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The spectrum of a twisted commutative algebra

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Abstract

A twisted commutative algebra is (for us) a commutative **Q**-algebra equipped with an action of the infinite general linear group. In such algebras, the "**GL**-prime" ideals assume the duties fulfilled by prime ideals in ordinary commutative algebra, and so it is crucial to understand them. Unfortunately, distinct **GL**-primes can have the same radical, which obstructs one from studying them geometrically. We show that this problem can be eliminated by working with super vector spaces: doing so provides enough geometry to distinguish **GL**-primes. This yields an effective method for analyzing **GL**-primes.

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1 | INTRODUCTION

A *twisted commutative algebra* (tca) is a commutative **Q**-algebra equipped with an action of the infinite general linear group $\mathbf{GL}_\infty$ under which it forms a polynomial representation; at least, that will be our definition for the moment. tcas have been effectively used to study asymptotic

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problems in algebra (see, for example, [5, 9, 11, 26]), and are closely related to many particular objects of interest (such as EFW complexes [10], determinantal varieties, and representations of infinite rank groups [21]); moreover, all evidence so far points to a rich internal theory. It is therefore sensible to study these objects in more detail. Although there have been many successes for particular tcas [5, 13, 16, 17, 22, 24, 25], there has really only been one significant result to date for general tcas, namely, Draisma's topological Noetherianity theorem [8]. In this paper, we take another step toward understanding the general case: we largely solve the problem of understanding the equivariant prime ideals of tcas.

1.1 | Equivariant commutative algebra

Let A be a tca. One can then formulate equivariant analogs of many familiar concepts from commutative algebra.[†]

- A **GL**-*ideal* of A is an ideal that is **GL**-stable.
- A **GL**-*prime* is a **GL**-ideal $\mathfrak{p}$ such that $VW \subset \mathfrak{p}$ implies $V \subset \mathfrak{p}$ or $W \subset \mathfrak{p}$, for subrepresentations $V, W \subset A$. Here VW denotes the image of the map $V \otimes W \to A$.
- The **GL**-*radical* of a **GL**-ideal I, denoted $\mathrm{rad}_{\mathbf{GL}} I$, is the sum of all subrepresentations V of A such that $V^n \subset I$ for some n. Here V^n denotes the image of the map $V^{\otimes n} \to A$. This is equal to the intersection of the **GL**-primes containing I (Proposition 2.8).
- The **GL**-*spectrum* of A, denoted $\mathrm{Spec}_{\mathbf{GL}}(A)$, is the set of **GL**-primes, endowed with the usual Zariski topology. The closed subsets of $\mathrm{Spec}_{\mathbf{GL}}(A)$ correspond bijectively to **GL**-radical ideals.

One can keep going, but this is all we need for the moment.

In ordinary commutative algebra, prime ideals are of central importance; this is no less true of **GL**-primes in tca. For instance, the support of an equivariant module is most naturally a subset of the **GL**-spectrum; the **GL**-primes can be used to generate the Grothendieck group of equivariant modules; and, under suitable hypotheses, one has an equivariant version of primary decomposition for **GL**-ideals. Therefore, to understand tcas, it is of crucial importance to understand their **GL**-primes.

Every **GL**-stable prime ideal of a tca is a **GL**-prime. However, the converse is not true. The following is an instructive example.

Example 1.1. Let $A = \bigoplus_{n \geqslant 0} \bigwedge^{2n}(\mathbf{Q}^\infty)$. This is a commutative subalgebra of the exterior algebra on $\mathbf{Q}^\infty$; note that even degree homogeneous elements of the exterior algebra are central. The group **GL** naturally acts on A, and this gives A the structure of a tca. As a tca, A is quite nice. For example, it is finitely generated in the equivariant sense: the **GL**-orbit of any nonzero element in $\bigwedge^2(\mathbf{Q}^\infty)$ generates it as a **Q**-algebra.

Suppose that V and W are nonzero subrepresentations of A. As exterior powers are irreducible, it follows that V contains $\bigwedge^i(\mathbf{Q}^\infty)$ and W contains $\bigwedge^j(\mathbf{Q}^\infty)$ for some i and j. Thus, VW contains $\bigwedge^{i+j}(\mathbf{Q}^\infty)$. We have therefore shown that if V and W are nonzero then so is VW. It follows that the zero ideal of A is **GL**-prime; in other words, A is **GL**-integral.

This example is rather shocking when one first encounters it: every positive degree homogeneous element of A is nilpotent, and yet A is a **GL**-domain! This example shows that (from our

[†] In fact, one can do this for commutative algebras in any tensor category.

current perspective) tcas do not have enough points to "see" their **GL**-primes: indeed, $\mathrm{Spec}(A)$ is a single point, and thus cannot distinguish the two **GL**-primes (0) and A_+ of A. Thus, **GL**-domains appear to be divorced from geometry, which might diminish our hopes of understanding **GL**-primes; fortunately, however, this appearance is deceiving.

1.2 | The key principle

The category $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$ of polynomial representations of $\mathbf{GL}_\infty$ is equivalent to the category **Pol** of polynomial functors of rational vector spaces; the equivalence is obtained by evaluating a functor on $\mathbf{Q}^\infty$. We can thus view a tca A as an algebra object of **Pol**. From this perspective, $A(\mathbf{Q}^\infty)$ can be seen as the "incarnation" of A in the category $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$. However, polynomial functors can be evaluated on objects in any **Q**-linear tensor category. We can thus form the "super incarnation" $A(\mathbf{Q}^{\infty|\infty})$ of A by evaluating on the super vector space $\mathbf{Q}^{\infty|\infty}$. This is an algebra object of $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}_{\infty|\infty})$. We can now succinctly express the point of this paper:

> **Key principle.** *The geometry of the super incarnation of a tca is sufficiently rich to detect its* **GL***-primes.*

This principle is borne out in the theorems stated below.

Example 1.2. Let A be the tca from Example 1.1, regarded in **Pol**. Then $A(\mathbf{Q}^{0|s}) = \bigoplus_{n\geqslant 0} \mathrm{Sym}^{2n}(\mathbf{Q}^s)$ is the second Veronese subring of $\mathrm{Sym}(\mathbf{Q}^s)$, which is a domain of Krull dimension s; in particular, its spectrum has plenty of points. For $r > 0$, the algebra $A(\mathbf{Q}^{r|s})$ is a nilpotent extension of $A(\mathbf{Q}^{0|s})$.

1.3 | Main results

We now state several precise theorems. In what follows, A is a tca (considered as a polynomial functor) and I and J are **GL**-ideals of A.

Theorem A. *We have* $\mathrm{rad}_{\mathbf{GL}} I \subset \mathrm{rad}_{\mathbf{GL}} J$ *if and only if* $\mathrm{rad}\, I(\mathbf{Q}^{\infty|\infty}) \subset \mathrm{rad}\, J(\mathbf{Q}^{\infty|\infty})$.

In other words, the theorem says that $\mathrm{rad}_{\mathbf{GL}} I \subset \mathrm{rad}_{\mathbf{GL}} J$ if and only if $\mathcal{V}(J(\mathbf{Q}^{\infty|\infty})) \subset \mathcal{V}(I(\mathbf{Q}^{\infty|\infty}))$; the latter condition is equivalent to $\mathcal{V}(J(\mathbf{Q}^{r|s})) \subset \mathcal{V}(I(\mathbf{Q}^{r|s}))$ for all r and s and thus (usually) reduces to a condition about finite-dimensional algebraic varieties. Note that as we are only concerned with vanishing loci here, we can pass to the reduced quotient of $A(\mathbf{Q}^{r|s})$, which is an ordinary (nonsuper) commutative ring. We emphasize that the theorem is false if one uses only ordinary vector spaces, as Example 1.1 shows.

Theorem B. *The ideal* $\mathrm{rad}_{\mathbf{GL}} I$ *is* **GL***-prime if and only if the ideal* $\mathrm{rad}\, I(\mathbf{Q}^{\infty|\infty})$ *is prime.*

Once again, we can verify the latter condition on finite-dimensional spaces. The theorem therefore reduces the problem of showing that $\mathrm{rad}_{\mathbf{GL}}(I)$ is **GL**-prime to showing that the algebraic varieties $\mathcal{V}(I(\mathbf{Q}^{r|s}))$ are irreducible.

In the two remaining theorems, we require a finiteness condition: we assume that A_0 is Noetherian and that A is finitely generated over A_0.

Theorem C. *We have the following.*

(a) *A has finitely many minimal* **GL***-primes, say $\mathfrak{p}_1, \dots, \mathfrak{p}_n$.*
(b) *$A(\mathbf{Q}^{\infty|\infty})$ has finitely many minimal primes, say $\mathfrak{q}_1, \dots, \mathfrak{q}_m$.*
(c) *$n = m$, and after applying a permutation we have $\mathfrak{q}_i = \operatorname{rad} \mathfrak{p}_i(\mathbf{Q}^{\infty|\infty})$ for all i.*

This theorem gives a useful way to find the minimal **GL**-primes, at least up to **GL**-radical, and thus the irreducible components of the **GL**-spectrum.

Theorem D. *The* **GL***-spectrum $\operatorname{Spec}_{\mathbf{GL}}(A)$ of A is a Noetherian topological space.*

This theorem is a strengthening of Draisma's topological Noetherianity theorem: indeed, Draisma's theorem only encompasses the **GL**-stable prime ideals, while this theorem accommodates all **GL**-primes. In fact, this theorem is easily deduced by combining Draisma's theorem with our other theorems; we do not have a new proof of Draisma's result.

1.4 | An example

In Section 5, we examine the tca $A = \operatorname{Sym}(\operatorname{Sym}^2(\mathbf{C}^\infty))$. Using our main theorems, we classify the **GL**-primes of A (they are the rectangular ideals) and the **GL**-radical ideals of A (they are the ideals generated by a single irreducible representation). Rectangular ideals have long been of interest (see, e.g., [3, 20]), and our results provide an elegant canonical description of this class. This example provides a good illustration of how our main theorems allow one to understand the equivariant commutative algebra of A through standard geometric means.

1.5 | Connection to other work

The notions of **GL**-prime and the **GL**-spectrum were discussed in [24, section 3] (with slightly different terminology). However, that paper only works with bounded tcas, and for these **GL**-primes are the same as **GL**-stable primes, so the chief difficulties disappear. (Note that the tca in Example 1.1 is not bounded. A tca A is *bounded* if there is some N such that if the Schur functor $\mathbf{S}_\lambda$ appears in A then λ has at most N rows. The exterior power $\bigwedge^n$ is the Schur functor $\mathbf{S}_{(1^n)}$, and the partition (1^n) has n rows. See Section 2 for background on Schur functors.)

In joint work with Rohit Nagpal [18, 19], we study the $\mathfrak{S}_\infty$-primes in the polynomial ring $\mathbf{C}[x_1, x_2, \dots]$ and manage to completely classify them. (Here $\mathfrak{S}_\infty$ denotes the infinite symmetric group.) However, there is no theory of $\mathfrak{S}_\infty$-primes for general $\mathfrak{S}_\infty$-algebras yet.

One can, and usually does, define tcas without any reference to super vector spaces. However, our results show that one is essentially forced to use super vector spaces to fully understand tcas. This is reminiscent of Deligne's theorem [7, Theorem 0.6], in which a natural class of tensor categories (defined without any reference to super vector spaces) is characterized using super objects. It would be interesting to find a direct connection between these results.

1.6 | Outline

In Section 2, we review relevant background material. In Section 3, we prove the key theorem, which is a certain special case of Theorem A. In Section 4, we deduce the main theorems stated above. Finally, in Section 5, we work out the $\mathrm{Sym}(\mathrm{Sym}^2)$ case in detail.

2 | BACKGROUND

2.1 | Super algebra

We briefly review some basic concepts from supermathematics. See [15, chapters 3 and 4] for more details.

Let K be a field of characteristic 0. A *super vector space* is a K-vector space V equipped with a $\mathbf{Z}/2$-grading $V = V_0 \oplus V_1$. For a homogeneous element $x \in V$, we write $|x| \in \mathbf{Z}/2$ for its degree. We write $K^{n|m}$ for the super vector space with even part K^n and odd part K^m. If V and W are super vector spaces, then $V \otimes W$ has the structure of a super vector space: if $x \in V$ and $y \in W$ are homogeneous then $x \otimes y$ is homogeneous of degree $|x| + |y|$.

A *commutative superalgebra* is an associative unital $\mathbf{Z}/2$-graded K-algebra A such that

$$xy = (-1)^{|x|\cdot|y|} \cdot yx$$

holds for all homogeneous elements $x, y \in A$. This relation implies that even degree elements are central, and odd degree elements square to 0. If I denotes the ideal of A generated by odd degree homogeneous elements, then A/I is concentrated in even degree. This shows that a general commutative superalgebra is a (locally) nilpotent extension of an ordinary commutative algebra.

Given $n, m \in \mathbf{N}$, there is a free commutative superalgbera $K[x_1, \dots, x_n, y_1, \dots, y_m]$ where the x's have degree 0 and the y's have degree 1. This algebra decomposes as $K[x_1, \dots, x_n] \otimes K[y_1, \dots, y_n]$. The first piece $K[x_1, \dots, x_n]$ is the ordinary polynomial ring on n generators, while the second piece $K[y_1, \dots, y_m]$ is the exterior algebra on m generators. Any finitely generated commutative superalgebra is a quotient of some $K[x_1, \dots, x_n, y_1, \dots, y_m]$.

Suppose A is a commutative superalgebra, and let I be the ideal generated by odd degree elements. Define $\mathrm{Spec}(A)$ to be the topological space $\mathrm{Spec}(A/I)$. This space is equipped with a a sheaf of commutative superalgebras derived from A. An *affine superscheme* is a locally superringed space isomorphic to $\mathrm{Spec}(A)$ for some commutative superalgebra A. One can define general superschemes by glueing affine ones, but we will not need them. We note that if X is a superscheme then its reduced subscheme X_{red} is an ordinary scheme. Thus, as with algebras, a superscheme can be viewed as a nilpotent thickening of an ordinary scheme.

An *affine supergroup* is a group object in the category of affine superschemes. The first example, and the only one we will use, is $\mathbf{GL}_{r|s}$. This can be defined by its functor of points: if T is a commutative superalgebra then $\mathbf{GL}_{r|s}(T)$ is the group of automorphisms of the T-module $T \otimes K^{r|s}$; such automorphisms are required to respect the grading. If T is concentrated in degree 0, then such an automorphism preserves $T \otimes K^r$ and $T \otimes K^s$; it follows that the reduced subgroup of $\mathbf{GL}_{r|s}$ is simply $\mathbf{GL}_r \times \mathbf{GL}_s$. In other words, $\mathbf{GL}_{r|s}$ is a nilpotent thickening of the familiar group $\mathbf{GL}_r \times \mathbf{GL}_s$. We can also understand $\mathbf{GL}_{r|s}$ in terms of its coordinate ring. Consider the $(r+s) \times (r+s)$ matrix

of indeterminates

$$\left(\begin{array}{ccc|ccc} x_{1,1} & \cdots & x_{1,n} & y_{1,n+1} & \cdots & y_{1,n+m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} & y_{n,n+1} & \cdots & y_{n,n+m} \\ \hline y_{n+1,1} & \cdots & y_{n+1,n} & x_{n+1,n+1} & \cdots & x_{n+1,n+m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ y_{n+m,1} & \cdots & y_{n+m,n} & x_{n+m,n+1} & \cdots & x_{n+m,n+m} \end{array}\right)$$

Let A be the free commutative superalgebra on these variables, where the x's have degree 0 and the y's have degree 1. Let $\delta_0, \delta_1 \in A_0$ be the determinants of the top left and bottom right blocks. Then the coordinate ring $\mathbf{Q}[\mathbf{GL}_{r|s}]$ of $\mathbf{GL}_{r|s}$ is $A[\delta_0^{-1}, \delta_1^{-1}]$. The group structure on $\mathbf{GL}_{r|s}$ induces a comultiplication and antipode on $\mathbf{Q}[\mathbf{GL}_{r|s}]$, which makes it a Hopf superalgebra.

Let SVec be the category super vector spaces; morphisms are linear maps preserving the grading. This is a monoidal category via the tensor product. In fact, it is naturally symmetric monoidal: if V and W are super vector spaces, the symmetry isomorphism $V \otimes W \to W \otimes V$ is defined by

$$x \otimes y \mapsto (-1)^{|x| \cdot |y|} y \otimes x$$

on homogeneous elements $x \in V$ and $y \in W$. Many algebra concepts, like "commutative algebra" or "module" or "Hopf algebra" make sense in any symmetric monoidal category. Our definition of commutative superalgebra is simply this general categorical definition in the setting of SVec.

2.2 | Polynomial representations

We recall some background material on polynomial representations. We refer to [23] and [17, section 2.2] for more details.

Regard $\mathbf{GL}_n$ as an algebraic group over $\mathbf{Q}$. By an algebraic representation of $\mathbf{GL}_n$, we mean a comodule over $\mathbf{Q}[\mathbf{GL}_n]$; the dimension is not required to be finite. We regard $\mathbf{GL}_\infty$ as the inductive system of algebraic groups $\bigcup_{n \geqslant 1} \mathbf{GL}_n$. An algebraic representation of $\mathbf{GL}_\infty$ is a vector space equipped with compatible algebraic representations of $\mathbf{GL}_n$ for all n. The basic example of such a representation is the standard representation $\mathbf{Q}^\infty = \bigcup_{n \geqslant 1} \mathbf{Q}^n$.

A *polynomial representation* of $\mathbf{GL}_\infty$ is one that occurs as a subquotient of a (possibly infinite) direct sum of tensor powers of the standard representation. We denote the category of such representations by $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$. It is clearly abelian and closed under tensor products. Recall that for a partition λ of n there is an associated irreducible representation S^λ of the symmetric group $\mathfrak{S}_n$, called the *Specht module* [12, chapter 4]. For any vector space V, the group $\mathfrak{S}_n$ acts on $V^{\otimes n}$ by permuting tensor factors, and we define $\mathbf{S}_\lambda(V)$ to be the S^λ-isotypic piece of $V^{\otimes n}$; the construction $\mathbf{S}_\lambda$ is called the *Schur functor*. There is a natural decomposition

$$V^{\otimes n} = \bigoplus_{\lambda \vdash n} S^\lambda \otimes \mathbf{S}_\lambda(V), \tag{2.1}$$

that is compatible with the $\mathfrak{S}_n$ and $\mathbf{GL}(V)$ actions on each side, see [12, Theorem 6.3]. As $\mathbf{S}_\lambda(\mathbf{Q}^N)$ is an irreducible representation of $\mathbf{GL}_N$ for N sufficiently large [12, Theorem 6.3], it follows that $\mathbf{S}_\lambda(\mathbf{Q}^\infty)$ is an irreducible representation of $\mathbf{GL}_\infty$. From (2.1), we thus see that $(\mathbf{Q}^\infty)^{\otimes n}$ is a semisimple representation. It therefore follows that $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$ is a semisimple abelian category, with the simple objects being the $\mathbf{S}_\lambda(\mathbf{Q}^\infty)$. In particular, we see that every polynomial representation V admits a canonical decomposition

$$V = \bigoplus_\lambda V_\lambda \otimes \mathbf{S}_\lambda(\mathbf{Q}^\infty)$$

where the V_λ are multiplicity spaces. We endow V with a grading by declaring the elements of $\mathbf{S}_\lambda(\mathbf{Q}^\infty)$ to be homogeneous of degree $|\lambda|$.

By an algebraic representation of $\mathbf{GL}_{r|s}$, we mean a comodule over the Hopf superalgebra $\mathbf{Q}[\mathbf{GL}_{r|s}]$. We then define polynomial representations of $\mathbf{GL}_{\infty|\infty}$ just as before, and denote the category by $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}_{\infty|\infty})$. It is again abelian and closed under tensor products. For a super vector space V, the group $\mathfrak{S}_n$ once again acts on $V^{\otimes n}$ by permuting tensor factors (and this action makes use of the symmetry isomorphism in SVec discussed at the end of Section 2.1), and we again define $\mathbf{S}_\lambda(V)$ as the S^λ-isotypic piece of $V^{\otimes n}$. For example, we have

$$\mathbf{S}_{(2)}(\mathbf{Q}^{r|s}) = \mathrm{Sym}^2(\mathbf{Q}^{r|s}) = \mathrm{Sym}^2(\mathbf{Q}^r) \oplus (\mathbf{Q}^r \otimes \mathbf{Q}^s)[1] \oplus \textstyle\bigwedge^2(\mathbf{Q}^s)[s],$$

where $[n]$ denotes a shift in super degree; this is the degree two part of the free commutative superalgebra on r even variables and s odd variables. The representation $\mathbf{S}_\lambda(\mathbf{Q}^{r|s})$ of $\mathbf{GL}_{r|s}$ is irreducible if r and s are sufficiently large compared to λ (see Proposition 2.3), and so it follows that $\mathbf{S}_\lambda(\mathbf{Q}^{\infty|\infty})$ is an irreducible representation of $\mathbf{GL}_{\infty|\infty}$. The identity (2.1) is still valid when V is a super vector space, and so we find that $(\mathbf{Q}^{\infty|\infty})^{\otimes n}$ is a semisimple representation. We therefore see that $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}_{\infty|\infty})$ is semisimple, with simple objects $\mathbf{S}_\lambda(\mathbf{Q}^{\infty|\infty})$. Warning: if V is a polynomial representation of $\mathbf{GL}_{\infty|\infty}$ then the shift $V[1]$ is typically not a polynomial representation, according to our definition. This disagrees with the convention used in [17].

Remark 2.2. Suppose V is an algebraic representation of $\mathbf{GL}_{r|s}$. A subspace W of V is a subrepresentation (i.e., a sub-comodule) if and only if for every commutative superalgebra T the space $T \otimes W$ is stable under the action of $\mathbf{GL}_{r|s}(T)$. One cannot simply check this condition for $T = \mathbf{Q}$ because $\mathbf{GL}_{r|s}$ is nonreduced.

For example, consider the representation $V = \mathrm{Sym}^2(\mathbf{Q}^{1|1})$ of $\mathbf{GL}_{1|1}$. Let x and y be a basis of $\mathbf{Q}^{1|1}$, with x even and y odd. Then x^2 and xy are a basis for V. The basis vectors are both eigenvectors for $\mathbf{GL}_{1|1}(\mathbf{Q}) = \mathbf{Q}^\times \times \mathbf{Q}^\times$. Consider the commutative superalgebra $T = \mathbf{Q}[\epsilon]$, where $|\epsilon| = 1$. Then $\mathbf{GL}_{1|1}(T)$ contains the matrix

$$\begin{pmatrix} 1 & \epsilon \\ 0 & 1 \end{pmatrix}.$$

This matrix takes xy to $\epsilon x^2 + xy$. It follows that the subrepresentation of V generated by xy must contain x^2, and so xy generates V. (Note: the subrepresentation generated by an element is the intersection of all subrepresentations containing that element.) Similar reasoning shows that x^2 generates V, which proves that V is an irreducible representation of $\mathbf{GL}_{1|1}$.

Consider the category Fun(Vec, Vec) of endofunctors of the category Vec of $\mathbf{Q}$-vector spaces. Let T_n be the functor given by $T_n(V) = V^{\otimes n}$. We say that an object of Fun(Vec, Vec) is *polynomial* if it is a subquotient of a (possibly infinite) direct sum of T_n's. We let **Pol** be the category of polynomial functors. It is abelian and closed under tensor products. The analog of (2.1) once again holds: we have a decomposition

$$T_n = \bigoplus_{\lambda \vdash n} S^\lambda \otimes \mathbf{S}_\lambda,$$

where $\mathbf{S}_\lambda$ is the Schur functor (regarded as an object of **Pol**). As $\mathbf{S}_\lambda(\mathbf{Q}^N)$ is an irreducible $\mathbf{GL}_N$-representation for N large, one can deduce that $\mathbf{S}_\lambda$ is a simple object of **Pol**. Thus, T_n is a semisimple object of **Pol**. It follows that **Pol** is a semisimple category with simple objects the $\mathbf{S}_\lambda$. The natural functors

$$\begin{array}{ccccc} \mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty) & \longleftarrow & \mathbf{Pol} & \longrightarrow & \mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}_{\infty\mid\infty}) \\ F(\mathbf{Q}^\infty) & \longleftarrow\!\shortmid & F & \shortmid\!\longrightarrow & F(\mathbf{Q}^{\infty\mid\infty}) \end{array}$$

are equivalences, as they induce bijections on the isomorphism classes of simple objects. These equivalences are compatible with tensor products, and respect algebras, modules, and ideals within the categories.

We will at times need to evaluate polynomial functors on finite-dimensional spaces. We recall the relevant result.

Proposition 2.3. *Let λ be a partition, and let $r, s \in \mathbf{N}$. If $\lambda_{r+1} \leqslant s$ then $\mathbf{S}_\lambda(\mathbf{Q}^{r\mid s})$ is an irreducible representation of $\mathbf{GL}_{r\mid s}$; otherwise it vanishes. Moreover, the irreducible representations of $\mathbf{GL}_{r\mid s}$ obtained in this way are mutually nonisomorphic.*

Proof. See [4, section 3.2.2]. □

We will require one additional simple result on polynomial representations. Define the *width* of a partition λ, denote $w(\lambda)$, to be λ_1. Define the *width* of a polynomial functor F, denoted $w(F)$, to be the supremum of $w(\lambda)$ over those λ for which $\mathbf{S}_\lambda$ occurs in F.

Proposition 2.4. *Let F be a polynomial functor. The following are equivalent.*

(a) $w(F) \leqslant N$.
(b) *For every n, every weight μ appearing in $F(\mathbf{Q}^n)$ satisfies $\mu_1 \leqslant N$.*

Proof. It suffices to treat the case where $F = \mathbf{S}_\lambda$ is a simple object; we then have $w(F) = \lambda_1$. For $n \gg 0$, the $\mathbf{GL}_n$ representation $F(\mathbf{Q}^n)$ is irreducible with highest weight λ. Thus, there is a weight μ in $F(\mathbf{Q}^n)$ with $\mu_1 = w(F)$ (namely, $\mu = \lambda$). Furthermore, as λ is a highest weight, we have $\mu_1 \leqslant \lambda_1 = w(F)$ for any other weight μ. This proves the result. □

Remark 2.5. We note that all the irreducible representations discussed above, such as S^λ or $\mathbf{S}_\lambda(\mathbf{Q}^N)$ with $N \gg 0$, are in fact absolutely irreducible, that is, they remain irreducible when one extends scalars to any extension field of $\mathbf{Q}$.

2.3 | Twisted commutative algebras

A tca is a commutative algebra object in one of the three equivalent categories $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$, $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}_{\infty|\infty})$, or **Pol**. For the moment, we work in $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$ to be definite.

Let A be a tca in $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$. Every polynomial representation carries a natural grading, and so A is canonically graded. This is compatible with the ring structure, that is, A is a graded ring. In particular, we can regard A as an algebra over is degree 0 piece A_0, which is an ordinary commutative ring.

By a "subrepresentation" of A, we mean a $\mathbf{Q}$-subspace that is stable by $\mathbf{GL}_\infty$. In practice, A will often be a $\mathbf{C}$-algebra, but we use $\mathbf{Q}$-subrepresentations nonetheless. One should think of a finite length subrepresentations of A as providing an equivariant substitute for the concept of element (or perhaps finite sets of elements).

Suppose $A \to B$ is a homomorphism of tcas. We say that B is *finitely* **GL**-*generated* over A if there is some finite length subrepresentation E of B such that the natural map $A \otimes \mathrm{Sym}(E) \to B$ is surjective. We typically apply this in the case $A = B_0$.

2.4 | Ideals in tcas

Let A be as above, that is, a tca in $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$. A **GL**-*ideal* of A is a **GL**-stable ideal of A. We say that a **GL**-ideal I is *finitely* **GL**-*generated* if there is a finite length subrepresentation E of A that generates I as an ideal. The sum and product of two finitely **GL**-generated ideals is again finitely **GL**-generated; for products, this relies on the fact that the tensor product of two finite length polynomial representations is again finite length.

We say that a **GL**-ideal $\mathfrak{p}$ is **GL**-*prime* if $VW \subset \mathfrak{p}$ implies $V \subset \mathfrak{p}$ or $W \subset \mathfrak{p}$, for subrepresentations V and W of A; here VW denotes the image of the map $V \otimes W \to A$. It is equivalent to ask the same condition with V and W finite length representations, or cyclic representations, or **GL**-ideals, or finitely generated **GL**-ideals. We define the **GL**-*radical* of a **GL**-ideal I, denoted $\mathrm{rad}_{\mathbf{GL}} I$ to be the sum of all subrepresentations V or A such that $V^n \subset I$ for some n; again, one can use ideals in place of subrepresentations. We say that I is **GL**-*radical* if $I = \mathrm{rad}_{\mathbf{GL}} I$. We note that every **GL**-prime is **GL**-radical.

Remark 2.6. A "prime **GL**-ideal" of A is a **GL**-ideal of A that is prime. This is potentially very different from a "**GL**-prime ideal" of A. Similarly, "radical **GL**-ideal" and "**GL**-radical ideal" are potentially very different.

We now establish some properties of the above definitions that are analogous to the classical situation.

Proposition 2.7. *Let I be a* **GL**-*ideal of A and let E be a finite length subrepresentation of* $\mathrm{rad}_{\mathbf{GL}}(I)$. *Then $E^n \subset I$ for some n. Similarly, if J is a finitely* **GL**-*generated ideal contained in* $\mathrm{rad}_{\mathbf{GL}}(I)$ *then $J^n \subset I$ for some n.*

Proof. By definition, we can write $\mathrm{rad}_{\mathbf{GL}}(I) = \sum_{i \in \mathcal{I}} W_i$ where W_i is a subrepresentation of A such that $W_i^{n(i)} \subset I$ for some $n(i)$. As E is contained in $\mathrm{rad}(I)$ and of finite length, there is some finite subset $\mathcal{J}$ of $\mathcal{I}$ such that $E \subset \sum_{i \in \mathcal{J}} W_i$. We thus have $E^n \subset I$ where $n = \#\mathcal{J} \cdot \max_{i \in \mathcal{J}} n(i)$. For the

ideal case, simply pick a finite length subrepresentation that generates and appeal to the previous argument. □

Proposition 2.8. *Let I be an ideal of A. Then* $\mathrm{rad}_{\mathbf{GL}}(I)$ *is the intersection of the* **GL***-primes containing I.*

Proof. Let $\mathcal{P}$ be the set of **GL**-primes containing I. Suppose $\mathfrak{p} \in \mathcal{P}$. Let $\mathfrak{a}$ be a finitely **GL**-generated **GL**-ideal contained in $\mathrm{rad}_{\mathbf{GL}}(I)$. Then $\mathfrak{a}^n \subset I \subset \mathfrak{p}$ for some n, and so $\mathfrak{a} \subset \mathfrak{p}$ because $\mathfrak{p}$ is **GL**-prime. As this holds for all $\mathfrak{a}$, we have $\mathrm{rad}_{\mathbf{GL}}(I) \subset \mathfrak{p}$. As this holds for all $\mathfrak{p}$, we have $\mathrm{rad}_{\mathbf{GL}}(I) \subset \bigcap_{\mathfrak{p}\in\mathcal{P}} \mathfrak{p}$.

We now prove the reverse inclusion. Let $\mathfrak{c}$ be a finitely **GL**-generated **GL**-ideal of A not contained in $\mathrm{rad}(I)$. Let S be the set of **GL**-ideals $\mathfrak{a}$ of A such that no power of $\mathfrak{c}$ is contained in $\mathfrak{a}$. Suppose that $\mathfrak{a}_1 \subset \mathfrak{a}_2 \subset \cdots$ is a chain in S, and let $\mathfrak{a}$ be its union. Then $\mathfrak{a}$ belongs to S too. Indeed, if $\mathfrak{c}^n$ belongs to $\mathfrak{a}$ then, because it is finitely generated, it belongs to some $\mathfrak{a}_i$, a contradiction. Let $\mathfrak{p}$ be a maximal element of S, which exists by Zorn's lemma. We claim that $\mathfrak{p}$ is prime. Indeed, suppose $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$, but $\mathfrak{a}, \mathfrak{b} \not\subset \mathfrak{p}$. Then $\mathfrak{p} + \mathfrak{a}$ and $\mathfrak{p} + \mathfrak{b}$ strictly contain $\mathfrak{p}$, and therefore do not belong to S. Thus, $\mathfrak{c}^n \subset \mathfrak{p} + \mathfrak{a}$ and $\mathfrak{c}^m \subset \mathfrak{p} + \mathfrak{b}$ for some n and m. Thus, $\mathfrak{c}^{n+m} \subset (\mathfrak{p} + \mathfrak{a})(\mathfrak{p} + \mathfrak{b}) \subset \mathfrak{p} + \mathfrak{a}\mathfrak{b} = \mathfrak{p}$, a contradiction. It follows that $\mathfrak{c} \not\subset \mathfrak{p}$, which completes the proof. Indeed, if $\bigcap_{\mathfrak{p}\in\mathcal{P}} \mathfrak{p}$ were strictly larger than $\mathrm{rad}_{\mathbf{GL}} I$, then we could find a finitely **GL**-generated $\mathfrak{c}$ contained in $\bigcap_{\mathfrak{p}\in\mathcal{P}} \mathfrak{p}$ but not contained $\mathrm{rad}_{\mathbf{GL}} I$, and the above argument would yield a contradiction. □

Proposition 2.9. *Every* **GL***-prime of A contains some minimal* **GL***-prime of A.*

Proof. An intersection of a descending chain of **GL**-primes is clearly **GL**-prime, so the claim follows from Zorn's lemma. □

The above concepts (**GL**-prime, **GL**-radical, etc.) are defined using only the language of the tensor category $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL})$. It follows that the same definitions can be made in $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}_{\infty|\infty})$ and **Pol**, and that the definitions agree on objects that correspond under the equivalences. Thus, the above propositions also hold in all three settings. In fact, one can formulate and prove these results for commutative algebra objects in quite general tensor categories.

One important construction that *cannot* be formulated using only the language of the tensor category is the ordinary radical. Suppose A is a tca in **Pol** and I is a **GL**-ideal in it. We can then consider their incarnations $I(\mathbf{Q}^\infty) \subset A(\mathbf{Q}^\infty)$ in $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$, and form $\mathrm{rad}(I(\mathbf{Q}^\infty))$. Similarly, we can consider their incarnations in $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}_{\infty|\infty})$ and form $\mathrm{rad}(I(\mathbf{Q}^{\infty|\infty}))$. There is no reason to expect these two radicals to be comparable in any way (except in tautological ways, e.g., both contain I). In fact, the point of this paper is that they really are not comparable, and the construction is better behaved on the super side.

2.5 | Minimal primes

We require the following result.

Proposition 2.10. *Let A be a tca in* $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$ *and let $\mathfrak{p}$ be a minimal prime of A. Then $\mathfrak{p}$ is* **GL***-stable.*

Proof. Consider the maps

$$A \xrightarrow{\Delta} A \otimes \mathbf{Q}[\mathbf{GL}_n] \xrightarrow{\pi\otimes\mathrm{id}} A/\mathfrak{p} \otimes \mathbf{Q}[\mathbf{GL}_n] \xrightarrow{\mathrm{id}\otimes\epsilon} A/\mathfrak{p}$$

where Δ is comultiplication, $\pi : A \to A/\mathfrak{p}$ is the quotient map, and $\epsilon : \mathbf{Q}[\mathbf{GL}_n] \to \mathbf{Q}$ is the counit. The composition is equal to π by the axioms for a comodule, and thus has kernel $\mathfrak{p}$. We thus see that $\mathfrak{q} = \ker((\pi \otimes \mathrm{id})\circ\Delta) \subset \mathfrak{p}$. However, $\mathfrak{p}$ is prime and $\mathbf{Q}[\mathbf{GL}_n]$ is a localization of a polynomial algebra over $\mathbf{Q}$, and so $A/\mathfrak{p} \otimes \mathbf{Q}[\mathbf{GL}_n]$ is a domain. Thus, $\mathfrak{q}$ is prime. As $\mathfrak{p}$ is minimal, we must have $\mathfrak{p} = \mathfrak{q}$. Now, let x be an element of $\mathfrak{p}$, and write $\Delta(x) = \sum_{i=1}^n a_i \otimes b_i$ where $a_i \in A$ and $b_i \in \mathbf{Q}[\mathbf{GL}_n]$ are $\mathbf{Q}$-linearly independent elements. Then $0 = (\pi \otimes \mathrm{id})(\Delta(x)) = \sum_{i=1}^n \pi(a_i) \otimes b_i$. As the b_i are linearly independent, it follows that $\pi(a_i) = 0$ for all i, and so $a_i \in \mathfrak{p}$ for all i. Thus, $\Delta(\mathfrak{p}) \subset \mathfrak{p} \otimes \mathbf{Q}[\mathbf{GL}_n]$, and so $\mathfrak{p}$ is $\mathbf{GL}_n$-stable. As this holds for all n, it follows that $\mathfrak{p}$ is $\mathbf{GL}_\infty$-stable. □

Remark 2.11. The analog of this statement for tcas in $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}_{\infty|\infty})$ does not hold: the above proof fails because $\mathbf{Q}[\mathbf{GL}_{r|s}]$ is not a domain.

2.6 | Radicals of GL-primes

We require the following result on **GL**-primes. See [23, section 8.6] for some similar results.

Proposition 2.12. *Let A be a tca in* $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}_{\infty|\infty})$ *and let $\mathfrak{p}$ be a* **GL***-prime of A. Then* $\mathrm{rad}(\mathfrak{p})$ *is prime.*

We require some preliminary work before proving the proposition. Let $\{e_i\}_{i\in\mathcal{I}}$ be a homogeneous basis for $\mathbf{V} = \mathbf{Q}^{\infty|\infty}$. The rows and columns of $\mathbf{GL}_{\infty|\infty}$ are thus indexed by $\mathcal{I}$. Let T be the diagonal subgroup of $\mathbf{GL}_{\infty|\infty}$. Given $\alpha \in \mathbf{Z}^{\oplus\mathcal{I}}$, there is an associated character $\chi_\alpha : T \to \mathbf{G}_m$. We define the *support* of α to be the finite set of indices $i \in \mathcal{I}$ for which α_i is nonzero.

If V is a polynomial representation of $\mathbf{GL}_{\infty|\infty}$ then it admits a weight space decomposition $V = \bigoplus_\alpha V_\alpha$, where V_α is the χ_α-eigenspace of T. (To see this, it suffices to verify the case where V is a tensor power of the standard representation, which is easy to do directly.) For $x \in V$, we write $x = \sum_\alpha x_\alpha$, and define the *support* of x to be the union of the supports of the α's for which x_α is nonzero. If V is realized as a subrepresentation of a sum of tensor powers of the standard representation, then the support of x simply records which of the standard basis vectors appear in x. We say that elements of V are *disjoint* if they have disjoint supports.

We let $\mathfrak{gl}_{n|m}$ be the Lie algebra of $\mathbf{GL}_{n|m}$ (see [4, section 1.1]), and we let $\mathfrak{gl}_{\infty|\infty}$ be the union of the $\mathfrak{gl}_{n|n}$'s. We define the *support* of an element X of $\mathfrak{gl}_{\infty|\infty}$ to be the set of indices i such that X has a nonzero entry in row i or column i. We say that an element of the universal enveloping algebra $\mathcal{U}(\mathfrak{gl}_{\infty|\infty})$ has *support contained in S* if it can be expressed in terms of elements of $\mathfrak{gl}_{\infty|\infty}$ having this property.

For an element $x \in A$, we let $\langle x\rangle$ be the **GL**-ideal it generates.

Lemma 2.13. *Suppose $x, y \in A$ are disjoint elements such that $xy = 0$. Then $\langle x\rangle \cdot \langle y\rangle = 0$.*

Proof. Let y' be an element of $\langle y\rangle$ that is disjoint from x. Write $y' = ay$ where $a \in \mathcal{U}(\mathfrak{gl}_{\infty|\infty})$. Now, the support of a may overlap with that of x, that is, we may use auxiliary basis vectors in the

process of building y' from y. However, it does not matter which auxiliary basis vectors we use, so we can modify a if necessary so that it is disjoint from x. More rigorously, choose a permutation σ of $\mathcal{I}$ that fixes the supports of y and y', and such that $\sigma a \sigma^{-1}$ is disjoint from x. Then $y' = \sigma a \sigma^{-1} y$ and $\sigma a \sigma^{-1}$ is disjoint from x. Now, applying a to the expression $xy = 0$, and using the fact that a commutes with x because it is disjoint from x, we find $x(ay) = 0$, that is, $xy' = 0$. (Note: the permutations used here are required to fix super degree; they belong to the group of permutation matrices in $\mathbf{GL}_\infty \times \mathbf{GL}_\infty \subset \mathbf{GL}_{\infty|\infty}$.)

Now let y' be an arbitrary element of $\langle y \rangle$. We can then write $y' = \sigma y''$ where σ is a permutation of $\mathcal{I}$ and y'' is disjoint from x. Let $E \in \mathfrak{gl}_\infty \times \mathfrak{gl}_\infty$ act by σ on the support of y'' and 0 on the remaining basis vectors. Then $Ex = 0$ and $Ey'' = y'$. As y'' is disjoint from x, we have $xy'' = 0$ by the previous paragraph. Applying E to this equation gives $xy' = 0$. This completes the proof. □

Lemma 2.14. *Let $x, y \in A$ be super homogeneous elements satisfying $xy = 0$. Then there exists $n \geqslant 0$ such that $\langle x^n \rangle \cdot \langle y \rangle = 0$.*

Proof. We claim that for any $a \in \mathcal{U}(\mathfrak{gl}_{\infty|\infty})$ there exists $n \geqslant 0$ such that $x^n \cdot ay = 0$. This is clear for $a = 1$. Suppose now it is true for a, and let us prove it for Ea, with $E \in \mathfrak{gl}_{\infty|\infty}$. It suffices to treat the case where E is super homogeneous. Let n be such that $x^n \cdot ay = 0$. Applying E, we find $nx^{n-1}Ex \cdot ay \pm x^n \cdot Eay = 0$, where the sign depends on super degrees. Multiplying by x and using the fact that $x^n \cdot ay = 0$, we find $x^{n+1} \cdot Eay = 0$. The claim now follows.

Now let $y' \in \langle y \rangle$ be disjoint from x and generate $\langle y \rangle$; for instance, one could take $y' = \sigma y$ for an appropriate permutation σ of $\mathcal{I}$. As $y' = ay$ for some $a \in \mathcal{U}(\mathfrak{gl}_{\infty|\infty})$, the previous paragraph gives $x^n y' = 0$ for some n. By Lemma 2.13, we find $\langle x^n \rangle \cdot \langle y \rangle = 0$, and so the result follows. □

Proof of Proposition 2.12. Passing to $A/\mathfrak{p}$, we assume $\mathfrak{p} = 0$ is **GL**-prime. We must show that $\mathrm{rad}(A)$ is prime; here we write $\mathrm{rad}(A)$ for the nilradical of A, that is, the radical of the zero ideal. As all odd elements of A are nilpotent, we have $A_1 \subset \mathrm{rad}(A)$. It thus suffices to show that if $xy \in \mathrm{rad}(A)$ with x and y even then $x \in \mathrm{rad}(A)$ or $y \in \mathrm{rad}(A)$. Thus, let even elements x and y be given such that xy is nilpotent, say $(xy)^k = 0$. As x and y are even, they commute, and so $x^k y^k = 0$. By Lemma 2.14, there exists $n \geqslant 0$ such that $\langle x^{nk} \rangle \cdot \langle y^k \rangle = 0$. As (0) is **GL**-prime, it follows that $x^{nk} = 0$ or $y^k = 0$. Thus, either x or y is nilpotent, which completes the proof. □

2.7 | Draisma's theorem

Suppose that a group G acts on a topological space X. We say that X is *G-Noetherian* if every descending chain of closed G-stable subsets stabilizes. Draisma [8, Corollary 3] proved the following important theorem in this context:

Theorem 2.15. *Let A be a tca in $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$ such that A_0 is Noetherian and A is finitely generated over A_0. Then $\mathrm{Spec}(A)$ is $\mathbf{GL}_\infty$-Noetherian.*

In fact, Draisma only states this theorem when A_0 is finitely generated over a field, but a slight modification in his proof yields the above statement. As it is not critical for this paper, we do not include details. We give a few corollaries of the theorem.

Corollary 2.16. *Let A be as in Theorem 2.15. Then every ascending chain of radical* **GL***-ideals in A stabilizes.*

Proof. This follows as radical **GL**-ideals of A correspond bijectively to **GL**-stable closed subsets of $\mathrm{Spec}(A)$. □

Corollary 2.17. *Let A be as in Theorem 2.15. Let $Y = \mathrm{Spec}(A)$ and let $X = Y^{\mathbf{GL}}$ be the subset consisting of $\mathbf{GL}_\infty$-stable prime ideals, endowed with the subspace topology. Then X is a Noetherian spectral space.*

Proof. Suppose that Z is a closed subset of X. Then Z has the form $W \cap X$ for some closed subset W of Y. As every point in Z is $\mathbf{GL}_\infty$-invariant, it follows that $Z = (gW) \cap X$ for any $g \in \mathbf{GL}_\infty$. Thus, $Z = W' \cap X$ where $W' = \bigcap_{g \in \mathbf{GL}_\infty} gW$ is a **GL**-stable closed subset of Y. It follows that $Z = \overline{Z} \cap X$, where $\overline{Z}$ is the closure of Z in Y, as $Z \subset \overline{Z} \subset W'$.

Now suppose that $\cdots \subset Z_2 \subset Z_1$ is a descending chain of closed sets in X. Then $\cdots \subset \overline{Z}_2 \subset \overline{Z}_1$ is a descending chain of **GL**-stable closed subsets of Y, and thus stabilizes by Theorem 2.15. As $Z_i = \overline{Z}_i \cap X$, it follows that the original chain stabilizes too. Thus, X is Noetherian.

Finally, let Z be an irreducible closed subset of X. Then $\overline{Z}$ is a **GL**-stable irreducible closed subset of Y. Its generic point is thus **GL**-stable, and therefore belongs to X. One easily sees that it is the unique generic point for Z. Thus, X is sober. As it is also Noetherian, it is spectral. □

Corollary 2.18. *Let A be as in Theorem 2.15. Then A has finitely many minimal primes.*

Proof. Let X be as in Corollary 2.17. As X is Noetherian, it has finitely many irreducible components. As it is sober, these components correspond to the minimal **GL**-stable prime ideals of X. There are thus finitely many of these. However, if $\mathfrak{p}$ is any minimal prime then it is **GL**-stable by Proposition 2.10, and thus obviously a minimal **GL**-stable prime. The result follows. □

3 | THE KEY RESULT

The following is the key theorem of this paper: it is the bridge that connects equivariant concepts to ordinary ones.

Theorem 3.1. *Let A be a tca in* **Pol** *that is finitely generated over* **Q**. *The following are equivalent.*

(a) *Every positive degree homogeneous element of $A(\mathbf{Q}^{r|s})$ is nilpotent, for all r, s.*
(b) *The ideal A_+ is nilpotent.*

We note that Example 1.1 shows that it is really necessary in (a) to evaluate on super vector spaces; indeed, for the algebra A discussed there, A_+ is not nilpotent, but every positive degree homogeneous element of $A(\mathbf{Q}^r)$ is nilpotent, for all $r \geqslant 0$.

Proof. It is clear that (b) implies (a). We prove the converse. We proceed by induction on the degree of generation of A. Thus, suppose A is generated in degrees $\leqslant d$ and satisfies (a), and that the theorem is true for tcas generated in degrees $< d$.

Before getting into the argument, we introduce a piece of notation. For a polynomial functor F, we let F' be the polynomial functor defined by $F'(V) = F(\mathbf{Q} \oplus V)$. We note that F' carries an action of $\mathbf{G}_m$, through its action on $\mathbf{Q}$. If $x \in F'(\mathbf{Q}^{r|s}) = F(\mathbf{Q}^{r+1|s})$ is a weight vector for $\mathbf{GL}_{r|s}$ of weight $(a_1, \dots, a_r; b_1, \dots, b_s)$ and simultaneously a weight vector for $\mathbf{G}_m$ of weight k then it is also a weight vector for $\mathbf{GL}_{r+1|s}$ of weight $(k, a_1, \dots, a_r; b_1, \dots, b_s)$.

Let $E \subset A_0 \oplus \cdots \oplus A_d$ be a subrepresentation generating A. Let $E' = E'_0 \oplus \cdots \oplus E'_d$ be the weight space decomposition for E' with respect to the $\mathbf{G}_m$ action; note that E'_i is a polynomial functor of degree $\leqslant d - i$. Let B be the subalgebra of A' generated by $E'_1, \dots, E'_d$, and let C be the subalgebra generated by E'_0. For $i > 0$, every weight vector of $E'_i(\mathbf{Q}^{r|s})$ is a weight vector of $E(\mathbf{Q}^{r+1|s})$ of nonzero weight (as the first component of the weight is i), and thus a positive degree homogenous element of $A(\mathbf{Q}^{r+1|s})$, and thus nilpotent. We thus see that the generators of $B(\mathbf{Q}^{r|s})$ are nilpotent, and so B satisfies (a). As B is finitely generated in degrees $< d$, we can apply the inductive hypothesis to conclude that $B_n = 0$ for $n \gg 0$. As the degree 0 generators of B are also nilpotent, it follows that B has finite length as a polynomial functor. In particular, only finitely many $\mathbf{G}_m$ weights appear in B; say that the largest one is N.

Let T be the maximal torus of $\mathbf{GL}_{n+1}$. Suppose that $\lambda = (\lambda_1, \dots, \lambda_{n+1})$ is a weight of T that appears in $A(\mathbf{Q}^{n+1}) = A'(\mathbf{Q}^n)$; we note that λ_1 records the action of $\mathbf{G}_m$. As $B(\mathbf{Q}^n)$ and $C(\mathbf{Q}^n)$ generate $A'(\mathbf{Q}^n)$ and are T-stable, it follows that λ can be written in the form $\mu + \nu$ where μ is a weight of T appearing in $B(\mathbf{Q}^n)$ and ν is one in $C(\mathbf{Q}^n)$. We have $\mu_1 \leqslant N$ by the definition of N. As C is generated by E'_0, on which $\mathbf{G}_m$ acts trivially, we see that $\nu_1 = 0$. Thus, $\lambda_1 \leqslant N$. As this holds for all weights in $A(\mathbf{Q}^{n+1})$ for any n, it follows that A has width $\leqslant N$ by Proposition 2.4.

Decompose A as $\bigoplus_\lambda A_\lambda \otimes \mathbf{S}_\lambda$ where A_λ is a multiplicity space. We have just shown that A_λ is only nonzero when $\lambda_1 \leqslant N$. Consider the superalgebra $A(\mathbf{Q}^{0|N}) = \bigoplus_\lambda A_\lambda \otimes \mathbf{S}_\lambda(\mathbf{Q}^{0|N})$. This is finitely generated, and every positive degree element is nilpotent by assumption. Thus, $A(\mathbf{Q}^{0|N})_n = 0$ for n sufficiently large, say $n > M$. We thus have $A_\lambda \otimes \mathbf{S}_\lambda(\mathbf{Q}^{0|N}) = 0$ for $|\lambda| > M$, and so $A_\lambda = 0$: indeed, we know this already if $\lambda_1 > N$, and otherwise $\mathbf{S}_\lambda(\mathbf{Q}^{0|N})$ is nonzero by Proposition 2.3. It follows that $A_n = 0$ for $n > M$ as well, and so A satisfies (b). This proves the theorem. □

Example 3.2. Suppose A is generated over $\mathbf{Q}$ by $E = \mathrm{Sym}^2$. The space $E'(\mathbf{Q}^n) = E(\mathbf{Q}^{n+1})$ has for a basis elements $x_{i,j}$ with $1 \leqslant i \leqslant j \leqslant n+1$. The degree of $x_{i,j}$ under the $\mathbf{G}_m$ action is simply the number of indices equal to 1. Thus, $E'_2(\mathbf{Q}^n)$ is spanned by $x_{1,1}$, while $E'_1(\mathbf{Q}^n)$ is spanned by the $x_{1,j}$ with $2 \leqslant j$, and $E'_0(\mathbf{Q}^n)$ is spanned by the $x_{i,j}$ with $2 \leqslant i, j$. Thus, $B(\mathbf{Q}^n)$ is generated by $x_{1,1}$, which is $\mathbf{GL}_n$ invariant, and the $x_{1,j}$ with $2 \leqslant j \leqslant n+1$, which generate a copy of the standard representation of $\mathbf{GL}_n$. We thus see that B is generated by $\mathrm{Sym}^0 \oplus \mathrm{Sym}^1$, which has degree $\leqslant 1$ as a polynomial functor. This shows that B can have degree 0 generators even if A does not. The algebra $C(\mathbf{Q}^n)$ is generated by the $x_{i,j}$ with $2 \leqslant i, j$, and so $\mathbf{G}_m$ acts trivially on it.

Corollary 3.3. *Let A be an arbitrary tca in* **Pol** *and let E be a finite length subrepresentation of A. The following are equivalent.*

(a) *Every element of $E(\mathbf{Q}^{r|s})$ is nilpotent, for all r and s.*
(b) *The space E is nilpotent, that is, the map $E^{\otimes n} \to A$ is zero for some n.*

Proof. Obviously (b) implies (a); we prove the converse. First suppose that E is generated in positive degrees. Let B be the sub tca of A generated over $\mathbf{Q}$ by E. Then $B(\mathbf{Q}^{r|s})$ is generated as

a subalgebra of $A(\mathbf{Q}^{r|s})$ by $E(\mathbf{Q}^{r|s})$, and so every positive degree element is nilpotent. Thus, by Theorem 3.1, we see that B_+ is nilpotent. As $E \subset B_+$, we see that it too is nilpotent.

In general, write $E = E_0 \oplus E_+$ where E_0 is the degree 0 piece of E and E_+ is the sum of the positive degree pieces of E. Then E_0 is finite-dimensional and every element is nilpotent, so E_0 is nilpotent, and E_+ is nilpotent by the previous paragraph. Thus, E is nilpotent. □

The proof of Theorem 3.1 is effective, in the following sense. For a finite length polynomial functor E, let $\mathcal{S}_k(E)$ be the class of tcas A such that (i) A contains a copy of E that generates it over $\mathbf{Q}$; and (ii) each weight space of $E(\mathbf{Q}^{r|s})$ of nonzero weight admits a basis consisting of k-nilpotent elements, for any r and s. Let $\eta_k(E)$ be the supremum of $\operatorname{maxdeg}(A)$ over $A \in \mathcal{S}_k(E)$, where maxdeg denotes the maximum nonzero degree. Theorem 3.1 simply states that $\operatorname{maxdeg}(A)$ is finite for $A \in \mathcal{S}_k(E)$. In fact, the proof yields a bound on $\eta_k(E)$. Let d be the degree of E, let E_i' be as in the proof of the theorem, and let P be the polynomial defined by $P(n) = \dim E(\mathbf{Q}^{0|n})$. Examining the proof, one finds

$$\eta_k(E) \leqslant kd \cdot P(d\eta_k(E_1' \oplus \cdots \oplus E_d') + kd \dim E(\mathbf{Q})).$$

This allows one to inductively obtain a bound on $\eta_k(E)$ because the argument to η_k on the right has smaller degree than E. Making the rough approximation $P(x) \approx x^d$, one finds

$$\eta_k(E) \lessapprox \prod_{i=1}^{d} (ik)^{d!/i!}.$$

This upper bound is quite large; for example, it is substantially larger than $k^{d!}2^{2^d}$. We do not know how close it is to the true behavior of $\eta_k(E)$.

4 | THE MAIN THEOREMS

We fix a tca A in $\operatorname{Rep}^{\mathrm{pol}}(\mathbf{GL}_{\infty|\infty})$ for this section. Consider the following condition.

(∗) A_0 is Noetherian and A is finitely generated over A_0.

We will sometimes require this condition, and sometimes not. Our goal now is to prove the main theorems stated in Subsection 1.3.

Proposition 4.1 (Theorem A). *Let I and J be $\mathbf{GL}$-ideals of A. Then $I \subset \operatorname{rad}_{\mathbf{GL}} J$ if and only if $I \subset \operatorname{rad} J$.*

Note that $I \subset \operatorname{rad}_{\mathbf{GL}} J$ if and only if $\operatorname{rad}_{\mathbf{GL}} I \subset \operatorname{rad}_{\mathbf{GL}} J$, and similarly for ordinary radicals, so this proposition is indeed equivalent to Theorem A.

Proof. We may replace I with $I + J$ without changing either condition. We may then check the conditions after passing to A/J. Thus, we may simply assume from the outset that $J = 0$.

If $I \subset \operatorname{rad}_{\mathbf{GL}}(A)$ then $I \subset \operatorname{rad}(A)$, as we have a containment $\operatorname{rad}_{\mathbf{GL}}(A) \subset \operatorname{rad}(A)$. Conversely, suppose that $I \subset \operatorname{rad}(A)$. Let E be a finite length subrepresentation of I. Then every element of E

is nilpotent, and so E is nilpotent by Corollary 3.3. Thus, $E \subset \mathrm{rad}_{\mathbf{GL}}(A)$. As this holds for all E, it follows that $I \subset \mathrm{rad}_{\mathbf{GL}}(A)$. □

We now introduce an auxiliary algebra that we will be helpful in what follows. Let $B = A/\mathrm{rad}(A)$. The ideal $\mathrm{rad}(A)$ is typically not $\mathbf{GL}_{\infty|\infty}$ stable (see the following remark), but it is stable by $\mathbf{GL}_\infty \times \mathbf{GL}_\infty \subset \mathbf{GL}_{\infty|\infty}$ because this group acts by algebra automorphisms of A. It is therefore also stable by the diagonal subgroup $\mathbf{GL}_\infty \subset \mathbf{GL}_\infty \times \mathbf{GL}_\infty$, and so this acts on B.

Remark 4.2. Let A be the tca from Example 1.1 in its super incarnation, so that $A_{2n} = \bigwedge^{2n}(\mathbf{Q}^{\infty|\infty})$. If $n > 0$ then A_{2n} contains nonzero nilpotent elements and nonnilpotent elements. As A_{2n} is an irreducible representation of $\mathbf{GL}_{\infty|\infty}$, it follows that $\mathrm{rad}(A)$ cannot be $\mathbf{GL}_{\infty|\infty}$ stable. Conceptually, this happens because elements of $\mathbf{GL}_{\infty|\infty}(T)$ (for a commutative superalgebra T) can mix nilpotent and nonnilpotent elements (see Remark 2.2).

Choose a surjection $A_0 \otimes \mathrm{Sym}(V) \to A$, for some representation $V \subset A$. The restriction of V to the diagonal $\mathbf{GL}_\infty$ has the form $W_0 \oplus W_1[1]$, where W_0 and W_1 are polynomial representations of $\mathbf{GL}_\infty$, and $[1]$ indicates the super grading. It follows that A is a quotient of $A_0 \otimes \mathrm{Sym}(W_0 \oplus W_1[1])$. As B has no odd part, we see that it is a quotient of $A_0 \otimes \mathrm{Sym}(W_0)$. In other words, B is a tca in the category $\mathrm{Rep}^{\mathrm{pol}}(\mathbf{GL}_\infty)$. If A satisfies $(*)$ then we can take V to be a finite length representation. It follows from basic properties of Schur functors that W_0 and W_1 are then of finite length as well, and so B is finitely generated over $B_0 = A_0$.

Proposition 4.3 (Theorem D). *Suppose* $(*)$ *holds. Then* $\mathrm{Spec}_{\mathbf{GL}}(A)$ *is Noetherian.*

Proof. It suffices to show that every ascending chain of $\mathbf{GL}$-radical ideals in A stabilizes. Thus, let $I_1 \subset I_2 \subset \cdots$ be such a chain. Then $\mathrm{rad}(I_1) \subset \mathrm{rad}(I_2) \subset \cdots$ is an ascending chain of $\mathbf{GL}_\infty$-stable radical ideals of A, and thus corresponds to an ascending chain of $\mathbf{GL}_\infty$-stable radical ideals of B. It therefore stabilizes by Corollary 2.16. By Proposition 4.1, it follows that the original chain stabilizes. □

Proposition 4.4 (Theorem C). *Suppose* $(*)$ *holds.*

(a) *A has finitely many minimal $\mathbf{GL}$-primes, say $\mathfrak{p}_1, \dots, \mathfrak{p}_n$.*
(b) *A has finitely many minimal primes, say $\mathfrak{q}_1, \dots, \mathfrak{q}_m$.*
(c) *$n = m$, and after applying a permutation we have $\mathfrak{q}_i = \mathrm{rad}(\mathfrak{p}_i)$ for all i.*

Proof.

(a) This is an immediate consequence of Proposition 4.3.
(b) The minimal primes of A correspond bijectively to those of B, and there are finitely many of these by Corollary 2.18.
(c) Consider a minimal prime $\mathfrak{q}_i$. We have

$$\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n = \mathrm{rad}_{\mathbf{GL}}(A) \subset \mathrm{rad}(A) \subset \mathfrak{q}_i.$$

As $\mathfrak{q}_i$ is prime, it follows that $\mathfrak{p}_j \subset \mathfrak{q}_i$ for some j. Thus, $\mathrm{rad}(\mathfrak{p}_j) \subset \mathfrak{q}_i$. As $\mathrm{rad}(\mathfrak{p}_j)$ is prime (Proposition 2.12) and $\mathfrak{q}_i$ is a minimal prime, we have $\mathrm{rad}(\mathfrak{p}_j) = \mathfrak{q}_i$. By Proposition 4.1, it follows that j is unique: indeed, if $\mathrm{rad}(\mathfrak{p}_j) = \mathrm{rad}(\mathfrak{p}_k)$ then $\mathfrak{p}_j = \mathfrak{p}_k$ and so $j = k$.

To complete the proof, it suffices to show that $\mathrm{rad}(\mathfrak{p}_j)$ is a minimal prime for all j. We know that $\mathrm{rad}(\mathfrak{p}_j)$ is prime. It therefore contains some minimal prime $\mathfrak{q}_i$. We have shown that $\mathfrak{q}_i = \mathrm{rad}(\mathfrak{p}_k)$ for some k. Thus, $\mathfrak{p}_k \subset \mathrm{rad}(\mathfrak{p}_j)$, and so $\mathfrak{p}_k \subset \mathrm{rad}_{\mathbf{GL}}(\mathfrak{p}_j) = \mathfrak{p}_j$ by Proposition 4.1. As $\mathfrak{p}_j$ is a minimal **GL**-prime, it follows that $\mathfrak{p}_j = \mathfrak{p}_k$. Hence $\mathrm{rad}(\mathfrak{p}_j) = \mathfrak{q}_i$ is a minimal prime. □

Proposition 4.5 (Theorem B). *Let I be a* **GL***-ideal of A. Then* $\mathrm{rad}_{\mathbf{GL}}(I)$ *is* **GL***-prime if and only if* $\mathrm{rad}(I)$ *is prime.*

Proof. If I is **GL**-prime then $\mathrm{rad}(I)$ is prime by Proposition 2.12. Now suppose that $\mathrm{rad}(I)$ is prime and A satisfies $(*)$. Then A/I has a unique minimal prime, and therefore a unique minimal **GL**-prime by Proposition 4.4. Thus, there is a unique minimal **GL**-prime $\mathfrak{p}$ over I, and so $\mathrm{rad}_{\mathbf{GL}}(I) = \mathfrak{p}$ is **GL**-prime. Finally, suppose that $\mathrm{rad}(I)$ is prime and A is arbitrary. Write $A = \bigcup A_i$ where $\{A_i\}$ is a directed family of sub tcas satisfying $(*)$. Then $\mathrm{rad}(I \cap A_i) = \mathrm{rad}(I) \cap A_i$ is prime, and so $\mathrm{rad}_{\mathbf{GL}}(I \cap A_i) = \mathrm{rad}_{\mathbf{GL}}(I) \cap A_i$ is **GL**-prime by the previous case. As $\mathrm{rad}_{\mathbf{GL}}(I) \cap A_i$ is prime for all i, it follows that $\mathrm{rad}_{\mathbf{GL}}(I)$ is prime by Lemma 4.6. □

Lemma 4.6. *Let A be a tca, and suppose that $A = \bigcup A_i$ for some directed family $\{A_i\}$ of sub tcas. Let $\mathfrak{p}$ be a* **GL***-ideal of A. If $\mathfrak{p} \cap A_i$ is* **GL***-prime for all i then $\mathfrak{p}$ is* **GL***-prime.*

Proof. Let V and W be finite length subrepresentations of A such that $VW \subset \mathfrak{p}$. As V and W are finite length, there is some i such that V and W are contained in A_i. Thus, $VW \subset \mathfrak{p} \cap A_i$. As $\mathfrak{p} \cap A_i$ is **GL**-prime, it follows that $V \subset \mathfrak{p} \cap A_i$ or $W \subset \mathfrak{p} \cap A_i$. Thus, $V \subset \mathfrak{p}$ or $W \subset \mathfrak{p}$, and so $\mathfrak{p}$ is **GL**-prime. □

5 | AN EXAMPLE

Let A be the tca in **Pol** given by $A(V) = \mathbf{C} \otimes \mathrm{Sym}(\mathrm{Sym}^2(V))$. Our goal is to classify the **GL**-prime and **GL**-radical ideals of A.

5.1 | The ideal lattice of A

The decomposition of A into irreducibles is well-known:

$$A = \bigoplus \mathbf{S}_{2\lambda},$$

where the sum is over all partitions λ, and $2\lambda = (2\lambda_1, 2\lambda_2, \ldots)$. See, for example, [14, section I.5, Example 5]. For a partition λ, let I_λ be the ideal of A generated by $\mathbf{S}_{2\lambda}$. The following result determines the ideal structure of A:

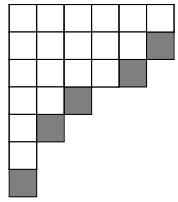
Proposition 5.1. *The ideal I_λ is the sum of those $\mathbf{S}_{2\mu}$ for which $\lambda \subset \mu$. In particular, $I_\lambda \subset I_\mu$ if and only if $\mu \subset \lambda$.*

Proof. This was originally proved in [1], but that is a difficult reference to obtain. The analogous result for $\mathrm{Sym}(\bigwedge^2)$ is proved in [2, Theorem 3.1]. That case actually implies this one, as

$\mathrm{Sym}(\mathrm{Sym}^2(V)) = \mathrm{Sym}(\bigwedge^2(V[1]))$. A complete proof in the case where λ is a rectangle also appears in [17, Corollary 2.8]. A closely related result appears in [6, Theorem 4.1]. □

The ideals generated by rectangular shapes will be particularly important, so we introduce some notation for them. We let $\rho(r,s)$ be the partition with r rows each of length s; thus the Young diagram for $\rho(r,s)$ is an $r \times s$ rectangle. We let $I_{r,s} = I_{\rho(r,s)}$. If $r = 0$ or $s = 0$ then $\rho(r,s)$ is an empty partition and $I_{r,s}$ is the unit ideal.

Let λ be a partition. By a *corner* of λ we mean a pair (r,s) such that λ has a box in the rth row and sth column, but no box below or to the right of this one. For example, in the following Young diagram the corners have been shaded:



The following observation illustrates the importance of the rectangular ideals.

Proposition 5.2. *Let C be the set of corners of λ. Then $I_\lambda = \bigcap_{(r,s)\in C} I_{r,s}$.*

Proof. Let μ be a partition. We have

$$\begin{aligned}
\mathbf{S}_{2\mu} \subset \bigcap\nolimits_{(r,s)\in C} I_{r,s} &\iff \rho(r,s) \subset \mu \text{ for all } (r,s) \in C \\
&\iff \bigcup\nolimits_{(r,s)\in C} \rho(r,s) \subset \mu \\
&\iff \lambda \subset \mu \\
&\iff \mathbf{S}_{2\mu} \subset I_\lambda.
\end{aligned}$$

The first and last step follow from Proposition 5.1, the second is trivial, and the third is simply the observation that $\lambda = \bigcup_{(r,s)\in C} \rho(r,s)$. The result thus follows. □

5.2 | The variety X

For $V = E \oplus F[1]$, let $B(V) = \mathrm{Sym}(\mathrm{Sym}^2(E) \oplus \bigwedge^2(F))$. Note that $B(V) = A(V)/\operatorname{rad}(A(V))$. We regard B as a 2-variable tca (in the variables E and F). Let $X(V) = \mathrm{Spec}(B(V))$, which we identify with $\mathrm{Sym}^2(E)^* \times \bigwedge^2(F)^*$. By a "closed subvariety" of X, we mean a subfunctor Y of X such that $Y(V)$ is a closed subvariety of $X(V)$ for all finite-dimensional V. Closed subvarieties of X correspond bijectively to $\mathbf{GL} \times \mathbf{GL}$ stable radical ideals of B. For $r, s \in \mathbf{N} \cup \{\infty\}$, let $X_{r,s}(V) \subset X(V)$ be the locus of pairs (ω, η) such that $\operatorname{rank}(\omega) \leqslant r$ and $\operatorname{rank}(\eta) \leqslant 2s$. Then $X_{r,s}$ is a closed subvariety of X in the above sense. We now show that these account for essentially all examples:

Proposition 5.3. *Let Y be a closed subvariety of X. Then there is a finite subset C of $(\mathbf{N} \cup \{\infty\})^2$ such that $Y = \bigcup_{(r,s)\in C} X_{r,s}$.*

Proof. By the *rank* of a point $(\omega, \eta) \in X(E,F)$, we mean the pair $(\mathrm{rank}(\omega), \frac{1}{2}\mathrm{rank}(\eta))$. Let $S \subset \mathbf{N}^2$ be the set of pairs (r,s) such that $Y(E,F)$ has a point of rank (r,s) for some E and F. We claim that $(r,s) \in S$ if and only if $X_{r,s} \subset Y$. It is clear that $X_{r,s} \subset Y$ implies $(r,s) \in S$. Conversely, suppose that $(r,s) \in S$. Then there exists some $(\omega_0, \eta_0) \in Y(E_0, F_0)$ of rank (r,s) for some E_0 and F_0. Let $(\omega, \eta) \in X_{r,s}(E,F)$ be given. By basic linear algebra, there are linear maps $\varphi : E \to E_0$ and $\psi : F \to F_0$ such that $\omega = \varphi^*(\omega_0)$ and $\eta = \psi^*(\eta_0)$. Thus, the map $X(E_0, F_0) \to X(E,F)$ defined by (φ, ψ) carries (ω_0, η_0) to (ω, η). As $(\omega_0, \eta_0) \in Y(E_0, F_0)$ and Y is a subfunctor of X, it follows that $(\omega, \eta) \in Y(E,F)$. This proves the claim.

It now follows that if $(r,s) \in S$ and $(r',s') \leqslant (r,s)$ then $(r',s') \in S$, where here $(r',s') \leqslant (r,s)$ means $r' \leqslant r$ and $s' \leqslant s$. A simple combinatorial argument now shows that there is a finite subset C of $(\mathbf{N} \cup \{\infty\})^2$ such that $(r',s') \in S$ if and only if $(r',s') \leqslant (r,s)$ for some $(r,s) \in C$. It follows that Y is the union of the $X_{r,s}$ with $(r,s) \in C$, which proves the proposition. □

Corollary 5.4. *Any irreducible closed subvariety of X is one of the $X_{r,s}$.*

5.3 | The vanishing locus of $I_{r,s}$

The goal of this section is to prove the following:

Proposition 5.5. *We have $\mathcal{V}(I_{r+1,s+1}(V)) = X_{r,s}(V)$ for $r, s \geqslant 0$.*

We break the proof into two lemmas.

Lemma 5.6. *We have the following:*

(a) *We have* $\mathrm{rad}(I_{r+1,s+1}(E)) = I_{r+1,1}(E)$ *in* $\mathrm{Sym}(\mathrm{Sym}^2(E))$.
(b) *We have* $\mathrm{rad}(I_{r+1,s+1}(F[1])) = I_{1,s+1}(F)$ *in* $\mathrm{Sym}(\bigwedge^2(F))$.
(c) *We have* $\mathcal{V}(I_{r+1,s+1}(V)) \subset X_{r,s}(V)$.

Proof.

(a) This is proved in [1], but we include an argument (due to the referee) to be self-contained. Let $A = \mathrm{Sym}(\mathrm{Sym}^2(E))$, and note that $\mathrm{Spec}(A) = \mathrm{Sym}^2(E)^*$ is the space of symmetric bilinear forms on E. The ideal $I_{r+1,1}(E)$ is the classical determinantal ideal cutting out the locus of forms of rank $\leqslant r$ (see [27, section 6.3]). Pick a basis for E, so that $\mathrm{Spec}(A)$ is the space of symmetric matrices, and A is the polynomial ring in the matrix entries.

Let $\delta \in A$ be the top left $(r+1) \times (r+1)$ minor of the matrix. This is a highest weight vector, in the sense that it is fixed by the group of upper triangular unipotent matrices. As δ has weight $\rho(r+1, 1)$, it follows that the **GL**-subrepresentation generated by δ is $\mathbf{S}_{\rho(r+1,1)}(E)$, and so the **GL**-ideal generated by δ is $I_{r+1,1}(E)$. Similarly, δ^{s+1} is a highest weight vector of weight $\rho(r+1, s+1)$, and so the **GL**-ideal it generates is $I_{r+1,s+1}(E)$. We therefore see $\delta \in \mathrm{rad}(I_{r+1,s+1}(E))$, and so $I_{r+1,1}(E) \subset \mathrm{rad}(I_{r+1,s+1}(E))$. By Proposition 5.1, we have $I_{r+1,s+1}(E) \subset I_{r+1,1}(E)$, and so we find $\mathrm{rad}(I_{r+1,s+1}(E)) \subset I_{r+1,1}(E)$ because $I_{r+1,1}(E)$ is prime.
(b) This is proved in [2, Theorem 5.1]. We can also argue analogously to the above.
(c) As $I_{r+1,s+1}(V)$ contains both $I_{r+1,s+1}(E)$ and $I_{r+1,s+1}(F[1])$, we see that its radical contains both $I_{r+1,1}(E)$ and $I_{1,s+1}(F[1])$ by (a) and (b), and thus the extensions of these ideals to $A(V)$.

It follows that $\mathcal{V}(I_{r+1,s+1}(V))$ is contained in the intersection of $\mathcal{V}(I_{r+1,1}(E)^e)$ and $\mathcal{V}(I_{1,s+1}(F[1])^e)$. Now, as above, $I_{r+1,1}(E)$ is the classical determinantal ideal: its vanishing locus in $\mathrm{Sym}^2(E)^*$ consists of those forms rank $\leqslant r$. The vanishing locus of $I_{r+1,1}(E)^e$ is thus $X_{r,\infty}(V)$. Similarly, the vanishing locus of $I_{1,s+1}(F[1])^e$ is $X_{\infty,s}(V)$. We thus find that $\mathcal{V}(I_{r+1,s+1}(V))$ is contained in $X_{r,\infty}(V) \cap X_{\infty,s}(V) = X_{r,s}(V)$. □

Lemma 5.7. *We have $X_{r,s}(V) \subset V(I_{r+1,s+1}(V))$.*

Proof. Let α be a nondegenerate symmetric bilinear form on $\mathbf{C}^r$. We obtain a natural map

$$\mathrm{Sym}^2(E) \to \mathrm{Sym}^2(\mathbf{C}^r) \otimes \mathrm{Sym}^2(E) \to \mathrm{Sym}^2(\mathbf{C}^r \otimes E),$$

where the first map comes from the inclusion $\mathbf{C} \to \mathrm{Sym}^2(\mathbf{C}^r)$ of the $\mathbf{O}_r$ invariant provided by α, and the second map comes from the Cauchy decomposition. The above map induces an algebra homomorphism $f\colon \mathrm{Sym}(\mathrm{Sym}^2(E)) \to \mathrm{Sym}(\mathbf{C}^r \otimes E)$. The induced map on spectra $f^*\colon \mathrm{Hom}(E, \mathbf{C}^r) \to \mathrm{Sym}^2(E)^*$ takes a linear map $\varphi\colon E \to \mathbf{C}^r$ to the form $\varphi^*(\alpha)$ on E, and thus surjects onto the locus of forms of rank $\leqslant r$.

Now let β be a nondegenerate symplectic form on $\mathbf{C}^{2s}$. A similar construction yields a homomorphism $g\colon \mathrm{Sym}(\bigwedge^2(F)) \to \mathrm{Sym}(\mathbf{C}^{2s} \otimes F)$ such that g^* surjects onto the locus of forms in $\bigwedge^2(F)^*$ of rank $\leqslant 2s$.

Finally, let γ be the nondegenerate orthosymplectic form on $\mathbf{C}^{r|2s}$ that restricts to α and β on the even and odd pieces. Once again, we get a natural algebra homomorphism $h\colon \mathrm{Sym}(\mathrm{Sym}^2(V)) \to \mathrm{Sym}(\mathbf{C}^{r|2s} \otimes V)$. One easily verifies that the following square commutes:

$$\begin{array}{ccc} \mathrm{Sym}(\mathrm{Sym}^2(V)) & \longrightarrow & \mathrm{Sym}(\mathrm{Sym}^2(E)) \otimes \mathrm{Sym}(\bigwedge^2(F)) \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f\otimes g} \\ \mathrm{Sym}(\mathbf{C}^{r|2s} \otimes V) & \longrightarrow & \mathrm{Sym}(\mathbf{C}^r \otimes E) \otimes \mathrm{Sym}(\mathbf{C}^{2s} \otimes F) \end{array}$$

Here the horizontal maps are the surjection of the left ring onto the quotient by its nilradical. It follows that h^* surjects onto $X_{r,s}(V)$.

Finally, observe that the multiplicity space of $\mathbf{S}_{\rho(r+1,s+1)}(V)$ in the algebra $\mathrm{Sym}(\mathbf{C}^{r|2s} \otimes V)$ is $\mathbf{S}_{\rho(r+1,s+1)}(\mathbf{C}^{r|2s})$ by the Cauchy decomposition, which vanishes by Proposition 2.3. Thus, $\ker(h)$ contains $I_{r+1,s+1}(V)$, and so $\mathcal{V}(I_{r+1,s+1}(V))$ contains the image of h^*. This proves the lemma. □

5.4 | The main theorem

We now come to our main result:

Theorem 5.8. *The* **GL***-primes of A are exactly the ideals $I_{r,s}$ with $r, s \geqslant 1$ and the zero ideal. The* **GL***-radical ideals of A are exactly the ideals I_λ and the zero ideal.*

Corollary 5.9. *Let $S = \mathbf{N}^2 \cup \{\infty\}$ equipped with the partial order described as follows: $(r, s) < \infty$ for all (r, s); and $(r, s) \leqslant (r', s')$ if $r \leqslant r'$ and $s \leqslant s'$. Endow S with the unique sober topology for which $\leq$ is the generalization order on points. Then $\mathrm{Spec}_{\mathbf{GL}}(A)$ is homeomorphic to S.*

Before proving the theorem, we require a lemma.

Lemma 5.10. *Suppose* $\dim(E) \geqslant r$ *and* $\dim(F) \geqslant 2s$. *Let* λ *be a partition such that* $\rho(r,s) \not\subset \lambda$. *Then* $\mathcal{V}(I_{r,s}(V)) \not\subset \mathcal{V}(I_\lambda(V))$.

Proof. By Proposition 5.2, we have $\mathcal{V}(I_\lambda(V)) = \bigcup_{(p,q)\in C} \mathcal{V}(I_{p,q}(V))$ where C is the set of corners of λ. For any $(p,q) \in C$ we have $\rho(r,s) \not\subset \rho(p,q)$, that is, $r > p$ or $s > q$. It follows that $V(I_{p,q}(V))$ does not contain any pair (ω, η) with $\operatorname{rank}(\omega) = r-1$ and $\operatorname{rank}(\eta) = 2s-2$. However, $\mathcal{V}(I_{r,s}(V))$ does contain such pairs. □

Proof of Theorem 5.8. As $\mathcal{V}(I_{r,s}(V)) = X_{r-1,s-1}(V)$ is irreducible for all super vector spaces V, we see that $\operatorname{rad}_{\mathbf{GL}}(I_{r,s})$ is **GL**-prime by Theorem B. If I is a **GL**-ideal of A that properly contains $I_{r,s}$ then it contains some $\mathbf{S}_{2\lambda}$ with $\rho(r,s) \not\subset \lambda$, and then $\mathcal{V}(I_{r,s}(V)) \not\subset \mathcal{V}(I(V))$ for large V by the lemma. As $\mathcal{V}(I_{r,s}(V)) = \mathcal{V}(\operatorname{rad}_{\mathbf{GL}}(I_{r,s})(V))$ for all V by Theorem A, it follows that $I_{r,s} = \operatorname{rad}_{\mathbf{GL}}(I_{r,s})$. Thus, $I_{r,s}$ is **GL**-prime.

We now show that the $I_{r,s}$ account for all the nonzero **GL**-primes of A. Thus, let I be some nonzero **GL**-prime ideal of A. Then $\mathcal{V}(I(V))$ is irreducible for all V by Theorem A. The rule $V \mapsto \mathcal{V}(I(V))$ defines an irreducible closed subvariety of X, and therefore coincides with $X_{r,s}$ for some $r, s \in \mathbf{N} \cup \{\infty\}$ by Corollary 5.4. As I contains $I_{\rho(r',s')}$ for some r' and s', it follows that $r, s < \infty$. Thus, $\operatorname{rad}(I(V)) = \operatorname{rad}(I_{r+1,s+1}(V))$ for all V, and so $I = I_{r+1,s+1}$ by Theorem A, as both I and $I_{r+1,s+1}$ are **GL**-radical.

As I_λ is an intersection of rectangular ideals (Proposition 5.2), it is therefore **GL**-radical. An argument similar to the one in the proof of Proposition 5.2 shows that any intersection of rectangular ideals is equal to I_λ for some λ, or the zero or unit ideal. As any **GL**-radical ideal is an intersection of **GL**-primes, the result follows. □

Remark 5.11. The idea that Theorem 5.8 should be true came out of joint work with Steven Sam.

Remark 5.12. There is an alternate method for proving that $I_{r,s}$ is **GL**-prime: explicitly compute the product ideal $I_\lambda I_\mu$ in A, for all λ and μ, and verify the primality condition directly. As far as we know, the computation of $I_\lambda I_\mu$ does not appear in the literature in this case. However, a closely related case (namely, that of $\operatorname{Sym}(V \otimes W)$ with $\mathbf{GL}(V) \times \mathbf{GL}(W)$ acting) is treated in [28].

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