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CONFIGURATIONAL CHARACTERIZATIONS OF COMMUTATIVITY IN PROJECTIVE SPACES

by

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CONFIGURATIONAL CHARACTERIZATIONS OF COMMUTATIVITY IN PROJECTIVE SPACES

by M. W. Al-Dhahir

The study of algebraic identities by configurational propositions was started by Hilbert's characterization of the commutativity of multiplication by the two-dimensional proposition of Pappus. H. F. Baker and K. Reidemeister have extended Hilbert's result to three dimensions where the "Eight Lines" theorem and the theorem of Möbius are proved to be two characterizations of commutativity. The problem of generalizing these configurational characterizations to higher spaces is dealt with in this dissertation.

The notion of "geometrical equivalence" is introduced to obtain the Pappus class of configurations corresponding to the Pappus class of propositions. This class is generalized to higher dimensions in two different ways. One way is proved to be an n-dimensional characterization of the commutativity of multiplication; the other way was of value only in three, four and five dimensions and it is also proved to characterize the commutativity of these spaces.

Again the concept of geometrical equivalence is applied to some configurations corresponding to three-dimensional propositions equivalent to that of Möbius and consequently the Möbius class of configurations is obtained in correspondence with the Möbius class of propositions; minor forms of the class are discussed. Then this class is extended to n dimensions and the extension is shown to be a characterization of the commutativity of multiplication. The Pappus and the Möbius classes are unified by specializing each into the other.

Two transformational characterizations of the commutativity of multiplication are obtained by configurational means, and the "Steinitz group" generated by null polarities is extended to general polarities.

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INTRODUCTION

A division ring (or a skew-field or a sfield) R is a collection of at least two elements in which two operations, called "addition" and "multiplication", are defined by the following axioms:

(1) R is an Abelian group under addition with the identity element 0.

(2) The elements of R, with 0 deleted, form a group under multiplication with the identity element 1.

(3) For any three elements $a, b, c \in R$, $a(b+c) = ab + ac$ and $(b+c)a = ba + ca$.

We shall say that R is a commutative division ring (or R is a field) if and only if for any two elements $a, b \in R$, $ab = ba$.

By a projective plane, S_2, we understand a set of elements called "points" and a set of elements called "lines" with a relation called incidence, satisfying the following propositions:

1. For every two distinct points, there exists a unique line, called their join, incident to them.

2. For every two distinct lines, there exists a unique point, called their meet, incident to them.

3. There exist three points which are not incident with one line.

4. For every line, there exist at least three points incident with it.

5. The Desargues proposition: If the joins of corresponding vertices of two triangles are incident to one point, then the meets of their corresponding sides are incident with one line.

Hereafter, we shall adopt the common terminology of geometry, like collinearity, concurrency etc.

An algebraic model for the projective plane may be described as follows:

We consider a division ring R. A point is represented by a triple

of elements (x_o, x_1, x_2), not all $x_i = 0$, $x_i \in R$ and a line is represented by a triple $[u_o, u_1, u_2]$, not all $u_i = 0$, $u_i \in R$, with the understanding that $(\lambda x_o, \lambda x_1, \lambda x_2)$ and $[u_o\lambda, u_1\lambda, u_2\lambda]$, $\lambda \in R \neq 0$, represent the same point and line respectively. The point $X(x_o, x_1, x_2)$ and the line $U[u_o, u_1, u_2]$ are incident provided that the "inner" product $\langle X \cdot U \rangle = x_o u_o + x_1 u_1 + x_2 u_2$ vanishes It follows that any point collinear with $X(x_o, x_1, x_2)$ and $Y(y_o, y_1, y_2)$ is represented by $X + \lambda Y = (x_o + \lambda y_o, x_1 + \lambda y_1, x_2 + \lambda y_2)$, $\lambda \neq 0$.

It is to be noted, after Hilbert [6][(1)], that on every line of the projective plane an algebra of points may be introduced where the operations of addition and multiplication may be defined by incidences; the resulting algebra forms a division ring. Because of this we shall identify the projective plane with its algebraic model and refer to either one as a projective plane defined over a division ring.

The configurational proposition of Pappus asserts the collinearity of the meets of the cross joins of two triples of collinear points lying on two coplanar lines. In our terminology, this is a two-dimensional proposition, i.e., a proposition whose corresponding configuration lies in one plane. In 1900, Hilbert [6] proved that for the proposition of Pappus to hold in a projective plane, it is necessary and sufficient that the division ring over which the plane is defined be commutative. This result of Hilbert is most important in the development of projective geometry and its foundations. Since that time it has been an interesting problem in the foundations of geometry to discover the propositions which require or are equivalent to commutativity. Moreover, the very same result was indeed the beginning of the study of algebraic identities by configurational propositions and vice versa. For example, we mention the character-

(1) Numbers in brackets refer to the bibliography at the end.

tization [15] of the identity $a + a = 0$ by the negation of the Fano proposition.

This paper concerns itself mainly with projective spaces of dimension greater than two. The projective n-space may be defined synthetically as follows [6a]: We consider sets of objects $S_0, S_1, \ldots, S_{n-1}$, called points, lines, ..., hyperplanes, with an incidence relation satisfying the following propositions (of incidence):

1. For every line, there exist three points incident with it.

2. If S_i is incident with S_j, $i < j$, then every point of S_i is incident with S_j.

3. If S_i is incident with S_j, $i < j$, and S_j is incident with S_k, $j < k$, then S_i is incident with S_k.

We shall term a set of m points to be dependent, if there is an S_{m-2} with which all of them are incident; otherwise the points are independent.

4. For every $i+1$ independent points, there is at least one S_i with which all of the points are incident.

Synonymous with incidence, in what follows, several terms will be used like, containing, on, lying in etc.

5. Any S_i contains at least one set of $i+1$ independent points and any S_j which contains the $i+1$ independent points contains S_i.

6. If $A_0, A_1, \ldots, A_i$ are $i+1$ independent points of an S_i, and $B_0, B_1, \ldots, B_j$ are $j+1$ independent points of an S_j, and if $i+j+2$ points $A_0, A_1, \ldots, A_i; B_0, B_1, \ldots, B_j$ are dependent, then there exists at least one point which is contained both in S_i and S_j.

7. There exist at least one set of $n+1$ independent points, but any set of m points, $m > n+1$, is dependent. We remark that if $n > 2$, then the Desargues proposition can be proved [16, p. 41] from these axioms, but if $n = 2$, this is not possible [6] and it should be added to the above

system as we did in the case of the projective plane. We also note that one may prove [6a, p. 211], from the above axioms, that the dimension r of the meet of two subspaces, S_p and S_q, connected with the dimension t of their join by the formula $r + t = p + q$. We shall need this formula in the sequel.

An algebraic model for the projective n-space, S_n, may be obtained as follows:

We consider a division ring R. We represent a point X by an (n+1)-tuple $(x_o, x_1, \ldots, x_n)$, not all $x_i = 0$, $x_i \in R$, and a hyperplane U by an (n+1)-tuple $[u_o, u_1, \ldots, u_n]$ not all $u_i = 0$, u_i R, with the understanding that $(\lambda x_o, \lambda x_1, \ldots, \lambda x_n)$ and $[u_o\lambda, u_1\lambda, \ldots, u_n\lambda]$ represent the same point and hyperplane respectively, provided $\lambda \neq 0$ is any element of R. A point and a hyperplane are incident if and only if the "inner" product $\langle X \cdot U \rangle = x_o u_o + \ldots + x_n u_n$ vanishes. Given any m + 1 points, they are independent if and only if, for every $\lambda_i \in R$, $i = 0, 1, \ldots, m$, $\lambda_o A_o + \lambda_1 A_1 + \ldots + \lambda_m A_m = 0$ implies $\lambda_i = 0$, $i = 0, 1, \ldots, m$. If the points are not independent, they are dependent. Let $X(x_o, x_1, \ldots, x_n)$ and $Y[y_o, y_1, \ldots, y_n]$ be two distinct points. The line joining them is represented by the totality of points $X + \lambda U = (x_o + \lambda y_o, X_1 + \lambda y_1, \ldots, x_n + \lambda y_n)$. Let $Z(z_o, z_1, \ldots, z_n)$ be another point not collinear with X and Y. Then the plane determined by the three points X, Y, Z is represented by the totality of points $X + \lambda U + \mu Z = (x_o + \lambda y_o + \mu z_o, \ldots, x_n + \lambda y_n + \mu z_n)$, $\lambda, \mu \neq 0 \in R$. We similarly represent any subspace of S_n. Because of the essential structural identity of the projective n-space and its algebraic model [6a], we identify one with the other and term either one as a projective n-space defined over a division ring. It follows that these "propositions of incidence" do not tell anything about the algebraic closure or the characteristic of R. However, these properties of R have also been characterized synthetically. But since

all our discussion will be concerned with a general division ring, except very few cases, which will be pointed out clearly, we omit these considerations and the reader may be referred to [15, 16] for discussions concerning these matters.

The problem of characterizing the commutativity of multiplication by means of n-dimensional configurational propositions, suggested by Professor Rainich, is fundamental in this dissertation. It arose in connection with Quasiprojective Geometry (Metaprojective Geometry; J. B. Wright, doctoral dissertation, Univ. of Michigan, 1950).

An approach to the solution of this problem has been made in recent years. In 1925, H. F. Baker established the first step in this direction He showed that the "Eight Lines" theorem, in 3-space, is equivalent to commutativity, generalizing to three dimensions the two-dimensional characterization of Hilbert. Another characterization in three dimensions was obtained in 1931 by K. Reidemeister and Schönhardt. Reidemeister observed that the three-dimensional configurational proposition of Möbius characterizes the commutativity of multiplication. Thus we have, besides the two-dimensional proposition of Pappus, two three-dimensional characterizations in 3-space. This was the situation when the present study was started. The problem of pushing the above mentioned researches of Hilbert, Baker and Reidemeister to higher dimensions appears natural and, in fact, desirable for the future development of higher projective geometries.

Several results are obtained in this paper. The proposition of Pappus is reformuled in several forms whose corresponding configurations are unified into the Pappus class of configurations corresponding to the Pappus class of propositions. A polarity associated with the Pappus configuration is discussed and a Desarguesian property of the configuration is obtained. A minor form of the class is introduced. Then this class is generalized to

higher dimensions in two different manners. One generalization is shown to be an n-dimensional configurational characterization of commutativity. The other generalization was of value in three, four and five dimensions, where it is proved to characterize the commutativity of these spaces; and a minor form of this generalization is shown to be equivalent to the first minor theorem of Pappus.

The Möbius proposition is given in several forms whose corresponding configurations are also unified into the Möbius class of configurations corresponding to the Möbius class of propositions. This class is related directly to the fundamental theorem of projective geometry, and minor forms of it are introduced. The Möbius class of propositions is extended to n-dimensions and consequently an n-dimensional characterization of commutativity is obtained. Then the Möbius and the Pappus classes are unified by specializing each into the other.

Two transformational characterizations of commutativity are obtained by configurational means and a self-dual extension of the Möbius class of configurations is established in odd-dimensional spaces. The Steinitz group generated by Möbius null polarities is extended to general polarities.

CHAPTER I

PAPPIAN CHAINS

The theorem of Pappus is restated in several forms in this paper. Two of the forms are given in the present chapter; and especially the first reveals some new algebraical and geometrical aspects of the Pappian configuration and its "minors".

1. Definition I. 1.

A triangle T_1 : 1 2 3 is said to be "inscribed" in another triangle T_2 : 1' 2' 3', if the vertices 1, 2, 3 of the first are incident to the sides (2' 3'), (1' 3'), (1' 2') respectively of the second. We may also, say T_2 circumscribes T_1. This relation between the two triangles is designated by writing $T_1 \subset T_2$ or $T_2 \supset T_1$. If one incidence is lacking we write $T_1 \dot{\subset} T_2$.

Definition I. 2.

A set of three triangles T_1, T_2, T_3 satisfying the relation, $T_1 \supset T_2 \supset T_3 \dot{\supset} T_1$ is called an "incomplete Pappian chain". If the lacking incidence is fulfilled, the chain is termed "complete".

It is known [16, p. 99] that the nine points and nine lines of the Pappus configuration may be arranged in sets of three triangles forming Pappian chains. However, the following formulation of the Pappus theorem does not seem to have been stated before:

Theorem I. 1.

Let S_2 be a projective plane over a division ring R. Let T_1: 1 2 3, T_2: 1'2'3', T_3: 1''2''3'' be three triangles such that $T_1 \supset T_2 \supset T_3 \supset T_1$, i. e., they form an incomplete Pappian chain. Then the chain can be completed, i. e., $T_1 \supset T_2 \supset T_3 \supset T_1$, if and only if R is a field.

In short, every incomplete Pappian chain can be completed if and only if the commutative identity of multiplication holds.

Proof:

A. Sufficiency:

Let R be a field. Choosing T_1 as a triangle of reference, i.e., $1 \quad 2 \quad 3$ $(1,0,0)$, $(0,1,0)$, $(0,0,1)$; we may represent the vertices of T_2 as follows:

$$\begin{matrix} 1' \\ 2' \\ 3' \end{matrix} \begin{pmatrix} 0 & 1 & a \\ b & 0 & 1 \\ 1 & c & 0 \end{pmatrix}, \quad \text{with } a, b, c \in R \text{ and } a, b, c \neq 0.$$

If the vertex 1'', of the third triangle T_3: 1''2''3'', is taken as unit point $(1,1,1)$, then the dependence of the triple (2'3'1'') gives the relation:

$$c(1 - b) = 1 \quad \ldots \quad (1).$$

The coordinates of the point 2'' may be computed from its incidences to the lines (1''3) and (1'3') as follows:

$(x,y,z) = (0,1,a) + q(1,c,0) = r(0,0,1) + s(1,1,1)$; $q, r, s \in R$.

Hence, $q = s = 1 + qc = a - r$; and consequenctly, $r = a - q = a - (c-1)^{-1}$, and $s = (c-1)^{-1}$, $c \neq 1$. Thus the point 2'' is represented by the vector $(1,1,-cba)$. By a similar computation we find the coordinates of the point 3'' from its incidences to the lines (1''2) and (1'2'); whence 3'' is represented by $(ab, b-1, ba)$. Now, the dependence of the triple (1 2'' 3'') is readily checked from the vanishing of the corresponding determinant.

B. Necessity:

Let R be a division ring. Let the proposition of Pappus hold. As before, let T_1 be the triangle of reference and 1'' the unit point. Let $a, b, \neq 0, 1$; be any two elements in R. We find the point $(1, a, 0)$ and call it 3', the point $(0.1, b)$ and call it 1'. Joining 1' with 2', by left-multiplication we find the point $(a-1, 0, a)$ and we call it 2'. We join 3' with 1'' and 2 with 1'' and let:

$(1'2') \cap (2\,1'') = 3''$ and $(1'3') \cap (3\,1'') = 2''$. Put $1 - a = c$.

By left-multiplication, we compute the coordinates of $2''$ and $3''$ as follows:

$(1,1,1) + \lambda(0,0,1) = (1,a,0) + \mu(0,1,b)$, which yields:

$1 = a + \mu$, $1 + \lambda = \mu b$ and hence, $\lambda = (1 - a)\,b - 1$. Thus the point $2''$ is represented by $(1,1,cb)$. We, similarily, compute the coordinates of $3''$ from the following left dependence:

$(1,1,1) + d(0,1,0) = x(0,1,b) + y(c,o,-a)$, which yields: $1 = yc$, $1 + d = x$, $yc + ya = xb$. Hence, $y\,(c + a)b^{-1} = x$, or $yb^{-1} = x = 1 + d$. Hence $d = c^{-1}b^{-1} - 1$ ($c \neq 0$ obviously). Hence, $3''$ is represented by $(bc,1,bc)$. Thus, we have exhibited an incomplete Pappian chain which is determined by the two "numbers" a and b. By the Pappus proposition the chain can be closed and hence the dependence of the triple $(1\ 2''3'')$ follows. This means $cb - bc = 0$, or $cb = bc$, yielding $(1 - a)b = b(1 - a)$. Hence, $ab = ba$. Since this holds for every pair of "numbers" in R, R is a field.

The configuration (for a precise and more general definition of a configuration than the usual see Chapter II) of the preceding proposition consists of 9 points and 9 lines with 3 points on each line and three lines on each point; it does not form a whole of a finite geometry. The high symmetry of the (Pappus) configuration is reflected in:

Corollary I.1.

If T_1: 1 2 3, T_2: 1'2'3', T_3: 1'' 2'' 3'' form a (complete) Pappian chain so are:

1 2 3'', 1'2'3, 1'' 2'' 3';
1 3 2'', 1'3'2, 1'' 3'' 2';
2 3 1'', 2'3'1, 2'' 3'' 1';
1 2'3'', 2 3'1'', 3 1'2'';
2 1'3'', 3 2'1'', 1 3'2''.

The above corollary is clear and, in it, is shown that the 9 points and the

9 lines of the Pappus configuration may be displayed in 6 different ways as Pappian chains.

Corollary I. 2.

In a commutative S_2, let $T_1 \supset T_2$. There exists a certain collection of triangles T_3 such that $T_1 \supset T_2 \supset T_3 \supset T_1$.

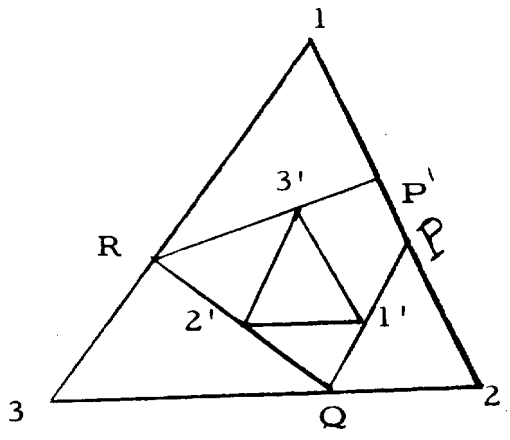
Proof: Let T_1 be 1 2 3 and T_2 be 1'2'3'. Let 1'' be any point $\neq$ 2', 3', on (2'3'). Let:
$(1'3') \cap (3\ 1'') = 2''$ and $(1'2') \cap (2\ 1'') = 3''$. Thus we have constructed the triangle T_3: 1''2''3'' with $T_1 \supset T_2 \supset T_3 \supset T_1$. By theorem I. 1, this incomplete Pappian chain can be closed, and hence $T_1 \supset T_2 \supset T_3 \supset T_1$. Since the point 1'' may be taken arbitrarily on (2'3'), the corollary is proven.

We note that commutativity was assumed for the validity of corollary I. 2, and upon assuming the ground field to be the field of the reals, the "number" of such triangles is infinite. However, in discussing a more general situation, some restrictions must be imposed on the field. This situation is illustrated in:

Theorem I. 2.

Let S_2 be a projective 2-space over a field R which is algebraically closed. Let T_1: 1 2 3 and T_2: 1'2'3' be two triangles in general position. There exists exactly two triangles, T_3 and T_3', such that $T_1 \supset T_3 \supset T_2$ and $T_1 \supset T_3' \supset T_2$.

Proof:



Let P be any point on a side of T_1, say (12). Consider the following sequence of perspectivities:

$$P \overset{1'}{\wedge} Q \overset{2'}{\wedge} R \overset{3'}{\wedge} P' .$$

The sequence of perspectivities defines a projectivity, $\Gamma : P \to P'$, on the line (12). Because R is algebraically closed, Γ must be hyperbolic, and hence it possesses two fixed points P_1 and P_2. Clearly, the two triangles $T_3 : P_1QR$ and $T_3' : P_2QR$ are the desired ones. This completes the proof of the theorem.

We remark that if R is not algebraically closed, Theorem I. 2 need not be true since Γ might be elliptic. If Γ is parabolic, one may say that the two triangles coincide. Finally, if $T_1 \supset T_2$, then Γ is the identity mapping and Theorem I. 2 reduces to Corollary I. 2. Thus, Theorem I. 2 may be considered as a generalization of Corollary I. 2.

It is worth noting that the method used in the proof of Theorem I. 2 is quite general, and may be applied to the general case when one has two simple n-points in general position, instead of two triangles which is a special case of it.

2. Definition I. 3.

Two triangles are said to be "perspective" from a point (line) if the joins (intersections) of their corresponding vertices (sides) concur (are collinear) at that point (line). We adopt the convention that when the triangles 1 2 3 and 4 5 6 are asserted to be perspective, the vertices correspond in the order in which they are written. The well-known theorem of Desargues states that if two triangles are perspective from a point, they are perspective from a line; such triangles are termed "Desarguesian".

The "first minor theorem of Pappus" arises when one requires three corresponding vertices, say (1, 1', 1''), to be collinear; the dual

of this proposition is called the "second minor theorem of Pappus."

In the following theorem, we introduce a "third minor theorem of Pappus" by requiring two of the three triangles, which form an incomplete Pappian chain, to be perspective from a point. The theorem of Pappus, in this minor form, turns out to be "simpler" in the sense that it does not depend on commutativity. This will be shown in

Theorem I. 3.

Let S_2 be a projective plane over a division ring R. Let, as before, T_1, T_2, T_3 be three triangles forming an incomplete Pappian chain, and such that $\{T_1, T_2\}_{0''}$, i.e., T_1 and T_2 are perspective from the point $0''$. Then the chain can be closed.

Proof:

Let $T_1 : 1\,2\,3$ be the triangle of reference, and the point $0''$ be the unit point. The vertices of $T_2 : 1'\,2'\,3'$ are represented as follows:

$$1' \qquad 2' \qquad 3'$$
$$(0,1,1)\,, \quad (1,0,1)\,, \quad (1,1,0)\,.$$

Let $1''$ be represented by the vector $(-a, b, c)$, where $a, b, c \neq 0$ are elements of R. By left-multiplication, one finds that the incidence $(1''\,2'\,3')$ yields the relation

$$a + b + c = 0 \cdots (1)\,.$$

The coordinates of $2''$ may be computed from its incidences to the lines $(1'\,3')$ and $(1''\,3)$ as follows: $\exists\, \lambda, \mu, \gamma \in R \ni (-a, b, c) + \lambda(0, 0, 1) = \mu(1, 1, 0) + \gamma(0, 1, 1)$, giving $-a = \mu$, $b = \mu + \gamma$, $c + \lambda = \gamma$. Because, $\lambda = b + a - c$, and $a + b + c = 0$, the point $2''$ is represented by $(a, -b, c)$. We, similarly, compute the coordinates of $3''$, by left-multiplication only, and this leads to the vector $(a, b, -c)$. Thus, we have obtained the following matrix of representation:

$$\begin{array}{c} 1 \\ 2'' \\ 3'' \end{array} \begin{pmatrix} 1 & 0 & 0 \\ a & -b & c \\ a & b & -c \end{pmatrix}$$

Assuming that $1 + 1 \neq 0$, we have:

$(a, -b, c) + (a, b, -c) = (2a, 0, 0) = 2a(1, 0, 0)$, proving the collinearity of the triple $(1\ 2''\ 3'')$.

We remark that, for the above computation to hold, we had to assume that the characteristic of R is $\neq 2$. However, if the characteristic is equal to 2, the above configuration does not exist, and the third minor theorem of Pappus holds vacuously.

Now, we derive the following interesting theorem which describes a Desarguesian property of the Pappian chain:

Theorem I.4.

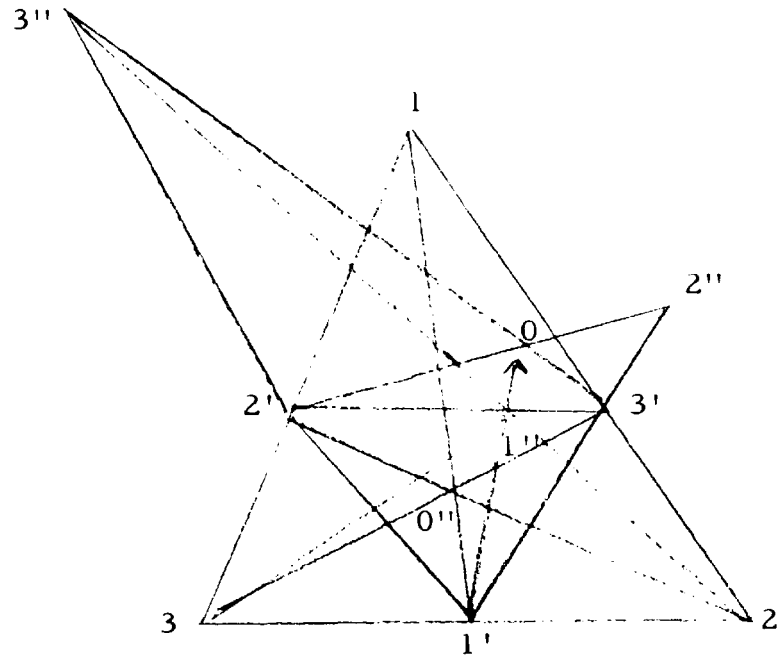
In a commutative S_2, let T_1, T_2, T_3 form a complete Pappian chain. If one pair of the three pairs $\{T_1, T_2\}$, $\{T_2, T_3\}$, $\{T_1, T_3\}$ is Desarguesian, so are the other two pairs.

Proof:

Let the pair $\{T_1, T_2\}$ be Desarguesian from the point $0''$. It is clear that a coordinate system may be chosen so that the following points have the corresponding coordinates.* We represent the point $1''$ by the vector $(-a, b, c)$. Since $1''$ is on the line $(2'3')$, we have, from the vanishing of the determinant corresponding to the triple $(2', 3', 1'')$, the relation:

$$a + b + c = 0 \quad \ldots\ldots \quad 1 \ .$$

The point $2''$ is incident to both $(1'\,3')$ and $(1''3)$, and hence its representing vector may be computed easily; and upon computing one finds that this vector is represented by $(a, -b, c)$. Similarly, the point $3''$ is incident to the lines $(1'\,2')$ and $(1''2)$, and hence one finds its representing vector may be written as $(a, b, -c)$. It is now a simple matter to check



1 2 3 1' 2' 3' 0'' *

(1, 0, 0), (0, 1, 0), (0, 0, 1), (0, 1, 1), (1, 0, 1), (1, 1, 0), (1, 1, 1) ;

the dependence of the triple (1 2''3'') by left-multiplication only. By direct computation, one finds that the following lines may be represented by the vectors corresponding to them as follows:

$$(1'\,1''): \begin{bmatrix} c-b, & -a, & a \\ b, & a-c, & -b \\ -c, & c, & b-a \end{bmatrix}$$
$$(2'\,2''):$$
$$(3'\,3''):$$

The two lines have the point (a^2, b^2, c^2) in common, and this point is on (3' 3'') as may be seen by simple calculation. Calling the point with the coordinates (a^2, b^2, c^2) by the symbol 0, we have $\{T_2, T_3\}_0$.

By a similar computation to the above, one finds that the lines (1 1''), (2 2''), (3 3'') have the following representation:

$$(1\,1''): \begin{bmatrix} 0, & c, & -b \\ -c, & 0, & a \\ b, & -a, & 0 \end{bmatrix}$$
$$(2\,2''):$$
$$(3\,3''):$$

, which forms a skew-symmetric determinant of order 3, i.e., odd. Hence, their corresponding determinant vanishes, and therefore the lines are concurrent from one point, say 0'. Thus we have $\{T_1, T_3\}_{0'}$. This com-

the proof of the theorem.

Definition I. 4.

Let S_n be a projective space over a field R. A one-to-one mapping between the points $P(x_o, x_1, \ldots, x_n)$ and the hyperplanes $H[\eta_o, \eta_1, \ldots, \eta_n]$ of S_n which preserves incidences is termed a correlation.

It is a known fact [16, p. 266] that every correlation may be represented analytically by the linear homogeneous form

$$\eta_i = \sum_{j=0}^{n} k_{ij} x_{ij}, \quad i = 0, 1, \ldots, n; \quad k_{ij}, x_i, \eta_i \in R$$

with the determinant $|k_{ij}| \neq 0$, and conversely. If the correlation is of period two, it is called a polarity.

Now, let Γ be a polarity mapping $x \rightarrow \eta$ and $y \rightarrow \xi$. We may write $\eta_i = \sum_{j=o}^{n} k_{ji} x_j$, $i = 0, \ldots n$ and $\xi_j = \sum_{i=o}^{n} k_{ji} y_i$, $y = 0, \ldots n$. If $x_j = y_j$, one finds that $\eta_i = \rho \xi_i$, with $\rho \neq 0$, and $\rho \notin R$. This implies $k_{ji} = \rho k_{ij}$, for all i, j. Repeating the last equality, we get $k_{ji} = \rho^2 k_{ji}$, yielding $\rho^2 = 1$, and hence $\rho = +1$. The case when $\rho = -1$ is possible only when n is odd, otherwise the determinant $|k_{ij}|$ vanishes. This mapping is called a null polarity and will be discussed in details in Chapter IV. The relation between commutativity and polarity as well as null polarity will be a problem of interest to us in Chapter IV.

But, when $\rho = 1$, we have $k_{ij} = k_{ji}$ which means that a polarity may be presented by a symmetric matrix. Conversely, if $k_{ij} = k_{ji}$, then the correlation Γ represented by $\eta = \sum_{j=o}^{n} k_{ij} x_j$, $i = 0, \ldots, n$, with the determinant $|k_{ij}| \neq 0$, is a polarity, because $(k_{ij}) = (k_{ij})^t$, where $(k_{ij})^t$ is the transpose of (k_{ij}), and hence the correlation Γ is equal to its own inverse, i.e., $\Gamma = \Gamma^{-1}$, yielding $\Gamma^2 = I$, where I is the identity mapping. This proves that Γ is of period two, which means that Γ is a polarity.

Consider any polarity $\Gamma : x \rightarrow \eta$ and $y \rightarrow \xi$. If it happened that

x is incident to ξ, i.e., $\langle x \cdot \xi \rangle = 0$, then y is incident to η, i e., $\langle y \cdot \eta \rangle = 0$. This is, indeed, the well-known property of "conjugacy" of a polarity. It says that if a point is incident to the polar of another point, then the second point is incident to the polar of the first point.

Before we prove our next theorem, concerning Pappian chains, we state two more definitions.

Definition I.5.

If η is the image of P under a polarity Γ, then any point of η is called a "conjugate" of P, and any hyperplane on P is called a "conjugate" of η.

Definition I. 6.

The "simplex" whose "faces" are the polar of the vertices of another simplex, is said to be a "polar" simplex of the second. If the faces of a simplex are the polars of its vertices, the simplex is termed "self-polar."

In this chapter, we restrict the above notions to 2-space, however, we shall need the general notions in the next chapters.

Theorem I. 5.

In S_2, let T_1, T_2, T_3 be three triangles forming a complete Pappian chain. If there exists a polarity Γ under which two of the triangles are polar, then the third triangle is self-polar under Γ.

Proof:

Let Γ be a polarity under which T_1 and T_2 are polar. We shall prove that T_3 is self-polar.

We choose a coordinate system so that the following points and lines have the corresponding coordinates:

1	2	3	(12)	(23)	(31)
$(1,0,0)$,	$(0,1,0)$,	$(0,0,1)$,	$[0,0,1]$,	$[1,0,0]$,	$[0,1,0]$.

Let Γ be represented by the symmetric matrix (k_{ij}), $i,j = 1,2,3$. Since

T_2 is polar to T_1, we have

$$\begin{matrix}(1'2') & (2'3') & (3'1')\\ [k_{13}, k_{23}, k_{33}], & [k_{11}, k_{12}, k_{13}], & [k_{12}, k_{22}, k_{24}]\end{matrix}.$$

Choosing 1'' to be (1, 1, 1), we have, by its incidence to (2' 3'):

$$k_{11} + k_{12} + k_{13} = 0 \ldots ① .$$

The coordinates of (1'' 3'') and (1'' 2'') are easily computed and they are:

$$\begin{matrix}(1''\ 2'') & (1''\ 3'')\\ [1, -1, 0], & [-1, 0, 1]\end{matrix}.$$

Hence 2'' is represented by $(-k_{23}, -k_{23}, k_{12} + k_{22})$ and 3'' may be represented by $(-k_{23}, k_{13} + k_{33}, -k_{23})$. Since (12), (1' 3') and (2' 3') are concurrent, we have

$$(k_{12})^2 - k_{11} k_{22} = 0 \ldots ②.$$

Similarly, the concurrency of (23,), (1' 3') and (1' 2') yields

$$(k_{23})^2 - k_{22} k_{33} = 0 \ldots ③;$$

and the concurrency of (13), (1' 2') and (2' 3') implies

$$(k_{13})^2 - k_{11} k_{33} = 0 \ldots ④ .$$

Now, upon applying Γ to 2'', we have, upon using 3 : $[-k_{11} k_{23} - k_{12} k_{23} + k_{13} k_{12} + k_{13} k_{22}, 0, k_{33} k_{12} - k_{13} k_{23}]$. Since 1 is on (2'' 3''), we have $k_{12} k_{33} + k_{13}(k_{12} + k_{22}) = 0 \ldots$ 5 ; by this last relation, one checks that the polar of 2'' passes through 1''. But the polar of 2'' clearly passes through 3'', therefore, it must be (1'' 3''). Similarly, one proves that the polar of 3'' is (1'' 2''). Hence, T_3 is a self-polar triangle under Γ. The theorem is proved.

It is interesting to note that Theorem I. 5 is susceptible of a simple synthetic proof as follows:

Consider the complete quadrangle $1''3''2'2''$. Join $2'$ with $2''$. The two opposite pairs $(1''2'')$ and $(2'3'')$ are conjugate under $'\ '$, because $(1''2'')$ is incident with 3, which is the pole of $(1'2')$. Also, the opposite pairs $(2''3'')$ and $(''2')$ are conjugate under Γ. Hence, by a theorem of von Staudt [16, p. 125], the remaining opposite sides, $(1''3'')$ and $(2''2')$, are conjugate under Γ. This means that the pole of $(1''3'')$ must be on $(2'2'')$. But, since $(1''3'')$ passes through the point 2, which is the pole of $(1'3')$, the pole of $(1''3'')$ must, also, be incident to $(1'3')$, or, which is the same thing, $(1'2'')$. Hence, the point $2'' = (1'2'') \cap (2'2'')$ is the pole of $(1''3'')$. By symmetry, one proves that $3''$ is the pole of $(1''2'')$; simply by considering, in this case, the complete quadrangle $2''3''1''3'$. Hence, $1''$ is the pole of $(2''3'')$, and consequently, T_3 is self-polar.

Combining Theorem I. 4 with Theorem I. 5 we obtain

Theorem I. 6.

In a commutative S_2, let $T_1 \supset T_2 \supset T_3 \supset T_1$. If there exists a polarity that interchanges the elements of one pair of the three pairs, $\{T_1, T_2\}$, $\{T_2, T_3\}$, $\{T_1, T_3\}$, (and consequently leaves the third self-polar), there exist two other polarities interchanging the members in each of the other two pairs.

Proof:

Let $\{T_1, T_2\}$ be polar with respect to a polarity Γ_1. By a theorem of Chale's [16, p. 125], $\{T_1, T_2\}$ are Desarguesian. Applying our Theorem I. 4, the two pairs $\{T_2, T_3\}$ and $\{T_1, T_3\}$ are each Desarguesian also. Hence, by a theorem of von Staudt [2, p. 52], there exist two polarities Γ_2 and Γ_3 under which the elements of the two pairs $\{T_2, T_3\}$ and $\{T_1, T_3\}$ are interchanged respectively. Hence Γ_2 and Γ_3 are the required polarities. The theorem is proved.

The three Desarguesian points $0, 0', 0''$, of Theorem I. 4, are not,

in general, collinear. The question as to what restriction, on the Pappian configuration, will make these three points collinear, will be answered in

Theorem I. 7.

In a commutative S_2 let T_1, T_2, T_3 be such that $T_1 \supset T_2 \supset T_3 \supset T_1$ and $(1, 1', 1'')$ are collinear, i.e., the three triangles form the first minor configuration of Pappus. If $\{T_1, T_2\}_{0''}$, then $\{T_2, T_3\}_0$, $\{T_1, T_3\}_{0'}$ and $(0, 0', 0'')$ are collinear.

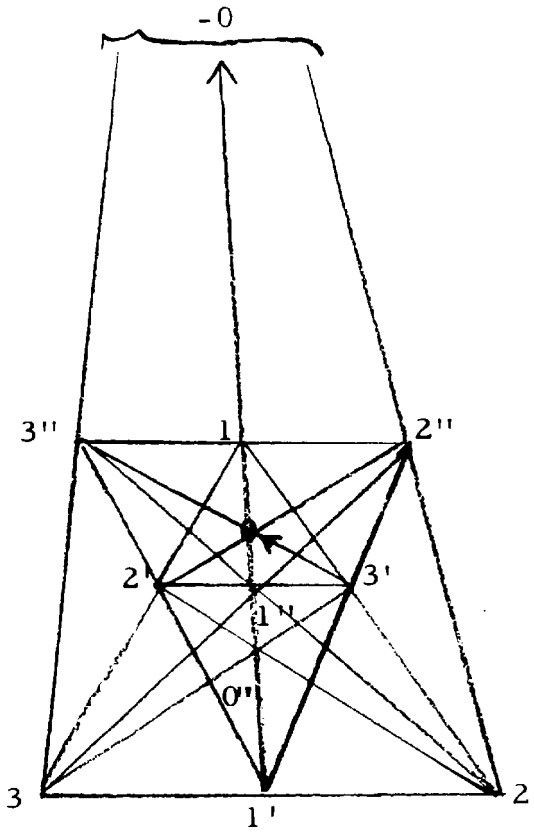
Proof:

Since T_1 : 1 2 3 and T_2 : 1' 2' 3' are perspective from the point 0'', there exists a polarity Γ under which the two triangles are polar to each other [2, p. 52]. Join 2' and 2'', 3' and 3'', and $(2'\,2'') \cap (3'\,3'') = 0$. We shall prove that 0 is incident to (1' 1''). By Theorem I. 5, the polarity Γ leaves the triangle T_3 self-polar. The pole of the line (11'') is the meet of (2' 3') and (2'' 3''). By hypothesis 1' is on (1 1''), therefore the pole of (1 1'') is incident to the line (23) because (2 3) is the polar of 1'. Consider the two triangles 3' 1'' 2'' and 3'' 1 2'; we have:

$(1''2'') \cap (1\,2') = 3$, $(2''\,3') \cap (2'\,3'') = 1'$, $(1\,3'') \cap (3'\,1'') = x$.

Since $(3, 1', x)$ are collinear, these two triangles, i.e., 3' 1'' 2'' and 3'' 1 2' are perspective from the line (2 3). Hence, they must be perspective from a point. That is to say, (3' 3''), (1 1'') and (2' 2'') are on one point, say 0. Hence, $\{T_2, T_3\}_{0}$. (See diagram next page).

In order to prove that $\{T_1, T_3\}$ are Desarguesian, consider the two triangles 1 2 3 and 1' 2'' 3''. We have: $(1\,2) \cap (1'\,2'') = 3'$, $(2\,3)\ (2''3'') = x$, $(1\,3) \cap (1'\,3'') = 2'$. Because the triple $(2', 3', x)$ is dependent, the three lines (2 2''), (3 3'') and (1 1') are concurrent on one point, say the point 0'. Hence, T_1 and T_3 are Desarguesian from 0', i.e., $\{T_1, T_3\}_{0'}$. Since the three Desarguesian points $0, 0', 0''$ lie on the line (1 1' 1''), the theorem is completely proven.


-0
3''
1
2''
2'
3'
1''
0''
3
1'
2

3. This section deals with another statement of the Pappus theorem. The statement involves a configuration which is associated with the Pappus configuration. This association calls for the introduction of a configurational notion which will prove useful in the following chapter.

Detinition I. 7.

Two triangles T_1: 1 2 3 and T_2: 1' 2' 3' are said to be "doubly perspective" [16, p. 100], if 1 2 3 is perspective with 1' 2' 3' and with 2' 3' 1'. They are called "triply perspective" if, in addition, 1 2 3 is perspective with 3' 1' 2' also.

Theorem I. 8.

The following proposition is equivalent to Pappus Theorem: If two triangles are doubly perspective, they are triply perspective.

Proof:

A. Let T_1: 1 2 3 and T_2: 1' 2' 3' be doubly perspective from the points 1" and 2" respectively. We shall prove that the two triangles are perspective from another point, say 3", by using the theorem of Pappus. That is to say, if 3" = (2 1')$\cap$(3 2'), we shall prove that 3" is on (1 3').

Consider the three triangles: 3 3' 2", 1' 1"2 and 1 2' 3". It is not difficult to check that these three triangles form the following incomplete Pappian chain:

$$(3\ 3'\ 2'') \supset (1'\ 1''\ 2) \supset (1\ 2'\ 3'') \supset (3\ 3'\ 2'').$$

By Theorem I. 1, i. e., Pappus, the chain can be closed. Hence 3' is incident with the line (1 3"), or the point 3" is on the line (1 3'), proving our contention.

B. Let T_1: 1 2 3, T_2: 1' 2' 3', T_3: 1" 2" 3" form an incomplete Pappian chain, where the incidence (1 2"3") is missing.

Join 1 with 1', 3 with 3", 1 with 1", 1' with 1", 3 with 3' and 3' with 3". Considering the two triangles 1 1' 1" and 3 3' 3", one may verify that they are doubly perspective from the two points 2' and 2'. By

the above proposition, they are triply perspective from a third point; and this point is clearly, the point 2''. This means that the point 1 is incident to the line (2''3''). Thus the chain is closed. This completes the proof of Theorem I 8.

We note that the configuration, corresponding to the preceding proposition, consists of 9 points and 15 lines with various incidences. It is different from the (Pappus) configuration of theorem I. 1. However, we observe that the present configuration may be derived from the Pappus configuration by allowable joins of points of the second. In other words, the Pappus configuration may be "extended" in a natural way to the present configuration. This phenomenon seems to occur quite often between some configurations, as we shall see in the next chapter where it will be discussed systematically.

CHAPTER II

THE MÖBIUS CLASS OF CONFIGURATIONS

The first objective of this chapter is to study the "direct" relations among the algebraic identity of the commutativity of multiplication and some geometrical propositions in projective 3-space.

The second aim of this chapter is to introduce the concept of "a natural extension" of a configuration. This notion will lead to the other notion of the "geometrical equivalence" of two configurations, by which some configurations, corresponding to equivalent propositions, are "unified" into one class corresponding to the class of their propositions. Some algebraical and geometrical results concerning this class will be obtained.

1. On Configurations.

Definition II. 1.

An n-configuration $A(a_0, a_1, \ldots, a_{n-1})$ is a finite collection of finite subspaces $a_0, a_1, \ldots, a_{n-1}$, in which an incidence relation R is defined. We call a_0 the set of points, a_1 the set of lines, ..., a_{n-1} the set of hyperplanes; any one of these sets may be empty.

Definition II. 2.

If each element of a_i is incident to the same number of elements of a_j, and this is true for all i and j, then the configuration is said to be regular.

Definition II. 3.

Two configurations $A(a_0, a_1, \ldots, a_{n-1})$ and $B(b_0, b_1, \ldots, b_{n-1})$ containing the same number of elements of corresponding kinds, are said to be "isomorphic", $A \cong B$, if there exist one-to-one mappings: $a_i \rightarrow b_i$, $i = 0, 1, \ldots, n-1$, such that the incidence relation R is preserved.

Definition II. 4.

A configuration $A^*(a_0, a_1, \ldots, a_{n-1}; x_0, x_1, \ldots, x_m)$ is said to be a natural extension of the configuration $A(a_0, a_1, \ldots, a_{n-1})$ if and only if each x_i, $i = 0, 1, \ldots, m$, may be obtained by a sequence of operations consisting of joins and intersections among the elements a_i and their derivatives. The configuration $X(x_0, x_1, \ldots, x_m)$ is called the complement of A with respect to A^*. We express the relation between A and A^* by writing $A \leq A^*$.

Definition II. 5.

Two configurations A and A' are said to be dual if there are one-to-one mappings between the points, lines, planes, ..., hyperplanes of A and the hyperplanes, $(n-2)$-spaces, ..., lines, points of A' such that all incidences are preserved.

A direct consequence of Definition II. 4 is

Corollary II. 1.

Every configuration is a natural extension of itself.

Lemma II. 1.

If $A \leq B$ and $B \leq C$, then $A \leq C$.

Proof:

Any element $c \in C$, is either an element of B or an element of the complement of B with respect to C. Since $A \leq B$, the element c must be either an element of A or an element derived from the elements of A by some sequence of natural extensions. Hence, the element c is an element of the complement of A with respect to C, if not an element of A. This, by Definition II. 4, means that $A \leq C$.

Definition II. 6.

Two configurations A and B are said to be geometrically equivalent if and only if there exists a configuration C such that $A \leq C$ and $B \leq C$. We denote this relation by writing $A \sim B$.

The Definition II. 6 gives immediately

Corollary II. 2.

Every configuration is geometrically equivalent to itself.

Lemma II. 2.

If $A \leq B$ and $A \leq C$, then there exists a configuration D such that $B \leq D$ and $C \leq D$.

Proof:

We construct a configuration D so that it consists of the configuration A and the complements of A with respect to both B and C We shall prove that D satisfies the required properties. Let $d \in D$. If $d \in A$, then $d \in B$ and $d \in C$ by hypothesis. Let $d \notin A$. Then d must be an element of either of the complements of A with respect to B and C. Suppose it is an element of the complement of A with respect to B. Then it must be determined by some sequence of natural extensions of A and hence of the elements of C. Thus, $C \leq D$. We, similarly, prove that $B \leq D$ if d is an element of the complement of A with respect to C. Hence, $B \leq D$ and $C \leq D$.

In the above proof we also proved

Corollary II. 3.

If $A \leq B$ and $A \leq C$, then $B \sim C$.

We now prove the following useful theorem.

Theorem II. 1.

The property of being "geometrically equivalent to" is an equivalence relation.

Proof:

Reflexivity and symmetry are clear. We shall prove that the relation is transitive.

Let $A \sim B$ and $B \sim C$. Then there exist D_1 and D_2 such that $A \leq D_1$ and $B \leq D_1$, and $B \leq D_2$ and $C \leq D_2$. If $D_1 \cong D_2$, the theorem

follows. Suppose $D_1 \neq D_2$. By Lemma II. 2, there exists a configuration D_3 such that $D_1 \leq D_3$ and $D_2 \leq D_3$. Applying Lemma II. 1, we have $C \leq D_3$ and $A \leq D_3$. By Definition II. 6, we have $A \sim C$, and the theorem is proved.

We may note that there exist some configurations which have no natural extensions, and hence they could not be geometrically equivalent to other configurations except themselves. For example, the following configurations have no natural extensions:

a point, a line and a point on it, a triangle.

It should be observed that the complement of a configuration with respect to a natural extension of it is not, in general, determined by a unique manner. In spaces of dimensionshigher than two, it can happen that the same complement may be obtained in two different processes. The following is an example in 3-space.

Consider the configuration consisting of four points not in one plane and such that no three of them are collinear. This configuration may be extended naturally to a tetrahedron by either forming the faces first and then the edges as intersections of them, or the edges first, as joins of points and the faces, second, as joins of edges.

2. The Möbius Proposition and Its Minor.

As long ago as 1828, F. Möbius [8] proved the following theorem: Let $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ be two tegrahedra such that the vertices B_1, B_2, B_3, B_4 are incident to the planes $(A_2A_3A_4)$, $(A_1A_3A_4)$, $(A_1A_2A_4)$, $(A_1A_2A_3)$. If, in addition, A_1 is incident to $(B_2B_3B_4)$, A_2 is incident to $(B_1B_3B_4)$ and A_3 is incident to $(B_1B_2B_4)$, then the theorem of Möbius claims that A_4 is incident to $(B_1B_2B_3)$.

Thus, we have altogether a configuration consisting of two tetrahedra which are mutually inscribed and circumscribed. This (Möbius) con-

figuration consists of two kinds of elements, points and planes, such that each element of one kind is on four elements of the other kind. It is, by Definition II. 2, regular and may be denoted by the symbol M(8, 0, 8). Hereafter, we shall refer to the Möbius proposition as the M-proposition, and to its corresponding configuration as the M-configuration.

Two tetrahedra satisfying the hypothesis of the M-proposition are said to be in an incomplete M-relation. The theorem of Möbius may, therefore, be stated as follows: Let S_3 be a projective 3-space over a division ring R. Every incomplete M-configuration is in fact complete if and only if R is a field.

A. Let R be a commutative division ring. Since A_1, A_2, A_3, A_4 are independent, by the hypothesis, we may write

$$\begin{aligned} B_1 &= k_{12}A_2 + k_{13}A_3 - A_4, \\ B_2 &= k_{21}A_1 + k_{23}A_3 - A_4, \\ B_3 &= k_{31}A_1 + k_{32}A_2 - A_4, \\ B_4 &= A_1 + A_2 + A_3; \quad \text{with } k_{ij} \in R. \end{aligned}$$

The above representation may be justified as follows: Since B_1 is on the plane $(A_2 A_3 A_4)$, B_4 is a linear combination of A_2, A_3, A_4, and by scalar multiplication we can make the coefficient of A_4 to be -1. Then we absorb the coefficient of B_1. We, similarly, justify the representations of B_2 and B_3. Since B_4 is a linear combination of A_1, A_2, A_3, we absorb their coefficients.

Now, we have A_1 on $(B_2 B_3 B_4)$, therefore $\exists$, not all zero elements, $\lambda_1, \lambda_2, \lambda_3 \ni$

$$\lambda_1(k_{21}A_1 + k_{23}A_3 - A_4) + \lambda_2(k_{31}A_1 + k_{32}A_2 - A_4) + \lambda_3(A_1 + A_2 + A_3) = A_1; \ \lambda_i \in R.$$

Because the A_i are independent, we obtain $\lambda_2 k_{32} + \lambda_3 = \lambda_1 k_{23} + \lambda_3 = \lambda_1 + \lambda_2 = 0$. By simple calculation we find $k_{23} = -k_{32}$. Similarly, from the other two incidencies in the hypothesis, we obtain (in fact, by left-

multiplication only) that $k_{31} = -k_{13}$ and $k_{12} = -k_{21}$. Thus far, without the use of commutativity, we have represented the incomplete M-relation, between the two tetrahedra, by a skew-symmetric matrix. As R is commutative, we have the determinant:

$$\langle A_4 B_1 B_2 B_3 \rangle = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & k_{12} & k_{13} & -1 \\ k_{21} & 0 & k_{23} & -1 \\ k_{31} & k_{32} & 0 & -1 \end{vmatrix} = \begin{vmatrix} 0 & k_{12} & k_{13} \\ k_{21} & 0 & k_{23} \\ k_{31} & k_{32} & 0 \end{vmatrix} .$$

The last determinant is skew-symmetric and of order three, hence it vanishes. This proves that A_4 is on $(B_1 B_2 B_3)$, and, therefore, the configuration is complete.

B. Let the M-proposition hold. Let a and b be any two elements of R different from 0, 1 and such that $(a - b) \neq 1$ (since otherwise the two elements trivially commute, and also a and $(a + 1)$ always commute). We shall prove that $ab = ba$. Let $T_1 : A_1 A_2 A_3 A_4$ be the tetrahedron of reference. We find the points B_1, B_2, B_3, B_4 :

$$\begin{aligned} B_1 &= \quad a A_2 + b A_3 + A_4 , \\ B_2 &= -a A_1 \quad + \quad A_3 + A_4 , \\ B_3 &= -b A_1 - A_2 \quad\quad + A_4 , \\ B_4 &= \quad - A_1 - A_2 - \quad A_3 \end{aligned}$$

Clearly, the points B_1, B_2, B_3, B_4 form a tetrahedron $T_2 \ni T_1 \supset T_2$. By left-multiplication, one checks that A_1 is on $(B_2 B_3 B_4)$ as follows:

$B_2 - B_3 + B_4 = (b - a - 1)A_1$. But $(b - a - 1) \neq 0$ by assumption, we have what we want. By similar straight-forward computation, it could be verified that A_2 is on $(B_1 B_3 B_4)$ and A_3 is on $(B_1 B_2 B_4)$. Thus, we have determined an incomplete M-configuration by using the two "numbers" a and b. By the M-proposition, this incomplete configuration can be completed, and hence $\exists \mu_1, \mu_2, \mu_3 \in R \ni$:

$A_4 = \lambda_1 B_1 + \lambda_2 B_2 + \lambda_3 B_3 = \lambda_1 (a A_2 + b A_3 + A_4) + \lambda_2 (-a A_1 + A_3 + A_4) + \lambda_3 (-b A_1 - A_2 + A_4)$.
Because A_1, A_2, A_3, A_4 are linearly independent, we have

$$-\lambda_2 a = \lambda_3 b, \quad \lambda_1 a = \lambda_3, \text{ and } \lambda_1 b + \lambda_2 = 0 .$$

Combining the first two equations and substituting in the third, we get $ab = ba$. Since every pair of elements in R, not trivially commutative, may be treated in the above manner, R is a commutative division ring, i.e., a field.

The above logical equivalence between the M-proposition and commutativity of multiplication was expected since 1929 when K. Reidemeister conjectured [11] that the M-proposition is equivalent to the Pappus proposition in the plane. Schönhardt proved [13] Reidemeister's conjecture in 1931. Our previous proof relates the M-proposition to commutativity directly.

The question arises whether it is possible to prove a special but non-degenerate form of the M-proposition without the use of the commutativity of multiplication. In the remaining part of this section, we obtain a theorem which we call the minor theorem of Möbius in a, not necessarily commutative, 3-space. This theorem may be considered as corresponding to our third minor theorem of Pappus, Theorem I. 3, which we obtained in the preceding chapter.

Theorem II. 3.

Let S_3 be a projective 3-space defined over a division ring R. Let T_1 : 0 1 2 3 and T_2 : 0' 1' 2' 3' be two tetrahedra in an incomplete M-relation, where the incidence (0' 1' 2' 3) is lacking. Let two adjacent pairs of the corresponding edges of the two tetrahedra, say $\{(0\,1), (0'\,1')\}$ and $\{(0\,2), (0'\,2')\}$, be dependent. Then the M-configuration is in fact complete.

Proof:

Let 0 1 2 3 be the reference tetrahedron. Then, according to the

discussion given in the proof of the M-theorem (which was independent of commutativity), we may represent the two tetrahedra as follows:

$$\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{array}{c} 0' \\ 1' \\ 2' \\ 3' \end{array}\begin{pmatrix} 0 & a & b & 1 \\ -a & 0 & c & 1 \\ -b & -c & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix};$$

where $a, b, c \neq 0 \in R$. Using the condition that (01) and (0'1') are dependent, we have

$$(1, 0, 0, 0) = \lambda_1(0, 1, 0, 0) + \lambda_2(0, a, b, 1) + \lambda_3(-a, 0, c, 1),$$

for $\lambda_i \in R$ and not all $\lambda_i = 0$. Hence, by simple calculation we find that $b = c$. Also, from the remaining condition, namely, (02) and (0'2') are dependent, we write

$$(1, 0, 0, 0) = \mu_1(0, 0, 1, 0) + \mu_2(0, a, b, 1) + \mu_3(-b, -c, 0, 1),$$

where not all $\mu_i = 0$ and $\mu_i \in R$. Again, by simple computation, one obtains $c = -a$. Hence, $b = c = -a$. Thus, the matrix representing the vertices of T_2', reduces to the following form:

$$\begin{array}{c} 0' \\ 1' \\ 2' \\ 3' \end{array}\begin{pmatrix} 0 & a & -a & 1 \\ -a & 0 & -a & 1 \\ a & a & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}.$$

Now, since we have

$$(a, a, 0, 1) + (-a, 0, -a, 1) - (0, a, -a, 1) = (0, 0, 0, 1),$$

the point 3 is incident with the plane (0'1'2'). It remains to be shown that the four points 0', 1', 2', 3' are independent, and then the theorem follows.

Let λ_i be some elements of R which are not all equal to zero, and $i = 1, 2, 3, 4$. Consider the following vanishing linear combination:

$\lambda_1(0, a, -a, 1) + \lambda_2(-a, 0, -a, 1) + \lambda_3(a, a, 0, 1) + \lambda_4(-1, -1, -1, 0) = 0$. This yields $-\lambda_2 a + \lambda_3 a = \lambda_1 a + \lambda_3 a = -\lambda_1 a - \lambda_2 a = \lambda_4$ and, $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Combining the first two, we have $(\lambda_1 + \lambda_2)a = 0$; and since $a \neq 0$ by assumption, we have $\lambda_1 = -\lambda_2$. Because $\lambda_1 + \lambda_2 + \lambda_3 = 0$, we have $\lambda_3 = 0$. Also, we have, from the second and the third, $(\lambda_3 - \lambda_2)a = 0$, giving $\lambda_3 = \lambda_2$. Hence, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$. This proves the independence of the four points $0', 1', 2', 3'$. The proof of the minor theorem of Möbius is complete.

Corresponding to Theorem I. 4, concerning the Pappus configuration, we establish the following theorem concerning the Möbius configuration.

Theorem II. 4.

Let S_3 be over a division ring R. Let T_1 and T_2 be two tetrahedra in an incomplete M-relation. If two pairs of the correspondingly adjacent edges are dependent, so are the oppositely remaining two such pairs.

Proof:

We first note that by Theorem II. 3, the M-configuration is in fact complete. However, the proof is independent of this fact.

According to Theorem II. 3, the vertices of the two given tetrahedra may be represented by the rows of the following two matrices:

$$\begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{array}{c} 0' \\ 1' \\ 2' \\ 3' \end{array}\begin{pmatrix} 0 & a & -a & 1 \\ -a & 0 & -a & 1 \\ a & a & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix},$$

where $a \in R$ and $a \neq 0$. In the above representation, we used the given condition that the pair $\{(01), (0'1')\}$ is dependent, and so is the pair $\{(02), (0'2')\}$. By left-multiplication only, we have:

$$a(-1, -1, -1, 0) + (a, a, 0, 1) - (0, 0, 0, 1) = (0, 0, -a, 0) = -a(0, 0, 1, 0),$$

and since $a \neq 0$, by assumption, we proved that the pair of edges $\{(2\,3), (2'\,3')\}$ is dependent. Also, we have

$$a(-1, -1, -1, 0) - (-a, 0, -a, 1) + (0, 0, 0, 1) = (0, -a, 0, 0) = -a(0, 1, 0, 0),$$

proving the dependence of the pair $\{(1\,3), (1'\,3')\}$. The theorem is completely proved.

3. On the Eight Lines Proposition (8L).

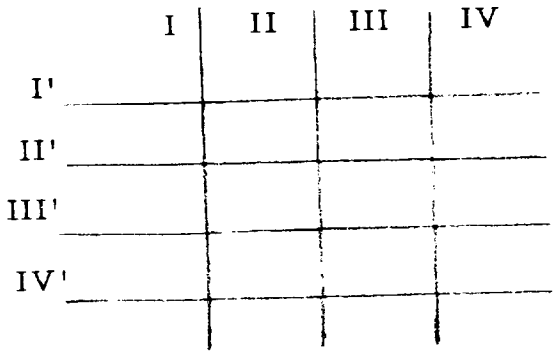
H. F. Baker has shown in 1925 [1a, I] that the proposition of Pappus (in the plane) is equivalent to the following (8L)-proposition, in 3-space, which it is convenient to formulate in the following self-dual form:

If fifteen of the sixteen pairs of lines that can be formed from two quadruples of mutually skew lines are non-skew, so is the sixteenth pair.

We remark that the term "non-skew" is self-dual. It means that the two lines are incident to a point and to a plane.

The close connection between the M-configuration and the (8L)-configuration, which is self-dual and consisting of 8 lines, seems to have been overlooked. In what follows we show the "direct" logical equivalence between this proposition and the M-proposition; then we prove the geometrical equivalence between their corresponding configurations.

We shall say that two propositions P_1 and P_2 are "directly" equivalent if each implies the other in the presence of the "axioms of incidence". A standard set of such axioms are to be found in [16] or [1a, I].



Let I, II, III, IV and I', II', III', IV' be two 4-tuples of mutually skew lines. Let $I \cap I' = 1$, $II \cap I' = 2$, $III \cap II' = 4$, $IV \cap II' = 3$, $I \cap III' = 2'$, $II \cap III' = 1'$, $III \cap IV' = 3'$. We shall prove, assuming the propositions of incidence, that IV meets IV' if and only if the M-proposition holds.

A. Let the M-proposition hold. Let $(1\,2\,3) \cap IV' = 4'$. Consider the two tetrahedra $1\,2\,3\,4$ and $1'\,2'\,3'\,4'$. By the hypothesis of the (8L)-proposition, the two tetrahedra are in an incomplete M-relation as it may be easily checked. By the M-proposition, the M-configuration is complete. That is to say, the point $4'$ is on the plane $(1'\,2'\,3')$. But we already have the point $4'$ on the plane $(1\,2\,3)$; therefore $4'$ is on $IV = (1\,2\,3) \cap (1'\,2'\,3)$. This proves the (8L)-theorem.

B. Let the (8L)-proposition hold. Let $1\,2\,3\,4$ and $1'\,2'\,3'\,4'$ be two tetrahedra in an incomplete M-relation, where the incidence $(4'\,1\,2\,3)$ is lacking. We shall prove that the M-relation is in fact complete. Consider the following two quadruples of mutually skew lines:

$$(1\,4'), \quad (4\,1'), \quad (3\,2'), \quad (2\,3') ;$$

$$(1\,4), \quad (1'\,4'), \quad (2\,3), \quad (2'\,3') .$$

Since the lines $(1\,4)$, $(1'\,4')$, $(2'\,3')$ intersect each of the lines $(1\,4')$, $(4\,1')$, $(3\,2')$, $(2\,3')$; and the line $(2\,3)$ intersects the lines $(4\,1')$, $(3\,2')$, $(2\,3')$, the above two sets are clearly in an incomplete (8L)-configuration. By the (8L)-proposition, the configuration is complete. This means that the line $(2\,3)$ intersects $(1\,4')$, or that the point $4'$ is on the plane $(1\,2\,3)$. This completes the proof.

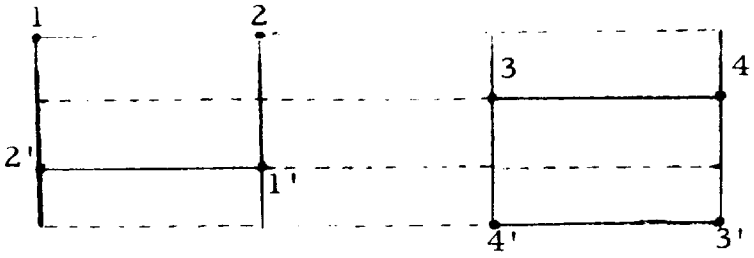
By analysis of the above proof, we are led to

Theorem II. 5.

The M-configuration is geometrically equivalent to the (8L)-configuration.

Proof:

We shall show the existence of a configuration which is a natural extension of both of the M and the (8L)-configurations. In the (8L)-configuration, consider all joins and meets of its eight elements. The result is a configuration, $(8L)^{e_1}$, consisting of 16 points, 8 lines and 16 planes, with 7 points on each plane and 7 planes on each point. By Definition II. 4, we have $(8L) \leq (8L)^{e_1}$. The eight lines (1 4), (1' 4'), (2 3), (2' 3'), (1 4'), (2' 3), (3' 2') are in one-to-one correspondence with the 8 lines of $(8L)^{e_1}$. The points of M together with the 8 points $(1\,2) \cap (3\,4')$, $(1\,2) \cap (4\,3')$, $(3\,4) \cap (2\,1')$, $(3\,4) \cap (1\,2')$, $(1'\,2') \cap (3\,4')$, $(1'\,2') \cap (4\,3')$, $(3'\,4') \cap (2\,1')$, and $(3'\,4')$ $(1\,2')$ are in one-to-one correspondence with the 16 points of $(8L)^{e_1}$.



Also, the 8 planes of the M-configuration together with the 8 planes, formed as above by replacing the operation $\cap$ by the operation $\cup$, i.e., join, are in one-to-one correspondence with the 16 planes of $(8L)^{e_1}$. Because the corresponding incidences are preserved under these correspondences, we have $M \leq (8L)^{e_1}$. Hence, $M \sim (8L)$. This proves the theorem.

In the proof of Theorem II. 5, we also proved

Theorem II. 6.

There exists a certain "number" of mutually inscribed tetrahedra whose vertices lie on a set of four mutually skew lines.

We note that if the space is defined over the field of the reals, this

"number" is ∞.

It is to be observed that the M-configuration may be put in three different ways as an (8L)-configuration as follows:

$$\begin{bmatrix} (1\,2'), & (2\,1'), & (3\,4'), & (4\,3') \\ (1\,2), & (1'\,2'), & (3\,4), & (3'\,4') \end{bmatrix},$$

$$\begin{bmatrix} (1\,3'), & (3\,1'), & (4\,2'), & (2\,4') \\ (1\,3), & (1'\,3'), & (2\,4), & (2'\,4') \end{bmatrix},$$

$$\begin{bmatrix} (1\,4'), & (4\,1'), & (2\,3'), & (3\,2') \\ (1\,4), & (1'\,4'), & (2\,3), & (2'\,3') \end{bmatrix}.$$

We close this section by the following algebraic argument which shows that the (8L)-proposition holds if and only if the commutative identity of multiplication holds, and which is direct and simple.

A. Let commutativity hold. Let I, II, III, IV and I', II', III', IV' be two quadruples of mutually skew lines in an incomplete (8L)-configuration. Let

$$A_1 = I \cap I', \quad A_2 = II \cap I', \quad A_3 = I \cap II', \quad A_4 = II \cap II';$$
$$B_1 = III \cap I', \quad B_2 = IV \cap I', \quad B_3 = III \cap II', \quad B_4 = IV \cap II',$$
$$B_5 = I \cap III', \quad B_6 = II \cap III', \quad B_7 = III \cap III', \quad B_8 = IV \cap III',$$
$$B_9 = I \cap IV', \quad B_{10} = II \cap IV', \quad B_{11} = III \cap IV', \quad B_{12} = IV \cap IV'.$$

We may write

$$B_1 = A_1 + k_1 A_2, \quad B_2 = A_1 + k_2 A_2, \quad B_3 = A_3 + {}_1 A_4, \quad B_4 = A_3 + {}_2 A_4,$$

$$B_5 = A_1 + m_1 A_3, \quad B_6 = A_2 + m_2 A_4, \quad B_9 = A_1 + n_1 A_3, \quad B_{10} = A_2 + n_2 A_4;$$

where k_i, ℓ_i, m_i, n_i are elements of some field R. For $p_1, p_2 \neq 0$ and $p_1, p_2 \in R$, we have $B_7 = B_5 + p_1 B_6 = B_1 + p_2 B_3$. By substitution this yields

$$A_1 + m_1 A_3 + p_1(A_2 + m_2 A_4) = A_1 + k_1 A_2 + p_2(A_3 + \ell_1 A_4).$$

Since A_1, A_2, A_3, A_4 are linearly independent, we get, upon equating coefficients of like points in the two sides of the above equation:

$$p_1 = k_1, \quad m_1 = p_2, \quad p_1 m_2 = p_2 \ell_1 .$$

and hence

$$k_1 m_2 = m_1 \ell_1 \quad \ldots \quad ①.$$

We, similarly, have from the equation $B_8 = B_5 + q_1 B_6 = B_2 + q_2 B_4$, for $q_1, q_2 \neq 0 \in R$, the equation

$$k_2 m_2 = m_1 \; 2 \quad \ldots \quad ②$$

Finally, from the equations $B_{11} = B_1 + r_1 B_3 = B_9 + r_2 B_{10}$, for $r_1, r_2 \neq 0 \in R$, we obtain

$$n_1 \ell_1 = k_1 n_2 \quad \ldots \quad ③$$

The equations ① and ② give $k_1 \ell_2 = \ell_1 k_2 \quad \ldots \quad ④$; and combining ④ with ③ we obtain

$$n_1 \ell_2 = n_2 k_2 \quad \ldots \quad ⑤.$$

Now, consider the two points $(B_2 + n_1 B_4)$ and $(-B_9 + k_2 B_{10})$, and the following linear combination of them:

$B_2 + n_1 B_4 - B_9 + k_2 B_{10} = A_1 + k_2 A_2 + n_1 (A_3 + \ell_2 A_4) - A_1 - n_1 A_3 + k_2 (A_2 + n_2 A_4) =$ $(n_1 \ell_2 - k_2 n_2) A_4$. This is equal to zero by ⑤. This proves the (8L)-theorem.

B. Let the (8L)-proposition hold. Let R be a division ring over which S_3 is defined. Let a and $b \neq 0, 1$ be any two elements of R. Consider a line in S_3 determined by the two points A_1 and A_2, and a second line, skew to the first, determined by the two points A_3 and A_4. We use these four points as reference system. We may find the following four points:

$$B_1 = a A_2 + b A_3 + A_4 ,$$
$$B_2 = -a A_1 + A_3 + A_4 ,$$
$$B_3 = -b A_1 - A_2 + A_4 ,$$
$$B_4 = -A_1 - A_2 - A_3 .$$

Now the quadruple of lines

$$(A_1 A_2),\ (A_3 A_4),\ (B_1 B_2),(B_3 B_4) \ldots\ (*)$$

consists of four mutually skew lines. To prove that we consider, for example, the pair $\{(A_1 A_2),(B_1 B_2)\}$ and the following vanishing linear combination of them:

$$\lambda_1 A_1 + \lambda_2 A_2 + \lambda_3 B_1 + \lambda_4 B_2 = 0, \qquad \lambda_i \in R$$

We shall have what we want if we prove $\lambda_i = 0$, $i = 1, 2, 3, 4$. By substitution, the left-hand side reduces to

$$\lambda_1 A_1 + \lambda_2 A_2 + \lambda_3(a A_2 + b A_3 + A_4) + \lambda_4(-a A_1 + A_3 + A_4) = 0 .$$

By left-multiplication, we find that $\lambda_1 - \lambda_4 a = 0$, $\lambda_2 + \lambda_3 a = 0$, $\lambda_3 b + \lambda_4 = 0$, $\lambda_3 + \lambda_4 = 0$. This implies that $\lambda_3(b - 1) = 0$. Since $b \neq 1$ by assumption, $\lambda_3 = \lambda_4 = 0$. Thus, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$. By a similar computation we prove that every pair is skew. Form the following quadruple of lines:

$$(A_1 B_2), (A_2 B_1), (A_4 B_3), (A_3 B_4) \ldots\ (**) .$$

This quadruple, again, may be shown by straight-forward computation using left multipliers only, that every pair in it is skew. We shall not prove this simple matter.

We claim that the two systems (*) and (**) form an incomplete (8L)-configuration. Clearly, the line $(A_1 A_2)$ intersects $(A_1 B_2)$ and $(A_2 B_1)$. In order to show that $(A_1 A_2)$ intersects $(A_4 B_3)$ and $(A_3 B_4)$, it is enough to show the left-dependence of the set of points A_1, A_2, A_4, B_3. In fact,

$$b A_1 + A_2 - A_4 + B_3 = b A_1 + A_2 - A_4 + (-b A_1 - A_2 + A_4) = 0 ,$$

proving the contention. In a similar way, one proves that each member of either system intersects each other, except that the intersection of $(A_4 B_3)$ and $(B_1 B_2)$ is perhaps missing. Thus we have constructed an

incomplete (8L)-configuration by using the two elements a and b. By the (8L)-proposition the configuration is complete. This means $\exists \mu_1, \mu_2, \mu_3 \in R$, not all $\mu_i = 0$, such that

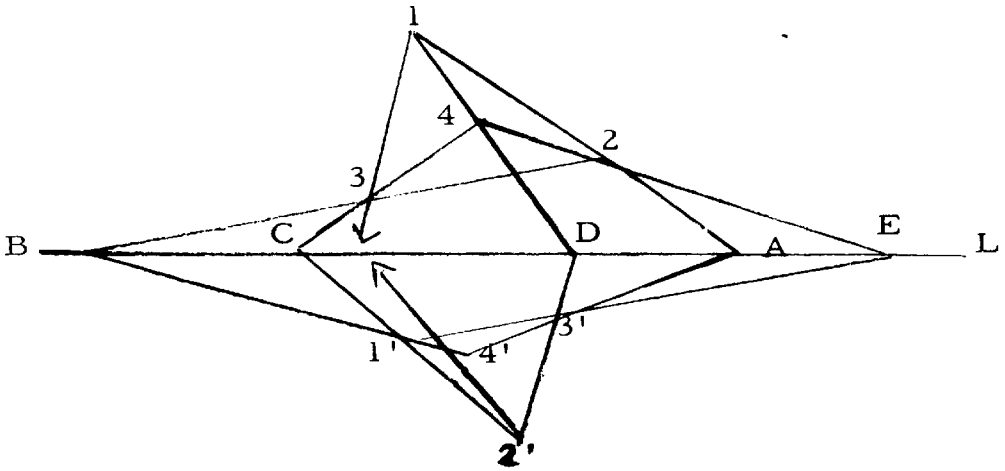
$$A_4 + \mu_1(a A_2 + b A_3 + A_4) + \mu_2(-a A_1 + A_3 + A_4) + \mu_3(-b A_1 - A_2 + A_4) = 0 .$$

Because the A_i, $i = 1, 2, 3, 4$, are independent, we have $-\mu_2 a = \mu_3 b$, $\mu_1 a = \mu_3$ and $\mu_1 b = -\mu_2$. From $\mu_2 = -\mu_1 b$ and $\mu_3 = \mu_1 a$, we obtain $\mu_1 b a = \mu_1 ab$, yielding $ba = ab$. Since this is true for every pair of "numbers," the division ring R is commutative.

4. On The Fundamental Theorem of Complete Quadrangles (F - Q).

It is known [16, p. 101] that if the Pappus proposition is assumed, it is possible for two complete quadrangles to be "oppositely" situated. This means that they correspond in such a way that the sides which meet at a point in one, are related with those which form a triangle in the other. However, the close relationship between this fact and the preceding two configurational propositions seem to have been overlooked. In what follows, we shall relate the present, (F - Q)-proposition, to the previous propositions, and discuss the relationships among their corresponding configurations.

Let 1 2 3 4 and 1' 2' 3' 4' be two complete quadrangles lying in two different planes. Let the two planes intersect in the line L. If the following five points: $(1\,2) \cap (3'\,4') = A$, $(2\,3) \cap (1'\,4') = B$, $(3\,4) \cap (1'\,2') = C$, $(1\,4) \cap (2'\,3') = D$, $(2\,4) \cap (1'\,3') = E$ are incident with L, so is $(1\,3) \cap (2'\,4')$.



Consider the two tetrahedra 1 2 3 4' and 1' 2' 3' 4. The following incidences are easily verified:

(1 2 3' 4'), (1' 2 3 4'), (1' 2' 3 4), (1 2' 3' 4), (1' 2 3' 4), (1 2 3 4), (1' 2' 3' 4').

Therefore, the two tetrahedra are in an incomplete M-relation. Applying the M-proposition, the relation is in fact complete, and hence the incidence (1 2' 3 4') follows. This means that (1 3) meets (2' 4') on L. The (F-Q)-proposition is proved.

Conversely, if T_1 : 1 2 3 4' and T_2 : 1' 2' 3' 4 are two tetrahedra in an incomplete M-relation, where the incidence (1' 2' 3 4) is missing, then by the (F-Q)-proposition the relation is complete. Indeed, the two complete quadrangles 1 2 3 4 and 1' 2' 3' 4' are easily seen to form an incomplete (F-Q)-configuration. That is to say, (1 2), (1 3), (1 4), (2 3), (2 4) intersect (3' 4'), (2' 4'), (2' 3'), (1' 4'), (1' 3') respectively. Therefore, (3 4) intersects (1' 2'). This implies the incidence (1' 2' 3 4), or that the M-configuration is complete.

Before we discuss the configurational relation between the M and the (F-Q) configurations, we give the following proof of the (F-Q)-theorem using the Grassmann algebra which is of interest in itself. This algebra assumes commutativity.

Let $A_1 A_2 A_3$ and $B_1 B_2 B_3$ be two triangles lying in two distinct or in the same plane. Let the lines $(A_1 A_2)$, $(A_2 A_3)$, $(A_3 A_1)$ meet a line L in C_1, C_2, C_3 respectively; and $(B_1 B_2)$, $(B_2 B_3)$, $(B_3 B_1)$ meet L in D_3, D_1, D_2 respectively. Let the lines $(B_1 C_2)$, $(B_2 C_3)$, $(B_3 C_1)$ meet at one point A_4. We shall prove that $(A_1 D_1)$, $(A_2 D_2)$, $(A_3 D_3)$ meet in one point, say B_4.

We may write $C_1 = A_1 - A_2$, $C_2 = A_2 - A_3$, $C_3 = A_3 - A_1$. Let $B_1 = A_4 + k_2 C_2$, $B_2 = A_4 + k_3 C_3$, $B_3 = A_4 + k_1 C_1$; k_i in some field R. Form the following points:

$$B_1 - B_2 = k_2 C_2 - k_3 C_3 ,$$
$$B_2 - B_3 = k_3 C_3 - k_1 C_1 ,$$
$$B_3 - B_1 = k_1 C_1 - k_2 C_2 .$$

Clearly, each of these points lie on L, since each is a linear combination of two points of L. Hence, $D_1 = B_2 - B_3$, $D_2 = B_3 - B_1$, $D_3 = B_1 - B_2$. Thus we have

$$D_1 = B_2 - B_3 = -(k_1 + k_3) A_1 + k_1 A_2 + k_3 A_3 ,$$
$$D_2 = B_3 - B_1 = k_1 A_1 - (k_1 + k_2) A_2 + k_2 A_3 ,$$
$$D_3 = B_1 - B_2 = k_3 A_1 + k_2 A_2 0 (k_2 + k_3) A_3 .$$

Upon multiplying the above points D_i, $i = 1, 2, 3$, by A_i, $i = 1, 2, 3$ respectively, we get

$$\langle A_1 D_1 \rangle = k_1 \langle A_1 A_2 \rangle + k_3 \langle A_1 A_3 \rangle ,$$
$$\langle A_2 D_2 \rangle = k_2 \langle A_2 A_1 \rangle + k_2 \langle A_2 A_3 \rangle ,$$
$$\langle A_3 D_3 \rangle = + k_3 \langle A_3 A_1 \rangle + k_2 \langle A_3 A_2 \rangle .$$

Because the Grassmann products are "anti-commutative" we have

$$\langle A_1 D_1 \rangle + \langle A_2 D_2 \rangle + \langle A_3 D_3 \rangle = 0.$$

That the (F-Q)-proposition directly implies the commutative law of multiplication may be shown as follows:

Let a, b be any two "numbers", $\neq 0, 1$ and such that $b \neq a + 1$, of a division ring R. Consider a complete quadrangle lying in some plane π. Let it be denoted by 1 2 3 4, and consider any point 4' not incident to it. Choose a coordinate system so that the following points have the corresponding coordinates:

1	2	3	4'	4
(1, 0, 0, 0) ,	(0, 1, 0, 0),	(0, 0, 1, 0),	(0, 0, 0, 1) ,	(1, 1, 1, 0) .

We find the following points:

$$1' \qquad 2' \qquad 3'$$

$$(0, a, b, -1)\,, \quad (-a, 0, 1, -1)\,, \quad (-b, -1, 0, -1)\,.$$

By left-multiplication only, one checks that (1 2) intersects (3'4'), (1 3) intersects (2'4'), (1 4) intersects (2'3'), (2 3) intersects (1'4'), (2 4) intersects (1'3') and (3 4) intersects (1'2'). We shall check one of them only, as follows:

$$(-a, 0, 1, -1) - (0, a, b, -1) + a(1, 1, 1, 0) - (a-b+1)(0, 0, 1, 0) = 0\,.$$

This proves that the line (3 4) meets the line (1'2'). The others may be verified similarly. Thus, the two elements a and b determine an incomplete (F-Q)-configuration. Assuming the (F-Q)-proposition, the configuration is complete. Hence, $\exists\ \lambda_1, \lambda_2, \lambda_3 \in R$, not all $\lambda_i = 0$, such that

$$(0, 0, 0, 1) + \lambda_1(0, a, b, -1) = \lambda_2(-a, 0, 1, 01) + \lambda_3(-b, -1, 0, -1)\,.$$

This yields $-\lambda_2 a = \lambda_3 b$, $\lambda_1 a = -\lambda_3$, $\lambda_1 b = \lambda_2$. Combining these equations we get $ab = ba$. Since this is true for any pair of non-trivially commutative elements, R is a field.

Summarizing our results, we have proved

Theorem II. 7.

Let S_3 be defined over a division ring R. The (F-Q)-proposition holds if and only if R is commutative.

The (F-Q)-configuration consists of 14 points, 13 lines and 2 planes; we denote it by (F-Q)(14, 13, 2). Consider the configuration $M^*(8, 12, 8)$ consisting of the elements of M together with the 12 edges of the involved two tetrahedra as complement. Clearly, we have $M \leq M^*$. By adding to the configuration (F-Q) the 6 planes formed by corresponding sides of the two complete quadrangles we get the configuration $(F\text{-}Q)^*(14, 13, 8)$. But the 6 collinear points in the configuration $(F\text{-}Q)^*$ are determined by the line elements of the configuration M^*. Moreover, the extra line L in the first may be obtained as an intersection of two plane elements of the second.

Thus $M^* \leq (F\text{-}Q)^*$. By Lemma II. 1. $M \leq (F\text{-}Q)^*$. Since $(F\text{-}Q) \leq (F\text{-}Q)^*$, by Definition II. 6. we have $M \sim (F\text{-}Q)$. Thus we have established Theorem II. 8.

The M-configuration is geometrically equivalent to the (F-Q)-configuration.

In the preceding pages we have been considering a class of configurational propositions, i. e., propositions of the form: "If A is an incomplete configuration, then A is in fact complete," and the configurations corresponding to them In all of them we found that the equivalence of two propositions was accompanied by the geometrical equivalence of their configurations. That if two propositions are equivalent, their corresponding configurations are not necessarily equivalent is shown by the following counter-example:

The M-proposition is equivalent to the Pappus proposition, but their corresponding configurations are clearly not equivalent. However, it remains problematical, whether it is possible to eliminate such counter-examples if the considerations are restricted to configurations consisting of the same kinds of elements.

So far we have been studying the algebraic identity ab = ba by means of certain configurational propositions. In the remaining part of this chapter we shall consider projective transformations. The relations of certain types of mappings to commutativity will be of considerable interest to us in Chapter IV.

5. On The Fundamental Theorem of Projective Geometry (F-P).

The so-called fundamental theorem of projective geometry may be stated as follows:

If a projectivity leaves each of three distinct points of a line invariant, it leaves every point of the line invariant. This proposition,

in fact, says that a projective transformation between two one-dimensional spaces is uniquely determined when three pairs of corresponding elements are given.

In [16, p. 148], this proposition is shown to be necessary and sufficient for the commutative law of multiplication to hold. It is a natural question to relate this one-dimensional proposition to the Möbius class of the three-dimensional ones. We shall do that in what follows.

A. Let I, II, III, IV and I', II', III', IV' be two quadruples of mutually skew lines forming an incomplete (8L)-configuration. Let I' intersect I, II, III, IV in 1, 2, 3, 4 respectively; II' in 1', 21, 3', 4'; and III' in 1'', 2'', 3'', 4'' respectively. Let IV' intersect I, II, III in 1''', 2''', 3''' respectively. Suppose that $(IV \cup III') \cap IV' = 4'''$. We shall prove, assuming the (F-P)-proposition, that 4''' is on IV. This will prove the (8L)-theorem.

Consider the following sequence of perspectivities:

$$1''' \; 2''' \; 3''' \overset{III'}{\wedge} 1 \; 2 \; 3 \overset{II'}{\wedge} 1''' \; 2''' \; 3''' \; \ldots \; (***) \; .$$

This means we have a projectivity, from IV' onto itself, which leaves the three distinct points 1''', 2''', 3''' invariant. By the (F-P)-proposition, it leaves every point of IV' invariant. Therefore it maps $4''' \rightarrow 4'''$, and consequently the point 4''' must be on the line IV.

B. Let the (8L)-proposition hold. We shall prove the (F-P)-theorem as follows:

Let Γ be a projectivity from any line IV' onto itself and having the three points 1''', 2''', 3''' fixed. Let I, II, III be three mutually skew lines through 1''', 2''', 3''' respectively. From any three distinct points of I, three lines may be drawn so that each of them intersects the two lines II and III. Clearly, these three lines, say I', II', III' form with IV' a quadruple of mutually skew lines. Let the points of intersections

of I', II', III' with each of the lines I, II, III be denoted by the same numbers given in the previous case. Thus the projectivity Γ is constructed by the sequence of perspectivities (***).

Let 4''' be any other point of the line IV'. Any line, through 4''', common to two lines of the triple I', II', III', intersects the third one, by the (8L)-proposition. Hence, upon following the mapping Γ defined by (***), we obtain the result that the point 4''' is invariant. Since this holds for every point of the line IV', Γ must be the identity mapping.

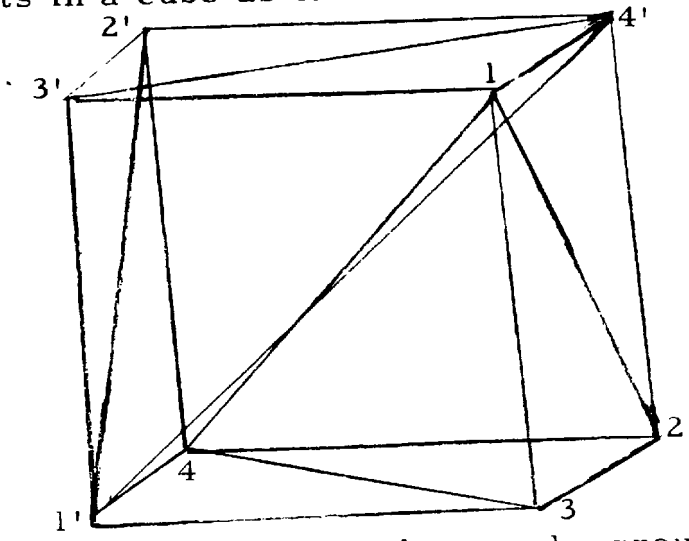
CHAPTER III

CONFIGURATIONAL CRITERIA FOR COMMUTATIVITY IN N DIMENSIONS

The M-class of propositions is generalized to higher spaces, and so is the P-class of propositions. Either generalization is proved to be a characterization of the commutativity of multiplication. The Pappus proposition is shown to be a special case of the M-class; and, also, the P_n-proposition is shown to specialize to the M-class. Thus the two classes are "unified".

1. A Generalization of the M-class.

It is known [13] that a pair of mutually inscribed and circumscribed tetrahedra exists in a cube as it is shown in the following figure.



In fact, the eight vertices of the cube may be grouped in four different ways as pairs of Möbius tetrahedra; one of the ways is shown in the above figure. The last statement is reflected in the fact that if 1 2 3 4 and 1' 2' 3' 4' be two tetrahedra in a complete M-relation, so are the following pairs:

$$[1\,2\,3'\,4', 1'\,2'\,3\,4],\ [1\,2'\,3\,4', 1'\,2\,3'\,4],\ [1\,2'\,3'\,4, 1'\,2\,3\,4'].$$

This fact may be justified as follows.

Consider the pair [1 2 3' 4', 1' 2' 3 4]. The point 1 is on the plane (2' 3 4) because, by hypothesis, the point 2' is on the plane (1 3 4). It may, similarly, be shown that each vertex of one tetrahedron is incident with the corresponding face of the other.

It was the above cube model for the Möbius configuration which suggested to the writer the following restatement of the M-proposition:

Let 1 2 3 4 be a plane complete quadrangle. Through each of the six sides of the quadrangle, let a plane, distinct from the plane of the quadrangle, be passed. The vertices of the quadrangle constitute four triangles; let the six planes be grouped correspondingly to determine four points. Then these four points are in one place.

An equivalent statement to this, is to be found in [1a; I, p. 62, IV, p. 58].

In what follows, we give a proof of the above statement.

Let the lines of the complete quadrangle be denoted by (1 2), (1 3), (1 4), (2 3), (2 4) and (3 4). Let the planes which pass through these lines be denoted by (1 2 4'), (1 3 2'), (1 4 3'), (2 3 1'), (2 4 1') and (3 4 2') respectively, where the numbers 1', 2', 3', 4' denote any set of four points. Consider the triangle 1 2 3 and let 4' be the point of intersection of the planes which pass through its sides. This means that each of the two sets (1 2' 3 4') and (1' 2 3 4') is coplanar. Next take the triangle 2 3 4 and let 1' be the common point to the three planes passing through its sides. This yields the dependence of the set (1' 2' 3 4). Similarly, by considering the remaining two triangles 1 3 4 and 1 2 4 and letting 2' and 3' be the common points of the planes passing through their sides respectively, we obtain the coplanar sets (1 2' 3' 4), (1 2 3' 4') and (1' 2 3' 4). Then, in accordance with the restatement of the M-theorem, the set (1' 2' 3' 4') is coplanar.

That the above statement involves a pair of mutually inscribed and circumscribed tetrahedra may be verified by taking, for example, the two tetrahedra: 1 2 3 4' and 1' 2' 3' 4. One sees that 1 is on (2' 3' 4') etc., and 1' is on (2 3 4') etc.

In what follows we give a proof of the above reformulation. Then we shall consider the generalization of the statement to higher spaces.

For the sake of computation, we designate the points 1, 2, 3, 4 by A_0, A_1, A_2, A_3 respectively; and 1', 2', 3', 4' by B_0, B_1, B_2, B_3 respectively. Since no three points of the A_i are collinear, and the B_i are not incident to the plane of the A_i, we may write:

$$B_i = P + \sum_{j=0}^{3} h_{ij} A_j, \quad i = 0, 1, 2, 3,$$

with $h_{jj} = 0$ and P is any auxiliary point not in the plane of the A_j. Because the A_j are in one plane, we may choose the notation so that $\sum_{j=0}^{3} A_j = 0$. The above mentioned incidences may be regarded as vanishing Grassmann products. Hence, we may compute one of the corresponding products as follows:

$$\begin{aligned} \langle A_0 A_1 B_2 B_3 \rangle &= h_{32} \langle A_0 A_1 A_2 P \rangle + h_{23} \langle A_0 A_1 A_2 P \rangle \\ &= (h_{32} + h_{23}) \langle A_0 A_1 A_2 P \rangle \\ &= 0 . \end{aligned}$$

Because the set A_0, A_1, A_2, P is independent, we obtain $h_{32} + h_{23} = 0$. Since the seven incidences in the hypothesis are similar, we have from their corresponding products the general relations among the coefficients:

$$(*) \qquad h_{ij} + h_{ji} = 0 \quad ; \; i, j = 0, 1, 2, 3 .$$

We consider the relations

$$B_i = \sum_{j=0}^{3} h_{ij} A_j + P, \quad i = 0, 1, 2, 3;$$

$$(**) \qquad 0 = \sum_{j=0}^{3} A_j + 0 \cdot P .$$

We form now the following determinant from the coefficients of the relations (**):

$$\begin{vmatrix} 0 & h_{01} & h_{02} & h_{03} & 1 \\ h_{10} & 0 & h_{12} & h_{13} & 1 \\ h_{20} & h_{21} & 0 & h_{23} & 1 \\ h_{30} & h_{31} & h_{32} & 0 & 1 \\ -1 & -1 & -1 & -1 & 0 \end{vmatrix} \qquad (1).$$

The determinant (1) is equal to the following determinant (2) because (2) is obtainable from (1) by subtracting the fourth column from the first, the second and the third successively:

$$\begin{vmatrix} -h_{03} & h_{01}-h_{03} & h_{02}-h_{03} & 1 \\ h_{10}-h_{13} & -h_{13} & h_{12}-h_{13} & 1 \\ h_{20}-h_{23} & h_{21}-h_{23} & -h_{23} & 1 \\ h_{30} & h_{31} & h_{32} & 1 \end{vmatrix} \qquad (2).$$

The determinant (2) expresses the relations of the B_i in terms of the independent points A_0, A_1, A_2 and P. This is obtained by using the second relation in (**) to substitute for A_3 in terms of A_0, A_1, A_2 in the first relation. Now the determinant (1) vanishes because it is skew-symmetric and of order three. Hence, the determinant (2) vanishes. This proves the dependence of the set (1', 2', 3', 4'). The theorem is proved.

The above reformulation of the M-proposition may be summarized in the following condensed form:

Let 0 1 2 3 and 0' 1' 2' 3' be two quadruples of points. If

(0 1 2 3), (0 1 2' 3'), (0 1' 2 3'), (0' 1 2 3'), (0' 1 2' 3), (0' 1' 2 3)

are dependent, so is the set $(0'\,1'\,2'\,3')$. One sees that this formulation is identical with the formulation of the (F - Q)-proposition which was obtained in II. 4. We shall come back to this point in the next section.

The formulation of the M-proposition given above suggests some possible generalizations. One generalization will be established in the following theorem; another one will be obtained in III. 4.

Theorem III. 1.

Let S_n, with $n>1$ any odd integer, be a projective space defined over a division ring R. Let $0, 1, \ldots, n$ be a dependent set of $(n+1)$ points of which no n are dependent. Let $0', 1', \ldots, n'$ be any other set of points such that all of the following $\frac{n(n+1)}{2}$ hold:

$(0'\,1'\,2\,3 \ldots n)$, $(0'\,1\,2'\,3\,4 \ldots n)$, $(0'\,1\,2\,3'\,4 \ldots n)$,

$(0\,1\,2 \ldots (n-2)\,(n-1)'\,n')$.

Then the set $(0'\,1' \ldots n')$ is dependent if and only if R is commutative.

Proof:

A. Sufficiency.

Let R be a field. We denote the unprimed numbers by $A_0, A_1, A_2, \ldots, A_n$; and the primed numbers by $B_0, B_1, \ldots, B_n$. Let P be any auxiliary point not incident with the hyperplane of the A_i. Since no n of the A_i are dependent, we may write:

$$B_i = \sum_{j=0}^{n} h_{ij} A_j + P \; ; \quad i = 0, 1, \ldots n,$$

$$(***) \qquad 0 = \sum_{j=0}^{n} A_j + 0.P \; ,$$

with $h_{jj} = 0$, $h_{ij} \in R$.

We translate the incidences given in the hypothesis into vanishing Grassmann products. Since the incidences are similar, we shall compute one of the corresponding products as follows:

$$\langle B_0 B_1 \rangle = h_{01} \langle P A_0 \rangle + h_{10} \langle A_1 P \rangle (\text{modulo } \langle A_2 A_3 \ldots A_n \rangle)$$

Hence, we have

$$\langle B_0 B_1 A_2 \ldots A_n \rangle = h_{01} \langle P A_0 A_2 \ldots A_n \rangle + h_{10} \langle A_1 P A_2 \ldots A_n \rangle = (h_{01} + h_{10}) \langle A_0 A_1 \ldots A_{n-1} P \rangle$$

Since the set $A_0, A_1, \ldots, A_{n-1}, P$ is independent by hypothesis we obtain $h_{01} + h_{10} = 0$. Hence, on account of the similarity of the products, we have the following general relation among the coefficients h_{ij}:

$$(*)^1 \qquad h_{ij} + h_{ji} = 0 \; ; \; i, j = 0, 1, \ldots, n.$$

Thus, we have the relations (***) with the conditions $(*)^1$ on the coefficients h_{ij}. By an argument similar to that given in the treatment of the three-dimensional case, one may show that the following vanishing determinant is necessary and sufficient for the B_i to be dependent.

$$\begin{vmatrix} 0 & h_{01} & h_{02} & \ldots & h_{0n} & 1 \\ h_{10} & 0 & h_{12} & \ldots & h_{1n} & 1 \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ h_{n0} & h_{n1} & h_{n2} & \ldots & 0 & 1 \\ -1 & -1 & -1 & \ldots & -1 & 0 \end{vmatrix} \qquad (D) .$$

The above determinant vanishes because it is skew-symmetric and of odd order. This implies that the set $(0' 1' \ldots n')$ is in one hyperplane.

B. Necessity:

We shall denote, for reasons which will appear in the next section, the configuration satisfying the hypothesis of theorem III. 1 an incomplete

$(F-Q)_n$-configuration. Hereafter, we adopt the convention that the integer which is attached to the symbol of a configuration or proposition indicates the number of dimensions of the space in which the configuration or the proposition is considered.

Now let $A_0 A_1 \ldots A_n$ be a complete hyperplanar $(n+1)$-point a. Let $A_0 = (1, 0, \ldots, 0)$, $A_1 = (0, 1, \ldots, 0)$, $\ldots$, $A_{n-1} = (0, 0, \ldots, 1, 0)$. Let $A_n = \sum_{i=0}^{n-1} A_i$. Let P be the point $(0, 0, \ldots, 1)$. Given any two elements, a and b of R such that $a, b \neq 0, 1$ and $(a - b + 1) \neq 0$, we find the following points:

$$\begin{aligned} B_0 &= -aA_1 - bA_2 + P \\ B_1 &= aA_0 - A_2 + P \\ B_2 &= bA_0 + A_1 + P \\ B_3 &= P \ . \end{aligned}$$

Consider any set of points $B_4, B_5, \ldots, B_n$ so that $B_0, B_1, \ldots, B_n$ and $A_0, A_1, \ldots, A_n$ form an incomplete $(F-Q)_n$-configuration where the incidence $(B_0 B_1 \ldots B_n)$ is hypothetically missing. Since the four points B_0, B_1, B_2, B_3 are already determined, the above consideration is possible if we prove, without using commutativity, that the following incidences hold:

$$(B_0 B_1 A_2 A_3 \ldots A_n),\ (B_0 A_1 B_2 A_3 \ldots A_n),\ (B_0 A_1 A_2 B_3 A_4 \ldots A_n),$$
$$(A_0 B_1 B_2 A_3 \ldots A_n),\ (A_0 B_1 A_2 B_3 A_4 \ldots A_n),\ (A_0 A_1 B_2 B_3 A_4 \ldots A_n).$$

It turns out that this is so and one can verify that by simple calculation. For example, $B_1 - B_0 + a(A_n - A_{n-1} - A_{n-2} \ldots - A_3) - (a - b + 1) A_2 = (a, 0, -1, 0, 0, \ldots, 0, 1) - (0, -a, -b, 0, 0, \ldots, 0, 1) - a(1, 1, 1, 0, \ldots, 0) - (a - b + 1)(0, 0, 1, 0, \ldots, 0) = 0$, proving the incidence $(B_0 B_1 A_2 \ldots A_n)$. One may check the other incidences similarly. Thus,

we have constructed an incomplete $(F - Q)_n$-configuration using the two given "numbers" a and b. By the $(F - Q)_n$- proposition, the configuration is complete, and consequently the set $(B_0 B_1 \ldots B_n)$ lies in one hyperplane β. The hyperplane β intersects the three space spanned by A_0, A_1, A_2, P in a plane π. The four points B_0, B_1, B_2, B_3 are in this 3-space and β. Hence, they are in π. This means $\exists \lambda_0, \lambda_1, \lambda_2 \neq 0 \in R \ni \lambda_0 B_0 + \lambda_1 B_1 + \lambda_2 B_2 + B_3 = 0$. This yields $\lambda_1 a + \lambda_2 b = 0$, $-\lambda_0 a + \lambda_2 = 0$, $-\lambda_0 b - \lambda_1 = 0$. Hence, $\lambda_2 = \lambda_0 a$, $\lambda_1 = \lambda_0 b$ and therefore $-\lambda_0 ba = -\lambda_0 ab$, yielding $ba = ab$. Because this is true for every pair of non-trivially commutative elements in R, Theorem III. 1 is completely proved.

It is to be noted that the $(F - Q)_n$- configuration consists of $2(n+1)$ points and $\frac{1}{2}(n^2 + n + 4)$ hyperplanes. It is non-self-dual except for the 3-dimensional case. Indeed, if $2(n+1) = \frac{1}{2}(n^2 + n + 4)$, then $n^2 - 3n = 0$, giving the case $n = 3$ and the trivial case $n = 0$.

2. Oppositely Situated Complete Hyperplanar (n+1)-points.

In this section we give a geometrical interpretation of theorem III. 1, by which it will appear a clear dimensional generalization of the $(F - Q)_3$-theorem.

By a complete planar 4-point we mean a set of 4 points in a plane, of which no three are collinear together with the lines joining any two of them. We shall generalize this notion in:

Definition III. 1.

Let $n>1$ be any integer. A complete hyperplanar (n+1)-point, in S_n, is a configuration of (n+1) points, in a hyperplane, of which no n are dependent with all subspaces which may be formed as joins of the points. In S_n, two hyperplanes meet in an S_{n-2}, and a line intersects a subspace in a point provided that the line is not a subset of the subspace.

The incidences in theorem III. 1, which are $\frac{n(n+1)}{2}$ in number, may be interpreted as follows. In each incidence, the line $(B_i B_j)$ intersects

the (n-2)-face $(A_k A_\ell \ldots A_2)$; i, j, k, ℓ, . . , w are distinct and they are elements of the set 0, 1, . . . , n; in a point common to the two hyperplanes $(A_0 A_1 \ldots A_n)$ and $(B_0 B_1 \ldots B_n)$. All these points of intersections lie in one (n-2)-space, which is the meet of the two hyperplanes. This discussion leads to

Theorem III. 2.

Let n>1 be any odd integer. Let S_n be defined over a division ring R. Let (0 1 2 . . . n) and (0' 1' . . . n') be two complete hyperplanar (n+1)-points which are so related-0 to 0', 1 to 1' etc - that all, but one of the $\frac{n(n+1)}{2}$ lines (i' j') meet the correspondingly opposite (n-2)-faces of the other. Then the remaining line of the first meets its corresponding face of the second if and only if R is a field.

Proof:

1. Assume the two complete hyperplanar (n+1)-points to be distinct.

Now the given intersections between the $\frac{n(n+1)}{2}$ lines of one and their corresponding faces of the other yield all the incidences in theorem III. 1, except one which is hypothetically missing. The set (0 1 2 . . . n) is dependent and no proper subset of it is dependent. Hence, the set (0' 1' . . . n') may be represented by (***) and by similar computation to that of theorem III. 1 the vanishing determinant of the set (0' 1' 2' . . . n') takes a form similar to D except for two elements, say h_{01} and h_{10}, which are symmetric with respect to the diagonal. Let $x = h_{01}$ and $y = h_{10}$, and denote this "incomplete" skew-symmetric determinant by D'. We shall show that the determinant D' is in fact complete, i. e., we shall prove that x = -y. Consider a determinant identical with D and in which $h_{01} = -h_{10} = x$. Without confusion we may denote this determinant by D. Since D is skew-symmetric and of odd order, it vanishes. Hence, D'' = D - D' = 0. Because all the columns of D, except the first, are identical

with their corresponding ones in D', the determinant D'' assumes the following form:

$$\begin{vmatrix} 0 & h_{01} & h_{02} & \cdots & h_{0n} & 1 \\ h_{01}+h_{10} & 0 & h_{12} & \cdots & h_{1n} & 1 \\ 0 & h_{21} & \cdot & \cdots & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & h_{n1} & h_{n2} & \cdots & 0 & 1 \\ 0 & -1 & -1 & \cdots & -1 & 0 \end{vmatrix} \qquad (D'') \ .$$

The determinant D'' may be expanded by the minors of the first column, and therefore we have

$$D'' = (h_{01} + h_{10})D''' \ ,$$

where the determinant D''' has the following form

$$\begin{vmatrix} h_{01} & h_{02} & \cdots & h_{0n} & 1 \\ h_{21} & 0 & \cdots & h_{2n} & 1 \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ h_{n1} & h_{n2} & \cdots & \cdot & \cdot \\ -1 & -1 & \cdots & -1 & 0 \end{vmatrix} \qquad (D''') \ .$$

By assumption, the set (0' 1' . . . n') has no proper subset which is dependent, therefore $D''' \neq 0$. In other words, if D''' were zero, this would mean the set $(B_0\ B_2\ B_3 \ldots B_n)$ is dependent because the condition $D''' = 0$ is necessary and sufficient for the projections of $B_0, B_2, \ldots, B_n$,

from A_0, to be dependent. Hence this would imply the set (B_0 B_2 ... B_n) itself is dependent. Thus $D''' \neq 0$, and consequently $h_{01} + h_{10} = 0$.

2. Let the two complete hyperplanar (n+1)-points be in the same hyperplane α. Let β be the (n-2)-subspace in which the common points lie. Let P be any point not on α. By projecting one of the two complete hyperplanar (n+1)-points from P into any hyperplane $\gamma \neq \alpha$ with $\gamma \supset \beta$, we have all the incidences given in the hypothesis preserved. Hence, this case is reduced to the first case and the missing incidence follows. Upon reversing the projection the second case follows. Thus in both cases, Theorem III. 2 is equivalent to Theorem III. 1. This completes the proof of Theorem III. 2.

The case when the two complete hyperplanar (n+1)-points lie in the same hyperplane may be considered as the complementary part of theorem III. 1 in "even"-dimensional spaces. This generalizes the two cases of the (F - Q)-proposition in the plane and 3-space.

It should be observed that the theorem III. 2 proves the existence of two oppositely situated complete hyperplanar (n+1)-points in higher spaces; and it, incidentally, generalizes the two-dimensional proposition of Veblen and Young [16, p. 101] to n dimensions.

We finally give the following synthetic proof which relates the $(F - Q)_n$-proposition, n is odd, to the $(F - Q)_3$-proposition directly.

Let $(A_0, A_1, \ldots, A_n)$ and $(B_0, B_1, \ldots, B_n)$ be two sets of (n+1)-points, no n of either set are dependent, forming an incomplete $(F - Q)_n$-configuration. In other words, we have all of the following incidences satisfied:

$(A_0A_1 \ldots A_n)$, $(B_0B_1A_2 \ldots A_n)$, $(B_0A_1B_2A_3 \ldots A_n)$, ... , ... , $(A_0A_1 \ldots A_{n-2}B_{n-1}B_n)$; while the incidence $(B_0B_1 \ldots B_n)$ is hypothetically missing. We shall prove the satisfaction of this incidence by using the $(F - Q)_3$- proposition.

Projecting the A_i from the line $(A_{n-1}A_n)$, by planes passing through it, into the next lower odd-dimensional space, S_{n-2}, which is spanned by the set $(B_0, B_1, \ldots, B_{n-2})$, we obtain an incomplete $(F-Q)_{n-2}$-configuration in S_{n-2}. This may be seen from the fact that the projecting planes, which passes through the line $(A_{n-1}A_n)$, lie in one hyperplane. Hence, the image set $(A'_0, A'_1, \ldots, A'_{n-2})$ is contained in the space common to S_{n-2} and S_{n-1}. That is to say, the image set lies in one hyperplane of S_{n-2}. Thus, the projection yields the following $[\frac{1}{2}(n-1)(n-2)+1]$ incidences, in S_{n-2}, by repeating the same argument:

$$(A'_0A'_1 \ldots A'_{n-2}),\ (B_0B_1A'_2 \ldots A'_{n-2}),\ \ldots,\ \ldots,$$

$$(A'_0A'_1 \ldots A'_{n-4}B_{n-3}B_{n-2}).$$

Now, projecting again from the line $(B_{n-1}B_n)$ into an arbitrary S_{n-2}, we have the above incidences preserved, after priming all the B_i and double-priming the A_i, and the set $(B'_0, B'_1, \ldots, B'_{n-2})$. These incidences clearly define an incomplete $(F-Q)_{n-2}$-configuration in S_{n-2}, where the dependence of the set $(B'_0, B'_1, \ldots, B'_{n-2})$ would imply the dependence of the original set $(B_0, B_1, \ldots, B_n)$. This means that the $(F-Q)_n$-proposition holds if the $(F-Q)_{n-2}$-proposition holds. Since n is odd, by hypothesis, successive projections of the above type will eventually lead to the $(F-Q)$-proposition. Thus, the $(F-Q)_n$-proposition may be reduced to the $(F-Q)_3$-proposition, and assuming the latter hold, the $(F-Q)_n$-proposition must hold. To see the projection clearly, we shall consider the passage from S_5 to S_3 as an example.

In S_5, we have the following 16 incidences:

$$(A_0A_1A_2A_3A_4A_5),\ (B_0B_1A_2A_3A_4A_5),\ (B_0A_1B_2A_3A_4A_5),$$
$$(B_0A_1A_2B_3A_4A_5),\ (B_0A_1A_2A_3B_4A_5),\ (B_0A_1A_2A_3A_4B_5),$$
$$(A_0B_1B_2A_3A_4A_5),\ (A_0B_1A_2B_3A_4A_5),\ (A_0B_1A_2A_3B_4A_5),$$
$$(A_0B_1A_2A_3A_4A_5),\ (A_0A_1B_2B_3A_4A_5),\ (A_0A_1B_2A_3B_4A_5),$$
$$(A_0A_1B_2A_3A_4B_5),\ (A_0A_1A_2B_3B_4A_5),\ (A_0A_1A_2B_3A_4B_5),$$

$(A_0A_1A_2A_3B_4B_5)$; while the incidence $(B_0B_1B_2B_3B_4B_5)$ is hypothetically missing. We shall prove that the $(F-Q)_5$-configuration is in fact complete by using the $(F-Q)_3$-proposition. Projecting from the line (A_4A_5), into the 3-space spanned by (B_0, B_1, B_2, B_3), we obtain, upon considering the incidences involving the line (A_4A_5), the following incidences in S_3:

$(A'_0A'_1A'_2A'_3)$, $(B_0B_1A'_2A'_3)$, $(B_0A'_1B_2A'_3)$, $(B_0A'_1A'_2B_3)$, $(A'_0B_1A'_2B_3)$, $(A'_0B_1B_2A'_3)$, $(A'_0A'_1B_2B_3)$, defining an incomplete $(F-Q)_3$-configuration. The second projection, into an arbitrary 3-space perserves these incidences and yields the set (B'_0, B'_1, B'_2, B'_3) where the B_i, $i = 0, 1, 2, 3$, are primed under the second projection. By the $(F-Q)_3$-proposition, the set (B'_0, B'_1, B'_2, B'_3) is dependent, and consequently their plane determines, with the line (B_4B_5), a four-space in which the set $(B_0, B_1, B_2, B_3, B_4, B_5)$ clearly lies. This proves that the $(F-Q)_3$-proposition implies the $(F-Q)_n$-proposition in any odd number of dimensions different from one.

3. The (F - Q)-proposition Specializes to the P-proposition.

Theorem III. 2 is interesting in the case when the two hyperplanes, of the two complete (n+1)-points, coincide. This case reveals an aspect of the (F - Q)-proposition by which the $(F-Q)_n$-proposition becomes a generalization of the P-proposition, i.e., Pappus proposition, also. Thus, theorem III. 2 may be considered an extension of the M-class as well as the P-class of propositions.

The $(F-Q)_2$-proposition may be stated as follows: Let 1 2 3 4 and 1' 2' 3' 4' be any two quadruples of points. Let L be any line in the (commutative) plane. If the following five points are incident to L

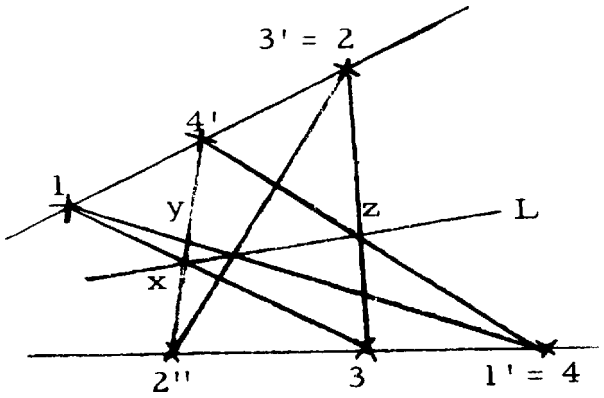
$(1\ 2)\cap(3'\ 4')$, $(1\ 3)\cap(2'\ 4')$, $(1\ 4)\cap(2'\ 3')$, $(3\ 4)\cap(1'\ 2')$, $(2\ 4)\cap(1'\ 3')$; so is the point $(2\ 3)\cap(1'\ 4')$.

If the two quadruples of points form two complete quadrangles then the above statement is the case n = 2 of Theorem III. 2. On the other hand,

if one, or both, of the complete quadrangles are degenerate, the statement becomes self-evident.

Now, consider the following special case:

Let the point 1 be incident with the line (4' 3') and the point 3 be incident with the line (1' 2'). Upon identifying the point 4 with the point 1' and 2 with 3', the above $(F - Q)_2$-proposition reduces to the following form:



Let (1 3'4') and (1' 2' 3) be two triples of collinear points. If $(1\ 3) \cap (2'4')$ = x and $(1\ 1') \cap (2'\ 3')$ = y determine the line L, then the point $(3\ 3') \cap (1'\ 4')$ = z is on L. This last statement is clearly the proposition of Pappus. Hence, the $(F - Q)_n$-proposition generalizes both the M- and P-classes of theorems.

4. A Generalization of the P- and the M-Classes.

In sections 1 and 2, we generalized the M-class of theorems to higher dimensional spaces, and we obtained a first configurational criterion for commutativity of multiplication in all odd-dimensional spaces. To deal with the other "half" of the problem, we went back to Pappus theorem in the plane. A generalization of the Pappus proposition is obtained in all dimensions. This generalization involves a statement in terms of the dual elements of the space, and it turns out to be a second configurational characterization of the commutativity of multiplication.

The theorem of Pappus was reformulated in two different forms in Chapter I. Another formulation may be given as follows: Let S_2 be a projective plane over a field R. Let (0 1 2) be a set of collinear points of which no two are dependent. Let (0' 1' 2') and (0'' 1'' 2'') be any two triples of points satisfying the following incidences:

(0 1' 2''), (0 1'' 2'), (0' 1'' 2), (0'' 1' 2), (0' 1 2''), (0'' 1 2') .

Then, if one of the two sets is dependent so is the other. We shall give a new proof of this statement; after that we shall consider the n-dimensional case.

Let P be any auxiliary point not incident to the line of the triple (0 1 2). Let 0, 1, 2 be denoted by A_0, A_1, A_2; 0', 1', 2' by B_0, B_1, B_2; 0'', 1'', 2'' by C_0, C_1, C_2 respectively. We may write

$$B_i = \sum_{j=0}^{2} k_{ij} A_j + P \quad , \quad k_{jj} = 0,$$

$$C_i = \sum_{j=0}^{2} \ell_{ij} A_j + P \quad \ell_{jj} = 0 \ ;$$

with k_{ij}, $\ell_{ij} \in R$ and i = 0, 1, 2. We choose the notation so that

$$0 = \sum_{j=0}^{2} A_j + 0 \cdot P \ .$$

We translate the above given incidences into Grassmann products, and since they are similar we shall compute one of them only as follows

$$\begin{aligned} \langle A_0 B_1 C_2 \rangle &= \langle A_0 \langle k_{10} A_0 + k_{12} A_2 + P \rangle \langle \ell_{20} A_0 + \ell_{21} A_1 + P \rangle \rangle \\ &= \langle k_{12} \langle A_0 A_3 \rangle + \langle A_0 P \rangle \langle \ell_{20} A_0 + \ell_{21} A_1 + P \rangle \rangle \ , \\ &= \ell_{21} \langle A_0 P A_1 \rangle + k_{12} \langle A_0 A_2 P \rangle \ , \\ &= (-\ell_{21} - k_{12}) \langle A_0 A_1 P \rangle \ . \end{aligned}$$

Since the set (A_0, A_1, P) is independent by assumption, $\langle A_0 A_1 P \rangle \neq 0$. As

the set $(0\,1'\,2'')$ is dependent, $\langle A_0 B_1 C_2 \rangle = 0$. This implies

$$\ell_{21} + k_{12} = 0.$$

By a similar way one finds the general formula

$$\ell_{ij} + k_{ji} = 0;\ i, j = 0, 1, 2.$$

We have

$$\begin{vmatrix} 0 & \ell_{01} & \ell_{02} & 1 \\ \ell_{10} & 0 & \ell_{12} & 1 \\ \ell_{20} & \ell_{21} & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & -k_{10} & -k_{20} & 1 \\ -k_{01} & 0 & -k_{21} & 1 \\ -k_{02} & -k_{12} & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}.$$

The last determinant is the negative transpose of the following determinant:

$$\begin{vmatrix} 0 & k_{01} & k_{02} & 1 \\ k_{10} & 0 & k_{12} & 1 \\ k_{20} & k_{21} & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}$$

which vanishes at the same time as the determinant

$$\begin{vmatrix} -k_{02} & k_{01} - k_{02} & 1 \\ k_{10} - k_{12} & -k_{12} & 1 \\ k_{20} & k_{21} & 1 \end{vmatrix},$$

because the latter is obtainable from the former by substracting the third column from the first and the second successively. The last determinant vanishes because it expresses the dependent set $(B_0 B_1 B_2)$ in terms of A_0, A_1, P. Hence, the set $(C_0 C_1 C_2)$ is dependent, or the set $(0''\,1''\,2'')$ is collinear. The above reformulation of the Pappus theorem leads to the following theorem in n dimensions.

Theorem III. 3.

Let $n > 1$ be any integer. Let S_n be defined over a division ring R. Let $0, 1, \ldots, n$ be a complete hyperplanar (n+1)-point. Let $0', 1', \ldots, n'$ and $0'', 1'', \ldots, n''$ be any two sets of (n+1) points satisfying the following incidences:

$(0'\ 1''\ 2\ 3 \ldots n)$, $(0''\ 1'\ 2\ 3 \ldots n)$, $(0\ 1'\ 2''\ 3 \ldots n)$, $(0\ 1''\ 2'\ 3 \ldots n)$, $\ldots$, $(0\ 1\ 2 \ldots (n-2)(n-1)'n'')$, $(0\ 1\ 2 \ldots (n-2)(n-1)''n')$.

Then, if one set is dependent so is the other and the dependence is of the same rank if and only if R is a field.

Proof:

A. Let R be a field. We denote $0, 1, \ldots, n$ by $A_0, A_1, \ldots, A_n$; $0', 1', \ldots, n'$ by $B_0, B_1, \ldots, B_n$; $0'', 1'', \ldots, n''$ by $C_0, C_1, \ldots, C_n$ respectively. Let P be any auxiliary point not incident with the hyperplane $(0\ 1\ 2 \ldots n)$. As in the two-dimensional case, we use a redundant coordinate system and write:

$$B_i = \sum_{j=0}^{n} k_{ij} A_j + P \quad , \quad k_{jj} = 0 \ ;$$

$$C_i = \sum_{j=0}^{n} \ell_{ij} A_j + P \quad , \quad \ell_{jj} = 0 \ ;$$

$$0 = \sum_{j=0}^{n} A_j + 0 \cdot P \ ; \quad k_{ij}, \ \ell_{ij} \in R, \quad i = 0, 1, \ldots, n.$$

The incidences given in the hypothesis are alike, and therefore we shall compute only one of them as follows

$$[0'\ 1''\ 3 \ldots n] = \langle B_0 C_1 A_2 A_3 \cdots A_n \rangle =$$

$$\langle\langle \sum_{j=0}^{n} k_{oj} A_j + P \rangle \langle \sum_{j=0}^{n} {}_{ij} A_j + P \rangle \ A_2 A_3 \cdots A_n \rangle =$$

$$(k_{01} + \ell_{10}) \langle P\, A_0 A_1 \cdots A_{n-1} \rangle .$$

Since $(P, A_0, A_1, \ldots, A_{n-1})$ are independent, by assumption. $\langle P A_0 A_{n-1} \rangle \neq 0$. But, $\langle B_0 C_1 A_2 A_3 \cdots A_n \rangle = 0$, therefore $k_{01} + \ell_{10} = 0$.

Hence, by similar calculation, we find the general formula

$$(*)'' \quad \ldots \quad \ell_{ij} = -k_{ji}; \; i,j = 0,1, \ldots, n.$$

Assuming the set $[0'\ 1'\ 2 \ldots n']$ to be in one hyperplane, the Grassmann product $\langle B_0 B_1 \ldots B_n \rangle = 0$, or that

$$\begin{vmatrix} 0 & k_{01} & k_{02} & \ldots & k_{0n} & 1 \\ k_{10} & 0 & . & \ldots & k_{1n} & 1 \\ . & . & . & & . & . \\ . & . & . & & . & . \\ . & . & . & & . & . \\ k_{n0} & k_{n1} & . & & 0 & 1 \\ 1 & 1 & . & & 1 & 0 \end{vmatrix} = 0.$$

The last determinant is equal to zero, because by subtracting the nth column from the first, the second, the third, ... , the (n-1)st successively, we obtain the following vanishing determinant which is necessary and sufficient for the B_i to be dependent:

$$\begin{vmatrix} -k_{0n} & k_{01}-k_{0n} & \ldots & k_{0n-1}-k_{0n} & 1 \\ k_{10}-k_{1n} & -k_{1n} & \ldots & k_{1n-1}-k_{1n} & 1 \\ . & & & . & . \\ . & & & . & . \\ . & & & . & . \\ k_{n0} & & & k_{nn-1} & 1 \end{vmatrix}$$

By the formula $(*)''$, the following determinant vanishes

$$\begin{vmatrix} 0 & \ell_{01} & \ldots & \ell_{0n} & 1 \\ \ell_{10} & 0 & \ldots & \ell_{1n} & 1 \\ . & . & & . & . \\ . & . & & . & . \\ . & & & . & . \\ \ell_{n0} & & & 0 & 1 \\ 1 & 1 & & 1 & 0 \end{vmatrix}.$$

This means the dependence of the set $(C_0 C_1 \ldots C_n)$ and hence the set $(0'' 1'' 2'', \ldots n'')$. Since the determinant $|\ell_{ij}|$ is the negative transpose of the $|k_{ij}|$ by the relations (*)'', the two dependences have the same rank. The sufficiency part is proved. Hereafter, we denote the configurational proposition which we just proved in part A, the "P_n-proposition".

B. Let R be a division ring. Let $a, b \neq 0, 1 \in R$. Consider a complete hyperplanar (n+1)-point: $A_0 A_1 \ldots A_n$. Let P be any point in S_n so that $A_0, A_1, \ldots, A_{n-1}, P$ be a simplex of reference where $A_0 = (1, 0, 0, \ldots, 0)$, $A_1 = (0, 1, 0, \ldots, 0), \ldots, P = (0, 0, \ldots, 0, 1)$. Let $A_n = \sum_{i=0}^{n-1} A_i$

We find the following points:

$$B_0 = P + aA_0 + cA_1$$

$$B_1 = P + A_0 + bA_1$$

$$B_2 = P$$

$$C_0 = P + (b-1)A_1$$

$$C_1 = P + (a-c)A_0$$

$$C_2 = P + aA_0 = bA_1; \; c \in R, \; c \neq 0, 1, a, b.$$

Let $P = C_i = B_i$, $3 \leq i \leq n$. Without using commutativity of multiplication, one may show that if $(B_0 B_1 B_2)$ are collinear then $c = ab$. For $\lambda_1 B_1 + \lambda_2 B_2 + B_0 = 0$ implies $\lambda_1 = -a$, $c = -\lambda_1 b$; and therefore $c = ab$. By a simple calculation, one checks that the above three sets of (n+1) points form an incomplete P_n-configuration where the collinearity of the triple (C_0, C_1, C_2) is hypothetically missing. This means we have to check the following incidences:

$(B_0 C_1 A_2 A_3 \ldots A_n)$, $(C_0 B_1 A_2 A_3 \ldots A_n)$, $(B_0 C_2 A_1 A_3 \ldots A_n)$, $(C_0 A_1 B_2 A_3 A_4 \ldots A_n)$, $(A_0 B_1 C_2 A_3 A_4 \ldots A_n)$, $(A_0 C_1 B_2 A_3 A_4 \ldots A_n)$.

For example, the first follows from the fact that $C_1 - B_0 - A_2 - A_3 - \ldots - A_{n-1} + A_n = 0$. The second follows because $B_1 - C_0 - A_n + A_2 + \ldots +$

$A_{n-1} = 0$, and the rest may be verified easily likewise. By the P_n-proposition, the P_n - configuration is complete and the triple C_0, C_1, C_2 is collinear. Hence, $\exists \mu_0, \mu_1, \mu_2 \in R \ni \mu_0 C_0 + \mu_2 C_2 = 0$. This, by simple computation, yields $-\mu_1 c = \mu_0 a$ and $\mu_0 = -\mu_1 b$. Thus $-\mu_1 c = -\mu b a$ and $c = ba$. But $c = ab$, therefore, we have $ab = ba$. Since every pair of non-trivially commutative elements in R may be used to determine a P_n-configuration as above, the division ring R is commutative Theorem III. 3 is completely proved.

It should be observed that the P_n-proposition solves the problem of obtaining an axiom, equivalent to the commutativity of multiplication, in terms of (the dual elements of the n-space) points and hyperplanes for a projective geometry in any number of dimensions.

We finally remark that the P_n-configuration consists of 3 (n+1) points and [n(n+1) + 3] hyperplanes with (n+1) points on each hyperplane and various number of hyperplanes on each point. It is non-self-dual except for n = 2, where we obtain the Pappian configuration. In fact, if 3 (n+1) = n(n+1) + 3, then n = 2 and the trivial case n= 0. It should be pointed out that Theorem III. 3 can be shown to be an alternative form of Theorem C of [7].

We close this chapter by relating the P_n-proposition to the $(F - Q)_n$-proposition, when $n \geq 3$ is any odd integer, in the following.

Theorem III. 4.

Let $n > 1$ be odd integer. The $(F - Q)_n$-proposition is a minor form of the P_n-proposition.

Proof:

Let $A_0, A_1, \ldots, A_n$ be a complete hyperplanar (n+1)-point in a projective space S_n over a field R. Let $B_0, B_1, \ldots, B_n$ and $C_0, C_1, \ldots, C_n$ be any two sets of (n+1) points forming an incomplete P_n-configuration. According to the proof of Theorem III. 3, we may write

$$B_i = P + \sum_{j=0}^{n} k_{ij} A_j, \quad k_{jj} = 0;$$

$$C_i = P + \sum_{j=0}^{n} \ell_{ij} A_j, \quad \ell_{jj} = 0;$$

$k_{ij}, \ell_{ij} \in R$, $i = 0, 1, \ldots, n$. By a calculation similar to that given in III. 3, we have

$$\ell_{ij} = -k_{ji}; \quad i, j = 0, 1, \ldots, n.$$

Upon identifying B_i with C_i, $i = 0, 1, \ldots, n$, we get from the above representation

$$\ell_{ij} = k_{ij}; \quad i, j = 0, 1, \ldots, n.$$

Hence, we obtain

$$k_{ij} = -k_{ji}; \quad i, j = 0, 1, \ldots, n,$$

which yield a skew-symmetric determinant, similar to D, which vanishes because it is of odd order. The $(F - Q)_n$-proposition is therefore a special form of the P_n-proposition.

This specialization shows that the P_n-theorem generalizes both the Pappus class of theorems in the plane and the Möbius class of theorems in three-space. Here, again, the two classes are unified.

CHAPTER IV

SOME RESULTS CONCERNING THE M-CONFIGURATION, TRANSFORMATIONS AND COMMUTATIVITY

Two transformational characterizations of the commutativity of multiplication are obtained in this chapter. Both characterizations are established by configurational means. The M-configuration is generalized in a self-dual form; and the "Steinitz" group generated by null polarities is extended to (general) polarities.

1. The First Transformational Criterion.

Definition IV. 1.

Let S_n be a projective n-space over a division ring R. By a null polarity we understand a one-to-one mapping between the points and the hyperplanes of S_n which satisfies the following properties:

(1) Each point P is incident to its image hyperplane ρ.

(2) If the point P is mapped into the hyperplane ρ, then the points Q, incident to ρ, are mapped into hyperplanes ψ incident to P.

We may remark that the above definition has no content for $n = 0$. If $n = 1$, i.e., S_n is a line, then the hyperplanes of the space are points also, and hence the null polarity is the identity mapping of the points of S_1 onto themselves. We shall exclude both cases throughout the remaining part of the present chapter.

Now let S_n be defined over a field R whose characteristic $\neq 2$. Consider a correlation given by

$$\rho_i = \sum_{j=o}^{n} k_{ij} x_j , \qquad i = 0, 1, \ldots, n;$$

k_{ij}, x_i, ρ_i are elements of R. Imposing the property (1) of Definition IV. 1, we get

$$\langle \rho_i \cdot x_i \rangle = \sum_{i,j=o}^{n} k_{ij} x_j x_i = 0 ,$$

for all points $x = (x_o, x_1, \ldots, x_n)$. This implies (omitting Σ):

$$k_{ij}(x_j + y_j)(x_i + y_i) = k_{ij}(x_j x_i + x_j y_i + y_j x_i + y_j y_i) = 0 \ ,$$

yielding

$$k_{ij} x_j y_i + k_{ij} y_j x_i = 0 \ .$$

By property (2) of Definition IV.1, $k_{ij} y_j x_i = 0$ if and only if $k_{ji} x_j y_i = 0$. Hence,

$$k_{ij} x_j y_i + k_{ji} x_j y_i = 0 \ ,$$

giving

$$(k_{ij} + k_{ji}) x_j y_i = 0 \ .$$

Because the last relation holds for all x and y, we have

$$k_{ij} + k_{ji} = 0; \qquad i, j = 0, 1, \ldots, n.$$

Thus, a correlation satisfying the properties (1) and (2) of Definition IV.1 assumes the following form:

$$\rho_i = \sum_{j=o}^{n} k_{ij} x_j, \quad k_{ij} + k_{ji} = 0 \ , \quad i = 0, 1, \ldots, n.$$

It is important to observe that n must be odd, since otherwise the determinant $|k_{ij}|$ would vanish. For $|k_{ij}| = (-1)^{n+1} |k_{ij}|$, and if n were even, then $|k_{ij}| + |k_{ij}| = 0$, yielding $|_{ij}| = 0$, which contradicts the fact that the given correlation is one-to-one.

Conversely, let a correlation be defined by

$$\rho_i = \sum_{j=o}^{n} k_{ij} x_j, \quad k_{ij} + k_{ji} = 0, \quad i = 0, 1, \ldots, n;$$

and n be odd. If $P = (x_o, x_1, \ldots, x_n)$ is any point, then its image hyperplane ρ satisfies the relation

$$\langle \rho \cdot p \rangle = \langle \rho_i \cdot x_i \rangle = \sum_{i,j=o}^{n} k_{ij} x_j x_i = 0,$$

proving property (1). Also, if $P(x_o, x_1, \ldots, x_n) \rightarrow \rho(\rho_o, \ldots, \rho_n)$, and $Q(y_o, \ldots, y_n) \rightarrow \psi(\psi_o, \ldots, \psi_n)$, then, assuming $\langle \rho \cdot p \rangle = \sum_{i,j=o}^{n} k_{ij} x_j x_i = 0$, we get $\langle \psi \cdot Q \rangle + \langle \rho \cdot Q \rangle = 0$. Because $\langle \psi \cdot Q \rangle = 0$, we have $\langle \rho \cdot Q \rangle = 0$,

proving property (2).

Summarizing our results, we have proved

Theorem IV. 1.

Let S_n be defined over a field R whose characteristic $\neq 2$. A necessary and sufficient condition for a correlation Γ: $\rho_i = \sum_{j=o}^{n} k_{ij} x_j$, $(i = 0, 1, \ldots, n)$, to be a null polarity is that $k_{ij} + k_{ji} = 0$ and n is odd. Theorem IV. 1 has been proved in a different way in [3] where the algebraic techniques of field theory are used.

Now let S_3 be commutative. Let T_1 : 0 1 2 3 and T_2 : 0' 1' 2' 3' be two tetrahedra in an incomplete M-relation. We may choose T_1 to be the tetrahedron of reference so that

$$\begin{array}{l} 0: \\ 1: \\ 2: \\ 3: \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The planes of the second tetrahedron, T_2, may be denoted as follows:

$$\begin{array}{cccc} (1'\,2'\,3') & (0'\,2'\,3') & (0'\,2'\,4') & (0'\,1'\,2') \\ [0, k_{12}, k_{13}, k_{14}], & [k_{21}, 0, k_{23}, k_{24}], & [k_{31}, k_{32}, 0, k_{34}], & [k_{41}, k_{42}, k_{43}, 0] \end{array}$$

with the conditions that $k_{21} = -k_{12}$, $k_{31} = -k_{13}$, $k_{41} = -k_{14}$. In the above representation we have used only the part of the hypothesis which says that T_1 is inscribed in T_2. Using the remaining part of the hypothesis, we get from the dependence of the four planes (0 1 2), (0' 2' 3'), (1' 2' 3'), (0' 1' 3'):

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & k_{12} & k_{13} & k_{14} \\ k_{21} & 0 & k_{23} & k_{24} \\ k_{31} & k_{32} & 0 & k_{34} \end{vmatrix} = k_{12} k_{31} (k_{23} + k_{32}) = 0 .$$

This yields $k_{23} = -k_{32}$, provided $k_{12}, k_{31} \neq 0$. Similarly, from the dependence of the four planes (0 1 3), (1' 2' 3'), (0' 2' 4'), (0' 1' 2'), we get $k_{24} = -k_{42}$, and from the dependence of (0 2 3), (1' 2' 3'), (0' 1' 3'), (0' 1' 2'), we get $k_{34} = -k_{43}$. Thus, the incomplete M-relation permits us to represent the planes of T_2 by the rows of the following skew-symmetric matrix:

$$M : \begin{pmatrix} 0 & k_{12} & k_{13} & k_{14} \\ k_{21} & 0 & k_{23} & k_{24} \\ k_{31} & k_{32} & 0 & k_{34} \\ k_{41} & k_{42} & k_{43} & 0 \end{pmatrix} .$$

To show that the M-relation is in fact complete, we must show that the quadruple (1 2 3), (0' 2' 3'), (0' 1' 3'), (0' 1' 2') is dependent. This is easily checked by verifying that the following determinant vanishes:

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ k_{21} & 0 & k_{23} & k_{24} \\ k_{31} & k_{32} & 0 & k_{34} \\ k_{41} & k_{42} & k_{43} & 0 \end{vmatrix} .$$

Consider the preceding matrix M. By direct calculation we find that the value of its determinant $= (k_{12}k_{34} + k_{13}k_{42} + k_{14}k_{23})^2 \neq 0$. The matirx M may be interpreted, on account of Theorem IV.1, as a null polarity mapping one (and hence each) tetrahedron onto the other. The converse is also true. That is to say, if a null polarity maps a tetrahedron onto another, the two tetrahedra are in a complete M-relation. For, if we take the given tetrahedron as a reference system, the plane coordinates of its image tetrahedron may be assigned the rows of the matrix M. Then it is an easy matter to verify that the given tetrahedron and its image are mutually inscribed.

The above discussion leads to the following criterion for commutativity

which is contained in

Theorem IV. 2.

Let S_3 be defined over a division ring R. The following three statements are equivalent:

(1) R is commutative.

(2) The (N-P)-proposition:

Given three non-collinear points and three non-dependent planes such that each plane is incident to one point and the meet of the planes is incident to the join of the points (but not on a line joining two points), there exists one, and only one, null polarity which takes the given points into their corresponding planes and the meet of the planes into the join of the points.

(3) Every incomplete M_3-configuration is in fact complete.

Proof:

We shall prove that (1) implies (2) implies (3) implies (1). To prove (1) implies (2), let R be a field. Let 0, 1, 2 be three non-collinear points and π_o, π_1, π_2 be three non-dependent planes incident to their corresponding points. We may choose a coordinate system so that the points and the planes have the following representation respectively:

$$0: \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}, \quad 1: \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}, \quad 2: \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\pi_o: \begin{pmatrix} 0 & k_{12} & k_{13} & k_{14} \end{pmatrix}, \quad \pi_1: \begin{pmatrix} k_{21} & 0 & k_{23} & k_{24} \end{pmatrix}, \quad \pi_2: \begin{pmatrix} k_{31} & k_{32} & 0 & k_{34} \end{pmatrix},$$

with $k_{21} = -k_{12}$ and $k_{31} = -k_{13}$.

Using the condition that the meet of π_o, π_1, π_2 is incident with the join of 0, 1, 2, and since the plane (0 1 2) has the coordinates [0, 0, 0, 1], the following determinant must vanish:

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & k_{12} & k_{13} & k_{14} \\ k_{21} & 0 & k_{23} & k_{24} \\ k_{31} & k_{32} & 0 & k_{34} \end{vmatrix}$$

Hence, $k_{12}k_{31}k_{23} + k_{21}k_{13}k_{32} = 0 = k_{12}k_{13}(k_{23} + k_{32})$. Assuming $k_{12}, k_{13} \neq 0$, we have $k_{23} = -k_{32}$.

Let 3 be denoted by $(0,0,0,1)$ so that we have 0 1 2 3 as tetrahedron of reference. We find the plane $|k_{41}, k_{42}, k_{43}, 0|$, with $k_{41} = -k_{14}$, $k_{42} = -k_{24}$, $k_{43} = -k_{34}$, and denote it by π_3. Thus we obtain the tetrahedron $\pi_o, \pi_1, \pi_2, \pi_3$ whose faces are represented by the rows of the skew-symmetric matrix M. We claim that M represents a null polarity mapping $0 \rightarrow \pi_o$, $1 \rightarrow \pi_1$, $2 \rightarrow \pi_2$, and $3 \rightarrow \pi_3$. In the first place, since $\pi_o, \pi_1, \pi_2, \pi_3$ are independent, the determinant $|k_{ij}| \neq 0$. By straightforward computation, one may check that M produces the above mapping, and therefore M represents a null polarity meeting the above conditions. To show that this null polarity is unique, let M and M' be two null polarities meeting these requirements. Then $M^{-1}M'$ is a collineation leaving $0, 1, 2, 0', 1'$ invariant. By the $(F\text{-}P)_3$-proposition, the collineation $M^{-1}M'$ is the identity. Hence, $M = M'$ and consequently (1) implies (2).

To prove that (2) implies (3), consider an incomplete M-configuration T_1 : 0 1 2 3 and T_2 : o' 1' 2' 3' where the incidence (0 1 2 3') is hypothetically missing. Let Γ be the null polarity affecting the following mapping: $1 \rightarrow (0'\,2'\,3')$, $2 \rightarrow (0'\,1'\,3')$, $3 \rightarrow (0'\,1'\,2')$, and $0' \rightarrow (1\,2\,3)$. By Definition IV.1, $1' \rightarrow$ into a plane containing the points $1', 2, 3$. But the plane $(1'\,2\,3)$ is the same as the plane $(0\,2\,3)$ by the definition of the incomplete M-relation. By a similar argument, one may prove that Γ takes $2' \rightarrow (0\,1\,2') = (0\,1\,3)$, and $0 \rightarrow (0'\,1'\,2') = (1'\,2'\,3')$. Since 3' is incident to the images of 0, 1 and 2, the point 3' is mapped into a plane

containing 0, 1, 2 by property (2) of Definition IV. 1. By property (1) of Definition IV. 1, the point 3' is incident to its image plane (0 1 2) and thus the incidence (0 1 2 3') follows.

Since (3) is the M-proposition and the M-proposition implies commutativity, (3) implies (1). Theorem IV. 2 is proved.

2. The Steinitz Group.

Definition IV. 2.

Two polarities Γ_1 and Γ_2 are said to be commutative provided the two collineations $\Gamma_1\Gamma_2$ and $\Gamma_2\Gamma_1$ are identical.

This definition leads to the following two corollaries.

Corollary IV. 1.

The two polarities Γ_1 and Γ_2 are commutative if and only if $\Gamma_1\Gamma_2$ (or $\Gamma_2\Gamma_1$) is an involutory collineation.

For, if Γ_1 and Γ_2 are commutative, then $\Gamma_1\Gamma_2 = \Gamma_2\Gamma_1$, yielding $(\Gamma_1\Gamma_2)(\Gamma_1\Gamma_2) = (\Gamma_1\Gamma_2\Gamma_1)\Gamma_2 = {}_2{}_2 = I$ (the identical collineation). On the other hand, if $(\Gamma_1\Gamma_2)(\Gamma_1\Gamma_2) = I$, then $\Gamma_1\Gamma_2 = (\Gamma_1\Gamma_2)^{-1} = \Gamma_2\Gamma_1$.

Corollary IV. 2.

Γ_1 and Γ_2 are commutative if and only if the set $\{\Gamma_1, \Gamma_2, \Gamma_1\Gamma_2, \Gamma_2\Gamma_1, I\}$ is a group.

For, if Γ_1 and Γ_2 are commutative, then $(\Gamma_1\Gamma_2)\Gamma_1 = \Gamma_2$. Similarly, one may prove that the product of any two elements of the set is again in the set, which proves the closure property of the set. The inverse of each element is itself by Definition IV. 1 and Corollary IV. 1. Because the identity is in the set, the set is a group.

Conversely, assuming $\Gamma_1 \neq \Gamma_2$, otherwise it is trivial, we have $(\Gamma_1\Gamma_2)\Gamma_1$ in the group. This product is a null polarity and it is different from Γ_1, therefore it must be equal to Γ_2. Hence, $(\Gamma_1\Gamma_2)\Gamma_1 = \Gamma_2$ giving $\Gamma_1\Gamma_2 = \Gamma_2\Gamma_1$.

Returning to the M-configuration, we have pointed out that if T_1 : 0 1 2 3

and T_2 : $0'1'2'3'$ are two tetrahedra in an M-relation, so are the three pairs: $013'3'$, $0'1'23$; $0'12'3$, $01'23'$; $0'123'$, $01'2'3$. We now associate, with each pair a null polarity so that we have four null polarities exhibited as follows:

$$\Gamma_1 : \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0' & 2' & 2' & 3' \end{bmatrix} \qquad \Gamma_2 : \begin{bmatrix} 0 & 1 & 2' & 3' \\ 0' & 1' & 2 & 3 \end{bmatrix}$$

$$\Gamma_3 : \begin{bmatrix} 0 & 1' & 2 & 3' \\ 0' & 1 & 2' & 3 \end{bmatrix} \qquad \Gamma_4 : \begin{bmatrix} 0 & 1' & 2' & 3 \\ 0' & 1 & 2 & 3' \end{bmatrix} .$$

In the above scheme for Γ_1, the point 0 is to be understood to correspond to the plane $(1'2'3')$, and etc.

Lemma IV. 1.

The null polarities $\{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\}$ are pairwise commutative.

For, consider any point, say 0, of the M-configuration. Upon applying Γ_1 on it we obtain the plane $(1'2'3')$ and applying Γ_2 to this plane we get the point $1'$. Thus, the point 0 is mapped onto the point $1'$ under the collineation $\Gamma_2\Gamma_1$. On the other hand, the point 0 goes to the plane $(1'23)$ under Γ_2 and this plane goes to the point $1'$ under Γ_1. Hence, the collineation $\Gamma_1\Gamma_2$ takes the point 0 to the point $1'$. On account of the symmetry of the M-configuration and the similar roles played by its elements, one concludes that Γ_1 and Γ_2 are commutative. In a similar manner, we can show that every pair of the four null polarities is commutative.

The above lemma immediately implies

Lemma IV. 2.

The product of any number of elements of the set $\{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\}$ is independent of the order of its factors.

We are now ready to prove

Theorem IV. 3.

The set of four mutually commutative null polarities Γ_i, $i = 1, 2, 3, 4$, generates an Abelian group consisting of eight collineations $\{\Gamma_i\Gamma_j, \Gamma_1\Gamma_2\Gamma_3\Gamma_4, I\}$ and eight correlations $\{\Gamma_i, \Gamma_i\Gamma_j\Gamma_k\}$; the eight collineations form a transitive subgroup of the whole group.

Proof:

Null polarities are of period two; hence each is its own inverse. Consequently, the inverse of each member of the set of collineations and correlations is in the set. They are associative by definition. By Lemma IV 2, the product of any two elements of the set is again in the set. Since the identity mapping is in the set, the set is an Abelian group. That the subset of the eight collineations is a subgroup is clear because the product of an even number of the Γ_i is a collineation which is equal to either $\Gamma_i\Gamma_j$, $\Gamma_1\Gamma_2\Gamma_3\Gamma_4$ or I. We shall show that this subgroup is transitive and then the theorem follows. By calculation we obtain the following table, in each column of which, every point of the M-configuration is present:

	0	1	2	3	0'	1'	2'	3'
I	0	1	2	3	0'	1'	2'	3'
$\Gamma_1\Gamma_2$	1'	0'	3'	2'	1	0	3	2
$\Gamma_1\Gamma_3$	2'	3'	0'	1'	2	3	0	1
$\Gamma_1\Gamma_4$	3'	2'	1'	0'	3	2	1	0
$\Gamma_2\Gamma_3$	3	2	1	0	3'	2'	1'	0'
$\Gamma_2\Gamma_4$	2	3	0	1	2'	3'	0'	1'
$\Gamma_3\Gamma_4$	1	0	3	2	1'	0'	3'	2'
$\Gamma_1\Gamma_2\Gamma_3\Gamma_4$	0'	1'	2'	3'	0	1	2	3

Theorem IV. 3. is completely proved.

Steinitz [14, p. 493] states a part of Theorem IV. 3 without proof; for this reason we call the projective group, associated with the M-configuration

the "Steinitz" group of the M-configuration In general, by a "Steinitz group" we mean any group generated by four mutually commutative null polarities.

The converse problem presents itself: given a Steinitz group, does it determine an M-configuration? The answer to this question is contained in

Theorem IV. 4.

Let $\{I, \Gamma_i \Gamma_j, \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4; \Gamma_i, \Gamma_i \Gamma_j \Gamma_k\}$ be a Steinitz group generated by the four mutually commutative null polarities, Γ_i, $i = 1, 2, 3, 4$. Let P be any point in S_3. The successive operations of the elements of the group on P yield a set of eight points: P, P_{ij}, P_{1234}, and eight planes π_i, π_{ijk}, which constitute an M-configuration.

Proof:

Applying the elements of the group on P, we obtain 8 points and 8 planes denoted as follows:

$$P,\ P_{12},\ P_{13},\ P_{14},\ P_{23},\ P_{24},\ P_{34},\ P_{1234};$$
$$\pi_1,\ \pi_2,\ \pi_3,\ \pi_4,\ \pi_{123},\ \pi_{124},\ \pi_{134},\ \pi_{234}.$$

One may verify directly, using Definition IV. 1, that we have the following table of incidences:

π_1,	π_2,	π_3,	π_4	are incident to	P,
π_1,	π_2,	π_{123},	π_{124}	" " "	P_{12},
π_1,	π_3,	π_{123},	π_{134}	" " "	P_{13},
π_1,	π_4,	π_{124},	π_{134}	" " "	P_{14},
π_2,	π_3,	π_{123},	π_{234}	" " "	P_{23},
π_2,	π_4,	π_{124},	π_{234}	" " "	P_{24},
π_3,	π_4,	π_{134},	π_{234}	" " "	P_{34},
π_{123},	π_{124},	π_{134},	π_{234}	" " "	P_{1234}.

Since the order of incidence of each point or plane is four, the above table

defines an M-configuration. We note that we could have started with a plane π instead of a point P; in each case the configuration is determined modulo a point or a plane. The theorem is proved.

The above mentioned connection between null polarity and the M-configuration suggests a self-dual generalization of the M-class of configurations to higher spaces which we give as follows:

Let S_n, n odd, be defined over a field R. Let E be the unit simplex. Upon applying a null polarity Γ on E, we obtain another simplex whose hyperplanes are represented by the rows of a skew-symmetric matrix M. It is clear that the simplex E is inscribed in the simplex represented by M. That the second is also inscribed in the first is readily seen from the fact that the determinants representing the necessary incidences can be written at once from the matrices E and M, and by inspection these determinants are equivalent to skew-symmetric determinants of odd order and therefore vanish. Hence we have

Theorem IV. 5.

Let S_n, n odd, be defined over a field R. There exists a pair of mutually inscribed simplexes.

The configuration corresponding to the above theorem consists of $(n + 1)$ points and $(n + 1)$ hyperplanes with $(n + 1)$ points on each hyperplane and $(n + 1)$ hyperplanes on each point.

It is important to observe that the converse of the Theorem IV. 5 is not true. In other words, the existence of the M_n-configuration, in any odd-dimensional space, does not imply the commutativity of the space. This is proved by the following counter-example:

Let $n = 3$ and assume S_3 to be defined over a division ring R. Let T_1 : 0 1 2 3 be the unit tetrahedron. Let T_2 : 0' 1' 2' 3' be another tetrahedron whose vertices are represented by the rows of the following skew-symmetric matrix:

$$\begin{array}{l} 0': \\ 1': \\ 2': \\ 3': \end{array} \begin{pmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

It is clear that T_1 is inscribed in T_2. Without using commutativity, one may show that T_2 is also inscribed in T_1. In fact, denoting T_1 by $A_o A_1 A_2 A_3$ and T_2 by $B_o B_1 B_2 B_3$, we have

$$A_o - B_1 + B_2 - B_3 = 0, \quad B_o - A_1 - B_2 + B_3 = 0,$$
$$B_o - B_1 - A_2 + B_3 = 0 \text{ and } B_o - B_1 + B_o - A_3 = 0,$$

proving our contention. To show that T_2 is non-degenerate, let $\lambda_o, \lambda_1, \lambda_2, \lambda_3$ be elements of R such that

$$\lambda_o B_o + \lambda_1 B_1 + \lambda_2 B_2 + \lambda_3 B_3 = 0.$$

We shall show that

$$\lambda_o = \lambda_1 = \lambda_2 = \lambda_3 = 0.$$

We have

$$\lambda_1 + \lambda_2 + \lambda_3 = 0, \quad \lambda_o - \lambda_2 - \lambda_3 = 0, \quad \lambda_o + \lambda_1 - \lambda_3 = 0 \quad \text{and}$$
$\lambda_o + \lambda_1 + \lambda_2 = 0$. Combining the first and the fourth, we get $\lambda_o = \lambda_3$. Hence $\lambda_1 = 0$ and $\lambda_2 = 0$. Thus $\lambda_o = \lambda_1 = \lambda_2 = \lambda_3 = 0$. It follows that the existence of the M-configuration, in any odd-dimensional space is not sufficient for the commutativity of multiplication. It remains problematical whether it is possible to generalize the M-proposition to higher spaces so as to obtain a self-dual configuration while the proposition remains as a criterion for commutativity. This problem deserves further investigations.

3. The Second Transformational Criterion and the M-polarities.

In Section 1 of the present chapter we have shown that the existence of a null polarity satisfying certain given conditions is a criterion for the commutativity of multiplication. We shall generalize this to (general)

polarities.

We proved in Chapter I, that a polarity may be represented by a matrix (k_{ij}) with $k_{ij} = k_{ji}$. Now let T_1 : 0 1 2 3 be denoted as follows:

0	1	2	3
(1, 0, 0, 0)	(0, 1, 0, 0)	(0, 0, 1, 0)	(0, 0, 0, 1)

Consider a polarity Δ represented by the symmetric matrix (k_{ij}), $i, j = 0, 1, 2, 3$; k_{ij} are elements of a field R. Let Δ leave T_1 self-polar. This implies that the polar plane of the point 0 is the plane (1 2 3) which has the coordinates $[1, 0, 0, 0,]$. Hence $k_{01} = k_{02} = k_{03} = 0$. Similarly, by considering the polars of 1 and 2 we obtain $k_{10} = k_{12} = k_{13} = k_{20} = k_{21} = k_{23} = k_{30} = k_{31} = k_{32} = 0$. It is a consequence that the polar of the remaining vertex of T_1 is its remaining face. Thus the polarity Δ assumes the form

$$\Delta : \begin{pmatrix} k_{oo} & 0 & 0 & 0 \\ 0 & k_{11} & 0 & 0 \\ 0 & 0 & k_{22} & 0 \\ 0 & 0 & 0 & k_{33} \end{pmatrix}.$$

Let $0', 1', 2'$ be three points incident to the faces (1 2 3), (0 2 3), (0 1 3) respectively, but not on an edge of T_1. We may represent these three points as follows:

$$\begin{matrix} 0': \\ 1': \\ 2': \end{matrix} \begin{pmatrix} 0 & a & b & 1 \\ c & 0 & d & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \qquad a, b, c, d, \neq 0 .$$

We may compute the values of the diagonal elements in Δ, by requiring the polarity Δ to leave the three points $0', 1', 2'$ pairwise conjugate. This requirement yields the following equations:

$$bd\,k_{22} + k_{33} = 0 \ldots \quad 1$$
$$a\,k_{11} + k_{33} = 0 \ldots \quad 2$$
$$c\,k_{00} + k_{33} = 0 \ldots \quad 3 .$$

By direct computation we find that $k_{oo} = bdc^{-1}k_{22'}$, $k_{11} = bda^{-1}k_{22'}$, $k_{33} = -bdk_{22'}$. Thus the polarity Δ is determined.

Now let T_1 : 0 1 2 3 and T_2 : 0' 1' 2' 3' be two tetrahedra in an incomplete M-relation where the incidence (0 1 2 3') is hypothetically missing. Consider a polarity Δ which leaves T_1 self-polar and the three points 0', 1', 2' pairwise conjugate. Because 0' is on (1 2 3) and 0' is conjugate with 1' and 2', by hypothesis, the polar of 0' is incident to 0, 1', 2, and hence it must be the plane (0 1' 2') or, which is the same thing, the plane (1' 2' 3'). By a similar argument we may prove that the polar of 1' is (0' 2' 3') and the polar of 2' is (0' 1' 3'). It follows that the polar of 3' is (0' 1' 2'). Since 3 is incident to (0' 1' 2'), by hypothesis, the polar of 3 must be incident to the pole of (0' 1' 2'). This means that 3' is on (0 1 2) and the configuration is in fact complete.

Summarizing our results, we have proved

Theorem IV. 6.

Let S_3 be a projective space over a division ring R. The following three statements are equivalent:

(1) R is commutative.

(2) Given a tetrahedron and three points incident to three of its faces, but not on an edge of it, there exists a unique polarity which leaves the tetrahedron self-polar and the three points pairwise conjugate.

(3) Every hypothetically incomplete M_3-configuration is in fact complete.

In the above discussion the following corollary is proved:

Corollary IV. 3.

Let S_3 be commutative. Given a pair of mutually inscribed tetrahedra, there exists a unique polarity which leaves both tetrahedra self-polar.

The Corollary IV. 3 enables us to associate with the M-configuration four (Möbius) polarities Δ_i, $i = 1, 2, 3, 4$, each of which leaves a pair of the four pairs of the Möbius tetrahedra, of which the M-configuration consists, self-polar. Thus we have the following arrangements:

$$\Delta_1: \begin{Bmatrix} 0 & 1 & 2 & 3 \\ 0' & 1' & 2' & 3' \end{Bmatrix} \qquad \Delta_2: \begin{Bmatrix} 0 & 1 & 2' & 3' \\ 0' & 1' & 2 & 3 \end{Bmatrix}$$

$$\Delta_3: \begin{Bmatrix} 0 & 1' & 2 & 3' \\ 0' & 1 & 2' & 3 \end{Bmatrix} \qquad \Delta_4: \begin{Bmatrix} 0 & 1' & 2' & 3 \\ 0' & 1 & 2 & 3' \end{Bmatrix}$$

In the above scheme for Δ_1 the point 0 is to be understood to correspond to the plane (1 2 3) etc. By direct argument and by analysis of the proof of Lemma IV. 1 and Theorem IV. 3 we observe that the property of self-conjugacy of a null polarity is not used. It follows that both Lemma IV. 1 and Theorem IV. 3 hold for the polarities Δ_i, $i = 1, 2, 3, 4$. Thus we obtain a projective group of 16 elements, 8 collineations and 8 correlations, generated by the four mutually commutative polarities Δ_i which extends Theorem IV. 3 to general polarities and consequently the Steinitz group generated by null polarities to that generated by polarities.

CHAPTER V

FURTHER CHARACTERIZATIONS OF COMMUTATIVITY IN HIGHER SPACES

A further generalization of the Pappus theorem to higher spaces is obtained in this chapter. Again, this extension is proved to be a criterion for the commutativity of multiplication. A "direct" relationship between the original theorem of Pappus and its present extension is established. A minor form of the generalization is proved independently of commutativity, and it is related to a minor form of the theorem of Pappus.

1. The Pappus Theorem in Three-dimensions and its Minor.

In its classical form, the plane proposition of Pappus may be stated as follows:

Let S_2 be a projective plane over a division ring R. Let (1 2 3) and (1' 2' 3') be two triples of collinear points. Let

$$(1\ 2') \cap (1'\ 2) = 3'',$$
$$(2\ 3') \cap (2'\ 3) = 1'',$$
$$(1\ 3') \cap (1'\ 3) = 2'';$$

then the triple (1'' 2'' 3'') is collinear if and only if R is commutative.

It is not difficult to see the relation between this formulation of the theorem of Pappus and the one which we gave in Theorem I.1. One, for example, sees that the three triangles (1 2' 3'), (1' 2'' 3''), (1'' 2 3) form a Pappian chain, where each triangle is inscribed in the following one. Hence, the two corresponding configurations are the same. However, the present statement leads to an extension to three-space which we give in

Theorem V.1.

Let S_3 be defined over a division ring R. Let L and L' be two skew-lines, in S_3, with two triples of points, (1 2 3) and (1' 2' 3') on them respectively. Any point P in S_3, determines with the three pairs of skew-

lines $\{1\ 2',\ 1'\ 2\}$, $\{2\ 3',\ 2'\ 3\}$, $\{1\ 3',\ 1'\ 3\}$, three lines L_3, L_1, L_2 respectively, each passes through P and intersects one of the three pairs. Then L_1, L_2, L_3 are in one plane, for arbitrary choice of P, if and only if R is a field.

Proof:

A. Suppose R is commutative. We may choose a coordinate system so that the following points have the corresponding coordinates:

1 2 3 1' 2' 3'

$(1,0,0,0)$, $(0,1,0,0)$, $(1,a,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, $(0,0,1,b)$;

where $a, b \neq 0$ are elements of R. We may choose P to be the unit point $(1,1,1,1)$. Since the three lines L_1, L_2, L_3 are incident to P, they will be determined upon finding one other point on each. We shall compute the coordinates of one such point as follows:

The line L_3 intersects the pair of skew-lines $\{1\ 2',\ 1'\ 2\}$ in two points which may be represented by:

$(1,0,0,0) + c(0,0,0,1)$ and $(0,1,0,0) + d(0,0,1,0)$ respectively; $c, d \neq 0$. These two points are collinear with P; hence $\exists \lambda_1, \lambda_2 \neq 0 \in R$, such that $(1,1,1,1) + \lambda_1(1,0,0,c) + \lambda_2(0,1,d,0) = 0$. This yields $\lambda_1 = -1$, $\lambda_2 = -1$, $d = 1$, $c = 1$. Hence, the point $(1,0,0,1)$ is another point, distinct from P, on the line L_3. By similar computations we find that the points $(b, ab, ab - a, 0)$ and $(0, 1 - a, 1, b)$ are on L_2 and L_1 respectively. For L_1, L_2, L_3 to be in one plane it is necessary and sufficient that the following four points are dependent:

$$\begin{array}{ll} P: & \\ P_3: & \\ P_2: & \\ P_1: & \end{array} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ b & ab & ab-a & 0 \\ 0 & 1-a & 1 & b \end{pmatrix} \circledast$$

Since the division ring is commutative, it is a simple matter to check

that the determinant of the above matrix vanishes.

B. Let R be a division ring. We shall call the theorem proved in the previous part a "skew-generalization of Pappus theorem", and denote it by the symbol $(S-P)_3$. Let $a, b \neq 0, 1$ be any two elements of R. Let the $(S-P)_3$-proposition hold. As above, let (1 2) and (1' 2') be two skew-lines in space. Denote 1 by (1, 0, 0, 0), 2 by (0, 1, 0, 0), 1' by (0, 0, 1, 0) and 2' by (0, 0, 0, 1). We find the points P = (1, 1, 1, 1); 3 = (1, a, 0, 0) and 3' = (0, 0, 1, b); and the three points

$$\begin{matrix} P_3: \\ P_2: \\ P_1: \end{matrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ b & ab & ab-a & 0 \\ 0 & 1-a & 1 & b \end{pmatrix}.$$

It can be shown, independently of the commutativity of multiplication, that the preceding ten points form an incomplete $(S-P)_3$-configuration. That is to say, we must verify that the line $(P\ P_1)$ intersects the lines (2 3') and (2'3), the line $(P\ P_2)$ intersects (1 3') and (1'3) and the line $(P\ P_3)$ intersects (1 2') and (1' 2). As an example, we show that $(P\ P_2)$ intersects (1 3') and (1' 3) as follows:

$$(b, ab, ab-a, 0) - ab(1, 1, 1, 1) + (ab - b)(1, 0, 0, 0) + a(0, 0, 1, b) = 0,$$

proving that P_2 is on (1' 3). On the other hand, the point P_2 is on (1' 3), therefore $(P\ P_2)$ intersects (1 3') and (1' 3). The others may be easily checked. By the $(S-P)_3$—proposition, the configuration is complete. Hence, $\exists\ \lambda_1, \lambda_2, \lambda_3 \neq 0$ such that:

$$\lambda_1(1, a, a-ab^{-1}, 0) + \lambda_2(1, 0, 0, 1) + \lambda_3(0, 1-a, 1, b) + (1, 1, 1, 1) = 0.$$

This implies:

$$\lambda_1 + \lambda_2 + 1 = 0 \quad ①,$$

$$\lambda_1 a + \lambda_3(1-a) + 1 = 0 \quad ②,$$

$$\lambda_1(a-ab^{-1}) + \lambda_3 + 1 = 0 \quad ③,$$

$$\lambda_2 = \lambda_3 b + 1 = 0 \quad ④.$$

Combining ② and ③ , we get $\lambda_1 = \lambda_3 \, ab\, a^{-1}$. Upon substituting in ① , we have

$\lambda_3 \, aba^{-1} - 1 - \lambda_3 b + 1 = 0$, giving $aba^{-1} = b$, or $ab = ba$.

Because every pair of non-trivially commutative elements of R may be used to determine an $(S-P)_3$-configuration, R is a commutative division ring. Theorem V.1 is proved.

Here, again, the question arises whether it is possible to prove a special, but non-degenerate, form of the $(S-P)_3$-proposition in a not necessarily commutative three-space. In this connection we deduce:

Theorem V.2.

Let S_3 be defined over a division ring R. Let: 1, 2, 3; 1', 2', 3'; P, P_1, P_2, P_3 form an incomplete $(S-P)_3$-configuration, and in addition, the two lines $(P\ P_2)$ and $(2\ 2')$ are coplanar. Then the configuration is in fact complete, i.e., the three lines $(P\ P_1)$, $(P\ P_2)$ and $(P\ P_3)$ are coplanar.

Proof :

Using the same representation given in the first part of Theorem V.1, one obtains, without commutativity, the matrix ⊛ representing the four points P, P_1, P_2, P_3. Imposing the additional condition that the quadruple $(P\ P_2\ 2\ 2')$ is dependent, we have for $\lambda_1, \lambda_2, \lambda_3 \neq 0$,

$$(b, ab, ab-a, 0) = \lambda_1(1,1,1,1) + \lambda_2(0,1,0,0) + \lambda_3(0,0,0,1) = 0.$$

This implies $\lambda_1 = -\lambda_3 = -b = a-ab$. Hence, $ab = a + b$. Thus the point P_2 may be represented by $(b,\ a+b,\ b, 0)$ and consequently the matrix ⊛ assumes the following form:

$$\begin{array}{c} P \\ P_3 \\ P_2 \\ P_1 \end{array} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ b & a+b & b & 0 \\ 0 & 1-a & 1 & b \end{pmatrix} \quad ⊛'$$

Since $(1,0,0,1) + (b, a+b, b, 0) + (0, 1-a, 1, b) - (1+b)(1,1,1,1) = 0$, the above "small" form of the $(S-P)_3$-theorem is proved.

There is a natural and direct relation between the $(S-P)_3$- and the P_2-proposition which we give in

Theorem V. 3.

The $(S-P)_3$-proposition is equivalent to the P_2-proposition.

Proof:

A. Suppose the P_2-proposition hold. Let L and L' be two skew-lines in space, with three points 1, 2, 3 and 1', 2', 3' on them respectively. Consider the three pairs of lines {1 2', 1' 2}, {1 3', 1' 3}, {2 3', 2' 3}. Let P be any point in space not incident to any one of these lines. Let L_3, L_2, L_1 be three lines through P, intersecting the above pairs respectively. We shall show that L_1, L_2, L_3 lie in one plane.

Consider a plane $\pi \subset S_3$ not incident to P. By projecting the configuration from the point P into π, by lines containing P, we get an incomplete P_2-configuration in π. This may be explained as follows. The lines determined by P and the triple (1, 2, 3) are contained in the plane determined by P and L. This plane intersects π in a line ℓ with three points A_1, A_2, A_3, which are the images of 1, 2, 3 respectively, on it. Similarly, the image set (B_1, B_2, B_3) of the triple (1', 2', 3') lie on a line $\ell' \subset \pi$. The above three pairs are mapped respectively onto the pairs $\{A_1B_2, B_1A_2\}$, $\{A_1B_3, B_1A_3\}$, $\{A_2B_3, B_2A_3\}$. Denoting the common points of the last three pairs by C_3, C_2, C_1 respectively, we obtain an incomplete P_2-configuration in π. By the P_2-proposition the triple $(C_1C_2C_3)$ lies on a line ℓ. The line ℓ and the point P determine a plane π'. Since each of the lines L_1, L_2, L_3 passes through the point P and one point of the triple C_1, C_2, C_3, the three lines are in the plane π'. This proves the completeness of the $(S-P)_3$-configuration.

B. Let the $(S-P)_3$-proposition hold. Let $A_1, A_2, A_3; B_1, B_2, B_3; C_1, C_2,$

C_3 be nine points, in a plane π, forming an incomplete P_2-configuration, i.e., the collinearity of the triple (C_1, C_2, C_3) is hypothetically missing.

Consider any point $P \subset S_3$ not incident to π. The point P with the line ℓ, of the triple $(A_1\ A_2\ A_3)$, determines a plane in which the three lines PA_1, PA_2, PA_3 are contained. Let L be any line in this plane; it cuts the lines PA_1, PA_2, PA_3 in 1, 2, 3 respectively. Similarly, a line in the plane, determined by the line ℓ', of the triple $(B_1\ B_2\ B_3)$, and the point P, cuts the three lines PB_1, PB_2, PB_3 in 1', 2', 3' respectively. Clearly, the line taken in the plane $(P\ell')$ may be taken to be skew to the first. The plane (PA_1B_2) contains the line PC_3 and so is the plane (PB_1A_2). But the plane (PA_1B_2) contains the line (1 2'), therefore the line PC_3 intersects the line (1 2'). By a similar argument, we show that the line PC_3 intersects (1' 2). Hence the line (PC_3) is the common line, through P, to (1 2') and (1' 2). We may, similarly, show that (PC_2) meets (1 3') and (1' 3) and (PC_1) meets (2 3') and (2' 3). Thus, we have constructed an incomplete $(S - P)_3$- configuration. By the $(S - P)_3$-proposition, (PC_1), (PC_2) and (PC_3) are contained in the plane π'. The plane π' cuts the plane π in a line, and since C_1, C_2, C_3 are on π, π', they are on their line of intersection. Hence, the P_2-configuration is complete. This completes the proof of Theorem V. 3.

The $(S - P)_3$-configuration consists of 13 points, 11 lines and one plane, with 3 points on each line and at least two lines on each point. It is to be noted that the $(S - P)_3$-theorem may be stated in the following form which is analogous to Pascal's form of the P_2-proposition.

Corollary V. 1.

If a skew-hexagon is inscribed in two skew-lines, then the three lines determined by each pair of its opposite sides and an arbitrary point, are coplanar.

It was remarked in Chapter I that the first minor theorem of

Pappus arises when the three points $(2, 2', 2'')$ are required to be collinear. This special form of the P_2-proposition does not depend on commutativity. We shall prove that in

Corollary V. 2.

In the presence of the "axioms of incidence", the minor theorem of $(S - P)_3$ is equivalent to the first minor theorem of Pappus.

Proof:

A. Suppose the first minor theorem of Pappus hold. Let L: 1 2 3, L': 1' 2' 3', L_1, L_2, L_3, P form a "small" incomplete $(S - P)_3$-configuration. Consider a plane π not incident with P. Projecting the configuration from Pinto π, we obtain an incomplete first minor configuration of Pappus. For, the points 1, 2, 3 are mapped onto collinear points A_1, A_2, A_3; and the points 1', 2', 3' are mapped onto another collinear triple B_1, B_2, B_3. Because (1 2 3) and (1' 2' 3') are in an incomplete $(S - P)_3$-configuration, their image sets $(A_1\ A_2\ A_3)$ and $(B_1\ B_2\ B_3)$ form an incomplete P_2-configuration. But, the two lines (2 2') and $(P\ P_2)$ are in one plane by hypothesis, therefore, the three points $A_2, B_2, C_2 = (A_1B_3) \cap (B_1A_3)$ are on the line of intersection of the plane $(2\ 2'\ P\ P_2)$ with π. By the first minor theorem of Pappus, the minor P_3-configuration is in fact complete. Hence, $C_1 = (A_2B_3) \cap (B_2A_3)$, $C_2 = (A_1B_3) \cap (B_1A_3)$, $C_3 = (A_1B_2) \cap (B_1A_2)$ are on one line. This line determines with P a plane in which the three lines L_1, L_2, L_3, clearly lie.

B. Conversely, let $A_1A_2A_3$, $B_1B_2B_3$, $C_1C_2C_3$ form an incomplete P_2-configuration with the additional incidence $(A_2B_2C_2)$. We shall prove that $(C_1C_3C_3)$ are dependent.

Let P be any point in the space not incident to the plane of the given configuration. The point P, with the line of the triple $(A_1A_2A_3)$, determines a plane in which the three lines PA_1, PA_2, PA_3 lie. Let L by any line in this plane cutting these three lines in 1, 2, 3 respectively.

Similarly, in the plane determined by P and the line $(B_1B_2B_3)$ we obtain a line L' on which the three points 1', 2', 3' lie. Now, the line $(P\,C_1)$ cuts both (2 3') and (2' 3); the line $(P\,C_3)$ cuts (1 2') and (1' 2) and $(P\,C_3)$ cuts (1 3') and (1' 3). Since, $(A_2B_2C_2)$ are on a line, the lines (2 2') and $(P\,C_2)$ are in the plane determined by P and the line $(A_2B_2C_2)$. Thus, we have constructed a small $(S-P)_3$-configuration, and by the "small" $(S-P)_3$-proposition, the configuration is complete. This means that the three lines $(P\,C_1)$, $(P\,C_2)$, $(P\,C_3)$ are in one plane. This plane cuts π in a line, and since C_1, C_2, C_3 are common to the two planes, they are on the line. Corollary V. 2 is proved.

2. The Pappus Theorem in Four-and Five-dimensions.

The extension of the Pappus theorem, given in the previous section, was based on the fact, which depends on the axioms of incidence only, that two skew-lines are incident with a unique line from an arbitrary point not incident to either one of them. We shall generalize this fact to n dimensions in

Lemma V. 1.

Let S_n be a projective space over a division ring. Given two lines L_1, L_2 and an (n-3)-space α, in general position in S_n, there exists a unique line L cutting L_1, L_2 and α.

Proof:

L_1 and L_2 are in general position and therefore determine a three-space β which intersects α in a point P. Since the point P in β the lemma follows.

The Lemma V. 1, in the case n= 4, says that three lines in general position have a unique line cutting them. This leads to the following generalization of theorem V. 1, and hence of the P_2-proposition to four dimensions.

Theorem V.4

Let S_4 be defined over a division ring R. Let θ and θ' be two skew-planes in S_4. Consider two hyperplanes α and α', containing θ and θ' respectively, intersecting in a plane θ''. Let 1, 2, 3; 1', 2', 3' be two triples of points on θ and θ' respectively. Let L be any line lying in θ''. The line L, taken with each pair of the three pairs of lines, $\{1\,2', 1'\,2\}$, $\{1\,3', 1'\,3\}$, $\{2\,3', 2'3\}$, determines three lines L_3, L_2, L_1 respectively. Then L_1, L_2, L_3 lie in the same hyperplane if and only if R is commutative.

Proof:

A. Assume R to be a field. Since L is in one 3-space with θ and another 3-space with θ', it intersects θ in a point 4 and θ' in a point 4'. We may choose a coordinate system so that the following points have the corresponding coordinates:

4':	1	0	0	0	0
2':	0	1	0	0	0
3':	0	0	1	0	0
4:	0	0	0	1	0
3:	0	0	0	0	1
2:	1	1	1	1	1
1:	1	1	1	c	d
1':	1	a	b	0	0

where $a, b, c, d \neq 0$ are elements of R and $d \neq 1$. Each of the lines L_1, L_2, L_3 is determined by two points on it. To find such pair of points, we shall consider, as an example, the line L_1 common to the triple $\{2\,3', 2'3, L\}$. Denoting the points of intersections of the line L_1 with the three lines (2 3'), (2' 3), L by A_1, B_1, C_1 respectively, we have for $\lambda, \mu, \nu \neq 0$,

$$A_1 = (1,1,1,1,1) + \lambda(0,0,1,0,0) ,$$
$$B_1 = (0,0,0,0,1) + \mu(0,1,0,0,0) ,$$
$$C_1 = (1,0,0,0,0) + \nu(0,0,0,1,0) .$$

Since A_1, B_1, C_1 are collinear, $\exists\, k_1, k_2, k_3 \neq 0$ such that

$$k_1A_1 + k_2B_1 + k_3C_1 = 0 .$$

By straight-forward computation, we get $k_1 = -k_2 = -k_3$, $\lambda = -\mu = -\nu = -1$. Thus $A_1 = (1,1,0,1,1)$, $B_1 = (0,1,0,0,1)$, $C_1 = (1,0,0,1,0)$. By a similar calculation, one finds that three points $A_3 = (b, a - ad + bd, b, cb, db)$, $B_3 = (1-d + bd, a-ad + bd, b, bd, bd)$, $C_3 = (b-bd + d-1, 0, 0, bc-bd, 0)$ are the points where the line L_3 meets the triple $\{1\ 2', 1'\ 2, L\}$ respectively. Because L is also determined by C_1 and C_3, it is sufficient to find one other point of the three points A_2, B_2, C_2 where the line L_2 meets the triple $\{1\ 3',\ 1'3,\ L\}$. A simple calculation shows that the point $B_2 = (1, a, b, 0, ad)$ is on both of the lines L_2 and $(1'\ 3)$. Finally, for the three lines L_1, L_2, L_3 to be in one hyperplane, it is necessary and sufficient that the following determinant must vanish:

B_1:	0	1	0	0	1
C_1:	1	0	0	1	0
A_3:	b	a-ad+bd	b	cb	db
C_3:	b+d-bd-1	0	0	bc-bd	0
B_2:	1	a	b	0	ad

It is now a simple matter to check that, and hence the sufficiency part is proved.

B. Let R be a division ring. Let the $(S - P)_4$-proposition hold. Given any two "numbers" $b, d \neq 0, 1$ in R, we let $1, 2, 3; 1', 2', 3'; 4, 4'$ be denoted by the notation given in part A, where $a, c \neq 0, 1$ are any two other elements of R. We find the five points B_1, C_1, A_3, C_3, B_2 with the coordinates given above. One may prove, without using commutativity,

that we have an incomplete $(S - P)_4$-configuration where the dependence of the set $(B_1C_1A_3C_3B_2)$ is hypothetically missing. In fact, we have computed the coordinates of these points, in part A, without using the commutativity of multiplication. Applying the $(S - P)_4$-proposition, the configuration must be closed, and hence the set $(B_1C_1A_3C_3B_2)$ is dependent. Now, projecting from the line (C_1C_3) or, what is the same thing, the line (4 4') into the plane $x_1 = 0$, $x_4 = 0$, i.e., the plane spanned by the points whose first and fourth coordinates vanish, we obtain the following representation for the images of A_3, B_1, B_2:

$$\begin{array}{ll} B_1': & \left(\begin{array}{ccc} 1 & 0 & 1 \\ a-ad+bd & b & db \\ a & b & ad \end{array} \right) . \\ A_3': & \\ B_2': & \end{array}$$

Because (B_1', A_3', B_2') lie in the space common to the hyperplane containing $(B_1C_1A_3C_3B_2)$ and the plane $x_1 = 0$, $x_4 = 0$, they must be collinear. This means a-a + ad - bd - 1 = ad - db - 1, yielding bd = db. Every pair in R, which are not trivially commutative, may be used to determine an incomplete $(S - P)_4$-configuration as we did with the pair b, d. Thus, Theorem V. 4 is completely proved.

There is a natural and direct connection between the above four-dimensional theorem and the two-dimensionsl theorem of Pappus which we exhibit as follows:

A. Let the P_2-proposition hold. Consider an incomplete $(S - P)_4$-configuration denoted by the same notation given in the statement of Theorem V. 4. Let π be any plane not intersection L.

Now, each point of the given incomplete $(S - P)_4$-configuration determines with L a plane intersecting π in a point. In this way the three points 1, 2, 3 are mapped onto three points A_1, A_2, A_3 respectively, in the plane π. Since the mapping planes pass through L, they are con-

tained in one 3-space which intersects π in a line ℓ. Hence, the three points A_1, A_2, A_3 are incident with ℓ. Similarly, the three points, 1', 2', 3' are mapped onto three points B_1, B_2, B_3 respectively, which are incident to one line ℓ'. The two lines ℓ and ℓ' together with the six points on them, form a configuration which may be extended naturally into an incomplete P_2-configuration. The three points $\{A_2B_3, B_2A_3\}$, $\{A_1B_3, B_1A_3\}$. $\{A_1B_2, B_1A_2\}$ are the points of intersections of the projecting planes, passing through the lines L_1, L_2, L_3, with π. Applying the P_2-proposition, these three points are contained in one line ℓ''. The lines $\{L, \ell''\}$ determine a 3-space containing the projecting planes passing through L_1, L_2, L_3. Hence, the 3-space, determined by the two lines L and ℓ'', contains L_1, L_2, L_3. This shows that the $(S - P)_4$-proposition is implied by the P_2-proposition.

B. Conversely, let the $(S - P)_4$-proposition hold. In some plane $\pi \subset S_4$, consider an incomplete P_2-configuration. That is to say, we have, in π, two lines ℓ and ℓ' with the two triples $(A_1A_2A_3)$ and $(B_1B_2B_3)$ on them respectively. These elements form a configuration which may be extended naturally to an incomplete P_2-configuration where the dependence of the cross-points C_1, C_2, C_3 is hypothetically missing. We shall prove the completeness of this plane configuration by using the $(S - P)_4$-proposition.

Consider a line L not in the same 3-space with π. The two lines L and ℓ determine a 3-space α. Each point of the triple $(A_1A_2A_3)$ forms, with L, a plane; and by taking a point on each of the resulting 3-planes, say 1, 2, 3, we obtain a plane θ spanned by (1, 2, 3). The line L and the plane θ are contained in α; hence L intersects θ in a point 4. By a similar construction, we get another plane $\theta' \subset \alpha'$ on which the triple of points (1', 2', 3') lies, where this triple has the same relation to the triple (B_1, B_2, B_3) as that of (1, 2, 3) to $(A_1A_2A_3)$. Join (1 2'), (1' 2),

(1 3'), (1' 3), (2 3'), (2' 3). The point C_1 forms with L a plane π_1 which is in the two 3-spaces {2 3', 4 4'}. Hence, π_1 intersects the two lines (2 3') and (2' 3) in two points whose joining line L_1 intersects the line L in a point since both lines are in π_1. In the same manner, the two points C_2 and C_3 determine with L the two planes π_2 and π_3 which yield the two lines L_2 and L_3. Thus we have obtained an incomplete $(S - P)_4$-configuration. By the $(S - P)_4$-proposition, the three lines L_1, L_2, L_3 are contained in one 3-space γ which intersects π in a line ℓ''. Because the three points C_1, C_2, C_3 are in both π and γ, they are in ℓ''. Hence, the P_2-configuration is complete.

Summarizing our results, we have proved

Theorem V. 5.

In the presence of the axioms of incidence, the P_2-proposition is equivalent to the $(S - P)_4$-proposition.

Now the problem of pushing the present extension to n-dimensions presents itself. From both formulations of the $(S - P)_3$- and the $(S - P)_4$-propositions, a formulation of a possible $(S - P)_n$-proposition would involve in the conclusion a dependence of three lines. But, three lines, in general position, in n-space, always span some five-space, since they are determined by six points in general position. Thus for $n > 5$, such an extension holds trivially. However, for $n = 5$, we prove

Theorem V. 6.

Let S_5 be a projective 5-space (defined by the propositions of incidence, i.e., defined over a division ring.) Let θ and θ' be two planes not in the same hyperlane. Let α and α' be two hyperplanes with $\alpha \supset \theta$ and $\alpha' \supset \theta'$. In each of θ and θ', three points are given, 1 2 3 and 1' 2' 3' respectively. The two hyperplanes α and α' intersect in a 3-space β. Let π be any plane in β. The plane π taken with each of the three pairs of lines {1 2', 1' 2}, {1 3', 1' 3}, {2 3', 2' 3}, (using Lemma V. 1),

yields three lines L_3, L_2, L_1 each of which intersects the plane π and one of the pairs. Then, the three lines L_1, L_2, L_3 are in the same hyperplane if and only if the P_2-proposition holds.

Proof:

A. Sufficiency:

Consider an arbitrary plane $\gamma \subset S_5$ such that γ and π are not in the same hyperplane. Each point of the given incomplete $(S - P)_5$-configuration determines with π a 3-space intersecting γ in a point. Under this mapping, the triple (1, 2, 3) is mapped into a triple (A_1, A_2, A_3) in γ. Because the two planes θ and π are contained in α, the projecting 3-spaces are contained in α also. Hence, the hyperplane α intersects γ in a line ℓ, and consequently, the triple (A_1, A_2, A_3) is contained in ℓ. By a similar argument, the triple (1', 2', 3') is mapped onto a triple (B_1, B_2, B_3) lying on a line $\ell' \subset \gamma$. The resulting plane configuration in γ, may be extended naturally into an incomplete P_2-configuration where the collinearity of the cross-points C_1, C_2, C_3 is hypothetically missing. By the P_2-proposition the triple (C_1, C_2, C_3) is on a line $\ell'' \subset \gamma$. The line ℓ'' determines with π a hyperplane α''. The three projecting 3-spaces which pass through π and the lines L_1, L_2, L_3 intersect γ in C_1, C_2, C_3. Hence, they are contained in the hyperplane α'', and consequently, the lines L_1, L_2, L_3 are in α''.

B. Necessity:

Consider, in a plane $\gamma \subset S_5$, an incomplete P_2-configuration. That is to say, we have the two lines ℓ and ℓ' with the two triples $(A_1 A_2 A_3)$ and $(B_1 B_2 B_3)$ on them respectively. Let

$$C_1 = A_2B_3 \cap B_2A_3,\ C_2 = A_1B_3 \cap B_1A_3,\ C_3 = A_1B_2 \cap B_1A_2.$$

We shall prove that C_1, C_2, C_3 are collinear by using the $(S - P)_5$-proposition.

Let π be an arbitrary plane in S_5 such that π and γ are not contained in one hyperplane. Let α and α' be the two distinct hyperplanes determined by π and each of ℓ and ℓ' respectively. The plane π determines, with each point of the triple $(A_1A_2A_3)$, a 3-space. We take a point on each 3-space so that the resulting 3 points, say 1, 2, 3, span a plane $\theta \subset \alpha$. In a similar way, we obtain the triple 1', 2', 3', corresponding to B_1, B_2, B_3, spanning the plane $\theta \subset \alpha'$. Join (1 2'), (1' 2), (1 3'), (13) (2 3'), (2' 3). The point C_1 determines with π a 3-space π_1 contained in the two 4-spaces $\{2\ 3',\ \pi\}$ and $\{2'3\ \pi\}$. Hence π_1 intersects (2 3') and (2' 3) in two points whose joining line L_1 intersects π also. We, similarly, obtain L_2 and L_3 intersecting $\{1\ 3',\ 1'\ 3,\ \pi\}$ and $\{1\ 2',\ 1'\ 2,\ \pi\}$ respectively. Thus we have built up an incomplete $(S - P)_5$-configuration. Using the $(S - P)_5$-proposition, the three lines L_1, L_2, L_3 are contained in one 4-space α''. The hyperplane α'' intersects γ in a line ℓ''. Since the triple (C_1, C_2, C_3) is contained in α'', it must be incident with ℓ''. This means the P_2-configuration is complete, and therefore Theorem V.6 follows.

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