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STUDIES OF ONE-PARAMETER SUBGROUPS OF CERTAIN TOPOLOGICAL AND MATRIX GROUPS

by

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Introduction

Several classes of abstract topological groups possess one important property of certain matrix groups. This property is the existence of a neighborhood of the identity in which each element belongs to one and only one one-parameter subgroup [5]*. This similarity of these groups with matrix groups is established, by recent researches in the theory of topological groups, by the use of certain extrinsic operations i.e. by the use of functions on the abstract groups to certain matrix groups.

Since the concept of a topological group was abstracted from observations on groups of transformations by recognizing certain operations as intrinsic, the question arises as to the possibility of the existence of certain intrinsic operations and properties of groups of transformations or certain matrix groups which will guarantee the existence of this property mentioned above in certain class of abstract topological groups.

It is with this question that this paper is concerned.

More precisely, this paper is concerned with investigating the possibility of characterizing by intrinsic properties a neighborhood of the identity with the property mentioned in the first paragraph. It is clear that the problem consists of two parts; (1) a scheme for defining one-parameter subgroups must be presented and (2) a neighborhood of the identity must be exhibited such that one-parameter subgroups are defined and are unique.

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*The numbers in square brackets refer to the entries in bibliography.

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This paper does not settle the question of general topological groups. However, in Part 1 of this paper, a method for defining the subgroups in abstract topological groups is presented. This method seems to be justified because of the apriori existence of canonical parameters in certain classes of them.

In Part II, various matrix groups are studied. The object of this study of matrix groups is to exhibit in each a neighborhood of the identity which has the desired property. Definitions and preliminary considerations of different norms on $M_n(c)$ are given in section I. In the absence of a complete solution for the most general matrix group, there is presented in section II a solution for this problem in two special cases. Certain relations between the eigen values and eigen directions of a transformation and those of a square root of a transformation are established in section III. Finally, the methods of the two special cases in section II are applied to the full linear group of order n in sections IV and section V, respectively. The powers of matrices in each case are defined using well known canonical forms of matrices. These canonical forms are those achieved by the affine group and by the unitary group, respectively.

Since such extensive use is made of the various canonical forms of an element of GL(n.c) the continuity of the canonical form of a matrix as a function of a matrix is discussed in the appendix.

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PART I

A set G of elements is called a <u>topological group</u> if

(1) G is an abstract group

(2) G is a topological space

(3) the group operations in G are continuous in the topological space G.

A neighborhood V of the identity in a topological group has a <u>one parameter</u> subgroup $g(t)$, $|t| \leq \alpha$, if in V an element $g(t)$ is given which depends continuously on a real parameter t and which is defined for all values t not exceeding α in absolute value, and if the following conditions are satisfied: $g(0) = e$ and if $|s| \leq \alpha$, $|t| \leq \alpha$, and $|t + s| \leq \alpha$, then $g(s) \cdot g(t)$ is defined and $g(s) \cdot g(t) = g(s + t)$.

Part 1 is concerned with proposing a method for defining one-parameter subgroups in certain classes of abstract topological groups.

§ 1.

The general method of this section is suggested by the process for the development of a left invariant Haar measure on locally compact topological groups. That process is based essentially on the study of sequences of ratios each of which compares the number of elements in a minimal covering by left translates of $V_K \in \{V_n\}$, a neighborhood system of the group, of a compact set with interior different from the null set to that of a fixed compact set having the same property. A description of the method for obtaining such sequences follows.

Let $\{V_n\}$ be a system of neighborhoods of the identity in a topological group G, and let A and B be two compact sets with the

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property mentioned above. For each V_k in V_n there exists a minimal covering of A and of B, respectively. In view of this fact there is defined for each V_k a rational number which is the ratio of the number of elements in a minimal covering of A to the number of elements of such a covering of B. The collection of all such ratios one for each V_k in $\{V_n\}$ defines a sequence of rational numbers which relates A to B. This process defines a function, relative to a fixed B, from the family of all compact sets in G to the set of bounded rational sequences.

The method proposed in this section is a generalization of the one just explained.

A covering of two elements x and y in a topological space S is a collection of open sets $\{O_\gamma\}$ in S such that x and y are contained in $\bigcup_\gamma O_\gamma$ and $\bigcup_\gamma O_\gamma$ is connected.

Let G be a locally connected topological group; then there is a system $\{V_n\}$ of neighborhoods of the identity in G such that each V_k is a connected set.

If $\mathcal{O}$ is a connected open set in G which contains e then for each $a \in \mathcal{O}$ there exists a covering of e and a for each V_k in $\{V_n\}$. For each V_k the set of all left translates of V_k for $x \in \mathcal{O}$ forms a covering of $\mathcal{O}$. Since $\mathcal{O}$ is connected, for any two elements x and y in $\mathcal{O}$, there is contained in this covering a simple chain from x to y. This simple chain is a covering of x and y in the sense defined above. In particular, there is a covering of e and a for any $a \in \mathcal{O}$.

Let $[a : V_k]$ denote the number of elements in a minimal covering of e and a by left translates of V_k. Suppose e, $a \in \mathcal{O}$, $\mathcal{O}$ defined above. Such a covering does exist and is finite.

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As in the Haar process for each two elements x and a in $\mathcal{O}$ and for each V_k there is defined a rational number which is the ratio of the number of elements in a minimal covering of e and x by left translates of V_k to the number of elements in such a covering of e and a. This rational number will be denoted by $\frac{[x : V_k]}{[a : V_k]}$. The collection of all such ratios, one for each V_k in V_n defines a sequence of rationals. If a is fixed element of $\mathcal{O}$ the above process defines a function from the elements of $\mathcal{O}$ to the set of all rational sequences. Denote this function by $\lambda_V(x) = \left\{\frac{[x:V_m]}{[a:V_m]}\right\}$.

The function $\lambda_V(x)$ satisfies the following relations:

(1) If $x, x^{-1} \in \mathcal{O}$, then $\lambda_V(x) = \lambda_V(x^{-1})$

(2) If xy, x, and $y \in \mathcal{O}$, then $\lambda_V(xy) \leq \lambda_V(x) + \lambda_V(y)$.

It is quite easily shown that the sequences obtained in the Haar process are uniformly bounded relative to the family of all equivalent system of neighborhoods in G. The results of an example to be worked out later will show that the boundedness of $\lambda_V(x_0)$ depends in some cases on the particular system of neighborhoods used. It is this variation of the boundedness of the sequence $\lambda_V(x_0)$ that motivates the proposed method for the definition of one parameter subgroups in a neighborhood of the identity in certain classes of topological groups.

The following facts are obvious consequences of the definition of $\lambda_V(x)$:

(1) If $\lambda_V(x)$ is bounded, then $\lambda_V(x^{-1})$ is bounded.

(2) If $\lambda_V(x)$ and $\lambda_V(y)$ are bounded, then $\lambda_V(xy)$ is bounded.

(3) If $\lambda_V(x)$ and $\lambda_V(xy)$ are bounded, then $\lambda_V(y)$ is bounded.

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Example 1. Let G be the additive group of vectors in R^2, and let $\mathcal{O}$ be R^2. Let $\Delta_{a,v}$ denote the set of all x's in R^2 such that $\left\{\frac{[x:V_n]}{[a:V_n]}\right\}$ is a bounded sequence for a fixed and $\{V_n\}$ a particular system of neighborhoods of e in G.

Consider the following equivalent systems of neighborhoods in G:

$$Q = \{Q_n\} = \{(x,y) \mid \rho[(x,y),e] < \tfrac{1}{n}\}$$

$$W = \{W_n\} = \{(x,y) \mid |x| < \tfrac{1}{n^2}, |y| < \tfrac{1}{n}\}$$

$$U = \{U_n\} = \{(x,y) \mid |x| < \tfrac{1}{n}, |y| < \tfrac{1}{n^2}\}$$

The succeeding work will lead to a description of $\Delta_{a,u}$, $\Delta_{a,w}$, and $\Delta_{a,Q}$ for $a = (a,0)$.

(a) If $x = (x,0)$, then $\lambda_u(x)$, $\lambda_w(x)$, and $\lambda_Q(x)$ are bounded sequences.

It is not difficult to see that $[x:U_n]$ must satisfy the following inequalities:

$$[x:U_n] \geq \frac{x}{\frac{1}{n}} \quad \text{and} \quad ([x:U_n] - 1) \leq \frac{x}{\frac{1}{n}} .$$

$[a:U_n]$ must also satisfy similar inequalities.

Therefore

$$nx \leq [x:U_n] \leq nx + 1 \qquad (A) \qquad \text{and}$$

$$na \leq [a:U_n] \leq na + 1 \qquad (B)$$

The result of combining these two inequalities is

$$\frac{nx}{na+1} \leq \frac{[x:U_n]}{[a:U_n]} \leq \frac{nx+1}{na}$$

which proves that $\lambda_u(x)$ is bounded. Since the inequalities (A) and (B) hold for $\lambda_Q(x)$ and $\lambda_W(x)$, these two sequences are also bounded.

(b) If $x = (0,x)$, then $\lambda_Q(x)$ and $\lambda_W(x)$ are bounded and $\lambda_u(x)$ is unbounded.

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The inequality

$$n^2x \leq [x:U_n] \leq n^2x + 1 \qquad (C)$$

follows in the same manner as in (A) of (a). (C) and (B) combine to give

$$\frac{n^2x}{na+1} \leq \frac{[x:u_m]}{[a:u_m]} \leq \frac{n^2x+1}{na}$$

This last inequality shows that $\lambda_u(x)$ is unbounded.

The properties of the boundedness of $\lambda_v(x)$ together with (a) and (b) show that

$$\Delta\, a,Q = R^2$$

$$\Delta\, a,W = R^2$$

$$\Delta\, a,U = \{(x,y) \mid (x,y) \in R^2 \text{ and } y = 0\}.$$

i.e. the straight line containing a.

This example illustrates clearly the following facts: (1) that Δ a,v, for a fixed, depends on V and (2) that there is a system of neighborhoods of R^2 such that a,u is the one-parameter subgroup containing a = (a,0).

It is not difficult to see that if a is any other element in R^2 then there is a neighborhood system V' which satisfies the condition that Δa,V' is a one parameter subgroup in R^2. Moreover these facts are obviously true in R^n, for any n.

The dependence of Δ a,v on the neighborhood system $V = \{V_n\}$ suggests the possibility of defining an order relation in the family $\mathcal{N}$ of all equivalent systems of neighborhoods of e in a topological group.

Denote the elements of $\mathcal{N}$ by X,Y,W,.... . Let $X\, r_a\, Y$ mean that Δ a,x is properly contained in Δ a,y. It is easy to see that Δ a,x is not empty for any group. This relation ra is transitive and r defines a partial ordering on $\mathcal{N}$.

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Z is a minimal element, relative to ra in $\mathcal{H}$, if there is no $X \in \mathcal{H}$ such that $\Delta a,x$ is properly contained in $\Delta a,z$. If $\Delta a,x = \Delta a,y$, then X is equivalent to Y.

The relation $\Delta a,x \cup y = \Delta ax \cap \Delta ay$ is a consequence of the definition of $\Delta a,v$.

Consider the group of example 1. Let $\mathcal{H}'$ be the family of all convex neighborhood systems in R^2. The system U is a minimal element relative to ra in $\mathcal{H}'$ and Q is equivalent to W. On the other hand, if a = (0,a) then U is not a minimal element and Q and W are not equivalent. This illustrates the dependence of ra on a. In particular, if a $\neq$ (0,a), then W is a minimal element in $\mathcal{H}'$ relative to ra.

In example 1, $\Delta a,u$ may be described in another manner. $\Delta a,u$ is the intersection of $\Delta a,v$ for all $V \in \mathcal{H}'$.

In view of the preceding considerations, the proposed method may be stated.

The conjecture regarding the proposed method is that in certain classes of topological groups a neighborhood $\mathcal{O}$ of the identity can be characterized such that for a $\in \mathcal{O}$ $\Delta a,v$ for a certain $V \in \mathcal{H}$ the family of all equivalent systems of neighborhoods, is a one-parameter subgroup containing a or $\bigcap_{V \in \mathcal{H}'} \Delta a,v$ where $\mathcal{H}'$ is a collection of elements of $\mathcal{H}$ which possess certain properties is the one parameter subgroup of a contained in $\mathcal{O}$. These two facts have been illustrated in the example 1.

There are many limitations to such a proposal. Some of these will now be pointed out.

The function $[x:V_k]$, defined in terms of "copies" of V_k which were contained in a covering of $\mathcal{O}$ by left translates of V_k depends on

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itself. In example 1, if $\mathcal{O}$ were not convex relative to the identity then one could not expect Δ a,v, for any V, to define a one-parameter subgroup for some a in the manner described above.

In example 1, U was a minimal element in $\mathcal{H}'$ relative to r_a and U was a system such that Δ a,u was the one parameter subgroup of a. A minimal element does not always define a one parameter subgroup as the following example will show. Let $G = R \times R/N$ and let $\mathcal{O}$ be $\{(-2 < t < 2) \times R/N\}$ and let a=(1,0). By the same method as in example 1, there exists a system Q such that Δ a,Q = $\{(-2,0) \leq (t,0) \leq (2,0)\}$. On the other hand there is another system Q' such that Δ a,Q' = $\{(-1,1) \leq (t,t) \leq (1,1)\}$ which is clearly a one parameter subgroup containing a. But $Q \cup Q'$ is a minimal system and

$$\Delta \text{ a, } Q \cup Q' = \{-a,e,a\}.$$

However, because of a relation between R^2 and $R \times R/N$ it is known that there is an Q in $R \times R/N$ in which this process will work [5].

The preceding discussion indicates that the first step in the development of this proposed method is to characterize a neighborhood $\mathcal{O}$ of the identity such that these difficulties at least will not be present.

The next step in this development must be to describe a system V such that Δ a,v will be the one parameter subgroup of a in $\mathcal{O}$. One observes, upon referring again to example 1, that the systems Q,W, and U, although topologically equivalent, differ in some respect, since the sets Δ a,u, Δ a,w, and Δ a,Q were not always the same. This difference may be regarded as due to the "shape" of the sequences. In particular, U is a "thin" sequence relative to a = (a,0) but not so if a = (0,a). A

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fundamental difference exists between a minimal sequence and a "thin" sequence. In the discussion of $R \times R/N$, $Q \cup Q'$ was a minimal sequence but $Q \cup Q'$ was not a "thin sequence. It is preferred that the concept of a "thin" sequence be reserved to systems which bound only a one parameter subgroup.

In order to substantiate further the proposal of this section, another example will be presented. In the discussion of this example, there will be exhibited a neighborhood V of the identity in which one parameter subgroups are defined and unique, and the existence of a neighborhood systems which will bound corresponding subgroups for given elements of this neighborhood will be established.

Example II. Let G be the special orthogonal group of order 3 i.e. G is composed of all 3 x 3 real orthogonal matrices with determinant equal to unity. This group is denoted by $SO(3)$ and is in 1-1 correspondence with the set of all rotations of three dimensional real vector space with angles satisfying

$$0 \leq \theta < 360^\circ.$$

If $M \in SO(3)$ then there exists an $\sigma \in SO(3)$ such that $M = \sigma P \sigma^{-1}$, where

$$P = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $0 \leq \theta < 360^\circ$.

Let $V = \{ M \mid M \in SO(3) \text{ and } |M - 1| < \epsilon < 1 \}$ *

*For definition of $|M - 1|$, see Part II, section 1.

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V is a symmetric neighborhood of 1 and the angle of M satisfies $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Any matrix of the form P has at most two distinct square roots in SO(3). Any square root of P in is of the form

$$Q = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

where $A \in SO(2)$. In this case, the angle of Q is $\frac{\theta}{2}$ or $\frac{\theta}{2} + \pi$. Hence, if $P \in V$, there is only one square root of P in V. Call this square root $P^{\frac{1}{2}}$.

Define P such that

$$P^{\lambda} = \begin{pmatrix} \cos \lambda\theta & -\sin \lambda\theta & 0 \\ \sin \lambda\theta & \cos \lambda\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

P^{λ} is uniquely defined in V and satisfies all properties of a one parameter subgroup in V.

If $M = O P O^{-1} = APA^{-1}$, then $CPC^{-1} = P$ where $C = A^{-1}O$. If $M \in V$, then $CP^{\frac{1}{2}}C^{-1} \in V$., and $CP^{\frac{1}{2}}C^{-1}$ is a square root of P in V. Since there is only one square root of P in V, it is clear that $CP^{\frac{1}{2}}C^{-1} = P^{\frac{1}{2}}$.

It follows from the preceding that $CP^{\lambda}C^{-1} = P^{\lambda}$, $|\lambda| \leq 1$. Define $M^{\lambda} = O P O^{-1}$, $|\lambda| \leq 1$. Then M^{λ} is a one parameter subgroup in V containing M.

Therefore V satisfied the conditions stipulated at the beginning of this discussion.

Consider the following function of λ to $M_3(R)$ i.e. the total matrix algebra over reals of order 3.

$$\frac{M^{\lambda} - 1}{\lambda} = O \frac{P^{\lambda} - 1}{\lambda} O^{-1}, \quad |\lambda| \leq I$$

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$$\lim_{\lambda \to 0} \frac{M^{\lambda}-1}{\lambda} = O H O^{-1} \text{ where}$$

$$H = \begin{pmatrix} 0 & -\theta & 0 \\ \theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It is clear that $O H O^{-1}$ is a continuous function of M, since H is a continuous function of P. Moreover $O H O^{-1}$ is skew symmetric.

This continuous mapping f of $M \to O H O^{-1}$ is 1-1 from V to $f(\bar{V}_{\epsilon})$ contained in the space of skew symmetric matrices. If $f(s) = f(t)$, then the point wise invariant line of S and T is the same. This is true since the null space of f(s) is of dimension 1 or dimension 3 and that line goes into this space. It is clear then that the angle of S = angle of T.

The space of skew symmetric matrices is isomorphic to R^3 and V_{ϵ} is homeomorphic to a neighborhood W of e in R^3 such that this neighborhood W is convex relative to e and the straight lines of W are the local homomorphic images of the one parameter subgroups of V.

In W, there exists a neighborhood system for each $a \in W$ that will bounded only the straight line containing a. If this homeomorphism is denoted by g then the image of this system under g will bound only the one-parameter subgroup containing g(a).

More generally, there is a large class of topological groups which possess a neighborhood of the identity that can be mapped topologically onto a convex relative to e neighborhood of R^n such that the segments of lines in R^n are the images of one parameter subgroups. In particular the property of matrix groups mentioned in the first paragraph of introduction is established in this way [1].

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This last fact motivates the study of these more concrete objects, i.e. Matrix groups. Part II of this paper is concerned with the characterization of neighborhoods of this type in $GL(n,c)$.

§ II.

In order to eliminate the second difficulty of the method in section 1, a new concept is defined. This new concept is to consider special types of covering of two elements, e and a, in an open connected set about the identity of certain classes of topological groups.

A neighborhood $\mathcal{O}$ of the identity is said to be <u>convex relative to e</u> if in each covering of $\mathcal{O}$ by left translates of V_k in $\{V_n\}$, there exists a straight chain of left translates from e to a, for each $a \in \mathcal{O}$. A <u>straight chain of left translates of V_k from e to a</u> is a set $\{t_k V_k, t^2{}_k V_k \ldots, t_k^{q_k} V_k\}$ such that $e \in t_k V_k$ and $a \in t_k^{q_k} V_k$.

<u>If G is locally compact and $\mathcal{O}$ is a neighborhood of e whose closure is compact and which has the property mentioned above, then $a \in \mathcal{O}$ there exists $a^{\frac{1}{2}} \in \bar{\mathcal{O}}$.</u>

If $a \in \mathcal{O}$, then it is clear that there exists a sequence $\{t_n\}$ such that $t_n \to e$ and $t_n^{k_n} \to a$, with $\{t_n^{k_n}\}$ also contained in $\mathcal{O}$.

Let $[k_n]_2$ denote the largest positive integer $\leq \frac{k_n}{2}$ and let $\{k_n\}_2$ denote the smallest positive integer $\geq \frac{k_n}{2}$.

The sequence $\{t_n^{[k_n]_2}\}$ has a subsequence $\{t_{n_m}^{[k_{n_m}]_2}\}$ which converges to some $a_o \in \bar{\mathcal{O}}$.

The sequence $\{t_{n_m}^{\{k_{n_m}\}_2}\}$ has a convergent subsequence $\{t_{n_{m_p}}^{\{k_{n_{m_p}}\}_2}\}$ that converges to $b_o \in \bar{\mathcal{O}}$.

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From the following relations

$$t_{nmp}^{[K_{nmp}]_2} \cdot t_{nmp}^{\{K_{nmp}\}_2} \longrightarrow a_0 \cdot b_0$$

$$t_{nmp}^{[K_{nmp}]_2} \cdot t_{nmp}^{-[K_{nmp}]} \longrightarrow e = a_0 \cdot b_0^{-1}$$

$$t_{nmp}^{[K_{nmp}]_2} \cdot t_{nmp}^{\{K_{nmp}\}_2} \longrightarrow a$$

it is clear that $a_0 = b_0$ and that $a_0 \circ a_0 = a$.

The hypothesis on θ is not sufficient to guarantee the uniqueness of this square root. The closure of the neighborhood of $R \times R/N$ described on page 18 will satisfy hypothesis but square roots are not unique there.

With regard to uniqueness of square roots in a neighborhood of e, if they exist, it is known that if G is abelian and there exists a neighborhood of the identity which contains no subgroup then there is a neighborhood V in which square roots, if defined there, are unique. This hypothesis has been shown to be sufficient for uniqueness in non-abelian groups for some special cases [6].

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PART II.

Groups of Matrices

Section 1

The total matrix algebra of order n, denoted by $M_n(C)$, consists of all n × n matrices with complex coefficients. The subset of this algebra composed of all matrices with determinant different from zero is a group under multiplication. This group is called the full linear group of order n and is denoted by $GL(n,C)$. An important subgroup of $GL(n,C)$ is the unitary group. An element of the full linear group belongs to this subgroup if it satisfies the relation $\bar{U}' U = U \bar{U}' = 1$, where U is the complex conjugate of U and U' is the transpose of U.

The unitary group, denoted by U_n, has several important properties which will be used in this paper. These properties follow:

1) Any unitary matrix is similar to a diagonal matrix by unitary matrices.
2) The eigen values of a unitary matrix are of absolute value 1.
3) Any matrix in $GL(n,C)$ is similar to a triangular matrix by unitary transformations. The diagonal elements of this triangular matrix are the eigen values of the given matrix [3].

Since the total matrix algebra, $M_n(C)$, is in 1-1 correspondence with C^{n^2}, the finite n-dimensional vector space over the complex field and since C^{n^2} is a metric space, $M_n(C)$ is a metric space in a natural manner. In particular, if $A = (a_{ij})$ is an element of $M_n(C)$, define

$$|A| = \sqrt{\sum_{i,j} a_{ij}\overline{a_{ij}}} .$$

Besides the usual properties of a norm, $|A|$ satisfies the additional

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property that

$$|AI\ |A|\ |B|.$$

This property implies the conity of multiplication in $M_n(C)$. In view of this fact and other pities, $M_n(C)$ is a topological algebra [4] and $GL(n,c)$ is a tgical group with multiplication as the group operation. The gro is a topological subgroup of $GL(n,c)$, since it is closed, and the tgical space of U_n is compact.

The norm, $|A|$ is inva under conjugation by unitary matrices. This norm may be rewritten as

$$|A\overline{\text{race } AA'}\ .$$

Since $\overline{U}' = U^{-1}$, $|UAU^{-1}| =$

On the other hand, one hink of the total matrix algebra, $M_n(C)$, as being the set of alnear transformations of an n-dimensional complex vector space into its It is a well known fact that there is a 1-1 correspondence betweis set of linear transformations, relative to a fixed basis, an set of all $n \times n$ matrices with complex coefficients. In viethis interpretation, a different norm, denoted by $\|A\|$, may efined on this set.

Let $\|A\| = \max_{|x|=1} |Ax$ where $|Ax|^2 = (Ax,Ax)$, with bracket denoting the ordinary inner pt (Hermitian) of a complex (finite dimensional) vector space. Thorm, $\|A\|$, also satisfies the additional property

$$\|AB\| \leq \|B\|.$$

In order to clarify anplify notation, the convention will be made that the symbol Ax de the effect of a linear transformation A operating on a vector The symbol XA denotes the matrix product

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of the matrix of the components of x by the matrix of the transformation A, both matrices given relative to a fixed basis over which 1-1 correspondence is defined. It is clear then that

$$(Ax,Ax) = (XA)(\overline{XA})' = XA\bar{A}'\bar{X}'.$$

This norm $\|A\|$ is invariant by left multiplication of the elements of the unitary group.

Let U be unitary. Then

$$(UAx,UAx) = (XAU)(\overline{XAU})'$$
$$= (XAU\bar{U}'\bar{A}'\bar{X}') = XA\bar{A}'\bar{X}'.$$

In particular $\|A\|$ gives rise to a left invariant metric on U_n but not on GL(n,c).

The norms, $|A|$ and $\|A\|$ are related by the following inequalities

(a) $|A| \leq \sqrt{n}\,\|A\|$.

(b) $\|A\| \leq |A|$.

The matrix $A\bar{A}'$ is Hermitian. $\overline{(AA')}' = (\bar{A}\bar{A}')' = (\bar{A}A')' = A\bar{A}'$. Since $H = A\bar{A}'$ is hermitian there exists a unitary matrix, O, such that $O\bar{A}'A\bar{O}'$ is a real diagonal matrix. The norm, $\|A\|$, is equal to the square root (positive) of the maximum eigen value of $\bar{A}'A$.

$$(Ax,Ax) = XH\bar{X}'$$
$$= X^* OH\bar{O}'\bar{X}^{*\prime}, \text{ where } X = X^*O.$$
and O is unitary.
$$= X^*(OH\bar{O}')X^{*\prime}, \text{ where } OHO^{-1} \text{ is a real diagonal matrix.}$$
$$= \lambda_1 x_1\bar{x}_1 + \lambda_2 x_2\bar{x}_2 + \cdots\cdots + \lambda_n x_n\bar{x}_n.$$

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λ_i's, are eigen values of H satisfying the additional property that $\lambda_i \geq 0$ for all i. Suppose $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ then

$$(Ax,Ax) \leq \lambda_1 (x_1\bar{x}_1 + x_2\bar{x}_2 + \cdots + x_n\bar{x}_n)$$
$$\leq \lambda_1, \quad \text{since } x^* = (x_1, x_2, \cdots x_n)$$

is a unit vector if x is a unit vector. On the other hand, there exists a unit vector such that (Ax,Ax) assumes λ_1. This gives the relation that

$$\|A\| = \sqrt{\lambda_1}.$$

Since O is unitary, $\bar{O}' = O^{-1}$. Using this fact one observes that H is similar to $OH\bar{O}'$. In this case,

$$\text{trace } H = \text{trace } OHO^{-1} \quad \text{i.e}$$
$$\sum_{i,j} a_{ij}\bar{a}_{ij} = \sum_{i=1}^{n} \lambda_i$$

$$\sum_{i,j} a_{ij}\bar{a}_{ij} \leq n\lambda_2$$

Therefore,

$$|A| \leq \sqrt{n}\,\|A\|.$$

$$\sum_{i,j} a_{ij}\bar{a}_{ij} = \sum_{i=1}^{n} \lambda_i \Rightarrow \sum_{i,j} a_{ij}\bar{a}_{ij} \geq \lambda_1$$

and

$$|A| \geq \|A\|.$$

<u>The eigen values of A are each less than or at most equal in absolute value to $|A|$ and to $\|A\|$.</u>

Let ρ be an eigen value of A and x an eigen vector belonging to ρ, then

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$$Ax = \rho x$$

$$\|A\|^2 \geq (Ax, Ax) = \rho\bar{\rho}(x,x) = |\rho|^2$$

Therefore

$$\|A\| \geq \rho.$$

$|A| \geq \rho$ follows from (a) or independently of this relation,

$$|A| = |UTU^{-1}| = |T|$$

where T is a triangular matrix. It is clear that

$\sqrt{\sum_{i=1}^{n} t_{ii}\,\bar{t}_{ii}} \leq |T|$, where the t_{ii}'s are the diagonal elements of T and also its eigen values.

Section 2

The remaining sections of this paper will be devoted to the consideration of special classes of topological groups, namely matrix groups. The objective of these sections is to exhibit a neighborhood V of the identity in these groups which satisfies the property that for each element M there is defined in V one and only one one-parameter subgroup. The method of definition to be employed here is to exhibit a set of powers of M denoted by M^{λ} for λ real and $|\lambda| \leq 1$ which satisfy the following properties;

(1) M^{λ} is continuous in λ.

(2) $M^1 = M$; $M^0 = 1$.

(3) If $|\lambda| \leq 1$, $|u| \leq 1$, and $|\lambda + u| \leq 1$, then $M^{\lambda} \cdot M^{u}$ is defined and $M^{\lambda} \cdot M^{u} = M^{\lambda+u}$.

Since the solution of this problem for the most general matrix group, i.e. GL(n,c), is not given in this paper, there will be presented

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in this section two special cases of matrix groups in which these properties are exhibited. Many special cases have been worked out. However, these two groups, namely GL(2,c) and U_n are presented here because of their significance to the remaining sections of this paper. In the first place, these examples will indicate in a general manner the steps necessary for such a construction in the most general case and, secondly, the methods of the solutions of each, which are essentially different, are used as a basis for generalization in the investigation of GL(n,c).

Example III. The full linear group of order 2, i.e. GL(2,c).

Let O be the set of all 2 x 2 non-singular matrices with eigen values λ_i satisfying the inequality $|\lambda_i - 1| < \frac{1}{\sqrt{2}}$. O is an open set containing the identity of GL(2,c) since each λ_i of a matrix M is a continuous function of M.

If $M \in GL(2,c)$, then M is similar under the affine group to one of the three canonical forms,

$$P_1 = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}, \quad P_2 = \begin{pmatrix} \rho & 0 \\ 0 & \rho \end{pmatrix}, \quad P_3 = \begin{pmatrix} \rho & 0 \\ 1 & \rho \end{pmatrix}$$

Consider the matrix equation $X^2 = P$

i.e.

$$\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}^2 = \begin{pmatrix} x^2 + yz & y(x+w) \\ z(x+w) & w^2 + yz \end{pmatrix} = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix}$$

Case (1) $P_2 = P_3 = 0$, $P_1 \neq P_4$

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$y(x + w) = 0$ implies that $y = 0$ or $(x + w) = 0$. If $x + w = 0$, then $x = -w$ and $P_1 = P_4$. Therefore $y = 0$ and $z = 0$. This proves that <u>any square root of a diagonal matrix with distinct eigen values is a diagonal matrix.</u> Case (2) $P_2 = P_3 = 0$, $P_1 = P_4$.

$y(x + w) = 0$ implies $y = 0$ or $x + w = 0$. Therefore $y = 0$ and $z = 0$ or $x + w = 0$. In this case it is clear that <u>any square root of a scalar matrix is either a diagonal matrix or a matrix with trace equal to zero or both.</u>

Since any matrix in O has trace different form zero, the only possible square roots of this type that are in O are diagonal matrices. Case (3) $P_2 = 0$, $P_1 = P_4$, $P_3 \neq 0$.

If $x + w = 0$, then $P_3 = 0$. Therefore $y = 0$ and <u>any square root of a matrix of type P_3 is a triangular matrix.</u>

From the preceding considerations one concludes that if P_1, P_2 or P_3 are in O then there is only one square root of P_i, $i = 1,2,3$, in O. Denote this matrix by $P_i^{\frac{1}{2}}$, $i = 1,2,3$. $P_i^{\frac{1}{2}}$ is the square root matrix whose coordinates are the principal square roots of eigen values of P_i.

If P_1, P_2 and P_3 are in O, define

$$P_1^{\lambda} = \begin{pmatrix} \rho^{\lambda} & 0 \\ 0 & \tau^{\lambda} \end{pmatrix} \quad P_2^{\lambda} = \begin{pmatrix} \rho^{\lambda} & 0 \\ 0 & \rho^{\lambda} \end{pmatrix} \quad P_3^{\lambda} = \begin{pmatrix} \rho^{\lambda} & 0 \\ \frac{\lambda}{\rho^{1-\lambda}} & \rho^{\lambda} \end{pmatrix}$$

P_i^{λ}, $i = 1,2$, and 3 for $|\lambda| \ll 1$ satisfies the conditions of the first paragraph of Section II.

Let $M \in O$. Suppose $M = APA^{-1} = OPO^{-1}$ with $P = P_i$, $i = 1,2$, or 3. Then $P = (A^{-1}O)P(A^{-1}O)^{-1}$. $(A^{-1}O)P^{\frac{1}{2}}(A^{-1}O)^{-1}$ is a square root of P and by definition of $P^{\frac{1}{2}}$ it is an element of O. In this case

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$$(A^{-1}\mathcal{O})P^{\frac{1}{2}}(A^{-1}\mathcal{O})^{-1} = P^{\frac{1}{2}}$$

The equation $(A^{-1}\mathcal{O})\ P^{\lambda}(A^{-1}\mathcal{O})^{-1} = P^{\lambda}$ follows by continuity.

Define $M^{\lambda} = AP^{\lambda}A^{-1}$ for $|\lambda| \leq 1$. M^{λ} is defined in $\mathcal{O} \cup \mathcal{O}^{-1} = V$ and satisfies all required properties.

Example II. The unitary group of dimension n.

Let $V = \{U \mid U \in U_n, |U-1| < \frac{1}{\sqrt{2}}\}$

V is symmetric, since $|e^{i\theta}-1| = |e^{i\theta}(1-e^{-i\theta})| = |1-e^{-i\theta}|$

Let P be a diagonal unitary matrix in V. The eigen values of P are in a neighborhood W of 1 in the complex number system where $W = \{z \mid z \in \mathbb{C},\ |z-1| < \frac{1}{\sqrt{2}}\}$.

If $P \in V$ it is clear that there is only one diagonal unitary matrix in V that is a square root of P. This square root of P is the diagonal unitary matrix whose diagonal elements are the principal square roots of those of P. These diagonal elements are in the same order as their corresponding square in P. Denote this matrix by $P^{\frac{1}{2}}$.

Since any square root of a diagonal matrix with distinct eigen values is a diagonal matrix with distinct eigen values, $P^{\frac{1}{2}}$ is the only square root of such a P in V, if $P \in V$. This is true by Theorem II, page 24.

The set of all diagonal unitary matrices with distinct eigen values is dense in the set of all diagonal matrices.

The operation of square root is uniformly continuous from this dense set to V. The eigen values of any such P are of the form $e^{i\theta}$. In the neighborhood W, $|e^{i\theta/2} - e^{i\phi/2}| \leq |e^{i\theta} - e^{i\phi}|$. This inequality implies that $|P^{\frac{1}{2}} - Q^{\frac{1}{2}}| \leq |P - Q|$, if $P,Q \in V$.

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In this case, the operation of square root can be extended uniquely and continuously to all diagonal unitary matrices in V.

The preceding proves that there is only one square root of a diagonal matrix P in V, if P is in V. This square root is $P^{\frac{1}{2}}$ defined on the preceding page.

It is easy to see that $P^{\lambda}, |\lambda| \leqslant 1$ is uniquely defined in V and that

$$|P^{\lambda} - 1| \leqslant |P - 1|, \quad \text{for } |\lambda| \leqslant 1.$$

Moreover, the set of all diagonal matrices in V satisfy the conditions (1), (2) and (3) of the first paragraph in this section.

Let U be any unitary matrix in V and suppose $U = APA^{-1} = OPO^{-1}$, with P a diagonal matrix. In this case, $(O^{-1}A)P(O^{-1}A)^{-1} = P$ and $(O^{-1}A)P^{\frac{1}{2}}(O^{-1}A)^{-1}$ is a square root of P. $(O^{-1}A)P^{\frac{1}{2}}(O^{-1}A)^{-1}$ is in V and therefore $(O^{-1}A)P^{\frac{1}{2}}(O^{-1}A)^{-1} = P^{\frac{1}{2}}$.

In general $(O^{-1}A)P^{n/2^{k}}(O^{-1}A)^{-1} = P^{n/2^{k}}$. By the continuity of $P^{n/2^{k}}$ and by the continuity of multiplication in U_n,

$$(O^{-1}A)P^{\lambda}(O^{-1}A)^{-1} = P^{\lambda}, \quad |\lambda| \leqslant 1.$$

Define $U^{\lambda} = OPO^{-1}$, $|\lambda| \leqslant 1$. U^{λ} is a continuous function of λ and satisfies all the properties (1) (2) and (3) mentioned above, since P^{λ} does.

Section 3

Certain relations between the eigen directions and the eigen values of a linear transformation f on a finite dimensional complex vector space V^n and those of a square root of f are established in this section.

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Definition. A transformation g is a square root of a transformation f if $g^2x = fx$ for all $x \in V^n$.

Theorem I. The eigen directions of a square root of f are eigen directions of f. Any eigen value of a square root g of f is the square root of an eigen value of f.

Proof: If x_0 is an eigen direction of g with eigen value λ, then

$$g[gx_0] = \lambda^2 x = fx.$$

The converse of the first part of this theorem is not in general true. The succeeding considerations will verify this fact.

Let f be the identity linear transformation on V^2. Then f is represented by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Every direction in V^2 is an eigen direction of f. On the other hand the matrices $g_\alpha = \begin{pmatrix} 0 & \alpha \\ \alpha^{-1} & 0 \end{pmatrix}$ are square roots of f for each complex α. In the coordinate system upon which g_α is represented by a matrix, (1,0) is not an eigen direction of g_α for any α.

Theorem II. If f is a linear transformation on V^n with n distinct eigen values, then any square root of f has the same eigen directions as those of f.

Proof: Denote the n distinct eigen directions of f which correspond to the n distinct eigen values by $x_1, x_2, \ldots x_n$. If g is a linear transformation on V^n then g has an eigen direction in V^n. If g is a square root of f then the eigen direction of g is an eigen direction of f. This eigen direction is one of the x_i's. Suppose that this direction is x_1.

$V^n - x_1$ is an invariant subspace of f, since it is generated by $x_2, x_3 \ldots x_n$.

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$V^n - X_1$ is also an invariant subspace of g. Suppose not i.e. there exists $z \varepsilon V^n - X$ such that $gz \notin V^n - X_1$. Then $gz = ax_1$ and $fz = a\lambda x_1$, where λ_1 is the eigen value of g that belongs to x_1. In this case $V^n - x_1$ is not an invariant subspace of f.

Since $V^n - X_1$ is an invariant subspace of g, g has an eigen direction in $V^n - X_1$. Suppose that this direction is x2. In the same manner as in the preceding paragraph $V^n - \{X_1 \cdot X_2\}$ is an invariant subspace of g. The bracket $\{x_1 x_2\}$ means the subspace of V^n generated by x_1 and x_2. In general, $V^n - \{X_1, X_2, \ldots X_K\}, K<n$ is an invariant subspace of g and this subspace has an invariant direction of g which belongs to the set $X_{K+1}, X_{K+2}, \ldots, X_n$.

In terms of matrices, the preceding theorem may be interpreted in the following manner: if P is a diagonal matrix with distinct diagonal elements then any square root of P is a diagonal matrix with distinct diagonal elements. The example following theorem 1 shows that this fact is not in general true for any diagonal matrix.

Section 4

Section 1 was concerned with the definition of powers of elements of two special subgroups of GL(n,c) and with the determination of a region about the identity of each in which these powers satisfy certain properties. Now the concepts and methods of that section will be extended to GL(n.c).

The powers of several canonical matrices will be defined and the properties of these powers will be discussed. These canonical matrices, which will be denoted by P_1, P_2, P_3, are elements of GL(n.c) such that

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any element of this group is similar under the affine group to one of them or to a matrix each of whose diagonal constituents are of the form P_1, P_2 or P_3.

Let P_1 be a diagonal matrix of order $k \leq n$ i.e.

$$P_1 = \begin{pmatrix} e^{p_1} & 0 & \cdots & 0 \\ 0 & e^{p_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e^{p_k} \end{pmatrix}$$

Define

$$P_1^{\lambda} = \begin{pmatrix} e^{\lambda p_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda p_2} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e^{\lambda p_k} \end{pmatrix}$$

P_1^{λ}, as defined, is a single valued function for λ real, $|\lambda| \leq 1$ and it is clear that

$$P_1^{\lambda} \cdot P_1^{u} = P_1^{\lambda+u} .$$

Let P_2 be the matrix given by

$$P_2 = \begin{pmatrix} e^{p} & 0 & \cdots & 0 \\ pe^{p} & e^{p} & \cdots & 0 \\ \frac{p^2e^{p}}{2!} & pe^{p} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \frac{p^{n-1}e^{p}}{(n-1)!} & \frac{p^{n-2}e^{p}}{(n-2)!} & \cdots & e^{p} \end{pmatrix}$$

where $P \neq 0$.

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Let $P_2(\lambda)$ be the matrix formed from P_2 by replacing P by λP, λ real, wherever P occurs in A, either as an exponent or as a base of a power. More precisely,

$$P_2(\lambda) = \begin{pmatrix} e^{\lambda P} & 0 & 0 & \cdots & 0 \\ \lambda e^{\lambda P} & e^{\lambda P} & 0 & \cdots & 0 \\ \frac{(\lambda P)^2 e^{\lambda P}}{2!} & \lambda P e^{\lambda P} & e^{\lambda P} & \cdots & 0 \\ \vdots & & & & \\ \frac{(\lambda P)^{n-1} e^{\lambda P}}{(n-1)!} & \cdots & \cdots & \cdots & e^{\lambda P} \end{pmatrix}$$

From these considerations, it is clear that P_2 and $P_2(\lambda)$ may be described in the following manner:

$P_2 = (a_{ij})$ (a) $a_{ij} = \frac{\rho^{i-j} e^{\rho}}{(i-j)!}$ $i \geq j$

(b) $a_{ij} = 0$ $i < j$

$P_2(\lambda) = (b_{ij})$ (a) $b_{ij} = \frac{(\lambda\rho)^{i-j} e^{\lambda\rho}}{(i-j)!}$ $i \geq j$

(b) $b_{ij} = 0$ $i < j$

Proposition 1. $P_2(\lambda)\, P_2(u) = P_2(\lambda+u)$.

Proof: Let $P_2(\lambda) = (b_{ij})$, $P_2(u) = (c_{ij})$, then

$P_2(\lambda)P_2(u) = \sum_{k=1}^{n} (b_{ik} \cdot c_{kj}) = (d_{ij})$. It is necessary to prove (a). $d_{ij} = 0$, $j > i$ (b) $d_{ij} = \frac{[(\lambda+u)\rho]^{i-j} e^{(\lambda+u)\rho}}{(i-j)!}$, $i \leq j$

Consider terms of form.

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$$d_{ij} = \sum_{k=1}^{n} b_{ik} c_{kj} \quad , \quad j > 1 \qquad (1)$$

In any term of this sum either $k \geq j$ or $j > k$, since $1 \leq k \leq n$ for i,j fixed integers. In any term of (1) with $j > k$, the factor $C_{kj} = 0$ by definition of $P_2(\lambda)$ hence the term equals zero. In any term of (1) with $k \geq j$, the factor $b_{ik} = 0$, since $k \geq j$ implies $k > i$. It is clear then that d_{ij} is a sum of terms each equal to zero.

To prove (b)

Consider $$d_{ij} = \sum_{k=1}^{n} b_{ik} c_{kj} \quad , \quad i \leq j$$

From the definition of (b_{ij}) and (c_{ij}) it is necessary to consider only those terms of d_{ij} which are different from zero. These terms are those for which k satisfies the inequality $j \leq k \leq i$. Any term in d_{ij} for which k does not satisfy this relation will have a zero factor. Therefore

$$d_{ij} = \sum_{k=1}^{n} b_{ik} c_{kj} = \sum_{k=j}^{i} b_{ik} c_{kj}$$

$$= \sum_{k=j}^{i} \frac{(\lambda\rho)^{i-k}}{(i-k)!} \cdot \frac{(\mu\rho)^{k-j}}{(k-j)!} e^{(\lambda+\mu)\rho}$$

$k = j+m \quad , \quad m = 0, 1, 2, \cdots, i-j.$

$$= \left[\sum_{m=0}^{i-j} \frac{(\lambda\rho)^{(i-j)-m}}{[(i-j)-m]!} \cdot \frac{(\mu\rho)^{m}}{m!} \right] e^{(\lambda+\mu)\rho}$$

$$= \frac{1}{(i-j)!} [(\lambda+\mu)\rho]^{i-j} \cdot e^{(\lambda+\mu)\rho} .$$

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If n is any integer, $P_2(n) = P_2^n$. P_2^n is ordinary matrix product of P_2 by itself n times. $P_2^2 = P_2 \cdot P_2 = P_2^{(2)}$ by proposition 1. Assume $P_2(k) = P_2^k$. $P_2^{k+1} = P_2^k \cdot P_2 = P_2(k) P_2 = P_2(k+1)$. Therefore $P_2(n)$ P_2^n for any positive integer. $P_2(-n)P_2(n) = P_2(0) = 1 = P_2(-n)P_2^n$. Therefore $P_2(-n) = (P_2^n)^{-1} = P_2^{-n}$ by definition of negative powers of matrices.

Since the definition of $P_2(\lambda)$ is compatible with that of P_2^λ for λ an integer, we denote

$$P_2(\lambda) \text{ by } P_2^\lambda .$$

The succeeding work will lead to a proof of the fact that for any degenerate non-singular linear transformation, α, (i.e. α has only one eigen direction) with eigen value e^ρ, $\rho \neq 0$, there exists a basis, ξ, in V_n such that relative to this basis α can be represented by a matrix of the form P_2; or, equivalently, there exists a non-singular T such that $T B T^{-1} = P_2$, with

$$B = \begin{pmatrix} e^\rho & 0 & \cdots & 0 \\ 1 & e^\rho & \cdots & 0 \\ 0 & 1 & & \\ \vdots & 0 & & \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & 1 \; e^\rho \end{pmatrix}$$

i.e. B is the Jordan or classical canonical form for α.

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Theorem 3. If T is a linear transformation on an n-dimensional (complex) vector space and T has no eigen values different from zero, then $T^q = 0$ for $q \geq n$.

Proof: T satisfies its Hamilton-Cayley equation, i.e. there exists complex numbers $a_1, a_2, \ldots a_{n-1}$ such that $T^n + a_1 T^{n-1} + a_2 T^{n-2} + \ldots + a_n = 0$. In this case the eigen values of T satisfy the equation $x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n = 0$. Since all roots of this equation are zero, $a_i = 0$ for $i = 1,2,\ldots n$. This implies that $T^n = 0$.

Theorem 4. Let $W = (w_{ij})$ be any n x n matrix satisfying the following conditions:

(a) $w_{ij} = 0, \quad j \geq i$

(b) $w_{i,i-1} \neq 0,$

then there exists an x_0 in W_n such that $\{x_0, Wx_0, W^2x_0, \ldots W^{n-1}x_0\}$ is a basis for V_n. The same x_0 satisfies the conclusion for any W defined above.

Proof: Let $x_0 = (x_i)$, $x_i = 1$ for $i = 1$, $x_i = 0$, $i \neq 1$.

This theorem will be proved in the following form:

$$W^q x_0 = \left(X_i^{(q)}\right), \quad X_i^{(q)} = 0, \quad 1 \leq i \leq q$$
$$X_{q+1}^{(q)} \neq 0$$

for any $q \leq n - 1$.

If $q = 1$, the theorem is true.

$$W x_0 = \sum_{j=1}^{n} \omega_{ij} x_j = \left(X_i^{(1)}\right)$$

$$X_1' = \sum_{j=1}^{n} \omega_{1j} X_j = 0 \qquad \text{since } W_{1j} = 0 \text{ for } 1 \leq j \leq n.$$

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$x_2^{(1)} = \sum_{j=1}^{n} \omega_{2j}\, x_j \neq 0$ since $w_{21} \neq 0$, $x_1 \neq 0$

and $\omega_{2j} = 0$ for $j \neq 1$, by definition of W.

Assume theorem true for q = k, i.e.

$$W^k X_0 = \left(X_i^{(k)}\right) = X^{(k)} \; ; \; X_i^{(k)} = 0, \; 1 \leq i \leq k$$

$$X_{k+1}^{(k)} \neq 0$$

The theorem is true for q k 1. It is necessary to prove that

$$W^{k+1} X_0 = \left(X_i^{(k+1)}\right) \quad , (1) \; X_i^{(k+1)} = 0, \; 1 \leq i \leq k+1$$

$$(2) \; X_{k+2}^{(k+1)} \neq 0$$

To show that (1) is satisfied, i.e.

$$X_i^{(k+1)} = \sum_{j=1}^{n} \omega_{ij}\, X_j^{(k)} = 0 \text{ for } i \leq k+1.$$

For each i,

$\omega_{ij}\, X_j^k = 0$ for $1 \leq j \leq k$, by definition of $X^{(k)}$

$\omega_{ij}\, X_j^k = 0$ for $k+1 \leq j \leq n$, since in this case

$i \leq k+1$ and $k+1 \leq j$ and this implies $i \leq j$. By definition of W,

$\omega_{ij} = 0$.

To show that (2) is satisfied, i.e.

$$X_{k+2}^{(k+1)} = \sum_{j=1}^{n} \omega_{k+2,j}\, X_j^{(k)} \neq 0$$

If $1 \leq j \leq k$, then $\omega_{k+2,j}\, X_j^k = 0$ since $X_j^{(k)} = 0$ for $1 \leq j \leq k$.

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If $j = k+1$, then $\omega_{k+2,k+1} \neq 0$ by definition of W

and $X^{(k)}_{k+1} \neq 0$, by definition of $X^{(k)}$.

This implies that $\omega_{k+2,k+1} X^{(k)}_{k+1} \neq 0$

If $k+1 < j < n$, then $\omega_{k+2,j} X^{k}_{j} = 0$ since in this case

$j \geq i$ and by definition of W

$\omega_{k+2,j} = 0$ for all j's.

Therefore $$X^{(k+1)}_{k+2} = \sum_{j=1}^{n} \omega_{k+2,j} X^{(k)}_{j} = \omega_{k+2,k+1} X^{k}_{k+1} \neq 0$$

The preceding argument shows that no vector in the set

$\{ X_0, X_0^{(1)}, X_0^{(2)}, \cdots\cdots X_0^{(n-1)} \}$ is a zero vector.

It is clear that this set is a basis for V_n.

Theorem 5. If U and V are linear transformations satisfying the hypothesis of Theorem 4, then there exists an automorphism T of V_n such that $TVT^{-1} = U$.

Proof:

$\{ X_0, UX_0, U^2X_0, \cdots U^{n-1}X_0 \}$ and $\{ X_0, VX_0, V^2X_0, \cdots V^{n-1}X_0 \}$ are two sets of bases in V_n.

Define T such that

$$T: \sum_{k=0}^{n-1} \alpha_k U^k X_0 \longrightarrow \sum_{k=0}^{n-1} \alpha_k V^k X_0$$

T is an automorphism of V_n.

Let x be an arbitrary vector in V_n, then x may be represented as

$$x = \sum_{k=0}^{n-1} \beta U^k X_0$$

$$TVT^{-1}x = TVT^{-1}\left(\sum_{k=0}^{n-1} \beta_k U^k X_0\right) = TV\left(\sum_{k=0}^{n-1} \beta_k V^k X_0\right)$$

$$= \sum_{k=0}^{n-1} \beta_k U^{k+1} X_0$$

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Proposition II. The matrix P_2 is similar to the matrix B.

Proof: Let $\bar{A}$ be the n x n scalar matrix with eigen values e^P.

Let $C = P_2 - \bar{A}$. Then $C = (c_{ij})$ satisfies hypothesis of Theorem 4.

Let $D = B - \bar{A}$. Similarly, $D = (d_{ij})$ satisfies hypothesis of Theorem 4.

By Theorem 5, there exists a non-singular T such that

$$TDT^{-1} = C$$

$$T(B - \bar{A})T^{-1} = A - \bar{A}$$

Since A is a scalar matrix,

$$TBT^{-1} \quad A.$$

Finally, let P_3 be the matrix

$$P_3 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & \\ 1/2! & 1 & 1 & & \\ \vdots & \vdots & \vdots & & \vdots \\ 1/(n-1)! & 1/(n-2)! & \cdots & & 1 \end{pmatrix}$$

Define

$$P_3^{\lambda} = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ \lambda & 1 & 0 & \cdots & \\ \lambda^2/2! & \lambda & 1 & 0 & \cdots \\ \vdots & \vdots & & & \vdots \\ \lambda^{n-1}/(n-1)! & \lambda^{n-2}/(n-2)! & & & 1 \end{pmatrix}$$

If A is the Jordan canonical form for a linear transformation with only one eigen value and this eigen value equal to one and with only one

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eigen direction, then, by theorem 4, P_3 is similar to A. By reasoning the same as in Proposition 1,

$$P_3^{\lambda} P_3^{\mu} = P_3^{\lambda+u}.$$

It is important to remark that the set of all P_3^{λ} for λ real is a one parameter subgroup in GL(n,c) which has the property that each one of its elements is similar to A defined above.

If Q is an n x n matrix, such that

$$Q = \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & 0 \cdots & 0 \\ \vdots & 0 & & \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & \alpha_k \end{pmatrix}$$

with α_i, $i = 1,2,\ldots k$ some one of the canonical matrices P_1, P_2 or P_3, then define

$$Q = \begin{pmatrix} \alpha_1^{\lambda} & 0 & \cdots & 0 \\ 0 & \alpha_2^{\lambda} & 0 \cdots & 0 \\ \vdots & 0 & & \vdots \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & \alpha_k^{\lambda} \end{pmatrix}$$

Any matrix, M, in GL(n,c) is similar to a matrix of the type Q, i.e. $\exists$ a C non-singular such that $M = CQC^{-1}$. In this case powers are defined for all elements of GL(n,c) but, of course, not uniquely.

Let $\mathcal{O}$ be the set of all n x n matrices in GL(n,c) whose eigen values lie in a spherical neighborhood, V, of the identity, 1, of the non-zero complex numbers. This neighborhood is defined by

$$V = \{z \mid |z - 1| < 1\}.$$

$\mathcal{O}$ is an open set in GL(n,c).

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Let M_n be a sequence of matrices in the compliment of $\mathcal{O}$ that converges to an element $\mathcal{D}$ in GL(n,c).

$M_n = U_n Q_n U_n^{-1}$, where U_n is unitary and Q_n is a triangular matrix.

There exists a subsequence $\{U_{n_k}\}$ of $\{U_n\}$ and a unitary matrix U such that

$$U_{n_k} \longrightarrow U.$$

Since the norm, defined by $|A|$, is invariant under conjugation by unitary matrices, $\{Q_{n_k}\}$ and $\{U_{n_k}^{-1} D U_{n_k}\}$ are asymptotic sequences. By continuity of multiplication, and also by the above fact

$$Q_{n_k} \longrightarrow U^{-1} D U, \text{ where } U^{-1} D U \text{ is triangular.}$$

In view of this, the eigen values of Q_{n_k} converge to those of $U^{-1}DU$. This proves that D is also in the compliment of $\mathcal{O}$.

The set $\mathcal{O} \cup \mathcal{O}^{-1}$ is an open set containing the identity in GL(n,c).

If P_1, P_2, and P_3 are in $\mathcal{O} \cup \mathcal{O}^{-1}$, then P_i^{λ}, $|\lambda| \leq 1$ is in $\mathcal{O} \cup \mathcal{O}^{-1}$. Moreover, since the coordinate functions of P_i^{λ} are uniquely defined and since they are also continuous functions of λ, P_i^{λ} is a continuous function of λ. Similarly, if Q, defined on page 34, is in $\mathcal{O} \cup \mathcal{O}^{-1}$, Q is uniquely defined and Q^{λ} is a continuous function of λ.

Any matrix in $\mathcal{O} \cup \mathcal{O}^{-1}$ is of the form CQC^{-1}, for some Q and some C. If

$(CQC^{-1})^{\lambda}$ is defined to be $CQ^{\lambda}C^{-1}$, $|\lambda| \leq 1$ then

$(CQC^{-1})^{\lambda}(CQC^{-1})^{\mu} = (CQC^{-1})^{\lambda+u}$,

if $|\lambda|$, $|u|$, and $|\lambda+u| \leq 1$. $(CQC^{-1})^{\lambda}$ is a continuous function of λ, by definition of Q^{λ} and continuity of multiplication in GL(n,c).

The preceding discussion shows that any matrix in $\mathcal{O} \cup \mathcal{O}^{-1}$ is contained in a one-parameter subgroup. However, the uniqueness of this one parameter subgroup is not established.

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If $M = AQA^{-1} = BQB^{-1}$, it remains to be proved that

$AQ^{\lambda}A^{-1} = BQ_{B}^{-1}$ if M^{λ} is to be defined in $\Theta_{U}\Theta^{-1}$.

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Section 5

Any matrix is similar under the unitary group to a triangular matrix. In Section 5 such a canonical form (triangular matrix) for matrices with distinct eigen values is given. The consideration of such matrices indicates a second approach to the problem of determining a region of unique power of elements in GL(n,c).

Let T be a linear transformation on V^3 with distinct eigen values λ_1, λ_2, and λ_3. The following unit vectors in V_3

$$x_1 = i, \quad x_2 = a_1 i + a_2 j, \quad x_3 = b_1 i + b_2 j + b_3 k$$

can be brought into unit eigen vectors of T by a unitary transformation of coordinates of V_3. Suppose that the eigen values λ_1, λ_2, and λ_3 belong to X_1, X_2, and X_3, respectively. In this case,

$$T i = \lambda_1 i$$

$$T(a_1 i + a_2 j) = \lambda_2 (a_1 i + a_2 j)$$

$$T j = \frac{a_1}{a_2} (\lambda_2 - \lambda_1) i + \lambda_2 j$$

$$T(b_1 i + b_2 j + b_3 k) = \lambda_3 (b_1 i + b_2 j + b_3 k)$$

$$T k = \left[\frac{b_1}{b_3} (\lambda_3 - \lambda_1) - \frac{b_2}{b_3} \left\{ \frac{a_1}{a_2} (\lambda_2 - \lambda_1) \right\} \right] i + \frac{b_2}{b_3} (\lambda_3 - \lambda_2) j + \lambda_3 k.$$

In this case, the matrix for T relative to this basis is

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$$T = \begin{pmatrix} \lambda_1 & 0 & 0 \\ \frac{a_1}{a_2}(\lambda_2-\lambda_1) & \lambda_2 & 0 \\ \frac{b_1}{b_2}(\lambda_3-\lambda_1)-\frac{b_2}{b_3}\left(\frac{a_1}{a_2}\{\lambda_2-\lambda_1\}\right), & \frac{b_2}{b_3}(\lambda_3-\lambda_2), & \lambda_3 \end{pmatrix}$$

If T^ρ is defined to be the linear transformation on V_3 with eigen values λ_1^ρ, λ_2^ρ and λ_3^ρ with the same eigen directions as those of λ_1, λ_2 and λ_3, then T^ρ has a similar triangular form

$$T^\rho = \begin{pmatrix} \lambda_1^\rho & 0 & 0 \\ a_1/a_2(\lambda_2^\rho-\lambda_1^\rho) & \lambda_2^\rho & 0 \\ \frac{b_1}{b_3}(\lambda_3^\rho-\lambda_1^\rho)-\frac{b_2}{b_3}\left(\frac{a_1}{a_2}\{\lambda_2^\rho-\lambda_1^\rho\}\right), & \frac{b_2}{b_3}(\lambda_3^\rho+\lambda_2^\rho), & \lambda_3^\rho \end{pmatrix}$$

If T^ρ, T^σ, and $T^{\rho+\sigma}$ are defined in the same way, then

$$T^\rho T^\sigma = T^{\rho+\sigma}$$

$$T^\rho[T_i^\sigma] = \lambda_1^{\rho+\sigma} i$$

$$T^\rho[T_j^\sigma] = \frac{a_1}{a_2}\left(\lambda_2^{\rho+\sigma} - \lambda_1^{\rho+\sigma}\right) i + \lambda_2^{\rho+\sigma} j.$$

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$$T^{\rho}[T^{\sigma}k] = \left[\frac{b_1}{b_3}\left(\lambda_3^{\rho+\sigma} - \lambda_1^{\rho+\sigma}\right) - \frac{b_2}{b_3}\left\{\frac{a_1}{a_2}\left(\lambda_2^{\rho+\sigma} - \lambda_1^{\rho+\sigma}\right)\right\}\right] i$$

$$+ \frac{b_2}{b_3}\left(\lambda_3^{\rho+\sigma} - \lambda_2^{\rho+\sigma}\right) j + \lambda_2^{\rho+\sigma} k.$$

By definition of $T^{\rho+\sigma}$

$$T^{\rho}[T^{\sigma}x] = T^{\rho+\sigma}x \qquad \text{for all } x$$

and

$$T^{\rho}T^{\sigma} = T^{\rho+\sigma} \qquad \text{wherever } T^{\rho}\text{'s are}$$

defined uniquely.

It is clear by Theorem II that any square root of T a triangular matrix with distinct eigen values is a triangular matrix with distinct eigen values. As in the canonical form of the preceding section for transformations with this property, there are only a finite number of square roots of such matrices.

If V is a neighborhood of the identity in which $T^{\lambda}, |\lambda| \leq 1$ is defined uniquely, T a triangular matrix with distinct eigen values, then powers of any other triangular matrix may be defined by continuity. The set of all matrices with distinct eiger values is dense in the set of all triangular matrices. Moreover, T^{λ_0} is a continuous function of T, for λ_0 a fixed real number such that $|\lambda_0| \leq 1$ in V. This is true since each coordinate of T is a continuous function λ_0.

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The neighborhood described in the preceding section i.e. $V = \Theta \cup \Theta^{-1}$ is such a neighborhood of existence and uniqueness for powers of T. Moreover, this is true for any matrix in V with distinct eigen values. Again uniqueness for all elements in V is not established.

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APPENDIX 1

Remarks on the Continuity of the Canonical Form of a Matrix.

In the definition of powers of a matrix, the canonical forms of a mattix were used extensively. The operation of attaching to a matrix a canonical form, either classical or triangular form by unitary matrices, is in general a multiple valued function. However, it is the continuity of such an operation, both as a single valued function and as a multiple valued function, that this section will be devoted to.

If an n x n matrix has the canonical form (classical) corresponding to a linear transformation with only one eigen value and only one eigen direction, then this canonical form is unique. It is easy to see that this function could be made single valued everywhere on $M_n(C)$ by simply ordering eigen values relative to absolute value and, if necessary, ordering eigen values relative to angle.

The classical canonical form is not a continuous function of matrices. The sequence $\{M_n\} = \begin{pmatrix} 1 & 0 \\ 1/n & 1 \end{pmatrix}$ converges to $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. On the other hand $M_n \sim \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, for each n, under affine group. It is clear that the canonical forms $\{Q_n\}$ of M_n will not converge to 1. This fact is true for matrices of any order.

The triangular canonical form (by unitary matrices) is a continuous function on $M_2(C)$.

One important property of unitary matrices is that any matrix is similar to a triangular matrix, i.e. for any M there exists U unitary

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such that UMU^{-1} is triangular. However, this triangular matrix is not unique. In the two dimensional case, it is easy to define a unique canonical (triangular) form for any matrix.

(a) $\begin{pmatrix} a & 0 \\ c & b \end{pmatrix} \sim \begin{pmatrix} a & 0 \\ d & b \end{pmatrix}$ by unitary matrices

iff $|c| = |d|$.

The invariance of the norm under conjugation by unitary matrices gives if part.

If $|c| = |d|$, then $|c| = |c|\ e^{i\phi}$ and $d = |c|\ e^{i\theta}$

If $U = \begin{pmatrix} 1 & 0 \\ 0 & e^{i(\theta-\phi)} \end{pmatrix}$ then $U \begin{pmatrix} a & 0 \\ c & b \end{pmatrix} U^{-1} = \begin{pmatrix} a & 0 \\ d & b \end{pmatrix}$

(b) If $A = \begin{pmatrix} a & 0 \\ |c| & b \end{pmatrix}$ and $A' = \begin{pmatrix} b & 0 \\ |c| & a \end{pmatrix}$, then $A \sim A'$ by unitary matrices.

Write $\begin{pmatrix} a & 0 \\ |c| & b \end{pmatrix} = \begin{pmatrix} a & 0 \\ \frac{q}{s}(b-a) & b \end{pmatrix}$ where $q = \frac{|c|}{\sqrt{|b-a|+|c|}}$, $s = \frac{b-a}{\sqrt{|b-a|+|c|}}$

Then $U = \begin{pmatrix} q & s \\ -\bar{s} & \bar{q} \end{pmatrix}$ is unitary and $UAU^{-1} = A'$.

If M is a matrix with a triangular canonical form $\begin{pmatrix} a & 0 \\ c & b \end{pmatrix}$, let the matrix

$$T = \begin{pmatrix} a & 0 \\ |c| & b \end{pmatrix}$$ with $|a| \geq |b|$ or if $|a| = |b|$ and the angle of a is greater than the angle of b.

be the canonical form of M. T as described is unique in the equivalence class of M.

If $M_n \to \mathcal{M}$ and $M_n = U_n T_n U_n^{-1}$, then it is clear that $T_n \to$ the canonical form of $\mathcal{M}$.

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This method of proving continuity for two dimensional cases is, of course, not easily generalized. However, it is clear that a canonical form as a multiple valued function is continuous at the identity in GL(n,c). One observes easily that the canonical form of a unitary matrix for any dimensional can be described and this function is continuous.

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SUMMARY

The problem of defining intrinsically one-parameter subgroups or powers of an element in a neighborhood of the identity of certain topological groups, in which they are known to exist, consists essentially of two parts: (1) a method for the definition of such must be given and (2) a region or neighborhood of the identity must be defined in which these powers are defined and unique.

Part I of this paper suggested a possible intrinsic method for the definition of one parameter subgroups in certain classes of abstract topological groups. This method is very general and its many limitations are pointed out here. However, a more restrictive hypothesis is considered and with this the existence of square roots of elements is assured, but not uniqueness of such.

Part II presented an investigation of certain matrix groups which have this property of the existence of one parameter subgroups and only one for each element in a neighborhood of the identity. This investigation was made in order to abstract facts from these groups which could be applicable to the more general groups of Part I. In section 1, two norms (equivalent) are defined on the total matrix algebra together with relations between the norm of an element and the eigen values of the element.

Two subgroups of GL(n,c) are discussed in detail in Section II with regard to the property mentioned at the beginning of this paragraph. In the case of GL(2,c) the neighborhood defined there was unbounded and defined in terms of the eigen values of elements. Moreover, this neighborhood contained a subgroup. The unitary group of dimension n has a spherical

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neighborhood which satisfies all properties. In section II, certain relations between the eigen values and eigen directions of a transformation T and those of a square root of a transformation are given. The method of the example III i.e. that of G (2,c) is applied to GL(n,c) in section IV. In this section an explicit method for defining powers for an element is presented. A region of existence of one parameter subgroups is determined but uniqueness is not established. In section IV, the classical canonical form is used. Following the example IV, i.e. that of the unitary group, use of the triangular canonical forms by unitary transformations is made in section V. This section contains a definition of powers of matrices with distinct eigen values. Powers of all other elements are defined by continuity.

The classical canonical form as a function of a matrix is not continuous. In one special case, the continuity of the canonical form by unitary transformations is established. Remarks concerning these facts are contained in the appendix.

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