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**UPPER AND LOWER BOUNDS FOR THE NORMS
OF SOLUTIONS OF SYSTEMS OF DIFFERENTIAL
EQUATIONS.**

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UPPER AND LOWER BOUNDS FOR THE NORMS OF SOLUTIONS
OF SYSTEMS OF DIFFERENTIAL EQUATIONS

by

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of the requirements for the degree of
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PREFACE

I would very much like to thank my advisor, Professor Wilfred Kaplan, for his advice, encouragement, and patience while I have been making this study.

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1. Introduction

In this paper the asymptotic behavior of the solutions of the matrix differential equations

$$(H) \quad x' = A(t)x$$

$$(NH) \quad x' = A(t)x + f(t), \quad \text{and}$$

$$(NL) \quad x' = A(t)x + f(x, t)$$

is considered. Here $' = d/dt$, the n dimensional vector x has real or complex components, the n by n matrix function $A(t)$ and the n by 1 vector function $f(t)$ have real or complex valued elements which are defined and continuous for $T_0 \leq t < +\infty$, and the n by 1 vector function $f(x, t)$ has real or complex valued elements which are defined and continuous for all (x, t) contained in some suitable domain D . Theorem 2.1 is a well known result (see references 8, 15, 24, and 26) which gives upper and lower bounds for the norms of the solutions of (H); this paper expands upon Theorem 2.1.

The homogeneous equation. Equation (H) is considered in Section 2. Theorem 2.1 seems to have been first stated in this form by Ważewski (reference 24).

Theorem 2.1. Let $x(t; x_0, t_0)$ be the solution of (H) such that $x(t_0; x_0, t_0) = x_0$ and let $\Lambda(t)$ and $\lambda(t)$ be the supremum and infimum, respectively, of the characteristic roots of $\frac{1}{2}(A^*(t) + A(t))$ ($*$ denotes complex conjugate transpose). Then for $t \geq t_0 \geq T_0$,

$$|x_0| \exp \int_{t_0}^t \lambda(s) ds \leq |x(t; x_0, t_0)| \leq |x_0| \exp \int_{t_0}^t \Lambda(s) ds,$$

(for a vector x , $|x| = (x^*x)^{1/2}$; for a matrix A , $|A| = \sup_{|x|=1} |Ax|$).

The corollaries of Theorem 2.1 translate the bounds of the theorem into the language of stability theory, i.e., they state conditions on $\int_{t_0}^t \Lambda(s) ds$ and $\int_{t_0}^t \lambda(s) ds$ which imply that $x = 0$ is a stable (or

uniformly stable, or equiasymptotically stable, or exponentially stable) solution of (H), or that the norms of the nonzero solutions of (H) are bounded away from zero or increase exponentially. Corollary 2 is typical of these corollaries.

Corollary 2. If $\int_0^\infty \Lambda(t)dt = -\infty$, then $x = 0$ is an equiasymptotically stable solution of (H).

Direct converses of these corollaries are not possible, but Theorem 2.2 gives a weak converse for three of them. For example, the weak converse of Corollary 2 would be: there exists a class of n by n matrices, ρ , such that the substitution $x = P(t)y$, $P(t) \in \rho$, transforms (H) into a new equation

$$(1.1) \quad y' = C(t)y$$

and that 1) if $y = 0$ is an equiasymptotically stable solution of (1.1), then $x = 0$ is an equiasymptotically stable solution of (H); and 2) if $x = 0$ is an equiasymptotically stable solution of (H), then there exists a $P(t) \in \rho$ such that for the corresponding (1.1), $\int_0^\infty \Lambda_C(t)dt = -\infty$ ($\Lambda_C(t)$ is the supremum of the characteristic roots of $\frac{1}{2}(C^*(t) + C(t))$).

If Theorem 2.1 yields no information when applied directly to (H), and there are cases where it does not, Theorem 2.2 shows the possibility of transforming (H) into an equation to which Theorem 2.1 may be applied with success. The proof of Theorem 2.2 involves transforming (H) into an equation with a diagonal coefficient matrix; such transformations have been studied by Diliberto [11].

Theorems 2.3 and 2.4 both deal with the second order differential equation. The theorems are mild improvements of known results and the proofs illustrate methods of transforming (H) and then applying Theorem 2.1.

Theorem 2.5 is used in Section 3 and 4 to extend the results of Theorem 2.1 to equations (NH) and (NL).

Theorem 2.5. Let $X(t)$ be the fundamental matrix of (H) such that $X(t_0)$ is the identity matrix and let $\Lambda(t)$ and $\lambda(t)$ be as defined in Theorem 2.1. Then for $t \geq t_0 \geq T_0$,

$$\exp \int_{t_0}^t \lambda(s) ds \leq |X(t)| \leq \exp \int_{t_0}^t \Lambda(s) ds \quad \text{and} \\ \exp \int_{t_0}^t -\Lambda(s) ds \leq |X^{-1}(t)| \leq \exp \int_{t_0}^t -\lambda(s) ds.$$

The nonhomogeneous equation. Equation (NH) is considered in Section 3. Let $\Lambda(t)$ and $\lambda(t)$ be as defined in Theorem 2.1 and let (H:nh) denote the homogeneous equation obtained from (NH) by suppressing $f(t)$.

Theorem 3.1. If $x(t; x_0, t_0)$ is the solution of (NH) such that $x(t_0; x_0, t_0) = x_0$ and $X(t)$ is the fundamental matrix of (H:nh) such that $X(t_0)$ is the identity matrix, then for $t \geq t_0 \geq T_0$,

$$|x_0| \exp \int_{t_0}^t \lambda(s) ds - \int_{t_0}^t |f(s)| \exp \int_s^t \lambda(u) du ds \leq |x(t; x_0, t_0)| \leq \\ \leq |x_0| \exp \int_{t_0}^t \Lambda(s) ds + \int_{t_0}^t |f(s)| \exp \int_s^t \Lambda(u) du ds.$$

The proof follows from applying Theorem 2.5 to

$$x(t; x_0, t_0) = X(t)x_0 + \int_{t_0}^t X(t)X^{-1}(s)f(s)ds.$$

Upper bounds for $|x(t; x_0, t_0)|$ may be deduced from Theorem 3.1; this is done in the corollary (lower bounds are discussed later). Two of the results are stated here.

Corollary. Let K_1, K_2, K_3, K_4 be constants.

1) If $\int_{t_0}^{\infty} |f(t)| dt \leq K_1$ and $\int_s^t \Lambda(u) du \leq K_2$ for $t \geq s \geq T_0$, then the solutions of equation (NH) are bounded.

2) If $|f(t)| \leq K_3$ and $\Lambda(t) \leq -K_4 < 0$ for $t \geq T_0$, then for all x_0 and $t \geq t_0 \geq T_0$,

$$|x(t; x_0, t_0)| \leq K_3/K_4 + (|x_0| - K_3/K_4) \exp(-K_4(t - t_0))$$

(this second result is extended in the corollary to Theorem 3.3).

Bellman (reference 3) has shown that if an $A(t)$, bounded and continuous for $t \geq T_0$, is given, then (NH) has all its solutions bounded for any $f(t)$ such that $\int_0^\infty |f(t)| dt < \infty$ if and only if $|X(t)X^{-1}(s)| \leq c$ for some positive constant c and all $t \geq s \geq T_0$ (here $X(t)$ is any fundamental matrix of $(H:nh)$). Since $|X(t)X^{-1}(s)| \leq \exp \int_s^t \Lambda(u) du$, the condition of (1) of the corollary ($\int_s^t \Lambda(u) du \leq K$ for all $t \geq s \geq T_0$) implies that $|X(t)X^{-1}(s)| \leq e^K$ for $t \geq s \geq T_0$. The converse is not true (as is shown by example in Section 3).

Perron (and Bellman, references 3 and 21) has proved that if $A(t)$, bounded and continuous for $t \geq T_0$, is given, then (NH) has all its solutions bounded for any $f(t)$ which is bounded and continuous for $t \geq T_0$ if and only if $x = 0$ is an exponentially stable solution of $(H:nh)$. The condition of part (2) of the corollary ($\Lambda(t) \leq -K < 0$ for $t \geq T_0$) implies, by Theorem 2.1, Corollary 3, that $x = 0$ is an exponentially stable solution of $(H:nh)$. The converse is not true (as is shown by example in Section 3).

There is a specialization of the Perron theorem by Bridgland (reference 6). Let $X(t)$ be the fundamental matrix of $(H:nh)$ such that $X(t_0) = I$; let $Y(t, t_0) = \int_{t_0}^t X(t)X^{-1}(s)ds$; and let $x(t; x_0, t_0)$ be the solution of (NH) such that $x(t_0; x_0, t_0) = x_0$. A vector or matrix function converges if it approaches a constant vector or matrix element-wise. The theorem states: given $A(t)$, bounded and continuous for $t \geq T_0$, every solution of (NH) converges for every convergent $f(t)$ if and only if a) every solution of (NH) is bounded for every bounded $f(t)$ and b) the matrix $Y(t, T_0)$ converges. Moreover,

$$\lim_{t \rightarrow \infty} x(t; x_0, t_0) = \left[\lim_{t \rightarrow \infty} Y(t, t_0) \right] \left[\lim_{t \rightarrow \infty} f(t) \right].$$

If, in addition to the hypotheses of Bridgland's theorem,

$\Lambda(t) \leq -K < 0$ for $t \geq T_0$, then

$$|X(t)X^{-1}(s)| \leq \exp \int_s^t \Lambda(u) du \leq \exp[K(s - t)] \quad \text{and}$$

$$|Y(t, t_0)| \leq \int_{t_0}^t |X(t)X^{-1}(s)| ds \leq \int_{t_0}^t \exp[K(s - t)] ds = K^{-1}[1 - \exp(K(t_0 - t))].$$

Taking norms in Bridgland's result,

$$|\lim_{t \rightarrow \infty} x(t; x_0, t_0)| \leq |\lim_{t \rightarrow \infty} Y(t, t_0)| |\lim_{t \rightarrow \infty} f(t)| \leq K^{-1} |\lim_{t \rightarrow \infty} f(t)|.$$

This is the bound obtained in the corollary to Theorem 3.3.

Compared with these earlier results, Theorem 3.1 and its corollary have three disadvantages: 1) they give only sufficient conditions; 2) in two instances, they require a stronger hypothesis; 3) they do not detect the convergence of a solution. But the theorem and its corollary also have advantages: 1) they do not require that $A(t)$ and $f(t)$ be bounded; 2) they give bounds for the norms of the solutions even if $f(t)$ does not converge; 3) they may be applied to equations not covered by the theorems of Perron and Bellman (For example, assume that (NH) is given with $A(t)$ and $f(t)$ specified, but that $x = 0$ is not an exponentially stable solution of (H:nh) and $\int_0^\infty |f(t)| dt = \infty$. In such a case, and such a case is given in Example 3.4, the theorems of Perron and Bellman would not apply, but the equation could have bounded solutions. The theorems of Section 3 would apply and might detect the boundedness.); 4) it is easier, at least for systems of order two or three, to compute $\Lambda(t)$ than it is to determine whether $x = 0$ is an exponentially stable solution of (H:nh) or whether $|X(t)X^{-1}(s)| \leq c$ for $t \geq s \geq T_0$.

If equation (NH) is restricted, the inequalities given in Theorem 3.1 may be examined in more detail. In Theorems 3.2 and 3.3 it is assumed that $\Lambda(t) \leq 0$ for $t \geq T_0$ (Theorem 3.2 permits a slightly weaker assumption) and that $f(t)$ is such that there exists a $G(t)$, defined,

non-negative, bounded, and continuous a.e. for $t \geq T_0$, with $|f(t)| = -G(t)\Lambda(t)$ (If $\Lambda(t) \leq -c < 0$ and $|f(t)| \leq M$ for $t \geq T_0$, then $G(t) = |f(t)|/-\Lambda(t)$. The definition of $G(t)$ becomes complicated in neighborhoods of points where $\Lambda(t) = 0$; this is discussed further in Section 3.). Roughly speaking, Theorems 3.2, 3.3 and the corollary to Theorem 3.3 describe the behavior of $|x(t; x_0, t_0)|$ relative to whether $|x(t; x_0, t_0)|$ is greater than or less than $G(t)$: if $|x(t; x_0, t_0)| \geq G(t)$ for $T_0 \leq t_1 \leq t \leq t_2 \leq \infty$, then $|x(t; x_0, t_0)|$ is monotonic non-increasing for $t_1 \leq t \leq t_2$; if $|x(t_1; x_0, t_0)| \leq G(t_1)$ for $T_0 \leq t_0 \leq t_1 < \infty$, then $|x(t; x_0, t_0)| \leq \sup\{G(s) : t_1 \leq s \leq t\}$ for $t \geq t_1$; and if $\int_{t_0}^{\infty} \Lambda(t) dt = \infty$ and $\limsup_{t \rightarrow \infty} G(t) = U$, U a non-negative constant, then $\limsup_{t \rightarrow \infty} |x(t; x_0, t_0)| \leq U$ for all x_0 and $t_0 \geq T_0$.

Theorems 3.2 and 3.3 analyzed the right inequality in Theorem 3.1; Theorems 3.4 and 3.5 analyze the left inequality in a similar fashion.

The nonlinear equation. Equation (NL) is considered in Section 4. Let $\Lambda(t)$ and $\lambda(t)$ be as defined in Theorem 2.1 and let $(H:n\ell)$ denote the homogeneous equation obtained from (NL) by suppressing $f(x, t)$. If $X(t)$ is the fundamental matrix of $(H:n\ell)$ such that $X(t_0)$ is the identity matrix and if $x(t; x_0, t_0)$ is a solution of (NL) such that $x(t_0; x_0, t_0) = x_0$, then $x(t; x_0, t_0)$ satisfies the integral equation

$$x(t; x_0, t_0) = X(t)x_0 + \int_{t_0}^t X(t)X^{-1}(s)f[x(s; x_0, t_0), s]ds$$

(assuming that $x(t; x_0, t_0)$ exists for all t under discussion).

Starting from this point, the reasoning used in the five theorems of Section 3 is repeated and five completely analogous theorems are obtained for (NL) (In fact, the work of Section 3 may be considered as an important special case of the work in the first part of Section 4). Only in dealing with (NL), care must be taken in all theorems to insure

that $x(t; x_0, t_0)$ exists wherever it is discussed. The work of Bellman, Perron, and Bridgland also carries over to (NL) (Bridgland's work requires a few more restrictions) and the relation between their results and the results of this paper is about the same as that discussed for (NH). All this constitutes the first five theorems of Section 4.

In the remainder of Section 4 it is assumed that $|f(x, t)| \leq g(t)|x|$, where $g(t)$ is defined, non-negative, and continuous for $t \geq T_0$, for all $(x, t) \in D$. Then Theorem 4.6 follows from Theorem 4.1.

Theorem 4.6. If $f(x, t)$ is as just described and if $x(t; x_0, t_0)$ is a solution of (NL) such that $x(t_0; x_0, t_0) = x_0$ and $x(t; x_0, t_0) \in D$ for $T_0 \leq t_0 \leq t \leq t_1 \leq \infty$, then
 $|x_0| \exp \int_{t_0}^t \{\lambda(s) - g(s)\} ds \leq |x(t; x_0, t_0)| \leq |x_0| \exp \int_{t_0}^t \{\lambda(s) + g(s)\} ds$
for $t_0 \leq t \leq t_1$.

If, in addition to the hypotheses of Theorem 4.6, $f(0, t) = 0$ for $t \geq T_0$ (and hence $x = 0$ is a solution of (NL)), then Theorem 4.6 may be used to discuss the stability (or uniform stability, or equiasymptotic stability, or exponential stability) of $x = 0$ as a solution of (NL). These results are stated in Theorem 4.7.

Theorem 4.7. Let equation (NL) be given and assume that the hypotheses of Theorem 4.6 are satisfied. In addition, assume that $f(0, t) = 0$ for $t \geq T_0$.

1) If $\int_{t_0}^t \{\lambda(s) + g(s)\} ds \leq L(t_0) < \infty$ for $t \geq t_0 \geq T_0$, then $x = 0$ is a stable solution of (NL).

2) If in (1), $L(t_0) \leq L$ for all $t_0 \geq T_0$, then $x = 0$ is a uniformly stable solution of (NL).

3) If the hypotheses of (1) are satisfied and if $\int_{t_0}^{\infty} \{\lambda(t) + g(t)\} dt = -\infty$, then $x = 0$ is an equiasymptotically stable solution of (NL).

- 4) If $\Lambda(t) + g(t) \leq 0$ for $t \geq T_0$ and $x(t; x_0, t_0) \in D$ for $t \geq t_0 \geq T_0$, then $|x(t; x_0, t_0)|$ is monotonic non-increasing for $t \geq t_0$.
- 5) If $\Lambda(t) + g(t) \leq -c$, c a positive constant, for $t \geq T_0$, then $x = 0$ is an exponentially stable solution of (NL).

In the following discussion of equation (NL) and of Lyapunov functions, it is assumed that x' , $A(t)$, x , and $f(x, t)$ have real valued elements.

Theorem 4.7 is similar to the work of several other authors. One line of investigation was started by Perron (in reference 21) and later developed by Persidskii and Malkin. The work of all three is summarized in reference 16, Chapter VII, and the following is the most convenient theorem of their approach.

Theorem. Let equation (NL) be given and assume that $A(t)$ is bounded for $t \geq T_0$ and that $|f(x, t)| = o(|x|)$ uniformly in t , i.e. $|f(x, t)| \leq |x|r(|x|)$ where $r(|x|) \rightarrow 0$ as $|x| \rightarrow 0$ for all $t \geq T_0$. If $(H:n\ell)$ has a Lyapunov function, $V(x, t)$, such that $V'(x, t)$ is negative definite, then $x = 0$ is an asymptotically stable solution of (NL).

A Lyapunov function for $(H:n\ell)$ is a real scalar function, $V(x, t)$, which is defined, positive definite, and has continuous first partial derivatives with respect to x and t for all x and $t \geq T_0$. Further, $V(0, t) = 0$ for $t \geq T_0$ and $V'(x, t) = V_x \cdot x' + V_t = V_x \cdot Ax + V_t$. The same definition holds for (NL) except $V(x, t)$ is defined only on D and $V'(x, t) = V_x \cdot (Ax + f) + V_t$.

The use of Lyapunov functions in stability theory has been extensively investigated, but most theorems require that $V'(x, t) \leq 0$. This is more than is required in Theorem 4.7. Recently Sell (reference 22) has studied Lyapunov functions with weaker restriction on $V'(x, t)$.

He considers the differential equation

$$(1.2) \quad u' = H(u, t)$$

where $H(u, t)$ is a real valued scalar function of the real scalar variables u and t which is defined for $(u, t) \in \bar{D} = \{(u, t): |u| < \bar{r}, t \geq T_0\}$. Further, $H(0, t) = 0$ for $t \geq T_0$ and $H(u, t)$ is such that (1.2) satisfies the conditions for the existence and uniqueness of solutions on $\bar{D}$.

An extended Lyapunov function, $V(x, t)$, for (H) or (NL) is the same as a Lyapunov function with the addition that

$$V'(x, t) \leq H(V(x, t), t)$$

for some $H(u, t)$ as defined in (1.2).

Theorem. Let equation (NL) be given and assume that 1) the solutions of (NL) are unique; 2) $f(0, t) = 0$ for $t \geq T_0$; and 3) an extended Lyapunov function exists for (NL). If $u = 0$ is a stable (or uniformly stable, or asymptotically stable, or equiasymptotically stable) solution of (1.2), then $x = 0$ is a stable (or uniformly stable, or asymptotically stable, or equiasymptotically stable) solution of (NL) (For uniform stability it is also necessary to assume that $V(x, t)$ has an infinitesimal upper bound.).

If (NL) is a real valued equation and if the solutions of (NL) are unique, the (1), (2), and (3) of Theorem 4.7 may be derived from the preceding theorem. Let $V(x, t) = |x| = (x^T x)^{\frac{1}{2}}$ (x^T is x transpose); it is easy to verify that $|x|$ is a Lyapunov function. It follows from a derivation similar to the proof of Theorem 2.1 that

$$|x|' \leq (\wedge(t) + g(t)) |x| \quad \text{for all } (x, t) \in D.$$

Then $|x|$ is an extended Lyapunov function and it is only necessary to study the null solution of

$$(1.3) \quad u' = (\wedge(t) + g(t)) u.$$

The general solution of this equation is

$$u(t) = u(t_0) \exp \int_{t_0}^t (\Lambda(s) + g(s)) ds \quad : \quad t \geq t_0 \geq T_0$$

and conditions (1), (2), and (3) of Theorem 4.7 are seen to be precisely the conditions which imply that $u = 0$ is a stable, uniformly stable, or equiasymptotically stable solution of (1.3).

Corollaries 1 and 2 of Theorem 2.1 may also be derived in this manner by simply repeating the preceding discussion with $g(t) = 0$ for $t \geq T_0$. Also, some of the results of Section 5 may be obtained by using $|P^{-1}x|$ (P^{-1} a nonsingular, constant matrix which is defined in Section 5) as an extended Lyapunov function.

Cesari (reference 9, p.94) and Golomb (reference 12) prove further results for (NL). The following theorem is one of Cesari's results; it is similar to (2) of Theorem 4.7.

Theorem. Assume the general hypotheses of Theorem 4.7 are satisfied. If $\int_{T_0}^{\infty} g(t) dt < \infty$ and $x = 0$ is a uniformly stable solution of (H: nℓ), then $x = 0$ is a uniformly stable solution of (NL).

Theorem 4.7 deals with the right inequality of Theorem 4.6; the left inequality is treated in Theorem 4.8.

Variations about a constant matrix. Several ideas in this section arose from remarks made by W. Kaplan. Some of these may be found in reference 15, sections 8-10 -- 8-15.

In Section 5, the special cases of (H), (NH), and (NL) are considered where $A(t) = A + B(t)$, A a constant matrix and $B(t)$ a continuous matrix function. The theorems of Sections 2, 3, and 4 are used to describe the asymptotic behavior of the solutions of these special systems in terms of the characteristic roots of A and the matrix $B(t)$.

The technique used is a substitution, $y = P^{-1}x$, where $P^{-1}AP = J$

and J is in Jordan form. This substitution transforms (H) from $x' = (A + B(t))x$ into $y' = (J + P^{-1}B(t)P)y = (J + F(t))y = C(t)y$. Equations (NH) and (NL) are transformed in a similar fashion.

A series of lemmas show that

- 1) the theorems of Sections 2, 3, and 4 may be applied to the transformed equations;
- 2) the asymptotic behavior of the original equations can be deduced directly from the asymptotic behavior of the transformed equations;
- 3) if Λ_X and λ_X are the supremum and infimum, respectively, of the characteristic roots of $\frac{1}{2}(X^* + X)$, where $X = J, F(t)$, or $C(t)$, then $\Lambda_C(t) \leq \Lambda_J + \Lambda_F(t)$ and $\lambda_C(t) \geq \lambda_J + \lambda_F(t)$;
- 4) if R and r are the maximum and minimum, respectively, of the real parts of the characteristic roots of A and if $\epsilon > 0$ is arbitrary, then P may be chosen such that $|\Lambda_J - R| < \epsilon$ and $|\lambda_J - r| < \epsilon$; and
- 5) $-\lambda_F(t)$ and $\Lambda_F(t) \leq |P^{-1}| |P| \sum_{i,j=1}^n |b_{ij}(t)|$.

Using these lemmas, several theorems may be obtained for the special cases of (H), (NH), and (NL), but for general A and $B(t)$ these theorems have been proved by other authors. Part (1) of Theorem 5.1 is typical of these results.

Theorem 5.1. 1) Let the equation

$$x' = (A + \epsilon B(t))x, \quad \epsilon, 0 \leq \epsilon < \infty, \text{ a parameter,}$$

be given and assume that $|B(t)| \leq M$ for $t \geq T_0$. If $x = 0$ is an exponentially stable solution of this equation when $\epsilon = 0$, then there exists an $\epsilon_0 > 0$ such that for $0 \leq \epsilon < \epsilon_0$, $x = 0$ is still an exponentially stable solution of this equation.

As A and $B(t)$ are restricted, the results of this method are improved.

One restriction is to assume that the characteristic roots of A are distinct; another restriction is to assume that

$$x' = (A + B(t))x$$

is a real n -th order equation written in system form, i.e.

$$\begin{bmatrix} x'_1 \\ \vdots \\ x'_n \end{bmatrix} = \left\{ \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & & & \ddots & 0 \\ & & & & 1 \\ -a_n & -a_{n-1} & \dots & \dots & -a_1 \end{bmatrix} + \begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \\ -b_n(t) \dots -b_1(t) \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

where $a_1, \dots, a_n$ are real constants and $b_1(t), \dots, b_n(t)$ are real valued functions, defined and continuous for $t \geq T_0$. For this last case, Theorem 5.3, (4) shows that if the characteristic roots of $(A + B(t))$ are "close enough" to the characteristic roots of A , then

$$x' = Ax \quad \text{and} \quad x' = (A + B(t))x$$

have the same asymptotic behavior.

Finally, the real, second order differential equation

$$x'' + (a_1 + b_1(t))x' + (a_2 + b_2(t))x = 0,$$

where a_1 and a_2 are real constants and $b_1(t)$ and $b_2(t)$ are real valued functions, defined and continuous for $t \geq T_0$, is studied. For this equation, the computations involved in reducing A to Jordan form can be carried out in detail. The bounds for $\Lambda_C(t)$ and $\lambda_C(t)$ (C is the coefficient matrix of the transformed equation) which result are given in Lemmas 5.14 - 5.16. For example, Lemma 5.14 states: if the characteristic roots of $\lambda^2 + a_1\lambda + a_2 = 0$ --call them λ_1 and λ_2 -- are complex, then the matrix P may be chosen such that

$$\Lambda_C(t) \leq -\frac{1}{2}(a_1 + b_1(t)) + |\lambda_2 - \lambda_1|^{-1} |\lambda_1 b_1(t) + b_2(t)|.$$

The efficacy of these bounds as a tool in stability theory is then demonstrated in Theorems 5.7 - 5.9.

2. Basic Definitions and the Homogeneous Linear System

Definition 2.1. Let

$$(H) \quad x' = A(t)x : t \geq T_0,$$

$$(NH) \quad x' = A(t)x + f(t) : t \geq T_0, \text{ and}$$

$$(NL) \quad x' = A(t)x + f(x, t) : (x, t) \in D$$

be systems of differential equations in matrix notation. Here $x' = \text{Col} (dx_1/dt, \dots, dx_n/dt)$ and $x = \text{Col} (x_1, \dots, x_n)$ are vectors; $A(t) = (a_{ij}(t))$, $i, j = 1, \dots, n$, is a matrix function of the real variable t with elements which are defined and continuous for $T_0 \leq t < +\infty$; $f(t) = \text{Col} (f_1(x, t), \dots, f_n(x, t))$, where $(x, t) = (x_1, \dots, x_n, t)$, is a vector function of (x, t) with components which are defined and continuous for $(x, t) \in D$. The components of x may be real or complex and the functions $a_{ij}(t)$, $f_i(t)$, and $f_i(x, t)$, $i, j = 1, \dots, n$, may be real or complex valued. The set $D \subset E^n \times R$ is open and connected and is such that there exists a constant, $r_0 > 0$, for which $\{x: x \in E^n \text{ and } |x| < r_0\} \times R \subset D$. Here E^n is an n -dimensional Euclidean or unitary vector space, depending on whether x has real or complex components, and R is the real interval $[T_0, \infty)$. Let $R_0 = \sup\{|x|: (x, t) \in D\}$; R_0 may equal $+\infty$.

Remarks on Definition 2.1. The homogeneous linear system, (H), and the nonhomogeneous linear system, (NH), satisfy the conditions for the existence and uniqueness of solutions for $t \geq T_0$; the nonlinear system, (NL), satisfies the conditions for the existence of solutions on the set D . The vector $x = 0$ is an equilibrium solution of equation (H); if $f(0, t) = 0$ for $t \geq T_0$, the vector $x = 0$ is an equilibrium solution of equation (NL).

Definition 2.2. 1) If $x \in E^n$, $\bar{x}$ is the complex conjugate

transpose of x (or simply the transpose if x is real) and $|x| = (x^*x)^{\frac{1}{2}}$.

2) If $A = (a_{ij})$, $i, j = 1, \dots, n$, then A^* is the complex conjugate transpose of A (or simply the transpose if A is real), $|A| = \sup_{|x|=1} |Ax|$, and $\|A\| = \sum_{i,j=1}^n |a_{ij}|$. Further, Λ_A and λ_A are the supremum and infimum, respectively, of the characteristic roots of $\frac{1}{2}(A^* + A)$. If $A = A(t)$, i.e. A is a matrix function of t , then $\Lambda_A = \Lambda_A(t)$ and $\lambda_A = \lambda_A(t)$ are functions of t .

Lemma 2.1. If $A = (a_{ij})$ and $B = (b_{ij})$, $i, j = 1, \dots, n$, $x = \text{Col}(x_1, \dots, x_n)$, and c is a scalar, then

- 1) $|AB| \leq |A||B|$;
- 2) $|A + B| \leq |A| + |B|$;
- 3) $|cA| = |c||A|$;
- 4) $|A| = |A^*|$;
- 5) $|Ax| \leq |A||x|$;
- 6) if $A = A^*$, then $|A|$ equals the maximum of the absolute values of the characteristic roots of A ;
- 7) $|A| \leq \|A\|$.

Proof. These are well known properties of the norm; proofs of (1) - (6) may be found in reference 13, pp. 176 - 181. To prove (7), note that (by Schwarz inequality)

$$\begin{aligned} |Ax| &= \left(\sum_{i=1}^n \left| \sum_{j=1}^n a_{ij} x_j \right|^2 \right)^{\frac{1}{2}} \leq \left[\sum_{i=1}^n \left(\sum_{j=1}^n |a_{ij}|^2 \right) \left(\sum_{j=1}^n |x_j|^2 \right) \right]^{\frac{1}{2}} = \\ &= |x| \left(\sum_{i,j=1}^n |a_{ij}|^2 \right)^{\frac{1}{2}} \leq \|A\| |x|. \quad \text{Q.E.D.} \end{aligned}$$

The following definition gives some of the usual terms used to describe the asymptotic behavior of solutions of differential equations. The terms are defined for (NL), but they also apply to (H), with $r_0 = \infty$. The boundedness terms also apply to (NH), with $r_0 = \infty$. In the definition, $x(t; x_0, t_0)$ is a solution of (NL) such that $x(t_0; x_0, t_0) = x_0$.

In (2), (3) and (4) of the following definition, assume that $f(0, t) = 0$ for $t \geq T_0$.

Definition 2.3. 1) If there exists a $\delta_0 > 0$ such that $|x_0| < \delta_0$ implies $x(t; x_0, t_0) \in D$ for $t \geq t_0 \geq T_0$ and if, given δ , $0 < \delta \leq \delta_0$, and $t_0 \geq T_0$, there exists an $M(\delta, t_0) > 0$ such that $|x_0| < \delta$ implies $|x(t; x_0, t_0)| \leq M$ for $t \geq t_0$, then the solutions of (NL) are locally bounded. If $M = M(\delta)$, then the solutions of (NL) are uniformly locally bounded.

2) If, given ϵ , $0 < \epsilon < r_0$, and $t_0 \geq T_0$, there exists a $\delta(\epsilon, t_0)$, $0 < \delta < r_0$, such that $|x_0| < \delta$ implies $|x(t; x_0, t_0)| < \epsilon$ for $t \geq t_0$, then $x = 0$ is a stable solution of (NL). If $\delta = \delta(\epsilon)$, then $x = 0$ is a uniformly stable solution of (NL). For (H), local boundedness is equivalent to the stability of $x = 0$ and uniform local boundedness is equivalent to the uniform stability of $x = 0$.

3) If $x = 0$ is a stable solution of (NL) and if, for $t_0 \geq T_0$, there exists a $\delta_c(t_0)$, $0 < \delta_c < r_0$, such that $|x_0| < \delta_c$ implies $\lim_{t \rightarrow \infty} x(t; x_0, t_0) = 0$, then $x = 0$ is an asymptotically stable solution of (NL). If this limit is uniform for all x_0 , $|x_0| < \delta_c$, then $x = 0$ is an equiasymptotically stable solution of (NL). For (H), equiasymptotic stability is equivalent to asymptotic stability.

4) If there exists a constant $c > 0$, and if, given ϵ , $0 < \epsilon < r_0$, and $t_0 \geq T_0$, there exists a $\delta(\epsilon)$, $0 < \delta < r_0$, such that $|x_0| < \delta$ implies $|x(t; x_0, t_0)| < \epsilon \exp(-c(t - t_0))$ for $t \geq t_0$, then $x = 0$ is an exponentially stable solution of (NL).

5) If, given $(x_0, t_0) \in D$, there exists a $\delta(x_0) > 0$ such that $x(t; x_0, t_0) \in D$ implies $|x(t; x_0, t_0)| \geq \delta$ for $t \geq t_0$, then $|x(t; x_0, t_0)|$ is bounded away from zero.

6) If there exists a positive constant c and if, given ϵ , $0 < \epsilon < r_0$, and $t_0 \geq T_c$ there exists a $\delta(\epsilon)$, $0 < \delta < r_0$, such that $|x_0| > \delta$ and $(x(t; x_0, t_0), t) \in D$ implies that $|x(t; x_0, t_0)| > \epsilon \exp(c(t - t_0))$ for $t \geq t_0$, then $|x(t; x_0, t_0)|$ increases exponentially. This property is of interest primarily when $r_0 = +\infty$.

The following theorem gives upper and lower bounds for the norms of the solutions of (H). In the corollaries to the theorem, these bounds are translated into the terminology of Definition 2.3.

Theorem 2.1 (References: 8, Sec. 2; 15, p. 500; 24, Sec. 2; 26, Append.). Let $x(t)$ be a solution of (H) with $x(t_0) = x_0$; let $\Lambda(t) = \Lambda_A(t)$ and $\lambda(t) = \lambda_A(t)$. Then for $t \geq t_0 \geq T_c$,

$$|x_0| \exp \int_{t_0}^t \lambda(s) ds \leq |x(t)| \leq |x_0| \exp \int_{t_0}^t \Lambda(s) ds.$$

Proof. If $x_0 = 0$, the uniqueness of the solutions of (H) implies that $x(t) = 0$ for $t \geq t_0 \geq T_c$ and the theorem holds. Assume $x_0 \neq 0$ and thus $x(t) \neq 0$ for $t \geq t_0 \geq T_c$.

Let $u^2(t) = x^*(t)x(t)$, i.e. $u(t) = |x(t)|$; then for $t \geq t_0 \geq T_c$,

$$2u(t)u'(t) = x^*(t)x'(t) + [x^*(t)]'x(t), \text{ or}$$

$$u'(t) = \left\{ (2|x(t)|^2)^{-1} (x^*(t)[A^*(t) + A(t)]x(t)) \right\} u(t) = H(x(t), t)u(t).$$

Therefore, $u'(t)$ is defined and continuous for $t \geq t_0$.

The function $H(x, t)$ is a Hermitean form evaluated on the unit sphere and is continuous in x and t . Thus $\sup_{|x| \neq 0} H(x, t)$ exists and is continuous in x and t and $\Lambda(t) = \sup_{|x| \neq 0} H(x(t), t)$ exists and is continuous in t for $t \geq t_0$. Therefore,

$$u'(t) = \Lambda(t)u(t) + g(t): \quad g(t) \leq 0 \text{ and continuous for } t \geq t_0,$$

$$u'(t) - \Lambda(t)u(t) = g(t).$$

The function $\exp \int_{t_0}^t -\Lambda(s) ds$ exists and is an integrating factor for this equation; the solution is

$$u(t) = u(t_0) \exp \int_{t_0}^t \Lambda(s) ds + \int_{t_0}^t g(s) \left[\exp \int_s^t \Lambda(u) du \right] ds.$$

Since $g(t) \leq 0$ for $t \geq t_0$,

$$|x(t)| \leq |x_0| \exp \int_{t_0}^t \Lambda(s) ds.$$

In a similar fashion, taking $\inf_{|x| \neq 0} H(x, t) = \lambda(t)$, it follows that

$$|x(t)| \geq |x_0| \exp \int_{t_0}^t \lambda(s) ds. \quad \text{Q.E.D.}$$

Corollary 1. If $\int_{t_0}^t \Lambda(s) ds \leq M(t_0) < \infty$ for $T_0 \leq t_0 \leq t$, then $|x(t)| \leq |x_0| \exp M(t_0)$, the solutions of (H) are locally bounded, and $x = 0$ is a stable solution of (H). If $M(t_0) \leq M$ for all $t_0 \geq T_0$, then $|x(t)| \leq |x_0| e^M$, the solutions of (H) are uniformly locally bounded, and $x = 0$ is a uniformly stable solution of (F).

Proof. The proof follows directly from the theorem and from Definition 2.3, (1) and (2).

Corollary 2. If $\int_{t_0}^{\infty} \Lambda(t) dt = -\infty$, then $x = 0$ is an equiasymptotically stable solution of (H).

Proof. Since $\Lambda(t)$ is continuous for $t \geq T_0$ and $\int_{t_0}^{\infty} \Lambda(t) dt = -\infty$, for $t_0 \geq T_0$ there exists an $M(t_0)$ such that $\int_{t_0}^t \Lambda(s) ds \leq M(t_0)$ for $t_0 \leq t$. By Corollary 1, $x = 0$ is a stable solution of (H). Further, $|x(t)| \leq |x_0| \exp \int_{t_0}^t \Lambda(s) ds \rightarrow 0$ as $t \rightarrow \infty$ for all $|x_0| < \delta_0$ (δ_0 an arbitrary positive constant). This limit is uniform with respect to x_0 for all $|x_0| < \delta_0$. Q.E.D.

Corollary 3. If $\Lambda(t) \leq -c$, c a positive constant, for $t \geq T_0$, then $x = 0$ is an exponentially stable solution of (H).

Proof. Given $\epsilon > 0$ and $t_0 \geq T_0$, let $\delta(\epsilon) = \epsilon$. By the theorem, if $|x_0| < \delta$, then $|x(t)| \leq |x_0| \exp \int_{t_0}^t \Lambda(s) ds \leq \epsilon \exp(-c(t - t_0))$. Q.E.D.

Corollary 4. If $\int_{t_0}^t \lambda(s) ds \geq m(t_0) > -\infty$ for $T_0 \leq t_0 \leq t$, then $|x(t)| \geq |x_0| \exp m(t_0)$. If $m(t_0) \geq m$ for all $t_0 \geq T_0$, then $|x(t)| \geq |x_0| e^m$ and the norm of every nonzero solution of (H) is bounded away from zero.

Proof. The proof follows directly from the theorem and from Definition 2.3, (5).

Corollary 5. If $\int_{t_0}^{\infty} \lambda(t) dt = \infty$, then the norm of every nonzero solution of (H) is unbounded as $t \rightarrow \infty$. If $\lambda(t) \geq c$, c a positive constant, for $t \geq T_0$, then the norm of every nonzero solution of (H) increases exponentially as $t \rightarrow \infty$.

Proof. The first part follows directly from the theorem. If $\lambda(t) \geq c$ for $t \geq T_0$ and if $\epsilon > 0$, $t_0 \geq T_0$ are given, let $\delta(\epsilon) = \epsilon$. Then if $|x_0| \geq \delta$, $|x(t)| \geq |x_0| \exp \int_{t_0}^t \lambda(s) ds > \epsilon \exp(c(t - t_0))$. Q.E.D.

If $A(t)$ is periodic in t with period T , i.e. $A(t + T) = A(t)$ for $t \geq T_0$, then the hypotheses of these corollaries may be stated in more convenient form. In each of the following corollaries it is assumed that $A(t)$ is periodic in t with period T .

Corollary 1'. If $\int_{t_0}^{t_0+T} \Lambda(t) dt \leq 0$, then $x = 0$ is a uniformly stable solution of (H).

Proof. If $\int_{t_0}^{t_0+T} \Lambda(t) dt \leq 0$, then $\int_{t_0}^t \Lambda(s) ds \leq (T) \max(\Lambda(t): T_0 \leq t \leq T_0 + T)$ for $t \geq t_0 \geq T_0$ and, by Corollary 1, $x = 0$ is a uniformly stable solution of (H). Q.E.D.

Corollary 2' - 3'. If $\int_{t_0}^{t_0+T} \Lambda(t) dt < 0$, then $x = 0$ is an exponentially stable solution of (H).

Proof. If $\int_{t_0}^{t_0+T} \Lambda(t) dt < 0$, then $\int_{t_0}^{t_0+T} \Lambda(t) dt = -cT$, c a positive constant. Let $x(t; x_0, t_0)$ be the solution of (H) such that $x(t_0; x_0, t_0) = x_0$. Since the solutions of (H) depend continuously upon their initial conditions, for any $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ (δ is independent of t_0 since $A(t)$ is periodic) such that $|x_0| < \delta$ and $t_0 \geq T_0$ implies that $|x(\bar{t}; x_0, t_0)| < \epsilon e^{-cT}$ for $t_0 \leq \bar{t} \leq t_0 + T$. For any $t \geq t_0$, $t = \bar{t} + nT$ (n a non-negative integer). Then by Theorem 2.1,

$$\begin{aligned}
 |x(t; x_0, t_0)| &\leq |x(\bar{t}; x_0, t_0)| \exp \int_{\bar{t}}^{t+nT} \Lambda(t) dt \leq (e^{-cT})(e^{-cnT}) = \\
 &= e^{-c(n+1)T} \leq e^{-c(t-t_0)}. \quad \text{Q.E.D.}
 \end{aligned}$$

Corollary 4'. If $\int_{t_0}^{t_0+T} \Lambda(t) dt \geq 0$, then the norms of the nonzero solutions of (H) are bounded away from zero as $t \rightarrow \infty$.

Proof. The proof is similar to the proof of Corollary 1'.

Corollary 5'. If $\int_{t_0}^{t_0+T} \Lambda(t) dt > 0$, then the norms of the nonzero solutions of (H) increase exponentially as $t \rightarrow \infty$.

Proof. The proof is similar to the proof of Corollary 2' - 3'.

Direct converses for these corollaries are not possible (Example 3.1 provides one counterexample), but weak converses are (except for Corollaries 4 and 5). These will now be developed.

Definition 2.4. If $(H) \in \mathcal{B}_1$, then $x = 0$ is a stable solution of (H); if $(H) \in \mathcal{B}_2$, then $x = 0$ is a uniformly stable solution of (H); if $(H) \in \mathcal{B}_3$, then $x = 0$ is an asymptotically stable solution of (H); if $(H) \in \mathcal{B}_4$, then $x = 0$ is an exponentially stable solution of (H); if $(H) \in \mathcal{B}_5$, then the norms of the nonzero solutions of (H) are bounded away from zero for all t ; if $(H) \in \mathcal{B}_6$, then the norms of the nonzero solutions of (H) increase exponentially as $t \rightarrow \infty$.

Let $L_A(t_0) = \limsup_{t \rightarrow \infty} \int_{t_0}^t \Lambda_A(s) ds$ and $\ell_A(t_0) = \liminf_{t \rightarrow \infty} \int_{t_0}^t \lambda_A(s) ds$.

If $(H) \in \mathcal{C}_1$, then $L_A(t_0) < \infty$; if $(H) \in \mathcal{C}_2$, then $L_A(t_0) \leq L$, a constant, for all $t_0 \geq T_0$; if $(H) \in \mathcal{C}_3$, then $L_A(t_0) = -\infty$; if $(H) \in \mathcal{C}_4$, then $\Lambda_A(t) \leq -c < 0$, c constant, for $t \geq T_0$; if $(H) \in \mathcal{C}_5$, then $\ell_A(t_0) \geq \ell$, a constant, for all $t_0 \geq T_0$; if $(H) \in \mathcal{C}_6$, then $\lambda_A(t) \geq c > 0$, c constant, for $t \geq T_0$. ($\mathcal{B}$ denotes behavior, $\mathcal{C}$ denotes condition).

Now Corollaries 1 - 5 of Theorem 2.1 may be written as one corollary (Corollary 1 has been split into two parts).

Corollary. If $(H) \in \mathcal{C}_1$, then $(H) \in \mathcal{B}_1$ ($i = 1, \dots, 6$).

Definition 2.5. Let $\mathcal{P} = \{P(t): P(t) \text{ is an } n \text{ by } n \text{ matrix which is defined, nonsingular, and has a continuous derivative for } t \geq T_0\}$; let $\mathcal{P}_1 = \{P(t): P(t) \in \mathcal{P} \text{ and } |P(t)| \text{ is bounded for } t \geq T_0\}$; let $\mathcal{P}_2 = \{P(t): P(t) \in \mathcal{P} \text{ and } |P^{-1}(t)| \text{ is bounded for } t \geq T_0\}$; let $\mathcal{P}_3 = \mathcal{P}_1 \cap \mathcal{P}_2$.

A matrix $P(t)$ will be called admissible only in connection with some equation (H) and only if that $(H) \in \mathcal{B}_i$ for some i , $1 \leq i \leq 6$. If $(H) \in \mathcal{B}_1$, $i = 1, 3$, and $P(t)$ is admissible, then $P(t) \in \mathcal{P}_1$; if $(H) \in \mathcal{B}_1$, $i = 2, 4, 6$, and $P(t)$ is admissible, then $P(t) \in \mathcal{P}_3$; if $(H) \in \mathcal{B}_5$ and $P(t)$ is admissible, then $P(t) \in \mathcal{P}_2$.

If $P(t) \in \mathcal{P}$, then the substitution $y = P^{-1}(t)x$ transforms (H) into

$$(2.1) \quad y' = P^{-1}(t)(A(t)P(t) - P'(t))y = B(t)y$$

where $B(t)$ is continuous for $t \geq T_0$. If $x(t; x_0, t_0)$ is the solution of (H) such that $x(t_0; x_0, t_0) = x_0$, then $y(t; y_0, t_0) = P^{-1}(t)x(t; x_0, t_0)$ is the solution of (2.1) such that $y(t_0; x_0, t_0) = y_0 = P^{-1}(t_0)x_0$. If $X(t, t_0)$ is the fundamental matrix of (H) such that $X(t_0, t_0)$ is the identity matrix, then $Y(t, t_0) = P^{-1}(t)X(t, t_0)P(t_0)$ is the fundamental matrix of (2.1) such that $Y(t_0, t_0)$ is the identity matrix.

Lemma 2.2. Assume the substitution $y = P^{-1}(t)x$, $P(t) \in \mathcal{P}$, has been made to transform (H) into (2.1). If $(2.1) \in \mathcal{B}_1$, and if $P(t)$ is an admissible matrix, then $(H) \in \mathcal{B}_1$, $i = 1, \dots, 6$.

Proof. Since $y(t; y_0, t_0) = P^{-1}(t)x(t; x_0, t_0)$ and $x(t; x_0, t_0) = P(t)y(t; y_0, t_0)$,

$$|P^{-1}(t)|^{-1}|y(t; y_0, t_0)| \leq |x(t; x_0, t_0)| \leq |P(t)||y(t; y_0, t_0)|.$$

Assume $(2.1) \in \mathcal{B}_1$, then $P(t) \in \mathcal{P}_1$ and $|P(t)| \leq M$ for $t \geq T_0$. Let $\epsilon > 0$ and $t_0 \geq T_0$ be given; let $\epsilon' = \epsilon/M$. Since $(2.1) \in \mathcal{B}_1$,

there exists a $\delta'(\epsilon', t_0) = \delta'(\epsilon, t_0) > 0$ such that $|y_0| < \delta'$ implies $|y(t; y_0, t_0)| < \epsilon'$ for $t \geq t_0$. Let $\delta(\delta', t_0) = \delta(\epsilon, t_0) = |P^{-1}(t_0)|^{-1} \delta'$ (Note that even if δ' did not depend on t_0 , δ would depend on t_0). Then $|x_0| < \delta$ implies $|y_0| \leq |P^{-1}(t_0)| |x_0| < \delta'$ which implies that $|y(t; y_0, t_0)| < \epsilon'$ or $|x(t; x_0, t_0)| \leq |P(t)| |y(t; y_0, t_0)| < \epsilon$ for $t \geq t_0$, i.e. $(H) \in \mathcal{B}_1$.

If $(2.1) \in \mathcal{B}_2$, then $P(t) \in \mathcal{P}_3$ and $|P(t)| \leq M$, $|P^{-1}(t)| \leq N$ for $t \geq T_0$. Now repeat the preceding proof but note that $\delta' = \delta'(\epsilon)$ and let $\delta = \delta(\delta') = \delta(\epsilon) = N^{-1} \delta'$.

If $(2.1) \in \mathcal{B}_1$, $i = 3$ or 4 , the proofs are much the same as the preceding proofs.

If $(2.1) \in \mathcal{B}_5$, then $P(t) \in \mathcal{P}_2$ and $|P^{-1}(t)| \leq N$ for $t \geq T_0$. Let (x_0, t_0) be given; this determines $y_0 = P^{-1}(t_0)x_0$. Since $(2.1) \in \mathcal{B}_5$, there exists a $\delta'(y_0) = \delta'(x_0) > 0$ such that $|y(t; y_0, t_0)| \geq \delta'$ for $t \geq t_0$. Then $|x(t; x_0, t_0)| \geq |P^{-1}(t)|^{-1} |y(t; y_0, t_0)| \geq N^{-1} |y(t; y_0, t_0)| \geq N^{-1} \delta' = \delta(x_0)$ for $t \geq t_0$ and $(H) \in \mathcal{B}_5$.

If $(2.1) \in \mathcal{B}_6$, the proof is similar to preceding proofs. Q.E.D.

Lemma 2.3. The substitution $y = P^{-1}(t)x$, $P(t) \in \mathcal{P}$, transforms (H) into

(2.2) $y' = D(t)y$, where $D(t) = \text{Diag}(d_1(t), \dots, d_n(t))$
is defined and continuous for $t \geq T_0$,
if and only if for $t \geq t_0 \geq 0$,

$$X(t, t_0) = P(t) \exp \int_{t_0}^t D(s) ds P^{-1}(t_0)$$

where $X(t, t_0)$ is the fundamental matrix of (H) such that $X(t_0, t_0)$ is the identity matrix and $D(t)$ is defined and continuous for $t \geq T_0$.

Proof. If $y = P^{-1}(t)x$, $P(t) \in \mathcal{P}$, transforms (H) into (2.2), then the discussion of equation (2.1) applies. First $D(t)$ is defined and continuous for $t \geq T_0$ and second

$$X(t, t_0) = P(t)Y(t, t_0)P^{-1}(t_0) = P(t)\exp \int_{t_0}^t D(s)ds P^{-1}(t_0).$$

Now assume $D(t)$ is defined and continuous for $t \geq T_0$. Fix $t_0 \geq T_0$; this determines $X(t; t_0)$. Let $P(t_0)$ be a constant, nonsingular matrix. Then

$$P(t) = X(t, t_0)P(t_0)\exp \int_{t_0}^t -D(s)ds$$

is defined and continuous for $t \geq T_0$ and $P'(t)$ is defined and continuous for $t \geq T_0$, i.e. $P(t) \in \mathcal{P}$. Further, $y = P^{-1}(t)x$ transforms (H) into (2.1) and since $Y(t, t_0) = P^{-1}(t)X(t, t_0)P(t_0) = \exp \int_{t_0}^t D(s)ds$, this substitution must transform (H) into (2.2). Q.E.D.

Theorem 2.2. If 1) $(H) \in \mathcal{B}_1$, $i = 1, 3$; or
2) $(H) \in \mathcal{B}_1$, $i = 2, 4$, and $\int_{t_0}^{\infty} \text{Tr } A(s)ds > -\infty$, then there exists an admissible $P(t)$ (which is not unique) such that the substitution $y = P^{-1}(t)x$ transforms (H) into (2.2) and $(2.2) \in \mathcal{C}_1$.

Proof. In each section of the proof a diagonal matrix, $D(t)$, is constructed which is defined and continuous for $t \geq T_0$. Then $P(t)$ is defined as

$$P(t) = X(t, T_0)\exp \int_{T_0}^t -D(s)ds$$

where $X(t, T_0)$ is the same as in Lemma 2.3 (This definition of $P(t)$ is taken from Lemma 2.3 with $t_0 = T_0$ and $P(t_0)$ the identity matrix.). By Lemma 2.3, $P(t) \in \mathcal{P}$ and the substitution $y = P^{-1}(t)x$ transforms (H) into (2.2). It will then only be necessary to show that $P(t)$ is contained in the proper $\mathcal{P}_1$ and (2.2) is contained in the proper $\mathcal{C}_1$.

1) Assume $(H) \in \mathcal{B}_1$; then $|X(t, T_0)| \leq M$ for $t \geq T_0$. Let $D(t)$ be the zero matrix; $D(t)$ is defined and continuous for $t \geq T_0$. Since $\Lambda_D(t) = 0$, $L_D(T_0) = 0$ and $(2.2) \in \mathcal{C}_1$. Also,

$$|P(t)| \leq |X(t, T_0)| \left| \exp \int_{T_0}^t -D(s)ds \right| \leq M \text{ for } t \geq T_0$$

and $P(t) \in \mathcal{P}_1$.

2) If $(H) \in \mathcal{B}_2$, then the preceding proof will apply if it can be shown that $|P^{-1}(t)| \leq N$ for $t \geq T_0$. Since $|P(t)| \leq M$ for $t \geq T_0$, it will only be necessary to show that $\text{Det}(P(t)) \geq c > 0$, c constant, for $t \geq T_0$. Now $\text{Det} P(t) = \text{Det} X(t, T_0) \times \text{Det} \exp \int_{T_0}^t -D(s) ds = \exp \int_{T_0}^t \text{Tr} A(s) ds$ and since $\int_{T_0}^{\infty} \text{Tr} A(s) ds > -\infty$, $\text{Det} P(t) \geq c > 0$ for $t \geq T_0$.

3) Assume $(H) \in \mathcal{B}_3$. Let $x_i(t)$ be the vector corresponding to the i -th column of $X(t, T_0)$ ($i = 1, \dots, n$); then $|x_i(t)|$ is defined, $|x_i(t)| \neq 0$, and $|x_i(t)|' = \frac{1}{2} |x_i(t)|^{-1} (x_i^* x_i' + x_i'^* x_i)$ is defined and continuous for $i = 1, \dots, n$ and $t \geq T_0$.

Let $d(t) = (\sum_{i=1}^n |x_i(t)|)^{-1} (\sum_{i=1}^n |x_i(t)|)'$, then $d(t)$ is defined and continuous for $t \geq T_0$,

$\int_{T_0}^t d(s) ds = \log(\sum_{i=1}^n |x_i(t)|) - \log(n)$, and $\exp \int_{T_0}^t d(s) ds = \frac{\sum_{i=1}^n |x_i(t)|}{n}$. Let $D(t) = \text{Diag}(d(t), \dots, d(t))$, then $D(t)$ is defined and continuous for $t \geq T_0$ and

$|P(t)| \leq |X(t, T_0)| |\exp \int_{T_0}^t -D(s) ds| = |X(t, T_0)| (\sum_{i=1}^n |x_i(t)|)^{-1} n \leq n^2$, i.e. $P(t) \in \mathcal{P}_1$.

Since $\Lambda_D(t) = d(t)$ and $|x_i(t)| \rightarrow 0$ as $t \rightarrow \infty$ for $i = 1, \dots, n$, $L_D(T_0) = \lim_{t \rightarrow \infty} \sup \int_{T_0}^t \Lambda_D(s) ds = \lim_{t \rightarrow \infty} \sup \log(\sum_{i=1}^n |x_i(t)|) - \log n = -\infty$ and $(2.2) \in \mathcal{C}_3$.

4) Assume $(H) \in \mathcal{B}_4$. Let $x_i(t)$ be as defined in part (3) of this proof. Then there exist positive constants K and c such that

$$|x_i(t)| \leq K e^{-c(t-T_0)} \quad \text{for } t \geq T_0 \text{ and } i = 1, \dots, n.$$

Let $D(t) = \text{Diag}(-c, \dots, -c)$; $D(t)$ is defined and continuous for $t \geq T_0$ and $\Lambda_D(t) = -c$ for $t \geq T_0$, i.e. $(2.2) \in \mathcal{C}_4$. Also $|P(t)| \leq |X(t, T_0)| \times \exp \int_{T_0}^t -D(s) ds = |X(t, T_0)| e^{c(t-T_0)} \leq nK$ for $t \geq T_0$ and $P(t) \in \mathcal{P}_1$. The reasoning of part (2) of this proof may again be applied to show that $|P^{-1}(t)| \leq N$ for $t \geq T_0$. Hence $P(t) \in \mathcal{P}_3$. Q.E.D.

Remarks on Theorem 2.2. This theorem does not really provide a

constructive method for finding $P(t)$ since to solve for $P(t)$ explicitly it is necessary to know $X(t, t_0)$. However, if Theorem 2.1 yields no information when applied directly to (H), and there are cases where it does not, Theorem 2.2 shows the possibility of transforming (H) into an equation to which Theorem 2.1 may be applied with success.

The process of diagonalization used in Lemma 2.3 has been studied by Diliberto (reference 11).

The following two theorems demonstrate the process of transforming a given equation and then applying Theorem 2.1.

Definition 2.6. For a function $f(t)$, let $f^+(t) = \sup (f(t), 0)$ and $f^-(t) = \inf (f(t), 0)$.

Consider the equation

$$x'' + p(t)x' + q(t)x = 0: \quad p(t) \text{ and } q(t) \text{ are defined, real-valued, and continuous for } t \geq T_0.$$

This equation may be written as the system

$$(2.3) \quad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -q(t) & -p(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{i.e.} \quad x' = A(t)x.$$

Theorem 2.3. Assume there exists a positive constant c such that $\int_0^\infty |c - q(t)| dt < \infty$. If $\int_0^\infty p^-(t) dt > -\infty$, then the solutions of (2.3) are uniformly locally bounded; if $\int_0^\infty p^+(t) dt < +\infty$, then the norms of the nonzero solutions of (2.3) are bounded away from zero as $t \rightarrow \infty$.

Proof. Let $y = P^{-1}x$ where $P = \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$, k an arbitrary real positive constant. This substitution transforms (2.3) into

$$(*) \quad \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & k^{-1} \\ -kq(t) & -p(t) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \text{i.e.} \quad y' = B(t)y.$$

Since $B(t)$ is continuous for $t \geq T_0$, Theorem 2.1 may be applied to

equation (*) and since $P \in \mathcal{P}_3$, Lemma 2.2 implies that if $(*) \in \mathcal{B}_1$, $1 = 2$ or 5 , then $(H) \in \mathcal{B}_1$, $1 = 2$ or 5 . It is therefore only necessary to study equation (*).

First form

$$\frac{1}{2}[B^*(t) + B(t)] = \begin{bmatrix} 0 & \frac{1}{2}(k^{-1} - kq(t)) \\ \frac{1}{2}(k^{-1} - kq(t)) & -p(t) \end{bmatrix}.$$

The characteristic roots of this matrix are

$$\lambda_1 = \frac{1}{2} \left\{ -p(t) \pm \sqrt{[p(t)]^2 + [k^{-1} - kq(t)]^2} \right\}, \quad 1 = 1, 2.$$

Hence,

$$\Lambda_B(t) \leq \frac{1}{2} \left\{ -p(t) + |p(t)| + k|k^{-2} - q(t)| \right\} = -p^-(t) + \frac{1}{2}k|k^{-2} - q(t)|,$$

$$\lambda_B(t) \geq \frac{1}{2} \left\{ -p(t) - |p(t)| - k|k^{-2} - q(t)| \right\} = -p^+(t) - \frac{1}{2}k|k^{-2} - q(t)|.$$

If $k = c^{-\frac{1}{2}}$ (i.e. $k^{-2} = c$) and the hypotheses are satisfied, then

$$\int_{t_0}^{\infty} |k^{-2} - q(t)| dt < \infty. \text{ Further, } \int_{t_0}^t |k^{-2} - q(s)| ds \leq \int_{t_0}^t |k^{-2} - q(s)| ds,$$

$$\int_{t_0}^t p^+(s) ds \leq \int_{t_0}^t p^+(s) ds, \text{ and } \int_{t_0}^t p^-(s) ds \geq \int_{t_0}^t p^-(s) ds \text{ for } t \geq t_0 \geq T_0.$$

If $\int_{t_0}^{\infty} p^-(t) dt > -\infty$, then $\int_{t_0}^t \Lambda_B(s) ds \leq M$ for $t \geq t_0 \geq T_0$ and by

Corollary 1 of Theorem 2.1 the solutions of (*) are uniformly locally bounded; if $\int_{t_0}^{\infty} p^+(t) dt < \infty$, then $\int_{t_0}^t \lambda_B(s) ds \geq m$ for $t \geq t_0 \geq T_0$ and by Corollary 4 of Theorem 2.1 the norms of the nonzero solutions of (*) are bounded away from zero as $t \rightarrow \infty$. Q.E.D.

Remarks on Theorem 2.3. Included in Theorem 2.3 is the known result: if $\int_{t_0}^{\infty} |c - q(t)| dt < \infty$ and $\int_{t_0}^{\infty} |d - p(t)| dt < \infty$ ($c > 0$, $d \geq 0$ are constants), then the solutions of (2.3) are bounded (reference 9, p. 36).

If k is negative or complex, it appears in the expressions for $\Lambda(t)$ and $\lambda(t)$ as $|k|$; hence k is assumed real and positive.

If Theorem 2.1 is applied directly to equation (2.3), i.e. let $k = 1$, the results are meager. For instance, with $p(t) = 3$ and $q(t) = 2$ for $t \geq T_0$, in which case (2.3) has constant coefficients and real, negative

characteristic roots, Theorem 2.1 gives no information when used directly. But a simple substitution transforms (2.3) into an equation to which Theorem 2.1 may be applied profitably.

Theorem 2.4. Let equation (2.3) be given and assume that $q'(t)$ exists and is continuous and that $0 < \bar{q} \leq q(t) \leq Q$, $\bar{q}$ and Q constant, for $t \geq T_0$. Let $F(t) = p(t) + q'(t)/2q(t)$. 1) If $\int_{T_0}^{\infty} F^-(t)dt > -\infty$, then $x=0$ is a uniformly stable solution of (2.3). 2) If $\int_{T_0}^{\infty} F^+(t)dt < +\infty$, then the norms of the nonzero solutions of (2.3) are bounded away from zero for all t .

Proof. Let $y = P^{-1}(t)x$ where $P(t) = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{q(t)} \end{bmatrix}$. This

substitution transforms (2.3) into

$$(*) \quad \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \left\{ \begin{bmatrix} 0 & \sqrt{q(t)} \\ -\sqrt{q(t)} & -p(t) \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & q'(t)/2q(t) \end{bmatrix} \right\} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

i.e. $y' = P^{-1}(t)(A(t)P(t) - P'(t))y = B(t)y.$

Again, $B(t)$ is continuous for $t \geq T_0$ and $P(t) \in \mathcal{P}_3$ and, as in the proof of Theorem 2.3, to prove the theorem it is sufficient to study equation (*).

Now $\frac{1}{2}(B^*(t) + B(t)) = \begin{bmatrix} 0 & 0 \\ 0 & -p(t) - q'(t)/2q(t) \end{bmatrix}$

and $\Lambda_B(t) = \sup (0, -p(t) - q'(t)/2q(t)) = -(p(t) + q'(t)/2q(t))^-$,

$\lambda_B(t) = \inf (0, -p(t) - q'(t)/2q(t)) = -(p(t) + q'(t)/2q(t))^+.$

Then $\int_{t_0}^t \Lambda_B(s)ds \leq - \int_{t_0}^t (p(s) + q'(s)/2q(s))^- ds$ for $t \geq t_0 \geq T_0$ and

(1) follows from Corollary 1 of Theorem 2.1. In a similar fashion, (2) follows from Corollary 4 of Theorem 2.1. Q.E.D.

Remarks on Theorem 2.4. This $P(t)$ was selected so that $\frac{1}{2}(B^*(t) + B(t))$ would be diagonal and thus make $\Lambda_B(t)$ and $\lambda_B(t)$ trivial to compute.

In part (1) of the theorem, if only $0 < q(t) \leq Q$ for $t \geq T_0$ holds, then $P(t) \in \mathcal{P}_1$. The same proof then shows that $x = 0$ is a stable

solution of (2.3).

Contained in Theorem 2.4 is the result of Starzinskii (reference 23): if equation (2.3) is given and if $0 < q(t) \leq Q$ and $p(t) > -\frac{1}{2}q'(t)/q(t)$ for $t \geq T_0$, then $x = 0$ is a stable solution of (2.3).

Starzinskii proved his result using Lyapunov functions and, as is mentioned in Section 1 and in the remarks on Theorem 4.7, the results of this section could also be proved by this method. It is necessary to use extended Lyapunov functions (see the remarks following Theorem 4.7) and in this case $|y| = |P^{-1}(t)x|$ can be considered as such a function.

The next theorem is used in Sections 3 and 4 to extend the results of Theorem 2.1 to equations (NH) and (NL).

Theorem 2.5. For $t_0 \geq T_0$, let $X(t)$ be the fundamental matrix of (H) such that $X(t_0)$ is the identity matrix and let $X^{-1}(t)$ be the inverse of $X(t)$. If $\Lambda_A(t) = \Lambda(t)$ and $\lambda_A(t) = \lambda(t)$, then for $t \geq t_0$

$$\begin{aligned} \exp \int_{t_0}^t \lambda(s) ds &\leq |X(t)| \leq \exp \int_{t_0}^t \Lambda(s) ds \text{ and} \\ \exp \int_{t_0}^t -\Lambda(s) ds &\leq |X^{-1}(t)| \leq \exp \int_{t_0}^t -\lambda(s) ds. \end{aligned}$$

Proof. If $t_0 \geq T_0$, then $X(t)x_0$ is the solution of equation (H) such that $X(t_0)x_0 = x_0$. By Theorem 2.1,

$$|x_0| \exp \int_{t_0}^t \lambda(s) ds \leq |X(t)x_0| \leq |x_0| \exp \int_{t_0}^t \Lambda(s) ds$$

and since $|X(t)| = \sup_{|x_0|=1} |X(t)x_0|$,

$$\exp \int_{t_0}^t \lambda(s) ds \leq |X(t)| \leq \exp \int_{t_0}^t \Lambda(s) ds.$$

To prove the second part, note that

$$\begin{aligned} \Lambda_{-A^*}(t) &= -\lambda_{A^*}(t) = -\lambda_A(t) \text{ and} \\ \lambda_{-A^*}(t) &= -\Lambda_{A^*}(t) = -\Lambda_A(t); \end{aligned}$$

further, by Lemma 2.1, (4),

$$|[X^{-1}(t)]^*| = |X^{-1}(t)|.$$

Now $[X^{-1}(t)]^*$ is the fundamental matrix for the adjoint equation of
 (H) (i.e. $x' = -A^*(t)x$) such that $[X^{-1}(t_0)]^*$ is the identity matrix.

Then by the first part of the theorem

$$\exp \int_{t_0}^t \lambda_{-A^*}(s) ds \leq |[X^{-1}(t)]^*| \leq \exp \int_{t_0}^t \wedge_{-A^*}(s) ds$$

and hence

$$\exp \int_{t_0}^t -\wedge_A(s) ds \leq |X^{-1}(t)| \leq \exp \int_{t_0}^t -\lambda_A(s) ds. \quad \text{Q.E.D.}$$

3. The Nonhomogeneous Linear System

In this section the results of Section 2 will be used to describe the asymptotic behavior of the solutions of the nonhomogeneous linear system.

Consider the system of differential equations

$$(NH) \quad x' = A(t) + f(t) \quad : t \geq T_c ,$$

as defined in Section 2. Let $\Lambda(t)$ and $\lambda(t)$ be the supremum and infimum, respectively, of the characteristic roots of $\frac{1}{2}(A^*(t) + A(t))$.

Definition 3.1. The homogeneous equation associated with (NH) is the equation obtained by suppressing $f(t)$. Denote this equation by (H:nh).

Theorem 3.1. If $x(t; x_0, t_0)$ is the solution of (NH) such that $x(t_0; x_0, t_0) = x_0$, then for $T_0 \leq t_0 \leq t$

$$|x_0| \exp \int_{t_0}^t \lambda(s) ds - \int_{t_0}^t |f(s)| \exp \int_s^t \lambda(u) du ds \leq |x(t; x_0, t_0)| \leq \\ |x_0| \exp \int_{t_0}^t \Lambda(s) ds + \int_{t_0}^t |f(s)| \exp \int_s^t \Lambda(u) du ds .$$

Proof: For $t_0 \geq T_0$, let $X(t)$ be the fundamental matrix of (H:nh) such that $X(t_0)$ is the identity matrix. Then (see reference 9, p. 19),

$$(*) \quad x(t; x_0, t_0) = X(t)x_0 + \int_{t_0}^t X(t)X^{-1}(s)f(s) ds .$$

Now $X(t)X^{-1}(s)$ is the fundamental matrix of (H:nh) such that $X(s)X^{-1}(s)$ is the identity matrix. Using Theorem 2.5

$$|x(t; x_0, t_0)| \leq |X(t)||x_0| + \int_{t_0}^t |X(t)X^{-1}(s)||f(s)| ds \leq \\ |x_0| \exp \int_{t_0}^t \Lambda(s) ds + \int_{t_0}^t |f(s)| \exp \int_s^t \Lambda(u) du ds .$$

Starting again from (*) and using Theorem 2.5,

$$X^{-1}(t)x(t; x_0, t_0) = x_0 + \int_{t_0}^t X^{-1}(s)f(s) ds , \\ |X^{-1}(t)||x(t; x_0, t_0)| \geq |x_0 + \int_{t_0}^t X^{-1}(s)f(s) ds| \\ |x(t; x_0, t_0)| \exp \int_{t_0}^t -\lambda(s) ds \geq |x_0| - \left| \int_{t_0}^t X^{-1}(s)f(s) ds \right| \\ |x(t; x_0, t_0)| \exp \int_{t_0}^t -\lambda(s) ds \geq |x_0| - \left| \int_{t_0}^t X^{-1}(s)f(s) ds \right| \geq$$

$$\begin{aligned} & \geq |x_0| - \int_{t_0}^t |X^{-1}(s)| |f(s)| ds \\ |x(t; x_0, t_0)| \exp \int_{t_0}^t -\lambda(s) ds & \geq |x_0| - \int_{t_0}^t \exp \int_{t_0}^s -\lambda(u) du |f(s)| ds \\ |x(t; x_0, t_0)| & \geq |x_0| \exp \int_{t_0}^t \lambda(s) ds - \int_{t_0}^t |f(s)| \exp \int_s^t \lambda(u) du ds. \text{ Q.E.D.} \end{aligned}$$

Corollary. Let $K_1, 1 = 1, \dots, 7$, be constants. If $\int_{t_0}^t \Lambda(s) ds \leq K_1$ and $\int_{t_0}^t |f(s)| \exp \int_s^t \Lambda(u) du ds \leq K_2$ for $t \geq T_0$, then the solutions of equation (NH) are locally bounded. In particular

- 1) if $\int_{t_0}^{\infty} |f(t)| dt \leq K_3$ and $\int_s^t \Lambda(u) du \leq K_4$ for $t \geq s \geq T_0$; or
- 2) if $|f(t)| \leq K_5$ and $\int_{t_0}^t \exp \int_s^t \Lambda(u) du ds \leq K_6$ for $t \geq T_0$,

then the solutions of equation (NH) are locally bounded.

- 3) As a special case of (2), if $|f(t)| \leq K_5$ and $\Lambda(t) \leq -K_7 < 0$ for $t \geq T_0$, then

$$|x(t; x_0, t_0)| \leq (K_5/K_7) + (|x_0| - (K_5/K_7)) \exp(-K_7(t - t_0))$$

for $t \geq t_0 \geq T_0$ (The corollary to Theorem 3.3 gives a second version of this part (3).).

Proof: The proof follows directly from the theorem.

Remarks on Theorem 3.1 and its corollary. Bellman (reference 3) has shown that given $A(t)$, bounded and continuous for $t \geq T_0$, equation (NH) has all its solutions bounded for any $f(t)$ such that $\int_{t_0}^{\infty} |f(t)| dt < \infty$ if and only if $|X(t)X^{-1}(s)| \leq c$ for some positive constant c and all $t \geq s \geq T_0$ (Here $X(t)$ is any fundamental matrix of (H:nh).). Since $|X(t)X^{-1}(s)| \leq \exp \int_s^t \Lambda(u) du$, the condition of (1) of the corollary ($\int_s^t \Lambda(u) du \leq K$ for all $t \geq s \geq T_0$) implies that $|X(t)X^{-1}(s)| \leq e^K$ for $t \geq s \geq T_0$. The converse is not true as is shown in Example 3.1.

Perron (and Bellman, references 3 and 21) has proved that given an $A(t)$, bounded and continuous for $t \geq T_0$, (NH) has all its solutions bounded for any $f(t)$ which is bounded and continuous for $t \geq T_0$ if and only if $x = 0$ is an exponentially stable solution of (H:nh). The

converse is not true as is shown in Example 3.1.

There is a specialization of the Perron theorem by Bridgland (reference 6). Let $X(t)$ be the fundamental matrix of $(H:nh)$ such that $X(t_0)$ is the identity matrix; let $Y(t, t_0) = \int_{t_0}^t X(t)X^{-1}(s) ds$; and let $x(t; x_0, t_0)$ be the solution of (NH) such that $x(t_0; x_0, t_0) = x_0$. A vector or matrix function converges if it approaches a constant vector or matrix element-wise. The theorem states: given $A(t)$, bounded and continuous for $t \geq T_0$, every solution of (NH) converges for every convergent $f(t)$ if and only if a) every solution of (NH) is bounded for every bounded $f(t)$ and b) the matrix $Y(t, T_0)$ converges. Moreover,

$$\lim_{t \rightarrow \infty} x(t; x_0, t_0) = \left(\lim_{t \rightarrow \infty} Y(t, t_0) \right) \left(\lim_{t \rightarrow \infty} f(t) \right),$$

If, in addition to the hypotheses of Bridgland's theorem,

$\Lambda(t) \leq -K < 0$ for $t \geq T_0$, then

$$|X(t)X^{-1}(s)| \leq \exp \int_s^t \Lambda(u) du \leq \exp(K(s - t)) \text{ and}$$

$$|Y(t, t_0)| \leq \int_{t_0}^t |X(t)X^{-1}(s)| ds \leq \int_{t_0}^t \exp(K(s - t)) ds = \frac{1}{K}(1 - \exp(K(t_0 - t))).$$

Taking norms in Bridgland's result,

$$\left| \lim_{t \rightarrow \infty} x(t; x_0, t_0) \right| \leq \left| \lim_{t \rightarrow \infty} Y(t, t_0) \right| \left| \lim_{t \rightarrow \infty} f(t) \right| \leq K^{-1} \left| \lim_{t \rightarrow \infty} f(t) \right|.$$

If $|f(t)| \leq \left| \lim_{t \rightarrow \infty} f(t) \right|$ for $t \geq T_0$, this is the bound given by (3)

of the corollary of Theorem 3.1; otherwise, this bound will be obtained in the corollary to Theorem 3.3.

Compared with these earlier results, Theorem 3.1 and its corollary have three disadvantages: 1) they give only sufficient conditions; 2) in two instances, they require a stronger hypothesis; 3) they do not detect the convergence of a solution. But the theorem and its corollary also have three advantages: 1) they do not require that $A(t)$ and $f(t)$ be bounded; 2) they may be applied to equations (even with bounded $A(t)$ and $f(t)$) to which the theorems of Perron and Bellman

do not apply (see Example 3.4); and 3) they give bounds for the norms of the solutions even if $f(t)$ does not converge.

Example 3.1(Illustrating two preceding remarks).

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : t \geq T_0 = 0.$$

A fundamental matrix is $X(t) = \begin{bmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{bmatrix}$ and

$$X(t)X^{-1}(s) = \begin{bmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{bmatrix} \begin{bmatrix} 2e^s & e^s \\ -e^{2s} & -e^{2s} \end{bmatrix} = \begin{bmatrix} 2e^{s-t}e^{-2s-2t} & e^{s-t}e^{-2s-2t} \\ -2e^{s-t}e^{-2s-2t} & -e^{s-t}e^{-2s-2t} \end{bmatrix}.$$

The vector $x = 0$ is an exponentially stable solution of this equation and $|X(t)X^{-1}(s)| \leq c < \infty$ for $t \geq s \geq 0$, but

$$\frac{1}{2}(A^* + A) = \begin{bmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & -3 \end{bmatrix}, \quad \Lambda(t) = \frac{1}{2}(\sqrt{10} - 3) \text{ for } t \geq 0, \text{ and } \int_0^\infty \Lambda(t)dt = \infty.$$

Next, by restricting $f(t)$, Theorem 3.1 will be extended.

Lemma 3.1(Reference 15, p. 496). Let $u(t)$ and $v(t)$ be defined, real valued, and continuous for $T_0 \leq t_0 \leq t \leq t_1 \leq \infty$; let $v(t) \geq 0$ and c be constant. If

$$u(t) \leq c + \int_{t_0}^t u(s)v(s)ds, \text{ then}$$

$$u(t) \leq c \exp \int_{t_0}^t v(s)ds : t_0 \leq t \leq t_1.$$

Proof: Let $x(t) = u(t) + w(t) = c + \int_{t_0}^t u(s)v(s)ds$, where $w(t)$ is defined, continuous, and non-negative for $t_0 \leq t \leq t_1$. The integral equation

$$x(t) = c + \int_{t_0}^t (x(s) - w(s))v(s)ds$$

is equivalent to the differential equation and initial condition

$$x' - v(t)x(t) = -w(t)v(t) : x(t_0) = c.$$

The solution to the differential equation is

$$x(t) = c \exp \int_{t_0}^t v(s)ds - \int_{t_0}^t v(s)w(s) \exp \int_s^t v(u)du ds.$$

Since $w(t) \geq 0$ and $v(s)w(s)\exp\int_s^t v(u)du \geq 0$,

$$u(t) = x(t) - w(t) \leq x(t) \leq c \exp\int_{t_0}^t v(s)ds. \text{ Q.E.D.}$$

Theorem 3.2. Let equation(NH) be given and let $x(t; x_0, t_0)$ be the solution such that $x(t_0; x_0, t_0) = x_0$. Assume that $f(t)$ is such that there exists a $G(t)$, defined, bounded, non-negative, and continuous a.e. for $t \geq T_0$, with $|f(t)| = G(t)|\wedge(t)|^{\frac{1}{2}}$. If $G(t) \leq |x(t; x_0, t_0)|$ for $T_0 \leq t_0 \leq t \leq t_1 \leq \infty$, then

$$|x(t; x_0, t_0)| \leq |x_0| \exp\int_{t_0}^t (|\wedge(s)| + \wedge(s)) ds.$$

If, in addition, $\wedge(t) \leq 0$ for $t_0 \leq t \leq t_1$, then $|x(t; x_0, t_0)|$ is monotonic non-increasing for $t_0 \leq t \leq t_1$.

Proof: Under the hypotheses of this theorem, Theorem 3.1 gives

$$|x(t; x_0, t_0)| \leq |x_0| \exp\int_{t_0}^t \wedge(s)ds + \int_{t_0}^t G(s)|\wedge(s)| \exp\int_s^t \wedge(u)du ds,$$

$$|x(t; x_0, t_0)| \exp\int_{t_0}^t -\wedge(s)ds \leq |x_0| + \int_{t_0}^t G(s)|\wedge(s)| \exp\int_s^t -\wedge(u)du ds.$$

If $G(t) \leq |x(t; x_0, t_0)|$ for $t_0 \leq t \leq t_1$, then

$$|x(t; x_0, t_0)| \exp\int_{t_0}^t -\wedge(s)ds \leq |x_0| + \int_{t_0}^t |\wedge(s)| |x(s; x_0, t_0)| \exp\int_s^t -\wedge(u)du ds$$

and by Lemma 3.1,

$$|x(t; x_0, t_0)| \exp\int_{t_0}^t -\wedge(s)ds \leq |x_0| \exp\int_{t_0}^t |\wedge(s)| ds \text{ for } t_0 \leq t \leq t_1.$$

Then $|x(t; x_0, t_0)| \leq |x_0| \exp\int_{t_0}^t (|\wedge(s)| + \wedge(s)) ds$ for $t_0 \leq t \leq t_1$.

If $\wedge(t) \leq 0$ for $t_0 \leq t \leq t_1$, consider t_2 and t_3 , $t_0 \leq t_2 < t_3 \leq t_1$.

Using the first part of the theorem, with $(x(t_2; x_0, t_0), t_2)$ as an initial

¹The existence of such a $G(t)$ implies certain restrictions on $f(t)$, e.g. if $\wedge(t) = 0$ for $t = \bar{t}$, then $f(\bar{t}) = 0$. It is not necessary to explicitly define $G(t)$ in either the statement or proof of the theorem; in a simple situation this definition would suffice: assume the zeros of $\wedge(t)$ are either isolated (i.e. if $\wedge(\bar{t}) = 0$, there exists a neighborhood of $\bar{t}$ in which $\wedge(t) \neq 0$) or form closed intervals; if $\wedge(t) \neq 0$, let $G(t) = |f(t)|/|\wedge(t)|$; if $\bar{t}$ is an isolated zero of $\wedge(t)$ and $\lim_{t \rightarrow \bar{t}^-} G(t) = L_1$ and $\lim_{t \rightarrow \bar{t}^+} G(t) = L_2$ exist, let $G(\bar{t}) = \min(L_1, L_2)$; if $\wedge(t) = 0$ for $t_1 \leq t \leq t_2$ and $\lim_{t \rightarrow t_1^-} G(t) = L_1$ and $\lim_{t \rightarrow t_2^+} G(t) = L_2$ exist, let $G(t) = \min(L_1, L_2)$ for $t_1 \leq t \leq t_2$. Here $f(t)$ is restricted so that the limits exist.

point,

$$|x(t_3; x_0, t_0)| \leq |x(t_2; x_0, t_0)| \exp \int_{t_2}^{t_3} (|\Lambda(s)| + \Lambda(s)) ds = |x(t_2; x_0, t_0)|.$$

Q.E.D.

Theorem 3.3. Given equation (NH), let $x(t; x_0, t_0)$ be the solution such that $x(t_0; x_0, t_0) = x_0$. Assume $\Lambda(t) \leq 0$ for $t \geq T_0$ and assume $f(t)$ is such that there exists a $G(t)$, defined, non-negative, bounded, and continuous a.e. for $t \geq T_0$, with $|f(t)| = -G(t)\Lambda(t)^1$; let $H(t_1, t_2) = \sup\{G(t) : T_0 \leq t_1 \leq t \leq t_2 \leq \infty\}$.

- 1) For $T_0 \leq t_0 \leq t$, $|x(t; x_0, t_0)| \leq (|x_0| - H(t_0, t)) \exp \int_{t_0}^t \Lambda(s) ds + H(t_0, t)$.
- 2) If $|x_0| \leq H(t_0, t_0) = G(t_0)$, then $|x(t; x_0, t_0)| \leq H(t_0, t)$ for $t \geq t_0$.
- 3) If $H(t_0, t_0) < |x_0| < H(t_0, \infty)$, then there exists a T , $t_0 < T < \infty$, such that $|x(t; x_0, t_0)| \leq H(t_0, t)$ for all $t \geq T$.
- 4) If $H(t_0, \infty) \leq |x_0|$, then there exists a T , $t_0 \leq T \leq \infty$, such that $|x(t; x_0, t_0)|$ is greater than $G(t)$ and non-increasing for $t_0 \leq t \leq T$ and $|x(t; x_0, t_0)| \leq H(T, t)$ for $t \geq T$ (if $T < \infty$).
- 5) If $\int_{t_0}^{\infty} \Lambda(t) dt = -\infty$, then $\lim_{t \rightarrow \infty} \sup |x(t; x_0, t_0)| \leq H(t_0, \infty)$.
- 6) If $\int_{t_0}^{\infty} \Lambda(t) dt = -\infty$ and $\lim_{t \rightarrow \infty} \sup G(t) = U$, a non-negative constant, then $\lim_{t \rightarrow \infty} \sup |x(t; x_0, t_0)| \leq U$.

Proof: 1) Under the hypotheses on $f(t)$, Theorem 3.1 gives

$$|x(t; x_0, t_0)| \leq |x_0| \exp \int_{t_0}^t \Lambda(s) ds + \int_{t_0}^t G(s) (-\Lambda(s)) \exp \int_s^t \Lambda(u) du ds.$$

Since $(-\Lambda(s)) \exp \int_s^t \Lambda(u) du \geq 0$,

$$|x(t; x_0, t_0)| \leq |x_0| \exp \int_{t_0}^t \Lambda(s) ds + H(t_0, t) \int_{t_0}^t (-\Lambda(s)) \exp \int_s^t \Lambda(u) du ds,$$

$$|x(t; x_0, t_0)| \leq (|x_0| - H(t_0, t)) \exp \int_{t_0}^t \Lambda(s) ds + H(t_0, t) \quad \text{for } t \geq t_0 \geq T_0.$$

- 2) For $t \geq t_0 \geq T_0$, $H(t_0, t)$ is a monotonic non-decreasing function

¹See the footnote on page 33.

of t . If $|x_0| \leq H(t_0, t_0)$, then $|x_0| \leq H(t_0, t)$, or $|x_0| - H(t_0, t) \leq 0$, for $t \geq t_0$ and, from (1), $|x(t; x_0, t_0)| \leq H(t_0, t)$ for $t \geq t_0$.

3) If $H(t_0, t_0) < |x_0| < H(t_0, \infty)$, then there exists a $T > t_0$ such that $|x_0| \leq H(t_0, t)$ for $t \geq T$; (3) now follows from (1).

4) If $|x(t; x_0, t_0)| > G(t)$ for $t_0 \leq t \leq T \leq \infty$, then by Theorem 3.2, $|x(t; x_0, t_0)|$ is monotonic non-increasing for $t_0 \leq t \leq T$. If $|\bar{x}| = |x(T; x_0, t_0)| = G(T) = H(T, T)$ for $T, t_0 \leq T < \infty$, then by (2) of this theorem, $|x(t; \bar{x}, T)| = |x(t; x_0, t_0)| \leq H(T, t)$ for $t \geq T$.

5) If $\int_{t_0}^{\infty} \Lambda(t) dt = -\infty$, then

$$\lim_{t \rightarrow \infty} \sup |x(t; x_0, t_0)| \leq \lim_{t \rightarrow \infty} \left(\left\{ |x_0| - H(t_0, t) \right\} \exp \int_{t_0}^t \Lambda(s) ds + H(t_0, t) \right) = H(t_0, \infty).$$

6) Let $x_0, t_0 \geq T_0$, and ϵ be given. Under the hypotheses, there exists a $T_1(\epsilon) \geq T_0$ such that for $t \geq T_1$, $G(t) < U + \frac{1}{2}\epsilon$; hence $H(T_1, \infty) \leq U + \frac{1}{2}\epsilon$. Let $x_1 = x(T_1; x_0, t_0)$ and apply (4) to $x(t; x_1, T_1) = x(t; x_0, t_0)$. There exists a $T_2(\epsilon, x_1, T_1) = T_2(\epsilon, x_0, t_0) \geq T_1$ such that for $t \geq T_2$, $|x(t; x_1, T_1)| < H(T_1, \infty) + \frac{1}{2}\epsilon \leq U + \epsilon$, i.e., $\lim_{t \rightarrow \infty} \sup |x(t; x_0, t_0)| \leq U$. Q.E.D.

Corollary. If $\lim_{t \rightarrow \infty} \sup \Lambda(t) = -c < 0$ and $\lim_{t \rightarrow \infty} \sup |f(t)| = L \geq 0$, then $\lim_{t \rightarrow \infty} \sup |x(t; x_0, t_0)| \leq L/c$.

Proof. If $\lim_{t \rightarrow \infty} \sup \Lambda(t) = -c < 0$, then $\int_{T_0}^{\infty} \Lambda(t) dt = -\infty$. If, additionally, $\lim_{t \rightarrow \infty} \sup |f(t)| = L \geq 0$, then for $\epsilon > 0$ choose $0 < \epsilon_1 < 1$ and $\epsilon_2 > 1$ such that $(\epsilon_2/\epsilon_1 - 1) < \epsilon c/L$, i.e.

$\epsilon_2 L / \epsilon_1 c < L/c + \epsilon$. There exists a $T(\epsilon_1, \epsilon_2) = T(\epsilon) \geq T_0$ such that for $t > T$, $\Lambda(t) \leq -\epsilon_1 c$ and $|f(t)| \leq \epsilon_2 L$; then $G(t) = |f(t)| / -\Lambda(t)$ is defined for $t \geq T$ and $G(t) \leq \epsilon_2 L / \epsilon_1 c < L/c + \epsilon$ for $t \geq T$, i.e.

$\lim_{t \rightarrow \infty} \sup G(t) \leq L/c$. The corollary now follows from (6) of the theorem.

Example 3.2. (In this equation, Theorem 3.1 is used to determine

the boundedness of the solutions; it is an example to which Theorem 3.3 does not apply.)

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -t & t \sin t \\ 0 & -t - t \cos t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} : t \geq T_0 = 0.$$

Let $t_0 = 0$, then the solution is

$$x_1(t; x_1^0, 0) = x_1^0 \exp(-t^2/2) + \int_0^t (u \sin u \times x_2(u; x_2^0, 0) + 1) \exp(-\frac{1}{2}(t^2 - u^2)) du,$$

$$x_2(t; x_2^0, 0) = x_2^0 \exp(-t^2/2 - t \sin t - \cos t + 1);$$

as $t \rightarrow \infty$, $x_1(t; x_1^0, 0)$ and $x_2(t; x_2^0, 0)$ are bounded.

Now $\frac{1}{2}(A^*(t) + A(t)) = \begin{bmatrix} -t & \frac{1}{2}t \sin t \\ \frac{1}{2}t \sin t & -t - t \cos t \end{bmatrix}$ and the

characteristic roots of this matrix are given by

$$\lambda^2 + \lambda(2t + t \cos t) + (t^2 + t^2 \cos t - t^2 \sin^2 t/4) = 0;$$

$$\lambda_1 = \frac{1}{2}(-(2t + t \cos t) \pm \sqrt{t^2 \cos^2 t + t^2 \sin^2 t}), \quad i = 1, 2;$$

$$\lambda_1 = \frac{1}{2}(-2t - t \cos t \pm t), \quad i = 1, 2.$$

Then $\Lambda(t) = \frac{1}{2}(-t - t \cos t)$.

Since $|f(t)|/(-\Lambda(t)) = 2/(t + t \cos t)$ is unbounded for $t = (2k + 1)\pi$, $k = 0, 1, \dots$, Theorem 3.3 will not apply; instead use Theorem 3.1.

First, $\int_0^\infty \Lambda(t) dt = -\infty$; second, for $t \geq 2$, $\Lambda(t) \leq -1 - \cos t$ and $\int_0^t |f(s)| \exp \int_s^t \Lambda(u) du ds \leq \int_0^2 \exp \int_s^t \Lambda(u) du ds + \int_2^t \exp \int_s^t (-1 - \cos u) du ds \leq 2 + \int_2^t e^{s-t} \times e^{\sin s - \sin t} ds \leq 2 + e^2(1 - e^{-t})$. Using Theorem 3.1, $|x(t; x_0, t_0)| \leq |x_0| \exp \int_{t_0}^t \Lambda(s) ds + \int_{t_0}^t |f(s)| \exp \int_s^t \Lambda(u) du ds < \infty$ as $t \rightarrow \infty$.

Example 3.3. (illustrating Theorem 3.3, (2) - (6)).

$$dx/dt + x = \sin t + \cos t : t \geq T_0 = 0.$$

Let $t_0 = 0$, then $x(t; x_0, 0) = x_0 e^{-t} + \sin t$. Here $\Lambda(t) = -1$ and $G(t) = |\sin t + \cos t|$; $H(0, 0) = 1$ and $H(0, \pi/4) = H(0, \infty) = \sqrt{2}$.

2) If $|x_0| \leq H(0, 0) = 1$, then: for $0 \leq t \leq .9$, $e^{-t} \leq \cos t$ and $|x(t; x_0, t_0)| \leq |\cos t + \sin t| = G(t)$; for $.9 \leq t$, $e^{-t} \leq .4$ and

$$|x(t; x_0, t_0)| \leq .4 + 1 < \sqrt{2} = H(0, t).$$

3) If $H(0, 0) = 1 < |x_0| < \sqrt{2} = H(0, \infty)$ and $t \geq T = 1.24$, then $|x_0|e^{-t} < \sqrt{2} e^{-t} < .4$ and $|x(t; x_0, t_0)| \leq .4 + 1 < \sqrt{2} = H(0, t)$ for $t \geq T$.

4) Assume $x_0 > H(0, \infty) = \sqrt{2}$: if $x(t; x_0, 0) \geq G(t) = |\sin t + \cos t| \geq \sin t + \cos t$ for $0 \leq t \leq T$ and $x(T; x_0, 0) = G(T)$, then

i) $x(t; x_0, 0) \geq 0$, $x(t; x_0, 0) = x_0 e^{-t} + \sin t \geq \sin t + \cos t$ or $x'(t; x_0, 0) = -x_0 e^{-t} + \cos t \leq 0$, and hence $|x(t; x_0, t_0)|$ is monotonic non-increasing for $0 \leq t \leq T$; and

ii) for $t \geq T$, (2) of Theorem 3.3 applies.

Assume $x_0 < -H(0, \infty) = -\sqrt{2}$: if $x(t; x_0, 0) \leq -G(t) = -|\sin t + \cos t| \leq \sin t + \cos t$ for $0 \leq t \leq T$ and $x(T; x_0, 0) = G(T)$, then

i) $x(t; x_0, 0) \leq 0$, $x(t; x_0, 0) = x_0 e^{-t} + \sin t \leq \sin t + \cos t$, or $x'(t; x_0, 0) = -x_0 e^{-t} + \cos t \geq 0$, and hence $|x(t; x_0, t_0)|$ is monotonic non-increasing for $0 \leq t \leq T$; and

ii) for $t \geq T$, (2) of Theorem 3.3 applies.

5 and 6) Since $\int_{T_0}^{\infty} \Lambda(t) dt = -\infty$, $H(0, \infty) = \sqrt{2}$, and $\lim_{t \rightarrow \infty} \sup G(t) = \sqrt{2}$, $\lim_{t \rightarrow \infty} \sup |x(t; x_0, t_0)| \leq \sqrt{2}$ for all $t_0 \geq T_0$ and x_0 .

Example 3.4. (An example where the theorems of Perron and Bellman do not apply, but where Theorem 3.3 does apply.)

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -1 & t^{-2} \\ 0 & -t^{-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ t^{-1} \end{bmatrix} : t \geq T_0 = 1.$$

The solution is

$$\begin{aligned} x_1(t; x_1^0, t_0) &= x_1^0 e^{-(t-t_0)} + \int_{t_0}^t u^{-2} e^{-t+u} x_2(u; x_2^0, t_0) du, \\ x_2(t; x_2^0, t_0) &= (x_2^0 - 1)t_0 t^{-1} + 1. \end{aligned}$$

Since $x = 0$ is not an exponentially stable solution of the homogeneous equation associated with this equation, the theorem of Perron does not apply; since $\int_{T_0}^{\infty} f(t) dt = \int_1^{\infty} t^{-1} dt = \infty$, the theorem of Bellman does

not apply.

Now $\frac{1}{2}(A^*(t) + A(t)) = \begin{bmatrix} -1 & \frac{1}{2}t^{-2} \\ \frac{1}{2}t^{-2} & -t^{-1} \end{bmatrix}$ and the characteristic

roots of this matrix are

$$\lambda_1 = \frac{1}{2}(-(1 + t^{-1}) \pm \sqrt{(1 - t^{-1})^2 + t^{-4}}), \quad i = 1, 2.$$

Then $\Lambda(t) \leq \frac{1}{2}(-(1 + t^{-1}) + (1 - t^{-1}) + t^{-2}) = -t^{-1} + \frac{1}{2}t^{-2}$ and

$G(t) = |f(t)| / -\Lambda(t)$ and $t^{-1}/(t^{-1} - \frac{1}{2}t^{-2}) = 1/(1 - (2t)^{-1})$. Since $\lim_{t \rightarrow \infty} \sup G(t) = 1$ and $\int_0^\infty \Lambda(t) dt = -\infty$, $\lim_{t \rightarrow \infty} \sup |x(t; x_0, t_0)| \leq 1$ for all $t_0 \geq T_0$ and x_0 .

Starting again from Theorem 3.1, the preceding work may be paraphrased to develop lower bounds for the solutions of (NH).

Lemma 3.2. Let $u(t)$ and $v(t)$ be defined, real-valued, and continuous for $t \geq T_0$; assume $v(t) \geq 0$. If

$$u(t) \geq u(t_0) - \int_{t_0}^t u(s)v(s)ds \quad \text{for } t \geq t_0 \geq T_0,$$

then

$$u(t) \geq u(t_0) \exp \int_{t_0}^t -v(s)ds.$$

Proof. Let $u(t)$, satisfying the hypotheses, and $t_0 \geq T_0$ be given. Let $\hat{u}(t)$ be the solution of

$$\begin{aligned} \hat{u}(t) &= u(t_0) - \int_{t_0}^t \hat{u}(s)v(s)ds, \quad \text{or} \\ \hat{u}'(t) &= -\hat{u}(t)v(t) : \quad \hat{u}(t_0) = u(t_0). \end{aligned}$$

The solution of this differential equation is defined and unique for $t \geq t_0$;

$$\hat{u}(t) = u(t_0) \exp \int_{t_0}^t -v(s)ds.$$

If $u(t) \geq \hat{u}(t)$ for $t \geq t_0$, the lemma is proved; assume there exist t_1 and t_2 , $t_0 \leq t_1 < t_2 < \infty$, such that $u(t_1) = \hat{u}(t_1)$ and $u(t) < \hat{u}(t)$ for $t_1 < t \leq t_2$. Then

$$u(t_2) - \hat{u}(t_2) \geq (u(t_1) - \int_{t_1}^{t_2} u(s)v(s)ds) - (\hat{u}(t_1) - \int_{t_1}^{t_2} \hat{u}(s)v(s)ds);$$

this follows from the hypotheses on $u(t)$ and by writing $\hat{u}(t)$ in its

integral form. Then

$$u(t_2) - \hat{u}(t_2) \geq \int_{t_1}^{t_2} (\hat{u}(s) - u(s))v(s)ds \geq 0$$

since $\hat{u}(t) \geq u(t)$ and $v(t) \geq 0$ for $t_1 \leq t \leq t_2$; this contradicts $u(t_2) < \hat{u}(t_2)$. Q.E.D.

Theorem 3.4. Given equation (NH), let $x(t; x_0, t_0)$ be the solution such that $x(t_0; x_0, t_0) = x_0$. Assume $f(t)$ is such that there exists a $g(t)$, defined, non-negative, bounded, and continuous a.e. for $t \geq T_0$, with $|f(t)| = g(t)|\lambda(t)|^1$. If $g(t) \leq |x(t; x_0, t_0)|$ for $T_0 \leq t_0 \leq t \leq t_1 < \infty$, then $|x(t; x_0, t_0)| \geq |x_0| \exp \int_{t_0}^t (\lambda(s) - |\lambda(s)|)ds$. If, in addition, $\lambda(t) \geq 0$ for $t_0 \leq t \leq t_1$, then $|x(t; x_0, t_0)|$ is monotonic non-decreasing for $t_0 \leq t \leq t_1$.

Proof. (The proof is similar to the proof of Theorem 3.2.)

If $f(t)$ satisfies the hypotheses, then Theorem 3.1 gives

$$|x(t; x_0, t_0)| \geq |x_0| \exp \int_{t_0}^t \lambda(s)ds - \int_{t_0}^t g(s)|\lambda(s)| \exp \int_s^t \lambda(u)du ds,$$

$$|x(t; x_0, t_0)| \exp \int_{t_0}^t -\lambda(s)ds \geq |x_0| - \int_{t_0}^t g(s)|\lambda(s)| \exp \int_{t_0}^s -\lambda(u)du ds.$$

If $g(t) \leq |x(t; x_0, t_0)|$ for $t_0 \leq t \leq t_1$, then

$$|x(t; x_0, t_0)| \exp \int_{t_0}^t -\lambda(s)ds \geq |x_0| - \int_{t_0}^t |\lambda(s)| |x(s; x_0, t_0)| \exp \int_{t_0}^s -\lambda(u)du ds$$

and, using Lemma 3.3,

$$|x(t; x_0, t_0)| \exp \int_{t_0}^t -\lambda(s)ds \geq |x_0| \exp \int_{t_0}^t -|\lambda(s)|ds$$

$$|x(t; x_0, t_0)| \geq |x_0| \exp \int_{t_0}^t (\lambda(s) - |\lambda(s)|)ds, \quad t_0 \leq t \leq t_1.$$

If, in addition, $\lambda(t) \geq 0$ for $t_0 \leq t \leq t_1$ and if $t_0 \leq t_2 < t_3 \leq t_1$, then, using the first part of the theorem with $(x(t_2; x_0, t_0), t_2)$ as an initial point,

$$|x(t_3; x_0, t_0)| \geq |x(t_2; x_0, t_0)| \exp \int_{t_2}^{t_3} (\lambda(s) - |\lambda(s)|)ds \geq |x(t_2; x_0, t_0)|. \text{ Q.E.D.}$$

Theorem 3.5. Given equation (NH), let $x(t; x_0, t_0)$ be the solution such that $x(t_0; x_0, t_0) = x_0$. Assume $\lambda(t) \geq 0$ for $t \geq T_0$ and assume $f(t)$ is such that there exists a $g(t)$, defined, non-negative, bounded,

¹The footnote of page 33 applies here too.

and continuous a.e. for $t \geq T_0$ with $|f(t)| = g(t) \lambda(t)$.¹ Let $h(t_1, t_2) = \sup(g(t): T_0 \leq t_1 \leq t \leq t_2 \leq \infty)$.

1) For $T_0 \leq t_0 \leq t$, $|x(t; x_0, t_0)| \geq (|x_0| - h(t_0, t)) \exp \int_{t_0}^t \lambda(s) ds + h(t_0, t)$.

2) If $t_0 \geq T_0$ and $|x_0| \geq h(t_0, \infty)$, then $|x(t; x_0, t_0)| \geq h(t_0, t)$ and $|x(t; x_0, t_0)|$ is monotonic non-decreasing for $t \geq t_0$.

3) If $t_0 \geq T_0$, $|x_0| > h(t_0, \infty)$, and $\int_{t_0}^{\infty} \lambda(t) dt = \infty$, then $\lim_{t \rightarrow \infty} |x(t; x_0, t_0)| = \infty$.

4) If $\limsup_{t \rightarrow \infty} g(t) < u$, u a positive constant, then there exists a $T(u) \geq T_0$ such that for $t \geq t_0 \geq T$, $h(t_0, t) \leq u$ (then (2) and (3) may be applied to $\{(x, t): |x| \geq u \text{ and } t \geq T\}$).

5) Under the hypotheses of (4), if $T_0 \leq \bar{t} < T$, then there exists a $k(u, \bar{t}) \geq u$ such that if $|\bar{x}| \geq k$, then $|x(t; \bar{x}, \bar{t})| \geq u$ for $\bar{t} \leq t \leq T$ (and (3) may be applied for $t \geq T$).

Proof. (The proof is similar to the proof of Theorem 3.3).

1) If $f(t)$ satisfies the hypotheses, then Theorem 3.1 gives

$$|x(t; x_0, t_0)| \geq |x_0| \exp \int_{t_0}^t \lambda(s) ds - \int_{t_0}^t g(s) \lambda(s) \exp \int_s^t \lambda(u) du ds.$$

Since $\lambda(s) \exp \int_s^t \lambda(u) du \geq 0$,

$$|x(t; x_0, t_0)| \geq |x_0| \exp \int_{t_0}^t \lambda(s) ds - h(t_0, t) \int_{t_0}^t \lambda(s) \exp \int_s^t \lambda(u) du ds,$$

$$|x(t; x_0, t_0)| \geq |x_0| \exp \int_{t_0}^t \lambda(s) ds + h(t_0, t) (1 - \exp \int_{t_0}^t \lambda(s) ds),$$

$$|x(t; x_0, t_0)| \geq (|x_0| - h(t_0, t_0)) \exp \int_{t_0}^t \lambda(s) ds + h(t_0, t).$$

2) For $t \geq t_0$, $h(t_0, t)$ is a monotonic non-decreasing function of t . Hence $|x_0| \geq h(t_0, \infty)$ implies $|x_0| \geq h(t_0, t)$ for $t \geq t_0$ and by (1) of this theorem $|x(t; x_0, t_0)| \geq h(t_0, t)$ for $t \geq t_0$. In this case, $|x(t; x_0, t_0)| \geq g(t)$ too and using Theorem 3.4, $|x(t; x_0, t_0)|$ is monotonic non-decreasing for $t \geq t_0$.

¹The footnote of page 33 applies here too.

- 3) If $t_0 \geq T_0$, $|x_0| > h(t_0, \infty)$, and $\int_{t_0}^{\infty} \lambda(t) dt = \infty$, then

$$\lim_{t \rightarrow \infty} |x(t; x_0, t_0)| = \lim_{t \rightarrow \infty} ((|x_0| - h(t_0, t)) \exp \int_{t_0}^t \lambda(s) ds + h(t_0, t)) = \infty.$$
- 4) If $\lim_{t \rightarrow \infty} \sup g(t) < u$, u a positive constant, then there exists a $T(u)$ such that $g(t) \leq u$ for $t \geq T$ and hence $h(t_0, t) \leq u$ for $t \geq t_0 \geq T$.
- 5) Assume the hypotheses of (4), let $T_0 \leq \bar{t} < T$, and $k(u, \bar{t}, T) = k(u, \bar{t}) = u + h(\bar{t}, T)$. Since $h(\bar{t}, t) \geq 0$ for $t \geq \bar{t}$ and $\lambda(t) \geq 0$,
 $|\bar{x}| \geq k$ implies $|\bar{x}| \geq \{(u - h(\bar{t}, t)) \exp \int_{\bar{t}}^t \lambda(s) ds + h(\bar{t}, t)\}$ for $\bar{t} \leq t \leq T$.
 This implies that $\{(|\bar{x}| - h(\bar{t}, t)) \exp \int_{\bar{t}}^t \lambda(s) ds + h(\bar{t}, t)\} \geq u$ for
 $\bar{t} \leq t \leq T$ and hence, by (1) of this theorem, $|x(t; \bar{x}, \bar{t})| \geq u$ for
 $\bar{t} \leq t \leq T$. Q.E.D.

4. The Nonlinear System

In this section the results of Section 2 will be used to describe the asymptotic behavior of the solutions of a nonlinear system.

Consider the system of differential equations

$$(NL) \quad x' = A(t)x + f(x, t) : (x, t) \in D,$$

as defined in Section 2. Let $\Lambda(t)$ and $\lambda(t)$ be the supremum and infimum, respectively, of the characteristic roots of $\frac{1}{2}(A^*(t) + A(t))$. The set D and the numbers r_0 and R_0 are the same as in Definition 2.1.

The first five theorems repeat the reasoning of Section 3 and they are, in fact, generalizations of the theorems of that section. The difference is that the solutions of equation (NL) may escape from the domain D . The following lemma is used in this connection.

Lemma 4.1. Let $x(t; x_0, t_0)$ be a solution of equation (NL) such that $x(t_0; x_0, t_0) = x_0$ and let $I(t_0) \subset [t_0, \infty)$ be the maximum interval such that 1) $t_0 \in I(t_0)$ and 2) if $t \in I(t_0)$, then $x(t; x_0, t_0) \in D$. If $x(t; x_0, t_0)$ is such that $t \in I(t_0)$ implies that $|x(t; x_0, t_0)| \leq r_1(t; x_0, t_0)$, where $r_1(t; x_0, t_0)$ is defined, continuous, and $r_1(t; x_0, t_0) < r_0$ for $t \geq t_0$, then $I(t_0) = [t_0, \infty)$ and $|x(t; x_0, t_0)| \leq r_1(t; x_0, t_0)$ for all $t \geq t_0$.

Proof. Consider $[t_0, \infty)$ with the usual topology. The set $I(t_0)$ is an open set and is not empty since $t_0 \in I(t_0)$. Let $J(t_0) \subset [t_0, \infty)$ be the maximum interval such that 1) $t_0 \in J(t_0)$ and 2) if $t \in J(t_0)$, then $|x(t; x_0, t_0)| \leq r_1(t)$. The set $J(t_0)$ is closed since $|x(t; x_0, t_0)|$ and $r_1(t)$ are continuous and $J(t_0) \subset I(t_0)$. If $x(t; x_0, t_0)$ is such that $t \in I(t_0)$ implies $|x(t; x_0, t_0)| \leq r_1(t)$, then $I(t_0) \subset J(t_0)$. In this case, $I(t_0) = J(t_0)$ and $I(t_0)$ is a non-empty set which is both open and closed, i.e. $I(t_0) = [t_0, \infty)$. Q.E.D.

Definition 4.1. The homogeneous equation associated with equation (NL)

is the equation obtained by suppressing $f(x, t)$. Denote this equation by $(H:n\ell)$.

Theorem 4.1. If $x(t; x_0, t_0)$ is a solution of equation (NL) such that $x(t_0; x_0, t_0) = x_0$ and if $x(t; x_0, t_0) \in D$ for $T_0 \leq t_0 \leq t \leq t_1 \leq \infty$, then $|x_0| \exp \int_{t_0}^t \lambda(s) ds - \int_{t_0}^t |f(x(s; x_0, t_0), s)| \exp \int_s^t \lambda(u) du ds \leq |x(t; x_0, t_0)| \leq |x_0| \exp \int_{t_0}^t \lambda(s) ds + \int_{t_0}^t |f(x(s; x_0, t_0), s)| \exp \int_s^t \lambda(u) du ds$ for $t_0 \leq t \leq t_1$.

Proof. Let $X(t)$ be the fundamental matrix of $(H:n\ell)$ such that $X(t_0)$ is the identity matrix. If $x(t; x_0, t_0) \in D$ for $t_0 \leq t \leq t_1$, then (see reference 9, p. 91)

$$x(t; x_0, t_0) = X(t)x_0 + \int_{t_0}^t X(t)X^{-1}(s)f(x(s; x_0, t_0), s)ds$$

The proof now proceeds the same as the proof of Theorem 3.1.

Corollary. Let equation (NL) be given and assume that $|f(x, t)| \leq F(t)$, where $F(t)$ is defined and continuous and non-negative for $t \geq T_0$, for all $(x, t) \in D$.

1) If $\int_{t_0}^t \lambda(s) ds \leq K_1(t_0) < \infty$ and $\int_{t_0}^t F(s) \exp \int_s^t \lambda(u) du ds \leq K_2(t_0) < r_0$ for all $t \geq t_0 \geq T_0$ and if $|x_0| < [\exp(-K_1(t_0))](r_0 - K_2(t_0))$, then $|x(t; x_0, t_0)| \leq |x_0| \exp[K_1(t_0)] + K_2(t_0) < r_0$ for $t \geq t_0$.

2) If $K_1(t_0) \leq K_3$ for all $t_0 \geq T_0$, $\int_{t_0}^{\infty} F(s) ds \leq K_4 < r_0 e^{-K_3}$, and $|x_0| < e^{-K_3} r_0 - K_4$, then $|x(t; x_0, t_0)| \leq e^{K_3}(|x_0| + K_4) < r_0$ for $t \geq t_0 \geq T_0$.

3) If $|F(t)| \leq K_5$ and $\lambda(t) \leq -K_6 < 0$, where $K_5/K_6 < r_0$, for $t \geq T_0$ and if $t_0 \geq T_0$, $|x_0| < r_0$, then

$$|x(t; x_0, t_0)| \leq \max(|x_0|, K_5/K_6) < r_0 \text{ for } t \geq t_0.$$

(The corollary to Theorem 4.3 improves this part (3).)

Proof. If $f(x, t)$ satisfies the hypotheses of the corollary, let $r_1(t; x_0, t_0) = |x_0| \exp \int_{t_0}^t \lambda(s) ds + \int_{t_0}^t F(s) \exp \int_s^t \lambda(u) du ds$; then

$r_1(t; x_0, t_0)$ is defined and continuous for $t \geq t_0$. If, additionally, $|x_0| < r_0$, then the interval $I(t_0)$, as defined in Lemma 4.1, exists and by Theorem 4.1, if $t \in I(t_0)$, then $|x(t; x_0, t_0)| \leq r_1(t; x_0, t_0)$. Therefore, if $r_1(t; x_0, t_0) < r_0$ for $t \geq t_0$, then $|x(t; x_0, t_0)| \leq r_1(t; x_0, t_0)$ for $t \geq t_0$.

1) If the hypotheses of (1) are satisfied, then $|x_0| < r_0$ and $r_1(t; x_0, t_0) \leq |x_0| e^{K_1(t-t_0)} + K_2(t_0) < r_0$ for $t \geq T_0$.

2) The proof of (2) is similar to the proof of (1).

3) If the hypotheses of (3) are satisfied, then $|x_0| < r_0$ and $r_1(t; x_0, t_0) \leq K_5/K_6 + e^{-K_6(t-t_0)}(|x_0| - K_5/K_6)$ for $t \geq t_0$. Further, $\{K_5/K_6 + e^{-K_6(t-t_0)}(|x_0| - K_5/K_6)\} \leq \max(|x_0|, K_5/K_6) < r_0$ for $t \geq t_0$. Q.E.D.

Theorem 4.2. Given equation (NL), let $x(t; x_0, t_0)$ be a solution such that $x(t_0; x_0, t_0) = x_0$. Assume $f(x, t)$ is such that there exists a $G(t)$, defined, non-negative, bounded, and continuous a.e. for $t \geq T_0$, with $|f(x, t)| \leq G(t)|\Lambda(t)|$ for all $(x, t) \in D$.¹ If $G(t) \leq |x(t; x_0, t_0)|$ and $x(t; x_0, t_0) \in D$ for $T_0 \leq t_0 \leq t \leq t_1 \leq \infty$, then

$$|x(t; x_0, t_0)| \leq |x_0| \exp \int_{t_0}^t \{ \Lambda(s) + |\Lambda(s)| \} ds.$$

If, in addition, $\Lambda(t) \leq 0$ for $t_0 \leq t \leq t_1$, then $|x(t; x_0, t_0)|$ is monotonic nonincreasing for $t_0 \leq t \leq t_1$.

Proof. With Theorem 4.1 used in place of Theorem 3.1, the proof is the same as the proof of Theorem 3.2.

Theorem 4.3. Given equation (NL), let $x(t; x_0, t_0)$ be a solution such that $x(t_0; x_0, t_0) = x_0$. Assume $\Lambda(t) \leq 0$ for $t \geq T_0$ and assume $f(x, t)$ is such that there exists a $G(t)$, defined, non-negative, bounded and continuous a.e. for $t \geq T_0$, with $|f(x, t)| \leq -G(t)\Lambda(t)$ for all $(x, t) \in D$ ¹; let $H(t_1, t_2) = \sup\{G(t): t_1 \leq t \leq t_2\}$.

¹The existence of such a $G(t)$ restricts $f(x, t)$ in a manner similar to that described in the footnote of page 33.

1) If $x(t; x_0, t_0) \in D$ for $T_0 \leq t_0 \leq t \leq t_1 \leq \infty$, then

$$|x(t; x_0, t_0)| \leq \left\{ |x_0| - H(t_0, t) \right\} \exp \int_{t_0}^t \Lambda(s) ds + H(t_0, t).$$

If $t_0 \geq T_0$, $|x_0| < r_0$, and $H(t_0, \infty) < r_0$, then $|x(t; x_0, t_0)| < r_0$ for $t \geq t_0$.

2) If $t_0 \geq T_0$, $|x_0| \leq H(t_0, t_0) = G(t_0)$, and $H(t_0, \infty) < r_0$, then $|x(t; x_0, t_0)| \leq H(t_0, t)$ for $t \geq t_0$.

3) If $t_0 \geq T_0$ and $H(t_0, t_0) < |x_0| < H(t_0, \infty) < r_0$, then there exists a T , $t_0 < T < \infty$, such that $|x(t; x_0, t_0)| \leq H(t_0, t)$ for $t \geq T$.

4) If $t_0 \geq T_0$ and $H(t_0, \infty) \leq |x_0| < r_0$, then there exists a T , $t_0 \leq T \leq \infty$, such that $|x(t; x_0, t_0)| \geq G(t)$ and $|x(t; x_0, t_0)|$ is monotonic nonincreasing for $t_0 \leq t \leq T$ and $|x(t; x_0, t_0)| \leq H(T, t)$ for $t \geq T$ (if $T < \infty$).

5) If $t_0 \geq T_0$, $|x_0| < r_0$, $H(t_0, \infty) < r_0$, and $\int_{t_0}^{\infty} \Lambda(t) dt = -\infty$, then $\lim_{t \rightarrow \infty} \sup |x(t; x_0, t_0)| \leq H(t_0, \infty)$.

6) If $t_0 \geq T_0$, $|x_0| < r_0$, $H(t_0, \infty) < r_0$, $\int_{t_0}^{\infty} \Lambda(t) dt = -\infty$, and $\lim_{t \rightarrow \infty} \sup G(t) = U$, a non-negative constant, then $\lim_{t \rightarrow \infty} \sup |x(t; x_0, t_0)| \leq U$.

Proof. 1) If $x(t; x_0, t_0) \in D$ for $T_0 \leq t_0 \leq t \leq t_1 \leq \infty$, the proof of the first part of (1) is the same as the proof of Theorem 3.3, (1).

Now assume $t_0 \geq T_0$, $|x_0| < r_0$, and $H(t_0, \infty) < r_0$. Let $I(t_0) = [t_0, t_1)$ be as defined in Lemma 4.1 and let $r_1(t; x_0, t_0) = \max(H(t_0, \infty), |x_0|)$; $r_1(t; x_0, t_0)$ is defined, continuous, and less than r_0 for $t \geq t_0$. Using the inequality just proved, if $t \in I(t_0)$

$$|x(t; x_0, t_0)| \leq \begin{cases} \left\{ |x_0| - H(t_0, t) \right\} + H(t_0, t) = |x_0| & \text{if } |x_0| \geq H(t_0, t) \\ H(t_0, t) & \text{if } |x_0| < H(t_0, t) \end{cases} \leq r_1(t);$$

by Lemma 4.1 then, $|x(t; x_0, t_0)| \leq r_1(t) < r_0$ for all $t \geq t_0$.

The second part of (1) may be used in (2) - (6) to insure that $x(t; x_0, t_0) \in D$ for $t \geq t_0$. Then the proofs of (2) - (6) are the same

as the proofs of Theorem 3.3, (2) - (6).

Corollary. Using the notation of the theorem, if there exist positive constants L and c with $L/c < r_0$ such that $|f(x, t)| \leq L$ for all $(x, t) \in D$ and $\Lambda(t) \leq -c$ for $t \geq T_0$ and if $|x_0| < r_0$, $t_0 \geq T_0$, then $\lim_{t \rightarrow \infty} \sup |x(t; x_0, t_0)| \leq L/c$.

Proof. Let $G(t) = (\sup_x \{ |f(x, t)| : (x, t) \in D \}) (-\Lambda(t))^{-1}$; under the hypotheses $G(t)$ is defined, non-negative, continuous and less than or equal to L/c for $t \geq T_0$. Then $H(T_0, \infty) \leq L/c < r_0$ and $\int_{T_0}^{\infty} \Lambda(t) dt = -\infty$ and hence the corollary follows from (5) of the theorem.

Remarks on Theorem 4.1 and 4.3 and their corollaries.

Perron (reference 21) has proved (e) of the following theorem; Bellman (reference 3) has proved (a) and (b). Let the equations

$$i) \quad x' = A(t)x + f(x, t) \quad \text{and}$$

$$ii) \quad x' = A(t)x + F(t)$$

be as defined in (NL) and (NH) in Definition 2.1. In addition, assume that $A(t)$ is bounded for $t \geq T_0$ and that $D = \{(x, t) : |x| < \infty, t \geq T_0\}$.

Theorem: A necessary and sufficient condition that the solutions of (i) are bounded for every $f(x, t)$ such that

$$a) \quad |f(x, t)| \text{ is bounded for all } (x, t) \in D;$$

$$b) \quad |f(x, t)| \leq g(t) \text{ where } \int_{T_0}^{\infty} g(t) dt < \infty \text{ for all } (x, t) \in D$$

is that (ii) has only bounded solutions for all $F(t)$ such that

$$a) \quad |F(t)| \text{ is bounded for all } t \geq T_0;$$

$$b) \quad \int_{T_0}^{\infty} |F(t)| dt < \infty.$$

Bridgland (reference 7) has proved the following theorem.

Theorem: If (a) every solution of (ii) is convergent (approaches a limit vector element-wise) for every convergent $F(t)$; (b) for sufficiently small $|x|$, $\lim_{t \rightarrow \infty} f(x, t) = f(x, \infty)$; (c) for sufficiently small

β , $|f(0,t)| \leq \beta$ for $t \geq T_0$; (d) for $\epsilon > 0$, there exists $\delta > 0$ and $T \geq T_0$ such that $|f(x_1,t) - f(x_2,t)| \leq \epsilon |x_1 - x_2|$ for $|x_i| \leq \delta$, $i = 1, 2$, and $t \geq T$; then for every vector c for which $|c|$ is sufficiently small a unique bounded solution $x = x(t; T, c)$ of (1) satisfying $x(T; T, c) = c$ exists on $[T, \infty)$ and all such solutions converge to the same limit vector, ξ , which may be determined uniquely as the solution of the equation $\xi = Y(\infty) f(\xi, \infty)$. (Here $Y(t) = \int_{T_0}^t X(t)X^{-1}(s)ds$ where $X(t)$ is the fundamental matrix of $(H:n\ell)$ such that $X(T_0)$ is the identity matrix.)

The comparison between these theorems and Theorems 4.1, 4.3 and their corollaries is much the same as that made in the remarks on Theorem 3.1. The theorems of this section give only sufficient conditions for the boundedness of the solution of (NL), in two instances they require a stronger hypothesis than the earlier theorems, and they do not detect the convergence of solution of (NL). However, the theorems of this section do not require that $A(t)$ and $f(t)$ be bounded, they apply to equations not covered by the Perron-Bellman theorem, and they determine bounds for the solutions of equation (NL) under hypotheses which are much weaker than those of the Bridgland theorem.

Example 4.1. (illustrating Theorem 4.3, (1) - (6)).

$$dx/dt + 3x = (1/3)x^{2/3} \cos t : t \geq T_0 = 0, |x| < r_0 = 27.$$

This is a Bernoulli equation; the method of solution is standard (see, for example, reference 14, p. 57).

Let $z = x^{1/3}$ or $x = z^3$, then

$$3z^2 dz/dt = dx/dt = (1/3)z^2 \cos t - 3z^3,$$

$$3z^2(dz/dt + z - (1/9)\cos t) = 0.$$

With $t_0 = 0$, either $z(t; 0, 0) = 0$ for $t \geq 0$, or

$z(t; z_0, 0) = z_0 e^{-t} + e^{-t} \int_0^t (e^u/9) \cos u \, du = z_0 e^{-t} + (1/18)(\sin t + \cos t) - e^{-t}/18$, and the solutions to the original equation are

$$x(t; 0, 0) = 0$$

$$x(t; x_0, 0) = [e^{-t}(x_0^{1/3} - 1/18) + (1/18)(\sin t + \cos t)]^3.$$

Now apply Theorem 4.3: with $\Lambda(t) = -3$, let $G(t) = |\cos t|$; then $|(1/3)x^{2/3} \cos t| \leq 3|\cos t| = -\Lambda(t)G(t)$ for all $t \geq 0$, $|x| < 27$.

1) If $t_0 = 0$ and $|x| < r_0 = 27$, then $H(0, \infty) = 1 < 27$ and (1) implies that $|x(t; x_0, t_0)| < 27$ for $t \geq 0$. That this is the case may be seen as follows: let $u(t, \pm 27) = e^{-t}[(\pm 27)^{1/3} - 1/18] + (1/18)(\cos t + \sin t)$. For $|x_0| < 27$, $[u(t, -27)]^3 < |x(t; x_0, 0)| < [u(t, 27)]^3$.

Now $u'(t, 27) = (-u + (1/9)\cos t) < 0$, and hence $u(t, 27)$ is monotonic decreasing for $0 \leq t \leq \pi/2$; $u(0, 27) = 3$. Further, for $t \geq \pi/2$, $u(t, 27) < [(0.23)(3 - 1/18) + \sqrt{2}/18] < 3$. Therefore, $u(t, 27) \leq 3$ for $t \geq 0$.

Now $u'(t, -27) = (-u + (1/9)\cos t) > 0$, and hence $u(t, -27)$ is monotonic increasing for $0 \leq t \leq \pi/2$; $u(0, -27) = -3$. Further, for $t \geq \pi/2$, $u(t, -27) > (0.23)(-3 - 1/18) - \sqrt{2}/18 > -3$. Therefore, $u(t, -27) \geq -3$ for $t \geq 0$.

2) If $t_0 = 0$, $H(0, \infty) = 1 < r_0 = 27$. Then $|x_0| < G(0) = 1$ implies that $|x(t; x_0, 0)| \leq H(0, t) = 1$ for $t \geq 0$. This may be verified by an analysis similar to that of (1).

3) Since $H(0, 0) = H(0, \infty)$, (3) does not apply, with $t_0 = 0$.

4) If $t_0 = 0$, $H(0, \infty) = 1 < r_0 = 27$. Then $1 \leq |x_0| < 27$ implies there exists a T , $0 \leq T \leq \infty$, such that $|x(t; x_0, 0)| \geq G(t)$ and $|x(t; x_0, 0)|$ is monotonic non-increasing for $0 \leq t \leq T$ and $|x(t; x_0, 0)| \leq H(T, t) = \sup \{ |\cos u| : T \leq u \leq t \}$ for $t \geq T$. To verify this let $x(t; x_0, 0) = [e^{-t}(x_0^{1/3} - 1/18) + (1/18)(\sin t + \cos t)]^3 = [u(t; x_0)]^3$.

Then $x'(t; x_0, 0) = 3[u(t; x_0)]^2[-u(t, x_0) + (1/9)\cos t]$. If $|u(t, x_0)| > (1/9)|\cos t|$, $|x(t; x_0, 0)|$ is monotonic non-increasing.

If $|x(t; x_0, t_0)| \geq G(t) = |\cos t|$ for $0 \leq t \leq T$, then $|u(t, x_0)| \geq |\cos t|$ (since $|\cos t| \leq 1$) and $|x(t; x_0, t_0)|$ is monotonic non-increasing for $0 \leq t \leq T$. If $|x(T; x_0, t_0)| = |\cos T|$, then for $t \geq T$, either $|x(t; x_0, t_0)| \leq |\cos t|$ or, as above, $|x(t; x_0, t_0)|$ is monotonic non-increasing. From this it follows (omitting the proof) that $|x(t; x_0, t_0)| \leq H(T, t)$ for $t \geq T$.

5 and 6) With $t_0 = 0$, $H(t_0, \infty) = \lim_{t \rightarrow \infty} \sup G(t) = 1$. Then $\int_{t_0}^{\infty} \Lambda(t) dt = -\infty$ implies that $\lim_{t \rightarrow \infty} \sup |x(t; x_0, t_0)| \leq 1$. In fact, from an inspection of the solution it follows that $\lim_{t \rightarrow \infty} \sup |x(t; x_0, t_0)| \leq (\sqrt{2}/18)^3$.

Starting again from Theorem 4.1, the preceding work may be paraphrased to develop lower bounds for the solutions of (NL).

Theorem 4.4. Given equation (NL), let $x(t; x_0, t_0)$ be a solution such that $x(t_0; x_0, t_0) = x_0$. Assume $f(x, t)$ is such that there exists a $g(t)$, defined, non-negative, bounded, and continuous a.e. for $t \geq T_0$, with $|f(x, t)| \leq g(t)|\lambda(t)|$ for all $(x, t) \in D$.¹ If $g(t) \leq |x(t; x_0, t_0)|$ and $x(t; x_0, t_0) \in D$ for $T_0 \leq t_0 \leq t \leq t_1 \leq \infty$, then

$$|x(t; x_0, t_0)| \geq |x_0| \exp \int_{t_0}^t \{ \lambda(s) - |\lambda(s)| \} ds \text{ for } t_0 \leq t \leq t_1.$$

If, in addition, $\lambda(t) \geq 0$ for $t_0 \leq t \leq t_1$, then $|x(t; x_0, t_0)|$ is monotonic non-decreasing for $t_0 \leq t \leq t_1$.

Proof. With Theorem 4.1 used in place of Theorem 3.1, the proof is the same as the proof of Theorem 3.4.

Theorem 4.5. Given equation (NL), let $x(t; x_0, t_0)$ be a solution such that $x(t_0; x_0, t_0) = x_0$. Assume $\lambda(t) \geq 0$ for $t \geq T_0$; assume $f(x, t)$

¹See the footnote on page 33.

is such that there exists a $g(t)$, defined, non-negative, bounded, and continuous a.e. for $t \geq T_0$, such that $|f(x, t)| \leq g(t) \lambda(t)$ for all $(x, t) \in D$.¹ Let $h(t_1, t_2) = \sup\{g(t) : T_0 \leq t_1 \leq t \leq t_2 \leq \infty\}$.

- 1) If $x(t; x_0, t_0) \in D$ for $T_0 \leq t_0 \leq t \leq t_1 \leq \infty$, then

$$|x(t; x_0, t_0)| \geq (|x_0| - h(t_0, t)) \exp \int_{t_0}^t \lambda(s) ds + h(t_0, t).$$
- 2) If $t_0 \geq T_0$, $|x_0| \geq h(t_0, \infty)$, and $x(t; x_0, t_0) \in D$ for $t_0 \leq t \leq t_1$, then $|x(t; x_0, t_0)| \geq h(t_0, t)$ and $|x(t; x_0, t_0)|$ is monotonic non-decreasing for $t_0 \leq t \leq t_1$.
- 3) If $t_0 \geq T_0$, $h(t_0, \infty) < |x_0|$, $\int_{t_0}^{\infty} \lambda(t) dt = \infty$, and $R_0 < \infty$, then $x(t; x_0, t_0)$ escapes from D .
- 4) If $\lim_{t \rightarrow \infty} \sup g(t) < u$, u a positive constant, then there exists a $T(u) \geq T_0$ such that for all $t \geq t_0 \geq T$, $h(t_0, t) \leq u$ (and (2) and (3) may be applied to $\{(x, t) : |x| \geq u, t \geq T\}$).
- 5) Under the hypotheses of (3), if $T_0 \leq \bar{t} < T$, then there exists a $k(u, \bar{t}) \geq u$ such that if $k \leq |\bar{x}|$ and $x(t; \bar{x}, \bar{t}) \in D$ for $\bar{t} \leq t \leq T$, then $|x(t; \bar{x}, \bar{t})| \geq u$ for $\bar{t} \leq t \leq T$ (and (4) may be applied for $t \geq T$).

Proof. With Theorem 4.1 used in place of Theorem 3.1, the proof is the same as the proof of Theorem 3.5 with the exception of part (3).

In part (3), if $x(t; x_0, t_0) \in D$ for all $t \geq t_0$, then as in Theorem 3.5, (3), $\lim_{t \rightarrow \infty} |x(t; x_0, t_0)| = \infty$; but this contradicts $R_0 < \infty$.

If it is assumed that $|f(x, t)| \leq g(t)|x|$ for all $(x, t) \in D$ ---an often used technique for equation (NL)---further results may be obtained from Theorem 4.1.

Theorem 4.6. Given equation (NL), let $x(t; x_0, t_0)$ be a solution such that $x(t_0; x_0, t_0) = x_0$. If $f(x, t)$ is such that there exists a $g(t)$, defined, non-negative, and continuous for $t \geq T_0$, with $|f(x, t)| \leq g(t)|x|$

¹See the footnote of page 33.

for all $(x, t) \in D$ and if $x(t; x_0, t_0) \in D$ for $T_0 \leq t_0 \leq t \leq t_1 \leq \infty$, then $|x_0| \exp \int_{t_0}^t \{\lambda(s) - g(s)\} ds \leq |x(t; x_0, t_0)| \leq |x_0| \exp \int_{t_0}^t \{\lambda(s) + g(s)\} ds$.

Proof. If $|f(x, t)| \leq g(t)|x|$, where $g(t)$ is as defined in the hypotheses, for all $(x, t) \in D$ and if $x(t; x_0, t_0) \in D$ for $t_0 \leq t \leq t_1$, then the right inequality of Theorem 4.1 gives

$$|x(t; x_0, t_0)| \leq |x_0| \exp \int_{t_0}^t \lambda(s) ds + \int_{t_0}^t g(s) |x(s; x_0, t_0)| \exp \int_s^t \lambda(u) du ds, \\ |x(t; x_0, t_0)| \exp \int_{t_0}^t -\lambda(s) ds \leq |x_0| + \int_{t_0}^t g(s) |x(s; x_0, t_0)| \exp \int_s^t -\lambda(u) du ds.$$

Using Lemma 3.1,

$$|x(t; x_0, t_0)| \exp \int_{t_0}^t -\lambda(s) ds \leq |x_0| \exp \int_{t_0}^t g(s) ds \\ |x(t; x_0, t_0)| \leq \exp \int_{t_0}^t \{\lambda(s) + g(s)\} ds.$$

Starting from the left inequality of Theorem 4.1 and using Lemma 3.3, a derivation similar to the one just complete yields

$$|x(t; x_0, t_0)| \geq |x_0| \exp \int_{t_0}^t \{\lambda(s) - g(s)\} ds. \quad \text{Q.E.D.}$$

The next theorem discusses the solution $x = 0$ of equation (NL) so it is necessary to assume $f(0, t) = 0$ for $t \geq T_0$.

Theorem 4.7. Given equation (NL), let $x(t; x_0, t_0)$ be a solution such that $x(t_0; x_0, t_0) = x_0$. Assume $f(x, t)$ is such that there exists a $g(t)$, defined, non-negative, and continuous for $t \geq T_0$, with $|f(x, t)| \leq g(t)|x|$ for all $(x, t) \in D$ (and hence $f(0, t) = 0$ for $t \geq T_0$).

1) If $\int_{t_0}^t \{\lambda(s) + g(s)\} ds \leq L(t_0) < \infty$ for all $t \geq t_0 \geq T_0$ ($L(t_0) \geq 0$) and $|x_0| < r_0 e^{-L(t_0)}$, then $|x(t; x_0, t_0)| < r_0$ for $t \geq t_0$. Further, $x = 0$ is a stable solution of (NL).

2) If in (1), $L(t_0) \leq L$ for all $t_0 \geq T_0$, then $x = 0$ is a uniformly stable solution of (NL).

3) If the hypotheses of (1) are satisfied and if $\int_{t_0}^{\infty} \{\lambda(t) + g(t)\} dt = -\infty$, then $x = 0$ is an equiasymptotically stable solution of (NL).

4) If $\lambda(t) + g(t) \leq 0$ for $t \geq T_0$ and $|x_0| < r_0$, then $|x(t; x_0, t_0)|$

is monotonic non-increasing for $t \geq t_0 \geq T_0$.

5) If $\Lambda(t) + g(t) \leq -c$, c a positive constant, for $t \geq T_0$, then $x = 0$ is an exponentially stable solution of (NL).

Proof. If the hypotheses are satisfied and if $x(t; x_0, t_0) \in D$ for $t \in I(t_0) = [t_0, t_1)$ then Theorem 4.6 gives

$$(*) \quad |x(t; x_0, t_0)| \leq |x_0| \exp \int_{t_0}^t \{\Lambda(s) + g(s)\} ds \quad \text{for } t \in I(t_0).$$

1) Let $t_0 \geq T_0$ be given, assume $\int_{t_0}^t \{\Lambda(s) + g(s)\} ds \leq L(t_0)$ for $t \geq t_0$, and let $|x_0| < r_0 e^{-L(t_0)}$ be given. Let $r_1(t) = |x_0| e^{L(t_0)}$; then $r_1(t)$ is defined, continuous, and $r_1(t) < r_0$ for $t \geq t_0$. With $I(t_0)$ as defined above and $t \in I(t_0)$, (*) gives

$$|x(t; x_0, t_0)| \leq |x_0| \exp \int_{t_0}^t \{\Lambda(s) + g(s)\} ds \leq |x_0| e^{L(t_0)} = r_1(t).$$

Then by Lemma 4.1, $|x(t; x_0, t_0)| \leq r_1(t) < r_0$ for all $t \geq t_0$.

If ϵ , $0 < \epsilon < r_0$, and $t_0 \geq T_0$ are given, let $\delta(\epsilon, t_0) = \epsilon e^{-L(t_0)} < r_0 e^{-L(t_0)}$. Then $|x_0| < \delta$ implies i) $|x(t; x_0, t_0)| < r_0$ for $t \geq t_0$ (this was just proved) and ii) from (*), $|x(t; x_0, t_0)| \leq |x_0| e^{L(t_0)} < \epsilon$ for $t \geq t_0$. Hence $x = 0$ is a stable solution of (NL).

2) If in (1), $L(t_0) \leq L$ for all $t_0 \geq T_0$, then the proof of (1) may be repeated using L in place of $L(t_0)$ and it will follow that $x = 0$ is a uniformly stable solution of (NL).

3) If the hypotheses of (1) are satisfied, then $x = 0$ is a stable solution of (NL) and for $t_0 \geq T_0$ and $|x_0| < r_0 e^{-L(t_0)}$, $x(t; x_0, t_0) \in D$ for $t \geq t_0$. Then using (*),

$$\lim_{t \rightarrow \infty} |x(t; x_0, t_0)| \leq \lim_{t \rightarrow \infty} \left\{ |x_0| \exp \int_{t_0}^t (\Lambda(s) + g(s)) ds \right\} = 0;$$

further, this limit is uniform for all $|x_0| < \delta_0$. Therefore, $x = 0$ is an equiasymptotically stable solution of (NL).

4) If $\Lambda(t) + g(t) \leq 0$ for $t \geq T_0$, then (1) may be applied with $L(t_0) = 0$; if $t_0 \geq T_0$ and $|x_0| < r_0$, then $x(t; x_0, t_0) \in D$ for $t \geq t_0$.

Further, if t_2 and t_3 , $t_0 \leq t_2 < t_3 < \infty$, are given, then using (*)

$$|x(t_3; x_0, t_0)| \leq |x(t_2; x_0, t_0)| \exp \int_{t_2}^{t_3} \{\Lambda(t) + g(t)\} dt \leq |x(t_2; x_0, t_0)|.$$

5) If $\Lambda(t) + g(t) \leq -c$ for $t \geq T_0$ and $|x_0| < r_0$, $t_0 \geq T_0$, then by (4), $x(t; x_0, t_0) \in D$ for $t \geq t_0$. If ϵ , $0 < \epsilon < r_0$, is given, let

$\delta(\epsilon) = \epsilon$; then from (*), $|x_0| < \delta$ implies

$$|x(t; x_0, t_0)| \leq |x_0| \exp \int_{t_0}^t \{\Lambda(s) + g(s)\} ds < \epsilon e^{-c(t-t_0)}$$

and $x = 0$ is an exponentially stable solution of (NL). Q.E.D.

Remarks on Theorem 4.7. Three common restrictions on $f(x, t)$ of the type used in Theorem 4.7 are 1) $|f(x, t)| = o(|x|)$ uniformly with respect to t , i.e. there exists an $r(|x|)$ where $r(|x|) \rightarrow 0$ as $|x| \rightarrow 0$ such that for $t \geq T_0$, $|f(x, t)| \leq |x|r(|x|)$; 2) $|f(x, t)| \leq k|x|$ for some constant $k > 0$ and given $\epsilon > 0$, there exists a $\delta > 0$ and a $T \geq T_0$ such that for $|x| < \delta$ and $t \geq T$, $|f(x, t)| \leq \epsilon|x|$; and 3) $|f(x, t)| \leq g(t)|x|$ where $g(t)$ is defined, non-negative, and continuous for $t \geq T_0$. In order of strength, $(1) \rightarrow (2) \rightarrow (3)$.

In the following discussion of equation (NL) and of Lyapunov functions, it is assumed that x' , $A(t)$, x , and $f(x, t)$ have real elements.

For the case of (NL) with $A(t)$ constant for $t \geq T_0$, if $x = 0$ is an asymptotically stable solution of (H:nl) and if $|f(x, t)| = o(|x|)$ uniformly with respect to t , i.e. $|f(x, t)| \leq |x|r(|x|)$ where $r(|x|) \rightarrow 0$ as $|x| \rightarrow 0$, for all $t \geq T_0$, then $x = 0$ is an asymptotically stable solution of (NL). This was proved by Lyapunov and Perron (see references 9, p.92 and 10, p.314). This theorem does not carry over to variable $A(t)$; a counterexample was provided by Perron (reference 21). The question then arises, what conditions are necessary to insure the asymptotic stability of $x = 0$ in the case of variable $A(t)$?

This question was answered by Perron in the same paper (reference 21)

and further investigations along this line were made by Persidskii and Malkin. The work of all three is summarized in reference 16, chapter VII and the following is the most convenient theorem of their approach.

Theorem. Let equation (NL) be given and assume, additionally, that $A(t)$ is bounded for $t \geq T_0$ and that $|f(x, t)| = o(|x|)$ uniformly in t . If $(H: n\ell)$ has a Lyapunov function, $V(x, t)$, such that $V'(x, t)$ is negative definite, then $x = 0$ is an asymptotically stable solution of (NL).

Here a Lyapunov function for $(H: n\ell)$ is a real scalar function, $V(x, t)$, which is defined, positive definite, and has continuous first partial derivatives with respect to x and t for all x and $t \geq T_0$. Further, $V(0, t) = 0$ for $t \geq T_0$ and $V'(x, t) = V_x \cdot x' + V_t = V_x \cdot Ax + V_t$. The same definition holds for (NL) except $V(x, t)$ is defined only on D and $V'(x, t) = V_x \cdot (Ax + f) + V_t$. Several types of stability (stability, uniform stability, asymptotic stability, equiasymptotic stability) may be deduced for (H) or (NL) by using Lyapunov functions, but the theorems usually require that $V'(x, t) \leq 0$ (For a survey of this topic see reference 1.).

Recently, Sell (reference 22) has studied Lyapunov functions with weaker continuity conditions (which will not be considered here) and with weaker restrictions on $V'(x, t)$. He considers the differential equation

$$(*) \quad u' = H(u, t)$$

where u and t are real scalar variables and $H(u, t)$ is a real scalar function which is defined for $(u, t) \in \bar{D} = \{(u, t): |u| < \bar{r}, t \geq T_0\}$. Further, $H(0, t) = 0$ for $t \geq T_0$ and $H(u, t)$ is such that equation $(*)$ satisfies the conditions for the existence and uniqueness of solutions on $\bar{D}$.

An extended Lyapunov function, $V(x, t)$, for (H) or (NL) is the same

as a Lyapunov function ($V'(x,t)$ is defined with respect to (H) or (NL)) with the addition that there exists an $H(u,t)$, as defined for equation (*), such that

$$V'(x,t) \leq H(V(x,t),t) .$$

There then follows a stability theorem (Sell states this theorem for systems more general than (NL)).

Theorem. Let equation (NL) be given and assume that: 1) the solutions of (NL) are unique; 2) $f(0,t) = 0$ for $t \geq T_0$; 3) an extended Lyapunov function exists for (NL). If $u = 0$ is a stable (or uniformly stable, or asymptotically stable, or equiasymptotically stable) solution of (*), then $x = 0$ is a stable (or uniformly stable, or asymptotically stable, or equiasymptotically stable) solution of (NL) (For uniform stability it is also necessary to assume that $V(x,t)$ has an infinitesimal upper bound, i.e. there exists a real valued function, $u(r)$, which is defined, continuous, and strictly increasing for $0 \leq r < R$ with $u(0) = 0$, such that $V(x,t) \leq u(|x|)$ for all $(x,t) \in D$).

If $x', x, A(t)$, and $f(x,t)$ have real valued elements and if the solutions of (NL) are unique, then (1), (2), and (3) of Theorem 4.7 may be derived from Sell's Theorem. Let $V(x,t) = |x| = (x^T x)^{\frac{1}{2}}$ (x^T is x transpose); then $V(x,t)$ is an extended Lyapunov function for

- 1) $V(x,t)$ is a real scalar function which is defined for all $(x,t) \in D$;
- 2) $V(x,t)$ is positive definite;
- 3) $V(0,t) = 0$ for $t \geq T_0$;
- 4) $V_x(x,t)$ and $V_t(x,t)$ exist and are continuous for all $(x,t) \in D$;
- 5) $V'(x,t) = |x|' \leq (\wedge(t) + g(t)) |x|$ for all $(x,t) \in D$ (This statement is derived in a manner similar to the proof of Theorem 2.1; the uniqueness of solutions is used here.);

6) $|x|$ has an infinitesimal upper bound.

If $|x|$ is considered as an extended Lyapunov function, it is only necessary to study the null solution of

$$(**) \quad u' = (\Lambda(t) + g(t)) u.$$

The general solution of this equation is

$$u(t) = u(t_0) \exp \int_{t_0}^t (\Lambda(s) + g(s)) ds \quad : t \geq t_0 \geq T_0$$

and conditions (1), (2), and (3) of Theorem 4.7 are seen to be precisely the conditions which imply that $u = 0$ is a stable, uniformly stable, or equiasymptotically stable solution of (**).

Corollaries 1 and 2 of Theorem 2.1 may also be derived from Sell's theorem by using the preceding discussion with $g(t) = 0$ for $t \geq T_0$. Theorems 2.3 and 2.4 and the work of Section 5 may also be interpreted in terms of extended Lyapunov functions using $V(x, t) = |y| = |P^{-1}x|$.

Cesari (reference 9, p. 94) gives the following theorem.

Theorem. Assume the general hypotheses of Theorem 4.7. If $\int_{t_0}^{\infty} g(t) dt < \infty$ and if $x = 0$ is a uniformly stable solution of $(H: n\ell)$, then $x = 0$ is a uniformly stable solution of (NL) . If $x = 0$ is a uniformly asymptotically stable solution of $(H: n\ell)$, then $x = 0$ is a uniformly asymptotically stable solution of (NL) .

This theorem is an improvement on Theorem 4.7, (2) in one sense, for $\int_s^t \Lambda(u) du \leq M < \infty$ for all $t \geq s \geq T_0$ (which together with $\int_{t_0}^{\infty} g(t) dt < \infty$ satisfies the hypotheses of Theorem 4.7, (2)) implies, by Corollary 1 of Theorem 2.1, that $x = 0$ is a uniformly stable solution of $(H: n\ell)$. The converse is not true as is shown by example in Section 3. But if $\int_{t_0}^t \Lambda(s) ds \rightarrow \infty$ as $t \rightarrow \infty$, then Theorem 4.7, (2) permits a weakening of the restrictions on $g(t)$.

Golomb (reference 12) gives further theorems along this line in

which it is assumed that $x = 0$ is a restrictively stable solution of $(H: n\ell)$ (i.e. $x = 0$ is a stable solution of $(H: n\ell)$ and of the adjoint equation of $(H: n\ell)$).

Theorem 4.7 deals with the right inequality of Theorem 4.6. In Theorem 4.8 a similar analysis is made of the left inequality and lower bounds for solutions are studied.

Example 4.2. (illustrating Theorem 4.7, (3))

$$dx/dt + \frac{1}{2}\sqrt{2} x = x^2 \cos t : t \geq T_0 = 0, |x| < r_0 = 1.$$

Since $|(x_1^2 \cos t - \frac{1}{2}\sqrt{2} x_1) - (x_2^2 \cos t - \frac{1}{2}\sqrt{2} x_2)| = |x_1 - x_2| |(x_1 + x_2) \cos t - \frac{1}{2}\sqrt{2}| < 3|x_1 - x_2|$ for all $x_1, x_2 \in D$, a Lipschitz condition is satisfied and the solutions are unique. As in Example 4.1, this is a Bernoulli equation; the substitution $z = x^{-1}$ ($x \neq 0$) gives

$$dz/dt - \frac{1}{2}\sqrt{2} z = -\cos t$$

with solution

$$z(t; z_0, t_0) = z_0 \exp\left[\frac{1}{2}\sqrt{2}(t - t_0)\right] - \int_{t_0}^t \cos u \exp\left[\frac{1}{2}\sqrt{2}(t - u)\right] du.$$

With $t_0 = 0$, the solution of the original equation is

$$x(t; 0, 0) = 0$$

$$x(t; x_0, 0) = \left[(x_0^{-1} - \sqrt{2}/3) e^{\frac{1}{2}\sqrt{2}t} + (2/3)(\frac{1}{2}\sqrt{2} \cos t - \sin t) \right]^{-1}, x_0 \neq 0.$$

$$\begin{aligned} \text{Now } x'(t; x_0, 0) &= - \left[(x_0^{-1} - \sqrt{2}/3) e^{\frac{1}{2}\sqrt{2}t} + (2/3)(\frac{1}{2}\sqrt{2} \cos t - \sin t) \right]^{-2} \times \\ &\quad \times \left[\frac{1}{2}\sqrt{2}(x_0^{-1} - \sqrt{2}/3) e^{\frac{1}{2}\sqrt{2}t} + (2/3)(-\frac{1}{2}\sqrt{2} \sin t - \cos t) \right] \\ x'(t; x_0, 0) &= [x(t; x_0, 0)]^2 \left[(-\frac{1}{2}\sqrt{2}) \left\{ (x_0^{-1} - \sqrt{2}/3) e^{\frac{1}{2}\sqrt{2}t} + (2/3) \times \right. \right. \\ &\quad \left. \left. \times (\frac{1}{2}\sqrt{2} \cos t - \sin t) \right\} + \cos t \right], \end{aligned}$$

$$x'(t; x_0, 0) = [x(t; x_0, 0)] [x(t; x_0, 0) \cos t - \frac{1}{2}\sqrt{2}].$$

If $\frac{1}{2}\sqrt{2} > x(t; x_0, 0) > 0$, then $x'(t; x_0, 0) < 0$; if $0 > x(t; x_0, 0) > -\frac{1}{2}\sqrt{2}$, then $x'(t; x_0, 0) > 0$; in both cases $|x(t; x_0, 0)|$ is monotonic decreasing.

Further, by the uniqueness of the solutions, no solution crosses the solution $x = 0$. Therefore, $x = 0$ is a stable solution of this example.

Further, $\lim_{t \rightarrow \infty} x(t; x_0, t_0) = 0$ and this limit is uniform in x_0 for all $|x_0| < \delta_0$, δ_0 a constant. Therefore, $x = 0$ is an equiasymptotically stable solution of this example.

Now apply Theorem 4.7; $\Lambda(t) = -\frac{1}{2}\sqrt{2}$ and $|x^2 \cos t| \leq |x| |\cos t|$ or $g(t) = |\cos t|$. Then $\int_0^\infty \{\Lambda(s) + g(s)\} ds = \lim_{n \rightarrow \infty} \int_0^{2n\pi} \left\{ -\frac{1}{2}\sqrt{2} + |\cos t| \right\} dt = \lim_{n \rightarrow \infty} \left\{ -\sqrt{2} n\pi + 4n \right\} = \lim_{n \rightarrow \infty} n(4 - \sqrt{2} \pi) = -\infty$. By (3) of Theorem 4.7, $x = 0$ is an equiasymptotically stable solution of this example.

Theorem 4.8. Given equation (NL), let $x(t; x_0, t_0)$ be a solution such that $x(t_0; x_0, t_0) = x_0$. Assume $f(x, t)$ is such that there exists a $g(t)$, defined, non-negative, and continuous for $t \geq T_0$, with $|f(x, t)| \leq g(t)|x|$ for all $(x, t) \in D$.

1) For $t_0 \geq T_0$, if $\int_{t_0}^t \{\lambda(s) - g(s)\} ds \geq m > -\infty$ for $t \geq t_0$ and if $x(t; x_0, t_0) \in D$ for $t_0 \leq t \leq t_1 \leq \infty$, then $|x(t; x_0, t_0)| \geq |x_0| e^m$ for $t_0 \leq t \leq t_1$.

2) If $\int_{t_0}^\infty \{\lambda(s) - g(s)\} ds = \infty$, $R_0 = \sup\{|x| : (x, t) \in D\} < \infty$, $t_0 \geq T_0$, and $|x_0| \neq 0$, then $x(t; x_0, t_0)$ escapes from D .

3) If $\lambda(t) - g(t) \geq c$, a positive constant, $x(t; x_0, t_0) \in D$ for $T_0 \leq t_0 \leq t \leq t_1 \leq \infty$, and $x_0 \neq 0$, then $|x(t; x_0, t_0)|$ increases exponentially for $t_0 \leq t \leq t_1$.

Proof. If the hypotheses on $f(x, t)$ are satisfied and if $x(t; x_0, t_0) \in D$ for $T_0 \leq t_0 \leq t \leq t_1 \leq \infty$, then Theorem 4.6 gives

$$|x(t; x_0, t_0)| \geq |x_0| \exp \int_{t_0}^t \{\lambda(s) - g(s)\} ds \quad \text{for } t_0 \leq t \leq t_1.$$

1) The proof of (1) is immediate.

2) If $x(t; x_0, t_0) \in D$ for $t_0 \leq t < \infty$, then $\lim_{t \rightarrow \infty} |x(t; x_0, t_0)| \geq \lim_{t \rightarrow \infty} |x_0| \exp \int_{t_0}^t \{\lambda(s) - g(s)\} ds = \infty$; this contradicts $R_0 < \infty$.

3) Given $\epsilon > 0$, let $\delta = \epsilon$. For $t_0 \geq T_0$, $|x_0| < \delta$, and $t_0 \leq t \leq t_1$, $|x(t; x_0, t_0)| \geq |x_0| \exp \int_{t_0}^t \{\lambda(s) - g(s)\} ds \geq \epsilon e^{c(t-t_0)}$. Q.E.D.

5. Variations about a Constant Matrix

In Sections 2, 3, and 4 the asymptotic behavior of (H), (NH) and (NL) was discussed in terms of the functions $\Lambda(t)$ and $\lambda(t)$. Another way to describe this behavior is to use the characteristic roots of the matrix $A(t)$. In this section the two methods will be combined; the special case of $A(t) = A + B(t)$, where A is constant and $B(t)$ is a "variation" about A , is considered and $\Lambda(t)$ and $\lambda(t)$ described in terms of the characteristic roots of A and the matrix $B(t)$.

The method involves a change of variables, $y = P^{-1}x$, P^{-1} is constant and nonsingular, which transforms A to Jordan form. Since P^{-1} is constant and nonsingular, the original and the transformed equations should have the same qualitative behavior and it should be necessary to study only the latter equation. The first two lemmas show this is so. The notation of Section 2 is used.

Lemma 5.1. Let $y = P^{-1}x$, where P^{-1} is a constant, nonsingular, n by n matrix, be a mapping of E^n into E^n . This mapping transforms (H), (NH) and (NL) into equations to which the theorems of Sections 2, 3, and 4 may still be applied. Further, if $x(t)$ is a solution of one of the original equations, then $y(t) = P^{-1}x(t)$ is a solution of the transformed equation.

Proof. The mapping $y = P^{-1}x$, as described in the hypotheses, transforms (H) into

$$y' = P^{-1}A(t)Py$$

and (NH) into

$$y' = P^{-1}A(t)Py + P^{-1}f(t).$$

Since P and P^{-1} are constant, $P^{-1}A(t)P$ and $P^{-1}f(t)$ are continuous for $t \geq T_0$ and the transformed equations satisfy the same hypotheses as the

original equations.

Equation (NL) becomes

$$y' = P^{-1}A(t)Py + P^{-1}f(Py, t).$$

Since P and P^{-1} are constant and nonsingular, $P^{-1}A(t)P$ is continuous for $t \geq T_0$ and $P^{-1}f(Py, t) = g(y, t)$ is defined and continuous for all $(y, t) \in D' = \{(y, t): (Py, t) \in D\}$. Further, D' is open and connected; $\{y: |y| < r_0(|P|)^{-1} = r'_0\} \times \mathbb{R} \subset D'$, since $|y| < r_0(|P|)^{-1}$ implies $|Py| \leq |P||y| < r_0$; $g(0, t) = 0$ for $t \geq T_0$ if and only if $f(0, t) = 0$ for $t \geq T_0$; and $R'_0 = \sup\{|y|: (y, t) \in D'\} < \infty$ if and only if $R_0 < \infty$.

Therefore, the hypotheses satisfied by (NL) are also satisfied by the transformed equation.

If $x(t)$ is a solution of one of the original equations, then $y(t) = P^{-1}x(t)$ is a solution of the transformed equation, as may be checked by substitution. Q.E.D.

Lemma 5.2: Let $y = Qx$, where Q is a constant, nonsingular, n by n matrix, be a mapping of E^n into E^n . Then $|x|_Q = |Qx|$ defines a norm for E^n and $(|Q^{-1}|)^{-1}|x| \leq |x|_Q \leq |Q||x|$ (Since Q is nonsingular, $|Q| \neq 0 \neq |Q^{-1}|$). Further, the norm $|x|_Q$ may be used in place of the norm $|x|$ in Definition 2.3.

Proof: A norm for E^n is a function, ν , mapping E^n into the non-negative real numbers, which is such that for $x, y \in E^n$ and c a scalar:

- i) $\nu(x) = 0$ if and only if $x = 0$;
- ii) $\nu(x + y) \leq \nu(x) + \nu(y)$;
- iii) $\nu(cx) = |c| \nu(x)$.

The function $|x|$, defined in Definition 2.2, is a norm for E^n ; hence $|x|_Q = |Qx|$ maps E^n into the non-negative reals. Further, since Q is nonsingular,

- 1) $|x|_Q = 0$ if and only if $x = 0$;
 11) $|x + y|_Q = |Q(x + y)| = |Qx + Qy| \leq |Qx| + |Qy| = |x|_Q + |y|_Q$;
 111) $|cx|_Q = |cQx| = |c||Qx| = |c||x|_Q$.

Therefore, $|x|_Q$ is a norm for E^n .

Using Lemma 2.1,

$$|x|_Q = |Qx| \leq |Q||x|$$

and

$$|x| = |Q^{-1}Qx| \leq |Q^{-1}||Qx| = |Q^{-1}||x|_Q.$$

The assertion about Definition 2.3 follows from the preceding inequalities. The details will be worked out for one case, the rest may be proved in a similar fashion.

Using the $|x|$ norm, assume the solutions of (NL) are locally bounded. There exists a $\delta_0 > 0$ such that $|x_0| < \delta_0$ implies $x(t; x_0, t_0) \in D$ for $t \geq t_0 \geq T_0$ and given δ , $0 < \delta < \delta_0$, and $t_0 \geq T_0$, there exists an $M(\delta, t_0) > 0$ such that $|x_0| < \delta$ implies $|x(t; x_0, t_0)| \leq M$ for $t \geq t_0$. Let $\delta'_0 = (|Q^{-1}|)^{-1} \delta_0$; then $|x_0|_Q < \delta'_0$ implies $|x_0| \leq |Q^{-1}||x_0|_Q < \delta_0$ which implies that $x(t; x_0, t_0) \in D$ for $t \geq t_0 \geq T_0$. Further, given $0 < \delta' \leq \delta'_0$ and $t_0 \geq T_0$, let $\delta = |Q^{-1}|\delta' < \delta_0$. With $M(\delta, t_0) = M(\delta', t_0)$ as just described, let $M'(\delta', t_0) = |Q|M$. Then $|x_0|_Q < \delta'$ implies $|x_0| < \delta$ which implies that $|x(t; x_0, t_0)| \leq M$ and hence $|x(t; x_0, t_0)|_Q \leq M'$ for $t \geq t_0$. Therefore, the solutions of (NL) are locally bounded in the $|x|_Q$ norm.

A similar proof shows that if the solutions of (NL) are locally bounded in the $|x|_Q$ norm, then they are locally bounded in the $|x|$ norm. Hence, either norm may be used in the definition of local boundedness. Q.E.D.

Lemma 5.3. If $A(t) = B(t) + C(t)$, where $A(t)$, $B(t)$ and $C(t)$ are matrix functions, defined and continuous for $t \geq T_0$, and if Λ_X and λ_X

are the supremum and infimum, respectively, of the characteristic roots of $\frac{1}{2}(X^* + X)$, for $X = A(t)$, $B(t)$, or $C(t)$, then

$$\Lambda_A(t) \leq \Lambda_B(t) + \Lambda_C(t) \text{ and } \lambda_A(t) \geq \lambda_B(t) + \lambda_C(t) \text{ for } t \geq T_0.$$

Proof: For the functions $f(t)$ and $g(t)$, $\sup(f + g) \leq \sup(f) + \sup(g)$ and $\inf(f + g) \geq \inf(f) + \inf(g)$. Q.E.D.

Remarks on Lemmas 5.1 - 5.3. Lemmas 5.1 and 5.2 make it possible to transform $x' = [A + B(t)]x$ into

$$y' = [P^{-1}AP + P^{-1}B(t)P]y = C(t)y$$

and to determine the asymptotic behavior of the former equation from that of the latter. By Lemma 5.3, instead of $\Lambda_C(t)$ it is sufficient to compute $\Lambda_{P^{-1}AP}$ and $\Lambda_{P^{-1}BP}$ (or $\lambda_{P^{-1}AP}$ and $\lambda_{P^{-1}BP}$ instead of λ_C).

The matrix P will be chosen so that $P^{-1}AP$ is in Jordan form.

The next step is to determine bounds for the characteristic roots of $\frac{1}{2}[(P^{-1}AP)^* + (P^{-1}AP)]$ and $\frac{1}{2}[(P^{-1}B(t)P)^* + (P^{-1}B(t)P)]$ for such P .

The matrices in the following lemmas may have either real or complex entries.

Lemma 5.4. If the matrix $A = (a_{ij})$ ($i, j = 1, \dots, n$) is given, then there exists a nonsingular matrix $P = (p_{ij})$ ($i, j = 1, \dots, n$), with $P^{-1} = (p_{ij}^{(-1)})$ ($i, j = 1, \dots, n$) as its inverse, such that $P^{-1}AP = J$ where J is in Jordan canonical form. The matrix J may be written as

$$J = \text{Diag}(J_1, J_2, \dots, J_u)$$

where $J_1 = \text{Diag}(\lambda_{11}, \lambda_{12}, \dots, \lambda_{1v(1)})$ and

$$J_1 = \begin{bmatrix} \lambda_{11} & R_1 & 0 & \dots & 0 \\ 0 & \lambda_{12} & R_1 & & \vdots \\ \vdots & 0 & \lambda_{13} & & 0 \\ \vdots & \vdots & 0 & & R_1 \\ 0 & 0 & 0 & \dots & \lambda_{1v(1)} \end{bmatrix}, \quad i = 2, \dots, u.$$

Here R_1 ($i = 2, \dots, u$) are arbitrary nonzero real numbers (P is a function

of these constants) and $\lambda_{11} = \dots = \lambda_{1v(1)}$ ($i = 2, \dots, u$). The $\lambda_{11}, \dots, \lambda_{1v(1)}$ are characteristic roots corresponding to elementary divisors of A whose exponents are 1; $\lambda_{11}, \dots, \lambda_{1v(1)}$ ($2 \leq i \leq u$) is a characteristic root corresponding to an elementary divisor of A with exponent $v(i) > 1$ (This sentence will not be elaborated upon; for a complete treatment of this topic the reader is referred to reference 4, chapter X. It will be noted, however, that the number of times a characteristic root appears on the main diagonal of J is equal to its multiplicity as a root of the characteristic equation of A). These two classes include all of the characteristic roots of A and $\sum_{i=1}^u \sum_{j=1}^{v(i)} 1 = n$.

Proof: If $R_1 = 1$ for $i = 2, \dots, u$, the lemma is well known (see reference 4, chapter X) and will be assumed as proved. This case will be written as

$$\hat{P}^{-1} \hat{A} \hat{P} = \hat{J} = \text{Diag}(\hat{J}_1, \dots, \hat{J}_n).$$

Given a set of nonzero real numbers $R_2, \dots, R_u$, construct the following matrices: D_1 equals a $v(1)$ by $v(1)$ identity matrix; for $i = 2, \dots, u$, $D_i = \text{Diag}(1, R_i, R_i^2, \dots, R_i^{v(i)-1})$ (where u and $v(i)$ are the same as in the statement of the lemma). Each D_i ($i = 1, \dots, u$) is nonsingular and its inverse can be constructed; D_1^{-1} is an identity matrix and

$$D_i^{-1} = \text{Diag}(1, R_i^{-1}, \dots, R_i^{1-v(i)}) \quad (i = 2, \dots, u).$$

Then $D_1^{-1} \hat{J}_1 D_1 = \hat{J}_1 = J_1$ and

$$D_i^{-1} \hat{J}_i D_i = \begin{bmatrix} \lambda_{i1} & R_i & 0 & \dots & 0 \\ 0 & \lambda_{i2} & R_i & & \vdots \\ \vdots & \vdots & \lambda_{i3} & & \\ \vdots & \vdots & \vdots & & R_i \\ 0 & 0 & 0 & \dots & \lambda_{iv(i)} \end{bmatrix} = J_i, \quad i = 2, \dots, u.$$

where $J_1, \dots, J_u$ are defined in the statement of the lemma. Let

$$D = \text{Diag}(D_1, \dots, D_u) \text{ and } P = \hat{P} D,$$

then D is nonsingular, P is nonsingular,

$$D^{-1} = \text{Diag}(D_1^{-1}, \dots, D_u^{-1}), \quad P^{-1} = D^{-1} \hat{P}^{-1}, \text{ and}$$

$$P^{-1} A P = D^{-1} \hat{P}^{-1} A \hat{P} D = D^{-1} \hat{J} D = \text{Diag}(D_1^{-1} \hat{J}_1 D_1, \dots, D_u^{-1} \hat{J}_u D_u) = \text{Diag}(J_1, \dots, J_u) = J$$

where J is defined in the statement of the lemma. Q.E.D.

Lemma 5.5 (References 5 and 12). If λ is a characteristic root of $A = (a_{ij})$ ($i, j = 1, \dots, n$), then

$$|\lambda - a_{ii}| \leq \sum_{j=1, j \neq i}^{n'} |a_{ij}| \text{ for some } i, 1 \leq i \leq n$$

(where ' indicates that $j = i$ is omitted from the summation).

Further, $|\lambda| \leq \sup_{(i)} \sum_{j=1}^n |a_{ij}|$.

Proof: Let $x = \text{Col}(x_1, \dots, x_n)$ be a characteristic vector corresponding to λ . Then $\lambda x_i = \sum_{j=1}^n a_{ij} x_j$,

$$(\lambda - a_{ii}) x_i = \sum_{j=1, j \neq i}^{n'} a_{ij} x_j : i = 1, \dots, n.$$

If $|x_M| = \max_{(i)} |x_i|$, then

$$|\lambda - a_{MM}| |x_M| \leq \sum_{j=1, j \neq M}^{n'} |a_{Mj}| |x_j| \leq |x_M| \sum_{j=1}^{n'} |a_{Mj}|,$$

$$|\lambda - a_{MM}| \leq \sum_{j=1}^{n'} |a_{Mj}| \text{ (since } |x_M| \neq 0 \text{)}.$$

The second estimate follows directly from the first. Q.E.D.

Lemma 5.6. Let $\lambda_1, \dots, \lambda_n$ be the characteristic roots of $A = (a_{ij})$ ($i, j = 1, \dots, n$) and let $L_i = \text{Re}(\lambda_i)$ ($i = 1, \dots, n$); assume that if $L_i = 0$ ($1 \leq i \leq n$), then λ_i corresponds to a simple elementary divisor of A (In particular, this holds if λ_i is a simple root of the characteristic equation of A). Let J be a matrix in Jordan form which is similar to A and let $\mu_1, \dots, \mu_n$ be the characteristic roots of $\frac{1}{2}(J^* + J)$. Given c , an arbitrary positive constant, the matrix J may be chosen such that

- 1) if λ_i ($1 \leq i \leq n$) corresponds to a simple elementary divisor of A , then there exists a μ_j ($1 \leq j \leq n$) such that $\mu_j = L_i$;
- 2) if λ_i ($1 \leq i \leq n$) corresponds to an elementary divisor of

multiplicity $m(1) > 1$ (hence $L_1 \neq 0$), then there exist m μ_j 's ($1 \leq j \leq n$) such that

$$|L_1| [\text{Sign}(L_1) - c] \leq \mu_j \leq |L_1| [\text{Sign}(L_1) + c];$$

3) the two preceding classes account for all μ_j 's ($1 \leq j \leq n$).

Proof: Following Lemma 5.4, let P be such that

$$P^{-1}AP = J = \text{Diag}(J_1, J_2, \dots, J_u).$$

Then $\frac{1}{2}(J^* + J) = \text{Diag}[\frac{1}{2}(J_1^* + J_1), \dots, \frac{1}{2}(J_u^* + J_u)]$ and

$$\text{Det}[\frac{1}{2}(J^* + J) - \mu I] = \text{Det}[\frac{1}{2}(J_1^* + J_1) - \mu I] \times \dots \times \text{Det}[\frac{1}{2}(J_u^* + J_u) - \mu I]$$

where each I is an identity matrix of the same size as the J .

The roots of the equation

$$\text{Det}[\frac{1}{2}(J^* + J) - \mu I] = 0$$

coincide with the roots of the system of equations

$$(*) \quad \text{Det}[\frac{1}{2}(J_i^* + J_i) - \mu I] = 0 \quad (i = 1, \dots, u).$$

Let $\text{Re}(\lambda_{1j}) = L_{1j}$ ($i = 1, \dots, u; j = 1, \dots, v(i)$). For $i = 1$,

$$\frac{1}{2}(J_1^* + J_1) = \text{Diag}(L_{11}, \dots, L_{1v(1)})$$

and the roots of equation $(*)$ are $L_{11}, \dots, L_{1v(1)}$. These are the characteristic roots described in (1) of this lemma.

For $i = 2, \dots, v$,

$$\frac{1}{2}(J_i^* + J_i) = \begin{bmatrix} L_{11} & \frac{1}{2}R_1 & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{2}R_1 & L_{12} & & & & \\ 0 & \frac{1}{2}R_1 & & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ \cdot & \cdot & & & & \\ 0 & 0 & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \quad \text{where } L_{11} = \dots = L_{1v(1)} = L_1.$$

Using Lemma 4.5, equation $(*)$ has $v(i)$ roots satisfying

$$|L_1 - \mu| \leq |R_1|.$$

With $L_1 \neq 0$ and c an arbitrary positive constant, let $R_1 = cL_1$

($i = 2, \dots, u$). Then

$$L_1 - c|L_1| \leq \mu \leq L_1 + c|L_1|$$

$$|L_1|[\text{Sign}(L_1) - c] \leq \mu \leq |L_1|[\text{Sign}(L_1) + c] \quad (1 = 2, \dots, u).$$

These are the characteristic roots described in (2) of this lemma.

Further, this accounts for all of the characteristic roots of $\frac{1}{2}(J^* + J)$.

Q.E.D.

Remark on Lemma 5.6. There is a simpler, but less accurate subdivision of the characteristic roots of A. All of the simple roots of the characteristic equation of A fall under (1) of the lemma; the multiple roots may fall under (1) or (2), but as long as the real part of the root is not zero the rougher estimate of (2) holds. Hence if λ_1 ($1 \leq i \leq n$) is a root of the characteristic equation of multiplicity $M(i) > 1$ and if $L_1 \neq 0$, then there exist M μ_j 's ($1 \leq j \leq n$) satisfying the inequalities of (2).

Definition 5.1. With $A = (a_{ij})$ ($i, j = 1, \dots, n$), $M|CR(A)|$ is the maximum of the absolute values of the characteristic roots of A. If the characteristic roots of A are real, then $MCR(A)$ is the maximum characteristic root of A and $mCR(A)$ is the minimum characteristic root of A.

Lemma 5.7. If $F = (f_{ij})$ ($i, j = 1, \dots, n$), then

$$MCR \left\{ \frac{1}{2}(F^* + F) \right\} \leq M|CR \left\{ \frac{1}{2}(F^* + F) \right\}| = \left| \frac{1}{2}(F^* + F) \right| \leq |F| \leq \|F\|$$

and

$$mCR \left\{ \frac{1}{2}(F^* + F) \right\} \geq -M|CR \left\{ \frac{1}{2}(F^* + F) \right\}| = -\left| \frac{1}{2}(F^* + F) \right| \geq -|F| \geq -\|F\|.$$

If $F = P^{-1}BP$, where P, B and P^{-1} are n by n matrices, then

$$MCR \left\{ \frac{1}{2}(F^* + F) \right\} \leq |F| \leq |P^{-1}| |B| |P| \leq |P^{-1}| \|B\| |P|$$

and

$$mCR \left\{ \frac{1}{2}(F^* + F) \right\} \geq -|F| \geq -|P^{-1}| |B| |P| \geq -|P^{-1}| \|B\| |P|.$$

If P is unitary (or orthogonal, if P is real), then $|P| = |P^*| = 1$ (or $|P| = |P^T| = 1$).

Proof. First, the derivation of the bound for $MCR \left\{ \frac{1}{2}(F^* + F) \right\}$:

the first inequality is trivial; the equality follows from Lemma 2.1, (6); the next inequality follows from Lemma 2.1, (2), (3) and (4), for

$$|\frac{1}{2}(F^* + F)| \leq \frac{1}{2}(|F^*| + |F|) = |F|;$$

the last inequality follows from Lemma 2.1, (7).

The derivation of the bound for $\text{mCR} \left\{ \frac{1}{2}(F^* + F) \right\}$ is similar to the derivation just completed.

If $F = P^{-1}BP$, then using Lemma 2.1

$$|F| = |P^{-1}BP| \leq |P^{-1}| |B| |P| \leq |P^{-1}| \|B\| |P|.$$

The remark on orthogonal and unitary P 's is well known. Q.E.D.

The preceding lemmas will now be used to study the homogeneous linear system. First, a lemma which improves the wording of a subsequent theorem.

Lemma 5.8. Given the system of equations:

(*) $x' = Ax$, $A = (a_{ij})$ ($i, j = 1, \dots, n$) is a constant matrix (for a definition of the matrix equation $x' = Ax$, see (H), Section 2), let $\lambda_1, \dots, \lambda_n$ be the characteristic roots of A and $L_1 = \text{Re}(\lambda_1)$ ($i = 1, \dots, n$).

1) The vector $x = 0$ is an exponentially stable solution of (*) if and only if $L_1 < 0$ for $i = 1, \dots, n$.

2) The vector $x = 0$ is a stable solution of (*) if and only if
 i) $L_1 \leq 0$ for $i = 1, \dots, n$ and ii) if $L_1 = 0$ ($1 \leq i \leq n$), then λ_1 corresponds to a simple elementary divisor of A .

3) The norms of all the nonzero solutions of (*) are bounded away from zero if and only if $L_1 \geq 0$ for $i = 1, \dots, n$.

4) The norms of all the nonzero solutions of (*) increase exponentially if and only if $L_1 > 0$ for $i = 1, \dots, n$.

Proof. This is a well known result. A proof of parts (1) and (2)

may be found in reference 9, p. 20; the proof of parts (3) and (4) is similar to the proof of the first parts.

The asymptotic properties of the homogeneous linear system with constant coefficients obtained in Lemma 5.8 prompts a similar attempt for the system with variable coefficients. This attempt meets with two complications: the transformation to Jordan form introduces new, unwieldy terms (the substitution $y = P^{-1}(t)x$, where $P^{-1}(t)A(t)P(t) = J(t)$, transforms $x' = A(t)x$ into $y' = [J(t) - P^{-1}(t)P'(t)]y$); and the properties of Lemma 5.8 do not carry over to the equation with variable coefficients.¹ An alternative approach is to start with a constant matrix and see what sort of variable terms may be added without changing the asymptotic behavior.

Consider the system of differential equations

$$(5.1) \quad x' = [A + \epsilon B(t)]x$$

where $A = (a_{ij})$ ($i, j = 1, \dots, n$) is a constant matrix, $B(t) = (b_{ij}(t))$ ($i, j = 1, \dots, n$) is a matrix function of t which is defined and continuous for $t \geq T_0$, and ϵ , $0 \leq \epsilon < \infty$, is a real parameter (this matrix notation for systems of differential equations is defined in Section 2). In particular, equation (5.1) is a special case of equation (H) and the theorems of Section 2 apply to equation (5.1).

Theorem 5.1. Given equation (5.1), assume that $|B(t)| \leq M$, M a

¹For example, the equation $x'' + (1/4t)x' + (1/8t^2)x = 0$: $t \geq 1$ has $x = \sqrt{t}$ as a solution and yet the characteristic roots of this equation, $(-1 \pm \sqrt{-7})/8t$, have negative real parts. As a second example, the system

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -1 & te^t \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad : \quad t \geq 0$$

has (-1) as a double characteristic root while $x_1 = (t - 1) + e^{-t}$, $x_2 = e^{-t}$ is a solution.

constant, for $t \geq T_0$.

1) If $x = 0$ is an exponentially stable solution of equation (5.1) when $\epsilon = 0$, then there exists an $\epsilon_0 > 0$ such that for $0 \leq \epsilon < \epsilon_0$, $x = 0$ is still an exponentially stable solution of equation (5.1).

2) If $x = 0$ is a stable solution of equation (5.1) when $\epsilon = 0$ (which is equivalent to its being a uniformly stable solution) and if $\int_{T_0}^{\infty} |B(t)| dt < \infty$, then $x = 0$ is a uniformly stable solution of equation (5.1) for any $\epsilon \geq 0$.

3) If the norms of all the nonzero solutions of equation (5.1) are bounded away from zero when $\epsilon = 0$ and if $\int_{T_0}^{\infty} |B(t)| dt < \infty$, then the norms of all the nonzero solutions of equation (5.1) are bounded away from zero for all $\epsilon \geq 0$.

4) If the norms of the nonzero solutions of equation (5.1) increase exponentially when $\epsilon = 0$, then there exists an $\epsilon_0 > 0$ such that for $0 \leq \epsilon < \epsilon_0$, the norms of the nonzero solutions of equation (5.1) still increase exponentially.

Proof. Let equation (5.1) be given. By Lemma 5.4, there exists a constant, nonsingular P such that $P^{-1}AP = J$ where J is in Jordan form. By Lemma 5.1, the mapping $y = P^{-1}x$ transforms equation (5.1) into

$$(*) \quad y' = [J + \epsilon P^{-1}B(t)P]y = [J + F(t, \epsilon)]y = C(t, \epsilon)y$$

where the theorems of Section 2 may be applied to equation (*). Further, if $x(t)$ is a solution of equation (5.1), then $y(t) = P^{-1}x(t)$ is a solution of equation (*) and, by Lemma 5.2, if any of the conclusions of (1) - (4) of this theorem is satisfied by the solutions of equation (*), then it is also satisfied by the solutions of equation (5.1). It is therefore necessary only to consider equation (*).

1) Let $\lambda_1, \dots, \lambda_n$ be the characteristic roots of A and let

$L = \max_{(1)} \operatorname{Re}(\lambda_1)$; the hypotheses of (1) and Lemma 5.8, (1) imply that $L < 0$. Let c_1 , $0 < c_1 < 1$, be an arbitrary constant; by Lemma 5.6, the matrix P may be chosen such that if μ is a characteristic root of $\frac{1}{2}(J^* + J)$, then $\mu \leq (1 - c_1)L$. Let $\epsilon_0 = (c_1 - 1)L(M|P^{-1}||P|)^{-1}$; if $0 \leq \epsilon < \epsilon_0$, then $\epsilon = c_2 \epsilon_0$ where $0 \leq c_2 < 1$. Using Lemma 5.7, if $0 \leq \epsilon < \epsilon_0$ and $\pi(t, \epsilon)$ is a characteristic root of $\frac{1}{2}(F^*(t, \epsilon) + F(t, \epsilon))$, then $|\pi(t, \epsilon)| \leq |F(t, \epsilon)| = |\epsilon P^{-1}B(t)P| \leq c_2 \epsilon_0 |P^{-1}||B(t)||P|$, or $|\pi(t, \epsilon)| \leq c_2(c_1 - 1)L$. Then by Lemma 5.3, $\Lambda_C(t, \epsilon) \leq \Lambda_J + \Lambda_F(t, \epsilon) \leq (1 - c_1)L + c_2(c_1 - 1)L = (1 - c_1)(1 - c_2)L < 0$ and by Corollary 3 of Theorem 2.1, $y = 0$ is an exponentially stable solution of equation (*).

2) Let $\lambda_1, \dots, \lambda_n$ be the characteristic roots of A , $L_1 = \operatorname{Re}(\lambda_1)$ ($i = 1, \dots, n$), and $L = \max_{(1)} L_1$. The hypotheses of (2) and Lemma 5.8, (2) imply that $L \leq 0$ and that if λ_1 is a characteristic root corresponding to a multiple elementary divisor of A , then $L_1 < 0$ ($1 \leq i \leq n$). By Lemma 5.6, the matrix P may be chosen such that if μ is a characteristic root of $\frac{1}{2}(J^* + J)$, then $\mu \leq 0$. By Lemma 5.7, if $\pi(t, \epsilon)$ is a characteristic root of $\frac{1}{2}(F^*(t, \epsilon) + F(t, \epsilon))$, then

$$|\pi(t, \epsilon)| \leq |F(t, \epsilon)| = |\epsilon P^{-1}B(t)P| \leq \epsilon |P^{-1}||P||B(t)|.$$

By Lemma 5.3,

$$\Lambda_C(t, \epsilon) \leq \Lambda_J + \Lambda_F(t, \epsilon) \leq \epsilon |P^{-1}||P||B(t)|$$

and since $\int_{t_0}^t |B(u)| du \leq \int_{t_0}^{\infty} |B(t)| dt < \infty$ for all $t \geq t_0 \geq T_0$, Corollary 1 of Theorem 2.1 implies that $x = 0$ is a uniformly stable solution of (*).

3 and 4) The proofs of (3) and (4) are similar, respectively, to the proofs of (2) and (1) and will be omitted. Q.E.D.

Remarks on Theorem 5.1. The matrix P , defined in the theorem, is not unique and for parts (1) and (4) it would be nice to specify an

"optimal" P . This is difficult; for instance in part (1) it would involve minimizing the function

$$U(c_1, P, \epsilon) = \text{l.u.b.} \left\{ \Lambda_J(c_1) + \Lambda_F(t, \epsilon, P(c_1)) : T_0 \leq t < \infty \right\}$$

(The notation is taken from part (1) of the proof.).

Parts (1) and (2) of Theorem 5.1 have been proved by several authors (see, for example, reference 2) and a good summary may be found in reference 9, pp. 33 - 37.

For part (2), it has been shown that if $x(t)$ is a solution of (5.1) for some $\epsilon \neq 0$, then there exists a solution of (5.1) with $\epsilon = 0$, say $\bar{x}(t)$, such that $|x(t) - \bar{x}(t)| \rightarrow 0$ as $t \rightarrow \infty$ (see references 17 and 25).

If the characteristic roots of A are distinct, then Theorem 5.1 may be improved upon. Consider the system of differential equations

$$(5.2) \quad x' = [A + B(t)]x$$

(this is equation (5.1) with $\epsilon = 1$).

Theorem 5.2. Given equation (5.2), let $\lambda_1, \dots, \lambda_n$ be the characteristic roots of A ; let $R = \text{Max}_{(1)} \text{Re}(\lambda_1)$ and $r = \text{Min}_{(1)} \text{Re}(\lambda_1)$; and assume that A is similar to a diagonal matrix (This assumption will be satisfied if $\lambda_i \neq \lambda_j$ for $1 \leq i \neq j \leq n$). Let P be a constant, nonsingular matrix such that $P^{-1}AP = J$ where J is in Jordan (diagonal) form; let $P^{-1}B(t)P = F(t)$; let $\Lambda_F(t)$ and $\lambda_F(t)$ be the supremum and infimum, respectively, of the characteristic roots of $\frac{1}{2}(F^*(t) + F(t))$; and let $H(t_0) = \int_{t_0}^{\infty} \{R + \Lambda_F(t)\} dt$ and $h(t_0) = \int_{t_0}^{\infty} \{r + \lambda_F(t)\} dt$ for $t \geq T_0$.

First, assume $R \leq 0$.

1) If $H(T_0) < \infty$, then $x = 0$ is a stable solution of equation (5.2). If $H(t_0) \leq M$, M a constant, for all $t_0 \geq T_0$, then $x = 0$ is a uniformly stable solution of equation (5.2).

2) If $H(T_0) = -\infty$, then $x = 0$ is an equiasymptotically stable

solution of equation (5.2).

3) If $R + \Lambda_F(t) \leq -c$, c a positive constant, for $t \geq T_0$, then $x = 0$ is an exponentially stable solution of equation (5.2).

Now assume $r \geq 0$.

4) If $h(t_0) \geq m > -\infty$ for all $t_0 \geq T_0$, then the norms of the nonzero solutions of equation (5.2) are bounded away from zero.

5) If $h(T_0) = \infty$, then the norms of the nonzero solutions of equation (5.2) are unbounded as $t \rightarrow \infty$. If $r + \lambda_F(t) \geq c$, c a positive constant, for $t \geq T_0$, then the norms of the nonzero solutions of equation (5.2) increase exponentially as $t \rightarrow \infty$.

Proof: Repeating the first part of the proof of Theorem 5.1, equation (5.2) may be transformed into

$$(*) \quad y' = [J + P^{-1}B(t)P]y = [J + F(t)]y = C(t)y$$

and again, to prove the theorem for equation (5.2), it is sufficient to prove the theorem for equation (*). Here $P^{-1}AP = J = \text{Diag}(\lambda_1, \dots, \lambda_n)$.

By Lemma 5.6, $\text{MCR}\left\{\frac{1}{2}(J^* + J)\right\} = R$ and $\text{mCR}\left\{\frac{1}{2}(J^* + J)\right\} = r$; by Lemma 5.3,

$$\begin{aligned} \Lambda_C(t) &\leq \Lambda_J + \Lambda_F(t) = R + \Lambda_F(t) \quad \text{and} \\ \lambda_C(t) &\geq \lambda_J + \lambda_F(t) = r + \lambda_F(t) \quad \text{for } t \geq T_0. \end{aligned}$$

If these bounds are used in Corollaries 1 - 5 of Theorem 2.1, then (1) - (5) of the theorem follow. Q.E.D.

Remarks on Theorem 5.2. Again, it would be nice, but difficult, to specify an "optimal" P . For instance, in part (1) this would involve minimizing the integral

$$I(P) = \int_{T_0}^{\infty} \{R + \Lambda_F(t, P)\} dt.$$

The first advantage of this theorem over Theorem 5.1 is that $|B(t)|$ need not be bounded. For example, using part (1) and Lemma 5.7, if

$$\int_{T_0}^t |B(t)| dt \leq (-R/|P^{-1}||P|)(t - T_0) \quad \text{for } t \geq T \geq T_0,$$

T sufficiently large, then $x = 0$ is a uniformly stable solution of equation (5.2).

The second advantage is that in applications where P and P^{-1} may be computed the results may be further improved. For a useful example of this, consider the n -th order differential equation

$$x^{(n)} + [a_1 + b_1(t)]x^{(n-1)} + \dots + [a_n + b_n(t)]x = 0,$$

or in system notation

$$(5.3) \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix} = \left\{ \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & 1 \\ -a_n & -a_{n-1} & \dots & -a_1 \end{bmatrix} + \begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ \vdots & & \vdots \\ 0 & & 0 \\ -b_n(t) & \dots & -b_1(t) \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix},$$

$$\text{i.e. } x' = A + B(t)x.$$

Here $a_1, \dots, a_n$ are real or complex constants and $b_1(t), \dots, b_n(t)$ are real or complex valued functions which are defined and continuous for $t \geq T_0$. Equation (5.3) is a special case of equation (5.2).

The notation of Definition 5.2 will be used without further mention in Lemmas 5.9 - 5.12.

Definition 5.2. 1) With a_i and $b_i(t)$, $i = 1, \dots, n$, as just defined, let $Q(r) = r^n + a_1 r^{n-1} + \dots + a_n$ and $Q_b(r, t) = r^n + [a_1 + b_1(t)]r^{n-1} + \dots + [a_n + b_n(t)]$ where r is an indeterminate.

2) Let $\lambda_1, \dots, \lambda_n$ be the roots of $Q(r) = 0$ and $\mu_1(t), \dots, \mu_n(t)$ be the roots of $Q_b(r, t) = 0$. Given equation (5.3), the λ 's are the characteristic roots of A and the μ 's, the characteristic roots of $[A + B(t)]$.

3) For $i = 1, \dots, n$, let $D_i = [(\lambda_1 - \lambda_1) \dots (\lambda_1 - \lambda_{i-1}) \times (\lambda_1 - \lambda_{i+1}) \dots (\lambda_1 - \lambda_n)]$.

4) Let $\sigma_i(\lambda_n)$, $0 \leq i \leq n$, be the i -th elementary symmetric function of the letters $\lambda_1, \dots, \lambda_n$; i.e., $\sigma_0(\lambda_n) = 1$, $\sigma_1(\lambda_n) = \lambda_1 + \dots + \lambda_n$, $\sigma_2(\lambda_n) = \lambda_1 \lambda_2 + \dots + \lambda_1 \lambda_n + \dots + \lambda_{n-1} \lambda_n, \dots$, $\sigma_n(\lambda_n) = \lambda_1 \dots \lambda_n$. Let $\sigma_i(\lambda_n | \lambda_j)$ ($0 \leq i \leq n-1$, $1 \leq j \leq n$) be the i -th elementary symmetric function of the letters $\lambda_1, \dots, \lambda_{j-1}, \lambda_{j+1}, \dots, \lambda_n$.

Lemma 5.9. The coefficients in the polynomial equation $Q(r) = 0$ are given by $a_i = (-1)^i \sigma_i(\lambda_n)$, $i = 1, \dots, n$.

Proof. For a proof of this lemma, see reference 18, p. 86. Q.E.D.

Lemma 5.10. For $1 \leq i \leq n$,

$$Q_b(\lambda_i, t) = \sum_{j=1}^n b_j(t) \lambda_i^{n-j} = (\lambda_i - \mu_1(t)) \dots (\lambda_i - \mu_n(t)).$$

Proof. Note that $Q(\lambda_i) = 0$ for $i = 1, \dots, n$ and substitute λ_i into $Q_b(r, t) = r^n + [a_1 + b_1(t)]r^{n-1} + \dots + [a_n + b_n(t)] = (r - \mu_1(t)) \times \dots (r - \mu_n(t))$. Q.E.D.

Lemma 5.11. Given equation (5.3), assume that $\lambda_i \neq \lambda_j$ for $1 \leq i \neq j \leq n$ (i.e., the characteristic roots of A are distinct). Then $P^{-1}AP = J = \text{Diag}(\lambda_1, \dots, \lambda_n)$ where $P = (p_{ij}) = (\lambda_j^{i-1})$ and $P^{-1} = (p_{ij}^{(-1)}) = ((-1)^{n-j} D_i^{-1} \sigma_{n-j}(\lambda_n | \lambda_i))$; $i, j = 1, \dots, n$.

Proof. (The first half was proved by Perron in reference 20.)

If $P^{-1}AP = J = \text{Diag}(\lambda_1, \dots, \lambda_n)$, where A is given in equation (5.3), then

$$AP = PJ,$$

$$\left(\sum_{k=1}^n a_{ik} p_{kj} \right) = p_{ij} \lambda_j : i, j = 1, \dots, n;$$

since $a_{i, i+1} = 1$, $a_{ij} = 0$ for $i = 1, \dots, n-1$, $j \neq i+1$,

$$p_{i+1, j} = p_{ij} \lambda_j : i = 1, \dots, n-1; j = 1, \dots, n.$$

Let $p_{1j} = 1$ and $p_{ij} = \lambda_j^{i-1}$ for $i = 1, \dots, n; j = 1, \dots, n$.

Let $(q_{i,n-1}, \dots, q_{i0})$ be the i -th row of P^{-1} and let $F_i(\lambda) = q_{i0} \lambda^{n-1} + q_{i1} \lambda^{n-2} + \dots + q_{i,n-1}$, $i = 1, \dots, n$. Then $P^{-1}P = I$ may be written as

$$\left. \begin{array}{l} F_1(\lambda_1) = 1 \\ F_i(\lambda_j) = 0 \end{array} \right\} i = 1, \dots, n; j = 1, \dots, i-1, i+1, \dots, n.$$

Hence $F_1(\lambda) = q_{10} (\lambda - \lambda_1) \dots (\lambda - \lambda_{i-1}) (\lambda - \lambda_{i+1}) \dots (\lambda - \lambda_n)$ where $q_{10} = [(\lambda_1 - \lambda_1) \dots (\lambda_1 - \lambda_{i-1}) (\lambda_1 - \lambda_{i+1}) \dots (\lambda_1 - \lambda_n)]^{-1} = D_1^{-1}$, $i = 1, \dots, n$. Then

$$D_1 F_1(\lambda) = \lambda^{n-1} + (D_1 q_{11}) \lambda^{n-2} + \dots + (D_1 q_{1,n-1}), i = 1, \dots, n.$$

Adapting Lemma 5.9 to this polynomial,

$$D_1 q_{1j} = (-1)^j \sigma_j(\lambda_n | \lambda_1) : i = 1, \dots, n; j = 1, \dots, n-1;$$

also,

$$D_1 q_{10} = 1 = (-1)^0 \sigma_0(\lambda_n | \lambda_1).$$

Since $p_{ij}^{(-1)} = q_{i,n-j}$,

$$p_{ij}^{(-1)} = (-1)^{n-j} D_1^{-1} \sigma_{n-j}(\lambda_n | \lambda_1) : i, j = 1, \dots, n. \text{ Q.E.D.}$$

Lemma 5.12. With $B(t)$ as defined in equation (5.3) and P and P^{-1} as defined in Lemma 5.11,

$$P^{-1}B(t)P = (-D_1^{-1}Q_b(\lambda_j, t)), i, j = 1, \dots, n.$$

Proof. With $B(t)$ as defined in equation (5.3),

$$b_{ij} = 0 \text{ for } i = 1, \dots, n-1; j = 1, \dots, n; b_{nj} = -b_{n+1-j}, j = 1, \dots, n.$$

Then

$$P^{-1}B(t)P = (p_{ik}^{(-1)} b_{kl}(t) p_{lj}) = (p_{in}^{(-1)} \sum_{k=1}^n -b_{n+1-k} p_{kj}),$$

$$P^{-1}B(t)P = (-p_{in}^{(-1)} \sum_{k=1}^n b_k p_{n+1-k,j}) : i, j = 1, \dots, n.$$

By Lemmas 5.10 and 5.11,

$$P^{-1}B(t)P = (-D_1^{-1} \sum_{k=1}^n b_k \lambda_j^{n-k}) = (-D_1^{-1} Q_b(\lambda_j, t)) : i, j = 1, \dots, n. \text{ Q.E.D.}$$

Theorem 5.3. Given equation (5.3), let $\lambda_1, \dots, \lambda_n$ be the characteristic roots of A and $\mu_1(t), \dots, \mu_n(t)$ the characteristic roots

of $[A + B(t)]$; assume $\lambda_1 \neq \lambda_j$ for $1 \leq i \neq j \leq n$. Let

$$|\lambda_M| = \max_{(1)} |\lambda_1|, \quad |b_M(t)| = \sup_{(1)} |b_1(t)|, \quad D = \max_{(1)} \prod_{j=1}^n |\lambda_1 - \lambda_j|$$

(' denotes that $j = 1$ is omitted), and $d = \min_{(1)} \prod_{j=1}^n |\lambda_1 - \lambda_j|$.

$$\text{With } Q_b(\lambda_1, t) = \sum_{j=1}^n b_j(t) \lambda_1^{n-j} = \prod_{j=1}^n (\lambda_1 - \mu_j(t)), \quad 1 \leq i \leq n$$

(i.e. $Q_b(\lambda_1, t)$ is the characteristic polynomial of $A + B(t)$ evaluated

at λ_1), let $Q_b(t) = \sup_{(1)} |Q_b(\lambda_1, t)|$. Then Theorem 5.2 may be

applied to equation (5.3) using any of the following estimates for

$R + \Lambda_F(t)$ and $r + \lambda_F(t)$.

$$1) \quad R + \Lambda_F(t) \leq R + nd^{-1} Q_b(t) \quad \text{and}$$

$$r + \lambda_F(t) \geq r - nd^{-1} Q_b(t).$$

$$2) \quad R + \Lambda_F(t) \leq R + nd^{-1} \max(1, |\lambda_M|^{n-1}) \sum_{j=1}^n |b_j(t)| \quad \text{and}$$

$$r + \lambda_F(t) \geq r - nd^{-1} \max(1, |\lambda_M|^{n-1}) \times \sum_{j=1}^n |b_j(t)|.$$

$$3) \quad \text{If } |\lambda_M| \neq 1, \quad \begin{cases} R + \Lambda_F(t) \leq R + nd^{-1} ((1 - |\lambda_M|^n) / (1 - |\lambda_M|)) |b_M(t)| \\ r + \lambda_F(t) \geq r - nd^{-1} ((1 - |\lambda_M|^n) / (1 - |\lambda_M|)) |b_M(t)|. \end{cases}$$

$$\text{If } |\lambda_M| = 1, \quad \begin{cases} R + \Lambda_F(t) \leq R + n^2 d^{-1} |b_M(t)| \\ r + \lambda_F(t) \geq r - n^2 d^{-1} |b_M(t)|. \end{cases}$$

4) If, as an additional hypothesis, $|\mu_1(t) - \lambda_1| \leq |\lambda_j - \lambda_1|$ for $1 \leq i \neq j \leq n$ and $t \geq T_0$, then

$$R + \Lambda_F(t) \leq R + nd^{-1} 2^{n-1} D \left[\sup_{(1)} |\mu_1(t) - \lambda_1| \right] \quad \text{and}$$

$$r + \lambda_F(t) \geq r - nd^{-1} 2^{n-1} D \left[\sup_{(1)} |\mu_1(t) - \lambda_1| \right].$$

Proof. Assume equation (5.3) is given and let P and P^{-1} be as defined in Lemma 5.11. Then $P^{-1}AP$ is in Jordan form and, as in

Theorem 5.2, $P^{-1}B(t)P = F(t)$. Further, $F(t)$ is given by Lemma 5.12.

Let $\pi(t)$ be a characteristic root of $\frac{1}{2}[F^*(t) + F(t)]$ and apply Lemma 5.5:

$$|\pi(t)| \leq \sup_{(1)} \left\{ \frac{1}{2} \sum_{j=1}^n |D_1^{-1} Q_b(\lambda_j, t) + \overline{D_j^{-1} Q_b(\lambda_1, t)}| \right\},$$

$$|\pi(t)| \leq \frac{1}{2} \sup_{(1)} \left\{ \sum_{j=1}^n |D_1^{-1} Q_b(\lambda_j, t)| + |D_j^{-1} Q_b(\lambda_1, t)| \right\}$$

(here $D_1 = \prod_{j=1}^n (\lambda_1 - \lambda_j)$). Then

$$|\pi(t)| \leq \frac{1}{2} \left\{ d^{-1} \sum_{j=1}^n |\omega_b(\lambda_j, t)| + \omega_b(t) \sum_{j=1}^n |D_j^{-1}| \right\},$$

$$|\pi(t)| \leq nd^{-1} \omega_b(t).$$

Then $\Lambda_F(t) \leq nd^{-1} \omega_b(t)$ and $\lambda_F(t) \geq -nd^{-1} \omega_b(t)$ and (1) of this theorem follows.

Since $\omega_b(\lambda_1, t) = \sum_{j=1}^n b_j(t) \lambda_1^{n-j}$, it follows that

$$|\omega_b(\lambda_1, t)| \leq \max(1, |\lambda_1|^{n-1}) \sum_{j=1}^n |b_j(t)| \quad \text{and}$$

$$\omega_b(t) \leq \max(1, |\lambda_M|^{n-1}) \sum_{j=1}^n |b_j(t)|;$$

(2) then follows from (1).

From the same expression for $\omega_b(\lambda_1, t)$, it also follows that

$$|\omega_b(\lambda_1, t)| \leq |b_M(t)| \sum_{j=1}^n |\lambda_1|^{n-j} \quad \text{and}$$

$$\omega_b(t) \leq |b_M(t)| \sum_{j=1}^n |\lambda_M|^{n-j} = |b_M(t)| ((1 - |\lambda_M|^n)/(1 - |\lambda_M|)), \text{ or}$$

$$\omega_b(t) \leq n|b_M(t)| \quad \text{if } |\lambda_M| = 1; \quad (3) \text{ then follows from (1).}$$

Since $\omega_b(\lambda_1, t) = \prod_{j=1}^n (\lambda_1 - \mu_j(t))$, it follows that

$$\omega_b(\lambda_1, t) = (\lambda_1 - \mu_1(t)) \prod_{j=1}^{n-1} (\lambda_1 - \lambda_j + \lambda_j - \mu_j(t))$$

and $|\omega_b(\lambda_1, t)| \leq |\lambda_1 - \mu_1(t)| \prod_{j=1}^{n-1} |\lambda_1 - \lambda_j| (1 + |(\lambda_j - \mu_j(t))/(\lambda_1 - \lambda_j)|).$

If $|\mu_1(t) - \lambda_1| \leq |\lambda_j - \lambda_1|$ for $1 \leq i \neq j \leq n$, then

$$|\omega_b(\lambda_1, t)| \leq 2^{n-1} |\lambda_1 - \mu_1(t)| \prod_{j=1}^{n-1} |\lambda_1 - \lambda_j| \quad \text{and}$$

$$\omega_b(t) \leq 2^{n-1} D \sup_{(1)} |\lambda_1 - \mu_1(t)|;$$

(4) then follows from (1). Q.E.D.

Remarks on Theorem 5.3. Part (4) states that if the characteristic roots of $A + B(t)$ are "close enough" to the characteristic roots of A , then $x' = (A + B(t))x$ and $x' = Ax$ have the same asymptotic behavior. This holds for n -th order equations written in system form; as the second example of the footnote on page 68 shows, the statement is not true for general systems.

Example 5.1.(Illustrating Theorem 5.3).

$$x''' + 4x'' + 6x' + (4 + f(t))x = 0 : t \geq T_0 = 0.$$

Assume $f(t) = f(t + T)$ for all $t \geq 0$. Writing this equation in the form of equation (5.3),

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -6 & -4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -f(t) & 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The characteristic roots of the constant matrix are: $\lambda_1 = -1 + i$, $\lambda_2 = -1 - i$, $\lambda_3 = -2$; also $R = -1$, $n = 3$, $d = 2$, and $|\lambda_M| = 2$. By (2) of Theorem 5.3,

$$R + \Lambda_F(t) \leq (-1) + (3/2)(4)|f(t)| = (-1) + 6|f(t)|.$$

Then by a modified version of Theorem 5.2 (Theorem 5.2 is easily adapted to periodic systems by using Corollaries 1' - 5' of Theorem 2.1 in place of Corollaries 1 - 5.): if $\int_0^T |f(t)| dt \leq (T/6)$ (or $<$), then $\int_0^T \{R + \Lambda_F(t)\} dt \leq 0$ (or $<$) and $x = 0$ is a stable (or exponentially stable) solution of this example.

The methods just developed for the homogeneous linear equation will now be applied to the nonhomogeneous linear equation. Consider the equation

$$(5.4) \quad x' = (A + \epsilon B(t))x + f(t)$$

where this is the same as equation (5.1) with the addition that $f(t) = \text{Col}(f_1(t), \dots, f_n(t))$ is a vector function of t which is defined and continuous for $t \geq T_0$. Equation (5.4) is a special case of equation (NH) and thus the theorems of Section 3 apply to equation (5.4). Let (H:5.4) be the homogeneous equation associated with equation (5.4), i.e. the equation obtained by suppressing $f(t)$.

Theorem 5.4. Let equation (5.4) be given and assume that $|B(t)| \leq M$ and $|f(t)| \leq N$, M and N constants, for $t \geq T_0$.

1) If $x = 0$ is an exponentially stable solution of equation (H:5.4) when $\epsilon = 0$, then there exists an $\epsilon_0 > 0$ such that for $0 \leq \epsilon < \epsilon_0$, the limits superior (as $t \rightarrow \infty$) of the norms of all solutions of equation (5.4) are less than or equal to some constant, Q .

2) If $x = 0$ is a stable solution of (H:5.4) when $\epsilon = 0$, if $\int_0^\infty |B(t)| dt < \infty$, and if $\int_0^\infty |f(t)| dt < \infty$, then the solutions of equation (5.4) are locally bounded for any $\epsilon \geq 0$.

Proof: Repeating the beginning of the proof of Theorem 5.1, equation (5.4) may be transformed into

$$(*) \quad y' = (P^{-1}AP + \epsilon P^{-1}B(t)P)y + P^{-1}f(t) = (J + F(t, \epsilon))y + g(t) = C(t, \epsilon)y + g(t)$$

where J is in Jordan form. By Lemma 5.1, the theorems of Section 3 may be applied to equation $(*)$ and by Lemma 5.2, if either of the conclusions of (1) or (2) of this theorem holds for $(*)$, then it also holds for (5.4) (with the exception of the actual value of the constant in (1); it will have one value, Q' , for $(*)$ and another value, $Q \leq |P|Q'$, for equation (5.4)). Therefore, it is necessary only to study equation $(*)$.

1) Under the hypotheses of (1), the proof of Theorem 5.1, (1) may be repeated to show that there exists an $\epsilon_0 > 0$ such that if $0 \leq \epsilon < \epsilon_0$, then there exists a $d(\epsilon) > 0$ with $\Lambda_C(t, \epsilon) \leq -d$ for $t \geq T_0$. Further, $|g(t)| = |P^{-1}f(t)| \leq |P^{-1}||f(t)| \leq N|P^{-1}|$ for $t \geq T_0$. By the corollary of Theorem 3.3, if $0 \leq \epsilon < \epsilon_0$ and if $y(t)$ is a solution of $(*)$, then $\limsup_{t \rightarrow \infty} |y(t)| \leq N|P^{-1}|/d = Q'$.

2) Under the hypotheses of (2), the proof of Theorem 5.1, (2) may be repeated to show that $\int_{t_0}^t \Lambda_C(s, \epsilon) ds \leq F(\epsilon)$ for all $t \geq t_0 \geq T_0$ and $\epsilon \geq 0$. Further, $\int_{t_0}^\infty |g(t)| dt \leq |P^{-1}| \int_{t_0}^\infty |f(t)| dt < \infty$ and hence, by (1) of the Corollary of Theorem 3.1, the solutions of equation $(*)$ are locally bounded. Q.E.D.

Remarks on Theorem 5.4. With $\epsilon = 0$, parts (1) and (2) of this Theorem are, respectively, the "sufficient half" of the Perron theorem and the "sufficient half" of the Bellman theorem (both of which are discussed in the remarks on Theorem 3.1) for the special case of $A(t) = A$ constant.

The method of this section has been used in conjunction with just two of the theorems of Section 3. Several more theorems on the boundedness or the growth of the solutions of equation (5.4) may be obtained in the same manner from the remaining results of Section 3. Also, if the characteristic roots of A are distinct or if the equation is an n -th order equation written as a system, then improved versions of these theorems may be stated.

So too with the nonlinear equation; the method of this section may be combined with results of Section 4 to produce several theorems on the asymptotic behavior of solutions. One such theorem will be given here.

Consider the equation

$$(5.5) \quad x' = (A + \epsilon_1 B(t))x + \epsilon_2 f(x, t)$$

where $A = (a_{ij})$ ($i, j = 1, \dots, n$) is a matrix function of t which is defined and continuous for $t \geq T_0$, ϵ_1 and ϵ_2 , $0 \leq \epsilon_1, \epsilon_2 < \infty$, are real parameters and $f(x, t) = \text{Col}(f_1(x, t), \dots, f_n(x, t))$ is a vector function of (x, t) which is defined and continuous for $(x, t) \in D$ (D is the same as in Definition 2.1). In particular, equation (5.5) is a special case of equation (NL) and hence the theorems of Section 4 apply to equation (5.5).

Theorem 5.5. Let equation (5.5) be given and assume that $|B(t)| \leq M$, a constant, for $t \geq T_0$. Further, assume that for all $(x, t) \in D$,

$|f(x, t)| \leq F(t)|x|$ where $F(t)$ is defined, continuous, and $F(t) \leq N$, a constant, for $t \geq T_0$.

1) If $x = 0$ is an exponentially stable solution of the equation $x' = Ax$, then there exists an $\epsilon_0 > 0$ such that if $0 \leq \epsilon_1, \epsilon_2 < \epsilon_0$, then $x = 0$ is an exponentially stable solution of equation (5.5).

2) If $x = 0$ is a uniformly stable solution of the equation $x' = Ax$ and if $\int_{t_0}^{\infty} |B(t)| dt < \infty$ and $\int_{t_0}^{\infty} F(t) dt < \infty$, then $x = 0$ is a uniformly stable solution of equation (5.5) for all $\epsilon_1, \epsilon_2 \geq 0$.

Proof: The proof is much like earlier proofs and will just be given in brief. First, equation (5.5) may be transformed into

$$(*) \quad y' = (J + \epsilon_1 P^{-1} B(t) P) y + \epsilon_2 P^{-1} f(Py, t) = C(t, \epsilon_1) y + \epsilon_2 g(y, t)$$

$$\text{where } |g(y, t)| = |P^{-1} f(Py, t)| \leq |P^{-1}| F(t) |Py| \leq |P^{-1}| |P| F(t) |y| = G(t) |y|.$$

Second, it is sufficient to prove the theorem for equation (*).

1) As in the proof of Theorem 5.4, (1), there exists an $\epsilon_3 > 0$ such that if $0 \leq \epsilon_1 < \epsilon_3$, then $\Lambda_C(t, \epsilon_1) \leq -d(\epsilon_1) < 0$ for $t \geq T_0$. Let $\bar{d} = \text{g.l.b.} \{d(\epsilon_1) : 0 \leq \epsilon_1 < \epsilon_3/2\}$ (using $\epsilon_3/2$ insures $\bar{d} > 0$) and let $\epsilon_4 = \bar{d} / (2N |P^{-1}| |P|)$ and $\epsilon_0 = \min(\epsilon_3/2, \epsilon_4)$. Then $0 \leq \epsilon_1, \epsilon_2 < \epsilon_0$ implies that

$$\Lambda_C(t, \epsilon_1) + \epsilon_2 G(t) \leq -d(\epsilon_1) + \frac{1}{2} \bar{d} \leq -\frac{1}{2} \bar{d} < 0 \text{ for } t \geq T_0$$

and by Theorem 4.7, (5), $y = 0$ is an exponentially stable solution of (*).

2) As in the proof of Theorem 5.4, (2), $\int_{t_0}^t \Lambda_C(s, \epsilon_1) ds \leq F(\epsilon_1)$ for $t \geq t_0 \geq T_0$ and $\epsilon_1 > 0$. Also, $\int_{t_0}^{\infty} \epsilon_2 G(s) ds \leq \epsilon_2 |P^{-1}| |P| \int_{t_0}^{\infty} F(s) ds < \infty$ for $t \geq t_0 \geq T_0$ and by Theorem 4.7, (1), $y = 0$ is a stable solution of (*). Q.E.D.

This section ends with a discussion of the real second order differential equation. First, the equation provides another example for Theorem 5.3; second, the method of this section may be directly

applied to the second order equation and the computations may be carried out in detail. The result is three lemmas; these lemmas provide a method for determining the asymptotic behavior of the solutions of the second order equation.

For another discussion of the same technique, a discussion to which I am indebted, see reference 15, section 8-15.

Consider the equation

$$x'' + (a_1 + b_1(t))x' + (a_2 + b_2(t))x = 0$$

or, in system notation,

$$(5.6) \quad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \left\{ \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -b_2(t) & -b_1(t) \end{bmatrix} \right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

i.e. $x' = (A + B(t))x$,

where a_1 and a_2 are real constants and $b_1(t)$ and $b_2(t)$ are real valued functions which are defined and continuous for $t \geq T_0$.

Definition 5.3. If equation (5.6) is given, let λ_1 and λ_2 be the characteristic roots of A , i.e.

$$\lambda_1 = \frac{1}{2} \left\{ -a_1 - \sqrt{a_1^2 - 4a_2} \right\}, \quad \lambda_2 = \frac{1}{2} \left\{ -a_1 + \sqrt{a_1^2 - 4a_2} \right\} \text{ if } a_1^2 - 4a_2 \geq 0 \text{ and}$$

$$\lambda_1 = \frac{1}{2} \left\{ -a_1 - i\sqrt{4a_2 - a_1^2} \right\}, \quad \lambda_2 = \frac{1}{2} \left\{ -a_1 + i\sqrt{4a_2 - a_1^2} \right\} \text{ if } 4a_2 - a_1^2 > 0$$

and let $\mu_1(t)$ and $\mu_2(t)$ be the characteristic roots of $(A + B(t))$, i.e. if $(a_1 + b_1)^2 - 4(a_2 + b_2) \geq 0$, then

$$\mu_1 = \frac{1}{2} \left(-a_1 - b_1 - \sqrt{(a_1 + b_1)^2 - 4(a_2 + b_2)} \right),$$

$$\mu_2 = \frac{1}{2} \left\{ -(a_1 + b_1) + \sqrt{(a_1 + b_1)^2 - 4(a_2 + b_2)} \right\}$$

and if $(a_1 + b_1)^2 - 4(a_2 + b_2) < 0$, then

$$\mu_1 = \frac{1}{2} \left\{ -(a_1 + b_1) - i\sqrt{4(a_2 + b_2) - (a_1 + b_1)^2} \right\},$$

$$\mu_2 = \frac{1}{2} \left\{ -(a_1 + b_1) + i\sqrt{4(a_2 + b_2) - (a_1 + b_1)^2} \right\}.$$

Let $L = \lambda_2 - \lambda_1$ and $R = \max(\operatorname{Re}(\lambda_1), \operatorname{Re}(\lambda_2))$.

First Theorem 5.3 will be applied to equation (5.6); conditions

will be derived to insure that $R + \Lambda_F(t) \leq 0$ (Recall that $F = P^{-1}B(t)P$ where P is such that $P^{-1}AP = J$ is in Jordan form.). This is not one of the expression appearing in Theorem 5.2 (Theorem 5.3 is used in conjunction with Theorem 5.2), but its use simplifies writing and permits a geometric interpretation. Further, it is not difficult to modify this expression to obtain any of the expressions of Theorem 5.2. If $R + \Lambda_F(t) \leq 0$ for $t \geq T_0$, then the solutions of equation (5.6) are monotonic non-increasing. Lower bounds for solutions will not be discussed.

Theorem 5.6. Let equation (5.6) be given and adopt the notation of Definition (5.3); assume $\lambda_1 \neq \lambda_2$ and $R < 0$.

1) If $|b_1(t)| + |b_2(t)| \leq (-R|\lambda_2 - \lambda_1|)/(2\max(1, |\lambda_1|))$ for $t \geq T_0$, then $R + \Lambda_F(t) \leq 0$ for $t \geq T_0$.

2) If $\sup(|b_1(t)|, |b_2(t)|) \leq (-R|\lambda_2 - \lambda_1|)/(2(1 + |\lambda_1|))$ for $t \geq T_0$, then $R + \Lambda_F(t) \leq 0$ for $t \geq T_0$.

3) If $|\lambda_1| = 1$, the region defined in part (2) is a subset of the region defined in part (1); if $|\lambda_1| \neq 1$, then neither region is a subset of the other.

4) If $\sup_{(1)} |\mu_1(t) - \lambda_1| \leq \inf(-R/4, |\lambda_2 - \lambda_1|)$ for $t \geq T_0$, then $R + \Lambda_F(t) \leq 0$ for $t \geq T_0$.

Proof: If λ_1 and λ_2 are complex, $|\lambda_1| = |\lambda_2|$; if λ_1 and λ_2 are real, $\lambda_1 < \lambda_2 < 0$; hence, $\max(|\lambda_1|, |\lambda_2|) = |\lambda_1|$. Also,

$$D = \max_{(1)} \prod_{j=1}^n |\lambda_1 - \lambda_j| = |\lambda_2 - \lambda_1| = \min_{(1)} \prod_{j=1}^n |\lambda_1 - \lambda_j| = d.$$

1) From Theorem 5.3, (2),

$$R + \Lambda_F(t) \leq R + 2|\lambda_2 - \lambda_1|^{-1} \max(1, |\lambda_1|) (|b_1(t)| + |b_2(t)|).$$

Hence, if $|b_1(t)| + |b_2(t)| \leq (-R|\lambda_2 - \lambda_1|)/(2\max(1, |\lambda_1|))$, then

$$R + \Lambda_F(t) \leq 0.$$

2) From Theorem 5.3, (3),

if $|\lambda_1| \neq 1$, then $R + \Lambda_F(t) \leq R + 2|\lambda_2 - \lambda_1|^{-1}(1 - |\lambda_1|^2) \times$
 $\times (1 - |\lambda_1|)^{-1} \sup_{(1)} |b_1(t)|$. Hence, if $|\lambda_1| \neq 1$ and

$$\sup_{(1)} |b_1(t)| \leq (-R|\lambda_2 - \lambda_1|)/(2(1 + |\lambda_1|)),$$

then $R + \Lambda_F(t) \leq 0$.

If $|\lambda_1| = 1$, then

$$R + \Lambda_F(t) \leq R + 4|\lambda_2 - \lambda_1|^{-1} \sup_{(1)} |b_1(t)|.$$

Hence, if $|\lambda_1| = 1$ and

$$\sup_{(1)} |b_1(t)| \leq (-R|\lambda_2 - \lambda_1|)/4 = (-R|\lambda_2 - \lambda_1|)/(2(1 + |\lambda_1|)),$$

then $R + \Lambda_F(t) \leq 0$.

3) The region described in part (1) is a square in the b_1, b_2 - plane with vertices at

$$((\pm R|\lambda_2 - \lambda_1|)/(2\max(1, |\lambda_1|)), 0), (0, (\pm R|\lambda_2 - \lambda_1|)/(2\max(1, |\lambda_1|)));$$

the region described in part (2) is a square in the b_1, b_2 - plane

with vertices at

$$((\pm R|\lambda_2 - \lambda_1|)/(2(1 + |\lambda_1|)), (\pm R|\lambda_2 - \lambda_1|)/(2(1 + |\lambda_1|))).$$

Since $2(1 + |\lambda_1|) > 2\max(1, |\lambda_1|)$ (because $\lambda_1 \neq 0$), the region defined in part (2) does not contain the region defined in part (1).

Since $2\max(1, |\lambda_1|) \geq (1 + |\lambda_1|)$, with the equality holding if and only if $|\lambda_1| = 1$, i.e.

$$\frac{1}{2}(-R|\lambda_2 - \lambda_1|)/(2\max(1, |\lambda_1|)) \leq (-R|\lambda_2 - \lambda_1|)/(2(1 + |\lambda_1|)),$$

the region defined in (1) contains the region defined in (2) if and only if $|\lambda_1| = 1$.

4) If $\sup_{(1)} |\mu_1(t) - \lambda_1| \leq |\lambda_2 - \lambda_1|$, then Theorem 5.3, (4) applies to give

$$R + \Lambda_F(t) \leq R + 2|\lambda_2 - \lambda_1|^{-1}(2)|\lambda_2 - \lambda_1| \sup_{(1)} |\mu_1(t) - \lambda_1|.$$

Hence, if $\sup_{(1)} |\mu_1(t) - \lambda_1| \leq \inf(|\lambda_2 - \lambda_1|, -R/4)$,

then $R + \Lambda_F(t) \leq 0$. Q.E.D.

For results more precise than those of the preceding theorem, the method of this section may be applied directly to equation (5.6).

Recall that the method involves substituting $y = P^{-1}x$, where P is a matrix such that $P^{-1}AP = J$ is in Jordan form, into equation (5.6) and transforming it into

$$(5.7) \quad y' = (J + P^{-1}B(t)P)y = (J + F(t))y = C(t)y.$$

Since equations (5.6) and (5.7) have the same asymptotic behavior, it is necessary only to study the latter equation.

Lemma 5.13. Let equation (5.6) be given and adopt the notation of Definition 5.3.

1) If $\lambda_1 \neq \lambda_2$, equation (5.7) may be written as

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \left\{ \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} + L^{-1} \begin{bmatrix} b_1 \lambda_1 + b_2 & b_1 \lambda_2 + b_2 \\ -b_1 \lambda_1 - b_2 & -b_1 \lambda_2 - b_2 \end{bmatrix} \right\} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

2) If $\lambda_1 = \lambda_2 = \lambda$, equation (5.7) may be written as

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \left\{ \begin{bmatrix} \lambda & k \\ 0 & \lambda \end{bmatrix} + k^{-1} \begin{bmatrix} b_1 \lambda + b_2 & b_1 \lambda + b_1 k + b_2 \\ -b_1 \lambda - b_2 & -b_1 \lambda - b_1 k - b_2 \end{bmatrix} \right\} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

where k is an arbitrary constant.

Proof: 1) Let $P = \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix}$ and $P^{-1} = L^{-1} \begin{bmatrix} \lambda_2 & -1 \\ -\lambda_1 & 1 \end{bmatrix}$.

2) Let $P = \begin{bmatrix} 1 & 1 \\ \lambda & \lambda + k \end{bmatrix}$ and $P^{-1} = k^{-1} \begin{bmatrix} \lambda + k & -1 \\ -\lambda & 1 \end{bmatrix}$

(and note in the computation that $\lambda = -\frac{1}{2}a_1$). Q.E.D.

Lemma 5.14. Let equations (5.6) and (5.7) be given and adopt the notation of Definition 5.3. If λ_1 and λ_2 are complex, then the matrix P may be chosen such that

$$\begin{aligned} \lambda_C(t) &\leq -\frac{1}{2}(a_1 + b_1(t)) + L^{-1} |\lambda_1 b_1(t) + b_2(t)|, \\ \lambda_C(t) &\geq -\frac{1}{2}(a_1 + b_1(t)) - L^{-1} |\lambda_1 b_1(t) + b_2(t)| \quad \text{for } t \geq T_0 \end{aligned}$$

Proof: If $\lambda_1 \neq \lambda_2$, P may be chosen such that equation (5.7) has the form given in Lemma 5.13, (1). Since λ_1 and λ_2 are complex,

$$\frac{1}{2}(J^* + J) = \begin{bmatrix} -\frac{1}{2}a_1 & 0 \\ 0 & -\frac{1}{2}a_1 \end{bmatrix},$$

$$\frac{1}{2}(F^*(t) + F(t)) = \begin{bmatrix} -\frac{1}{2}b_1 & L^{-1}(b_1\lambda_2 + b_2) \\ L^{-1}(-b_1\lambda_1 - b_2) & -\frac{1}{2}b_1 \end{bmatrix}, \text{ and}$$

$$\frac{1}{2}(C^*(t) + C(t)) = \begin{bmatrix} -\frac{1}{2}(a_1 - b_1) & L^{-1}(b_1\lambda_2 + b_2) \\ L^{-1}(-b_1\lambda_1 - b_2) & -\frac{1}{2}(a_1 + b_1) \end{bmatrix}.$$

The characteristic roots of this matrix are given by

$$r^2 + (a_1 + b_1)r + ((a_1 + b_1)/2)^2 + L^{-2}(b_1\lambda_1 + b_2)(b_1\lambda_2 + b_2) = 0,$$

$$r_1 = -\frac{1}{2}(a_1 + b_1) \pm (-L^{-2}(b_1\lambda_1 + b_2)(b_1\lambda_2 + b_2))^{\frac{1}{2}}, \quad i = 1, 2,$$

$$r_1 = -\frac{1}{2}(a_1 + b_1) \pm (|L^{-1}(b_1\lambda_1 + b_2)|^2)^{\frac{1}{2}} = -\frac{1}{2}(a_1 + b_1) \pm |L^{-1}(b_1\lambda_1 + b_2)|,$$

for $i = 1, 2$. The lemma follows from this. Q.E.D.

Lemma 5.15. Let equations (5.6) and (5.7) be given and adopt the notation of Definition 5.3. If λ_1 and λ_2 are real and distinct, then the matrix P may be chosen such that

$$\begin{aligned} 1) \quad \Lambda_C(t) &\leq -\frac{1}{2}a_1 + \frac{1}{2}(|b_1(t)| - b_1(t)) + \frac{1}{2}|L^{-1}(2b_2(t) - a_1b_1(t)) - L|, \\ \Lambda_C(t) &\geq -\frac{1}{2}(a_1 + b_1(t)) - \frac{1}{2}|b_1(t)| - \frac{1}{2}|L^{-1}(2b_2(t) - a_1b_1(t)) - L|. \end{aligned}$$

If, in addition, $b_2(t) = 0$ for $t \geq T_0$ and λ_1 and λ_2 have the same sign, then the matrix P may be chosen such that

$$\begin{aligned} 2) \quad \Lambda_C(t) &\leq -\frac{1}{2}(a_1 + b_1(t)) + \frac{1}{2}|L + L^{-1}(a_1b_1)|, \\ \Lambda_C(t) &\geq -\frac{1}{2}(a_1 + b_1(t)) - \frac{1}{2}|L + L^{-1}(a_1b_1)|. \end{aligned}$$

Proof: If $\lambda_1 \neq \lambda_2$, P may be chosen such that equation (5.7) has the form given in Lemma 5.13, (1). Since λ_1 and λ_2 are real,

$$\begin{aligned} \frac{1}{2}(J^* + J) &= J, \\ \frac{1}{2}(F^*(t) + F(t)) &= L^{-1} \begin{bmatrix} b_1\lambda_1 + b_2 & \frac{1}{2}b_1(\lambda_2 - \lambda_1) \\ \frac{1}{2}b_1(\lambda_2 - \lambda_1) & -b_1\lambda_2 - b_2 \end{bmatrix}, \text{ and} \end{aligned}$$

$$\frac{1}{2}(C^*(t) + C(t)) = \begin{bmatrix} \lambda_1 + L^{-1}(b_1 \lambda_1 + b_2) & b_1/2 \\ b_1/2 & \lambda_2 - L^{-1}(b_1 \lambda_2 + b_2) \end{bmatrix}.$$

If r is a characteristic root of this matrix, then by Lemma 5.5

$$|r - (\lambda_1 + (-1)^{i+1} L^{-1}(b_1 \lambda_1 + b_2))| \leq |b_1|/2 \quad \text{for some } i, 1 \leq i \leq 2.$$

Then $\Lambda_C(t) \leq \sup_{(i)} (\lambda_1 + (-1)^{i+1} L^{-1}(b_1 \lambda_1 + b_2)) + |b_1|/2,$

and $\lambda_C(t) \geq \inf_{(i)} (\lambda_1 + (-1)^{i+1} L^{-1}(b_1 \lambda_1 + b_2)) - |b_1|/2.$

Since $\max(a, b) = \frac{1}{2}\{a + b + |a - b|\}$ and $\min(a, b) = \frac{1}{2}\{a + b - |a - b|\},$

$$\Lambda_C(t) \leq \frac{1}{2} \{ \lambda_1 + \lambda_2 - b_1 + |\lambda_1 - \lambda_2 + L^{-1}(b_1(\lambda_1 + \lambda_2) + 2b_2)| + |b_1|/2,$$

$$\Lambda_C(t) \leq -a_1/2 + \frac{1}{2}(|b_1| - b_1) + \frac{1}{2}|L^{-1}(2b_2 - a_1 b_1) - L|, \quad \text{and}$$

$$\lambda_C(t) \geq -\frac{1}{2}(a_1 + b_1) - \frac{1}{2}|b_1| - \frac{1}{2}|L^{-1}(2b_2 - a_1 b_1) - L|.$$

Now assume $b_2(t) = 0$ for $t \geq T_0$ and λ_1 and λ_2 have the same sign.

Then Lemma 5.13 has the form

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \left\{ \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} + L^{-1} \begin{bmatrix} b_1 \lambda_1 & b_1 \lambda_2 \\ -b_1 \lambda_1 & -b_1 \lambda_2 \end{bmatrix} \right\} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Let $Q = \begin{bmatrix} \sqrt{\lambda_2/\lambda_1} & 0 \\ 0 & 1 \end{bmatrix}$ and substitute $w = Q^{-1}y$ into the preceding

equation. Then

$$\begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \left\{ \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} + L^{-1} \begin{bmatrix} b_1 \lambda_1 & b_1 \sqrt{\lambda_1 \lambda_2} \\ -b_1 \sqrt{\lambda_1 \lambda_2} & -b_1 \lambda_2 \end{bmatrix} \right\} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \text{ i.e. } w' = G(t)w.$$

Then

$$\frac{1}{2}(G^*(t) + G(t)) = \begin{bmatrix} \lambda_1 + b_1 \lambda_1/L & 0 \\ 0 & \lambda_2 - b_1 \lambda_2/L \end{bmatrix} \quad \text{and}$$

$$\Lambda_G(t) \leq \frac{1}{2} \{ \lambda_1 + \lambda_2 - b_1 + |\lambda_1 - \lambda_2 + L^{-1}(b_1(\lambda_1 + \lambda_2))| =$$

$$= -\frac{1}{2}(a_1 + b_1) + \frac{1}{2}|L + L^{-1}(a_1 b_1)|,$$

$$\lambda_G(t) \geq -\frac{1}{2}(a_1 + b_1) - \frac{1}{2}|L + L^{-1}(a_1 b_1)|. \quad \text{Q.E.D.}$$

Lemma 5.16. Let equation (5.6) and (5.7) be given and adopt the notation of Definition 5.3. If λ_1 and λ_2 are real and identical ($\lambda_1 = \lambda_2 = \lambda$), then the matrix P may be chosen such that

$$1) \begin{cases} \lambda_C(t) \leq -\frac{1}{2}(a_1 + b_1(t)) + \frac{1}{2}|b_1(t) + k| + (2k)^{-1}|2b_2(t) - a_1b_1(t) + kb_1(t)| \\ \lambda_C(t) \geq -\frac{1}{2}(a_1 + b_1(t)) - \frac{1}{2}|b_1(t) + k| - (2k)^{-1}|2b_2(t) - a_1b_1(t) + kb_1(t)| \end{cases},$$

If, in addition, $b_2(t) \equiv 0$ for $t \geq T_0$ and k is such that $\lambda(\lambda + k) > 0$, then the matrix P may be chosen such that

$$2) \begin{cases} \lambda_C(t) \leq -\frac{1}{2}(a_1 + b_1(t)) + \frac{1}{2}|b_1(t) - k^{-1}a_1b_1(t)| + \frac{1}{2}k\sqrt{\lambda(\lambda + k)^{-1}} \\ \lambda_C(t) \geq -\frac{1}{2}(a_1 + b_1(t)) - \frac{1}{2}|b_1(t) - k^{-1}a_1b_1(t)| - \frac{1}{2}k\sqrt{\lambda(\lambda + k)^{-1}} \end{cases}.$$

Proof. If $\lambda_1 = \lambda_2 = \lambda$, then P may be chosen such that equation (5.7) has the form given in Lemma 5.13, (2). Then

$$\frac{1}{2}(J^* + J) = \begin{bmatrix} \lambda & k/2 \\ k/2 & \lambda \end{bmatrix},$$

$$\frac{1}{2}(F^*(t) + F(t)) = \begin{bmatrix} k^{-1}(b_1\lambda + b_2) & \frac{1}{2}b_1 \\ \frac{1}{2}b_1 & k^{-1}(-b_1\lambda - b_1k - b_2) \end{bmatrix},$$

$$\text{and } \frac{1}{2}(C^*(t) + C(t)) = \begin{bmatrix} \lambda + k^{-1}(b_1\lambda + b_2) & \frac{1}{2}(b_1 + k) \\ \frac{1}{2}(b_1 + k) & \lambda - k^{-1}(b_1\lambda + b_1k + b_2) \end{bmatrix}.$$

Proceeding in the manner of Lemma 5.15,

$$\lambda_C(t) \leq \frac{1}{2}\{2\lambda - b_1 + |k^{-1}(2(b_1\lambda + b_2) + b_1k)|\} + \frac{1}{2}|b_1 + k|,$$

$$\lambda_C(t) \geq \frac{1}{2}\{2\lambda - b_1 - |k^{-1}(2(b_1\lambda + b_2) + b_1k)|\} - \frac{1}{2}|b_1 + k|,$$

and since $\lambda = -\frac{1}{2}a_1$, (1) follows.

If $b_2(t) = 0$ for $t \geq T_0$ and k is such that $\lambda(\lambda + k) > 0$, then by proceeding as in the second part of the proof of Lemma 5.15, with

$$Q = \begin{bmatrix} \sqrt{\lambda^{-1}(\lambda + k)} & 0 \\ 0 & 1 \end{bmatrix}, \text{ the equation}$$

$$\begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} \lambda & k\sqrt{\lambda(\lambda+k)^{-1}} \\ 0 & \lambda \end{bmatrix} + \begin{bmatrix} k^{-1}b_1\lambda & k^{-1}b_1\sqrt{\lambda(\lambda+k)} \\ -k^{-1}b_1\sqrt{\lambda(\lambda+k)} & -k^{-1}b_1(\lambda+k) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

i.e., $w' = G(t)w$ is obtained.

Then $\frac{1}{2}(G^*(t) + G(t)) = \begin{bmatrix} \lambda + k^{-1}b_1\lambda & \frac{1}{2}k\sqrt{\lambda(\lambda+k)^{-1}} \\ \frac{1}{2}k\sqrt{\lambda(\lambda+k)^{-1}} & \lambda - k^{-1}b_1(\lambda+k) \end{bmatrix}$

and $\Lambda_G(t) \leq \frac{1}{2}\{2\lambda - b_1 + |k^{-1}(2b_1\lambda + b_1k)|\} + |\frac{1}{2}k\sqrt{\lambda(\lambda+k)^{-1}}|,$

$\lambda_G(t) \geq \frac{1}{2}\{2\lambda - b_1 - |k^{-1}(2b_1\lambda + b_1k)|\} - |\frac{1}{2}k\sqrt{\lambda(\lambda+k)^{-1}}|. \text{ Q.E.D.}$

Remarks on Lemmas 5.14, 5.15 and 5.16. The five pairs of inequalities of the preceding lemmas may be combined with the nine corollaries of Theorem 2.1 to produce 45 theorems on the asymptotic behavior of equation (5.6); this will be left to the reader. Some interesting special cases are considered here.

Consider the damped Hill equation

$$(5.8) \quad x'' + ax' + p(t)x = 0$$

where a is a real constant and $p(t)$ is a real-valued function which is defined and continuous for $t \geq T_0$ and $p(t+T) = p(t)$ for $t \geq T_0$.

Theorem 5.7. Let equation (5.8) be given.

1) Assume there exists a constant c , $c < a^2/4$, such that $\int_0^T |p(t) + c - \frac{1}{2}a^2| dt \leq \frac{1}{2}|a|T\sqrt{a^2 - 4c}$, (or $<$). If $a > 0$, then $x = 0$ is a uniformly (or exponentially) stable solution of equation (5.8); if $a < 0$, then the norms of the nonzero solutions of equation (5.8) are bounded away from zero (or increase exponentially).

2) Assume $\int_0^T |p(t) - a^2/4| dt \leq a^2T/8$ (or $<$). If $a > 0$, then $x = 0$ is a uniformly (or exponentially) stable solution of equation (5.8); if $a < 0$, then the norms of the nonzero solutions of equation (5.8) are bounded away from zero (or increase exponentially).

3) Assume there exists a constant c , $c > a^2/4$, such that

$$\int_0^T |p(t) - c| dt \leq \frac{1}{2}|a|T\sqrt{4c - a^2} \quad (\text{or } <).$$

If $a > 0$, then $x = 0$ is a uniformly (or exponentially) stable solution of equation (5.8); if $a < 0$, then the norms of the nonzero solutions of equation (5.8) are bounded away from zero (or increase exponentially).

Proof. Write equation (5.8) as

$$x'' + ax' + [(p(t) - c) + c]x = 0 : c \text{ constant}$$

and consider this equation as a special case of equation (5.6) with

$$a_1 = a, a_2 = c, b_1(t) = 0, \text{ and } b_2(t) = p(t) - c.$$

1) If $a^2 - 4c > 0$, then Lemma 5.15 applies and

$$\begin{aligned} \Lambda_c(t) &\leq -\frac{1}{2}a + (a^2 - 4c)^{-\frac{1}{2}}|(p(t) - c) - (\frac{1}{2}a^2 - 2c)| = -\frac{1}{2}a + \\ &\quad + (a^2 - 4c)^{-\frac{1}{2}}|p(t) + c - \frac{1}{2}a^2|, \end{aligned}$$

$$\begin{aligned} \lambda_c(t) &\geq -\frac{1}{2}a - (a^2 - 4c)^{-\frac{1}{2}}|(p(t) - c) - (\frac{1}{2}a^2 - 2c)| = \\ &= -\frac{1}{2}a - (a^2 - 4c)^{-\frac{1}{2}}|p(t) + c - \frac{1}{2}a^2|. \end{aligned}$$

If $a > 0$ and $\int_0^T |p(t) + c - \frac{1}{2}a^2| dt \leq \frac{1}{2}aT\sqrt{a^2 - 4c}$ (or $<$), then $\int_0^T \Lambda_c(t) dt \leq 0$ (or $<$) and by Corollary 1' (or Corollary 2' - 3') of Theorem 2.1, $x = 0$ is a uniformly (or exponentially) stable solution of equation (5.8).

If $a < 0$ and $\int_0^T |p(t) + c - \frac{1}{2}a^2| dt \geq \frac{1}{2}(-a)T\sqrt{a^2 - 4c}$ (or $>$), then $\int_0^T \lambda_c(t) dt \geq 0$ (or $>$) and by Corollary 4' (or Corollary 5') of Theorem 2.1, the nonzero solutions of equation (5.8) are bounded away from zero (or increase exponentially).

2) If $a^2 - 4c = 0$, then Lemma 5.16, (1) applies and

$$\left. \begin{aligned} \Lambda_c(t) &\leq -\frac{1}{2}a + \frac{1}{2}k + k^{-1}|p(t) - c| \\ \lambda_c(t) &\geq -\frac{1}{2}a - \frac{1}{2}k - k^{-1}|p(t) - c| \end{aligned} \right\} \text{ where } k \text{ is an arbitrary positive constant.}$$

Let $k = \frac{1}{2}|a|$; this is the optimal k for both cases.

If $a > 0$ and $\int_0^T |p(t) - a^2/4| dt \leq a^2T/8$ (or $<$), then $\int_0^T \Lambda_c(t) dt \leq 0$ (or $<$); if $a < 0$ and $\int_0^T |p(t) - a^2/4| dt \leq a^2T/8$ (or $<$),

then $\int_0^T \lambda_c(t) dt \geq 0$ (or $>$). As in part (1), the conclusion now follows.

3) If $a^2 - 4c < 0$, then Lemma 5.14 applies and

$$\lambda_c(t) \leq -\frac{1}{2}a + (4c - a^2)^{-\frac{1}{2}}|p(t) - c|,$$

$$\lambda_c(t) \geq -\frac{1}{2}a - (4c - a^2)^{-\frac{1}{2}}|p(t) - c|.$$

If $a > 0$ and $\int_0^T |p(t) - c| dt \leq \frac{1}{2}aT(4c - a^2)^{-\frac{1}{2}}$ (or $<$), then $\int_0^T \lambda_c(t) dt \leq 0$ (or $<$); if $a < 0$ and $\int_0^T |p(t) - c| dt \leq \frac{1}{2}(-a)T\sqrt{4c - a^2}$ (or $<$), then $\int_0^T \lambda_c(t) dt \geq 0$ (or $>$). The conclusion now follows as in part (1). Q.E.D.

Remarks on Theorem 5.7. The substitution $x = ye^{-\frac{1}{2}at}$, $a > 0$, reduces equation (5.8) to

$$(*) \quad y'' + (p(t) - a^2/4)y = y'' + q(t)y = 0$$

and further, if the solutions of equation (*) are bounded, then $x = 0$ is an exponentially stable solution of equation (5.8). Thus, theorems on the boundedness of the solutions of equation (*) may be compared to Theorem 5.7.

Lyapunov had proved that if

$$q(t) \neq 0 \text{ for } t \geq T_0, \quad \int_0^T q(t) dt \geq 0, \text{ and } \int_0^T |q(t)| dt \leq 4/T,$$

then the solutions of equation (*) are bounded, or if

$$p(t) \neq a^2/4 \text{ for } t \geq T_0, \quad \int_0^T p(t) dt \geq a^2T/4, \text{ and } \int_0^T |p(t) - a^2/4| dt \leq 4/T,$$

then $x = 0$ is an exponentially stable solution of equation (5.8).

This may be compared with Theorem 5.7, (2). (The preceding transformation and theorem may be found in Reference 15, section 8 - 12.)

If $p(t)$ is specified, the results of Theorem 5.7 may be improved. Consider the equation

$$(5.9) \quad x'' + ax' + (\delta + \cos^n t)x = 0$$

where a , δ and ϵ are real constants, n is a positive integer, $a > 0$,

and $t \gg T_0$.

Theorem 5.8. Let equation (5.9) be given and assume that n is a positive odd integer. If either

1) $\delta - |\epsilon| > 0$ and $\sqrt{\delta} \leq a \leq \sqrt{2(\delta - |\epsilon|)}$; or

2) $\delta \geq 0$ and $\sqrt{2(\delta + |\epsilon|)} \leq a$; or

3) $\sqrt{\max(0, 2(\delta - |\epsilon|))} < a < \sqrt{2(\delta + |\epsilon|)}$ and

$$\int_0^\theta \cos^n t \, dt - \theta \cos^n \theta \leq \frac{1}{2} |\epsilon|^{-1} \pi \delta$$

(here $\theta = \cos^{-1}(\frac{1}{2} |\epsilon|^{-1} (a^2 - 2\delta))^{1/n}$, $0 < \theta < \pi$) is satisfied, then $x = 0$ is a uniformly stable solution of equation (5.9).

If, additionally: in (1), $\sqrt{\delta} < a$; or in (2), $\delta > 0$; or in (3), $\int_0^\theta \cos^n t \, dt - \theta \cos^n \theta < \pi \delta / 2 |\epsilon|$, then $x = 0$ is an exponentially stable solution of equation (5.9).

Proof: Equation (5.9) is a special case of equation (5.8), hence Theorem 5.7 applies. The choice of an optimal c is difficult. Here $c = 0$ is used as a convenient choice and Theorem 5.7, (1) is applied. It is only necessary to prove the theorem for $\epsilon > 0$ since the substitution $\tau = t - \pi$ in equation (5.9) just changes the sign of ϵ .

If $I = \int_0^\pi |\delta + \epsilon \cos^n t - \frac{1}{2} a^2| dt \leq \frac{1}{2} a^2 \pi$ (or $<$), then $x = 0$ is a uniformly (or exponentially) stable solution of equation (5.9).

1) If $\delta - \frac{1}{2} a^2 \geq \epsilon > 0$, i.e. $\delta - \epsilon > 0$ and $\sqrt{2(\delta - \epsilon)} \geq a$, then $I = \int_0^\pi (\delta + \epsilon \cos^n t - \frac{1}{2} a^2) dt = (\delta - \frac{1}{2} a^2) \pi$. If, additionally, $a \geq \sqrt{\delta}$ (or $>$), then $a^2 \pi \geq \delta \pi$ (or $>$) and $\frac{1}{2} a^2 \pi \geq (\delta - \frac{1}{2} a^2) \pi = I$ (or $>$).

2) If $\frac{1}{2} a^2 - \delta \geq \epsilon > 0$, i.e. $a^2 \geq 2(\delta + \epsilon)$, then

$$I = -\int_0^\pi (\delta + \epsilon \cos^n t - \frac{1}{2} a^2) dt = (\frac{1}{2} a^2 - \delta) \pi.$$

If, additionally, $\delta \geq 0$ (or $>$), then $\delta + \epsilon > 0$ and $I \leq \frac{1}{2} a^2 \pi$ (or $<$).

3) If $|\delta - \frac{1}{2} a^2| < \epsilon$, i.e. $2(\delta - \epsilon) < a^2 < 2(\delta + \epsilon)$ or

$\sqrt{\max(0, 2(\delta - \epsilon))} < a < \sqrt{2(\delta + \epsilon)}$, and if

$\theta = \cos^{-1}(\frac{1}{2}\epsilon^{-1}(a^2 - 2\delta))^{1/n}$, $0 < \theta < \pi$, then

$$I = \int_0^\theta (\delta + \epsilon \cos^n t - \frac{1}{2}a^2) dt - \int_\theta^\pi (\delta + \epsilon \cos^n t - \frac{1}{2}a^2) dt,$$

$$I = (\delta - \frac{1}{2}a^2)(2\theta - \pi) + \int_0^\theta \epsilon \cos^n t dt - \int_\theta^\pi \epsilon \cos^n t dt,$$

$$I = -2\epsilon \theta \cos^n \theta - \pi \delta + \frac{1}{2}\pi a^2 + 2\epsilon \int_0^\theta \cos^n t dt.$$

If $-2\epsilon \theta \cos^n \theta - \pi \delta + \frac{1}{2}\pi a^2 + 2\epsilon \int_0^\theta \cos^n t dt \leq \frac{1}{2}\pi a^2$ (or $<$), or

$$2\epsilon (\int_0^\theta \cos^n t dt - \theta \cos^n \theta) \leq \pi \delta \quad (\text{or } <), \text{ or}$$

$$\int_0^\theta \cos^n t dt - \theta \cos^n \theta \leq \pi \delta / 2\epsilon \quad (\text{or } <),$$

then $I \leq \pi a^2 / 2$ (or $<$). Q.E.D.

Example 5.2 (Illustrating Theorem 5.8).

$$x'' + 2x' + (1 + \epsilon \cos t)x = 0 : t \geq 0.$$

Theorem 5.8 will be used to determine values of ϵ for which $x = 0$ is an exponentially stable solution of this equation.

1) Since $\sqrt{1} \leq 2 \not\leq \sqrt{2(1 - |\epsilon|)}$, (1) does not apply.

2) First, $\delta = 1 > 0$; second, $\sqrt{2(1 + |\epsilon|)} \leq 2$ if $|\epsilon| \leq 1$.

Hence for $|\epsilon| \leq 1$, $x = 0$ is an exponentially stable solution of this equation.

3) First, $\sqrt{\max(0, 2(1 - |\epsilon|))} < 2$ for all ϵ and $2 < \sqrt{2(1 + |\epsilon|)}$ if $|\epsilon| > 1$. The problem is now to find all ϵ , $|\epsilon| > 1$, such that

$$\sin \theta - \theta \cos \theta \leq \pi/2 |\epsilon|.$$

Let $F(\theta) = \sin \theta - \theta \cos \theta$; since $F(0) = 0$ and $F'(\theta) = \theta \sin \theta \geq 0$

(with $F'(\theta) = 0$ only at $\theta = \pi/2$), the problem is to find the θ_0 such that

$\sin \theta_0 - \theta_0 \cos \theta_0 = \pi/2 |\epsilon|$. Then $\sin \theta - \theta \cos \theta \leq \pi/2 |\epsilon|$ if and only

if $0 \leq \theta \leq \theta_0$. Let ϵ_0 be the value of ϵ corresponding to θ_0 and

assume that $0 < \theta_0 < \pi/2$. Then

$$\cos \theta_0 (\tan \theta_0 - \theta_0) = \pi/2 |\epsilon_0|, \text{ or}$$

$$|\epsilon_0|^{-1} (\tan \theta_0 - \theta_0) = \pi/2 |\epsilon_0| = |\epsilon_0|^{-1} (1.57). \quad .$$

Let $\theta_0 = 1.22$, then $(2.73) - (1.22) = 1.51 \sim 1.57$.

Therefore, $0 < \theta \leq \theta_0 = 1.22$,

$$1 > \cos \theta = |\epsilon|^{-1} \geq \cos \theta_0 = 0.34,$$

$$1 < |\epsilon| < 2.9.$$

If $1 < |\epsilon| < 2.9$, $x = 0$ is an exponentially stable solution of this equation; if $|\epsilon| = 2.9$, $x = 0$ is a uniformly stable solution.

Theorem 5.9. Let equation (5.9) be given and assume that n is a positive even integer. Let $N = (\frac{n-1}{n})(\frac{n-3}{n-2}) \dots (\frac{1}{2})$, $0 < N < 1$, and $\theta = \cos^{-1}(\frac{a^2 - 2\delta}{2\epsilon})^{1/n}$. If either

- 1) $\epsilon > 0$, $\delta > 0$, and $\sqrt{\delta + \epsilon N} \leq a \leq \sqrt{2\delta}$; or
- 2) $\epsilon > 0$, $\delta + \epsilon N \geq 0$ (hence $\delta + \epsilon > 0$), and $\sqrt{2(\delta + \epsilon)} \leq a$; or
- 3) $\epsilon > 0$, $\delta + \epsilon > 0$, $\sqrt{\max(0, 2\delta)} < a < \sqrt{2(\delta + \epsilon)}$,
and $\int_0^\theta \cos^n t \, dt - \theta \cos^n \theta \leq (\pi/4)(N + \delta/\epsilon)$; or
- 4) $\epsilon < 0$, $\delta + \epsilon N \geq 0$ (hence $\delta > 0$), and $\sqrt{2\delta} \leq a$; or
- 5) $\epsilon < 0$, $\delta + \epsilon > 0$ (hence $\delta - \epsilon N > 0$), and $\sqrt{\delta + \epsilon N} \leq a \leq \sqrt{2(\delta + \epsilon)}$;

or 6) $\epsilon < 0$, $\delta > 0$, $\sqrt{\max(0, 2(\epsilon + \delta))} < a < \sqrt{2\delta}$,
and $\int_0^\theta \cos^n t \, dt - \theta \cos^n \theta \leq \frac{\pi}{4}(N + \frac{\delta - a^2}{\epsilon})$

is satisfied, then $x = 0$ is a uniformly stable solution of equation (5.9).

If, additionally: in (1), $\sqrt{\delta + \epsilon N} < a$; or in (2), $\delta + \epsilon N > 0$;
or in (3), $\int_0^\theta \cos^n t \, dt - \theta \cos^n \theta < \frac{\pi}{4}(N + \delta/\epsilon)$; or in (4), $\delta + \epsilon N > 0$;
or in (5), $\sqrt{\delta + \epsilon N} < a$; or in (6), $\int_0^\theta \cos^n t \, dt - \theta \cos^n \theta < \frac{\pi}{4}(N + \frac{\delta - a^2}{\epsilon})$,
then $x = 0$ is an exponentially stable solution of equation (5.9).

Proof: Equation (5.9) is a special case of equation (5.8), hence Theorem 5.7 applies. Again, let $c = 0$ and apply Theorem 5.7, (1).
If $I = \int_0^{\frac{\pi}{2}} |\delta + \epsilon \cos^n t - \frac{1}{2}a^2| \, dt \leq \frac{a^2 \pi}{4}$ (or $<$), then $x = 0$ is a uniformly (or exponentially) stable solution of equation (5.9).

1) If $\delta > 0$, $\epsilon > 0$, and $\delta - \frac{1}{2}a^2 \geq 0$, i.e. $\delta > 0$, $\epsilon > 0$,

and $\sqrt{2\delta} \geq a$, then

$$I = (\delta - \frac{1}{2}a^2)(\pi/2) + \epsilon \int_0^{\pi/2} \cos^n t \, dt = (\delta - \frac{1}{2}a^2)(\pi/2) + \epsilon N(\pi/2).$$

If, additionally, $(\delta + \epsilon N - \frac{1}{2}a^2)(\pi/2) \leq (a^2\pi/4)$ (or $<$), or

$$\delta + \epsilon N \leq a^2 \text{ (or } < \text{), or } \sqrt{\delta + \epsilon N} \leq a \text{ (or } < \text{),}$$

then $I \leq (a^2\pi)/4$ (or $<$).

2) If $\epsilon > 0$ and $\delta - \frac{1}{2}a^2 \leq -\epsilon$, i.e. $\epsilon > 0$ and $2(\delta + \epsilon) \leq a^2$, then $I = (\frac{1}{2}a^2 - \delta)(\pi/2) - \epsilon N(\pi/2)$. If, additionally,

$$(\frac{1}{2}a^2 - \delta - \epsilon N)(\pi/2) \leq (a^2\pi/4) \text{ (or } < \text{), or}$$

$$\delta + \epsilon N \geq 0 \text{ (or } > \text{), then}$$

$$I \leq a^2\pi/4 \text{ (or } < \text{)}.$$

3) If $\epsilon > 0$ and $-\epsilon < \delta - \frac{1}{2}a^2 < 0$, i.e. $\epsilon > 0$, $\delta + \epsilon > 0$, and $\sqrt{\max(0, 2\delta)} < a < \sqrt{2(\delta + \epsilon)}$, then $0 < \theta = \cos^{-1}(\frac{\epsilon^2 - 2\delta}{2\epsilon})^{1/n} < \frac{\pi}{2}$

$$\text{and } I = \int_0^\theta (\delta + \epsilon \cos^n t - \frac{1}{2}a^2) dt - \int_\theta^{\pi/2} (\delta + \epsilon \cos^n t - \frac{1}{2}a^2) dt,$$

$$I = (\delta - \frac{1}{2}a^2)(2\theta - \frac{\pi}{2}) + \epsilon \int_0^\theta \cos^n t \, dt - \epsilon \int_\theta^{\pi/2} \cos^n t \, dt,$$

$$I = -2\epsilon \theta \cos^n \theta - \frac{1}{2}\delta\pi + (\pi a^2/4) + 2\epsilon \int_0^\theta \cos^n t \, dt - \epsilon \int_0^{\pi/2} \cos^n t \, dt,$$

$$I = 2\epsilon (\int_0^\theta \cos^n t \, dt - \theta \cos^n \theta) - (\frac{1}{2}\delta\pi - (\pi a^2/4) + \epsilon N(\pi/2)).$$

If, additionally, $2\epsilon (\int_0^\theta \cos^n t \, dt - \theta \cos^n \theta) - (\frac{1}{2}\delta\pi - (\pi a^2/4) + \epsilon N(\frac{\pi}{2})) \leq (\pi a^2/4)$ (or $<$), or

$$\int_0^\theta \cos^n t \, dt - \theta \cos^n \theta \leq (\pi/4)(N + \frac{\delta}{\epsilon}) \text{ (or } < \text{),}$$

$$\text{then } I \leq (a^2\pi/4) \text{ (or } < \text{)}.$$

4) If $\epsilon < 0$ and $\delta - \frac{1}{2}a^2 \leq 0$, i.e. $\epsilon < 0$ and $2\delta \leq a^2$, then $I = -\int_0^{\pi/2} (\delta + \epsilon \cos^n t - \frac{1}{2}a^2) dt$ and, as in the proof of part (2), $I \leq (a^2\pi/4)$ (or $<$) if $(\delta + \epsilon N) \geq 0$ (or $>$).

5) If $\epsilon < 0$, $\delta + \epsilon > 0$, and $-\epsilon \leq \delta - \frac{1}{2}a^2$, i.e. $\epsilon < 0$, $\delta + \epsilon > 0$, and $\sqrt{2(\delta + \epsilon)} \geq a$, then

$$I = \int_0^{\pi/2} (\delta + \epsilon \cos^n t - \frac{1}{2}a^2) dt$$

and, as in the proof of part (1), $I \leq (a^2\pi/4)$ (or $<$), if $\sqrt{\delta + \epsilon N} \leq a$ (or $<$).

6) If $\epsilon < 0$ and $0 < \delta - \frac{1}{2}a^2 < -\epsilon$, i.e. $\epsilon < 0$, $\delta > 0$,
and $\sqrt{\max(0, 2(\delta + \epsilon))} < a < \sqrt{2\delta}$, then

$$0 < \theta = \cos^{-1}\left(\frac{a^2 - 2\delta}{2\epsilon}\right)^{1/n} < (\pi/2) \text{ and}$$

$$I = -\int_0^\theta (\delta + \epsilon \cos^n t - \frac{1}{2}a^2) dt + \int_\theta^{\pi/2} (\delta + \epsilon \cos^n t - \frac{1}{2}a^2) dt$$

and using the proof of part (4),

$$I = -2\epsilon \left(\int_0^\theta \cos^n t dt - \theta \cos^n \theta \right) + \left(\frac{1}{2} \delta \pi - (\pi a^2/4) + \epsilon N(\pi/2) \right).$$

If, additionally,

$$-2\epsilon \left(\int_0^\theta \cos^n t dt - \theta \cos^n \theta \right) + \left(\frac{1}{2} \delta \pi - (\pi a^2/4) + \epsilon N(\pi/2) \right) \leq (\pi a^2/4) \text{ (or } < \text{)},$$

$$\text{or } \int_0^\theta \cos^n t dt - \theta \cos^n \theta \leq (\pi/4)(N + \frac{\delta}{\epsilon} - (a^2/\epsilon)) \text{ (or } < \text{)},$$

$$\text{then } I \leq (a^2 \pi/4) \text{ (or } < \text{)}. \text{ Q.E.D.}$$

These last theorems show how the method of this section turns
Theorem 2.1 into an effective tool for studying the asymptotic behavior
of particular differential equations.

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