleration of the liquid itself in the absence of the bubble is equal to

$$\frac{d}{dt}\left(\frac{\dot{R}}{r_1^2}\right) = -2\frac{\dot{R}}{r_1^3}\frac{dr}{dt} = -2\frac{\dot{R}}{r_1^3}\frac{dr_1}{dt} = -2\frac{\dot{R}}{r_1^3}\left(\frac{\dot{R}}{r_1^2}\right) = -2\frac{\dot{R}^2}{r_1^5}$$

By comparison it can be seen that the acceleration of the liquid may cause three times as much acceleration to the bubbles translating motion provided that the other two components in equation (67) are excluded from consideration.

The second term on the right side of equation (67) corresponds to the acceleration of the bubble's translating motion induced by the interaction between $\dot{R}$ the rate of bubble growth or collapse, and $(v_1 - \frac{\dot{R}}{r_1^2})$, the relative velocity between the liquid and the bubble. The third term is the acceleration caused by the liquid velocity alone. Since the latter is of the third power of the ratio R/r_1, its effect is indeed negleigible.

Equation (68) indicates the acceleration of the bubble surface. The quantity $\frac{1}{2}\left[\left(\frac{\dot{R}}{r_1^2}\right)^2 - \left(\frac{\dot{R}}{r_0^2}\right)^2\right]$ is the change in the liquid pressure when the bubble is located at two different positions: the initial position $r_1(0) = r_0$ and an instantaneous position $r_1(t) = r_1$. Therefore, the first term in equation (68) expresses the contribution of the change in the liquid pressure to the acceleration of the bubble surface. The second term $\frac{1}{4R}\left(\frac{\dot{R}}{r_1^2} - v_1\right)^2$ shows the effect of the relative velocity between the liquid and bubble motion, and the third term $-\frac{3}{2}\frac{\dot{R}^2}{R}$ is the effect of the rate of bubble growth or collapse on the acceleration of the bubble surface. If the ideal gas inside the bubble undergoes an adiabatic process then

through the aid of equation (54) the fourth term may be written as

$$\frac{P_{in} - P_o}{R} = \frac{P_{in}(0)}{R}\left(\frac{1}{R^{3\gamma}} - 1\right) + \frac{2}{R \cdot R(0)} \quad . \tag{69}$$

The fifth term $-2/R^2$ is the effect of surface tension and the last term indicates the effect of the liquid velocity. The latter effect is considered small since it involves the second power of R/r_1. Equations (67) and (68) were integrated simultaneously for r_1 and R subject to the initial conditions:

$$R(0) = 1 \ , \quad \dot{R}(0) = 0 \ . \tag{70}$$

The physical properties selected for the present computation are: room temperature = 77°F, the density ρ of water = 0.997044 gram/cm^3, and the surface tension σ for water-air system = 71.97 dynes/cm. If the initial bubble radius is $R(0) = 0.5$ cm, then the parameter $[\sigma/\rho R(0)]^{1/2}$ in (66) would be

$$\left(\frac{\sigma}{\rho R(0)}\right)^{1/2} = 0.393 \ \ ft/sec \tag{71}$$

For the liquid velocity of 80 ft/sec (approximately the same as the experiments reported in reference 5), dimensionless velocity is 204.

For a bubble travelling in a sink flow

$$P_{in}(0) = 4395$$

$$r_1(0) \equiv r_0 = 105$$

$$V_1(0) \equiv V_0 = -200$$

μ is selected as

$$\mu = -2.2(10^6)$$

such that the liquid velocity at the bubble's initial position $r_0 = 105$ is same as the bubble's translating velocity. The result is plotted in Figure 5. The calculation is carried out up to the position $r = 100$ where the liquid pressure reaches its saturated value $p = 195$. It is disclosed from Figure 5 that both the velocity V_1 and the radius R increase monotonously with the bubble's position.

For a bubble travelling in a source flow the following initial conditions are employed.

$$p_0 = 195$$

$$r_0 = 100$$

$$\mu = 2.2(10^6)$$

and liquid velocity at $r_1 = r_0$ is

$$\frac{\mu}{r_0^2} = 220.$$

Several different initial velocities for the bubble's translating motion were used in numerical calculations: ranging from that of the liquid, namely $V_0 = 220$, to higher values. In practice, the physical situation corresponding to the given conditions may be achieved by connecting different size of nozzles or pipes with a diffuser as shown in Figure 6. Figure 6(a) illustrates the case where a straight pipe is connected to a diffuser. With the neglect of friction, an air bubble formed upstream

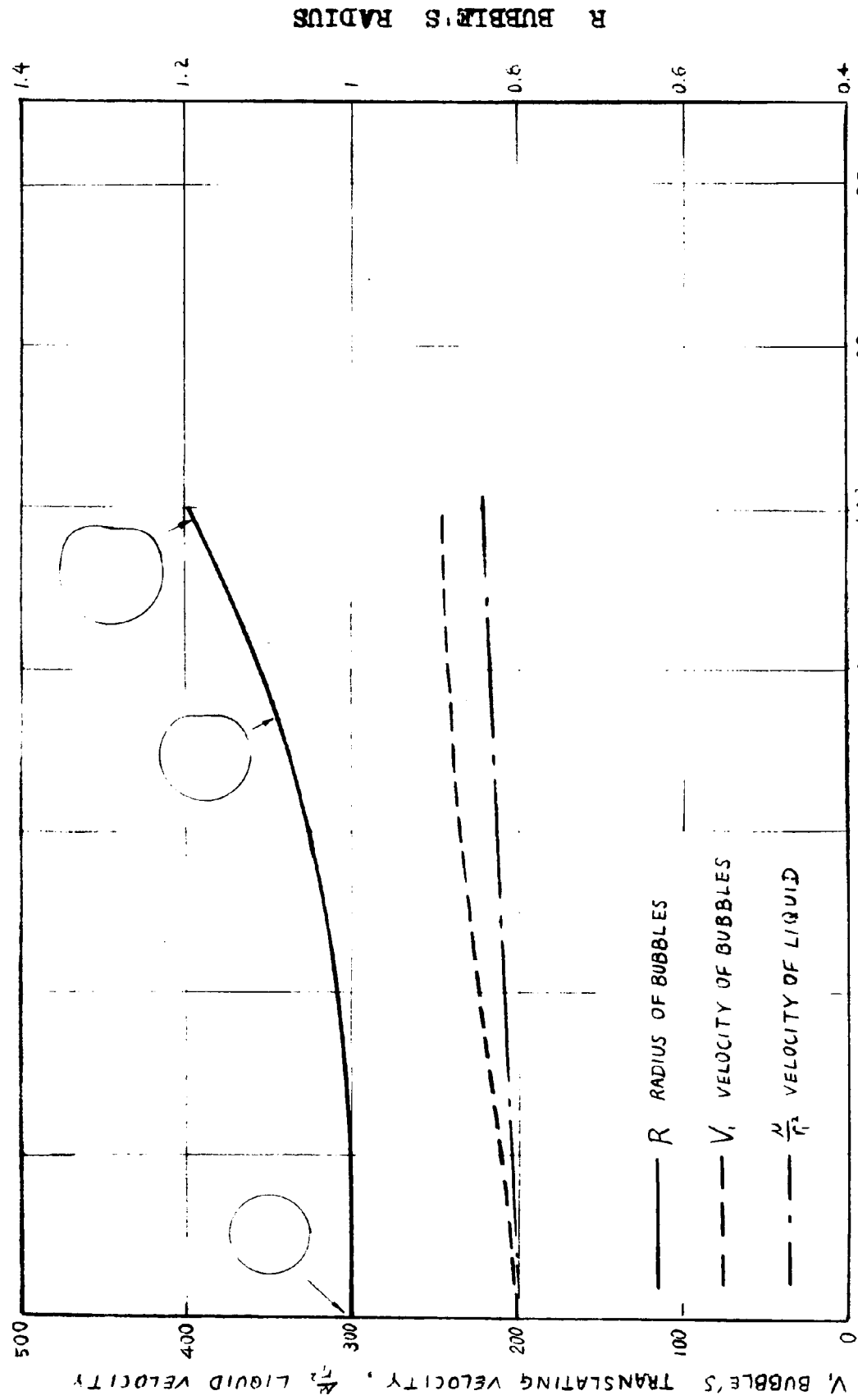


Figure 5. The velocities μ/r_1^2 and V_1 and the radius R of a spherical bubble moving in a sink flow with the initial conditions $r_1 = 1.05$, $R_0 = 1$, $V_0 = -.95$ and $\frac{\mu}{r_0^2} = .4395$. (The bubble shape is predicted by equations (52) and (57)

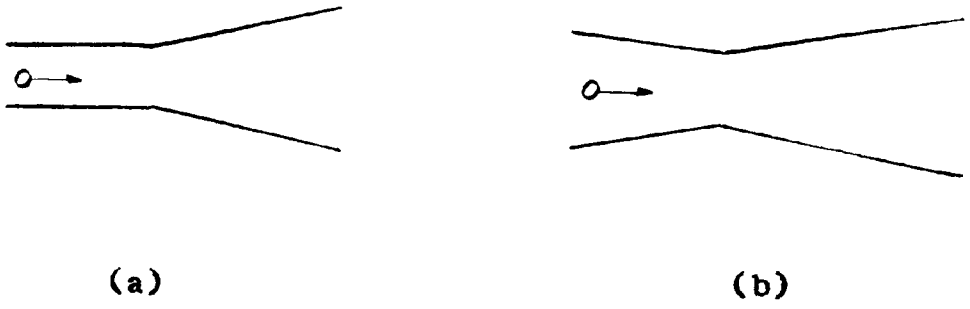


Figure 6. Geometrical arrangement of (a) a pipe and a diffuser and (b) a nozzle and a diffuser.

travels with the same velocity, say 220, as that of liquid in the pipe. Considering the entrance to the diffuser as an initial position in the present study the initial velocity of the bubble would be that of liquid namely $V_0 = 220$. Now, the straight pipe is replaced by a nozzle as in Figure 6(b), Then the bubble will be accelerated in the nozzle like a sink flow. So by the time the bubble reaches the entrance of the diffuser, its velocity is greater than that of the surrounding liquid. By varying the length or shape of the nozzle it is feasible to obtain a desired velocity of the bubble at the entrance of the diffuser. The conditions at the entrance are then employed as the initial conditions.

Figures 7 through 14 illustrate the changes in the bubble radius R and bubble's translating velocity V_1 with the bubble's position in the diffuser. An examination of these figures reveals that depending upon the magnitude of the initial velocity V_0 the bubble's translating velocity and size may vary in two different ways as schematically illustrated in Figures 15 and 16. There exists a critical value for V_{0cr} beyond which the bubble velocity and size may follow a different pattern: When V_0 is lower than the critical value V_{0cr} the bubble velocity and size vary following the pattern as shown in Figure 15, while when V_0 exceeds V_{0cr} these variations follow the other pattern as shown in Figure 16. For the parameters used in the present computations, the

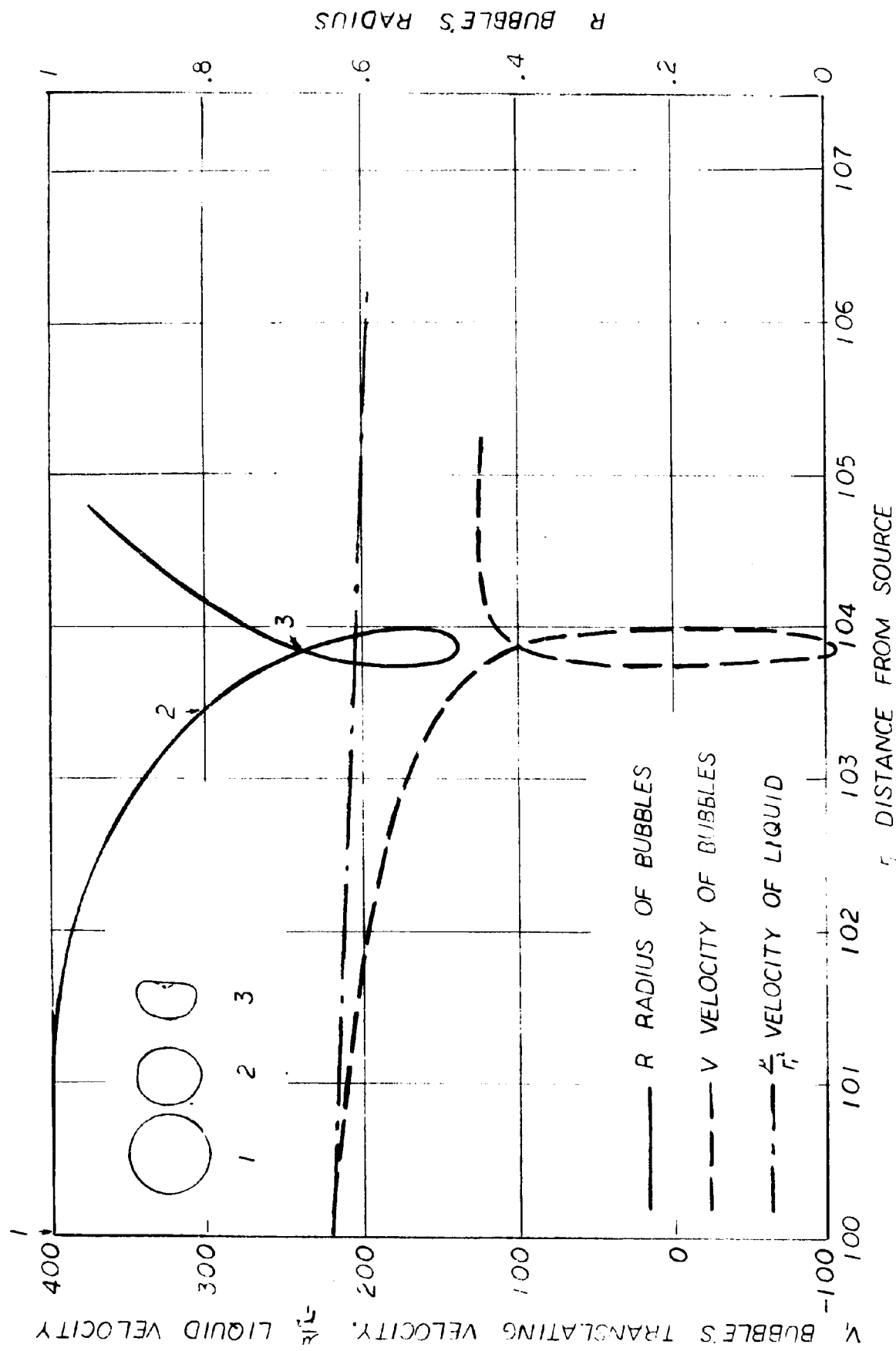


Figure 7. The velocities μ/r_1^2 and V_1 and the radius R of a spherical bubble moving in a source flow with the initial conditions $r_p=100$, $R(0)=1$, $V_0=220$ and $p_\infty(0)=195$. (The bubble shape is predicted by equations (52) and (57).)

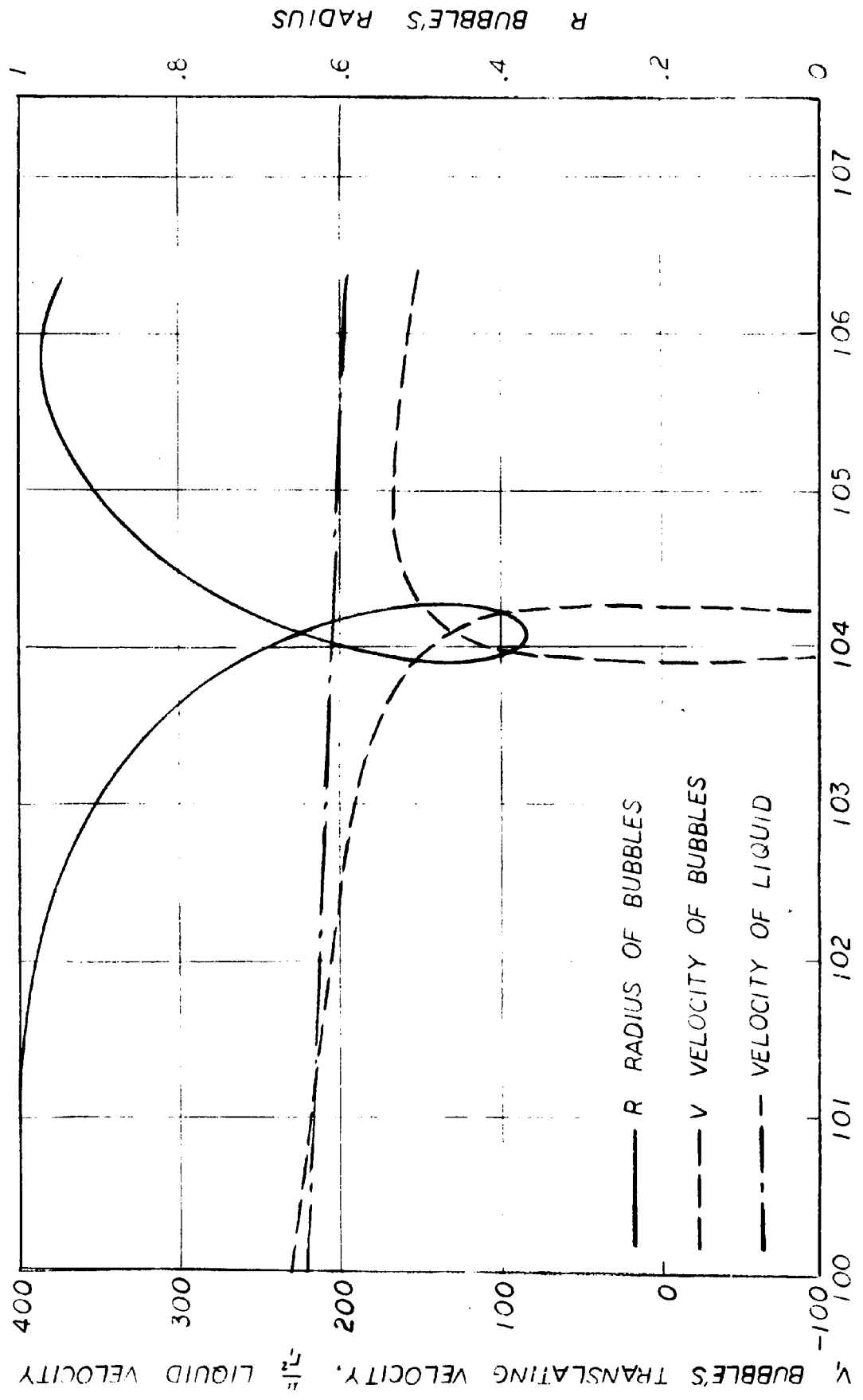


Figure 8. The velocities μ/r_i^2 and V_i and the radius R of a spherical bubble moving in a source flow with the initial conditions $r_0=100$, $R(0)=1$, $V_0=230$ and $P_{in}(0)=195$.

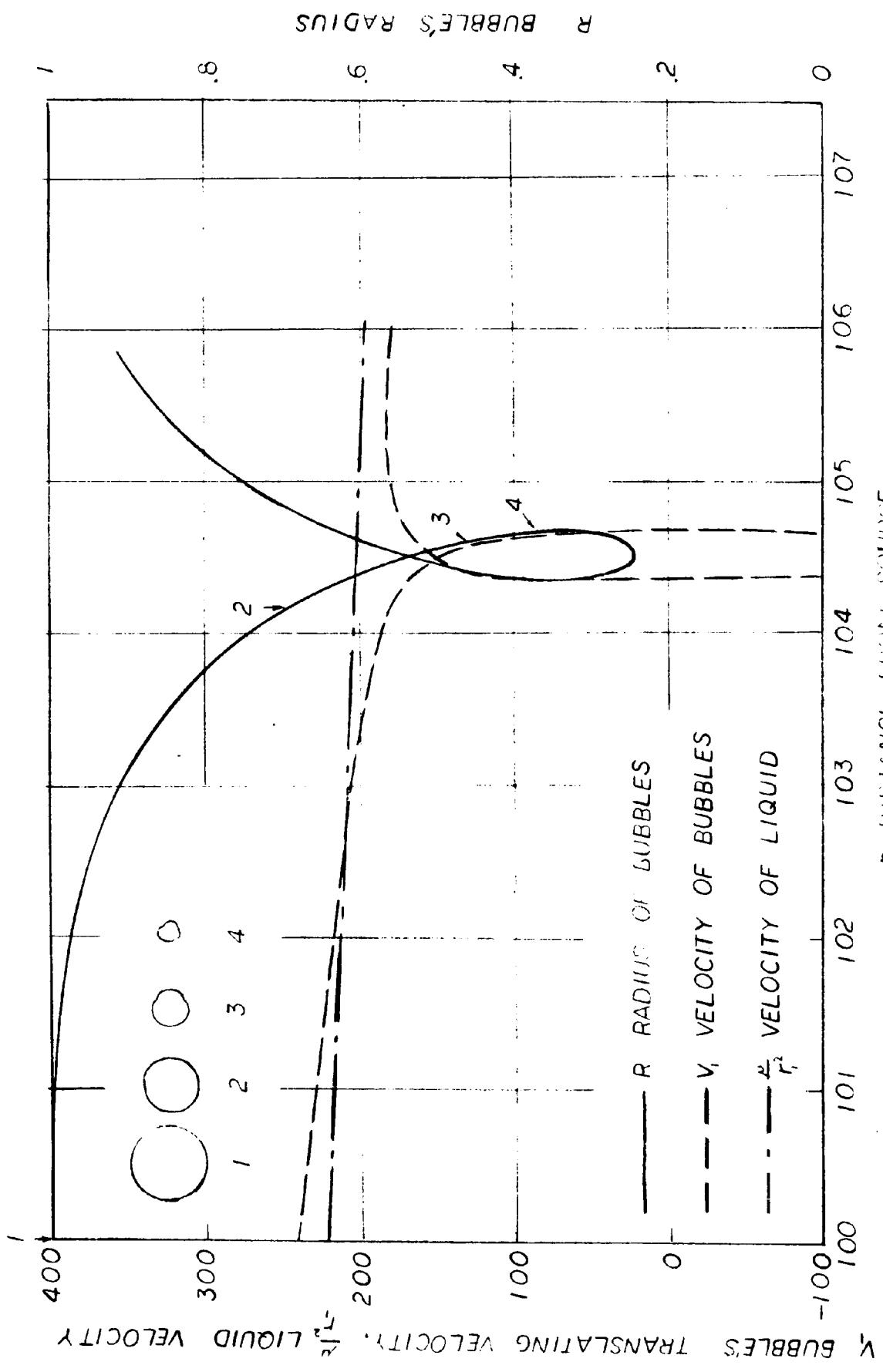


Figure 9. The velocities μ/r_1^2 and V_1 and the radius R of a spherical bubble moving in a source flow with the initial conditions $r_0=100$, $R(0)=1$, $V_0=240$ and $p_\infty(0)=195$. (The bubble shape is predicted by equations (52) and (57).)

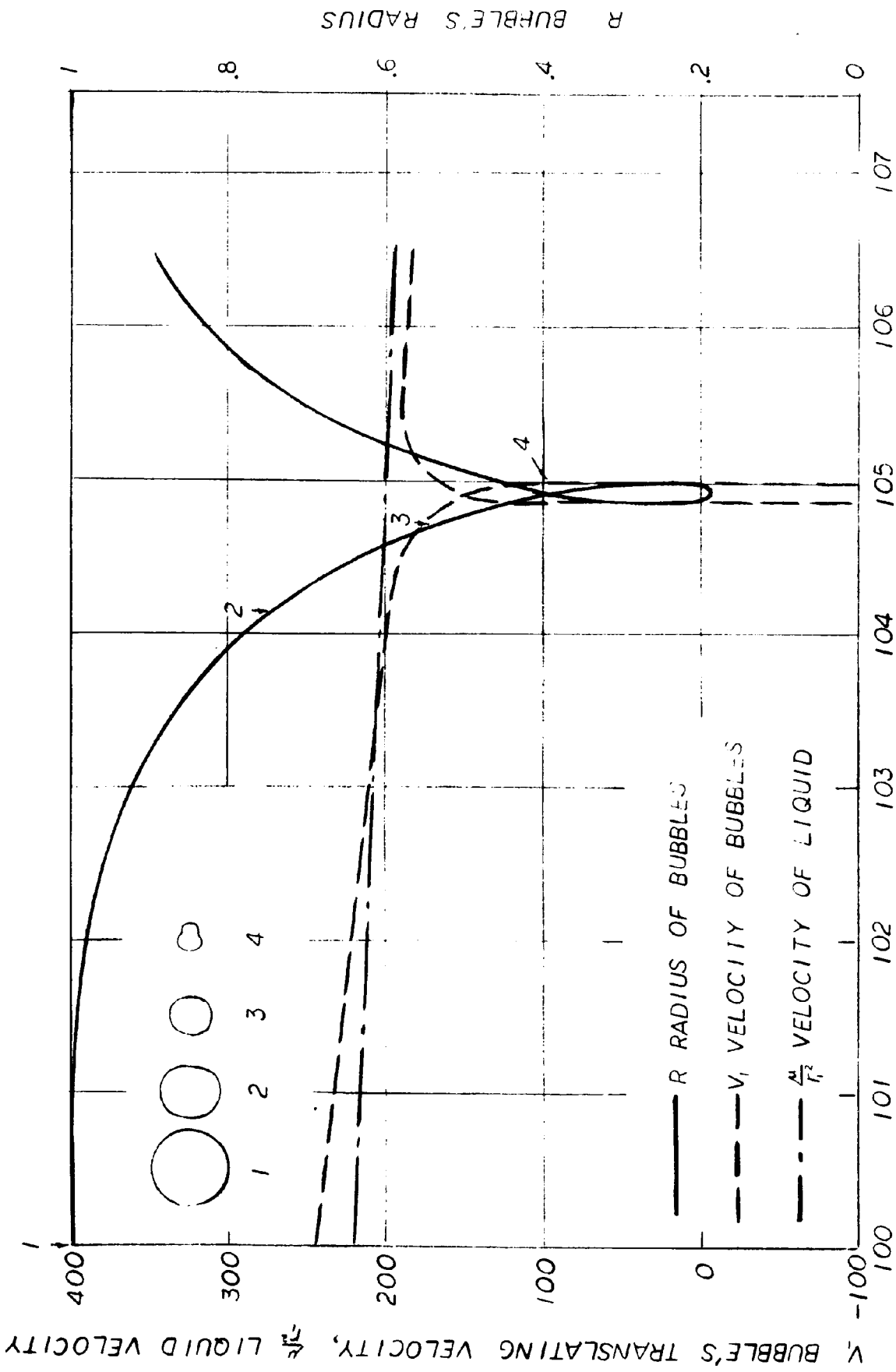


Figure 10. The velocities μ/r_1^2 and V_1 and the radius R of a spherical bubble moving in a source flow with the initial conditions $r_0=100$, $R(0)=1$, $V_0=245$ and $\mu_{in}(0)=195$. (The bubble shape is predicted by equations (52) and (57).)

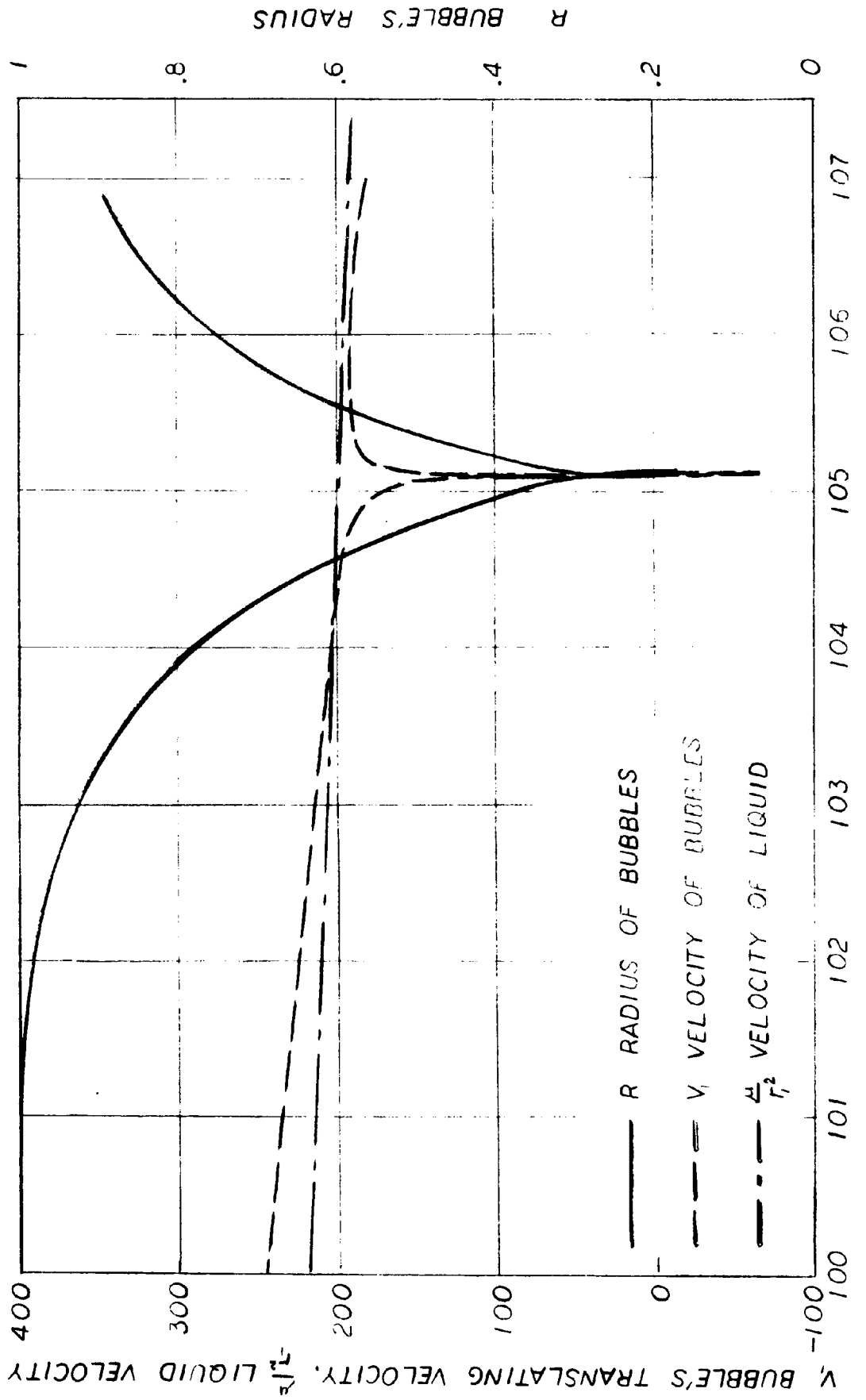


Figure 11. The velocities μ/r_1^2 and V_1 and the radius R of a spherical bubble moving in a source flow with the initial conditions $r_0 = 100$, $R(0) = 1$, $V_0 = .247$ and $\mu_{in}(0) = 199$.

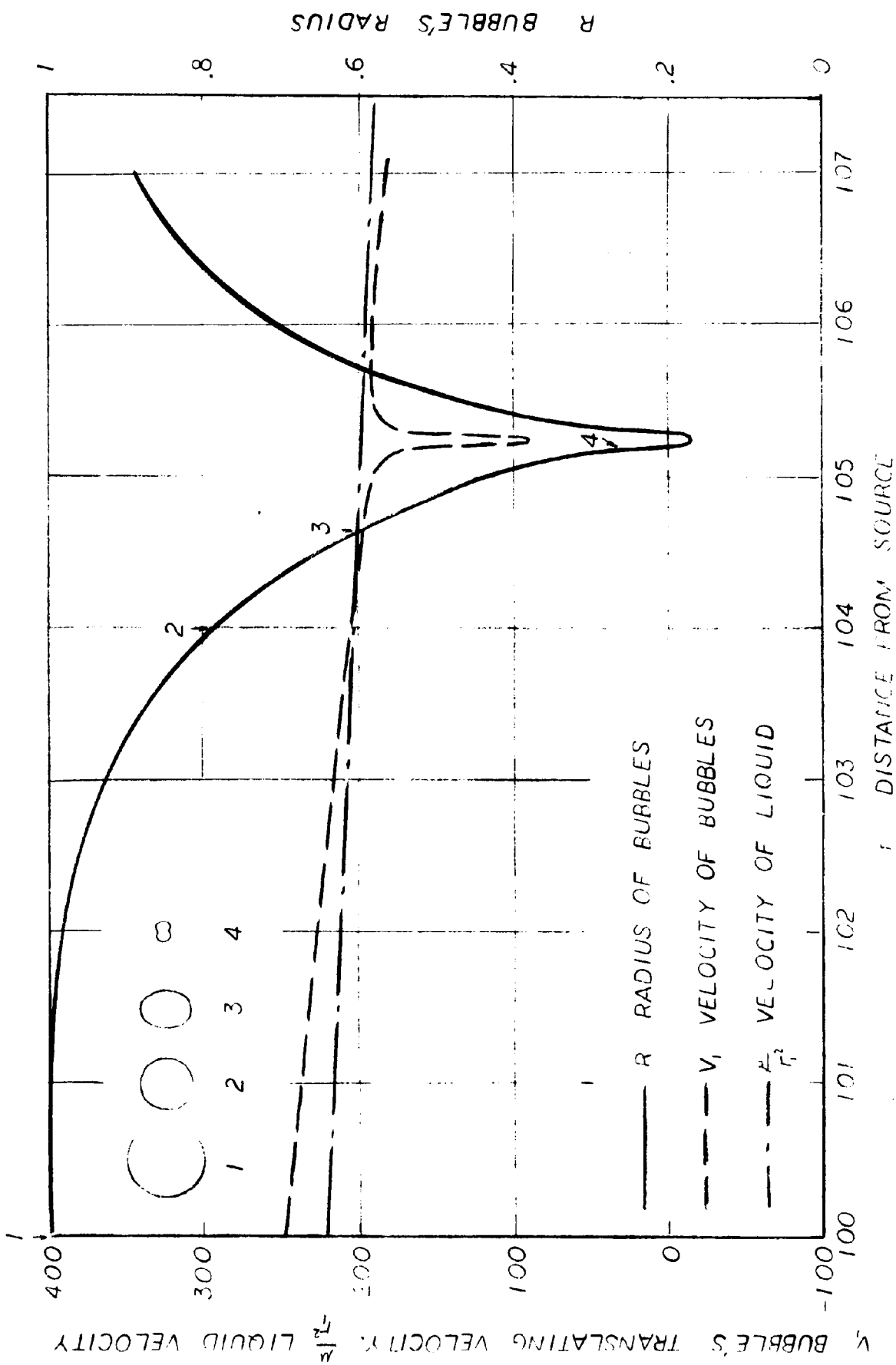


Figure 12. The velocities μ/r_1^2 and V_1 and the radius R of a spherical bubble moving in a source flow with the initial conditions $r_0=100$, $R(0)=1$, $V_0=248$ and $P_{in}(0)=195$. (The bubble shape is predicted by equations (52) and (57).)

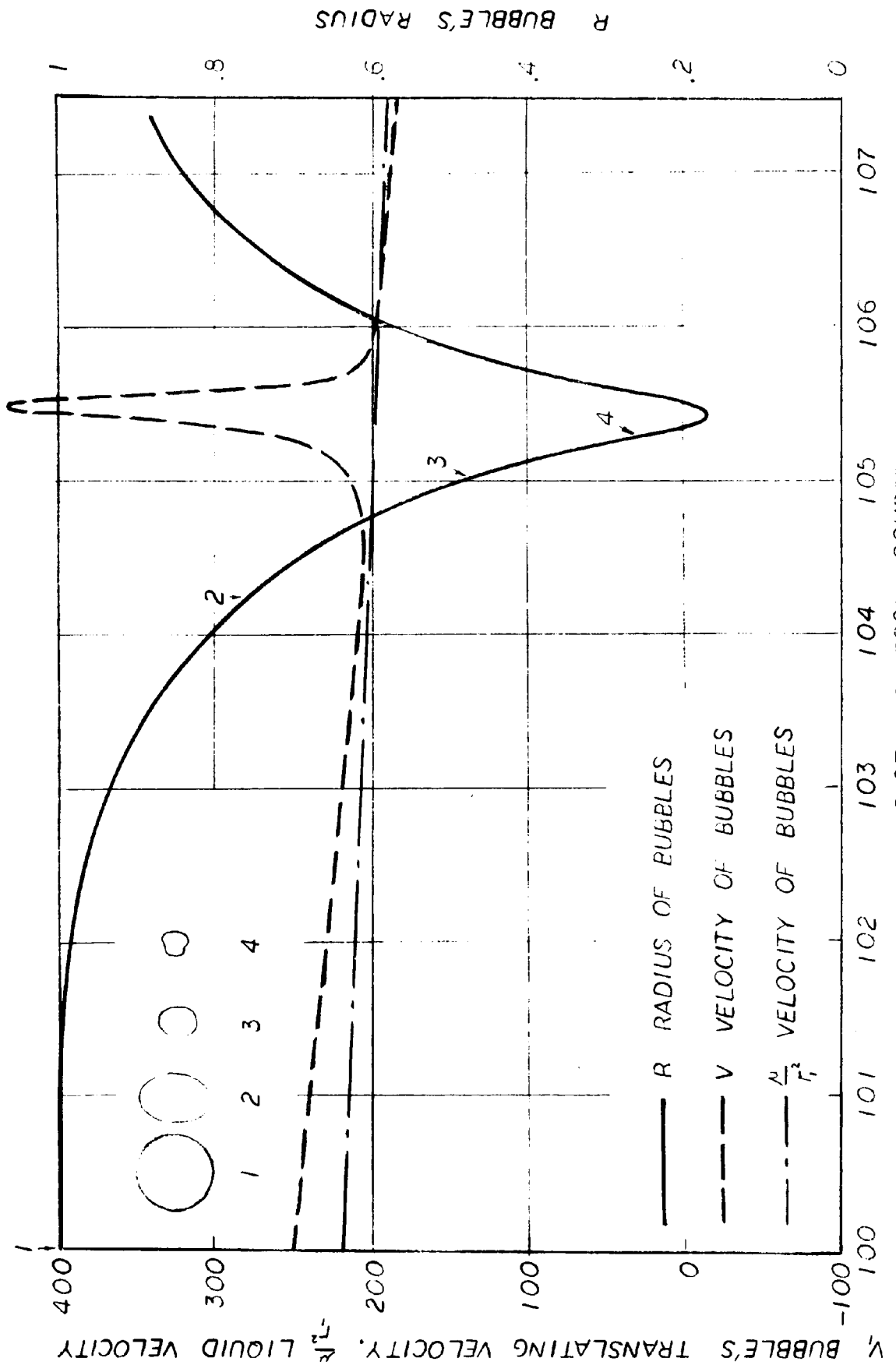


Figure 13. The velocities μ/r_1^2 and V_1 and the radius $\bar{R}$ of a spherical bubble moving in a source flow with the initial conditions $r_0=100$, $\bar{R}(0)=1$, $V_0=200$ and $p_{in}(0)=195$. (The bubble shape is predicted by equations (52) and (57).)

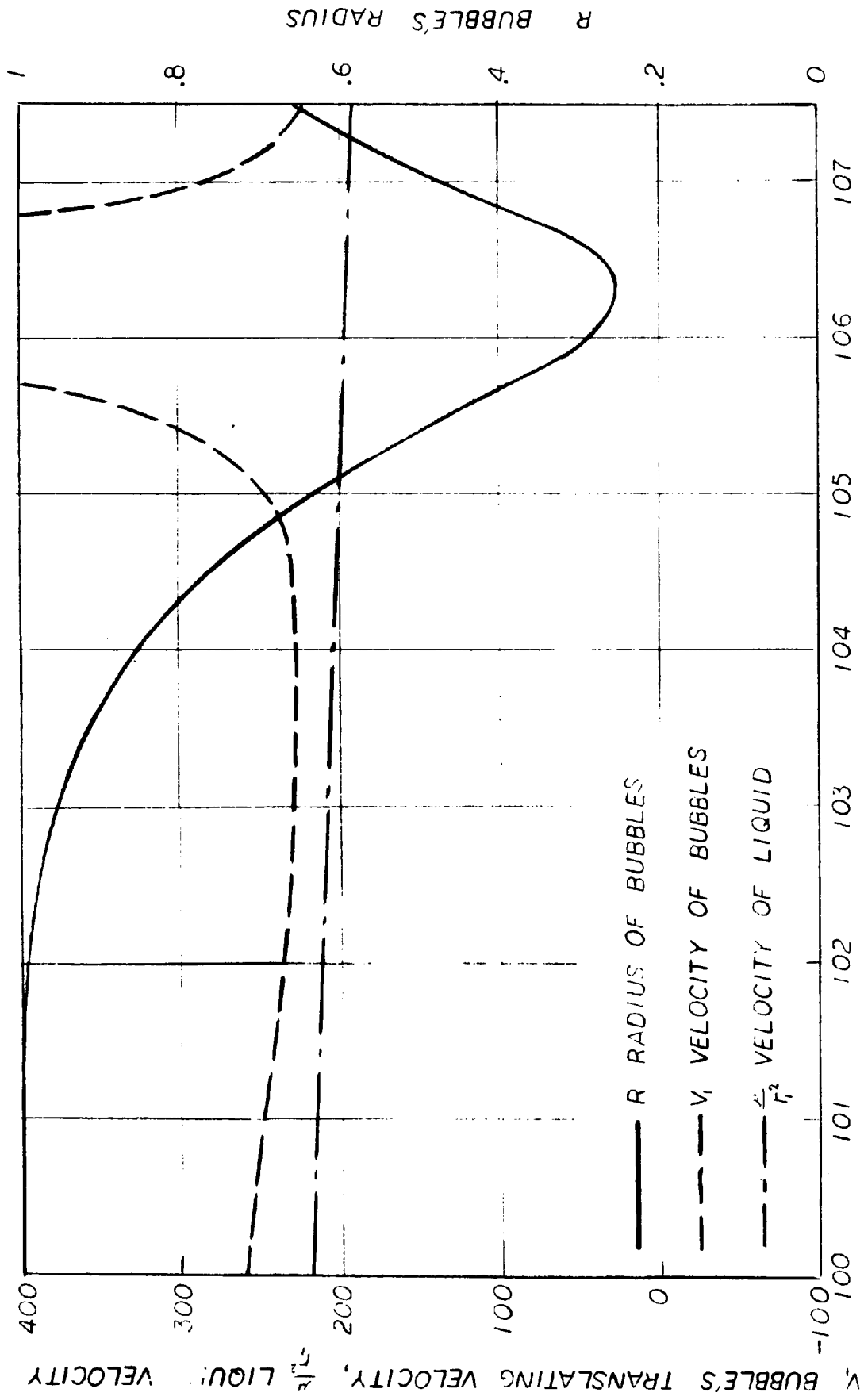


Figure 14. The velocities μ/r_i^2 and V_i and the radius R of a spherical bubble moving in a source flow with the initial conditions $r_o=100$, $R(0)=1$, $V_o=260$ and $p_n(0)=195$.

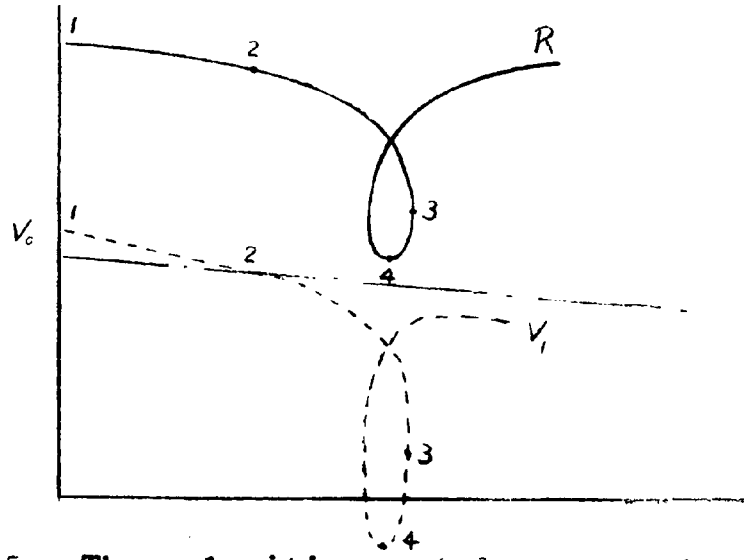


Figure 15. The velocities μ/r_1^2 and V_1 and the radius R of a spherical bubble moving in a source flow, initial bubble velocity $V_0 < V_{0cr}$.

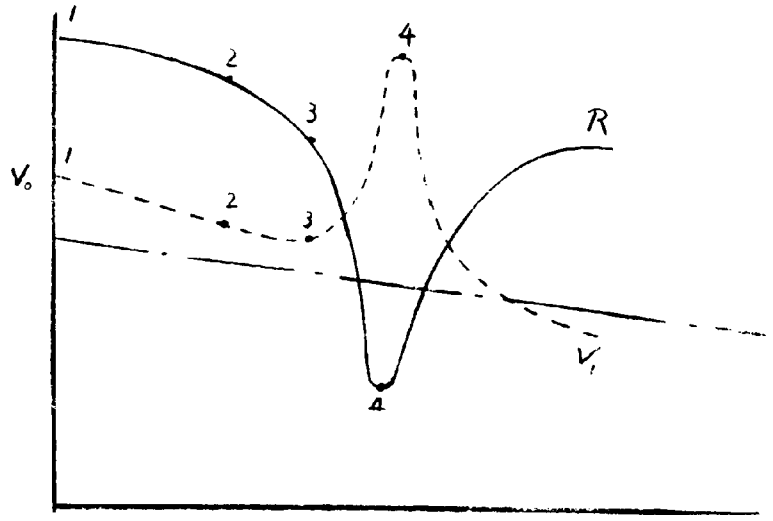


Figure 16. The velocities μ/r_1^2 and V_1 and the radius R of a spherical bubble moving in a source flow, initial bubble velocity $V_0 > V_{0cr}$.

critical initial velocity is 249

When the initial velocity V_o is smaller than V_{ocr} the bubble's translating motion follows the pattern shown in Figure 15. During the initial stage of the transient corresponding to the interval between points 1 and 2 in the figure the translating motion of the bubble is governed by the force induced by the deceleration of the liquid, namely the $-6\mu^2/r_1^2$ term, while the bubble collapse is governed by the force induced by the change in the liquid pressure, namely the $\frac{1}{2R}\left[\left(\frac{\mu}{r_1^2}\right)^2-\left(\frac{\mu}{r_0^2}\right)^2\right]$ term. However, as the bubble reaches point 2, the relative velocity $\left(V_1-\frac{\mu}{r_1^2}\right)$ changes its sign. As the result the force induced by the bubble collapse and the relative velocity $-3\frac{\dot{R}}{R}\left(V_1-\frac{\mu}{r_1^2}\right)$ in equation (67) becomes negative, which causes the bubble to further decelerate. As the bubble travels further downstream the collapse rate $\dot{R}$ continues to increase thus causes an increase in the quantity $-\frac{3}{2}\dot{R}^2/R$ in equation (68). Therefore, the increases in the collapse rate and relative velocity interact to cause the bubble to further decelerate. This is indicated by a very rapid increase in the term $-3\frac{\dot{R}}{R}\left(V_1-\frac{\mu}{r_1^2}\right)$ in equation (67). The translating velocity of bubble eventually becomes negative and the bubble starts to travel upstream when it reaches point 3 in Figure 15. After point 3 the term $\frac{1}{4R}\left(\frac{\mu}{r_1^2}-V_1\right)^2$ in equation (68) increases as the relative velocity increases. Since this term is always positive it exerts a force to retard the collapse

rate and eventually causes the bubble to stop collapsing when it reaches point 4 in Figure 15. The bubble then starts to grow and travels upstream. This phenomenon may be called "regrow" which was observed in experiments (5). At the moment of "regrow" the sign of $\dot{R}$ changes and the term $-3\frac{\dot{R}}{R}(V_1 - \frac{\mu}{r_1^2})$ in equation (67) becomes positive. This causes the bubble to accelerate so rapidly that the bubble's translating velocity V_1 soon becomes positive and moves downstream again. However as V_1 increases the relative velocity $(V_1 - \frac{\mu}{r_1^2})$ as well as the accelerating term $-3\frac{\dot{R}}{R}(V_1 - \frac{\mu}{r_1^2})$ tend to decrease. The latter term may eventually be exceeded by the decelerating terms in equation (67). Therefore the value of V_1 cannot exceed μ/r_1^2 in the accelerating stage.

When the initial velocity V_0 is greater than V_{0cr} the translating motion of the bubble follows the pattern shown in Figure 16. During the initial stage corresponding to the interval between points 1 and 2 in Figure 16, the translating motion of bubble is governed by the deceleration of the liquid, while the bubble collapse is governed by the change in the liquid pressure. As the bubble travels downstream, $\dot{R}$ increases and the $-3\frac{\dot{R}}{R}(V_1 - \frac{\mu}{r_1^2})$ term in equation (67) becomes important. Since this term is positive, it tends to reduce the deceleration of the bubble motion before the translating velocity of the bubble is reduced to that of the liquid, the magnitude of this term may become comparable to that of the term -6μ

in equation (67) and eventually the bubble becomes accelerated. This corresponds to point 3 in Figure 16. Meanwhile the term $-\frac{3}{2}\dot{R}^2/R$ in equation (68) governs the bubble collapse as in the previous case. As the bubble is accelerated, the relative velocity $(V_1 - \frac{\mu}{r_1^2})$ increases. So does the term $\frac{1}{4R}(V_1 - \frac{\mu}{r_1^2})^2$ in equation (68). As a result, the bubble's collapse rate $\dot{R}$ decreases At point 4 in Figure 16 the bubble stops collapsing and starts to grow. But due to the change in sign for the term $3\frac{\dot{R}}{R}(V_1 - \frac{\mu}{r_1^2})$ the bubble's translating velocity is retarded.

The gas pressure inside the bubble plays an important role only when the bubble size is small. Although the gas pressure may somewhat help the "regrow" of the bubble, for high liquid velocity as in those computations it is the $\frac{1}{4R}(V_1 - \frac{\mu}{r_1^2})^2$ term which plays more important role on the "regrow" phenomenon. As a matter of fact, even for the case of $p_{in} = 0$ the phenomenon of "regrow" still occurs as shown in Figures 17 and 18. This is how the "regrow" of the bubble is different from the rebound of a balloon which is solely due to its internal pressure.

The effect of surface tension is entirely negligible for the parameters employed in the present computations.

Numerical results were also obtained for the case where the bubble first moves in a sink flow for a certain distance then travels in a source flow of equal strength

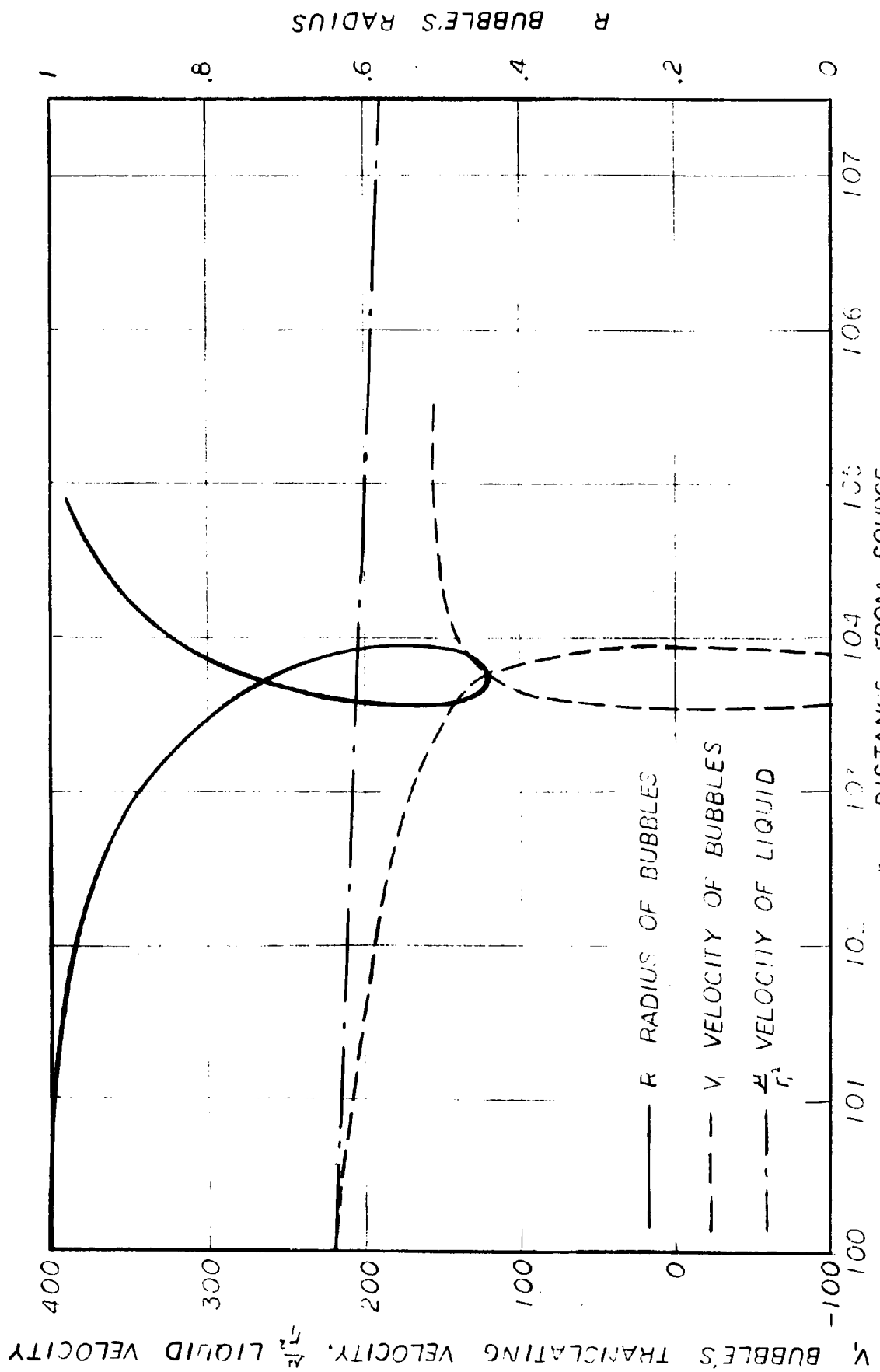


Figure 17. The velocities μ/r_1^2 and V_1 and the radius R of a spherical bubble moving in a source flow with the initial conditions r_0=100, R(0)=1, V_0=220 and $P_{in}(0)$=0.

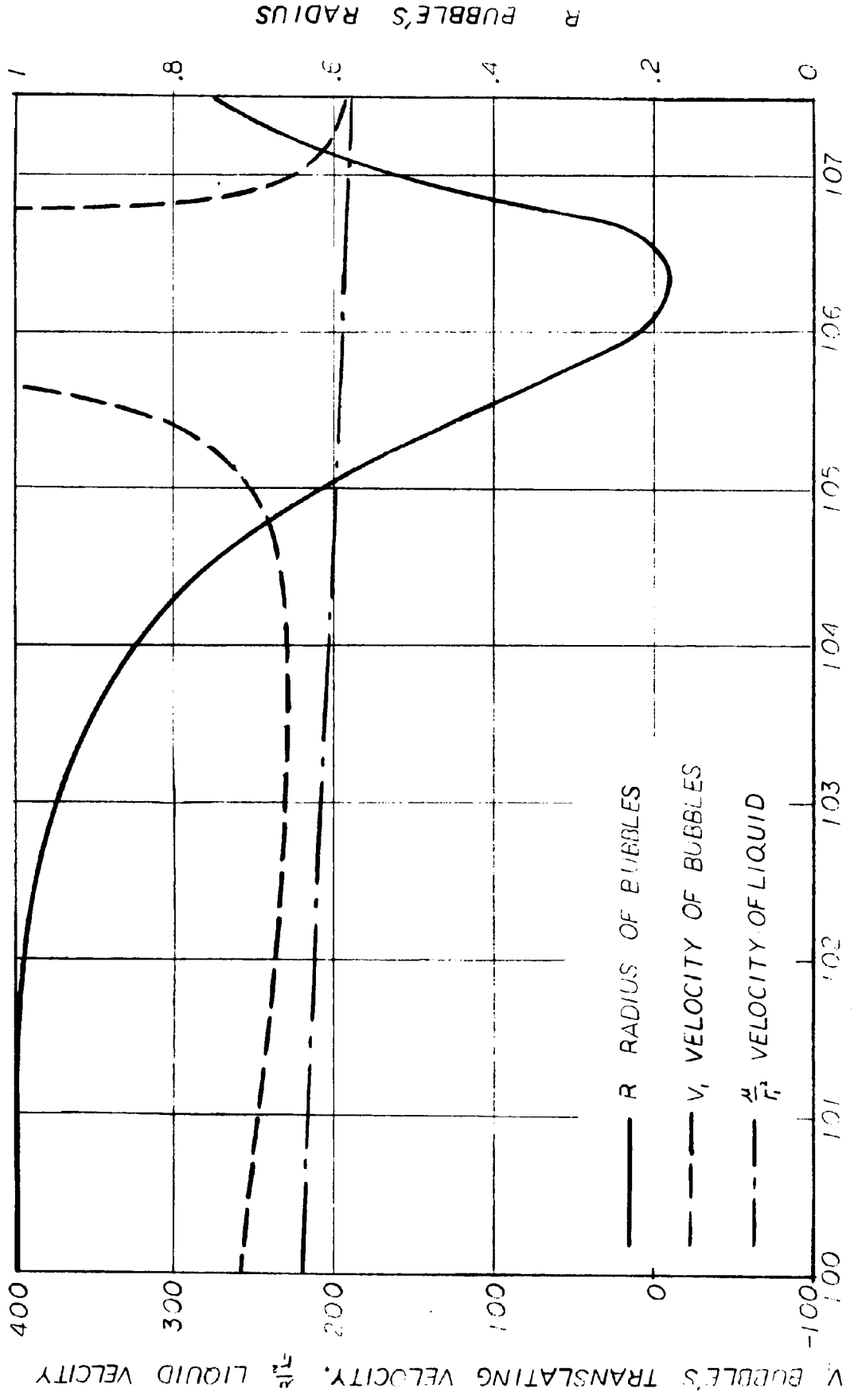


Figure 18. The velocities μ/r_1^2 and V_1 and the radius R of a spherical bubble moving in a source flow with the initial conditions $r_0=100$, $R(0)=1$, $V_0=260$ and $p_{in}(0)=0$.

as the sink. In practice this corresponds to the case where the bubble travels in a venturi tube. The results are presented in Figure 19. It is seen from this figure that the bubble continues to grow over a certain distance in the diffuser portion until location 2, then followed by collapsing.

If the bubble maintains a spherical shape, Figure 19 shows that it first grows, then collapses followed by regrowing. This process repeats as the bubble moves downstream. The deceleration and acceleration of bubble's translating motion also repeat. In practice the bubble cannot remain spherical. As the result its shape starts to shatter when the bubble collapses to a certain size.

IV-1-b The time history of bubble shape

Using the same initial conditions as the previous section, equations (52) and (57) were numerically integrated for r_1, R and a_n's by retaining the first twelve a_n's The bubble is initially spherical. That is at $t=0$, $a_n=0$ and $\dot{a}_n=0$ for all a_n. It is disclosed from the numerical results that the values of r_1, V_1, R and $\dot{R}$ thus obtained are very closed to those obtained with neglection of the a_n.

The values of the a_n for a typical case were presented in Appendix V, With the a_n available equation (9) was used to evaluate r_3. The values of P_n needed for evaluating equation (9) were obtained from the following formula

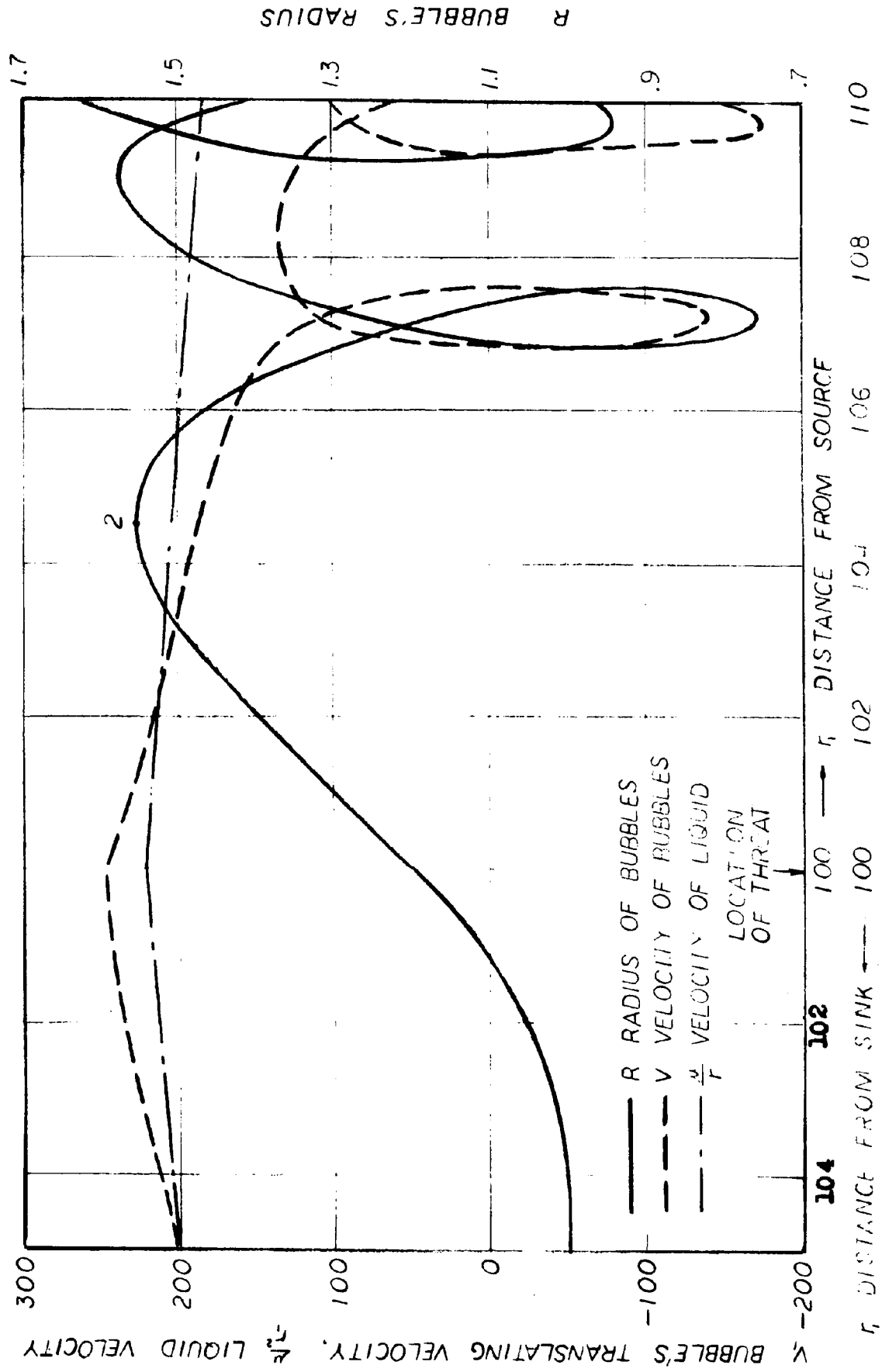


Figure 19. The velocities μ/r_i^2 and V_i and the radius R of a spherical bubble moving in a sink flow and then in a source flow with the initial conditions r_o=105, R(0)=1, V_o=-200, $p_{in}(0)$=4395.

$$P_n(x) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{1\cdot 2\cdot 3\cdots n}\left\{x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2}\right.$$

$$\left. + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot(2n-1)(2n-3)}x^{n-4} - \cdots\right\}$$

The results are presented in Table 1.

The time history of the bubble shape is presented in Figures 20 through 26. For a sink flow, the bubble retains its spherical shape for a distance and then starts to flatten out at the downstream side as shown in Figure 20.

For a source flow, however, the change in the bubble shape is rather complicated depending on the initial bubble velocity V_0. For small values of V_0 the bubble first changes its shape from a sphere to an ellipsoid, then starts to dent on the downstream side and eventually becomes a crater as shown in Figure 21.

As the initial velocity increases the indentation shifts upstream along the bubble surface as shown in Figure 22 to 25 for V_0 between 238 and 248. When the initial velocity is further increased the indentation occurs at the upstream side of the bubble surface as illustrated in Figure 26.

The numerical results indicate that in the vicinity of point 3 in Figures 15 and 16 where both $V_1 - \frac{dr_3}{dt}$ and $\dot{R}$ are large, the a_n increase very rapidly and soon become so

TABLE 1. Numerical values of Legendre polynomials

θ	$P_1 (=\cos\theta)$	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}
0°	1	1	1	1	1	1	1	1	1	1	1	1
15°	.9659	.8995	.8042	.6847	.5471	.3983	.2455	.0962	-.0428	-.1651	-.2655	-.3402
30°	.8660	.6250	.3248	.0234	-.2233	-.3740	-.4102	-.3388	-.1896	.0000	.1607	.2732
45°	.7071	.2500	-.1768	-.4063	-.3756	-.1484	.1271	.2983	.2855	.1151	-.1042	-.2467
60°	.5000	-.1250	-.4375	-.2891	.0898	.3232	.2231	-.0736	-.2679	-.1882	.0639	.2338
75°	.2588	-.3995	-.3449	.1434	.3427	.0431	-.2731	-.1702	.1595	.2316	-.0306	-.2275
90°	.0000	-.5000	.0000	.3750	.0000	-.3125	.0000	.2734	.0000	-.2461	.0000	.1610
105°	-.2588	-.3995	.3449	.1434	-.3427	.0431	.2730	-.1702	-.1595	.2316	.0305	-.2275
120°	-.5000	-.1250	.4375	-.2891	-.0898	.3232	-.2232	-.0736	.2679	-.1882	-.0639	.2338
135°	-.7071	.2500	.1768	-.4062	.3757	-.1485	-.1270	.2983	-.2855	.1151	.1042	-.2467
150°	-.8660	.6250	-.3248	.0235	.2233	-.3740	.4102	-.3388	.1896	.0000	-.1607	.2732
165°	-.9659	.8995	-.8042	.6847	-.5471	.3983	-.2456	.0962	.0427	-.1650	-.2654	-.3402
180°	+1	+1	-1	+1	-1	+1	-1	+1	-1	+1	-1	+1

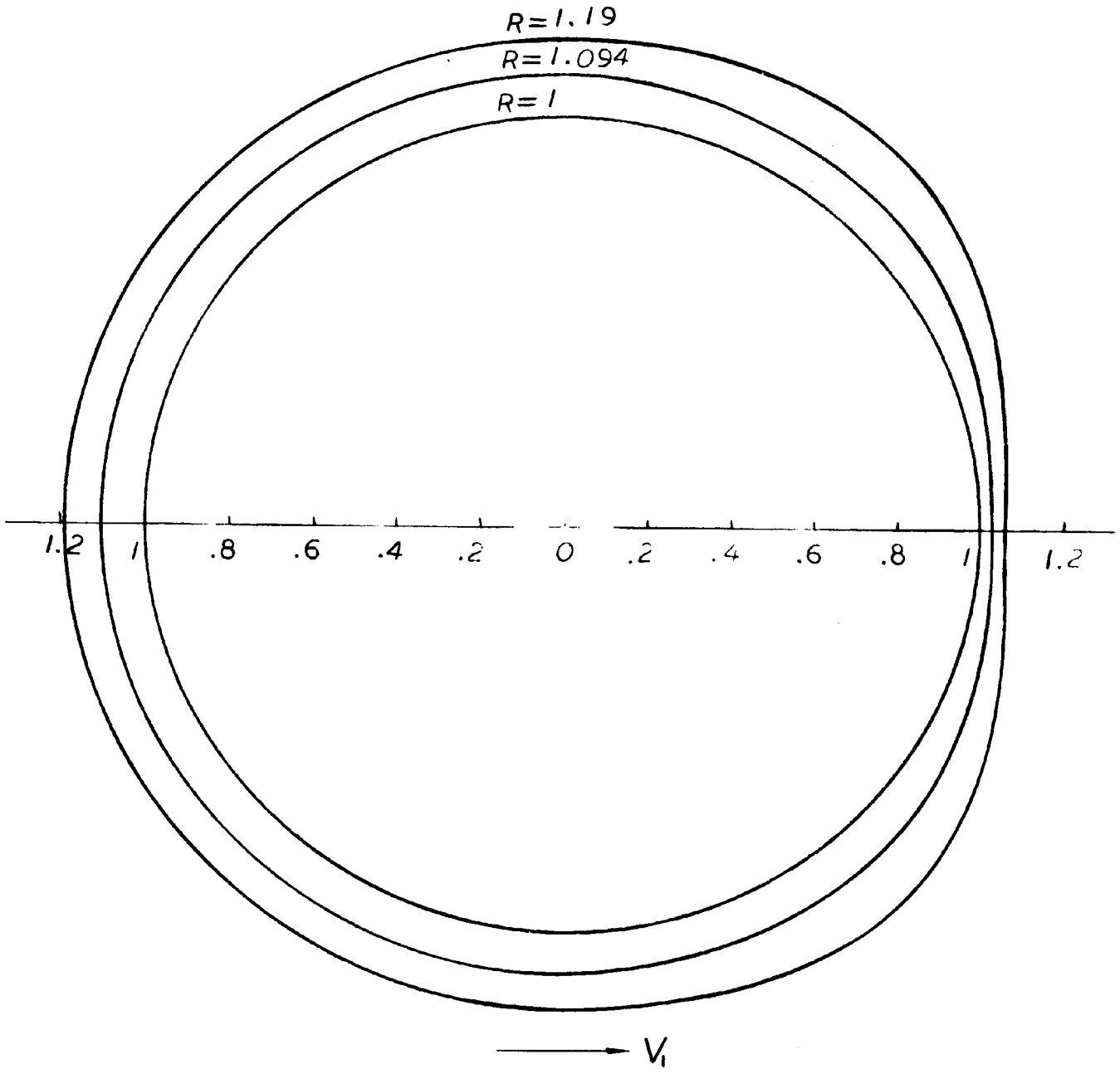


Figure 20. The transient shape of an initially spherical bubble moving in a sink flow with the initial conditions $r_0=105$, $R(0)=1$, $V_0=200$ and $p_{in}(0)=4395$.

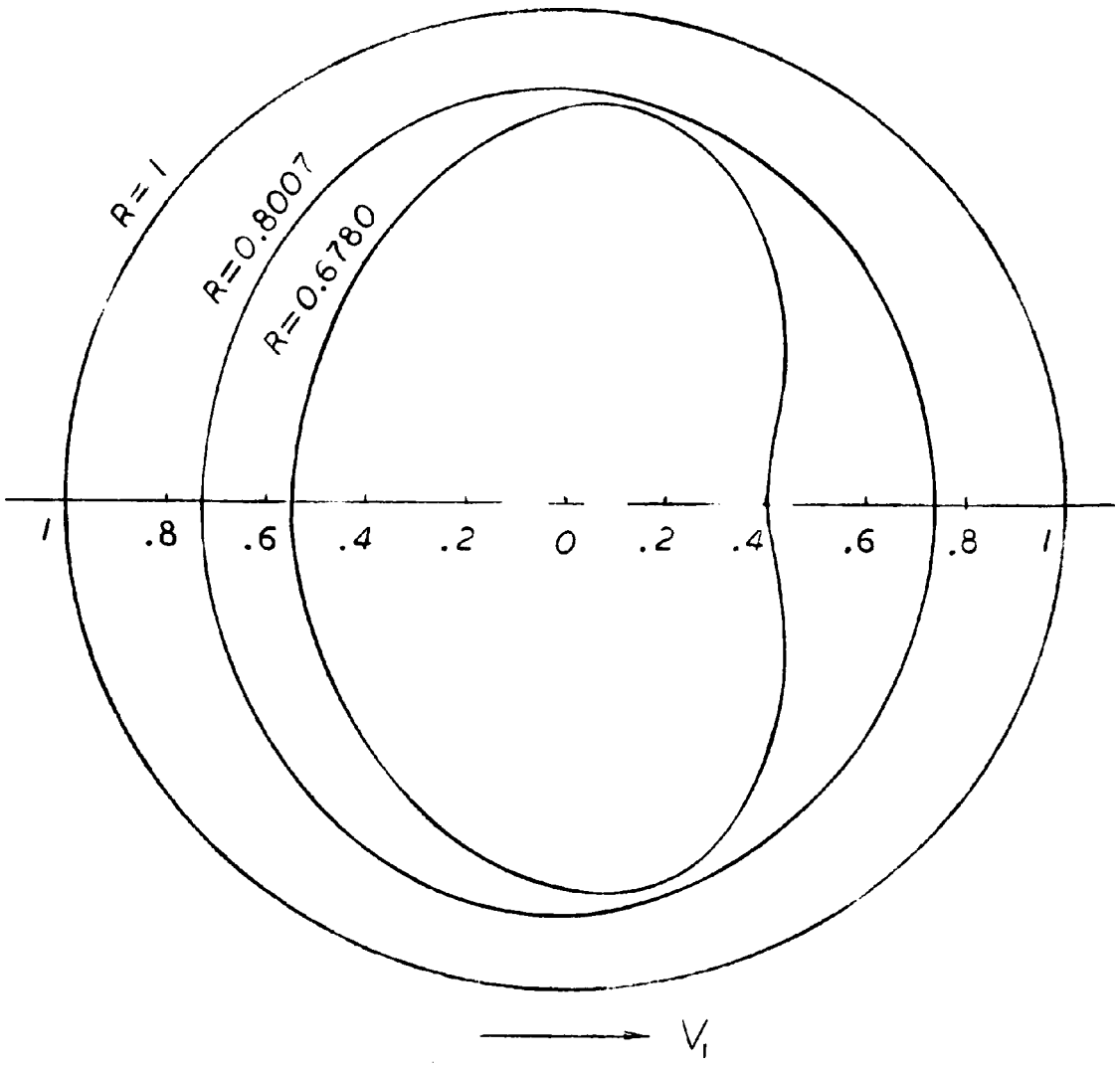


Figure 21. The transient shape of an initially spherical bubble moving in a source flow with the initial conditions $r_0=100$, $R(0)=1$, $V_0=220$ and $p_{in}(0)=195$.

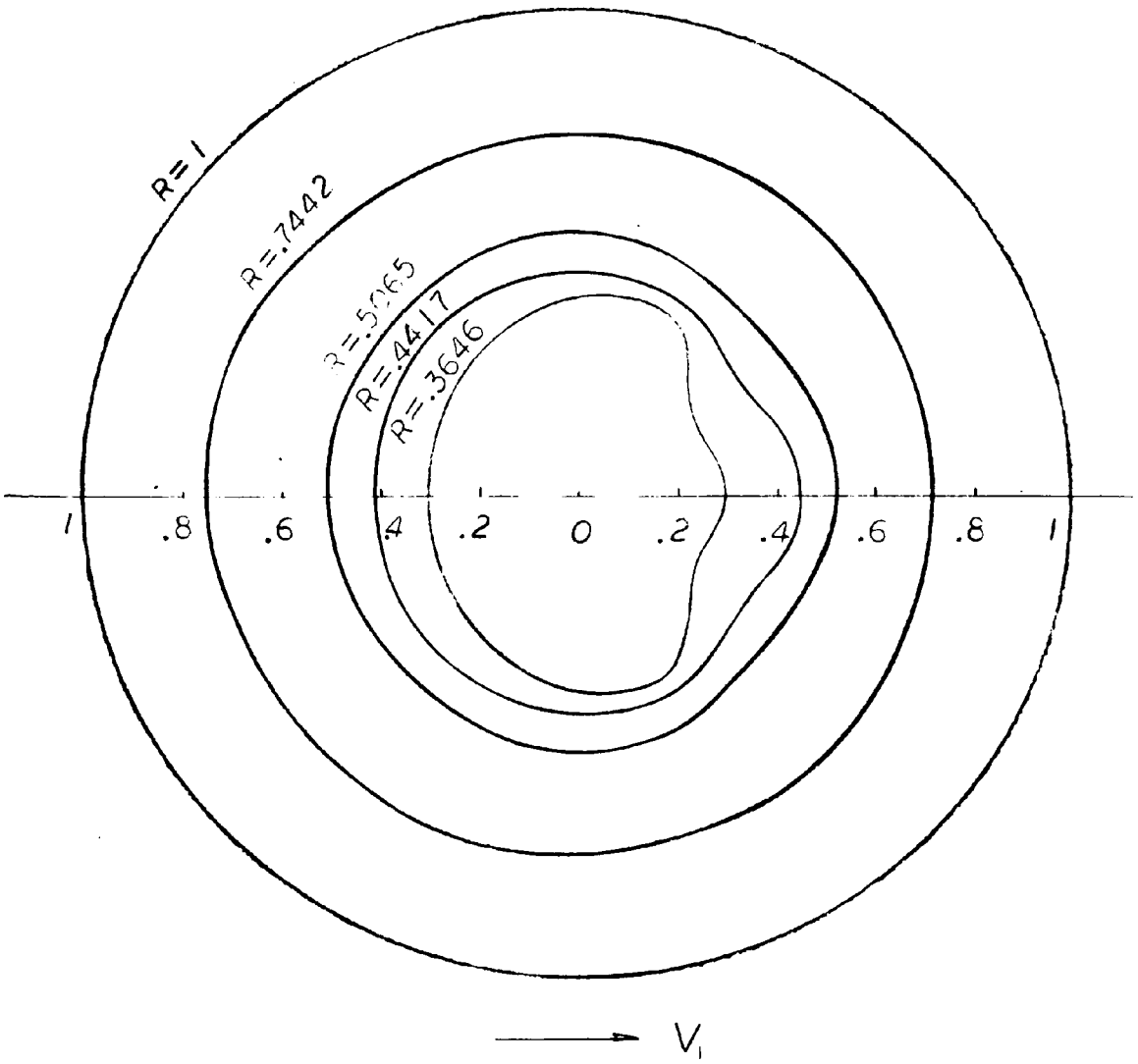


Figure 22. The transient shape of an initially spherical bubble moving in a source flow with the initial conditions $r_0 = 100$, $R(0)=1$, $V_0 = 238$ and $p_{in}(0)=195$.

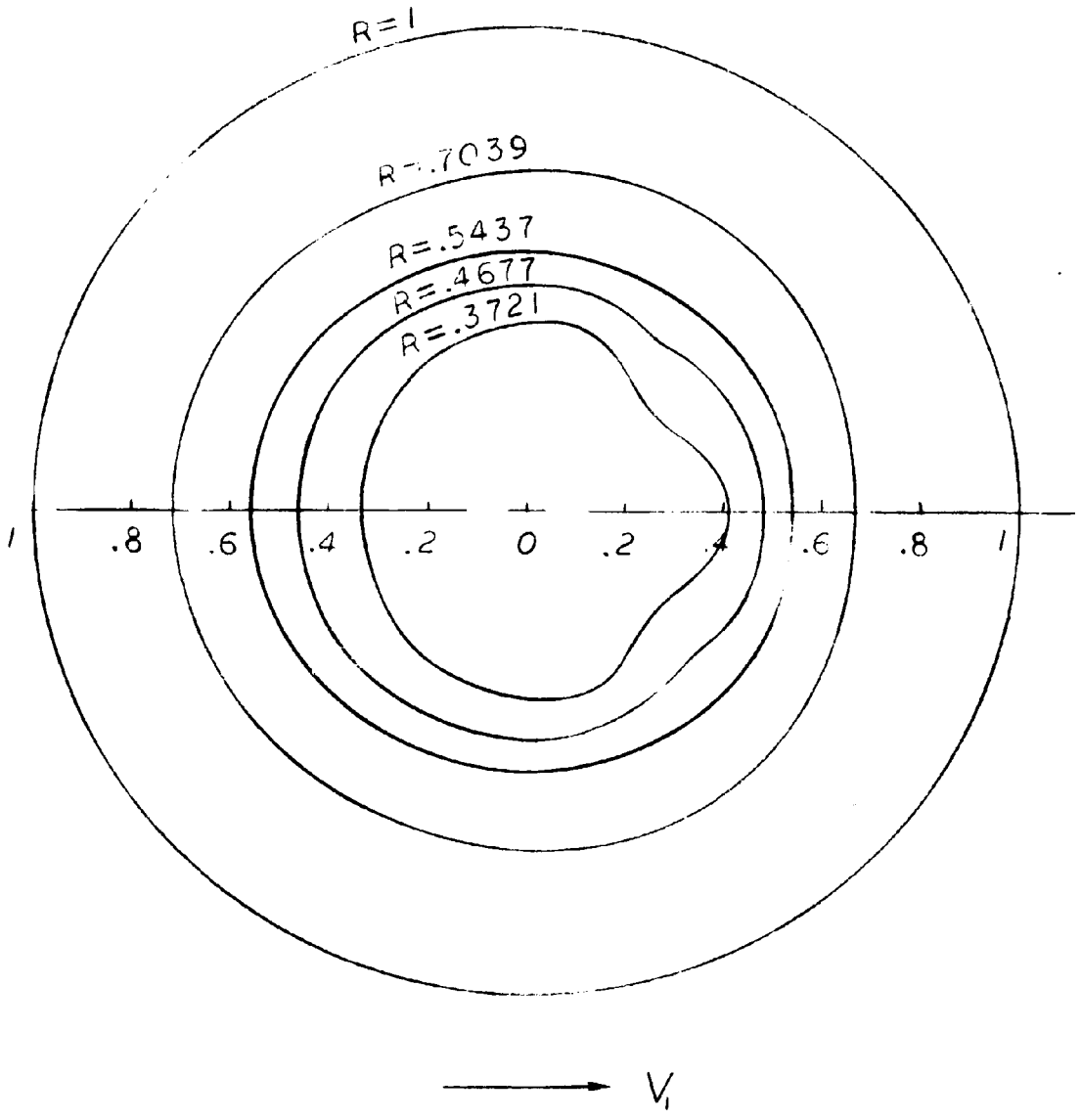


Figure 23. The transient shape of an initially spherical bubble moving in a source flow with the initial conditions $r_0 = 100$, $R(0)=1$, $V_0=240$ and $p_{in}(0)=195$.

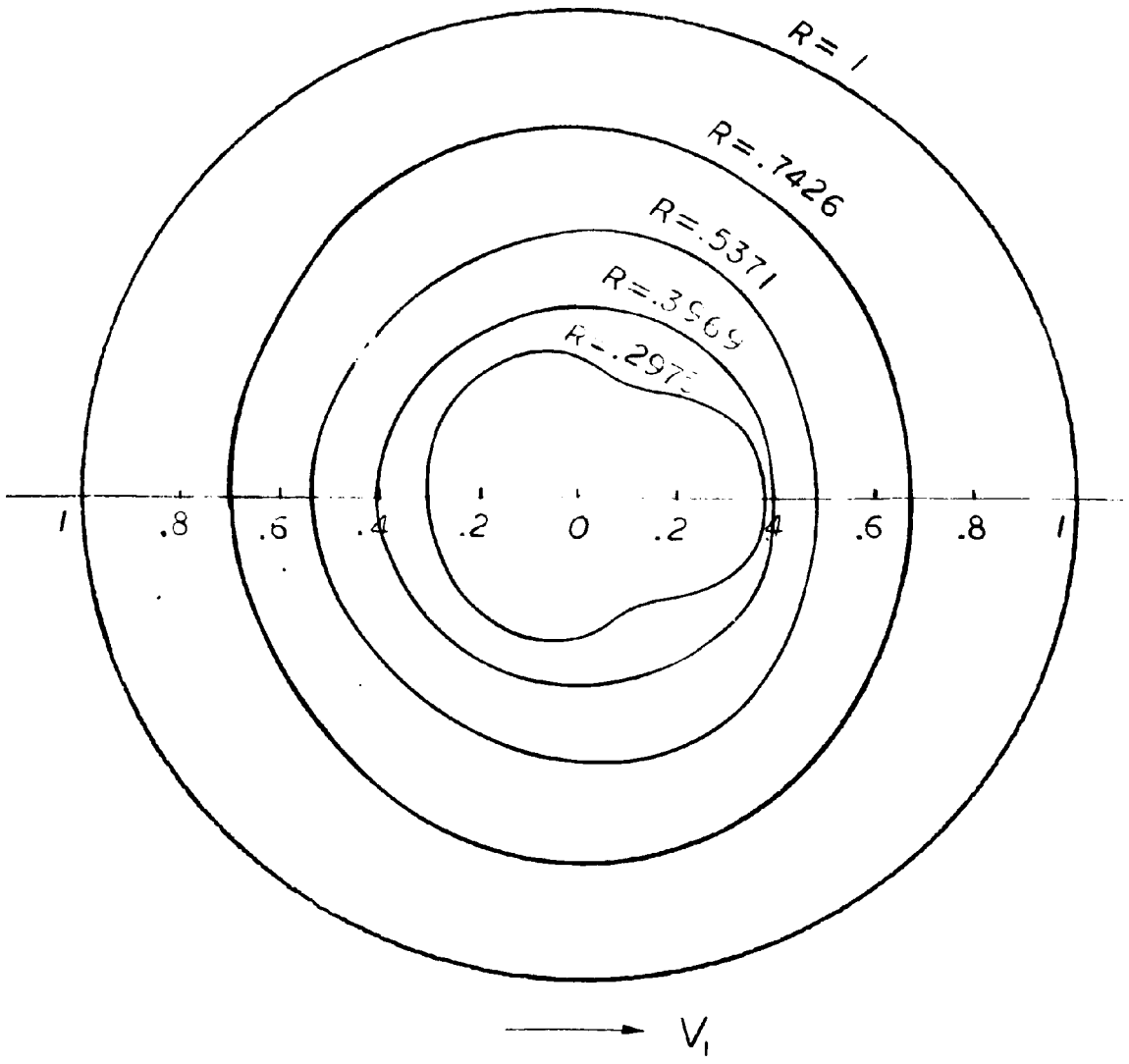


Figure 24. The transient shape of an initially spherical bubble moving in a source flow with the initial conditions $r_o = 100$, $R(0) = 1$, $V_o = 245$ and $p_{in}(0) = 195$.

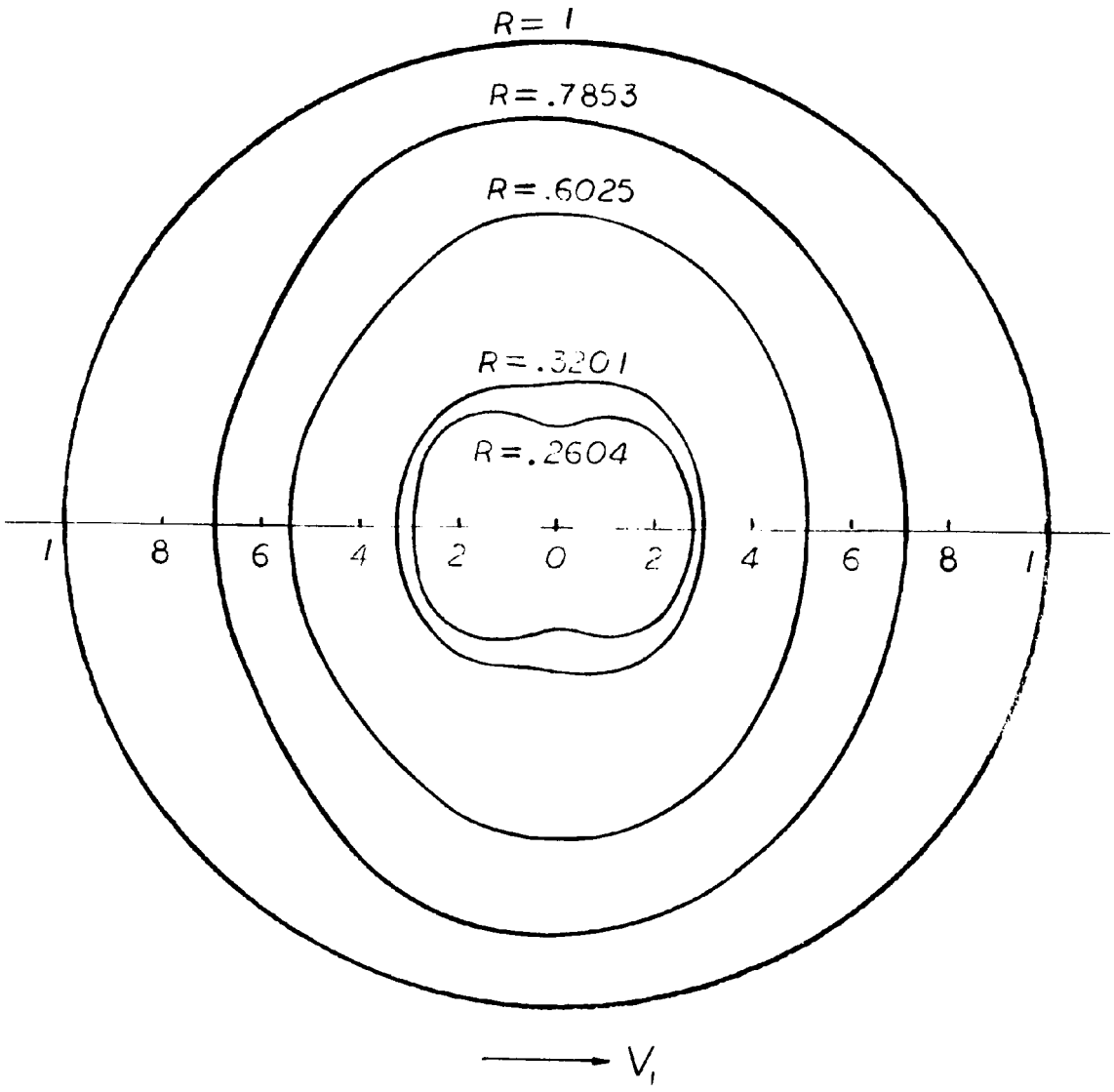


Figure 25. The transient shape of an initially spherical bubble moving in a source flow with the initial conditions $r_o=100$, $R(0)=1$, $V_o=2.48$ and $p_{in}(0)=19.5$.

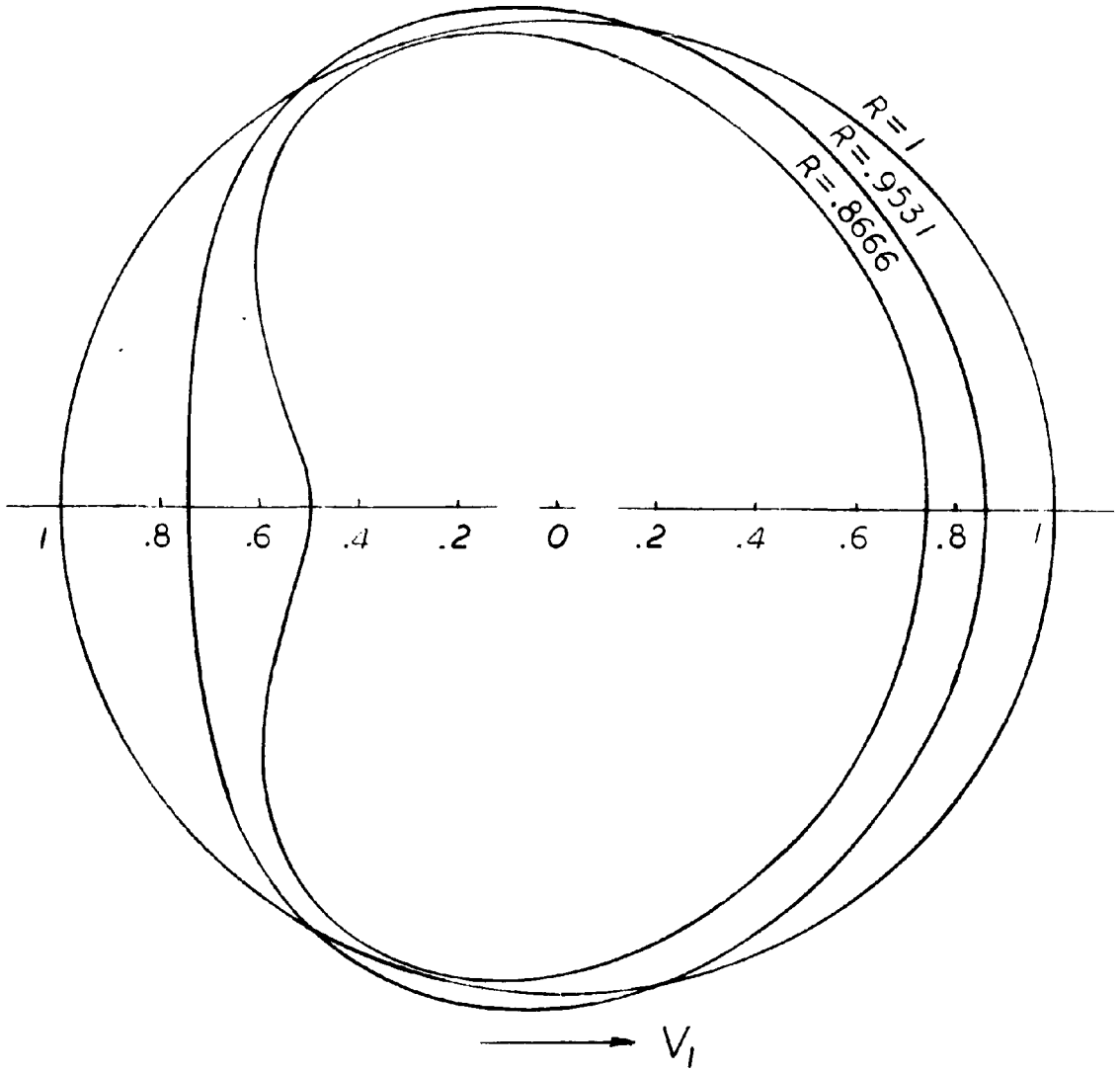


Figure 26. The transient shape of an initially spherical bubble moving in a source flow with the initial conditions $r_0=100$, $R(0)=1$, $V_0=260$ and $p_{in}(0)=195$.

large that the present analysis becomes invalid. This moment corresponds to the threshold of instability where the bubble becomes an irregular shape.

Photographic study for the dynamics of a cavitation bubble travelling through a $\frac{1}{4}$ inch venturi throat was performed in reference 5. Two typical results were reproduced here in Figures 27 and 28. The bubble's travelling velocity versus its position shown in Figures 27 and 28 is plotted in Figures 29 and 30, respectively. It must be noted that since the venturi is rectangular in cross section, the flow may be approximated as a two dimensional source or sink flow (that is from a line source or line sink). However, Figures 27 to 30 indicate that the instability in bubble shape begins at the moment when the rate change in the translating velocity is maximum. The mechanism is in agreement with the present analysis for the point source flow.

The instantaneous shape of the bubble is shown in Figures 5,7,9,10,12 and 13 as an indication of how they are related to the bubble velocity, size and location.

IV-2 Dynamics of a bubble moving in a quiescent liquid

Equations (64) and (65a) are , with the neglect of the a_n, rewritten here to show the relationship between R and V_1 :

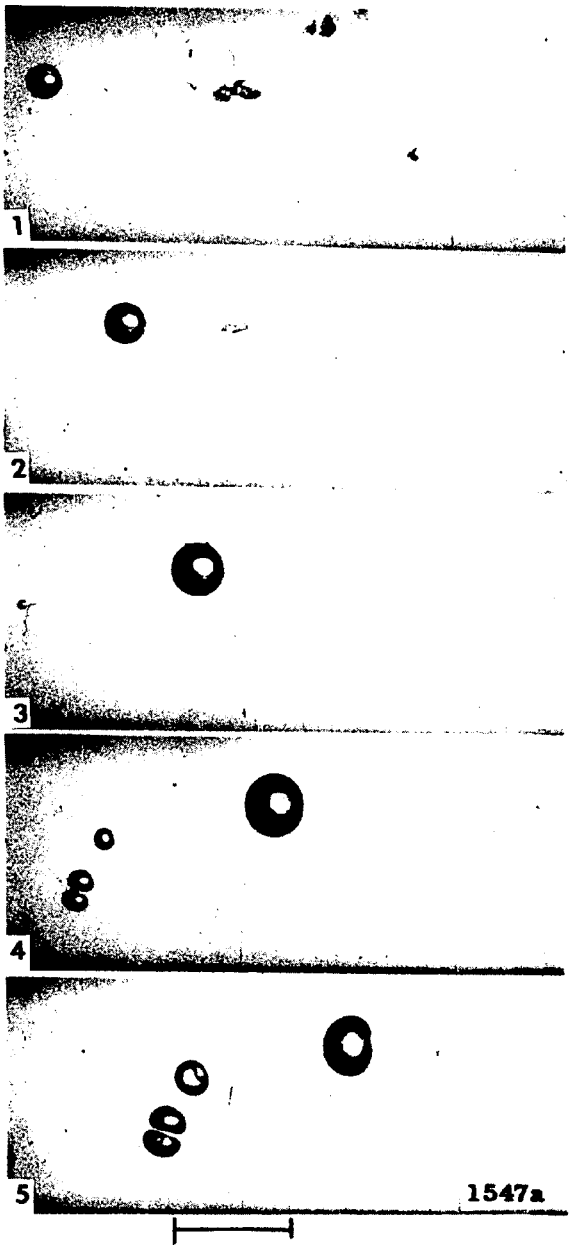


Figure 27. High speed photographs, 1/4 inch venturi throat, velocity 74.6 ft/sec, air content 2.35 vol. %, 157 microseconds per frame, scale length 0.25 in. (Courtesy of ASME "Cavitation bubble collapse observations in a venturi," Reference 5)

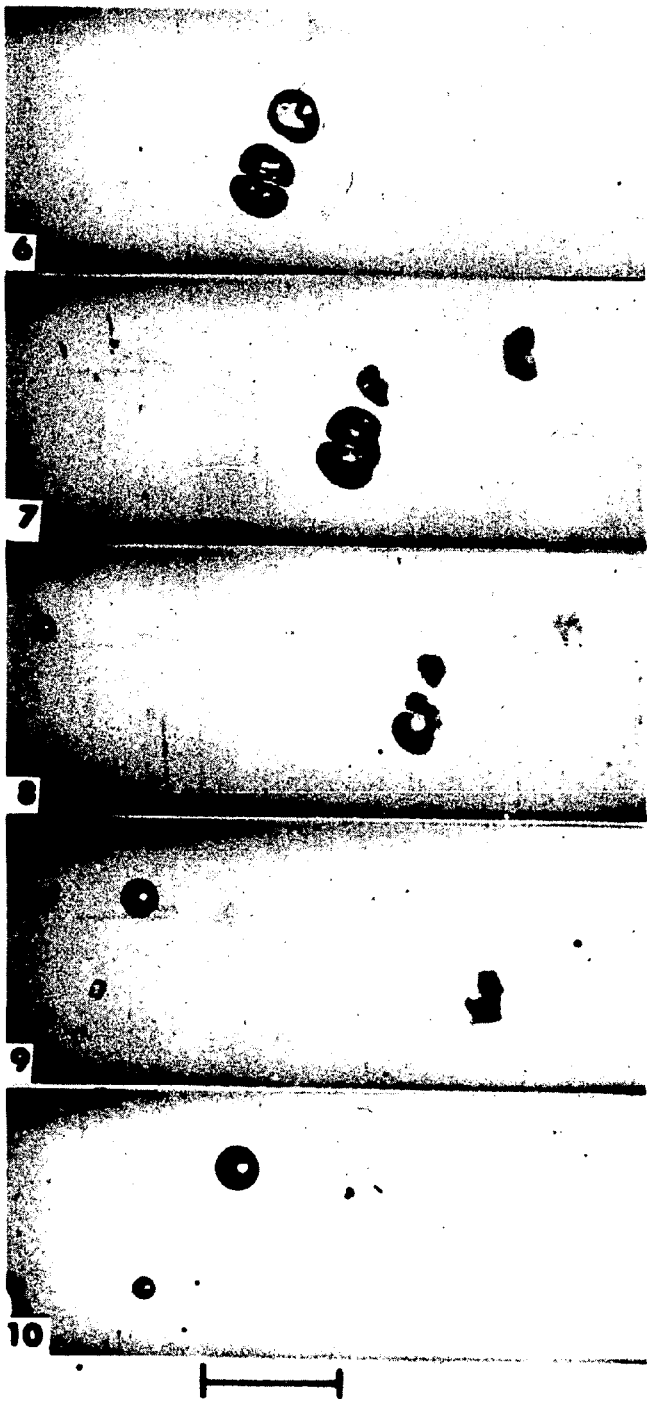


Figure 27. (Continued)

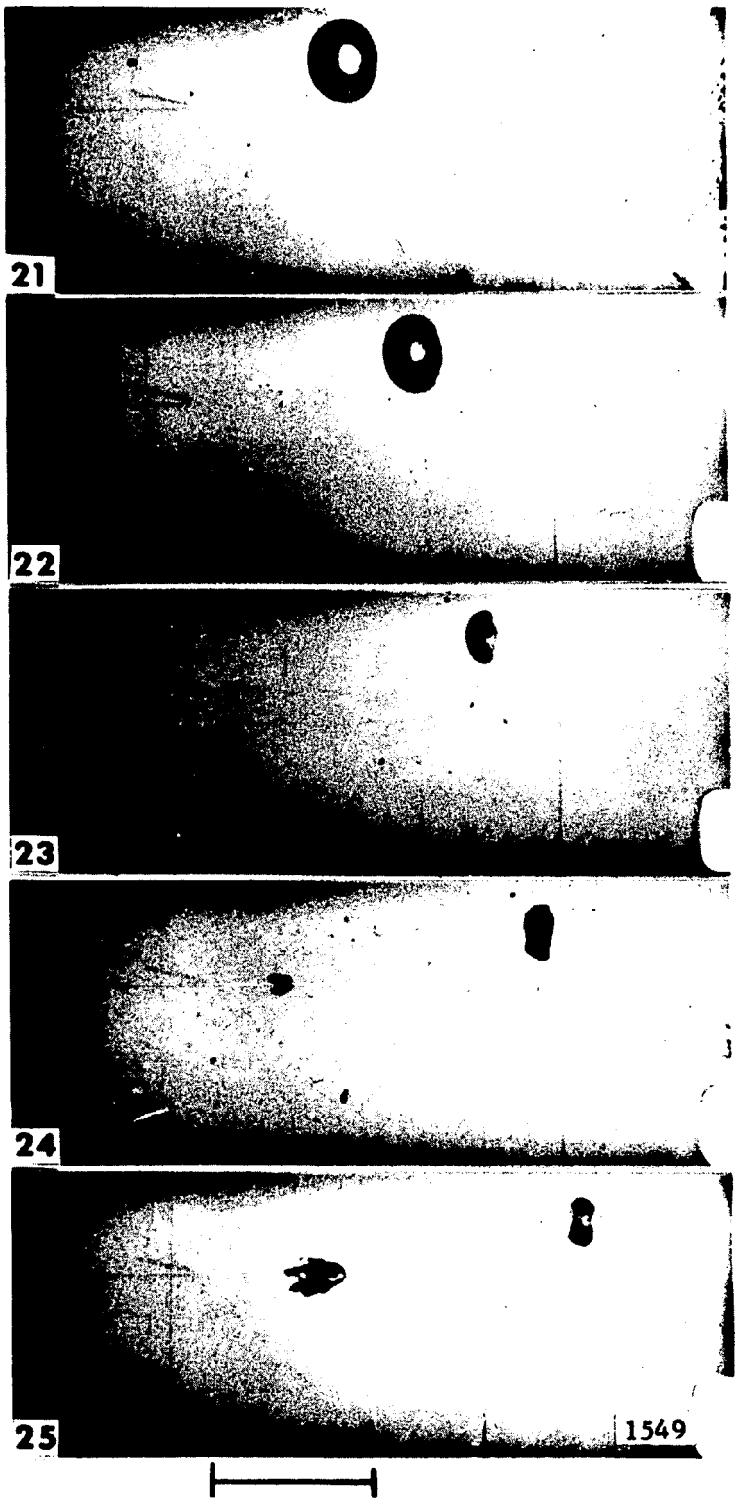


Figure 28. High speed photographs, 1/4 inch venturi throat velocity 74.6 ft/sec, air content 2.35 vol. %, 132 microseconds per frame, scale length 0.25 in. (Courtesy of ASME "Cavitation bubble collapse observations in a venturi," Reference 5)

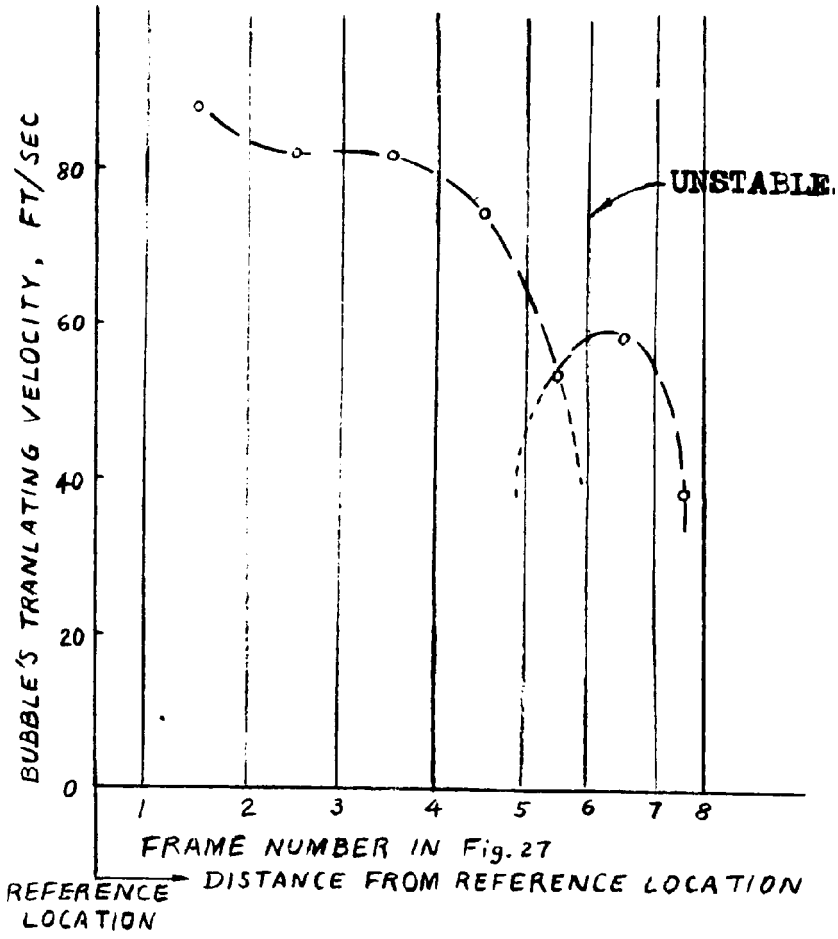


Figure 29. Bubble travelling velocity versus its position in $\frac{1}{4}$ inch venturi throat, Figure 27.

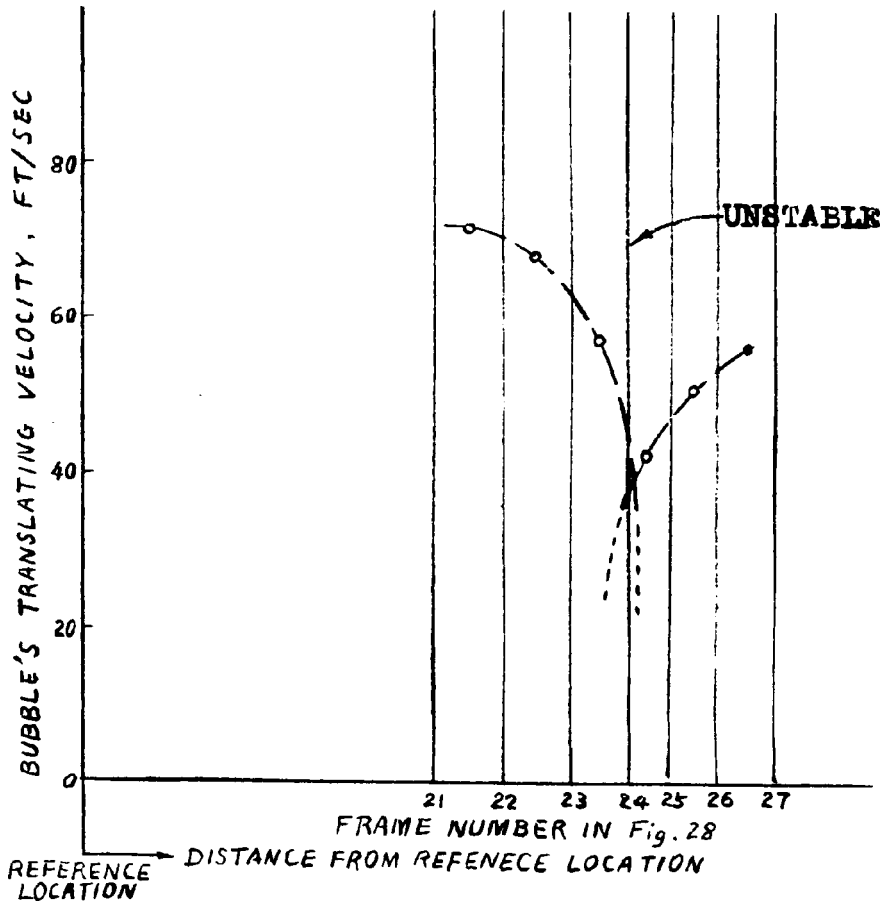


Figure 30. Bubble travelling velocity versus its position in $\frac{1}{4}$ inch venturi throat, Figure 28.

$$\frac{d^2x}{dt^2} = -3\frac{\dot{R}}{R}U \tag{72}$$

$$\ddot{R} = \frac{U^2}{4R} - \frac{3}{2}\frac{\dot{R}^2}{R} + \frac{P_{in} - P_\infty}{R} - \frac{2}{R^2} \tag{73}$$

Equation (72) indicates that the acceleration is induced by the interaction between $\dot{R}$ and bubble's translating velocity U $(= dx/dt)$. On the other hand, equation (73) shows that the acceleration of the bubble's surface displacement is induced by the forces $\frac{U^2}{4R}$, $-\frac{3}{2}\frac{\dot{R}^2}{R}$, $\frac{P_{in} - P_\infty}{R}$ and $-\frac{2}{R^2}$. By comparison it is found that the quantity U in equations (72) and (73) is equivalent to the relative velocity $(V_1 - \frac{M}{r_1^2})$ in equations (67) and (68). Therefore it is conjectured that the effect of the relative velocity on the accelerations $\frac{d^2r_1}{dt^2}$ or $\frac{d^2x}{dt^2}$ and $\ddot{R}$ we disclosed for the cases of sink or source flow and in a quiescent liquid may exist in any other type of liquid flow. If the bubble is not in translatory motion (i.e., $U=0$), equation (73) reduces to the so called "modified Rayleigh's equation".

For numerical computations, the same physical properties of water and air as the previous section were used. The gas pressure is

$$P_{in} = P_{in}(0)\frac{1}{R^{3\gamma}}$$

Two cases were considered. First, an air bubble travels at constant velocity $U_0 = 50$ in water following a step change in pressure from 195 to 7030. The initial conditions are $x(0) = 0$, $U(0) = U_0 = 50$, $R(0) = 1$ and $\dot{R}(0) = 0$. The results are illustrated in Figure 31.

In the initial stage the pressure force $(P_{\infty} - P_{\infty})/R$ which is negative dominates the collapse of the bubble. As the bubble travels downstream, it collapses while the bubble is accelerated. Until the $\frac{U^2}{4R}$ term eventually dominates the bubble dynamics the bubble stops collapsing and starts to grow, the phenomenon of "regrow".

For the second case, the pressure undergoes a step change from 7030 to 195. All other conditions remain the same. The results are shown in Figure 31. The time history of the bubble shape is illustrated in Figures 31 and 32. It is seen that for both cases the bubble dents at the rear stagnation point and eventually becomes a crater.

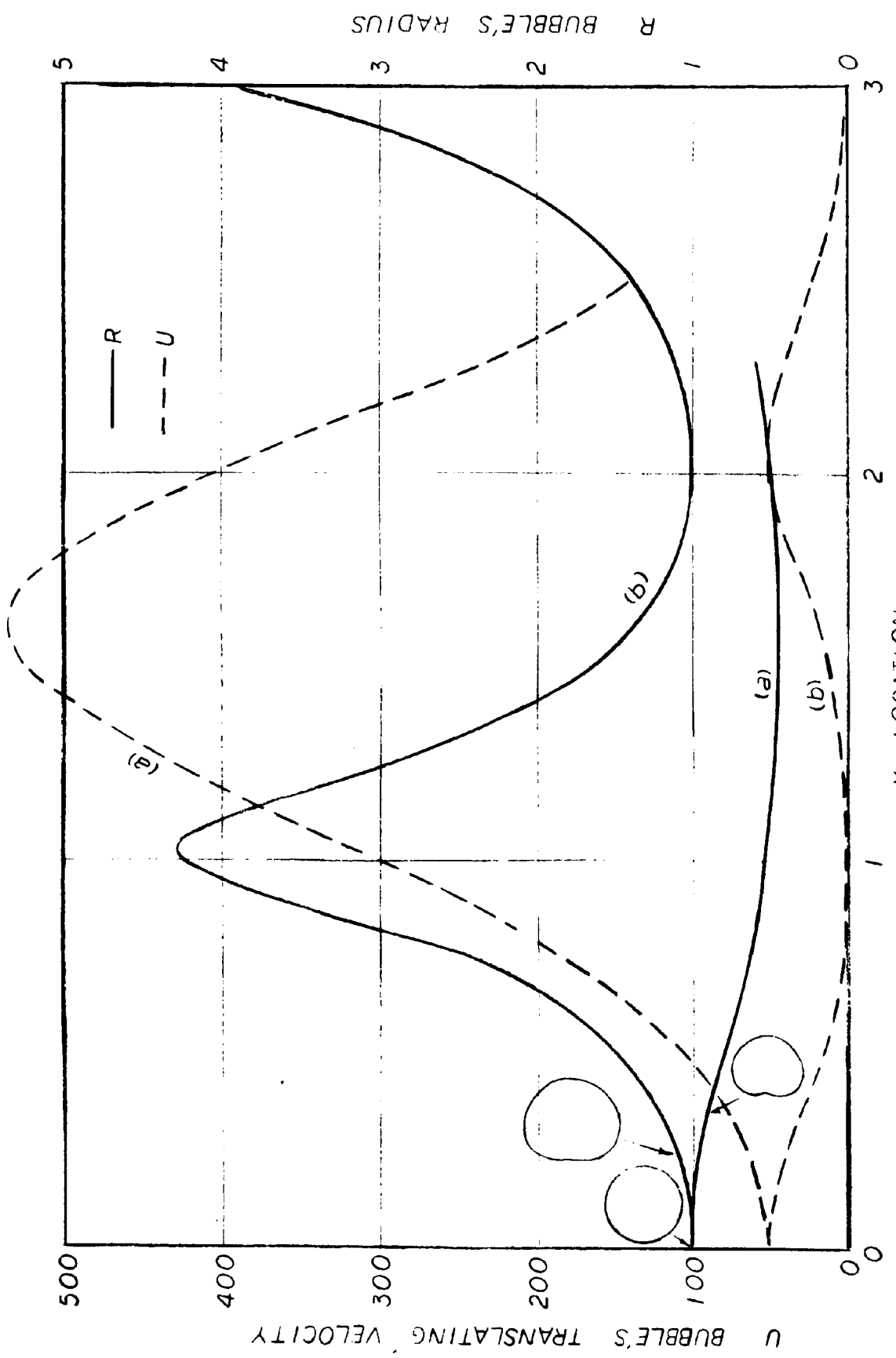


Figure 31. The velocity U and the radius R of a spherical bubble moving in a quiescent liquid with the initial conditions U_o=50, R(0)=1, R(0)=0, and (a) $p_{in}(0)$=195, p_∞=7030, (b) $p_{in}(0)$=7030, p_∞=195.

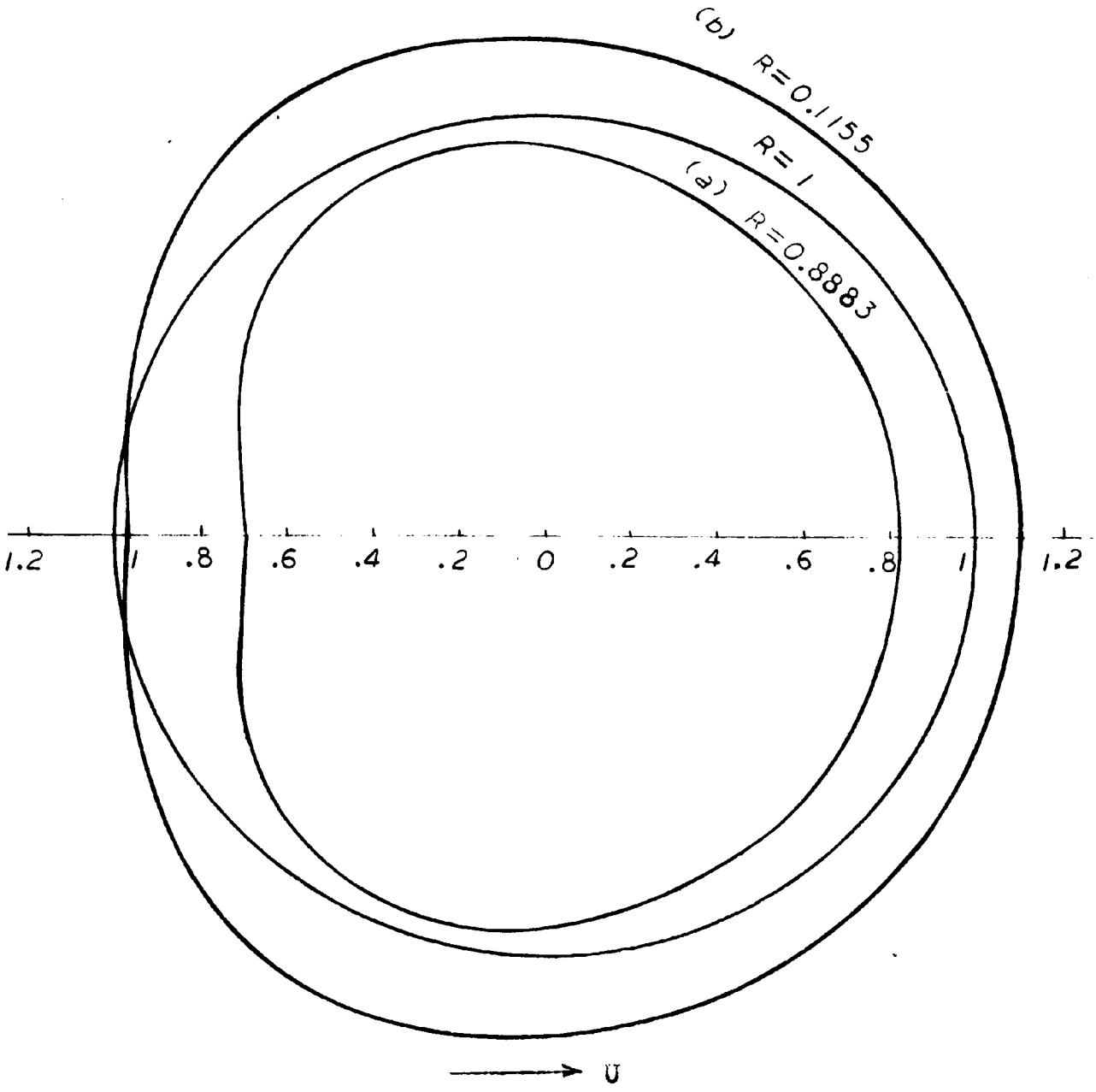


Figure 32. The transient shape of an initially spherical bubble moving in a quiescent liquid with the initial conditions $U_0=50$, $R(0)=1$, $\dot{R}(0)=0$, (a) $p_{in}(0)=195$, $p_\infty=7030$ and (b) $p_{in}(0)=7030$, $p_\infty=195$.

CHAPTER V

CONCLUSIONS

The following conclusions are made from the present study.

1. When a gas bubble is in translatory motion the bubble may be accelerated or decelerated as the result of the relative velocity between the bubble and the liquid and the rate of bubble collapse or growth. In addition, the relative velocity between the bubble and the liquid may accelerate the rate of growth. This is true for both cases: the case where the liquid is in a source or sink flow and the case where the bubble travels in a quiescent liquid subject to a step change in the liquid pressure. It is conjectured that the conclusion may also be applied to the case where the fluid is in any other type of flow.

2. The time history of the bubble's translating velocity in a sink flow is monotonically increasing while in a source flow it varies following two typical patterns depending on the initial velocity. When the initial velocity is smaller than a critical value, the translating motion of the bubble is decreased as it collapses. After the bubble comes to a complete stop it starts to travel upstream for a short distance then moves downstream again

regrowing. When the initial velocity is greater than the critical value the bubble's translating motion is first decreased as it collapses. The velocity is then increased over a short distance followed by a deceleration and a re-growth of the bubble.

In a sink flow, a bubble can maintain a nearly spherical shape over a rather long distance as it grows. The bubble is then flattened on the downstream side. In a source flow, the bubble shape varies with time in various ways depending upon the initial velocity. When the initial translating velocity is low, the initially spherical bubble becomes oblate in shape. It is then flattened at the downstream side. As the initial velocity increases, the indentation shifts upstream along the bubble surface. When the initial velocity is further increased the indentation occurs on the upstream side. During the collapse the bubble shape may become unstable when the rate change in the translatory velocity is large.

When an initially spherical bubble moving in a quiescent liquid is subject to a step change in the liquid pressure, the indentation starts at the rear stagnation point.

3. The characteristic features in the dynamic behavior of a translating gas bubble as described in the previous sections were also disclosed in the photographic study of a cavitation bubble in a rectangular venturi tube (reference 5). The validity of the present analysis is borne out in its agreement with experimental observations.

APPENDIX I

SURFACE CONDITION DF/Dt

The following derivation is essentially an extract from Lord Kelvin's work (100).

Let the bounding surface of two fluids be

$$F(x, y, z, t) = 0 \tag{A-1}$$

If there is no fluid moving across the surface, and a particle remains on the bounding surface, then the surface motion in the normal direction is equal to the normal component of the motion of a neighboring fluid particle.

Suppose a point P on the surface at time t moves to point P' at time $t + dt$. Since points P and P' are both on the surface, they satisfy equation (A-1). That is

$$F(x, y, z, t) = 0 \tag{A-2}$$

$$F(x', y', z', t + dt) = 0 \tag{A-3}$$

The direction cosines of the normal to the surface $\vec{n}$ are

$$\frac{F_x}{\sqrt{F_x^2 + F_y^2 + F_z^2}}, \quad \frac{F_y}{\sqrt{F_x^2 + F_y^2 + F_z^2}}, \quad \frac{F_z}{\sqrt{F_x^2 + F_y^2 + F_z^2}}$$

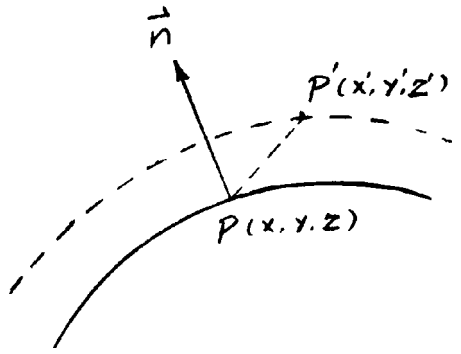


Figure 33. Bounding surface

Therefore the projection of PP' on the normal is

$$\frac{(x'-x)F_x}{\sqrt{F_x^2+F_y^2+F_z^2}} + \frac{(y'-y)F_y}{\sqrt{F_x^2+F_y^2+F_z^2}} + \frac{(z'-z)F_z}{\sqrt{F_x^2+F_y^2+F_z^2}} \qquad \text{(A-4)}$$

Since $(x'-x)$ etc are very small, expanding (A-3) in Taylor's series and make use of (A-2), one obtains

$$(x'-x)F_x + (y'-y)F_y + (z'-z)F_z + dt\,F_t = 0$$

Thus (A-4) becomes

$$\left(\begin{matrix}\text{projection of } \overbrace{PP'} \\ \text{on the normal}\end{matrix}\right) = -\frac{F_t\,dt}{\sqrt{F_x^2+F_y^2+F_z^2}} \qquad \text{(A-5)}$$

Now, the normal component of the motion of a particle is

$$\vec{v}\cdot\vec{n}\,dt = \frac{u\,dt\,F_x}{\sqrt{F_x^2+F_y^2+F_z^2}} + \frac{v\,dt\,F_y}{\sqrt{F_x^2+F_y^2+F_z^2}} + \frac{w\,dt\,F_z}{\sqrt{F_x^2+F_y^2+F_z^2}} \qquad \text{(A-6)}$$

Equating (A-5) and (A-6), there results

$$uF_x + vF_y + wF_z + F_t = 0$$

or

$$\frac{DF}{Dt} = 0$$

Further discussion on this subject can be found on p. 7 of reference 8.

APPENDIX II

RECURRENCE FORMULAR FOR LEGENDRE POLYNOMIAL

$$\sin^2\theta\, P_n' = \frac{n(n+1)}{2n+1}\left(-P_{n+1} + P_{n-1}\right)$$

$$\cos\theta\, P_n = \frac{n+1}{2n+1} P_{n+1} + \frac{n}{2n+1} P_{n-1}$$

$$\sin^2\theta\, P_2' P_n' = 3\frac{n(n+1)}{2n+1}\left[-\frac{n+2}{2n+3}P_{n+2} + \frac{2n+1}{(2n-1)(2n+3)}P_n + \frac{n-1}{2n-1}P_{n-2}\right]$$

$$P_2 P_n = \frac{3}{2}\frac{(n+1)(n+2)}{(2n+1)(2n+3)}P_{n+2} + \frac{n(n+1)}{(2n-1)(2n+3)}P_n + \frac{3}{2}\frac{(n-1)n}{(2n-1)(2n+1)}P_{n-2}$$

$$\sin^2\theta\, P_2' P_n = -3\frac{(n+1)(n+2)(n+3)}{(2n+1)(2n+3)(2n+5)}P_{n+3} + 3\frac{n+1}{2n+1}\left[\frac{2}{5} - \frac{3}{5}\frac{n(n+2)}{(2n-1)(2n+5)}\right]P_{n+1}$$
$$+ 3\frac{n}{2n+1}\left[\frac{2}{5} - \frac{3}{5}\frac{(n-1)(n+1)}{(2n-3)(2n+3)}\right]P_{n-1} - 3\frac{n(n+2)}{(2n-3)(2n-1)(2n+1)}P_{n-3}$$

$$\sin^2\theta\, P_2' P_n = -9\frac{(n+1)(n+2)(n+3)(n+4)}{(2n+1)(2n+3)(2n+5)(2n+7)}P_{n+4}$$
$$+ 9\frac{(n+1)(n+2)}{(2n+1)(2n+3)}\left[\frac{2}{5} - \frac{3}{5}\frac{n(n+2)}{(2n-1)(2n+5)} - \frac{(n+3)^2}{(2n+5)(2n+7)}\right]P_{n+2}$$
$$+ \frac{9}{5}\frac{1}{2n+1}\left[2\left(\frac{(n+1)^2}{2n+3} + \frac{n^2}{2n-1}\right) - 3\frac{n(n+1)}{(2n-1)(2n+3)}\cdot\right.$$
$$\left.\cdot\left(\frac{(n+1)(n+2)}{2n+5} + \frac{(n-1)n}{2n-3}\right)\right]P_n$$
$$+ 9\frac{(n-1)n}{(2n-1)(2n+1)}\left[\frac{2}{5} - \frac{3}{5}\frac{(n-1)(n+1)}{(2n-3)(2n+3)} - \frac{(n-2)^2}{(2n-5)(2n-3)}\right]P_{n-2}$$
$$- 9\frac{(n-3)(n-2)(n-1)n}{(2n-5)(2n-3)(2n-1)(2n+1)}P_{n-4}$$

$$\sin^2\theta\, P_n = -\frac{(n+1)(n+2)}{(2n+1)(2n+3)}P_{n+2} + \frac{2(n^2+n-1)}{(2n-1)(2n+3)}P_n - \frac{(n-1)n}{(2n+1)(2n-1)}P_{n-2}$$

The first and second formulas are obtained from reference 101, while the third formula is derived from them. The others are derived from reference 102.

$$P_m(x)\,P_n(x) = \sum_{r=0}^{m} \frac{A_{m-r}\,A_r\,A_{n-r}}{A_{n+m-r}} \left(\frac{2n+2m-4r+1}{2n+2m-2r+1}\right) \cdot$$

$$\cdot\, P_{n+m-2r}(x)$$

Where

$$A_m = \frac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{m!}$$

APPENDIX III

FORCE BALANCE ON BUBBLE SURFACE AND MEAN CURVATURE*

The radius of normal curvature at a point on a surface in general depends on the direction of the tangent line at that point (except for a sphere and a plane). The maximum and minimum values are called principal radii of normal curvatures. The corresponding direction of a tangent line is called the principal direction. From differential geometry, it can be proved that these two principal directions are orthogonal to each other.

Let us take a differential element on a bubble surface whose sides ΔS_1 and ΔS_2 are cut along the principal directions, and the corresponding principal radii of normal curvatures are R_1 and R_2 respectively. The force balance in the normal direction is

$$2\sigma\Delta S_2 \cdot \sin\frac{\Delta\theta_1}{2} + 2\sigma\Delta S_1 \cdot \sin\frac{\Delta\theta_2}{2} = (P_{in} - P_{out}) \cdot (\Delta A)_n \qquad \text{(A-7)}$$

where P_{in} and P_{out} are the pressure inside and outside of the bubble, respectively, and $(\Delta A)_n$ is the projectional

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* The sum of principal curvatures $\frac{1}{R_1} + \frac{1}{R_2}$ is called "mean curvature" by some authors while other authors define the half of the sum as mean curvature which seems more logical.

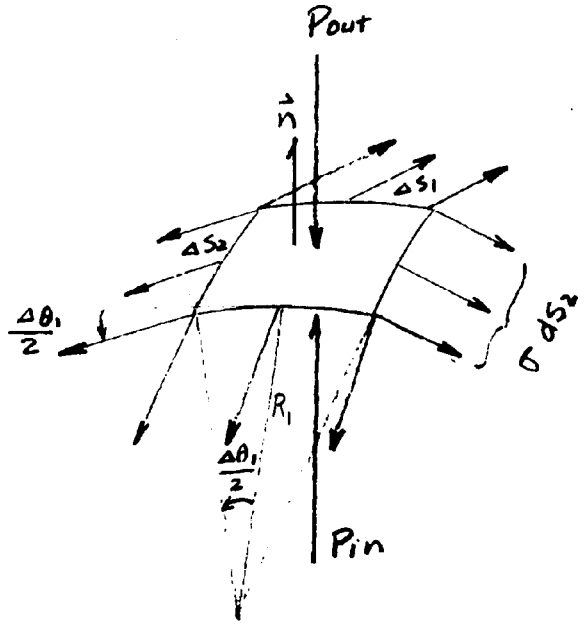


Figure 34. Forces on an element of bounding surface

area of the differential element on the tangent plane. Since ΔS_1 and ΔS_2 are in the principal directions, they are perpendicular to each other. Therefore the projected area is

$$(\Delta A)_n = (\Delta S_1 \cos\frac{\Delta\theta_1}{2}) \cdot (\Delta S_2 \cos\frac{\Delta\theta_2}{2}) \tag{A-8}$$

Furthermore

$$\Delta S_1 = R_1 \cdot \Delta\theta_1$$
$$\Delta S_2 = R_2 \cdot \Delta\theta_2 \tag{A-9}$$

Now, equations (A-8), and (A-9) are substituted into equation (A-7). Since $\Delta\theta_1$ and $\Delta\theta_2$ are so small that $\sin\frac{\Delta\theta}{2} \approx \frac{\Delta\theta}{2}$ and $\cos\frac{\Delta\theta}{2} \approx 1$. One then obtains

$$\sigma(R_2 + R_1) = (P_{in} - P_{out}) R_1 R_2$$

or

$$P_{in} - P_{out} = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \tag{A-10}$$

Let the equation of a surface be

$$F(x, y, z) = 0 \tag{A-11}$$

and the direction cosines of the normal to the surface $\vec{n}$ be n_1, n_2 and n_3 . Then from differential geometry one gets

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{\partial n_1}{\partial x} + \frac{\partial n_2}{\partial y} + \frac{\partial n_3}{\partial z} \tag{A-12}$$

The direction cosines of normal are

$$n_1 = \frac{F_x}{\sqrt{F_x^2+F_y^2+F_z^2}}, \quad n_2 = \frac{F_y}{\sqrt{F_x^2+F_y^2+F_z^2}}, \quad n_3 = \frac{F_z}{\sqrt{F_x^2+F_y^2+F_z^2}} \tag{A-13}$$

where the sugscripts x, y, z, indicate the differentiation with respective to x, y, z, respectively.

For the perturbed bubble surface, it yields

$$F_x = \frac{x}{r_s} + \frac{xz}{r_s^3}\sum_{n=1}^{\infty} a_n P_n' \tag{A-14}$$

$$F_y = \frac{y}{r_s} + \frac{yz}{r_s^3}\sum_{n=1}^{\infty} a_n P_n' \tag{A-15}$$

$$F_z = \frac{z}{r_s} + \frac{r_s^2-z^2}{r_s^3}\sum_{n=1}^{\infty} a_n P_n' \tag{A-16}$$

By neglecting the terms involving $(a_n)^2$ one obtains

$$F_x^2 = \frac{x^2}{r_s^2} + 2\frac{x^2 z}{r_s^4}\sum_{n=1}^{\infty} a_n P_n'$$

$$F_y^2 = \frac{y^2}{r_s^2} + 2\frac{y^2 z}{r_s^4}\sum_{n=1}^{\infty} a_n P_n'$$

$$F_z^2 = \frac{z^2}{r_s^2} - 2\frac{(r_s^2-z^2)z}{r_s^4}\sum_{n=1}^{\infty} a_n P_n'$$

Therefore

$$\sqrt{F_x^2+F_y^2+F_z^2} = \left[1 + 2\frac{z(x^2+y^2-r^2+z^2)}{r_s^4}\sum_{n=1}^{\infty} a_n P_n'\right]^{1/2} = 1$$

and the direction cosines of normal become

$$n_1 = F_x \quad , \quad n_2 = F_y \, , \qquad n_3 = F_z$$

With the aid of equations (A-14) to (A-16), the gradient of the normal can be obtained

$$\frac{\partial n_1}{\partial x} = F_{xx} = \frac{1}{r_s} - \frac{x^2}{r_s^3} + \left(\frac{z}{r_s^3} - 3\frac{x^2 z}{r_s^5}\right)\sum_{n=1}^{\infty} a_n P_n' - \frac{x^2 z^2}{r_s^5}\sum_{n=1}^{\infty} a_n P_n'' \tag{A-17}$$

$$\frac{\partial n_2}{\partial y} = F_{yy} = \frac{1}{r_s} - \frac{y^2}{r_s^3} + \left(\frac{z}{r_s^3} - 3\frac{y^2 z}{r_s^5}\right)\sum_{n=1}^{\infty} a_n P_n' - \frac{y^2 z^2}{r_s^5}\sum_{n=1}^{\infty} a_n P_n'' \tag{A-18}$$

$$\frac{\partial n_3}{\partial z} = F_{zz} = \frac{1}{r_s} - \frac{z^2}{r_s^3} + \left(\frac{z}{r_s^3} + \frac{2z}{r_s^3} - 3\frac{z^3}{r_s^5}\right)\sum_{n=1}^{\infty} a_n P_n' - \frac{-(x^2+z^2)z^2}{r_s^5}\sum_{n=1}^{\infty} a_n P_n'' \tag{A-19}$$

Adding these equations together, one obtains

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{2}{r_s} + \frac{2}{r_s^2}\sum_{n=1}^{\infty} a_n P_n' \cos\theta - \frac{1}{r^2}\sum_{n=1}^{\infty} a_n P_n'' \sin^2\theta \tag{A-20}$$

Now since Legendre polynomial P_n is the solution of the equation

$$(1-x^2)\,y'' - 2xy' + n(n+1)\,y = 0$$

it satisfies the relation

$$\sin^2\theta \cdot P_n'' - 2\cos\theta\, P_n' + n(n+1)\,P_n = 0 \tag{A-21}$$

By virtue of equation (A-21), equation (A-20) becomes

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{2}{r_s} + \frac{1}{r_s^2}\sum_{n=1}^{\infty} n(n+1)\,a_n P_n \tag{A-22}$$

Substituting equation (9) for r_s into equation (A-22) and neglecting the terms involving $(a_n)^2$ one finally gets

$$\frac{1}{R_1} + \frac{1}{R_2} = \frac{2}{R} + \sum_{n=1}^{\infty}\frac{(n-1)(n+2)}{R^2}\,a_n P_n \tag{A-23}$$

APPENDIX IV

PROGRAM FOR A BUBBLE MOVING IN A SOURCE OR SINK FLOW

For computer programming, equations (52) (with neglect of $\ddot{a}_1$ term) and (57) are rewritten in the following forms

$$\left(1 + \frac{4}{5}\frac{a_2}{R}\right)\ddot{r}_1 - 7\frac{a_1}{R}\ddot{R} = h_r \qquad \text{(A-24)}$$

$$\frac{1}{3}\frac{a_1}{R}\ddot{r}_1 + \ddot{R} = h_R \qquad \text{(A-25)}$$

$$\left(\frac{6}{5}\frac{a_2}{R} - 1\right)\ddot{r}_1 - \frac{a_1}{R}\ddot{R} + \ddot{a}_1 = h_{a_1} \qquad \text{(A-26)}$$

and for $m \geq 2$

$$\left[\frac{(m+1)^2}{2m+3}\left(1+\frac{m}{2}\right)\frac{a_{m+1}}{R} - \frac{1}{2}\frac{m(m+1)}{2m-1}\frac{a_{m-1}}{R}\right]\ddot{r}_1 - m\frac{a_m}{R}\ddot{R} + \ddot{a}_1 = h_{a_m} \qquad \text{(A-27)}$$

where h_r, h_R, h_{a_1} and h_{a_m} are functions of r_1, V_1, R $\dot{R}$, a_m and $\dot{a}_m$. Equations (A-24) through (A-27) are then solved for $\ddot{r}_1$, $\ddot{R}$ and $\ddot{a}_m$ in order to apply the Runge-Kutta method.

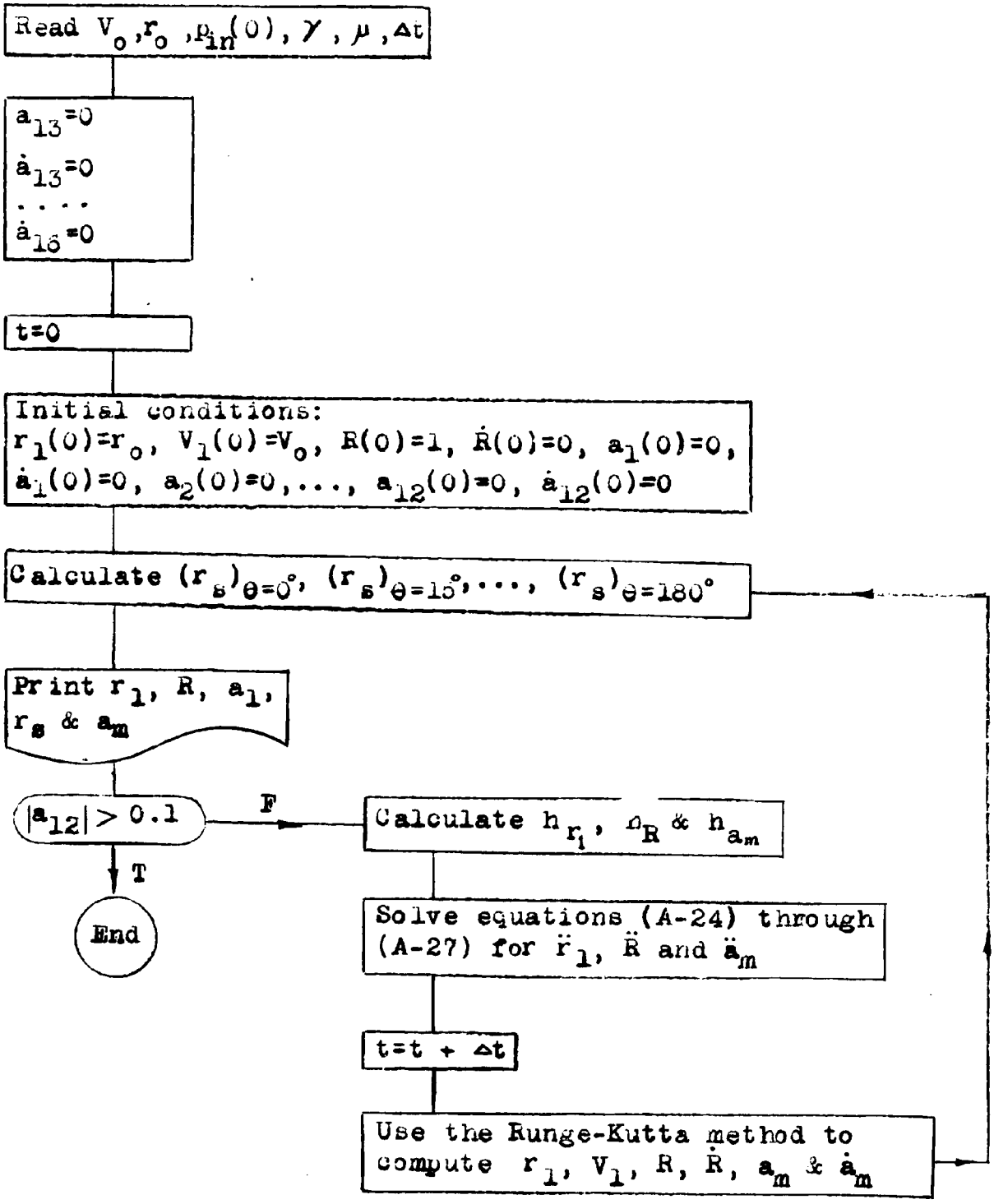


Figure 35. The flow diagram of computer program for a bubble moving in a source or sink flow.

PROGRAM FOR A BUBBLE MOVING IN A SOURCE OR SINK FLOW

```
          PRINT COMMENT $BUBBLE MOVING IN SOURCE OR SINK FLOW$                    *001
          DIMENSION A(3216,WA),AP(3216,WB),APP(3216,WC),                          *002
         1 BZERO(22),B1(22),B2(22),B3(22),B4(22),B5(22),                          *002
         2 R(201),RP(201),RPP(201),P(201),Y(28),F(28),Q(28),                      *002
         3 R1(201),V1(201),V1P(201),TA(201),TB(201),TC(201),                      *002
         4 LZERO(22),L1(22),L2(22),L3(22),L4(22),AA(20),BA(20),AMPP(20)           *002
         5 ,HA(20)                                                               *002
          VECTOR VALUES WA = 2,1,201                                              *003
          VECTOR VALUES WB = 2,1,201                                              *004
          VECTOR VALUES WC = 2,1,201                                              *005
          VECTOR VALUES RESULT = $1H ,6F20.6*$                                    *006
          INTEGER M,J, Z,V                                                        *007
START     READ AND PRINT DATA                                                     *008
          EXECUTE FTRAP.                                                          *009
          J=1                                                                     *010
S11       A(13,J)=0.                                                              *011
          AP(13,J)=0.                                                             *012
          A(14,J)=0.                                                              *013
          AP(14,J)=0.                                                             *014
          A(15,J)=0.                                                              *015
          AP(15,J)=0.                                                             *016
          A(16,J)=0.                                                              *017
          AP(16,J)=0.                                                             *018
          WHENEVER J .E. JMAX, TRANSFER TO S12                                    *019
          J=J+1                                                                   *020
          TRANSFER TO S11                                                         *021
S12       EXECUTE SETRKD. (28,Y(1),F(1),Q,X,STEP)                                 *022
          PRINT COMMENT $R1=Y(1), V1=Y(2), R=Y(3), RP=Y(4),                       *023
         1 A(1,J)=Y(5), AP(1,J)=Y(6),..., A(6,J)=Y(15), AP(6,J)=Y(16),            *023
         2 F(N)=DERIVATIVE OF Y(N)$                                               *023
          J=0                                                                     *024
          X=0                                                                     *025
          Y(1)=RZERO                                                              *026
          Y(2)=VZERO                                                              *027
          Y(3)=1.                                                                 *028
          Y(4)=0.                                                                 *029
          Y(5)=0.                                                                 *030
          Y(6)=0.                                                                 *031
          Y(7)=0.                                                                 *032
          Y(8)=0.                                                                 *033
          Y(9)=0.                                                                 *034
          Y(10)=0.                                                                *035
          Y(11)=0.                                                                *036
          Y(12)=0.                                                                *037
          Y(13)=0.                                                                *038
          Y(14)=0.                                                                *039
          Y(15)=0.                                                                *040
          Y(16)=0.                                                                *041
          Y(17)=0.                                                                *042
          Y(18)=0.                                                                *043
          Y(19)=0.                                                                *044
          Y(20)=0.                                                                *045
          Y(21)=0.                                                                *046
```

```
        Y(22)=0.                                                                *047
        Y(23)=0.                                                                *048
        Y(24)=0.                                                                *049
        Y(25)=0.                                                                *050
        Y(26)=0.                                                                *051
        Y(27)=0.                                                                *052
        Y(28)=0.                                                                *053
        PRINT COMMENT $R1, R, RSZERO, APP(1,J)$                                 *054
        PRINT COMMENT $RS15, RS30, RS45, RS60, RS75, RS90$                      *055
        PRINT COMMENT $RS105, RS120, RS135, RS150, RS165, RS180$                *056
        PRINT COMMENT $ A1, A2, A3, A4, A5, A6 $                                *057
        PRINT COMMENT $A7, A8, A9, A10. A11, A12$                               *058
        PRINT COMMENT $ER1, ER2, ER3, ER4, ER5, ER6 $                           *059
SAVE    J=J+1                                                                   *060
        P(J)=X                                                                  *061
        R1(J)=Y(1)                                                              *062
        V1(J)=Y(2)                                                              *063
        V1P(J)=F(2)                                                             *064
        R(J)=Y(3)                                                               *065
        RP(J)=Y(4)                                                              *066
        RPP(J)=F(4)                                                             *067
        A(1,J)=Y(5)                                                             *068
        AP(1,J)= Y(6)                                                           *069
        A(2,J)=Y(7)                                                             *070
        AP(2,J)=Y(8)                                                            *071
        A(3,J)=Y(9)                                                             *072
        AP(3,J)=Y(10)                                                           *073
        A(4,J)=Y(11)                                                            *074
        AP(4,J)=Y(12)                                                           *075
        A(5,J)=Y(13)                                                            *076
        AP(5,J)=Y(14)                                                           *077
        A(6,J)=Y(15)                                                            *078
        AP(6,J)=Y(16)                                                           *079
        A(7,J)=Y(17)                                                            *080
        AP(7,J)=Y(18)                                                           *081
        A(8,J)=Y(19)                                                            *082
        AP(8,J)=Y(20)                                                           *083
        A(9,J)=Y(21)                                                            *084
        AP(9,J)=Y(22)                                                           *085
        A(10,J)=Y(23)                                                           *086
        AP(10,J)=Y(24)                                                          *087
        A(11,J)=Y(25)                                                           *088
        AP(11,J)=Y(26)                                                          *089
        A(12,J)=Y(27)                                                           *090
        AP(12,J)=Y(28)                                                          *091
        APP(1,J)=F(6)                                                           *092
        APP(2,J)=F(8)                                                           *093
        APP(3,J)=F(10)                                                          *094
        APP(4,J)=F(12)                                                          *095
        APP(5,J)=F(14)                                                          *096
        APP(6,J)=F(16)                                                          *097
        APP(7,J)=F(18)                                                          *098
        APP(8,J)=F(20)                                                          *099
        APP(9,J)=F(22)                                                          *100
        APP(10,J)=F(24)                                                         *101
        APP(11,J)=F(26)                                                         *102
        APP(12,J)=F(28)                                                         *103
        TA(J)=MU/R1(J)/R1(J)-V1(J)                                              *104
        TR(J)=R(J)/R1(J)                                                        *105
        TC(J)=MU/R1(J)/R1(J)                                                    *106
```

```
  ER1=3*RP(J)/R(J)*TA(J)                                                *107
  ER2=6*TC(J)*TC(J)*TB(J)                                               *108
  ER3=-0.5/R(J)*((MU/RZERO/RZERO).P.2-TC(J)*TC(J))                      *109
  ER4=0.25/R(J)*TA(J)*TA(J)                                             *110
  ER5=-1.5/R(J)*RP(J)*RP(J)                                             *111
  ER6=-1/R(J)*(PZERO-PZERO/R(J).P.(3*GAMMA))                            *112
  RSZERO=R(J)+A(1,J)+A(2,J)+A(3,J)+A(4,J)+A(5,J)+A(6,J)                 *113
1 +A(7,J)+A(8,J)+A(9,J)+A(10,J)+A(11,J) +A(12,J)                        *113
  RS15= R(J)+.9659*A(1,J)+.8995*A(2,J)+.8041*A(3,J)                     *114
1 +.6846*A(4,J)+.5471*A(5,J)+.3983*A(6,J)+.2455*A(7,J)                  *114
2 +.096*A(8,J)-.0428*A(9,J)-.1651*A(10,J)-.2655*A(11,J)                 *114
3 -.3402*A(12,J)                                                        *114
  RS30=R(J)+.866*A(1,J)+.625*A(2,J)+.3248*A(3,J)                        *115
1 +.0234*A(4,J)-.2233*A(5,J)-.3740*A(6,J)-.4102*A(7,J)                  *115
2 -.3388*A(8,J)-.1896*A(9,J)-.0070*A(10,J)+.1607*A(11,J)                *115
3 +.2732*A(12,J)                                                        *115
  RS45=R(J)+.7071*A(1,J)+.2500*A(2,J)-.1768*A(3,J)                      *116
1 -.4053*A(4,J)-.3756*A(5,J)-.1484*A(6,J)+.1271*A(7,J)                  *116
2 +.2983*A(8,J)+.2855*A(9,J)+.1151*A(10,J)-.1042*A(11,J)                *116
3 -.2467*A(12,J)                                                        *116
  RS60=R(J)+.5000*A(1,J)-.1250*A(2,J)-.4375*A(3,J)                      *117
1 -.2891*A(4,J)+.0898*A(5,J)+.3232*A(6,J)+.2231*A(7,J)                  *117
2 -.0736*A(8,J)-.2679*A(9,J)-.1882*A(10,J)+.0639*A(11,J)                *117
3 +.2338*A(12,J)                                                        *117
  RS75=R(J)+.2588*A(1,J)-.3995*A(2,J)-.3449*A(3,J)                      *118
1 +.1434*A(4,J)+.3427*A(5,J)+.0431*A(6,J)-.2731*A(7,J)                  *118
2 -.1702*A(8,J)+.1595*A(9,J)+.2316*A(10,J)-.0306*A(11,J)                *118
3 -.2275*A(12,J)                                                        *118
  RS90=R(J)-.5*A(2,J)+.375*A(4,J)-.3125*A(6,J)                          *119
1 +.2734*A(8,J)-.2461*A(10,J)+.1610*A(12,J)                             *119
  RS105=R(J)-.2588*A(1,J)-.3995*A(2,J)+.3449*A(3,J)                     *120
1 +.1434*A(4,J)-.3427*A(5,J)+.0431*A(6,J)+.2730*A(7,J)                  *120
2 -.1702*A(8,J)-.1595*A(9,J)+.2316*A(10,J)+.0305*A(11,J)                *120
3 -.2275*A(12,J)                                                        *120
  RS120=R(J)-.5000*A(1,J)-.1250*A(2,J)+.4375*A(3,J)                     *121
1 -.2891*A(4,J)-.0898*A(5,J)+.3232*A(6,J)-.2232*A(7,J)                  *121
2 -.0736*A(8,J)+.2679*A(9,J)-.1882*A(10,J)-.0639*A(11,J)                *121
3 +.2338*A(12,J)                                                        *121
  RS135=R(J)-.7071*A(1,J)+.2500*A(2,J)+.1778*A(3,J)                     *122
1 -.4062*A(4,J)+.3756*A(5,J)-.1485*A(6,J)-.1270*A(7,J)                  *122
2 +.2983*A(8,J)-.2855*A(9,J)+.1151*A(10,J)+.1042*A(11,J)                *122
3 -.2467*A(12,J)                                                        *122
  RS150=R(J)-.8660*A(1,J)+.6250*A(2,J)-.3248*A(3,J)                     *123
1 +.0235*A(4,J)+.2233*A(5,J)-.3740*A(6,J)+.4102*A(7,J)                  *123
2 -.3388*A(8,J)+.1896*A(9,J)-.0071*A(10,J)-.1607*A(11,J)                *123
3 +.2732*A(12,J)                                                        *123
  RS165= R(J)-.9659*A(1,J)+.8995*A(2,J)-.8042*A(3,J)                    *124
1 +.6847*A(4,J)-.5471*A(5,J)+.3983*A(6,J)-.2456*A(7,J)                  *124
2 +.0962*A(8,J)+.0427*A(9,J)-.1650*A(10,J)+.2654*A(11,J)                *124
3 -.3402*A(12,J)                                                        *124
  RS180=R(J)-A(1,J)+A(2,J)-A(3,J)+A(4,J)-A(5,J)+A(6,J)                  *125
1 -A(7,J)+A(8,J)-A(9,J)+A(10,J)-A(11,J) +A(12,J)                        *125
  V=(J-1)/DELTAJ                                                        *126
  H=(J-1.)/DELTAJ                                                       *127
  WHENEVER H .E. V                                                      *128
  PRINT RESULTS R1(J), R(J), RSZERO, APP(1,J)                           *129      01
  PRINT FORMAT RESULT, RS15, RS30,RS45,RS60,RS75,RS90                   *130      01
  PRINT FORMAT RESULT, RS105  RS120,RS135,RS150,RS165,RS180             *131      01
  PRINT FORMAT RESULT, A(1,J.,A(2,J),A(3,J),A(4,J),A(5,J),              *132      01
1 A(6,J)                                                                *132
```

```
      PRINT FORMAT RESULT, A(7,J), A(8,J), A(9,J),A(10,J),                    *133      01
     1 A(11,J), A(12,J)                                                       *133
      PRINT FORMAT RESULT, ER1, ER2, ER3, ER4, ER5, ER6                       *134      01
      END OF CONDITIONAL                                                      *135      01
      WHENEVER  .ABS. A(12,J) .G. 0.1, TRANSFER TO START                      *136
      WHENEVER J .E.  JMAX, TRANSFER TO START                                 *137
      M=0                                                                     *138
      Z=0                                                                     *139
SW    MMM1=2.*M-1                                                             *140
      MMM3=2.*M-3                                                             *141
      MMM5=2.*M-5                                                             *142
      MMM7=2.*M-7                                                             *143
      MMM9=2.*M-9                                                             *144
      MMP1=2.*M+1                                                             *145
      MMP3=2.*M+3                                                             *146
      MMP5=2.*M+5                                                             *147
      MMP7=2.*M+7                                                             *148
      MMP9=2.*M+9                                                             *149
      MM1=M-1.                                                                *150
      MM2=M-2.                                                                *151
      MM3=M-3.                                                                *152
      MM4=M-4.                                                                *153
      MP1=M+1.                                                                *154
      MP2=M+2.                                                                *155
      MP3=M+3.                                                                *156
      MP4=M+4.                                                                *157
      MP5=M+5.                                                                *158
      WHENEVER Z .E. 0, TRANSFER TO SWZERO                                    *159
      WHENEVER Z .E. 1, TRANSFER TO SW11                                      *160
      WHENEVER Z .E. 2, TRANSFER TO SW22                                      *161
      WHENEVER Z .E. 3, TRANSFER TO SW33                                      *162
      WHENEVER Z .E. 4, TRANSFER TO SW44                                      *163
      WHENEVER Z .E. 5, TRANSFER TO SW55                                      *164
SWZERO DZERO=2.5*MP1*MP2*MP3/MMP3/MMP5/MMP7*(3.*M+MP2*MP3)                    *165
     1 *TA(J)*TC(J)/R1(J)*A(M+3,J)+25.*MP1*MP1*MP2*MP3*MP4                    *165
     2 /MMP3/MMP5/MMP7/MMP9*TC(J)*TB(J)*TC(J)/R1(J)*A(M+4,J)                  *165
      CZERO= MP1*MP2/MMP3/MMP5*(-5.*TC(J)*TB(J)*AP(M+2,J)+                    *166
     1 (-10.*TC(J)*RP(J)/R1(J)+0.75*MP1*MP2/R(J)*TA(J)*TA(J)                  *166
     2 +7.5*MP3*TC(J)/R1(J)*TA(J)+25.*(0.4-0.6*MP1*MP3/MMP1                   *166
     3 /MMP7-M*M/MMM1/MMP1-MP2*MP3/MMP3/MMP7)*TC(J)*TB(J)*TC(J)               *166
     4 /R1(J))*A(M+2,J))+DZERO                                                *166
      BZERO(M)=-2.5*MP2*MP1/MMP3*TC(J)*TB(J)*TC(J)*TB(J).P.MP1                *167
     1 +MP1/MMP3 *(-1.5*TA(J)*AP(M+1,J)+(       -3.*RP(J)/R(J)                *167
     2 *TA(J)+2.25*MP2/R(J)*TA(J)*TA(J)+(0.4*MP1/MMP3-7.5*MP1                 *167
     3 *MP2/MMP5/MMP3*(0.6*M/MMM1+MP1/MMP1+MP2*MP3/MMP3))                     *167
     4 *TA(J)*TC(J)/R1(J))*A(M+1,J))+CZERO                                    *167
      LZERO(M)=-5.*MP1*MP1*MP2*MP4/MMP3/MMP3/MMP1*TC(J)*TC(J)                 *168
     1 *TB(J).P.MP2*A(M+1,J)/R(J)                                             *168
      WHENEVER M .E. MMAX, TRANSFER TO SW1                                    *169
      M=M+1                                                                   *170
      TRANSFER TO SW                                                          *171
SW1   M=1                                                                     *172
      Z=1                                                                     *173
      TRANSFER TO SW                                                          *174
SW11  D1=MP1*M/MMP3*(-0.5*TA(J)*AP(M+1,J)+(TC(J)/R1(J)*V1(J)                  *175
     1            +2.5*MP1*M/MMP3/MMM1*TC(J)/R1(J)*TA(J))*A(M+1,J))          *175
     2 +5.*M*MP2/MMP3/MMP5*(-TC(J)*TB(J)*AP(M+2,J)+(TC(J)*TB(J)               *175
     3 /R1(J)*V1(J)-TC(J)*RP(J)/R1(J)+5.*MMP1/MMP3/MMM1*TC(J)                 *175
     4 *TB(J)*TC(J)/R1(J))*A(M+2,J))                                          *175
      C1=-2.25*MP1*MP1*MP2/MMP1/MMP3*TA(J)*TA(J)/R(J)*A(M,J)+7.5*M            *176
```

```
     1 *MP1/MMM1/MMP3*TC(J)/R1(J)*TA(J)*A(M,J)+(25.*MP1*MP1/MMP1          *176
     2 /MMP3*(0.4-0.6*M/MMM1*MP2/MMP5-3*MP2*MP2/MMP3/MMP5)+25*M*M         *176
     3 /MMM1/MMP1*(0.4-0.6*MP1*MM1/MMP3/MMM3-MP1*MP1/MMP3                 *176
     4 /MMP3/MMM1))*TC(J)*TB(J)*TC(J)/R1(J)*A(M,J)+D1                      *176
       B1(M)=-1.5*MP1*TA(J)*TC(J)*TB(J).P.M+3./MP1*RP(J)                  *177
     1 *AP(M,J)-5.*M/MMM1/MMP3*TC(J)*TB(J)*AP(M,J)                        *177
     2         +5.*M/MMM1/MMP3*(MP2*TC(J)*RP(J)/R1(J)*A(M,J)              *177
     3 -3.*TB(J)*TC(J)/R1(J)*V1(J)*A(M,J)+TB(J)*TC(J)*AP(M,J))            *177
     4 +(-5.*M*MP1/MMM1/MMP3-10.*M/MMM1/MMP3)*TC(J)*RP(J)/R1(J)           *177
     5 *A(M,J)+C1                                                         *177
       L1(M)=-3.*MP1*MP1*MP4/MMP3/MMP1*TA(J)*TC(J)                        *178
     1 *TB(J).P.M*A(M+1,J1/R(J)                                           *178
       WHENEVER M .E. MMAX, TRANSFER TO SW2                               *179
       M=M+1                                                              *180
       TRANSFER TO SW                                                     *181
SW2    M=2                                                                *182
       Z=2                                                                *183
       TRANSFER TO SW                                                     *184
SW22   C2=7.5*M/MMM1*(0.4*MM1 *M41/MMP1/MMP3-MP1/MMP3                     *185
     1 *(3.*MP1/MMP1+0.6*MM1/MMM3+M/MMM1))*TC(J)/R1(J)*TA(J)              *185
     2 *A(M-1,J)-7.5*MM1*MM1*MP1*M/MMM1/MMP1/MMP3*TA(J)*TC(J)             *185
     3 /R1(J)*A(M+1,J)                                                    *185
       B2(M)=MMP1*TC(J)*(-TB(J).P.M*V1(J)+M/MP1*TB(J).P.MM1.              *186
     1 *RP(J)-2.5*M/MMM1/MMP3*TB(J).P.M*TC(J))+(MM1*MP2/R(J)/R(J)         *186
     2 -0.75*M*M*MM1*MM1/MMP1/4MM1*TA(J)*TA(J)/R(J))*A(M,J)               *186
     3 +1.5*TA(J)*AP(M-1,J)+M/MMM1*(-0.5*V1P(J)+3.*MM1/M*RP(J)/R(J)       *186
     4 *TA(J)-1.5*TC(J)/R1(J)*V1(J)-2.25*MM1/R(J)*TA(J)*TA(J))            *186
     5 *A(M-1,J)+C2                                                       *186
       CL2=2.*MP1*MP4/MMP3*TC(J)*(-TB(J).P.M*V1(J)+M/MP1*TB(J)            *187
     1 .P.MM1*RP(J)-2.5*M/MMM1/MMP3*TB(J).P.M*TC(J))/R(J)*A(M+1,J)        *187
       L2(M)=5.*MP1*MP2*MM3*M/MMP3/MMM1/MMP1*TC(J)*TC(J)*TB(J)            *188
     1 .P.MP2/R(J)*A(M-1,J) +3.*MP1*MM3*M/MMM1/MMP1*TA(J)                 *188
     2 *TC(J)*TB(J).P.M/R(J)*A(M-1,J)-2.*MP3*M/MMM1*T((J)                 *188
     3 *(-TB(J).P.M*V1(J)+M/MP1*TB(J).P.MM1*RP(J)-2.5*M/MMM1              *188
     4 /MMP3*TB(J).P.M*TC(J))/R(J)*A(M-1,J)  +CL2                         *188
       WHENEVER M .E. MMAX, TRANSFER TO SW3                               *189
       M=M+1                                                              *190
       TRANSFER TO SW                                                     *191
SW3    M=3                                                                *192
       Z=3                                                                *193
       TRANSFER TO SW                                                     *194
SW33   C3=M/MMM3/MMM1*((15.*MM1/MP1+5.*MM2)*TC(J)*TB(J)                  *195
     1 *AP(M-2,J)+((15*MM1/MP1+10.*MM2)*TC(J)*RP(J)/R1(J)-15.             *195
     2 *MM1/MP1*TC(J)*TB(J)/R1(J)*V1(J)+2.25*MM1*MM1/R(J)*TA(J)           *195
     3 *TA(J)-7.5*MM1*MM2*TA(J)*TC(J)/R1(J)+25.*MM1*(0.4-0.6*MM2          *195
     4 *M/MMM5/MMP1-MP1*MP1/MMP1/MMP3*MM2*MM2/MMP5/MMM1-3.*M              *195
     5 /MMM1/MMP3) *TC(J)*TB(J)*TC(J)/R1(J))*A(M-2,J))                    *195
       B3(M)=1.5*TA(J)*TC(J)*TB(J).P.(M-2)*(M-1.)-25.*MM2*MM2             *196
     1 *M*M*MM1/MMM1/MMM1/MMP1/MMM3*TC(J)*TB(J)*TC(J)/R1(J)*A(M,J)        *196
     2 -2.5*MM1*MM1*MM2*MM2/MMM1/MMM1/MMM3*TC(J)/R1(J)*TA(J)              *196
     3 *A(M-1,J)+C3                                                       *196
       L3(M)=-3.*MM1*MP3*M/MMM1/MMP1*TA(J)*TC(J)*TB(J).P.MM2              *197
     1 /R(J)*A(M-1,J) +3.*MM1*MP1*MP4/MMP3/MMP1*TA(J)*TC(J)               *197
     2 *TB(J).P.MM2/R(J)*A(M+1,J)                                         *197
       WHENEVER M .E. MMAX, TRANSFER TO SW4                               *198
       M=M+1                                                              *199
       TRANSFER TO SW                                                     *200
SW4    M=4                                                                *201
       Z=4                                                                *202
       TRANSFER TO SW                                                     *203
```

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SW44        B4(M)=2.5*TC(J)*TB(J)*TC(J)*TB(J).P.(M-3)*(M-2.)*M                  *204
           1 /(2.*M-1)                                                         *204
           2 +7.5*MM2*MM1/MMM1/MMM3/MMM5*(M*M-3.)*TA(J)*TC(J)/R1(J)            *204
           3 *A(M-3,J)                                                         *204
             L4(M)=-5.*MM2*M*M*MM3/MMM1/MMM1/MMP1*TC(J)*TC(J)                  *205
           1 *TB(J).P.MM2/R(J)*A(M-1,J)+5.*MM2*M*MP1*MP4/MMM1/MMP3             *205
           2 /MMP1*TC(J)*TC(J)*TB(J).P.MM2/R(J)*A(M+1,J)                       *205
             WHENEVER M .E. MMAX, TRANSFER TO SW5                              *206
             M=M+1                                                             *207
             TRANSFER TO SW                                                    *208
SW5          M=5                                                               *209
             Z=5                                                               *210
             TRANSFER TO SW                                                    *211
SW55         B5(M)=25.*MM3*MM2*M*(4.*M-7)/MMM1/MMM3/MMM5/MMM7                  *212
           1 *TC(J)*TB(J)*TC(J)/R1(J)*A(M-4,J)                                 *212
             WHENEVER M .E. MMAX, TRANSFER TO SGV1                             *213
             M=M+1                                                             *214
             TRANSFER TO SW                                                    *215
SGV1         GV1= LZERO(0)+LZERO(1)+LZERO(2)+LZERO(3)+L1(1)+L1(2)              *216
           1 +L1(3)+L2(2)+L2(3)+L3(3)                                          *216
             M=4                                                               *217
SWGV1        GV1=GV1+LZERO(M)+L1(M)+L2(M)+L3(M)+L4(M)                          *218
             WHENEVER M .E. MMAX, TRANSFER TO  SWFV1                           *219
             M=M+1                                                             *220
             TRANSFER TO  SWGV1                                                *221
SWFV1        FV1=1.33333*(1.5*RP(J)*TA(J)-3.*TC(J)*TB(J)*V1(J))*A(2,J)         *222
           1 /R(J)+GV1                                                         *222
             EV1=2./7*(TC(J)*TB(J)/R1(J)*V1(J)-TC(J)/R1(J)*RP(J)               *223
           1 +3.*TC(J)*TC(J)*TB(J)/R1(J))*A(3,J)+4J./21*TC(J)/R1(J)            *223
           2 *TA(J)*A(4,J)+800./693*TC(J)*TC(J)*TB(J)/R1(J)                    *223
           3 *A(5,J)-0.2*TA(J)*TA(J)/R(J)*A(1,J)+8./3*(1.5*RP(J)*RP(J)         *223
           4 +PZERO+0.5*((MU/RZERO/RZERO).P.2-(MU/R1(J)/R1(J)).P.2)             *223
           5 -0.25*TA(J)*TA(J))*A(1,J)/R(J)+FV1                                *223
             DV1=4./15*(TC(J)/R1(J)*V1(J)          +TC(J)/R1(J)                *224
           1 *TA(J))*A(2,J)-4./7*TC(J)*TB(J)*AP(2,J)+4./35*(-10.               *224
           2 *TC(J)/R1(J)*RP(J)+4.5/R(J)*TA(J)*TA(J)+30.*TC(J)                 *224
           3 /R1(J)*TA(J)-85./9*TC(J)*TC(J)*TB(J)/R1(J))*A(3,J)                *224
           4 -2./7*TC(J)*TB(J)*AP(3,J)+EV1                                     *224
             CV1=(-8./3*TC(J)*RP(J)/R1(J)-1.2/R(J)*TA(J)*TA(J)                 *225
           1 +2.*TC(J)/R1(J)*TA(J))*A(1,J)-56./3*TC(J)*TC(J)*TB(J)/R1(J)       *225
           2 *A(1,J)-0.4*TA(J)*AP(2,J)+4./15*(       -3.*RP(J)/R(J)            *225
           3 *TA(J)+27./4/R(J)*TA(J)*TA(J)-4.55*TA(J)*TC(J)/R1(J))*            *225
           4 A(2,J)-2./15*TA(J)*AP(2,J)+DV1                                    *225
             HR1 =3./R(J)*(RP(J)*TA(J)-2.*TC(J)*TC(J)*TB(J)-2.*TC(J)           *226
           1 *TC(J)*TB(J)*TB(J)*TB(J)                    +(RP(J)               *226
           2 -0.6667*TC(J)*TB(J))*AP(1,J)                                      *226
           3 -0.6667*(3.*TC(J)/R1(J)*RP(J)*A(1,J)-3.*TC(J)*TB(J)               *226
           4 /R1(J)*V1(J)*A(1,J)+TB(J)*TC(J)*AP(1,J))+ CV1)                    *226
             HR  =-1/R(J)*(BZERO(0)+PZERO+0.5*((MU/RZERO/RZERO).P.2             *227
           1 +V1(J)*V1(J))+1.5*RP(J)*RP(J)-0.75*TA(J)*TA(J)-2./1               *227
           2 +2/R(J)-PZERO/R(J).P.(3*GAMMA)-TC(J)*V1(J))                       *227
             HA1  =-2./R(J)*(BZERO(1)+B1(1)                                    *228
           1 +1.5*RP(J)*TA(J)-3*TC(J)*TB(J)*V1(J))                             *228
             HA(2)=-3./R(J)*(BZERO(2)+B1(2)+B2(2)+0.75*TA(J)*TA(J))            *229
             HA(3)=-4. /R(J)*(BZERO(3)+B1(3)+B2(3)+B3(3))                      *230
             HA(4)=-5/R(J)*(BZERO(4)+B1(4)+B2(4)+B3(4)+B4(4))                  *231
             HA(5)=-6/R(J)*(BZERO(5)+B1(5)+B2(5)+B3(5)+B4(5)                   *232
           1 +B5(5))                                                           *232
             HA(6)=-7/R(J)*(BZERO(6)+B1(6)+B2(6) +B3(6)+B4(6)+B5(6))           *233
             HA(7)=-8./R(J)*(BZERO(7)+B1(7)+B2(7)+B3(7)+B4(7)+B5(7))           *234
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      HA(8)=-9./R(J)*(BZERO(8)+B1(8)+B2(8)+B3(8)+B4(8) +B5(8))                  *235
      HA(9)=-10./R(J)*(BZERO(9)+B1(9)+B2(9)+B3(9)+B4(9)+B5(9))                  *236
      HA(10)=-11./R(J)*( BZERO(10)+B1(10)+B2(10)+B3(10)                         *237
     I +B4(10)+B5(10))                                                          *237
      HA(11)=-12./R(J)*(BZERO(11)+B1(11)+B2(11)+B3(11)+B4(11)                   *238
     1 +B5(11))                                                                 *238
      HA(12)=-13./R(J)*(BZERO(12)+B1(12)+B2(12)+B3(12)+                         *239
     1 B4(12)+B5(12))                                                           *239
      AR1=1.+0.8*A(2,J)/R(J)                                                    *240
      AR=0.33333*A(1,J)/R(J)                                                    *241
      AA1=1.2*A(2,J)/R(J)-1.                                                    *242
      BR1=-7.*A(1,J)/R(J)                                                       *243
      BR=1.                                                                     *244
      BA1=-A(1,J)/R(J)                                                          *245
      DET=AR1*BR                -AR*BR1                                         *246
      M=2                                                                       *247
SAA   AA(M)=(M+1).P.2/(2.*M+3)*(1.+M/2.)*A(M+1,J)/R(J)                          *248
     1 -0.5*M*(M+1.)/(2.*M-1.)*A(M-1,J)/R(J)                                    *248
      BA(M)=-M*A(M,J)/R(J)                                                      *249
      AMPP(M)=(AR1*BR*HA(M)+AR*BA(M)*HR1+AA(M)*HR*BR1                           *250
     1 -HR1*BR*AA(M)-HR*BA(M)*AR1-HA(M)*AR*BR1)                                 *250
      WHENEVER M .E. MMAX, TRANSFER TO CALL                                     *251
      M=M+1                                                                     *252
      TRANSFER TO SAA                                                           *253
CALL  S=RKDEQ.(0)                                                               *254
      WHENEVER S .E. 1.0                                                        *255
      F(1)=Y(2)                                                                 *256          01
      F(2)= 1./DET*(HR1*BR                -HR*BR1)                              *257          01
      F(3)=Y(4)                                                                 *258          01
      F(4)=1./DET*(AR1*HR                 -AR*HR1)                              *259          01
      F(5)=Y(6)                                                                 *260          01
      F(6)=1./DET*(AR1*BR*HA1+AR*BA1*HR1+AA1*HR*BR1                             *261          01
     1 -HR1*BR*AA1-HR*BA1*AR1-HA1*AR*BR1)                                       *261
      F(7)=Y(8)                                                                 *262          01
      F(8)=AMPP(2)                                                              *263          01
      F(9)=Y(10)                                                                *264          01
      F(10)=AMPP(3)                                                             *265          01
      F(11)=Y(12)                                                               *266          01
      F(12)=AMPP(4)                                                             *267          01
      F(13)=Y(14)                                                               *268          01
      F(14)=AMPP(5)                                                             *269          01
      F(15)=Y(16)                                                               *270          01
      F(16)=AMPP(6)                                                             *271          01
      F(17)=Y(18)                                                               *272          01
      F(18)=AMPP(7)                                                             *273          01
      F(19)=Y(20)                                                               *274          01
      F(20)=AMPP(8)                                                             *275          01
      F(21)=Y(22)                                                               *276          01
      F(22)=AMPP(9)                                                             *277          01
      F(23)=Y(24)                                                               *278          01
      F(24)=AMPP(10)                                                            *279          01
      F(25)=Y(26)                                                               *280          01
      F(26)=AMPP(11)                                                            *281          01
      F(27)=Y(28)                                                               *282          01
      F(28)=AMPP(12)                                                            *283          01
      TRANSFER TO CALL                                                          *284          01
      END OF CONDITIONAL                                                        *285          01
      TRANSFER TO SAVE                                                          *286
      END OF PROGRAM                                                            *287
```

APPENDIX V

NUMERICAL EXAMPLE FOR A BUBBLE MOVING IN A SOURCE FLOW

The inputs are:

$$p_{in}(0) = 195, \quad \gamma = 1.4, \quad V_o = 240, \quad \mu = 2.2(10^6), \quad r_o = 100$$

The outputs are arranged in a sequence as

r_1 , R , $(r_s)_{\theta=0°}$, $\ddot{a}_1$

$(r_s)_{\theta=15°}$, $(r_s)_{\theta=30°}$, $(r_s)_{\theta=45°}$, $(r_s)_{\theta=60°}$, $(r_s)_{\theta=75°}$, $(r_s)_{\theta=90°}$

$(r_s)_{\theta=105°}$, $(r_s)_{\theta=120°}$, $(r_s)_{\theta=135°}$, $(r_s)_{\theta=150°}$, $(r_s)_{\theta=165°}$, $(r_s)_{\theta=180°}$

a_1 , a_2 , a_3 , a_4 , a_5 , a_6

a_7 , a_8 , a_9 , a_{10}, a_{11}, a_{12}

ER1 , ER2 , ER3 , ER4, ER5, Er6

where

$$ER1 = 3\frac{\dot{R}}{R}\left(\frac{\mu}{r_1^2} - V_1\right) \qquad ER2 = 6\frac{\mu^2 R}{r_1^5} \qquad ER3 = -\frac{1}{2R}\left[\left(\frac{\mu}{r_o^2}\right)^2 - \left(\frac{\mu}{r_1^2}\right)^2\right]$$

$$ER4 = \frac{1}{4R}\left(\frac{\mu}{r_1^2} - V_1\right)^2 \qquad ER5 = -\frac{3}{2R}\dot{R}^2 \qquad ER6 = -\frac{1}{R}p_{in}(0)\left(1 - \frac{1}{R^{3\gamma}}\right)$$

R1(1) =	100.000000,	R(1) =	1.000000,	RSZERO =	1.000000,	APP(1,1) = 1.8845176-37
1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
1.000000	1.000000	1.000000	1.000000	1.000000	1.000000	1.000000
.000000	.000000	.000000	.000000	.000000	.000000	.000000
.000000	.000000	.000000	.000000	.000000	.000000	.000000
-.000000	2903.999939	-.000000	100.000000	-.000000	.000000	
R1(6) =	100.179173,	R(6) =	1.000018,	RSZERO =	.999925,	APP(1,6) = 2.844228
.999924	.999926	.999943	.999986	1.000055	1.000124	
1.000153	1.000111	.999991	.999832	.999697	.999644	
.000000	-.000221	.000142	-.000012	-.000001	.000000	
.000000	-.000000	.000000	.000000	-.000000	.000000	
-1.378484	2878.173492	-172.663723	86.580893	-.000914	-.014337	
R1(11) =	100.356721,	R(11) =	1.000001,	RSZERO =	.999651,	APP(1,11) = 5.859511
.999644	.999647	.999710	.999882	1.000151	1.000417	
1.000523	1.000346	.999887	.999293	.998799	.998607	
.000002	-.000849	.000531	-.000023	-.000011	.000000	
.000000	.000000	-.000000	-.000000	.000000	-.000000	
4.631641	2852.757385	-342.248096	74.123573	-.012059	-.001049	
R1(16) =	100.532652,	R(16) =	.999848,	RSZERO =	.999105,	APP(1,16) = 8.231622
.999086	.999082	.999213	.999590	1.000184	1.000762	
1.000973	1.000565	.999573	.998329	.997312	.996922	
.000008	-.001829	.001116	-.000004	-.000033	-.000001	
.000000	.000000	-.000000	-.000000	.000000	.000000	
16.153411	2827.450073	-508.891720	62.701921	-.173395	.124449	
R1(21) =	100.706982,	R(21) =	.999457,	RSZERO =	.998217,	APP(1,21) = 10.043096
.998177	.998152	.998365	.999016	1.000049	1.001038	
1.001367	1.000633	.998945	.996885	.995233	.994605	

```
   .000018          -.003108           .001855          .000067          -.000069          -.000006
   .000001           .000000          -.000000         -.000000          -.000000           .000000
 31.408732        2801.966431        -672.797020        52.349912          -.785188           .445390

R1(26) =    100.879729,        R(26) =      .998730,        RSZERO =      .996920,    APP(1,26) =     11.369192
   .996848           .996781           .997080          .998064           .999641          1.001127
  1.001576          1.000425           .997906          .994909           .992546           .991655
   .000034          -.004632           .002714          .000204          -.000117          -.000015
   .000002           .000000           .000000         -.000000          -.000000           .000000
 48.744149        2776.035767        -834.233711        43.069646         -2.298595          1.045088

R1(31) =    101.050919,        R(31) =      .997569,        RSZERO =      .995149,    APP(1,31) =     12.288961
   .995032           .994897           .995276          .996642           .998855          1.000910
  1.001473           .999821           .996362          .992345           .989226           .988059
   .000056          -.006350           .003662          .000417          -.000175          -.000032
   .000002           .000001           .000000         -.000000          -.000000           .000000
 66.637712        2749.400970        -993.529800        34.838543         -5.310910          2.008777

R1(36) =    101.220577,        R(36) =      .995878,        RSZERO =      .992838,    APP(1,36) =     12.878120
   .992662           .992428           .992872          .994657           .997583          1.000269
  1.000933           .998707           .994227          .989136           .985237           .983790
   .000085          -.008214           .004678          .000706          -.000242          -.000058
   .000003           .000002           .000000         -.000000          -.000000           .000000
 83.689769        2721.816071       -1151.072098        27.616651        -10.567262          3.426379

R1(41) =    101.388731,        R(41) =      .993563,        RSZERO =      .989922,    APP(1,41) =     13.201920
   .989671           .989302           .989788          .992015           .995720           .999083
   .999835           .996976           .991416          .985221           .980539           .978812
   .000122          -.010177           .005745          .001071          -.000313          -.000094
   .000002           .000004           .000000         -.000000          -.000000           .000000
 98.599607        2693.045197       -1307.300858        21.353103        -18.970471          5.395927

R1(46) =    101.555409,        R(46) =      .990528,        RSZERO =      .986332,    APP(1,46) =     13.311731
   .985990           .985448           .985945          .988622           .993156           .997235
   .998061           .994521           .987846          .980537           .975079           .973078
   .000166          -.012194           .006850          .001505          -.000389          -.000140
  -.000001           .000006           .000001         -.000000          -.000000          -.000000
110.129857        2662.859314       -1462.717026        15.991788        -31.601077          8.028115

R1(51) =    101.720633,        R(51) =      .986678,        RSZERO =      .981996,    APP(1,51) =     13.242976
   .981549           .980794           .981261          .984384           .989780           .994604
   .995493           .991241           .983432          .975019           .969802           .966536
   .000217          -.014223           .007985          .002001          -.000468          -.000198
  -.000005           .000009           .000002         -.000000          -.000000          -.000000
117.059071        2631.035706       -1617.873749        11.476402        -49.750009         11.451933

R1(56) =    101.884427,        R(56) =      .981914,        RSZERO =      .976841,    APP(1,56) =     13.013991
   .976272           .975264           .975654          .979204           .985480           .991068
   .992014           .987033           .978091          .968593           .961644           .959123
   .000276          -.016224           .009143          .002547          -.000551          -.000266
  -.000012           .000013           .000003         -.000000          -.000000          -.000000
118.121868        2597.352386       -1773.399658         7.754922        -74.967314         15.822305

R1(61) =    102.046806,        R(61) =      .976135,        RSZERO =      .970784,    APP(1,61) =     12.626304
   .970081           .968780           .969039          .972981           .980137           .986500
   .987503           .981793           .971734          .961185           .953534           .950772
   .000342          -.018159           .010319          .003130          -.000639          -.000345
  -.000020           .000017           .000004         -.000000          -.000000          -.000000
111.931074        2561.588226       -1929.990936         4.783407       -109.132165         21.329880
```

```
 R1(66) =    102.207795,          R(66) =      .969238,        RSZERO =      .964746,     APP(1,66) =    12.105558
    .963860          .962032          .961618          .965231          .972857          .980242
    .981929          .975921          .964713          .952761          .944046          .940895
    .000416         -.019297          .012489          .003259         -.000984         -.000404
    .000000          .000025          .000004         -.000001         -.000000         -.000000
  96.846241i      2523.521851     -2088.445251         2.529144      -154.518578        28.212247

 R1(71) =    102.367392,          R(71) =      .961109,        RSZERO =      .956917,     APP(1,71) =    11.187223
    .955868          .953624          .952813          .956485          .964807          .973053
    .975024          .968457          .956178          .943142          .933682          .930272
    .000497         -.020776          .014005          .003718         -.001187         -.000488
    .000003          .000032          .000006         -.000001         -.000000         -.000000
   70.895041      2482.910553     -2249.662872          .978385      -214.047817        36.781066

 R1(76) =    102.525599,          R(76) =      .951630,        RSZERO =      .947950,     APP(1,76) =     9.913616
    .946731          .944037          .942742          .946347          .955276          .964384
    .966706          .959657          .946363          .932277          .922092          .918429
    .000583         -.022032          .015587          .004126         -.001426         -.000574
    .000010          .000041          .000007         -.000001         -.000001         -.000000
  31.559133      2439.511749     -2414.682922          .142446      -291.331875        47.437282

 RI(81) =    102.682418,          R(81) =      .940670,        RSZERO =      .937726,     APP(1,81) =     8.186973
    .936335          .933165          .931299          .934694          .944111          .954070
    .956826          .949396          .935158          .920063          .909174          .905266
    .000675         -.023022          .017231          .004454         -.001706         -.000655
    .000023          .000051          .000008         -.000002         -.000001          .000000
 -24.489400      2393.059784     -2584.725922          .063903      -391.043800        60.716640

 R1(86) =    102.837824,          R(86) =      .928089,        RSZERO =      .926110,     APP(1,86) =     5.885918
    .924550          .920887          .918368          .921394          .931148          .941931
    .945216          .937533          .922444          .906388          .894818          .890671
    .000772         -.023697          .018927          .004666         -.002029         -.000725
    .000043          .000061          .000008         -.000003         -.000001          .000000
-101.778275      2343.266235     -2761.224976          .831149      -519.301842        77.345564

 R1(91) =    102.991789,          R(91) =      .913725,        RSZERO =      .912947,     APP(1,91) =     2.862819
    .911223          .907064          .903818          .906302          .916209          .927763
    .931687          .923909          .908085          .891128          .878900          .874520
    .000871         -.024009          .020659          .004724         -.002395         -.000774
    .000072          .000071          .000008         -.000004         -.000001          .000000
-206.624611      2289.808167     -2945.915100         2.599719      -684.268196        98.328909

 R1(96) =    103.144264,          R(96) =      .897398,        RSZERO =      .898051,     APP(1,96) =    -1.057618
    .896176          .891531          .887499          .889259          .899095          .911338
    .916016          .908341          .891929          .874142          .861283          .856677
    .000972         -.023902          .022397          .004585         -.002797         -.000791
    .000109          .000079          .000006         -.000005         -.000001          .000000
-348.075714      2232.319611     -3140.918457         5.627520      -897.055145       125.087772

R1(101) =    103.295184,         R(101) =      .878899,        RSZERO =      .881200,     APP(1,101) =    -6.078596
    .879194          .874093          .869234          .870085          .879587          .892393
    .897946          .890610          .873798          .855272          .841813          .836987
    .001072         -.023315          .024098          .004198         -.003221         -.000764
    .000155          .000083          .000004         -.000006         -.000001          .000000
-539.431480      2170.376190     -3348.893982        10.335094     -1173.132095       159.682718

R1(106) =    103.444446,         R(106) =      .857980,        RSZERO =      .862127,     APP(1,106) =   -12.424290
    .860015          .854510          .848813          .848577          .857439          .870624
    .877163          .870453          .853482          .834338          .820317          .815279
    .001169         -.022183          .025693          .003509         -.003640         -.000678
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      .000203          .000081          -.000000          -.000007          -.000001          .000000
   -800.830811      2103.478027      -3573.228394         17.413315      -1534.577560       205.188902

 RI(111) =    103.591900,    R(111) =      .834348,    SZERO =      .840496,    APP(1,111) =    -20.317792
      .838313          .832484          .825976          .824495          .832372          .845673
      .853285          .847540          .830726          .811125          .796596          .791355
      .001257         -.020434          .027080          .002465         -.004009         -.000520
      .000247          .000072         -.000004         -.000007         -.000001          .000000
 -1163.790817      2031.022873      -3818.351440         28.023364      -2013.809296       266.352669

 RI(116) =    103.737348,    R(116) =      .807641,    RSZERO =      .815880,    APP(1,116) =    -29.918540
      .813675          .807633          .800395          .797551          .804068          .817124
      .825835          .821450          .805208          .785383          .770423          .764992
      .001334         -.017990          .028105          .001020         -.004260         -.000284
      .000272          .000056         -.000007         -.000006         -.000001          .000000
 -1679.585052      1952.267670      -4090.231812         44.188423      -2660.016052       350.797367

 RI(121) =    103.880488,    R(121) =      .777408,    RSZERO =      .787723,    APP(1,121) =    -41.170886
      .785555          .779460          .771646          .767392          .772161          .784486
      .794210          .791620          .776509          .756803          .741534          .735938
      .001394         -.014781          .028537         -.000847         -.004291          .000021
      .000259          .000034         -.000005         -.000006         -.000001          .000000
 -2435.553284      1866.276199      -4397.146729         69.608222      -3550.777740       471.368382

 RI(126) =    104.020885,    R(126) =      .743069,    RSZERO =      .755270,    APP(1,126) =    -53.443757
      .753218          .747286          .739152          .733557          .736228          .747188
      .757639          .757278          .744047          .724991          .709628          .703911
      .001429         -.010755          .028025         -.003085         -.003963          .000347
      .000189          .000020          .000001         -.000006         -.000001          .000000
 -3588.936279      1771.834061      -4751.035156        111.478376      -4814.260864       651.020905

 RI(131) =    104.157866,    R(131) =      .703853,    RSZERO =      .717460,    APP(1,131) =    -64.724277
      .715629          .710157          .702104          .695423          .695762          .704594
      .715146          .717313          .706955          .689416          .674361          .668619
      .001434         -.005934          .026039         -.005484         -.003116          .000581
      .000056          .000034          .000008         -.000010         -.000001          .000001
 -5442.657532      1667.316605      -5169.925415        184.922424      -6674.537781       933.910942

 RI(136) =    104.290360,    R(136) =      .658699,    RSZERO =      .672730,    APP(1,136) =    -70.010157
      .671260          .666661          .659299          .652084          .650139          .656041
      .665546          .670089          .663824          .649267          .635351          .629808
      .001402         -.000541          .021798         -.007476         -.001663          .000502
     -.000078          .000102          .000001         -.000019          .000001          .000002
 -8631.655762      1550.467789      -5682.646851        324.999695      -9551.993530      1413.434814

 RI(141) =    104.416562,    R(141) =      .606079,    RSZERO =      .618647,    APP(1,141) =    -59.225159
      .617728          .614584          .608839          .602116          .598501          .600968
      .607619          .613204          .612097          .603044          .592149          .587397
      .001331          .004665          .014250         -.007721          .000072         -.000180
      .000016          .000200         -.000052         -.000023          .000007          .000003
 -14637.363159      1418.008148      -6338.902405        623.869667      -14309.372192      2313.883179

 RI(146) =    104.533177,    R(146) =      .543666,    RSZERO =      .551187,    APP(1,146) =    -18.812188
      .551073          .550210          .547487          .543047          .539445          .539102
      .540844          .543569          .546754          .547133          .543897          .541728
      .001231          .007411          .002549         -.003420          .000286         -.001284
      .000783          .000080         -.000140          .000000          .000020          .000005
 -27433.690430      1264.904633      -7233.428101       1367.285263      -22934.965332      4279.085754

 RI(151) =    104.633612,    R(151) =      .467703,    RSZERO =      .463329,    APP(1,151) =    132.033508
      .464265          .466825          .469252          .470160          .470109          .470490
      .467553          .459749          .458401          .470497          .486056          .492466
      .001127          .000978         -.010949          .009545         -.006608          .000299
      .001860         -.000698         -.000091          .000074          .000092         -.000005
 -59863.527832      1082.956345      -8574.391235       3665.806580      -40732.758301      9726.578369

 RI(156) =    104.702723,    R(156) =      .372067,    RSZERO =      .342896,    APP(1,156) =   4022.600861
      .344490          .351256          .362436          .373430          .384912          .394401
      .399290          .371799          .336015          .341314          .383722          .410116
      .001277         -.035583         -.006650          .027833         -.028739          .013812
      .000376         -.003365          .000734          .002306         -.000608         -.000564
 -165288.689453       858.673828    -10921.477295      13618.527344      -83588.181641      32803.398438

 RI(161) =    104.692848,    R(161) =      .255329,    RSZERO =     -.394204,    APP(1,161) =  1.569237E+04
      .408089         -.060812          .442936          .021618          .394351          .350748
      .293966          .592950         -.044157          .109595          .055046          .078895
      .007789         -.209496          .138329          .027685         -.130801          .128744
     -.117477          .011880          .550009         -.189468         -.684400         -.182328
 -581800.429688       589.537834    -15885.097656     108584.608398     -129887.858398     235345.941406
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