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TITLE A CHARACTERIZATION OF CERTAIN TOPOLOGICAL SPACES BY MEANS OF THEIR GROUPS OF HOMEOMORPHISMS

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# A CHARACTERIZATION OF CERTAIN TOPOLOGICAL SPACES BY MEANS OF THEIR GROUPS OF HOMEOMORPHISMS

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## PREFACE

The principal theme of this thesis is the study of groups of homeomorphisms of topological spaces when these groups are topologized with the point-open function space topology. The chief result is that among a certain class of highly homogeneous topological spaces, which includes all manifolds of dimension two or greater, the homeomorphism group under the point-open topology characterizes the space.

To Professor Hans Samelson, whose guidance and encouragement have made this work possible, the author is deeply indebted and wishes to express his sincere thanks.

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## Introduction

Every topological space F has associated with it a group G(F) consisting of all the homeomorphisms of that space onto itself. The elements of the group G(F) may at the same time be organized into a space by the introduction of one of the several function space topologies. This paper is a study of the connections between G(F) and F, where by G(F) we shall always mean the group-space obtained by the introduction of the point-open topology. Two spaces will be said to have the "same" group of homeomorphisms if their groups are topologically isomorphic, that is, if there exists a mapping between the two groups which is at once an isomorphism and a homeomorphism; two spaces will often be said to be the "same" if they are homeomorphic.

A question, which almost asks itself, is whether there exist "different" (i.e., non-homeomorphic) topological spaces which have the same group of homeomorphisms. The answer comes nearly as easily - yes. Examples are close at hand. The open interval and the closed, the ordinary open two ended cylinder and a sphere with two points identified are two pairs among many which share the same groups of homeomorphisms. We shall verify these statements in part III below. Another example is afforded by any topologically rigid space, that is, a space which admits no homeomorphisms onto itself other than the identity homeomorphism. The group of homeomorphisms of any such space would be indistinguishable from the group of the space consisting of a single point. Besicovitch[1] has exhibited a continuum

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1. Besicovitch, A. S.; Proc. Cambridge Philos. Soc.; 41, 96-103 (1945).

which is topologically rigid.

Thus the initial picture would not look too encouraging to some one setting out to study, and perhaps to distinguish, topological spaces by means of their groups of homeomorphisms. Nevertheless, it might still be possible, by making suitable restrictions on the spaces to be considered, for instance, to make the rather tidy statement, "If two spaces have the same group of homeomorphisms, they are the same."

It is this type of questioning that has led to the work and results which are here presented. In brief, a family $\mathcal{F}$ of topological spaces will be defined and it will be proved that any two spaces which have isomorphic groups of homeomorphisms are homeomorphic. The interest of this theorem is, in one sense, measured by the breadth of the family $\mathcal{F}$. A lower bound for this measurement is provided by the fact that, as defined below, $\mathcal{F}$ is broad enough to include all manifolds of dimension greater than one as a proper sub-family. Other than the definition given below no classification of the spaces in the family $\mathcal{F}$ has been achieved. However, an example of a space in $\mathcal{F}$ which is not a manifold is afforded by the space consisting of all the rational points in the plane in their sub-space topology.

No proof is given here that $\mathcal{F}$ is the widest class of spaces which are characterized by their groups of homeomorphisms; the author believes that it is probable that $\mathcal{F}$ can be enlarged. Another unanswered question, perhaps of greater interest, is the possibility

of obtaining a characterization theorem using the compact-open topology in the group of homeomorphisms. The latter question is likely to be a difficult one, for the point-open topology seems to lend itself more naturally to the recovery of the points and topology of F from its group G(F).

One question that is settled by this work is the matter of the existence of outer automorphisms of the group G(F), where F belongs to the family $\mathcal{F}$. It will be proved, along with several other collateral results, that every automorphism of G(F) is an inner-automorphism.

## I. The Nature of the Group of Homeomorphisms

### A. Introduction

Our job in this section will be to establish a few useful facts about G(F) (often just G for short), to define the family $\mathcal{F}$, and to end up with a theorem that will play a crucial role in the later argument.

Regarding the general nature of the group-space G, it should be pointed out at the outset that the point-open topology in G has certain weaknesses. For one thing G(F) is not a topological group for any of the spaces in $\mathcal{F}$. In this respect the compact-open topology displays a more respectable behavior. For the important class of compact Hausdorff spaces that topology makes the groups of homeomorphisms into topological groups. The compact open topology in the group of homeomorphisms of a locally compact regular topological space will cause group multiplication to be simultaneously continuous in both variables; if the space is non-compact inversion will not be continuous unless the space is locally connected.[2] Arens[3] has defined a modification of the compact open topology under which the group of homeomorphisms of any locally compact Hausdorff space becomes a topological group.

Another illustration of the shortcomings of the point-open topology is that, although for orientable manifolds the compact-open topology keeps the orientation preserving maps and the orientation

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2. Arens, Richard; Amer. J. Math.; 68, 593-610 (1946).

3. Ibid.

reversing ones in separate components, the point-open topology allows these two classes of maps to exist in a state of scrambled proximity. Still another failing of the point-open topology is that, though G in this topology is a group of transformations of G, it is not, in the currently accepted use of the term, a transformation group of F. Yet, for all this, we remain faithful to the point-open topology. We shall in fact find that even in the point-open topology the algebraic and topological structures of G are related and that the relationship is quite strong enough for the present purposes.

We must mention here that in the section on collateral results it will be shown that G(F) is connected whenever F is a manifold of dimension greater than one. G(F) ought to be connected whenever F is a connected member of $\mathcal{F}$, but the means of showing this have thus far eluded the author.

B. Relation between Algebra and Topology in G.

The point-open topology in G is defined by prescribing that the family of all sets of the form

$$W_{x_i;U_i} = W_{x_1,x_2,\ldots,x_n;U_1,U_2,\ldots,U_n} = \{g \in G \mid g(x_i) \in U_i, \forall i\},$$

where each $x_i$ is a point of F and each $U_i$ is an open set of F, be a basis for the open sets of G. As used above, the symbol $\forall i$ is to be taken to mean "for all permissable i", in this case for $i = 1,\ldots,n$. It is important to note that consideration of basis elements in which for some $i < j$, $x_i = x_j$ is unnecessary because for

such a set

$$W_{x_1,x_2,\ldots,x_n;U_1,\ldots,U_n} = W_{x_1,\ldots,\hat{x}_j,\ldots x_n;U_1,\ldots,U_{i-1},U_i\wedge U_j,\ldots\hat{U}_j,\ldots U_n},$$

where the symbol $\wedge$ indicates omission. Therefore, it is to be understood that no basic open set will be represented in a manner which involves repetitions among the $x_i$. Much of our work will concern n-tuples such as the $x_1,x_2,\ldots x_n$ above, and it will be convenient for us to adopt the name "proper n-tuple" for an n-tuple of objects in which no repetitions occur.

We now proceed to establish the relationship between the algebraic and topological structures of G. Until otherwise restricted, F is to be taken as an arbitrary topological space.

Lemma 1: $$g(W_{x_i;U_i}) = W_{x_i;g(U_i)}$$

(Note: Here and in the following the letters g, h, and f will represent elements of G; x, y, and z will represent elements of F.)

Proof: $$g\{h|h(x_i) \in U_i, \forall i\} = \{gh|h(x_i) \in U_i, \forall i\}$$
$$= \{f|g^{-1}f(x_i) \in U_i, \forall i\} = \{f|f(x_i) \in g(U_i), \forall i\}$$
$$= W_{x_i;g(U_i)}.$$ q.e.d.

Lemma 2: $$(W_{x_i;U_i})g = W_{g^{-1}(x_i);U_i}$$

Proof: $$\{h|h(x_i) \in U_i, \forall i\}\, g = \{hg|h(x_i) \in U_i, \forall i\} = \{f|fg^{-1}(x_i) \in U_i, \forall i\}$$
$$= W_{g^{-1}(x_i);U_i}.$$ q.e.d.

**Theorem 1:** All left and right translations in G are homeomorphisms.

**Proof:** These mappings are one to one onto. Lemma 1 yields that left translations are open, and, since the inverse of a left translation is a left translation, continuity in both directions follows. The proof for right translations proceeds similarly from Lemma 2.

q.e.d.

C. Relations between G(F) and F.

We begin by defining the map $\mu_x$ from G to F by the rule $\mu_x(g) = g(x)$, where x, here and throughout this section, is a fixed but arbitrary point of F. The following lemma holds for arbitrary F.

Lemma 3: $\mu_x$ is continuous.

Proof: Suppose U is open in F.

Then $\mu_x^{-1}(U) = \{g | g(x) \in U\} = W_{x;U}$ which is open in G.

q.e.d.

This completes the statements we will be able to make about G(F) with arbitrary F. From now on the symbol F will represent a space in the family $\mathcal{F}$, which family we now proceed to define. A topological space will be said to be n-homogeneous if for any pair of proper n-tuples $(x_1,\ldots,x_n)$ and $(y_1,\ldots,y_n)$ of points of the space, there exists a homeomorphism g of the space onto itself with the property, that $g(x_i) = y_i$, $\forall$ i. A space which is n-homogeneous for every n will be called $\omega$-homogeneous.

**Definition:** $\mathcal{F}$ is the totality of topological spaces, $\mathcal{F} = \{F\}$, which satisfy the following three conditions:

(a) F is $\omega$-homogeneous

(b) F is non-discrete

(c) F contains at least two non-empty and disjoint open sets.

In the light of condition (a), G(F) is called $\omega$-transitive over F.

It follows at once from property (c) and the fact that F is 2-homogeneous that F is Hausdorff; for if x and y are any two points of F and U and V are the two open sets provided in (c), there will exist a homeomorphism g such that $g(x)$ is in U and $g(y)$ is in V so that x will be in $g^{-1}(U)$ and y in $g^{-1}(V)$. $g^{-1}(U)$ and $g^{-1}(V)$ are disjoint open sets because $g^{-1}$ is a homeomorphism.

The idea of the homogeneity of a topological space is a familiar one. Some simple facts about homogeneous spaces might be mentioned. If a 1-homogeneous space has a property in the neighborhood of one of its points, it has that property in the neighborhood of each of its points. If a 2-homogeneous space is not totally disconnected it is connected. The line is 2-homogeneous but not 3. The circle is 3-homogeneous but not 4. The author has not been able to produce an example of a space which is 4-homogeneous but not 5.

The next lemma is trivial, but since it crops up frequently it will be convenient to have it stated in the form of a lemma.

Lemma 4: A topological space which is homogeneous, non-discrete, and Hausdorff has the property that every non-empty open set contains infinitely many distinct points.

The proof is omitted.

Definition: $G_x \equiv \{g \in G | g(x) = x\}$ , for a fixed x.

$G_x$ is clearly a subgroup of G. It has been called variously the isotropy group or the stability group. For us it will be simply the subgroup of a point. Since it is equal to the complement of the open set $\{g|g(x) \in F-x\}$, $G_x$ is a closed subgroup.

We direct ourselves now toward the proof of the theorem that $G/G_x$, the space of left cosets of $G_x$, is homeomorphic in a natural way to the space F.

First we note that the coset $h\,G_x = \{g|g(x) = h(x)\}$ , for if $g(x) = h(x)$ then $h^{-1}g(x) = x$ or $h^{-1}g = \bar{g}$ an element of $G_x$, whence $g = h\bar{g}$ an element of $h\,G_x$, and if $\bar{h} = hg$ an element of $h\,G_x$, then $\bar{h}(x) = hg(x) = h(x)$. This makes possible the

Definition: $\theta: G/G_x \longrightarrow F$ by $\theta(h\,G_x) = h(x)$.

In addition let $\nu$ be the natural mapping of G onto $G/G_x$, and take the usual factor space topology in $G/G_x$. This topology is defined by considering those subsets of $G/G_x$ open if and only if their inverse image under $\nu$ is open. At this point we can already see that $G/G_x$ is a space in which points are closed since we know that $G_x$ is a closed subgroup.

Note that the functional relation $\mu_x = \theta\nu$ obtains.

**Lemma 5:** $\mu_x$ is open.

**Proof:** Let $W_{x_1,\ldots,x_n;U_1,\ldots,U_n} = W_{x_i;U_i} = W$ be any basic neighborhood. Then

$$\mu_x(W) = \{y \in F \mid y = g(x) \text{ for some } g \text{ in } W\} = W(x),$$

where the last equality is taken as a definition of $W(x)$. We will frequently write the symbol $K(x)$ to mean the collection of all images of $x$ under the homeomorphisms of $K$ a subset of $G$.

**Case I:** $x = x_k$ for some $k$.

But for every $g$ in $W$, $g(x_k)$ is in $U_k$.

And, since $G$ is n-transitive, every point in $U_k$ will be covered.

Thus $W(x_k) = U_k = \mu_x(W)$.

q.e.d. Case I.

**Case II:** $x \neq x_i$, for all $i$.

We prove that $W(x) = F$, an open set.

Let $y$ be any point of $F$.

By lemma 4 we may choose a proper n-tuple $(y_1,\ldots,y_n)$ of points of $F$ such that $y_i$ is in $U_i$ and $y \neq y_i$ for all $i$. Then, by the n+1 transitivity of $G$, there exists a homeomorphism $g$ with the property that $g(x_i) = y_i$ for all $i$ and $g(x) = y$. Hence $g$ is in $W$ and $y$ is in $W(x) = \mu_x(W)$.

q.e.d.

Theorem 2: $\theta: G/G_x \longrightarrow F$ is a homeomorphism.

Proof: $\theta$ is onto since $\theta(h\ G_x) = h(x)$ and G is transitive over F.

$\theta$ is one to one, for if $\theta(h_1\ G_x) = \theta(h_2\ G_x)$ then $h_1(x) = h_2(x)$ and $h_1$ and $h_2$ belong to the same coset, i.e. $h_1\ G_x = h_2\ G_x$.

$\theta$ is continuous: Let U be open in F

$\theta^{-1}(U)$ is open in $G/G_x$ if and only if $\nu^{-1}(\theta^{-1}(U))$ is open in G.

But $\theta\nu = \mu_x$ so that $\nu^{-1}\theta^{-1} = \mu_x^{-1}$

By lemma 4 $\mu_x$ is continuous. Thus $\mu_x^{-1}(U) = \nu^{-1}(\theta^{-1}(U))$ is open in G and hence $\theta^{-1}(U)$ is open in $G/G_x$. †

$\theta$ is open: Say $\mathcal{W}$ is open in $G/G_x$.

Then $\nu^{-1}(\mathcal{W})$ is open in G.

Also $\theta(\nu[\nu^{-1}(\mathcal{W})]) = \theta(\mathcal{W})$

But $\theta\nu = \mu_x$, so $\theta(\mathcal{W}) = \mu_x(\nu^{-1}(\mathcal{W}))$ which is open in F by lemma 5.

q.e.d.

## II. Proof of the Main Theorem

### A. Statement of Main Theorem

The task of proving the principal result of this work must now be faced. But first we state it.

Theorem 3 - Main Theorem:

If F' and F are two spaces belonging to the class $\mathcal{F}$, and if G(F') and G(F) are topologically isomorphic, then F' and F are homeomorphic.

It will pay us to chart our course before we set out across the sea of detail which separates us from our goal. So suppose there be given two spaces F' and F satisfying the hypothesis of our theorem. That means that there exists a map

$$\Phi : G(F') \longrightarrow G(F)$$

which is onto and which is a bicontinuous isomorphism. Theorem 2 of part I provides us with the fact that $G(F')/G_{x'}$ is isomorphic to F' (x' being an arbitrary point of F').

Define the subgroup H of G(F) to be the image of $G_{x'}$ under $\Phi$, i.e., $H = \Phi(G_{x'})$. Then $\Phi$ induces a homeomorphism

$$\Phi_{x'} : G(F')/G_{x'} \longrightarrow G(F)/H,$$

so that G(F)/H is homeomorphic to F'. If we could only show that, under the conditions we have caused to prevail, there exists some x in F such that $H = G_x$, then $G(F)/H = G(F)/G_x$ would also be homeo-

morphic to F and the main theorem would be established. We can and will show that such an x must exist. It is this demonstration which is the main work of this thesis and which we now begin.

B. The Conditions on H

The situation may be distilled to the point where we consider just one space F in $\mathcal{F}$ and its group of homeomorphisms G. The problem stated above reduces itself to the question whether a subgroup H of G, which is intrinsically indistinguishable from the subgroup $G_x$ of a point x in F, is itself the subgroup of a point. The procedure, of course, will be to pick out a certain crucial set of properties of $G_x$ and to prove that any subgroup possessing these properties must be the subgroup of a point.

Before setting down these properties it is necessary to point out that if H is any subgroup of G, then G operates on the space G/H of left cosets as a (not necessarily effective) group of homeomorphisms. Calling the homeomorphism corresponding to $\bar{g}$, $\mathcal{T}_{\bar{g}}$ we have $\mathcal{T}_{\bar{g}}(gH) = \bar{g}(gH) = \bar{g}gH$. We talk of the topology induced in G by H. $\mathcal{T}$ may be considered as a map of G into the space of functions $G/H^{G/H}$ by $\mathcal{T}(g) = \mathcal{T}_g$. Then if we define a subset of G to be open if and only if it is the inverse image under $\mathcal{T}$ of an open set in $G/H^{G/H}$ we have the topology induced by H. (The point-open topology is to be assumed in all function spaces.)

From here on out let H be a subgroup of G having the following properties:

1. H is a proper, closed subgroup of G.
2. G operates on G/H with $\omega$-transitivity.
3. The topology induced in G by H agrees with the original topology in G.

These conditions are necessary for H to be the subgroup of a point, for —

1. $G_x$ is proper and closed, being the complement of the open set $W_{x;F-x}$.
2. The elements of $G/G_x$ are left cosets of $G_x$. The result of part I was that these cosets are in one to one bicontinuous correspondence with the points of F. The coset [y] corresponding to y in F consists (see part I, section C) of all those homeomorphisms which carry x into y. To see how $\tau_g$ operates on [y], let g be a homeomorphism such that $g(x) = y$, then $[y] = gG_x$ and $\tau_{\bar{g}}([y]) = \tau_{\bar{g}}(gG_x) = \bar{g}gG_x = [\bar{g}(y)]$. Finally, if $([x_1], [x_2], \ldots, [x_n])$ and $([y_1], [y_2], \ldots, [y_n])$ are any pair of proper n-tuples of elements of $G/G_x$, the $\omega$-transivity of G over F insures the existence of a g in G with $g(x_i) = y_i$ and hence $\tau_g([x_i]) = [g(x_i)] = [y_i]$, $\forall i$, since $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are also proper n-tuples.
3. Since $G/G_x$ is a homeomorph of F, the topology induced by $G_x$ must coincide with that induced by F.

Thus the subgroup of a point satisfies the conditions. We now turn to the preparations for the proof of sufficiency.

C. The Geometric Definition of H

In this section we "tool up" for the proof to follow. As ever, H will be a subgroup of G satisfying the conditions set down in section IIB above.

By property 1 there exists at least one basic open set W of G which is entirely contained in the complement of H. Now every basic open set $W_{x_1,\ldots,x_m;U_1,\ldots,U_m}$ is based on a finite number $m = N(W_{x_i;U_i})$ of distinct points of F. Let n be the minimum of the set $\{N(W) \mid W \subset CH\}$ of positive integers. Thus n depends on H and may be any natural number including 1.

Now choose, and from here on keep fixed, a particular basic open set $W_{x_i;U_i}$ lying in the complement of H and having the property that $N(W_{x_i;U_i}) = n$. Call $\vec{x} = (x_1,\ldots,x_n)$ and $\vec{U} = (U_1,\ldots,U_n)$. Then $\vec{x}$ is an element of $F^n$, the n-fold Cartesian product of F with itself, and $\vec{U}$ is a basic open set of $F^n$.

Throughout the remainder of the proof of the main theorem $\vec{x}$ and $\vec{U}$ shall retain the meanings just given them. Note that G operates on $F^n$ as a group of homeomorphisms by $g(\vec{y}) = g(y_1,\ldots,y_n) = (g(y_1),\ldots,g(y_n))$. From this point of view we may write $W_{x_i;U_i} = W_{\vec{x};\vec{U}}$. Also, we will designate by $F^n_*$ that subspace of $F^n$ consisting of all the proper n-tuples in $F^n$. Thus $\vec{x}$ is in $F^n_*$.

The conditions on H imply that H is maximal. Theorem 4 establishes this fact in a strong way.

Lemma 6: $\tau : G \longrightarrow G/H^{G/H}$ is one to one, that is, G operates effectively on G/H.

Proof: The fact that F is Hausdorff yields that G is Hausdorff, for suppose $g_1 \neq g_2$, then there exists an x in F such that

$$g_1(x) \neq g_2(x).$$

Choose $U_1$ and $U_2$ open and disjoint such that $g_1(x)$ is in $U_1$ and $g_2(x)$ is in $U_2$.

Then $W_{x,U_1}$ and $W_{x,U_2}$ are two disjoint open sets of G such that $g_1$ is in $W_{x,U_1}$ and $g_2$ is in $W_{x,U_2}$. In other words, G is Hausdorff.

Now suppose G did not operate effectively on G/H, that is, that there exists $h_1$ and $h_2$ such that $h_1 \neq h_2$ but $h_1(fH) = h_2(fH)$ for all f in G.

But the topology induced by H must make G Hausdorff. If $h_1$ and $h_2$ agree on all cosets of H no pair of open sets will separate $\tau(h_1)$ and $\tau(h_2)$ in $G/H^{G/H}$, which means that the induced topology could not separate $h_1$ and $h_2$ in G, a contradiction.

q.e.d.

Lemma 7: G/H is an infinite set.

Proof: G is infinite because F is (lemma 4) and G is transitive on F.

If G/H were finite the infinite set G could not operate effectively on G/H. Lemma 6 rules out the latter possibility.

q.e.d.

Theorem 4: If g is in G but not in H, then the set HgH is dense in G.

Proof: Condition 3 on H will be used strongly. The inverse image of a basic open set of $G/H^{G/H}$ under the map $\tau$ is a basic open set in G and is of the form $\{f \in G \mid f(f_iH) \in \mathcal{W}_i, i=1,\ldots,k\}$ where $(f_1H, f_2H, \ldots, f_kH)$ is a proper k-tuple of elements of G/H, and the $\mathcal{W}_i$ are open sets of G/H. We will write $W_{f_iH;\,\mathcal{W}_i}$ for this open set.

Now let $W_{f_iH;\,\mathcal{W}_i}$ be any non-empty basic open set. The fact that $W_{f_iH;\,\mathcal{W}_i}$ is non-empty means, in the light of the $\omega$-transitivity of G over G/H, that there exists a proper k-tuple $(f_1'H, f_2'H, \ldots, f_k'H)$ such that $f_i'H$ is in $\mathcal{W}_i$ for each i.

Choose a g not in H.

Then choose $h_1$ in H such that $h_1(f_iH) \neq g^{-1}(H)$ for all i. Such an $h_1$ exists because H operates $\omega$-transitively on G/H - H, because G/H contains infinitely many elements, and because, if for some j $f_jH = H$, then $h_1(f_jH) = h_1(H) = H$ and $H \neq g^{-1}(H)$ since g is not in H.

Finally, choose $h_2$ in H such that

$$h_2gh_1(f_1H) = f_1'H, \qquad \text{for each i.}$$

Such an $h_2$ must exist because $gh_1(f_iH) \neq x$, for each i, and because H is $\omega$-transitive on G/H - H.

If, for some j, $f_j'H$ should equal H, choose the original $h_1$ so that $h_1(f_jH) = g^{-1}(H)$.

Thus $h_2gh_1$ is an element of HgH and lies in the arbitrary open set $W_{f_iH;\,\mathcal{W}_i}$. In other words, HgH is dense.

q.e.d.

Definition: The orbit $\mathcal{O}$ of H is a subset of $F^n$ (actually, of $E_\pi^n$) defined by

$$\mathcal{O} = H(\vec{x}) = \{y \in F^n \mid \vec{y} = g(\vec{x}) \quad \text{for some } g \text{ in } H\}$$

Note that $W_{\vec{x};\vec{U}} \cap H = \emptyset$ implies that $\mathcal{O} \cap \vec{U} = \emptyset$.

The next theorem contains the geometric definition of H.

Theorem 5: $$H = \{g \mid g(\mathcal{O}) = \mathcal{O}\}$$

Proof: By the definition of $\mathcal{O}$, H is contained in $\{g \mid g(\mathcal{O}) = \mathcal{O}\}$.

If inclusion in the other direction did not hold, there would exist a g not in H with the property that $g(\mathcal{O}) = \mathcal{O}$. Then, by theorem 4, HgH would be dense in G. In particular, HgH would have to meet $W_{\vec{x};\vec{U}}$. But this is impossible since $HgH(x) = Hg(\mathcal{O}) = H(\mathcal{O}) = \mathcal{O}$ and $\mathcal{O} \cap \vec{U} = \emptyset$. Hence H must contain $\{g \mid g(\mathcal{O}) = \mathcal{O}\}$.

q.e.d.

If g is not in H then $g(\mathcal{O})$ is not identical with $\mathcal{O}$. We shall be interested in the set $\Omega$ of all distinct images of $\mathcal{O}$ under the elements of G. $g(\mathcal{O})$ will be called a co-orbit, and $\Omega$ the set of co-orbits. It is clear that G operates on $\Omega$ as a group of permutations. The importance of $\Omega$ is first indicated by the following

Corollary to Theorem 5:

$$\bar{g}H = \{g \mid g(\mathcal{O}) = \bar{g}(\mathcal{O})\}$$

Proof: Suppose that g is in $\bar{g}H$, then $g = \bar{g}h$ for some h in H.

So $g(\mathcal{O}) = \bar{g}h(\mathcal{O}) = \bar{g}(\mathcal{O})$ and we have that $\bar{g}H$ is included in $\{g|g(\mathcal{O}) = \bar{g}(\mathcal{O})\}$.

Now suppose that $g(\mathcal{O}) = \bar{g}(\mathcal{O})$, then $\bar{g}^{-1}g(\mathcal{O}) = \mathcal{O}$ or, by Theorem 5, $\bar{g}^{-1}g$ is in H.

This means that g is in $\bar{g}H$ or that $\{g|g(\mathcal{O}) = \bar{g}(\mathcal{O})\}$ is included in $\bar{g}H$.

q.e.d.

This corollary forms the basis for much of the work to follow. It makes evident a natural one to one correspondence between G/H and $\Omega$. Let us call this correspondence T, so that $T(\bar{g}H) = \bar{g}(\mathcal{O})$. Since G acts as a set of operators on both the domain and range of T we may state the

Theorem 6: T is an operator correspondence, i.e.,

$T \cdot g = g \cdot T$, for all g in G.

Proof: $T \cdot g(gH) = T(g\bar{g}H) = g\bar{g}(\mathcal{O}) = g(\bar{g}(\mathcal{O})) = g \cdot T(\bar{g}H)$

q.e.d.

Many statements about G/H will now be immediately translatable into statements about $\Omega$. For instance, related to $\Omega$, condition 2 states that G acts with $\omega$-transitivity on $\Omega$.

To achieve an intuitive grasp of the role of the concepts that have been introduced in this section it is useful to interpret them in the case where $H = G_x$. Here the basic neighborhood $W_{\vec{x};\vec{U}}$ that was chosen above for H must be of the form $W_{x;U}$, where x is the fixed point of $G_x$ and U is some open set in F not meeting x. n is equal to 1. We have that $H(x) = \mathcal{O} = x$, the orbit is just x

itself. The co-orbits become the other points of F. Theorem 5 just restates the definition of $G_x$. The corollary of theorem 5 says that a coset of $G_x$ is exactly the entirety of homeomorphisms carrying x into a given point y. T is the map $\theta$. To sum up, then, $\mathcal{O}$ and $\Omega$ play the same role toward H and G/H as x and F do to $G_x$ and $G/G_x$.

At this point we continue our proof by contradicting the main theorem and assuming that $\Omega$ is not a subset of $F^1$ consisting of only one point; that is, we assume that for no x is $H = G_x$. Immediately we have the

Theorem 7: $\mathcal{O}$ cannot be a subset of $F^1$ consisting of a finite number (greater than one, of course) of points.

Proof: Suppose $\mathcal{O} = \{x_1, x_2, \ldots, x_k\}$. Then, since G is $\omega$-transitive on F, every set of k distinct points will be a co-orbit. In particular, we can find an $\mathcal{O}_1 = \{y_1, \ldots, y_n\}$ such that $y_1 = x_1$ and $\mathcal{O} \neq \mathcal{O}_1$. Since F is infinite there will also exist an $\mathcal{O}_2 = \{y_1', y_2', \ldots, y_k'\}$ which does not meet $\mathcal{O}$. By the $\omega$-transitivity of G on $\Omega$ there must exist a g in G such that $g(\mathcal{O}) = \mathcal{O}$ and $g(\mathcal{O}_1) = \mathcal{O}_2$. This last is impossible, though, since it requires that $g(x_1)$ be contained in each of the mutually disjoint sets $\mathcal{O}$ and $\mathcal{O}_2$.

q.e.d.

We are left then with the two possibilities, $\mathcal{O}$ is an infinite subset of $F^n$, n = 1, or a subset of $F^n$ for some n greater than 1.

D. A Property of $\Omega$

We devote this section to a proof of the following crucial theorem:

Theorem 8: $\vec{x}$ is contained in infinitely many co-orbits.

By $\vec{x}$ we mean, of course, that element of $F_*^n$ which generates $\mathcal{O}$, $\mathcal{O} = H(x)$. Since the n-transitivity of G over F implies that G operates transitively on $F_*^n$, an immediate corollay to theorem 8 is that every point of $F_*^n$ is contained in infinitely many co-orbits. Furthermore, since G operates on $\Omega$ $\omega$-transitively, any k co-orbits may be mapped homeomorphically onto a set of k co-orbits (whose existence is guaranteed by theorem 8) which have a non-empty intersection. Thus theorem 8 is equivalent with the statement that $\Omega$ possesses the finite intersection property.

Any k-tuple defines a set whose elements are the coordinates of the k-tuple. Two k-tuples will be called independent if their sets do not intersect. A collection of k-tuples will be called independent if they are pair-wise independent. Finally, a k-tuple will be said to be independent of a given set if the set and the set of the k-tuple are disjoint.

The work in proving theorem 8 comes in demonstrating that $\mathcal{O}$ must contain at least two independent points. In order to show this we must first establish

Lemma 8: If A is a subset of $F^k$ such that $\bar{A}$, the closure of A, contains an open set, then, given any finite set B of elements of F, there exists a point $\vec{y}$ in A such that $\vec{y}$ is independent of B.

Proof: Let $V_1 \times V_2 \times \ldots \times V_k = \vec{V}$ be a basic open set of $F^k$ which is contained in $\bar{A}$; $\vec{V} \subset \bar{A}$.

Since points are closed in F, the sets $V_i' = V_i - B$ are open, and the set $\vec{V}' = V_1' \times V_2' \times \ldots \times V_k'$ is open and $\vec{V}' \subset \vec{V} \subset \bar{A}$.

Since (lemma 4) each non-empty open set of F contains infinitely many points, the set $\vec{V}'$ is not empty.

Now $\vec{V}' \cap A$ is not empty (A is dense in $\vec{V} \supset \vec{V}'$) and any point $\vec{y}$ in that intersection will be independent of the set B. q.e.d.

By $(x_1,\ldots,\hat{x}_p,\ldots,x_n)$ we will mean the n-1-tuple which agrees with $\vec{x}$ except that the $p^{th}$ coordinate of $\vec{x}$ is omitted. The fact that $W_{\vec{x},\vec{U}}$ was chosen with $n = N(W_{\vec{x},\vec{U}})$ minimal implies that $H(x_1,\ldots,\hat{x}_p,\ldots,x_n)$ is dense in the space $F^{n-1}$. This is so because if not $H(x_1,\ldots,\hat{x}_p,\ldots,x_n)$ would miss some open set $\vec{V} = V_1 \times V_2 \times \ldots \times V_{n-1}$ of $F^{n-1}$ so that the basic open set $W' = \{g \mid g(x_i) \in V_i$ for $i < p$ and $g(x_i) \in V_{i-1}$ for $i > p\}$ would be in the complement of H and $N(W') = n-1$ in contradiction to the fact that $N(W_{\vec{x};\vec{U}})$ is minimal. Therefore, as a consequence of lemma 4, we know that, given any finite set of elements of F, there will exist a point of $H(\vec{x})$ such that any previously chosen n-1 coordinates are disjoint from that set. This last is the result which we use in proving

Lemma 9: There exist two independent points in the orbit $\mathcal{O}$.

Proof: First note that the lemma is trivially true in the case that $n = 1$.

Let us assume the opposite - i.e., that every two points of $\mathcal{O}$ are dependent.

We use the notation $\vec{x} = (x_1, x_2, \ldots, x_n)$.

By lemma 8 and following, we may choose a $\vec{y}$ in $\mathcal{O}$ such that $(y_1, \ldots, \hat{y}_n)$ is independent of $\vec{x}$.

Then, since $\vec{y}$ and $\vec{x}$ are dependent, $y_n = x_\alpha$ for some $\alpha \leq n$.

It is necessary to consider two cases: $\alpha = n$ and $\alpha \neq n$.

In case $\alpha = n$, choose $\vec{w}$ in $\mathcal{O}$ such that $(w_1, \ldots, \hat{w}_\beta, \ldots, w_n)$ is independent of both $\vec{x}$ and $\vec{y}$ and such that $\beta \neq n$.

Since $\vec{w}$ must be dependent on both $\vec{x}$ and $\vec{y}$, it must intersect them both in $w_\beta$. But $\vec{x}$ and $\vec{y}$ have only one point in common, namely $x_n = y_n$. Therefore

$$w_\beta = x_n = y_n.$$

If $\alpha \neq n$, then let $\alpha = \beta$ and $x = w$.

Now choose n-1 points $\{\vec{z}^m\} \subset \mathcal{O}$ in the following manner:

$(z_1^1, \ldots, \hat{z}_\beta^1, \ldots, z_n^1)$ independent of $\vec{w}$ and $\vec{y}$.

$(z_1^m, \ldots, \hat{z}_\beta^m, \ldots, z_n^m)$ independent of $\vec{w}$, $\vec{y}$, $\vec{z}^1, \ldots, \vec{z}^k$

for $k < m \leq n-1$.

This gives us a collection of points of $\mathcal{O}$, $\vec{w}$, $\vec{y}$, $\vec{z}^1, \ldots, \vec{z}^{n-1}$, which are independent except in the $n^{th}$ coordinate of $\vec{y}$ and the $\beta^{th}$ coordinates of $\vec{w}$, $\vec{z}^1, \ldots, \vec{z}^{n-1}$. By our assumption these points must be pair-wise dependent, and the only way for this to happen is for

(a) $$w_\beta = y_n = z_\beta^1 = z_\beta^2 = \ldots = z_\beta^{n-1}.$$

Now since $\vec{w}$, $\vec{y} \in \mathcal{O} = H(x)$, there exists a $g \in H$ such that $g(\vec{w}) = \vec{y}$,

that is $g(w_i) = y_i$ all $i \leq n$.

In particular,

(b) $g(w_\beta) = y_\beta$

and $g(w_n) = y_n = w_\beta$.

Applying this same transformation to $\vec{z}^1$ we obtain a new point $g(\vec{z}^1)$ in $\mathcal{O}$. We will prove that $g(\vec{z}^1)$ cannot be dependent on <u>all</u> of the points $\vec{w}$, $\vec{z}^1$, $\vec{z}^2$,...,$\vec{z}^{n-1}$, thus arriving at the desired contradiction.

First we show that the $\beta^{th}$ coordinate of $g(\vec{z}^1)$ does not appear anywhere in $\vec{w}$, $\vec{z}^1$,...,$\vec{z}^n$.

By (a) and (b) $g(\vec{z}^1)_\beta = g(z^1_\beta) = g(w_\beta) = y_\beta$

But $y_\beta \in \{y_1,...,\hat{y}_n\}$ which as stated above is disjoint from $(w_1,...,\hat{w}_\beta,...,w_n)$, $(z^1_1,...,\hat{z}^1_\beta,...,z^1_n)$,...,$(z^{n-1}_1,...,\hat{z}^{n-1}_\beta,...,z^{n-1}_n)$, so that $g(\vec{z}^1)_\beta$ cannot agree with any of the non-$\beta$ coordinates of the points $\vec{w},\vec{z}^1,...,\vec{z}^{n-1}$.

Now the $\beta^{th}$ coordinates of all these points are all equal to $y_n$. But $g(\vec{z}^1)_\beta = y_\beta \neq y_n$ since $\beta \neq n$ and $\vec{y}$ is a proper n-tuple.

So $g(\vec{z}^1)_\beta$ is disjoint from $\vec{w},\vec{z}^1,...,\vec{z}^{n-1}$.

Next we show that the $\beta^{th}$ coordinates of $\vec{w},\vec{z}^1,...,\vec{z}^n$ are disjoint from $g(\vec{z}^1)$. All these coordinates are equal to $y_n$. We know that $g(w_n) = y_n$, and we know that $w_n$ is disjoint from $\vec{z}^1$. Since g is a one to one transformation of F it is impossible that any point of

$\vec{z}^1$ have the same image under g as $w_n$. Thus $g(\vec{z}^1)$ is disjoint from $g(w_n) = y_n = w_\beta = z^1_\beta = z^2_\beta = \ldots = z^{n-1}_\beta$.

We have thus succeeded in showing that the dependence of $g(\vec{z}^1)$ on the set $\vec{w}, \vec{z}^1, \ldots, \vec{z}^{n-1}$ can be achieved only if $(g(z)_1, \ldots, \widehat{g(z^1)}_\beta, \ldots, g(z^1)_n)$ is dependent on each of the n-1-tuples $(w_1, \ldots, \hat{w}_\beta, \ldots, w_n)$, $(z^1_1, \ldots, \hat{z}^1_\beta, \ldots, z_n), \ldots, (z^{n-1}_1, \ldots, \hat{z}^{n-1}_\beta, \ldots, z^n)$. But this is impossible since it requires that a set of n-1 elements have representatives in each of n disjoint sets.

q.e.d.

We may now proceed to the

Proof of Theorem 8: Lemma 9 assures us the existence of two independent points $\vec{w}_1$ and $\vec{w}_2$ in $\mathcal{O}$. Since $\vec{x}$ is also in $\mathcal{O}$, there will be an element g of H such that $g(\vec{w}_1) = \vec{x}$. Let $g(\vec{w}_2) = \vec{y} \in \mathcal{O}$. $\vec{x}$ and $\vec{y}$ are independent.

The theorem is now proved with an induction.

$\vec{x}$ is clearly in one co-orbit, namely .

Suppose x were in k co-orbits $\mathcal{O} = \mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_k$.

Two cases arise:

I. There exists a $\vec{z} \in C(\bigcup_{i=1}^{k} \mathcal{O}_i)$ such that $\vec{z}$ is independent of $\vec{x}$ and $\vec{y}$.

In this case we can find an element g in G such that

$$g(\vec{x}) = \vec{z} \qquad \text{and} \qquad g(\vec{y}) = \vec{z}.$$

Consider, then, the co-orbit $g(\mathcal{O}_1)$.

Since $g(\vec{z}) = \vec{z}$, $\vec{z}$ is in $g(\mathcal{O}_1)$.

Since $\vec{y}$ is in $\mathcal{O}_1$, $g(\vec{y}) = \vec{z}$ is in $g(\mathcal{O}_1)$.

Since z is not in $\mathcal{O}_i$, $\forall i \leq k$, $g(\mathcal{O}_1) \neq \mathcal{O}_i$, $\forall i \leq k$.

Then we have shown that $\vec{x}$ belongs to the k+1 co-orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_k, g(\mathcal{O}_1)$ proving the theorem in case I.

II. Suppose now there exists no z in $C(\bigcup_{i=1}^{k} \mathcal{O}_i)$ which is independent of both $\vec{x}$ and $\vec{y}$.

Lemma 8 assures us that $F^n_x$ contains an infinite number of points independent from each other and any initially given finite set of points. That means that some $\mathcal{O}_p$ must contain an infinite number of independent points, and since each pair of orbits is equivalent under a homeomorphism of $F^n$ we may conclude that $\mathcal{O}_1$ contains an infinite number of points mutually independent and independent of $\vec{x}$.

Thus we may certainly choose k points $y_1, \ldots, y_k$ in $\mathcal{O}_1$, which are mutually independent and independent of x.

The complement of $\mathcal{O}$, and hence of every co-orbit, contains an open set of $F^n$. This fact together with Lemma 8 makes possible the choice of k points $\vec{z}_i$ such that each $\vec{z}_i$ is in $C\,\mathcal{O}_i$ and $\vec{z}_1$ is independent of $\vec{x}$, and $\vec{z}_i$ is independent of each $\vec{z}_p$ when $p \leq i$.

The independence properties of the two sets $\{\vec{x}, \vec{y}_i\}$ and $\{\vec{x}, \vec{z}_i\}$ and the $\omega$-transitivity of G yield the existence of an element $g \in G$ with the property that

$$g(\vec{x}) = \vec{x} \qquad \text{and} \qquad g(\vec{y}_i) = \vec{z}_i, \quad \forall i.$$

Consider $g(\mathcal{O}_1)$.

Clearly $\vec{x}$ is in $g(\mathcal{O}_1)$.

Also $g(\mathcal{O}_1) \neq \mathcal{O}_i$, $\forall i$, because $g(\mathcal{O}_1)$ contains $g(\vec{y}_i) = \vec{z}_i$ and $\vec{z}_i \notin \mathcal{O}_i$.

Hence $\vec{x}$ belongs to k+1 orbits.

q.e.d.

E. Topology in G/H.

G/H is given the usual factor space topology (see section IB). Let $\mathcal{W}$ denote the generic open set in G/H. The goal of this section is

Theorem 9: G/H is non-discrete.

The fact that H is a closed subgroup of G guarantees that points are closed sets in G/H. This, the homogeneity of G/H, and Theorem 9 yield the

Corollary to Theorem 9: Any non-empty open set $\mathcal{W}$ of G/H contains an infinite number of elements.

It is this corollary which we will need for the final stage of the proof of the Main Theorem.

Lemma 10: Every open set of G which contains the identity generates an $\omega$-transitive, and hence dense, subgroup of G.

Proof: An open set of G containing the identity must of course contain a basic neighborhood of the identity, and this will be of the form $W_{y_i;V_i}$, $1 \leq i \leq m$.

Since F is Hausdorff we may find open sets $V_i \subset V_i'$ such that $y_i$ is in $V_i$, $\forall$ i, and $V_i \cap V_j = \emptyset$, $i \neq j$.

Then $W_{y_i;V_i} \subset W_{x_i;V_i'}$, and it is with $W = W_{y_i;V_i}$ that we will work.

Let $(z_1,\ldots,z_k)$ be any proper k-tuple of points of F ordered so that the first p points are points of the m-tuple $(y_1,\ldots y_m)$, where $0 \leq p \leq k$ or m if $m < k$.

Let $(w_1,\ldots,w_k)$ be a second proper k-tuple.

We will find an element of $W \cdot W$ which carries $\vec{z}$ into $\vec{w}$.

First, there will exist a g in W with the property that $g(z_i) \neq z_j$ for all i, j with $0 \leq i \leq p$ and $0 \leq j \leq k$, and $g(z_i) = z_i$ for all i such that $p < i$.

The existence of this g is assured by the $\omega$-transitivity of G and the fact that each $V_i$ contains infinitely many points.

Note that $g(\vec{z})$ is independent of $\vec{y}$.

Second, there will, for the same reasons, exist a g' in W with the property that $g'g(z_i) = w_i \qquad 0 \leq i \leq k$.

Thus $W \cdot W$ is $\omega$-transitive on F, implying of course that the group generated by the original open set containing W is also $\omega$-transitive.

q.e.d.

Corollary to Lemma 10: H is nowhere dense in G.

Proof: If not, H, being closed, would contain an open set and hence be open. By our lemma, H would be dense in G and since H is closed it would have to be equal to all of G - which is a contradiction to property 1 of H.

q.e.d.

Proof of Theorem 9: If G/H were discrete the element H in G/H would be an open set. This means that the inverse image $\nu^{-1}(H) = H$ in G must be open in G. But that is impossible by the corollary to lemma 10.

q.e.d.

F. The Two Topologies in G.

In this section we use the third condition on H in a particularly strong way, forcing H to emerge finally as the subgroup of a point.

Theorem 10: H is the subgroup of some point in F.

The basic open sets of G in the topology induced by G/H are of the form

$$W = W_{g_1H, g_2H, \ldots, g_kH; \mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_k} .$$

where $(g_1H, \ldots, g_kH)$ is a proper k-tuple of points of G/H and the $\mathcal{W}_i$ are open sets of G/H.

Let $W = W_{x_i; U_i} = W_{\vec{x}; \vec{U}}$, where $\vec{x} \in F^n$ and $U_i$ open in F are as defined in section IIC.

That is, $H(\vec{x}) = \mathcal{O}$, $\mathcal{O} \cap \vec{U} = \emptyset$, and $W(\vec{x}) = \vec{U}$.

Proof of Theorem 10: We proceed under the assumptions that H is not the subgroup of a point in $F^1$ and that it satisfies conditions 1 through 3.

Condition 3 states that the topologies induced in G by F and by H are equivalent. We shall show that under our assumptions on H no basic neighborhood in the topology induced by H is completely contained within the particular neighborhood $W_{\vec{x}; \vec{U}}$.

Let $W^* = W_{g_iH; \mathcal{W}_i}$ be any basic neighborhood in the new topology on G.

Since (Theorem 8) $\vec{x}$ is contained in infinitely many co-orbits, there exists a co-orbit $g(\mathcal{O})$ which contains $\vec{x}$ and which is distinct from all the co-orbits $T(g_iH) = g_i(\mathcal{O})$, i.e.

$$g^*(\mathcal{O}) \neq g_i(\mathcal{O}), \quad \forall i$$

We may now select a homeomorphism g lying in $W^*$ and having the additional property that

$$g[g^*(\mathcal{O})] = \mathcal{O}, \text{ i.e. } \quad g[g^*H] = H.$$

That such a g can be found is insured by the $\omega$-transitivity of G over G/H (condition 2) and the fact that each $\mathcal{W}_i$ contains infinitely many elements (Corollary to Theorem 9). The only trouble we might have run into here is that every proper k-tuple of cosets $(f_1H,\ldots,f_kH)$ such that $f_iH$ is an element of $\mathcal{W}_i$ for each i might contain H as one of its coordinates. This possibility is ruled out by the fact that each $\mathcal{W}_i$ is an infinite set. Looking at g we see that although it <u>is</u> contained in $W^*$ it is <u>not</u> contained in W because $\vec{x} \in g^*(\mathcal{O})$, and $g[g^*(\mathcal{O})] = \mathcal{O}$, and $\mathcal{O} \cap \vec{U} = \emptyset$. That is, $g(\vec{x})$ is not an element of $\vec{U}$ so that g is not an element of $W_{\vec{x};\vec{U}}$.

Thus, if H is to satisfy conditions 1, 2, and 3, the orbit $\mathcal{O}$ cannot be of the above type; that is to say, it cannot be a subset of $F^n$ for n 1, nor can it be an infinite subset of $F^1$, or a finite set of more than one point of $F^1$. The orbit $\mathcal{O}$ must be a set in $F^1 = F$ consisting of a single point. In other words, there is a point $\vec{x}$ in F such that $\mathcal{O} = x$, and, by Theorem 5,

$$H = \{g | g(\mathcal{O}) = \mathcal{O}\} = \{g | g(x) = x\} = G_x$$

q.e.d.

G. Proof of the Main Theorem

Referring back to section IIA, we see that $\Phi(G_{x'}) = H \subset G(F)$ lies in G(F) exactly as $G_{x'}$ lies in G(F'). Since $G_{x'}$ satisfies the conditions 1 through 3 (section IIB), so also must $\Phi(G_{x'}) = H$, for conditions 1 to 3 are intrinsic in G. Therefore, Theorem 10 applies the $\Phi(G_{x'})$ and tells that $\Phi(G_{x'}) = G_x \subset G(F)$ where x is some point of F.

Let $\theta'$ be the homeomorphism

$$\theta': \quad G(F')/G_{x'} \longrightarrow F', \qquad \text{(see section IC)}$$

Then $\theta' \Phi_{x'}^{-1} \theta^{-1}: \quad F \rightarrow F'$ is the composition of three homeomorphisms and hence is itself a homeomorphism. The Main Theorem is proved.

## III. Collateral Results

### A. Determining Subgroups of G(F):

Though it hasn't handicapped us in any way, we have through all the preceding work been carrying a bit of extra baggage. G(F) was defined to be the group all homeomorphisms of F onto F. Yet the only property of G(F) which has been used is its $\omega$-transitivity over F. Thus we could have begun with the definition of a class of determining subgroups $\{G^d(F)\}$ of G(F) having the property that $G^d(F)$ operates on F with $\omega$-transitivity. Every statement in parts I and II of this treatise remains true if the group $G^d(F)$ is substituted for G(F) throughout. For instance, the main theorem would read: If F' and F are members of $\mathcal{F}$ with the property that a determining subgroup of G(F') is topologically isomorphic to a determining subgroup of G(F), then F' and F are homeomorphs.

Whenever F is a manifold this fact has immediate implications. Assuming F to be orientable, the subgroup $G^+(F)$ consisting of all the orientation preserving homeomorphisms of F onto itself is $\omega$-transitive on F. We may conclude that $G^+(F)$ characterizes F among the spaces in $\mathcal{F}$. Assuming F to be sufficiently differentiable, it may be shown that $G^r(F)$, the group of all r-fold differentiable homeomorphisms of F onto F, is $\omega$-transitive on F and hence characterizes F among the spaces of $\mathcal{F}$.

A problem, so far unsolved, is whether, if F is an analytic manifold, the group of analytic homeomorphisms characterizes F in our sense.

Related to the problem of characterizing spaces by means of their groups of homeomorphisms is that of characterizing those groups which occur as groups of homeomorphisms. Every group-space in which left translations are homeomorphisms acts as a group of homeomorphisms on any of its coset spaces. In the general case, though, we cannot expect the topology in G to be the topology that G would receive as a subset of the space of functions from the coset space into itself. The theorem which we now state is a weak one in that it dodges the issue by assuming that the original topology in G is recovered when G is given the function space topology over the coset space in question.

Theorem 11: If a group space G contains a subgroup H satisfying the conditions 1, 2, and 3, section IIB, and itself has the properties

(a) left-translations in G are continuous

(b) every neighborhood of the identity in G generates a dense subgroup of G

(c) G contains at least two non-empty and disjoint open sets,

then G is a determining group of homeomorphisms of the space G/H and the space G/H is a member of the family $\mathcal{F}$.

Proof: Property (a) of G guarantees that the elements of g, when acting by left multiplication on G/H, will be homeomorphisms.

Property (b) of G implies that the space G/H cannot be discrete. This implication follows the proof of Theorem 9 (section IIE) above.

Property (c) of G is needed to insure the existence in G/H of two disjoint non-empty open sets. This is accomplished as follows:

According to condition 3 on H, the topology in G is generated by basis elements of the form $W_{g_iH;\mathcal{W}_i}$, where $g_1H, g_2H, \ldots, g_nH$ is a proper n-tuple of cosets and the $\mathcal{W}_i$ are open sets in G/H. If G contains two disjoint non-empty open sets it will contain two disjoint non-empty basic open sets. Let these be $W_{g_iH;\mathcal{W}_i} = W_1$ and $W_{f_jH;\mathcal{V}_j} = W_2$. The non-emptiness of these sets implies the non-emptiness of the sets $\mathcal{W}_i$ and $\mathcal{V}_j$. Properties (b) of G and (1) of H tell us that H, and hence each coset of H, is nowhere dense in G. From this we may conclude that no open set $\mathcal{W}$ of G/H can contain only a finite number of cosets of H, because the union of a finite number of nowhere dense sets cannot be open. Hence the $\mathcal{W}_i$ and $\mathcal{V}_j$ are infinite sets and we are now in a position to state that in order for $W_1$ and $W_2$ to be disjoint there must be a pair of integers k and m such that $g_kH = f_mH$ and furthermore $\mathcal{W}_k \cap \mathcal{V}_m = \emptyset$. Thus property (c) reflects itself in G/H and completes the proof that G/H is a space of $\mathcal{F}$. Since G is assumed to be $\omega$-transitive over G/H and has the point-open topology over G/H, G is a determining subgroup of the group of all homeomorphisms of G/H.

q.e.d.

B. Induced Homeomorphisms

In this section we shall prove that an isomorphism between G(F') and G(F) induces a "natural" homeomorphism between F' and F, and that the automorphisms of G(F) are all inner.

We shall need the auxiliary concept of the set $\mathcal{G}(F)$ of all the subgroups of points of F, $\mathcal{G}(F) = \mathcal{G} = \{G_y | y \in F\}$.

As a result of the homogeneity of F, $\mathcal{G}$ consists of the set of all subgroups of G(F) which are conjugate to $G_x$, where x is a given point of F.

Let us consider a fixed topological isomorphism $\Phi$ carrying G(F') onto G(F). Our main theorem tells us that $\Phi$ induces a map from $\mathcal{G}(F')$ to $\mathcal{G}(F)$. It will be convenient to denote this induced map by $\Phi$ also, then $\Phi : \mathcal{G}(F') \longrightarrow \mathcal{G}(F)$ where the image of $G_{y'}$ is simply $\Phi(G_{y'})$. $\Phi$ is clearly one to one from $\mathcal{G}(F')$ to $\mathcal{G}(F)$.

In this way $\Phi$ induces a one to one map $\phi$ of F' onto F. $\phi$ is defined by the equation

(i) $$(G_{y'}) = G_{\phi(y)}.$$

Let $\Phi_{x'}$ be the map from $G(F')/G_{x'}$ to $G(F)/G_{\phi(x)}$ induced by $\Phi$. That is,

(ii) $$\Phi_{x'}(h'G_{x'}) = \Phi(h')\,\Phi(G_{x'}) = \Phi(h')\,G_{\phi(x')}.$$

Finally $\Phi_{x'}$ induces $\phi_{x'}: F' \longrightarrow F$ by

(iii) $$\phi_{x'}(y') = \Phi_{x'}\left[\theta'^{-1}_{x'}(y')\right]\left(\phi(x')\right)$$

where $\theta'_{x'}: G(F')/G_{x'} \longrightarrow F'$ by $\theta'_{x'}(h'G_{x'}) = h'(x')$.

Note here that $\Phi_{x'}\left[\theta'^{-1}_{x'}(y')\right]$ is an element of $G(F)/G_{\phi(x')}$, that is a coset of $G_{\phi(x')}$ and as such maps $\phi(x')$

into a unique point of F. Thus if $h' \in \theta'^{-1}_{x'}(y')$ then $h'(x') = y'$,

$$\Phi(h') \in \Phi_{x'}\left(\theta'^{-1}_{x'}(y')\right), \quad \text{and}$$

(iv) $$\phi_{x'}(y') = \Phi(h')(\phi(x')).$$

Lemma 11: $$\phi = \phi_{x'}$$

Proof: We shall accomplish the proof by proving that $\phi_{x'}$ satisfies equation (i), i.e.

$$G_{y'} = {}^{G}\phi_{x'}(y') .$$

To do this let $h'$ be an element of $G(F')$ such that $h'(x') = y'$, that is $h' \in \theta'^{-1}_{x'}(y')$.

Then using (ii),

(v) $$\Phi_{x'}\left[\theta'^{-1}_{x'}(y')\right] = \Phi_{x'}\left[h'G_{x'}\right] = \Phi(h')G_{\phi(x')} .$$

With this in mind, we write,

$$\Phi(G_{y'}) = \Phi(h'G_{x'}h'^{-1}) = \Phi(h')G_{\phi(x')}\left(\Phi(h')\right)^{-1}$$

$$= {}^{G}\Phi(h')(\phi(x')) = {}^{G}\phi_{x'}(y') .$$

The final equality follows from (iv).

q.e.d.

Theorem 12: The homeomorphisms $\phi_{x'}$ are independent of $x'$.

Proof: This is a corollary to the preceding lemma, since the point x' was quite arbitrary. Thus, for each x' in F', $\phi_{x'} = \phi$. The map induced by a given isomorphism $\Phi$ is natural and we may write simply $\phi$ for this map.

q.e.d.

Suppose now that $\Phi$ is a topological automorphism of G(F), i.e.

$$\Phi : G \longrightarrow G.$$

Theorem 12 tells us that the induced homeomorphism of F onto itself is unique and given by the formula

$$\Phi(G_y) = G_{\phi(y)} .$$

Let the inner automorphism of G generated by the element g be represented by the symbol $\Phi_g$. Thus $\Phi_g(h) = ghg^{-1}$.

Theorem 13: If $\Phi$ is an automorphism of G, it is an inner-automorphism.

Proof: We prove that $\Phi = \Phi_\phi$

Let h be any element of G and x any element of F, and let $h(x) = y$.

By equation (iv) above we have

$$\phi_x(y) = \Phi(h)\left(\phi(x)\right) .$$

But by theorem 12, $\phi_x(y) = \phi(y)$,

so $\phi(y) = \Phi(h)\left(\phi(x)\right)$.

Using $y = h(x)$ we get

(vi) $$\phi(h(x)) = \Phi(h)\left(\phi(x)\right) .$$

Since x was arbitrary (vi) holds true when x is replaced by the element $\phi^{-1}(x)$. This yields

$$\phi\, h\left(\phi^{-1}(x)\right) = \Phi(h)\left[\phi\left(\phi^{-1}(x)\right)\right] = \Phi(h)(x)$$

or

$$\Phi_{\phi}(h)(x) = \Phi(h)(x).$$

The last equation is true no matter what h and x are used, hence

$$\Phi_{\phi} = \Phi .$$

q.e.d.

As a last remark, the equation

$$\Phi_{g}(G_x) = G_{g(x)}$$

tells us that the inner-automorphism generated by a homeomorphism g re-induces g. In other words, the correspondences $\Phi \longrightarrow \phi$ and $g \longrightarrow \Phi_g$ are inverses and define a one to one correspondence between G and the group of automorphisms of G. This correspondence may easily be shown to be an isomorphism.

C. Some Non-homogeneous Spaces

Arens (Amer. J. Math., 68, 593-610 (1946)) states a theorem whose analog using the point-open topology we state here.

Theorem 14: The group of homeomorphisms $G(X)$ of an arbitrary topological space $X$ is topologically isomorphic to the sub-group $G_B(X \cup B)$ of the group of homeomorphisms $G(X \cup B)$ of the one point compactification $X \cup B$ of $X$ which leaves the ideal point $B$ fixed. (The point-open topology in the group of homeomorphisms is understood.)

Proof: If $X$ is compact, $B$ is isolated and the theorem follows easily.

That $G(X)$ and $G_B(X \cup B)$ are in natural one to one correspondence is clear.

This correspondence is an algebraic isomorphism.

The basic open set $W_{x_i;U_i}$ of $G(X)$ is also a basic open set of $G_B(X \cup B)$ and vice versa. Note that the special sub-basic neighborhood $W_{B,U}$ of $G_B(X \cup B)$ is either empty or all of $G_B(X \cup B)$ according as $B$ is not or is a point of $U$.

q.e.d.

The theorem establishes the validity of one of the examples cited in the introduction, namely, that the ordinary two ended cylinder (product of circle and infinite line) and the 2-sphere with two points identified have the same groups of homeomorphisms. The cylinder may be considered the space $X$. Then, the one point compactification $X \cup B$ is a 2-sphere with two points identified. In this case, then, every homeomorphism of $X$ $B$ onto itself must leave $B$ fixed so that $G_B(X \cup B) = G(X \cup B)$. Since Theorem 14 states that $G_B(X \cup B) \approx G(X)$ the validity of the example is established.

If we let $B$ be a finite subset of compact, locally connected, second countable space $F \in \mathcal{F}$ consisting of more than one

point, and by $G_B(F)$ denote the subgroup $\{g \in G(F) | g(B) = B\}$, then, while it is true that there is a natural algebraic isomorphism between the groups $G_B(F)$ and $G(F - B)$, it is not true that this isomorphism is topological. The image of the open set $W_{x_1,U}$ of $G_B(F)$ in $G(F - B)$, where $x_1 \in B$ and U is a neighborhood of $x_2 \in B$ such that $U \cap B - x_2 = \emptyset$, contains no basic open set of the group $G(F - B)$. $W_{x_1,U}$ consists of all those elements of $G_B(F)$ which map $x_1$ into $x_2$. In $G(F - B)$, $W_{x_1,U}$ corresponds to all those homeomorphisms which map the end[4] associated with $x_1$ into the end associated with $x_2$. Let $W_{y_i;U_i}$ be any basic open set of $G(F - B)$. Then of course $W_{y_i;U_i}$ is an open set of $G(F)$ and since $x_1$ is not one of the $y_i$ we may find a g in $W_{y_i;U_i}$ such that $g(x_1) = x_1$. Looked at in $G(F - B)$, g is an element of $W_{y_i;U_i}$ which maps the end associated with $x_1$ into itself, not into the end associated with $x_2$. Hence g is not in the image of $W_{x_i;U}$ and we have established the fact that the natural application of $G_B(F)$ on $G(F - B)$ is not a homeomorphism.

The closed line interval I is not a member of the family $\mathcal{F}$. Letting E be the set consisting of the two end points of I, we may ask whether $G_E(I) = G(I)$ is topologically isomorphic to $G(I - E)$, in contrast to the situation when we are dealing with spaces of $\mathcal{F}$. Again these groups are algebraically isomorphic. The isomorphism will be topological if the set of homeomorphisms leaving E point-wise fixed, which is an open set in $G(I)$, is open in $G(I - E)$. This question is settled by the

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4. For the theory of ends see Freudenthal, Hans, Ann. of Math. 43, 261-279 (1942) and Hopf, Heinz, Commentarii Math. Helvetici 16, 81-100 (1943).

Theorem 15: The group of homeomorphisms in the point-open topology of a line interval (open, closed, or half closed) is disconnected. In particular the set $A^+$ consisting of the orientation preserving maps and that $A^-$ of the orientation reversing ones are open.

Proof: Let the interval be denoted by X.

We proved (lemma 3) that $\mu_x: G(X) \to X$ by $\mu_x(g) = g(x)$ is a continuous map.

From this it follows that the mapping

$$\mu_{x_1,x_2}: G(X) \longrightarrow X \times X$$

by

$$\mu_{x_1,x_2}(g) = (g(x_1), g(x_2)) \quad \text{is continuous.}$$

It is an elementary fact that the diagonal of $X \times X$ separates $X \times X$.

Choose $x_1$ and $x_2$ both in the interior of X and such that $x_1 \neq x_2$.

Then for every element g of G(X) $g(x_1) \neq g(x_2)$, i.e., $\mu_{x_1,x_2}(G(X))$ does not meet the diagonal of $X \times X$.

The diagonal divides $X \times X$ into two open sets $L^+$ and $L^-$ where

$$L^+ = \{(x,y) | x < y\} \quad \text{and}$$
$$L^- = \{(x,y) | x > y\} .$$

Clearly the image of G(X) under $\mu_{x_1,x_2}$ meets both $L^+$ and $L^-$.

Hence $\mu_{x_1,x_2}(G(X))$ is disconnected and so also is G(X).

$\mu^{-1}_{x_1,x_2}$ will provide a disjunction of G(X) into two open sets.

Suppose $x_1 < x_2$. Then $\mu^{-1}_{x_1,x_2}(L^+) = A^+$ and $\mu^{-1}_{x_1,x_2}(L^-) = A^-$, proving our second contention. q.e.d.

Returning to the question of the continuity of the isomorphism between G(I) and G(I - E), we see that, since those homeomorphisms of I which leave E point-wise fixed are precisely the orientation preserving homeomorphisms they form an open set when considered to lie in E. Thus the image of <u>every</u> open set of G(I) in G(I - E) is open, and vice versa. The natural map between G(I) and G(I - E) is a topological isomorphism, establishing the example of the open interval and the closed one which was cited in the introduction.

Having discussed the homeomorphism group of the line, we may turn for a moment to the other one dimensional manifold, the circle. Let K stand for the circle space.

<u>Theorem 16</u>: G(K) is disconnected. The orientation preserving homeomorphism and the reversing ones form disjoint open sets.

<u>Proof</u>: Let $K^3$ be the 3-torus (3-fold Cartesian product of the circle with itself).

We define the diagonals of $K^3$

$$\Delta_1 = \{(x_1,x_2,x_3) \mid x_1 = x_2\}$$
$$\Delta_2 = \{(x_1,x_2,x_3) \mid x_2 = x_3\}$$
$$\Delta_3 = \{(x_1,x_2,x_3) \mid x_1 = x_3\}.$$

We call a point $\vec{x} = (x_1,x_2,x_3)$ even if the cyclic order of the points $x_1,x_2,x_3$ in K is the cyclic order $x_1,x_2,x_3$. We call a point $\vec{x}$ odd if the cyclic order of $x_1,x_2,x_3$ is not the order $x_1,x_2,x_3$ (i.e., if it is the order $x_2,x_1,x_3$). The collection of all even points of $K^3$ we call $L^+$, of all odd points $L^-$.

Then $K^3 = L^+ \cup L^- \cup \Delta_1 \cup \Delta_2 \cup \Delta_3$

$$L^+ \cap L^- = \emptyset, \quad L^+ \cap (\Delta_1 \cup \Delta_2 \cup \Delta_3) = \emptyset, \quad \text{and}$$
$$L^- \cap (\Delta_1 \cup \Delta_2 \cup \Delta_3) = \emptyset.$$

It is easy to verify that $L^+$ and $L^-$ are open sets.

Then $\Delta_1 \cup \Delta_2 \cup \Delta_3$ separates $K^3$.

Now $\mu_{x_1,x_2,x_3} : G(K) \longrightarrow K^3$ by $\mu_{x_1,x_2,x_3}(g) = (g(x_1),g(x_2),g(x_3))$ is a continuous map.

If $(x_1,x_2,x_3)$ is a proper 3-tuple $\mu_{x_1,x_2,x_3}(G(K))$ will have values in both $L^+$ and $L^-$, but none in $\Delta_1 \cup \Delta_2 \cup \Delta_3$.

Then $\mu^{-1}_{x_1,x_2,x_3}(L^+)$ and $\mu^{-1}_{x_1,x_2,x_3}(L^-)$ provides a disjunction of G(K).

If $x_1,x_2,x_3$ are in the order $x_1,x_2,x_3$ in K, then $\mu^{-1}_{x_1,x_2,x_3}(L^+)$ is the set of orientation preserving maps and $\mu^{-1}_{x_1,x_2,x_3}(L^-)$ the set of orientation reversing ones. q.e.d.

In the next section we shall indicate that the orientation preserving maps in both G(K) and the group of the line from the co ponent of the identity.

It should be noted that G(K) and G(X), where X is a line interval, are not algebraically isomorphic. This is established by the fact that G(K) contains finite cyclic subgroups of any order while G(X) does not.

D. Topological Structure of G(F)

Here we fulfill our promise to show that G(F) is connected whenever F is a manifold. On the way we prove

Theorem 15: If $F \in \mathcal{F}$ then if G(F) is not connected it is totally disconnected.

Proof: Suppose G(F) is not connected.

Then there exists a separation of G(F) into two disjoint open sets $W_1$ and $W_2$.

Choose a basic neighborhood in each of these,

$$W_{x_i;U_i} \subset W_1 \qquad \text{and} \qquad W_{y_j;V_j} \subset W_2 .$$

Let $g_1$ and $g_2$ be any two distinct elements of G(F).

Then there will be a point x on which they disagree, and our assumptions on F insure that there will be a neighborhood U of x such that $g_1(U)$ and $g_2(U)$ are disjoint.

$\omega$ -transitivity (and lemma 4) allows us to find an element $h_1$ of G(F) with the property that

$$h_1(x_i) \in U, \quad \forall\, i, \qquad \text{and} \qquad h_1(y_j) \in U, \quad \forall\, j\,.$$

Then $g_1h_1(\vec{x})$ is independent of $g_2h_1(\vec{y})$, where x and y are the n- and m-tuples of the basic open sets, $\vec{x} = (x_1,\ldots,x_i,\ldots,x_n)$ and $\vec{y} = (y_1,\ldots,y_j,\ldots,y_m)$.

Hence there will exist a homeomorphism $h_2$ such that $h_2g_1h_1(\vec{x}) \in \vec{U}$ and $h_2g_1h_1(\vec{y}) \in \vec{V}$, where $\vec{U} = U_1 \times \ldots \times U_i \times \ldots \times U_n$ and $\vec{V} = V_1 \times \ldots \times V_j \times \ldots \times V_m$.

Define the map $\psi : G(F) \to G(F)$ by $\psi(g) = h_2gh_1$.

is a homeomorphism onto.

Then $\psi(g_1) \in W_1$ and $\psi(g_2) \in W_2$.

Since $\psi$ is a homeomorphism, $\psi^{-1}(W_1)$ and $\psi^{-1}(W_2)$ will be a disjunction of G(F) into two open sets.

But $g_1 \in \psi^{-1}(W_1)$ and $g_2 \in \psi^{-1}(W_2)$ so that $g_1$ and $g_2$ are not in a connected subset of G(F).

Since $g_1$ and $g_2$ were arbitrary, we have proved that no two distinct points lie in the same connected subset of G(F), i.e., that G(F) is totally disconnected.

q.e.d.

Now we state the main result of this section.

Theorem 16: G(F) is a fibre bundle in the sense of Ehresmann-Feldbau, if F is a manifold.

Proof: For the definition of this type of bundle we quote from N. E. Steenrod, The Topology of Fibre Bundles, (Princeton University Press, 1951), page 18:

"A bundle, in the sense of Ehresmann-Feldbau (briefly, an E-F bundle), consists of B, p, X, and Y as usual. In addition there is given a group G (not topologized) of homeomorphisms of Y, and for each x a family of homeomorphisms $G_x$ of Y into $Y_x$ such that (i) $\xi, \xi' \in G_x$ implies $\xi^{-1}\xi' \in G$, and (ii) $\xi \in G_x$, $g \in G$ implies $\xi g \in G_x$. Finally, for each x, there exists a neighborhood V of x and a homeomorphism $\phi : V \times Y \longrightarrow p^{-1}(V)$ such that $p\phi(x,y) = x$, and $\phi_x \in G_x$ for each x."

Pick any point $x_o$ of F and keep it fixed for the remainder of this discussion.

The bundle B will be G(F).

The projection map p will be our $\mu_{x_o}$.

The base space X will be our F.

The fibre Y will be our $G_{x_o}$.

Before completing this correspondence we must prove

Lemma 12: G(F) admits a local cross-section.

Proof: Choose a neighborhood V of $x_o$ which is homeomorphic to the interior of the unit sphere in n-space.

We need to show the existence of a continuous map $\chi$ taking V into G(F) and such that for each x in V $\mu_{x_o}\chi(x) = x$.

Each element of V may be identified with a position vector of the interior of the unit sphere in n-space, and as such the elements of V may be added vectorially and each element will have a norm. The point $x_o$ may be identified with the zero vector.

We define $\chi(x)$ by $[\chi(x)](y) = y + x(1 - |y|)$, where x and y are in V and $|y|$ is the norm of y, and $[\chi(x)](y) = y$ if x is in V but y is not.

Then $\chi(x)$ is an element of G(F), and

$\chi(x)$: $V \to V$ homeomorphically onto.

Also $[\chi(x)](x_o) = x_o + x(1 - 0) = x$, so that

$\mu_{x_o}\chi(x) = [\chi(x)](x_o) = x$ according to the requirement.

Now for the continuity of $\chi$.

We need only to check whether the inverse images of sub-basic neighborhoods are open in F.

Consider the sub-basic set $W_{y,U}$.

Suppose y is not in V. Then every element of $\chi(V)$ will leave y fixed so that if $y \in U$ then $\chi^{-1}(W_{y,U}) = V$ and if $y \notin U$ then $\chi^{-1}(W_{y,U}) = \emptyset$ both of which are open sets.

Suppose y is in V. We investigate the set $S_y = \{z \in V \mid [\chi(x)](y) = z\}$. If z is any element of V such that $|y - z| < 1 - |y|$ then there will exist an x in V such that $[\chi(x)](y) = z$. Also for any x in V, $[\chi(x)](y) = z$ is such that $|y - z| = |y - y + x(1 - |y|)| = |x|(1 - |y|) < 1 - |y|$ since $|x| < 1$.

In short, $S_y$ consists of an open sphere of radius $1 - |y|$ with center at y.

Now if $U \cap S_y = \emptyset$ then $\chi^{-1}(W_{y,U}) = \emptyset$.

If not then $\chi^{-1}(W_{y,U}) = \{x \in V \mid [\chi(x)](y) \in S_y \cap U\}$, and the fact that the latter set is open is seen from the fact that $[\overline{\chi}(y)](x) = [\chi(x)](y) = y + x(1 - |y|)$ is a continuous function of x for each y, $(x,y \in V)$. In particular, $\chi^{-1}(W_{y,U}) = [\overline{\chi}^{-1}(y)](S_y \cap U)$.

Thus $\chi^{-1}(W_{y,U})$ is open in V for any y and U, and the cross-section $\chi$ is continuous.

q.e.d.

We return to the correspondence between the components of the definition of an E-F bundle and the bundle which we are constructing.

The fibre $Y_x$ over a point x will be our $[x] = \{g \in G(F) \mid g(x_o) = x\}$.

The group G of homeomorphisms of the fibre Y will be for us the group $G_{x_o}$ acting on itself by left multiplication.

The family $G_x$ of homeomorphisms from the fibre Y to the fibre over x will for us be the coset [x] acting on $G_{x_o}$ by left multiplication.

Finally the map $\phi: V \times Y \longrightarrow p^{-1}(V)$ will be determined for us by the map $X: V \times G_{x_o} \longrightarrow \mu_{x_o}^{-1}(V)$ by $X(x,g) = \chi(x)g$.

This completes the correspondence. We now check the various conditions of the definition.

(i) Suppose $g_1$ and $g_2$ are elements of the coset [x], i.e., $g_1(x) = g_2(x) = x$. Let $g_o$ be an arbitrary element of the fibre $G_{x_o}$. Then first applying the map associated with $g_1$ and then the inverse of the map associated with $g_2$, we have $g_2^{-1} g_1 g_o$.

The element $g_2^{-1}g_1$ belongs to $G_{x_o}$ since $g_2^{-1}g_1(x_o) = g_2^{-1}(x) = x_o$.

(ii) Let $g'$ be an element of $[x]$, i.e., $g'(x_o) = x$.

Let $g$ be an element of $G_{x_o}$.

Then $g'g$ belongs to $[x]$ and multiplication by it maps $G_{x_o}$ into $[x]$

Last we must verify that the map X is a homeomorphism.

X is onto:

Say h is in $\mu_{x_o}^{-1}(V)$.

Then $h(x_o) = x$ where $x \in V$.

Therefore $h \in [x] = \chi(x)\, G_{x_o}$, that is, there exists a $g$ in $G_{x_o}$ such that $h = \chi(x)g = X(x,g)$.

X is one to one:

Suppose $\chi(x_1)g_1 = \chi(x_2)g_2$, where $x_1, x_2 \in V$ and $g_1, g_2 \in G_{x_o}$.

Then $[\chi(x_1)g_1](x_o) = [\chi(x_2)g_2](x_o)$,

and $$[\chi(x_1)](x_o) = [\chi(x_2)](x_o)$$

But the image of $x_o$ uniquely determines $(x)$, so that $[\chi(x_1)](x_o) = x_1 = [\chi(x_2)](x_o)$ implies

$$x_1 = x_2 \qquad \text{and} \qquad \chi(x_1) = \chi(x_2).$$

Cancellation yields $g_1 = g_2$.

$\chi^{-1}$ is continuous:

Let $A = U \times \{g \mid g(x_o) = x_o \text{ and } g(x) \in U'\}$ be a basic open set of $V \times G_{x_o}$. U is an open subset of V. U' is an open set of F, and x is a point in F.

Then $X(A) = \{\chi(y)g \mid y \in U, g(x_o) = x_o, \text{ and } g(x) \in U'\}$.

Let us extend the definition of the set $S_y$ (defined in the proof of lemma 12) to points y not necessarily in V.

Thus, $S_y = \{z \mid \text{for some } w \text{ in } V, [\chi(w)](y) = z\}$.

If y is in V, then $S_y$ is an open sphere about y with radius $1 - |y|$. If y is not in V, then $S_y = y$.

We claim first that $S_{U'} = \bigcup_{y \in U'} (S_y)$ is an open set for any open U' in F.

That is, $S_{U'} = U' \cup \left( \bigcup_{y \in U' \cap V} (S_y) \right)$.

Finally we claim that

$$X(A) = \{g \mid g(x_o) \in U \text{ and } g(x) \in S_{U'}\}.$$

Say h is in X(A).

Then $h(x_o) = [\chi(y)](g(x_o)) = [\chi(y)](x_o) = y \in U$, and $h(x) = [\chi(y)](g(x)) = [\chi(y)](z)$, $z \in U'$.

But, by definition, $[\chi(y)](z)$ is in $S_z$ and $S_z \subset S_{U'}$.

This proves inclusion in one direction.

Now say that $h(x_o)$ is in U and $h(x)$ is in $S_{U'}$.

Then $h(x_o) = y \in U$, so that $h \in [y] = \{g \mid g(x_o) = y\}$.

But $[y] = \chi(y) G_{x_o}$, and therefore there is a g in $G_{x_o}$ such that $h = \chi(y)g$.

Now $h(x) = [\chi(y)](g(x)) = w \in S_{U'}$,

or $g(x) = [\chi^{-1}(y)](w) \in U'$.

Also $h(x_o) = [\chi(y)](x_o) = y \in U$,

and h is an element of X(A).

X is continuous:

Since every basic open set in G is the intersection of a finite number of sub-basic open sets, and since $X^{-1}$ preserves intersections we may confine ourselves to the study of $X^{-1}(W_{x,U})$.

Suppose first that $x = x_o$.

Then $X^{-1}(W_{x,U}) = (U \cap V) \times G_{x_o}$ open in $V \times G_{x_o}$.

For let y be in $U \cap V$ and g be in $G_{x_o}$, then $[X(y,g)](x_o) = [\chi(y)](g(x_o)) = [\chi(y)](x_o) = y \in U$, and if y is not in $U \cap V$ then $[X(y,g)](x_o)$ is not in U.

Now suppose $x \neq x_o$.

First note that the map $\overline{X}$: $V \times V \longrightarrow V$ by $X(y_1,y_2) = [\chi(y_1)](y_2) = y_2 + y_1(1 - |y_2|)$ is continuous.

Let (y,g) be an element of $X^{-1}(W_{x,U})$, then $[\chi(y)](g(x)) = w$ a point of U.

We see first that if w is not in V, then g(x) must equal w.

If w is not in $\bar{V}$, the closure of V, then the set $V \times \left(G_{x_o} \cap W_{x,U \cap C\bar{V}}\right)$ is an open set of $V \times G_{x_o}$ which contains (y,g) and which is contained in $X^{-1}(W_{x,U})$.

If w is in V, then by the continuity of X, there will exist a neighborhood $U_1$ of y and a neighborhood $U_2$ of g(x) (g(x) must be in V also) such that $X(U_1, U_2) \subset V \cap U$.

Then $U_1 \times \left(G_{x_o} \cap W_{x,U_2}\right)$ is an open set of $V \times G_{x_o}$ which contains $(y,g)$ and which is contained in $X^{-1}(W_{x,U})$, for if $(y',g')$ is in $U_1 \times \left(G_{x_o} \cap W_{x,U_2}\right)$ then

$$[X(y',g')](x) = [\chi(y')](g'(x)) = X(y', g'(x)) \in V \cap U.$$

Finally suppose $[\chi(y)](g(x)) = w$ is an element of the boundary of V in U, i.e., $w \in U \cap (\bar{V} - V)$.

We use the fact that $\bar{X}$ is continuous from $\bar{V} \times \bar{V}$ to $\bar{V}$ to find a neighborhood $U_1$ of y and another $U_2$ of $g(x)$ such that $\bar{X}(U_1 \times U_2) \subset V \cap U$.

($U_1$ and $U_2$ are open sets relative to $\bar{V}$)

Let $U_3 = U_2 \cup (U \cap C\bar{V})$, then $U_1 \times \left(G_{x_o} \cap W_{x,U_3}\right)$ is an open set of $V \times G_{x_o}$ which contains $(y,g)$ and which is contained in $X^{-1}(W_{x,U})$ for if $(y',g') \in U_1 \times \left(G_{x_o} \cap W_{x,U_3}\right)$, then

$$w' = [X(y',g')](x) = [\chi(y')](g'(x)) = \begin{cases} \bar{X}(y',g'(x)) & \text{if } g'(x) \in V \\ g'(x) & \text{if } g'(x) \notin V \end{cases}$$

and in either case w' is in U.

Therefore $X^{-1}(W_{x,U})$ is open.

This completes the proof of the last condition.

Suppose $\bar{x} \neq x_o$. We find the local product homeomorphism at $\bar{x}$ as follows.

Choose g such that $g(x_o) = \bar{x}$.

Then $g(V)$ is a neighborhood of $\bar{x}$.

Define $X_{\bar{x}}: g(V) \times G_{x_o} \longrightarrow \mu_{x_o}^{-1}(g(V))$ by $X_{\bar{x}}(y,h) = g\,X(g^{-1}(y),h)$.

We check the condition $\mu_{x_o}(X_\chi(y,h)) = y$.

$$\mu_{x_o}(X_\chi(y,h)) = \mu_{x_o}(g\, X(g^{-1}(y),h) = [g\, X(g^{-1}(y),h)](x_o)$$

$$= [g\, \chi(g^{-1}(y))](h(x_o)) = [g\, \chi(g^{-1}(y))](x_o) = g(g^{-1}(y)) = y.$$

The rest of the conditions at $\bar{x}$ go through in a similar and automatic way.

q.e.d.

It is to be noted that G(F) is not a bundle in the Steenrod sense. The important place where G(F) fails to be such a bundle is in condition (5) of Steenrod, page 7, that is, the group of the fibre $G_{x_o}$ is not a topological transformation group. $G_{x_o}$ is not even a topological group.

Theorem 17: If $F \in \mathcal{F}$ is a manifold (connected), then G(F) is connected.

Proof: Lemma 12 yields a cross-section function $\chi$, and the image $\chi(V)$ in G(F) is a connected set consisting of more than one point.

Hence G(F) is not totally disconnected so that, by Theorem 15, it must be connected.

q.e.d.

The proofs of Theorem 15 and lemma 12 in this section may be modified to apply to the circle and the line. Theorem 15 becomes the statement that the subgroup $G^+(X)$, where X may be the circle or the line, of orientation preserving homeomorphisms is either connected or totally disconnected. In the proof one must work with open sets $W_{x_i;U_i}$ where the $x_i$ are in the order $x_1 < x_2 < \ldots < x_n$. The basic

fact that must be used is that $G^+(X)$ is $\omega$-transitive in the sense that if $x_1,\ldots,x_n$ and $y_1,\ldots,y_n$ are two proper n-tuples such that $x_1 < x_2 < \ldots < x_n$ and $y_1 < y_2 < \ldots < y_n$, then there exists an element g of $G^+(X)$ such that $g(x_i) = y_i$, for all i. With this type of modification the proof follows in an analogous fashion. The existence of a local cross-section is established in a manner identical with that used for higher dimensional manifolds. The fact that lemma 12 and theorem 15 hold for $G^+(X)$ implies that $G^+(X)$ is connected.

E. Normal Subgroups

For F a member of $\mathcal{F}$, G(F) cannot have any non-trivial, proper, closed, normal subgroups. This fact follows as a corollary to the

Theorem 18: If g G(F) and g is not the identity, then the conjugate class of g in G(F) is $\omega$-transitive, and hence dense, in G(F).

Proof: The conjugate class of g in G(F) is the set

$$\left\{hgh^{-1} \mid h \quad G(F)\right\}.$$

Let $x_1,\ldots,x_n$ and $z_1,\ldots,z_n$ be two arbitrary, independent, proper n-tuples of points of F.

Let x be a point of F such that $g(x) \neq x$.

Choose U, a neighborhood of x, such that $g(U) \cap U = \emptyset$.

Choose a proper n-tuple $y_1,\ldots,y_n$ such that $y_i \in U$, for all i.

Then $g(y_1),\ldots,g(y_n)$ and $y_1,\ldots,y_n$ are independent n-tuples, so that we may find an h such that $h(y_i) = x_i$ and $h(g(y_i)) = z_i$ for all i.

With this choice we have,

$$hgh^{-1}(x_i) = hg(y_i) = z_i$$

q.e.d.

Clearly then, no non-trivial subgroup of G(F) can be non-dense and normal, and since a proper closed subgroup is not dense it cannot be normal. Similarly every normal subgroup of G(F) must be $\omega$-transitive. Also, if X is a topological space and G(X) in the point-open topology contains a non-trivial, closed, normal subgroup, then G(X) is not $\omega$-transitive over X. A problem which may be of some interest is whether the fact that G(X) is non-$\omega$-transitive implies the existence of a non-trivial, proper, closed, normal subgroup.

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