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CUBICAL CECH HOMOLOGY

by

Jozef Blass

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ABSTRACT

Cubical Čech Homology

by

Jozef Blass

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In the present work we construct a homology theory of the Čech type built on the cubical scheme. The basis of the construction is the functional assignment to every compact space X of a cubical polyhedron QX having the homotopy type of the space X.

We study the category of cubical polyhedra and cubical morphisms, and introduce the category of cubical polyhedra and contiguous classes of cubical morphisms. Applying the acyclical models technique, we define a functor C from the contiguous category of finite cubical polyhedra to the category of chain complexes and chain homotopy classes of chain homomorphisms. The combinatorial homology of the finite cubical polyhedron is defined as the homology of $C(V)$.

The cubical Čech homology of a compact space we define as the inverse limit of the combinatorial homologies

of finite polyhedra approximately QX. We show that the theory of homology constructed in the above manner satisfies the Eilenberg-Steenrod axioms for the continuous homology theory defined on the category of compact spaces.

The technique used in this study can be illustrated by the following example: To every finite system $A = \{(G_{-1}, G_1), (H_{-1}, H_1), \ldots\}$ of two-element open coverings of the space X we assign the cubical polyhedron NA. Given two finite systems of two-element open coverings A and B, such that A is a shrinking of B, the canonical cubical morphism $N(A \cup B) \to NA$ is a contiguous equivalence. (See II. 3.1.)

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INTRODUCTION

There are a number of reasons to construct a cubical Čech theory. First of all there exist simplicial and cubical singular theories, as well as the Čech theory based on the simplicial scheme, but there is no cubical Čech theory. This is certainly not the most important reason.

The simplicial singular homology is "simplicial", since for every space X there is Eilenberg's simplicial complex $S(X)$ and a weak homotopy equivalence $S(X) \to X$, which induces in a natural way an isomorphism on the homology level.

There is no analogous construction of a cubical complex $C(X)$ for the cubical singular theory.[1]

Let us consider now the simplicial Čech theory. We are given an open covering C_1 a space X and a refinement C_2 of C_1. The induced simplicial map of the nerve of C_2 into the nerve of C_1 is not uniquely determined. Because of this there is no construction of a simplicial

[1] We distinguish between cell complexes that consist of cubes and cubical complexes which are subcomplexes of cubes.

complex $S'(X)$, which is naturally homotopically equivalent to X .

But as we will show, there is a respective cubical Čech theory with a cubical complex QX , which is in a natural way homotopically equivalent to X for a compact space X . One could expect that in the case of Čech theory, the cubical scheme would give a more canonical construction than the simplicial one.[2] It is also clear that all constructions related to cartesian product in the cubical homology are simpler than in the simplicial case, for example, homotopy axiom, cup and cap products, Steenrod's squares.

In this paper we construct a cubical homology without any reference to the existence of a simplicial theory.

We begin from the combinatorial level--a category of cubical polyhedra is defined. We approximate a compact space by an inverse system of finite cubical polyhedra (with cubical projections!) and extend homology theory from the

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[2] There is a crucial combinatorial difference between the simplex and the cube. Every map of vertices of the simplex into E^n can be uniquely extended to a linear map of the simplex, but projections onto faces are not uniquely defined. In the case of a cube the situation is converse. Therefore, in a sense simplicial singular theory is more canonical than the cubical one.

combinatorial onto the topological level.

In a sense the category of finite cubical polyhedra is smaller than the category of simplicial polyhedra (there are less morphisms) and because of that there is no analogue of the simplicial approximation theorem. The important point is the fact that the finite cubical category is rich enough to approximate the category of compact spaces.

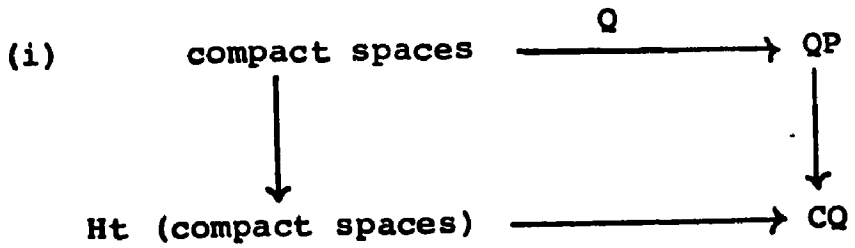
Here is an outline of the work.

CHAPTER I.

The category QP of cubical polyhedra is defined and for a space X , the cubical polyhedron QX is constructed. Ht and QoHt are shown to be naturally equivalent for compact spaces, where Ht is the homotopy functor.

CHAPTER II.

Here the contiguous category CQ is introduced. The main object of this section is to construct the functor Ht(QP) → CQ such that the following diagram commutes:



CHAPTER III.

At this stage, a functor from the contiguous category into combinatorial homology is defined. This functor is extended to infinite polyhedra and its composition with the functor $\downarrow_{\rightarrow} = {}^{\rightarrow}\downarrow$ [see diagram (i)] gives cubical Čech homology.

CHAPTER IV.

Eilenberg's-Steenrod's axioms for the combinatorial homology are shown to hold.

CHAPTER V.

Cubical Čech homology is constructed. E-S axioms for the continuous theory are to be valid.

CHAPTER I.

§1. The cubical category QP .

Throughout this paper we will use the following notations.

Id_A - the identity map of A onto A .

$f \bullet g$ - set-theoretical composition of f and g (composition in the category of sets).

$f \circ g$ - composition of f and g in a category.

$I = [-1;1]$ - the set of real x such that $-1 \leq x \leq 1$.

$F_{(a,\varepsilon)} = \{x=(x_\alpha)_{\alpha \in A} \in I^A : x_a = \varepsilon\}$ - a face of the cube I^A of codimension 1, for $\varepsilon = \pm 1$.

Let $\beta : B \to \{-1, 1\}$ be a function defined on a subset B of A . Then we define

$$F_\beta = \bigcap_{a \in B} F_{(a,\beta(a))}$$

and we call F_β a face of I^A. If $B=A$ then F_β is a single-element set consisting of a vertex of I^A. If $B=\emptyset$ then we define $F_\beta = I^A$. All faces are non-empty sets. If F_β, F_γ are faces of I^A, $\beta:B \to \{-1, 1\}$, $\gamma:C \to \{-1, 1\}$, $B \cup C \subset A$, then

$$F_\beta \subset F_\gamma \text{ iff } C \subset B \text{ and } \gamma = \beta|C .$$

If $X \subset I^A$ then carr X will denote the smallest face containing X. In the case of a single element set $X=\{x\}$ we will write carr x instead of carr$\{x\}$.

(1.1) Definition: A subset W of I^A is said to be a cubical polyhedron (or simply a polyhedron) iff

$$W = \bigcup \{\text{carr } x: \ x \in W\} .$$

A pair (W,V) consisting of polyhedra $V \subset W \subset I^A$ is said to be a polyhedral pair.

Thus every polyhedron $W \subset I^A$ is a union of faces of I^A. Obviously, the converse is also true. Every union of a family of faces of I^A is a polyhedron.

We will always identify a pair $(W,\emptyset)$ and W.

(1.2) Theorem: Let $W \subset I^A$ be a polyhedron and let F_β be

a face of I^A, where $\beta: B \to \{-1, 1\}$ for a subset B of A. Then

$$F_\beta \subset W \text{ iff } \bigcap \{G_{(a,\varepsilon)} : a \in A \setminus B \text{ or } \beta(a) = \varepsilon\} \neq \emptyset$$

where $G_{(a,\varepsilon)} = W \setminus F_{(a,-\varepsilon)}$ for $a \in A$, $\varepsilon = \pm 1$.

Proof. Suppose that $F_\beta \subset W$. Let

$$x_a = \begin{cases} 0 & \text{for } x \in A \setminus B \\ \beta(a) & \text{for } x \in B \end{cases}.$$

Obviously, $x \in G_{(a,\varepsilon)}$ if $a \in A \setminus B$ or $\beta(a) = \varepsilon$. Thus

(i) $$x \in \bigcap \{G_{(a,\varepsilon)} : a \in A \setminus B \text{ or } \beta(a) = \varepsilon\} \neq \emptyset .$$

Conversely, if (i) holds for an $x = (x_a)_{a \in A} \in I^A$, then $x_a \neq 1$ for $a \in A \setminus B$ and $x_a \neq -\beta(a)$ for $a \in B$.

Let $F_\gamma = \operatorname{carr} x$ for a function $\gamma: C \to \{-1, 1\}$. Then $C \subset B$ and $\gamma = \beta | C$. Thus $F_\beta \subset F_\gamma \subset W$. □

(1.3) Definition: Given polyhedral pairs (W,V) and (W_1, V_1), $V \subset W \subset I^A$ and $V_1 \subset W_1 \subset I^{A_1}$. A cubical morphism $q: (W,V) \to (W_1, V_1)$ is a function $q: A_1 \to A$ such that $f_q(W) \subset W_1$ and $f_q(V) \subset V_1$, where $f_q: I^A \to I^{A_1}$ is

a map given by $(f_q(x))_a = x_{q(a)}$ for every $a \in A_1$ and $x=(x_\alpha)_{\alpha \in A} \in I^A$. The composition $p \circ q:(W,V) \to (W_2, V_2)$ of cubical morphisms $q:(W,V) \to (W_1, V_1)$ and $p:(W_1, V_1) \to (W_2, V_2)$ is defined by

$$p \circ q = q \bullet p \ .$$

The unit morphism $1_{(W,V)}$ is defined by

$$1_{(W,V)} = Id_A \qquad (\text{for } V \subset W \subset I^A) \ .$$

Thus $f_{1_{(W,V)}} = Id_{I^A}$.

(1.4) Definition: The cubical category QP is the category of polyhedral pairs and cubical morphisms of such pairs.

In the next sections we will deal with "forgetful" functor $F_o:QP \to Top$ and the functor $F = Ht \circ F_o:QP \to HtTop$, where Top is the topological category of pairs, Ht is the homotopy functor and HtTop is the category of topological pairs with homotopy classes of mappings as morphisms. These functors are given by

(1.5) $$F_o(W,V) = F(W,V) = (W,V)$$

for every polyhedral pair (W,V) , and

(1.6) $\qquad F_o(q) = f_q \qquad$ (see def.(1.3))

and

(1.7) $\qquad F(q) = Ht(f_q) = [f_q]$.

The following simple proposition will be useful.

(1.8) Proposition: Let $q:A \to A_1$ be a function (i.e. q is a cubical morphism of I^{A_1} into I^A). Let $F_\beta \subset I^{A_1}$ be a face, where β is defined on $B_1 \subset A_1$. Then

$$\text{carr } f_q(F_\beta) = F_{\tilde{\beta}} \quad .$$

where $\tilde{\beta}$ is defined on $B=q^{-1}(B_1)$ by

$$\tilde{\beta} = \beta \bullet (q|B) \quad . \quad \square$$

(1.9) Proposition: Let $x=(x_a)_{a \in A} \in I^A$ and $B \subset A$ and $\beta:B \to \{-1, 1\}$. Then carr $x=F_\beta$ iff $B=\{a \in A: |x_a|=1\}$ and $\beta(a)=x_a$ for every $a \in B$.

(1.10) Proposition: Let $g:(W,W_o) \to (V,V_o)$ be a 1-1 continuous map of a polyhedral pair (W,W_o) onto (V,V_o)

(hence $g(W)=V$ and $g(W_o)=V_o$) and let g be induced by a cubical morphism $q:(W,W_0) \to (V,V_o)$, i.e., let $g=f_q=F_o(q)$. Then q is a cubical isomorphism of (W,W_o) and (V,V_o) and the inverse function g^{-1} is induced by q^{-1}, $g^{-1}=F_o(q^{-1})$ (Hence g is a homeomorphism). □

§2. The cubical nerves.

Let X be a topological space and let $T=T(X)$ be the family of all functionally open subsets X *). We will also use the following notation

$$A(X) = \{G = (G_{-1}, G_1) \quad T \times T: G_{-1} \cup G_1 = X\},$$

and we define $\pi_\varepsilon: A(X) \to T$ as

$$\pi_\varepsilon(G) = G_\varepsilon \quad \text{for every } G = (G_{-1}, G_1) \quad A(X), \varepsilon = \pm 1.$$

Next, for $B \subset A \subset A(X)$, $\beta: B \to \{-1, 1\}$ and $X_o \subset X$ we put

$$(2.1) \quad \text{supp } F_\beta = \bigcap \{\pi_\varepsilon(G): G \in A \setminus B \quad \text{or} \quad \beta(G) = \varepsilon\}$$

---

*) i.e. of sets of the form $f^{-1}(R \setminus \{0\})$, where $f: X \to R$ is a continuous real-valued function.

and

(2.2) $\quad \operatorname{supp}_{X_o} F_\beta = X_o \cap \operatorname{supp} F_\beta \ .$

We define also the nerve $N_{X_o} A$ of $A \subset A(X)$ in $X_o \subset X$, as a subpolyhedron of I^A such that

(2.3) $\quad F_\beta \subset N_{X_o} A \quad$ iff $\quad \operatorname{supp}_{X_o} F_\beta \neq \emptyset \ .$

For X and (X,X_o) we put

(2.4) $\quad NA = N_X A \quad$ and $\quad NA(X,X_o) = (NA, N_{X_o} A) \ .$

(2.5) Theorem: Let (W,W_o) be a polyhedral pair in I^A. Let $A' = \{G_a = (G_{(a,-1)}, G_{a,1}) : a \in A\}$, where $G_{(a,\varepsilon)} = W \setminus F_{(a,\varepsilon)}$. Next, let $i:A \to A'$ be given by $i(a)=G_a$ for $a \in A$. Then i is a cubical isomorphism of $NA'(W,W_o)$ and (W,W_o).

Proof. The theorem is an immediate consequence of Th.(1.2) and (2.3), (2.4). □

(2.6) Proposition: Let $X_o \subset X$ and $B \subset A \subset A(X)$. Then the inclusion map $i_B^A : B \to A$ is a cubical morphism from $NA(X,X_o)$ into $NB(X,X_o)$.

Proof: The map $f_i: I^A \to I^B$ for $i=i_B^A$ (see def.(1.3)) is the projection $p_B^A: I^A \to I^B$. Let $x=(x_G)_{G \in A} \in N_{X_o} A$. Then carr $x \subset N_{X_o} A$. By Prop. (1.9) that means

$$X_o \cap \bigcap \{\pi_\varepsilon(G): x_G \neq -\varepsilon \text{ and } G \in A\} \neq \emptyset \ .$$

Thus

$$X_o \cap \bigcap \{\pi_\varepsilon(G): x_G \neq -\varepsilon \text{ and } G \in B\} \neq \emptyset \ ,$$

and $p_B^A(x)=(x_a)_{a \in B} \in N_{X_o} B$. Thus $p_B^A(N_{X_o} A) \subset N_{X_o} B$ for any $X_o \subset X$. In particular, $p_B^A(NA) \subset NB$. Thus i_B^A is a cubical morphism of $NA(X,X_o)$ into $NB(X,X_o)$. □

Obviously,

(2.7) if $C \subset B \subset A \subset A(X)$ then $i_C^A = i_C^B \circ i_B^A$. □

(2.9) Proposition: Let $g:(X,X_o) \to (Y,Y_0)$ be a continuous map of topological pairs and let $A \subset A(X)$, $B \subset A(Y)$ be such sets that $g^{-1}(B) \subset A$. Then the function $N_B^A(g):A_1 \to A$ given by $(N_B^A(g))(G) = g^{-1}(G)$ *) is a cubical morphism of

*) $g^{-1}(B)=\{g^{-1}(G):G \in B\}$, where $g^{-1}(G) = (g^{-1}(G_{-1})$, $g^{-1}(G_1))$ for $G=(G_{-1}, G_1)$.

$NA(X,X_o)$ into $NB(Y,Y_o)$.

The proof is similar to the proof of Prop.(2.7) (which is a special case of the above (2.9)). □

Obviously,

(2.10) If composition of continuous maps $g \bullet f$ is defined as well as $N_B^A(f)$ and $N_C^B(g)$ then $N_C^A(g \bullet f)$ is defined and $N_C^A(g \bullet f) = N_C^B(g) \circ N_B^A(f)$.

(2.11) Remark. Let us put NX for $NA(X)$ and $N(X,X_o)$ for $NA(X)(X,X_o)$ and $Nf:N(X,X_o) \to N(Y,Y_o)$ for $N_{A(Y)}^{A(X)}f$ (for $f:(X,X_o) \to (Y,Y_0)$). Nf is well defined for every continuous map f . Thus, by (2.10), we have obtained a functor $N:Top \to QP$.

§3. The cubical functor Q .

Let $FinK$ denotes the family of all finite subsets of K .

(3.1) Definition: Let X_o be a subspace of X . We define $Q_X X_o = \bigcap \{(p_A^{A(X)})^{-1}(N_{X_o} A) : A \in FinA(X)\}$, $QX = Q_X Q$ and $Q(X,X_o) = (QX, Q_X X_o)$.

Evidently,

(3.2) $x=(x_G)_{G\in A(X)} \in Q_X X_o$ iff $X_o \{\pi_\varepsilon(G): x_G \neq -\varepsilon$ and $G \in A\} \neq \emptyset$ for every $A \in \mathrm{Fin}A(X)$.

(3.3) Proposition: Let $f:(X,X_o) \to (Y,Y_o)$ be a continuous map. Then the function $Q(f):A(Y) \to A(X)$ given by $(Q(f))(G) = f^{-1}(G)$ for every $G \in A(Y)$ is a cubical morphism of $Q(X,X_o)$ into $Q(Y,Y_o)$. □

(3.4) Proposition: If composition $g \bullet f$ of continuous maps of topological pairs is defined then $Q(g) \circ Q(f)$ is defined and $Q(g) \circ Q(f) = Q(g \bullet f) = Q(f) \bullet Q(g)$.

We have also

(3.5) $$Q(\mathrm{Id}_{(X,X_o)}) = 1_{Q(X,X_o)} .$$

Thus we have obtained a functor $Q:\mathrm{Top} \to QP$. We call this functor the cubical functor. Functor Q has the following important properties, which are direct consequences of def.(3.1) and Prop.(3.3).

(3.6) Property: $(Q(X,X_o),\ p_B^{A(X)}:B \in \mathrm{Fin}A(X))$ is a representation of $Q(X,X_o)$ as the inverse limit of the system $(N_B^A(X,X_o),\ p_B^A:B \subset A \in \mathrm{Fin}A(X))$ in the category

Top ($p^A_B = F_o(i^A_B)$ denotes the projection map $I^A \to I^B$) induced by the inclusion $i^A_B : B \to A$, $B \subset A$) . □

(3.7) Property: Let $g:(X,X_o) \to (Y,Y_o)$ be a continuous map. Then the map $f_{Q(g)} = F_o \circ Q(g) : Q(X,X_o) \to Q(Y,Y_o)$ is the inverse limit map in Top of maps $g^A_B : NA(X,X_o) \to NB(Y,Y_o)$, $A \supset f^{-1}(B)$, $g^A_B = F_o(N^A_B g)$. □

Using Prop.(1.10), we obtain from the above two properties the following analogous properties in Top

(3.8) Property: $(Q(X,X_o)$, $i^{A(X)}_B : B \in FinA(X))$ is a representation of $Q(X,X_o)$ as the inverse limit of the system $(N^A_B(X,X_o)$, $i^A_B : B \subset A \in FinA(X))$ in the cubical category QP . □

(3.9) Property: Let $g:(X,X_o) \to (Y,Y_o)$ be a continuous map. Then $Q(g):Q(X,X_o) \to Q(Y,Y_o)$ is the inverse limit morphism of morphisms $N^A_B g(f^{-1}(B)$ A FinA(X) and B FinA(Y)) in the cubical category QP .

(3.10) Remark: $N(X,X_o)$ is a subpair of $Q(X,X_o)$, i.e. $NX \subset QX$ and $N_{X X_o} \subset Q_{X X_o}$, for every topological pair (X,X_o) . In other words the identity functions $Id_{A(X)} : A(X) \to A(X)$ are the cubical morphisms

$Id_{A(X)}: N(X,X_o) \to Q(X,X_o)$. All of these morphisms $Id_{A(X)}: N(X,X_o) \to Q(X,X_o)$ form a natural transformation from the nerve functor N to the cubical functor Q .

§4. The natural transformation $\Phi: Ht \to F \circ Q$.

To each pair $G=(G_{-1}, G_1) \in A(X)$ we assign a continuous function $f_G: X \to I$ for every $G \in A(X)$ such that

(4.1) $\quad f_G | X \setminus G_\varepsilon = -\varepsilon \quad$ for $\quad \varepsilon = \pm 1$.

We also define the continuous function $f=f^X: X \to QX \subset I^{A(X)}$ as $f(x)=(F_G(x))_{G \in A}$ i.e. $f=\underset{G \in A}{\Delta} f_G$.

The following proposition shows that the mapping f is well defined (see remark (3.10)).

(4.2) Proposition:

$$f(X,X_o) \subset N(X,X_o) \quad \text{i.e.}$$

$$f(X) \subset NX \quad \text{and} \quad f(X_o) \subset N_X X_o .$$

Proof: We will show that $f(X_o) \subset N_X X_o$.

Suppose that $x \in X_o$. Notice that if $f_G(x) \neq -\varepsilon$ then $x \in G_\varepsilon$ and, therefore, $x \in X_o \{\pi_\varepsilon(G): f_G(X) = (f(x))_G \neq -\varepsilon , G \in A(X)\} \neq \emptyset$. Hence $f(x) \in N_X X_o$ (see (2.5)). Thus $f(X_o) \subset Q_X(X_o)$. In particular, $f(X) \subset N_X X = NX$.

The following assertion is evident.

(4.3) Proposition: Let $f' = \Delta_{G \in A} f'_G : X \quad X \subset I^A$ be another mapping such that $f'_G | X \setminus G_\varepsilon = -\varepsilon$ for every G A and $\varepsilon = \pm 1$. Then $f \simeq f' : (X, X_o) \to Q(X, X_o)$ and we have a canonical homotopy $h: (X, X_o) \times I \to Q(X, X_o)$ given by the formula

$$h(x,t) = (1-t)f(x) + tf'(x) \quad \text{for} \quad t \in I \quad \text{and} \quad x \in X .$$

Proof: Indeed, for $f_t = \Delta_{G \in A} f_{t,G}$, given by $f_t(x) = h(x,t)$, $f_{t,G} | X \setminus G_\varepsilon = -\varepsilon$ (see (4.1)).

Thus the homotopy class of $f: (X, X_o) \to Q(X, X_o)$ which satisfies (4.1), depends only on (X, X_o) , and we can define $\Phi: Ht \to F \circ Q$ by putting

$$\Phi(X, X_o) = [f] .$$

(4.4) Theorem: $\Phi: Ht \to F \circ Q$ is a natural transformation.

Proof: Let $\phi:(X,X_o) \to (Y,Y_o)$ be a continuous mapping and let $\phi(Y,Y_o) = [g]$ for some $g=\Delta_{G \in A(Y)} g_G$ such that (4.1) holds for every g_G and $G \in A(Y)$. Then let

$$f_{\phi^{-1}(G)} = g_G \circ \phi \quad \text{for every} \quad G \in A(Y)$$

and let f_G for $G \in A(X)$ $\phi^{-1}(A(Y))$, be a mapping such that (4.1) holds. Then for $f=\Delta_{G \in A(X)} f_G$ we have $\phi(X,X_o)=[f]$ and $g \circ \phi=(F_o \circ Q)(\phi) \circ f$. Thus $[g] \circ [\phi]=(F \circ Q)(\phi) \circ [f]$ and finally

$$\phi(Y,Y_o) \circ Ht(\phi) = (F \circ Q)(\phi) \circ \phi(X,X_o) \quad .$$

(4.5) Remark: One can observe that ϕ is a composition of a natural transformation $Ht \to F \circ N$, which is defined similarly to ϕ (see Prop.(4.2)), and of the natural transformation $F \circ N \to F \circ Q$, which is induced by the natural transformation $N \to Q$ from remark (3.10).

§5. The natural transformation $\Gamma_o: F_o \circ Q|Comp \to 1_{Comp}$.

For a compact space X we define the function g as $g=\Gamma_o(X):QX \to X$.

Let $x=(x_G)_{G \in A} \in QX$ and let $\mathcal{F}(x)$ be the filter in the lattice $T=T(X)$ (see beginning of §2) generated by the family

$$(5.1) \qquad B(x) = \{\pi_\varepsilon(G): x_G \neq -\varepsilon \ , \ G \in A(X)\} \ .$$

Thus, for every $G \in A(X)$, we have either $G_{-1} \in \mathcal{F}(x)$ or $G_1 \in \mathcal{F}(x)$. In other words, $G_{-1} \in \mathcal{F}(x)$ or $G_1 \in \mathcal{F}(x)$ for every $G_{-1}, G_1 \in T$ such that $G_{-1} \cup G_1 = X$. Thus, $\mathcal{F}(x)$ has the unique limit in X . We define $g:QX \to X$ by the formula

$$(5.2) \qquad g(x) = \lim \mathcal{F}(x) \ .$$

(5.3) Lemma: $g:QX \to X$ is a continuous mapping.

Let $x=(x_G)_{G \in A} \in QX$ and let V be a neighborhood of $g(x)$ in X.. Let $\phi:X \to I$ be a continuous mapping such that $\phi(g(x))=1$ and $\phi|X \setminus V=-1$. Then for $G_{-1} = \phi^{-1}([-1; \frac{1}{2}))$ and $G_1=\phi^{-1}((-\frac{1}{2}; 1])$ we have that

$G=(G_{-1}, G_1) \in A$ and $g(x) \overline{G}_{-1}$ and $\overline{G}_{-1} \subset V$. Thus $G_{-1} \notin \mathcal{F}(x)$ and $G_1 \in \mathcal{F}(x)$. The set $U=\{y=(y_H)_{H \in A} \in QX: y_G > -1\}$ is a neighborhood of x in QX $(X_G = 1)$ such that $g(U) \subset \overline{G}_1 \subset V$. □

Q.E.D.

(5.4) Lemma: Let X_o be a compact subspace of X. Then $g(Q_X(X_o)) \subset X_o$.

Proof: Suppose $x=(x_G)_{G \in A(X)} \in Q_X(X_o)$. Then the family $\{\pi_\varepsilon(G) \; X_o: G \in A(X) \text{ and } x_G \neq \varepsilon\}$ has the finite intersection property. Therefore, $g(x)=\lim \mathcal{F}(x) \in X_o$.

Using (4.4) we can consider g as a mapping $g:Q(X,X_o) \to (X,X_o)$ for every compact pair (X,X_o). Let Comp be the category of compact pairs. We have thus obtained a transformation $\Gamma_o: F_o \circ Q|\text{Comp} \to 1_{\text{Comp}}$ such that $\Gamma_o(X,X_o)=g:Q(X,X_o) \to (X,X_o)$.

(5.5) Theorem: $\Gamma_o:F_o \circ Q|\text{Comp} \to 1_{\text{Comp}}$ is a natural transformation. □

(5.6) Remark: Let $T_{3\frac{1}{2}}$ be the topological category of pairs of completely regular spaces. Similarly to Γ_o one can

define a natural transformation $F_o \circ N|T_{3\frac{1}{2}} \to 1_{T_{3\frac{1}{2}}}$.

It is the composition of Γ_o and of $\bar{N} \to Q$ from remark (3.10), if one restricted his attention to the category Comp .

§6. Φ and Γ as natural equivalences.

In this section we will prove the basic theorem of this paper.

(6.1) Theorem: Let $\Gamma : F \circ Q|Comp \to Ht|Comp$ be the natural transformation induced by Γ_o (see §5) and let $\Phi|Comp : Ht|Comp \to F \circ Q|Comp$ be the restriction of Φ (see §4). Then $\Phi|Comp = \Gamma^{-1}$, and consequently Γ and $\Phi|C$ are natural equivalences.

Proof: If (X,X_o) C and $[f]=\Phi(X,X_o)$ then

$$(6.2) \qquad \Gamma_o(X,X_o) \circ f = Id_{(X,X_o)} ,$$

and for $h(x,t) = (1-t)x+t \cdot (f \circ \Gamma_o(X,X_o))(x)$, $x \in Q(X,X_o)$, $t \in I$, if

$$(6.3) \qquad h((X,X_o] \times I \subseteq Q(X,X_o)$$

then

$$(6.4) \qquad f \circ \Gamma_o(X,X_o) \underset{h}{\simeq} 1_{Q(X,X_o)} \quad .$$

Thus it suffices to prove (6.3).

Let $x \in X_o$ and $0 < t < 1$ and let $g=\Gamma_o(X,X_o)$. Then (by (5.1))

$$B(h(x,t)) = B((1-t)x+t \cdot (f \circ g)(x)) = B(x) \cup B(f \circ g(x)).$$

Since the family $\{U \cap X_o : U \in B(x)\}$ has the finite intersection property, and $g(x) \in \overline{U \cap X_o}$ for every $U \in B(x)$, and $g(x) \in V$ for every $V \in B(f \circ g(x))$ it follows that $\{U \cap X_o : U \in B(h(x,t))\}$ has the finite intersection property. Thus, $h(x,t) \in Q_X X_o$. In particular, for $X = X_o$ we have $h(x,t) \in QX$.

CHAPTER II.

§1. Contiguous category.

In this section we will define the contiguous equivalence relation for cubical morphisms. We will obtain a projection category CQ and the projection functor from the category of cubical pairs QP into CQ.

(1.1) Definition: We say that a family of cubical morphisms

$$G = \{g_t : (V,V_0) \longrightarrow (W,W_0) : t \quad T\}$$

admits a common cubical carrier iff for every face $F_\beta \subset V$ there is a face $F_{\beta'} \subset W$ such that

a) $$\bigcup_{t \in T} f_{q_t}(F_\beta) \quad F_{\beta'}$$

and

b) $F_{\beta'} \subset W_0$ whenever $F_\beta \subset V_0$.

Such a carrier exists iff one can put $F_{\beta'}=\text{carr} \bigcup_{t \in T} f_{q_t}(F_\beta)$.

(1.2) Definition: Cubical morphisms $p,q:(V,V_0) \to (W,W_0)$ of finite polyhedra are said to be contiguous iff there exists a finite sequence of morphisms $p=q_0, q_1,\dots,q_n = q:(V,V_0) \to (W,W_0)$ such that every pair q_{i-1}, q_i admits a common cubical carrier for $i=1,\dots,n$.

(1.2') Definition: Cubical morphisms $p,q:(V,V_0) \to (W,W_0)$, where $V \subset I^A$ and $W \subset I^{A_1}$ are said to be contiguous iff for every finite $A_1' \subset A_1$ there exists a finite $A' \subset A$ such that $A' \supset p(A_1') \cup q(A_1')$ and such that the induced morphisms

$$p', q': p^{A}_{A'}(V,V_0) \longrightarrow p^{A_1}_{A_1'}(W,W_0)$$

are contiguous. (in the sense of the def. (1.2)).

(1.3) Remark: For finite cubical polyhedra both definitions (1.2) and (1.2') are equivalent.

The following two assertions are obvious.

(1.4) Proposition: The contiguity relation is an equivalence relation.

(1.5) Proposition: If $p', p'' : (U,U_0) \to (V,V_0)$ and $q', q'' : (V,V_0) \to (W,W_0)$ are pairs of contiguous morphisms then $q' \circ p'$ and $q'' \circ p''$ are contiguous morphisms. Thus cubical pairs and contiguous classes of cubical morphisms form a category CQ.

$Cg : Qp \to CQ$ denotes the canonical projection functor. $Cg(W,W_0) = (W,W_0)$ for every polyhedral pair (W,W_0), and $Cg(q)$ is the contiguous class of q for every cubical morphism q.

(1.6) Remark: If we release the definition (1.2) from the finiteness condition we will get the relation of strong-contiguity. This is stronger than the relation of contiguity (see def.(1.2')).

§2. Swelling operation.

(2.1) Definition: Let $\alpha : A \to A_1$ be a function and $W \subset I^{A_1}$ be a cubical polyhedron. We define a polyhedron $W^\alpha \subset I^A$ as follows

$$W^\alpha = \{F_{\alpha\circ\beta} : F_\beta \subset W_1\}$$ (for the definition of F_β see (I. 1.8)

(i.e. W^α is a sum of carriers of images of faces of W under morphism $\alpha: I^{A_1} \to I^A$.) We put also $(W_1, W_0)^\alpha = (W^\alpha, W_0{}^\alpha)$ in the case of surjective α, W^α is said to be a swollen polyhedron, and $(W, W_0)^\alpha$ a swollen pair. We have

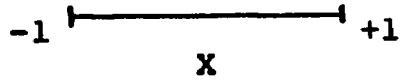
(2.2) Proposition: $F_\beta \subset W^\alpha$ iff

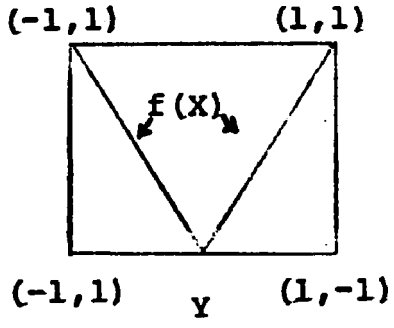
$$f_\alpha(W) \cap \bigcap \{G_{(a,\varepsilon)} : (a) = \varepsilon \quad \text{or} \quad a \in A \quad B\} \neq \emptyset$$

(compare (I. 1.2)).

(2.3) Example: Let $X=I$, $Y=I^2$ and let $f: X \to Y$ be given by

$$f(x) = (x, 2|x| - 1)$$

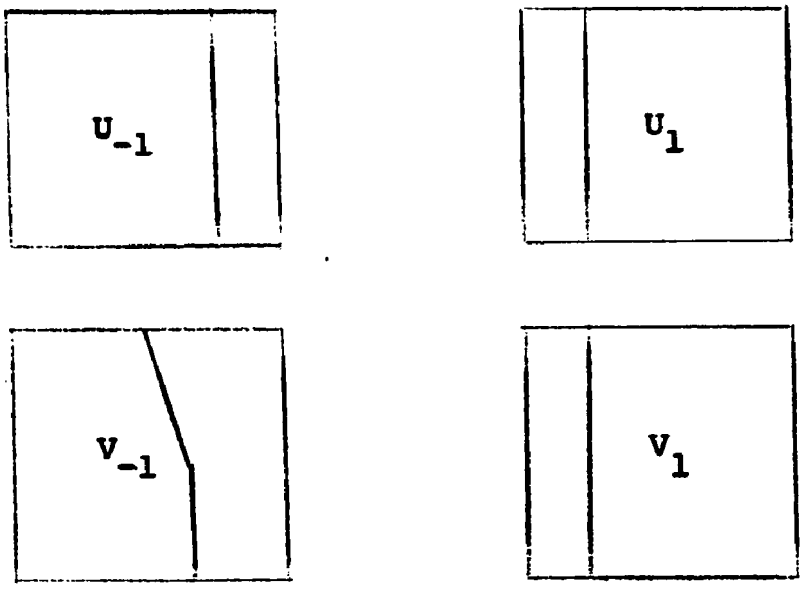




Let

$U_{-1} = \{(a,b): a < \frac{1}{2}\}$ $\qquad$ $U_1 = \{(a,b): a > -\frac{1}{2}\}$

$V_{-1} = \{(a,b): a < \frac{1}{2}$ and $2a+b < 1\}$ $\qquad$ $V_1 = U_1$



Let $A=(U,V)$, where $U=(U_{-1}, U_1)$ and $V=(V_{-1}, V_1)$.

Note that

$$f^{-1}(U) = f^{-1}(V) = ([-1; \tfrac{1}{2}), \quad (-\tfrac{1}{2}, 1])$$

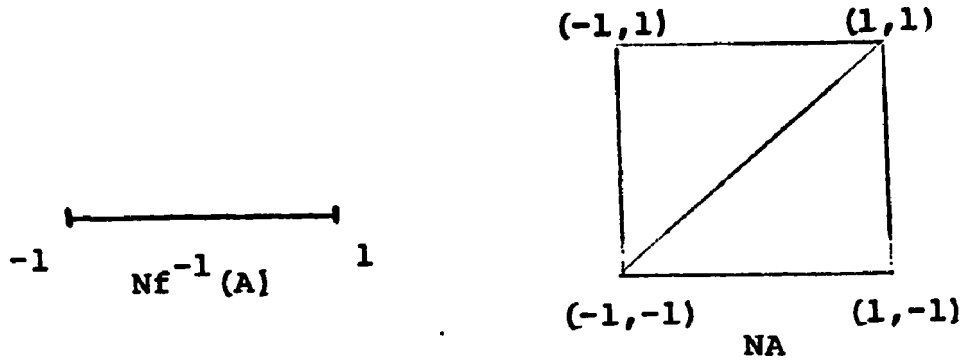
$$NA = I^2: Nf^{-1}(A) = I \quad .$$

We have the canonical map:

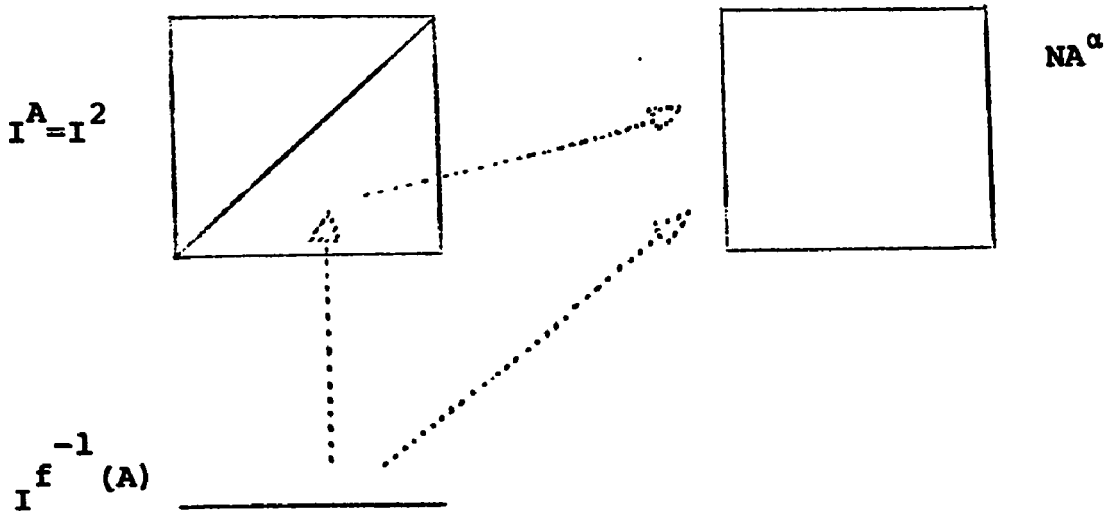
$$\alpha: A \to f^{-1}(A) \qquad \text{(taking inverse image)}.$$

Since $f^{-1}(A)$ consists of only one element α is surjective.

$f_\alpha : Nf^{-1}(A) \to NA$ is given by $f_\alpha(a)=(a,a)$.



Evidently $carr(f_\alpha(Nf^{-1}(A))=I^A= I^2$ and is two-dimensional.



This explains the motivation behind the definition of W^α .

It allows us to distinguish "different" coverings of X

which were actually different in Y .

(2.4) Proposition: A surjection $\alpha:A \to A_1$ is a morphism of W into W^α for every polyhedron W I^{A_1} .

(2.5) Theorem: The swelling operation $Cg(\alpha):(W,W_0) \to (W,W_0)^\alpha$ is an isomorphism (in the category CQ) for any cubical pair (W,W_0) .

Proof: Let $\alpha':A_1 \to A$ be a function such that $\alpha \cdot \alpha' = Id_{A_1}$. α' is a morphism of W^α into W .

Indeed every face of W^α is contained in a face of the form F_β is a face of W . (β is defined on B_1 A_1). By (I. 1.10) carr $\alpha'(F_\beta) = F_\gamma$, where γ is defined on $B'(\alpha \cdot \alpha)^{-1}(B)$ as follows:

$$\gamma = \tilde{\beta} \cdot (\alpha'|B') = \beta \cdot \alpha \cdot (\alpha'|B') = \beta$$

Thus image of any face of W^α under α' is contained in a face of W . (i.e. α' is a morphism of W^α into W). Therefore it makes a sense to write

(2.6) $$\alpha' \circ \alpha = Id_W \quad .$$

We will show now that $\alpha \circ \alpha'$ and Id_{W^α} admit common

cubical carrier. It suffices to show that $f_{\alpha \circ \alpha'}(F_{\tilde{\beta}})$ $F_{\tilde{\beta}}$ is contained in a face of W^{α} for every face F_{β} W. But

$$f_{\alpha \circ \alpha'}(F_{\tilde{\beta}}) = (f_{\alpha} \cdot f_{\alpha'})(\operatorname{carr} f_{\alpha}(F_{\beta}))\ \operatorname{carr}(f_{\alpha} \cdot f_{\alpha'} \cdot f_{\alpha})(F_{\beta}) = F_{\tilde{\beta}}.$$

This and (2.6) prove the theorem in the absolute case. But the extension onto the relative case is trivial.

The following factorization theorem is easy to prove.

(2.6) Theorem: Let $q:(W,W_0) \to (V,V_0)$ be a cubical morphism of cubical polyhedra $W_0 \subset W \subset I^{A}$ into $V_0 \subset V \subset I^{A_1}$. Then

$$[q:(W,W_0) \to (V,V_0)] =$$

$$= [Id_{A_1}:(W,W_0)^{q} \to (V,V_0)] \circ [q:(W,W_0) \to (W,W_0)^{q}] .$$

In particular, the above theorem asserts that $Id_{A_1}:(W,W_0)^{q} \to (V,V_0]$ is a cubical morphism.

§3. Homotopical invariance of contiguity.

In this section we will show homotopical invariance of contiguous category of compact pairs. More precisely, we will show that $\tilde{Cg} \circ Q|C = Cg \circ Ht|C$ (C denotes the category of compact pairs], where $\tilde{Cg}$ is a function from HtC into CQP .

(3.1) Lemma: Given a topological pair (X,X_0) and a pair of abstract sets A,A_0 let $i:A_0 \to A$ and $q:A \to A_0$ and $\alpha:A \to A(X)$ be functions such that

a) $q \cdot i = Id_{A_0}$, and

b) $\pi_\varepsilon(\alpha \cdot i \cdot q(a)) \subset \pi_\varepsilon(\alpha(a))$

then

i) i is a cubical morphism of $N\alpha(X,X_0)$ into $N\alpha_0(X,X_0)$. q is a cubical morphism of $N\alpha_0(X,X_0)$ into $N\alpha(X,X_0)$, where $\alpha_0 = \alpha \cdot i$, and

ii) $Cg(q): N\alpha_0(X,X_0) \to N\alpha(X,X_0)$ and $Cg(i):N\alpha(X,X_0) \to N\alpha_0(X,X_0)$ are isomorphisms in CQP , one inverse to another.

Proof: It is obvious that i is a morphism of

$N\alpha(X,X_0)$ into $N\alpha_0(X,X_0)$. Now let F_β be a face of $N_{X_0}\alpha_0$, where β is defined on $B_0 \subset A_0$. Then, by (I. 1.10) ,

$$\operatorname{carr} f_q(F_\beta) = F_{\tilde{\beta}} \ .$$

where $\tilde{\beta}$ is defined on $B=q^{-1}(B_0)$ by $\tilde{\beta} = \beta \cdot (q|B)$.

$$\alpha\text{-}\operatorname{supp}_{X_0} F = X_0 \cap \bigcap \{\pi_\varepsilon(\alpha(a)) : a \in A \setminus B \text{ or } \tilde{\beta}(a) = \varepsilon\}$$

$$= X_0 \cap \bigcap \{\pi_\varepsilon(\alpha(a)) : q(a) \in A_0 \setminus B_0 \text{ or } \tilde{\beta}(a) = \varepsilon\}$$

$$X_0 \cap \bigcap \{\pi_\varepsilon(\alpha \cdot i \cdot q(a)) : q(a) \in A_0 \setminus B_0 \text{ or } \tilde{\beta}(a) = \varepsilon\}$$

$$= X_0 \cap \bigcap \{\pi_\varepsilon(\alpha_0(q(a))) : q(a) \in A_0 \setminus B_0 \text{ or}$$

$$\beta(q(a)) = \varepsilon\} =$$

$$= X_0 \cap \bigcap \{\pi_\varepsilon(\alpha_0(a)) : a \in A_0 \setminus B_0 \text{ or } \beta(a) = \varepsilon\} =$$

$$= \alpha_0 - \operatorname{supp}_{X_0} F_\beta \neq \emptyset \ .$$

Thus $F_{\tilde{\beta}} \subset N_{X_0}\alpha$, and q is a morphism of $N_{X_0}\alpha_0$ into $N_{X_0}\alpha$. Since X_0 was an arbitrary subset of X , q is in particular a morphism of $N\alpha_0$ into $N\alpha$. Thus

q is a morphism of $N\alpha_0(X,X_0)$ into $N\alpha(X,X_0)$. (i) is proved. Now, by a) $Cg(i) \circ Cg(q)=1$. Next, let F_β be a face of $N_{X_0}\ \alpha$, β defined on $B \subset A$. Then

$$\operatorname{carr} f_{q \circ i}(F_\beta) = F_{\tilde{\beta}} \quad ,$$

where $\tilde{\beta}$ is defined on a $\tilde{B}=(q \circ i)^{-1}(B)=(i \cdot q)^{-1}(B)$ by $\tilde{\beta} = \beta \cdot (q \circ i|\tilde{B}) = \beta \cdot i \cdot (q|\tilde{B})$.

The intersection $\alpha\text{-supp}_{X_0} F_\beta \cap \alpha\text{-supp}_{X_0} F_{\tilde{\beta}}$ is the α-support in X_0 of a face of I^A which contains both F_β and $F_{\tilde{\beta}}$. Since, by a), $\alpha\text{-supp}_{X_0} F_\beta \cap \alpha\text{-supp}_{X_0} F_{\tilde{\beta}} =$

$$X_0 \cap \bigcap \{\pi_\varepsilon(\alpha(a)) : a \in A \setminus B \quad \text{or} \quad \beta(a) = \varepsilon\}$$

$$\cap \bigcap \{\pi_\varepsilon(\alpha(a)) : a \in A \setminus \tilde{B} \quad \text{or} \quad \beta(a) = \varepsilon\} \supset$$

$$X_0 \cap \bigcap \{\pi_\varepsilon(\alpha(a)) : a \in A \setminus B \quad \text{or} \quad \beta(a) = \varepsilon\} \cap$$

$$\cap \bigcap \{\pi_\varepsilon(\alpha \cdot i \cdot q(a)) : a \in A \setminus \tilde{B} \quad \text{or} \quad \beta(a) = \varepsilon\} =$$

$$= X_0 \cap \bigcap \{\pi_\varepsilon(\alpha(a)) : a \in A \setminus B \quad \text{or} \quad \beta(a) = \varepsilon\} \cap$$

$$\cap\bigcap\{\pi_\varepsilon(\alpha \cdot i \cdot q(a)) : i \cdot q(a) \in A \setminus B \quad \text{or} \quad \beta \cdot i \cdot q(a) = \varepsilon\} =$$

$$= X_0 \cap \bigcap\{\pi_\varepsilon(\alpha(a)) : a \in A \setminus B \quad \text{or} \quad \beta(a) = \varepsilon\} =$$

$$= \alpha\text{-supp}_{X_0} F_\beta \neq \emptyset$$

hence this face (which contains F_β $F_{\tilde{\beta}}$) is contained in $N_{X_0}\ \alpha$. Similarly, if $F_\beta \subset N\alpha$ then there exists a face of $N\alpha$ which contains F_β and $F_{\tilde{\beta}}$ for $F_{\tilde{\beta}} = \text{carr}\ f_{qoi}(F_\beta)$. Thus $1_{N\alpha(X,X_0)}$ and qoi admit a common cubical carrier, i.e.

$$Cg(q) \circ Cg(i) = 1_{N\alpha(X,X_0)} \qquad \square$$

We assume $(L,M) \times I = (L \times I, M \times I)$.

Let $i_t : (X,X_0) \to (X,X_0)$ I denote the embedding defined by $i_t(x) = (x,t)$ for t I . Next, for $A_0 \subset A(X)$ define $A_0 \times I \subset A(X \times I)$ to be the set

$$A_0 \times I = \{G \times I : G \in A_0\} .$$

(3.2) Lemma: Given $A \subset A(X \times I)$ and $A_0 \subset A(X)$, let $j : A_0 \to A \cup (A_0 \times I)$ and $q : A \cup (A_0 \times I) \to A_0$ be functions

such that $j(G)=G \times I$ for every $G \in A$, and $g \cdot j=Id_{A_0}$ and $\pi_\varepsilon(j \cdot q(G)) \subset \pi_\varepsilon(G)$ for every $G \in A$, $\varepsilon=\pm 1$. Then $A'=i_{-1}^{-1}(A) \cup i_1^{-1}(A) \cup A_0$ morphisms $N_A^{A'}(i-1)$ and $N_A^{A'}(i_1)$ are contiguous morphisms of $NA'(X,X_0)$ into $NA(X,X_0) \times I)$.

Proof: Since $N_A^{A'}(i_\varepsilon)=i_A^{A \cup (A_0 \times I)} \circ N_{A \cup (A_0 \times I)}^{A'}$,

hence it suffices to prove that $N^{A'}_{A \cup (A_0 \times I)}(i_{-1}) = N^{A'}_{A \cup (A_0 \times I)}(i_1)$. Let us remark that

$$(3.3) \qquad N^{A'}_{A \cup (A_0 \times I)}(i_\varepsilon) = N^{i_\varepsilon^{-1}(A) \cup A_0}_{A \cup (A_0 \times I)}(i_\varepsilon) \circ i^{A'}_{i_\varepsilon^{-1}(A) \cup A_0}$$

and that $j: N(A \cup (A_0 \times I))(X,X_0) \to NA_0(X,X_0)$ is the cubical morphism induced by the projection $Pr:(X,X_0) \times I \to (X,X_0)$ i.e. $j=N^{A \cup (A_0 \times I)}_{A_0}(pr)$. $pr \cdot i_\varepsilon = Id_{(X,X_0)}$ and therefore

$$(3.4) \qquad j \circ N^{i_\varepsilon^{-1}(A) \cup A_0}_{A \cup (A_0 \times I)}(i_\varepsilon) = i^{i_\varepsilon^{-1}(A) \cup A_0}_{A_0} .$$

Next, by Lemma (3.1) $Cg(j)$ is an isomorphism. Thus, by (3.3) and (3.4)

$$Cg(N^{A'}_{A \cup (A_0 \times I)}(i_\varepsilon)) =$$

$$= Cg(j)^{-1} \circ Cg(i^{i_\varepsilon^{-1}(A) \cup A_0}_{A_0}) \circ Cg(i^{A'}_{i_\varepsilon^{-1}(A) \cup A_0})$$

$$= Cg(j)^{-1} \circ Cg(i^{A'}_{A_0})$$

does not depend on ε .

(3.5) Theorem: The embeddings i_{-1} , $i_1=(X,X_0) \to (X,X_0) \times I$ induce contiguous morphisms $Q(i_{-1})$ and $Q(i_1):Q(X,X_0) \to ((X,X_0) \times I)$ for every compact pair (X,X_0) .

Proof: Let $A \in FinA(X \times I)$. Then there exists a finite sequence of real numbers $-1=t_0 < t_1 < \ldots < t_n=1$, a sequence $A_1,\ldots,A_n \in FinA(X)$ and a sequence of functions $p_k:A \to A_k$ for $k=1,\ldots,n$, such that

$$\pi_\varepsilon(p_k(G) \times [t_{k-1} ; t_k]) \subset \pi_\varepsilon(G)$$

for every $G \in A_k$, $k=1,\ldots,n$ and $\varepsilon=\pm 1$. Let us put

$$A'_k = i^{-1}_{t_{k-1}}(A) \cup i^{-1}_{t_k}(A) \cup A_k \ .$$

Then, by lemma (3.2), $N_A^{A'_k}(i_{t_{k-1}})$ and $N_A^{A'_k}(i_{t_k})$ are contiguous for every $k=1,\ldots,n$. Thus for $A'=\bigcup_{k=1}^{n} A'_k$, $N_A^{A'}(i_{t_{k-1}})$ and $N_A^{A'}(i_{t_k})$ are contiguous for $k=1,\ldots,n$. Consequently $N_A^{A'}(i_{-1})$ and $N_A^{A'}(i_1)$ are contiguous. Thus, by definition (1.2'), $Q(i_{-1})$ and $Q(i_1)$ are contiguous.

(3.6) Corollary: If f and $g:(X,X_0) \to (Y,Y_0)$ are homotopically equivalent and (X,X_0) is a compact pair, then Qf and $Qg:Q(X,X_0) \to Q(Y,Y_0)$ are contiguous.

(3.7) Corollary: If (X,X_0) is a compact pair then $Q(X,X_0)$ and $Q((X,X_0) \times I)$ are isomorphic in CQ (the canonical isomorphism is induced by the projection of $(X,X_0) \times I$ onto (X,X_0)).

CHAPTER III

§1. Incidence coefficients of permutations and orientation of faces.

We will introduce now notation which will lead us to definitions of oriented faces of cubical polyhedra and chain complexes of cubical polyhedra. Let B be a finite n-element set. Let $\phi:\{1,\ldots,n\} \to B$ and $\phi_1:\{1,\ldots,n\} \to B$ be two permutations of B. We define an incidence number

$$(1.1) \qquad [\phi:\phi_1] = \begin{cases} -1 \text{ when } \phi \text{ and } \phi_1 \text{ are of the opposite parity} \\ 1 \text{ when } \phi \text{ and } \phi_1 \text{ are of the same parity} \end{cases}$$

Let $B_0 \subset B$ be a subset of B such that $B \setminus B_0 = \{a\}$ and card $B_0 = k$. Let $\phi_0:\{1,\ldots,k\} \to B_0$ and $\phi:\{1,\ldots,k+1\}$ be permutations of B_0 and B respectively. Let a permutation ϕ_1 of the set B be given by

$$(1.2) \qquad \phi_1(i) = \begin{cases} a & \text{if } i=1 \\ \phi_0(i-1) & \text{if } 2 \leq i \leq k+1 \end{cases}$$

We define an incidence number $\{\phi:\phi_0\}$ as follows:

$$\{\phi:\phi_0\} \overset{df}{=} [\phi:\phi_1] \quad . \tag{1.3}$$

We observe, that $\{\phi:\phi_0\}$ depends only on the parity classes of ϕ and ϕ_0 and that a change of one of the parity classes into the opposite one will change the sign of $\{\phi:\phi_0\}$. Let $F_\beta \subset 1^A$ be a face of a cube given by $\beta:B \to \{-1, 1\}$. An orientation of a face F_β is defined as follows:

(1.4) Definition:

If $\dim F_\beta \leq 1$ then an orientation of F_β is a choice of 1, -1.

If $\dim F_\beta > 1$ then an orientation of F_β is a choice of a permutation class of the set $A \setminus B$.

We will use $\pi(F_\beta)$ to denote an oriented face.

§2. Incidence coefficient of faces.

Let $\pi(F_\beta)$ and $\pi'(F_{\beta'})$ be orientations of faces of the cube I^A, where β and β' are defined on B and B' respectively. We now define the so called incidence coefficient μ of the pair of orientations:

(2.1) $\mu = [\pi' : \pi] \neq 0$ iff $B' \subset B$, $\operatorname{card}(B \setminus B') = 1$ and $\beta | B = \beta'$.

If the above conditions are satisfied and if $B \setminus B' = \{a\}$ we define:

a) $\mu = \pi' \cdot \pi$ if $\dim F_\beta = 0$

b) $\mu = \pi \cdot \{\pi' : \pi_a\}$ if $\dim F_\beta = 1$

c) $\mu = \{\pi' : \pi\}$ if $\dim F_\beta \geq 2$.

Putting an orientation on a polyhedron $W \subset I^A$ is to assign to each of its faces an orientation. More precisely an orientation of the polyhedron I^A is a function γ from the set of faces of W into the set of their orientations such that $\gamma(F_\beta)$ is an orientation of F_β .

(2.2) Lemma. Let γ be an orientation of a cube I^A . Let F_β be a k-dimensional face of I^A and let $F_{\beta''}$ be a (k-2)-dimensional face of I^A . Then

$\sum_{F_{\beta'}} [\gamma(F_{\beta'}) : \gamma(F_{\beta'})] \cdot [\gamma(F_{\beta'}) : \gamma(F_{\beta''})] = 0$ where $F_{\beta'}$ are (k-1)-dimensional faces of I^A .

Proof. Assume that β and β'' are defined on B and B'' respectively. [Note that B might be empty.] In this case we can find $a_1 \neq a_2$ such that $a_1, a_2 \in A \setminus B$ and $B'' = \{a_1, a_2\} \cup B$, $\beta'' | B = \beta$. Then there are only two

non-zero elements F_{β_i} in the sum given by restriction of β'' to $\{a_i\} \cup B$ for $i=1, 2$. Hence,

$$\sigma = \sum_{F_{\beta'}} [\gamma(F_\beta) : \gamma(F_{\beta'})] \cdot [\gamma(F_{\beta'}) : \gamma(F_{\beta''})] =$$

$$= [\gamma(F_\beta) : \gamma(F_{\beta_1})] \cdot [\gamma(F_{\beta_1}) : \gamma(F_{B''})] + [\gamma(F_\beta) : \gamma(F_{\beta_2})] : [\gamma(F_{\beta''})]$$

Let us consider σ. It has a constant absolute value independent of the choice of

$$\pi = \gamma(F_\beta), \ \pi_1 = \gamma(F_{\beta_1}), \ \pi_2 = \gamma(F_{\beta_2}), \ \pi'' = \gamma(F_{\beta''}) \ .$$

Therefore, we may assume that the sets $A \setminus B''$, $A \setminus B'$ and $A \setminus B$ satisfy

$$A \setminus B'' \subset \{a_2\} \cup (A \setminus B'') = A \setminus B' \subset \{a_1, a_2\} \cup A \setminus B'' = A \setminus B$$

and that they are ordered so that

$$[\pi_2 : \pi''] = [\pi : \pi_2] = [\pi_1 : \pi''] = 1 \ .$$

It is easy to see that $[\pi : \pi_1] = -1$ and therefore $\sigma = 0$. This completes the proof of (2.2).

We assume that every face F_β of the cube I^A has been chosen a fixed orientation or (F_β). This assigns to every cubical polyhedron in I^A an orientation. We will use F_β to denote the pair $[F_\beta$, or $(F_\beta)]$.

(2.3) Definition:
The incidence coefficient $[F_{\beta'} : F_\beta]$ of the pair of faces of the cube I^A is given by:

$$[F'_\beta : F_\beta] \overset{df}{=} [\text{ or } (F_{\beta''}) : \text{ or } (F_\beta)]$$

§3. Chain complex of cubical polyhedron.

We define a group of k-dimensional chains $C_k(I^A)$ as follows:

if $k < 0$ then $C_k(I^A) = 0$

if $k \geq 0$ then $C_k(I^A)$ is the free group generated by k-dimensional faces of I^A with the chosen orientation.
We now define the boundary operation $\delta_k : C_k(I^A) \to C_{k-1}(I^A)$ as follows:

(3.1) $\quad \delta_k = 0 \quad$ if $\quad k \leq 0$

$$\delta_k(F_\beta) = \sum_{F_{\beta'}} [F_\beta : F_{\beta'}] \cdot F_{\beta'} \quad \text{if} \quad k > 0 .$$

Notice that only the generators of $C_{k-1}(I^A)$ can occur in the above sum with non-zero coefficients. We have

(3.2) Lemma. Let $F_\beta \in C_k(I^A)$ and let $F_{\beta''} \in C_{k-2}(I^A)$ and $k \geq 0$.

Then

$$\sum_{F_{\beta'} \in C_{k-1}(I^A)} [F_\beta : F_{\beta'}]\, [F_{\beta'} : F_{\beta''}] = 0 \quad .$$

Proof. Apply Lemma (2.2) for $\gamma =$ or . □

From Lemma (3.2) we obtain the familiar result:

(3.3) Theorem. $\delta_{k-1} \circ \delta_k = 0$. □

Let $V \subset I^A$ be a cubical polyhedron. We define $C_k(V)$ - a group of k-dimensional chains of V as a subgroup of $C_k(I^A)$ generated by (oriented) faces of V. Since $\delta_k(C_k(V)) \subset C_{k-1}(V)$ we have obtained a chain complex $C(V) = \{C_k(V);\ \delta_k\}$. If $V = \emptyset$ $C_k(V) = 0$ for every integer k. Let (V, V_0) be a pair of cubical polyhedra in I^A. We define $C_k(V, V_0)$ to be $C_k(V)/C_k(V_0)$ and we call $C_k(V, V_0)$ the group of k-dimensional chains of the pair (V, V_0). We will identify $C_k(V)$ and $C_k(V, \emptyset)$. Notice that the commutativity of the diagram

$$\begin{array}{ccc} C_k(V_0) & \longrightarrow & C_k(V) \\ \downarrow{\scriptstyle \delta_k} & & \downarrow{\scriptstyle \delta_k} \\ C_{k-1}(V_0) & \longrightarrow & C_{k-1}(V) \end{array}$$

implies that δ_k induces a homomorphism

$$\delta_k : C_k(V, V_0) \longrightarrow C_{k-1}(V, V_0) \quad .$$

We have thus assigned to a pair of cubical polyhedra (V, V_0) a chain complex

$$C(V, V_0) = \{C_k(V, V_0), \delta_k\} \quad .$$

We recall that a chain complex is acyclic in dimension n iff the sequence

$$C_{n+1} \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \quad \text{is exact.}$$

A chain complex is acyclic iff it is acyclic in all dimensions.

(3.3) Example: Let F_β be a vertex in I^A. Then $C_k(F_\beta) = 0$ for $k > 0$, hence $C(F_\beta)$ is acyclic in positive dimensions.

(3.4) Remark: For a chain complex $C(V)$ we can define an augmentation $\varepsilon : C(V) \to Z$ as follows:

$$\varepsilon(F_\beta) = \text{or}(F_\beta) \cdot \prod_{a \in B} \beta(a) \quad .$$

It is easy to notice that $\varepsilon \cdot \delta_1 = 0$. If V is connected and non-empty then ε and $-\varepsilon$ are unique homomorphisms of $C_0(V) \to Z$ making the augmented complex acyclic in dimension 0 .

§4. Cylinder.

Let A be a finite set and let $a \in A$. Let us assume that (V,V_0) is a pair of $(A \setminus \{a\})$ - cubical polyhedra. $(V_0 \subset V \subset I^{A \setminus \{a\}})$ and let $q : A \setminus \{a\} \to A$ be the inclusion.

The pair $I_a(V,V_0) = (f_q^{-1}(V), f_q^{-1}(V_0))$ of A-cubical polyhedra is said to be cylinder (a-cylinder) over (V,V_0) .

(4.1) Proposition: Inclusion $q : A \setminus \{a\} \to A$ is a cubical morphism of $I_a(V,V_0)$ into (V,V_0) for every pair of $(A \setminus \{a\})$ - polyhedra (V,V_0) .

Proof of (4.1) is obvious.

(4.2) A pair

$$I_a^{-1}(V,V_o) = F_{(a,-1)} \cap I_a(V,V_o) =$$

$$= (F_{(a,-1)} \cap f_q^{-1}(V),\ F_{(a,-1)} \cap f_q^{-1}(V_o))$$

is said to be the lower base of the cylinder $I_a(V,V_o)$. Similarly, $I_a^{+1}(V,V_o)=F_{(a,+1)} \cap I_a(V,V_o)$ is called the upper base.

We define the chain homomorphism $C(q):C(I_a(V,V_o)) \to C(V,V_o)$ as follows

$$(4.3) \qquad C(q)[F_\beta] = \begin{cases} 0 & a \in B \\ \beta(a) \cdot [or(F_\beta) : or\ (f_q(F_\beta))] \cdot f_q(F_\beta) & \text{*)} \end{cases}$$

Geometrically $C(q)$ is the projection of the faces of I^A onto $I^{A\setminus\{a\}}$ and it decreases the dimension of the faces not lying in the bases. For this reason we define $C(q)$ to be zero on these faces. Let F_β be a face of the polyhedron $V \subset I^{A\setminus\{a\}}$. We define a chain homomorphism

$$i_\varepsilon : C(V,V_0) \to C(I_a(V,V_0))$$

---

*) Note that $[or\ (F_\beta) : or\ (f_q(F_\beta))]$ is defined since $A\setminus\{a\}\setminus B = A\setminus B$.

as follows:

(4.4) $$i_\varepsilon(F_\beta) = \varepsilon \cdot [\text{or}\,(F_\beta) : \text{or}\,(F_{\beta_\varepsilon})] \cdot F_{\beta_\varepsilon}$$

where $\beta_\varepsilon : B \cup \{a\} \to \{-1, 1\}$ is given by

$$\beta_\varepsilon(x) = \begin{cases} \beta(x) & x \neq a \\ \varepsilon & x = a \end{cases}$$

(4.5) Theorem:

a) $$C(q) \circ i_\varepsilon = Id_{C(V, V_0)}$$

b) $$i_\varepsilon \circ C(q) \simeq Id_{C(I_a(V, V_0))}$$

for $\varepsilon = \pm 1$ ($\simeq$ denotes chain homotopy).

Before we prove (4.5) we will state three corollaries.

(4.6) Corollary 1. i_{+1} is chain homotopic to i_{-1}. □

This can be considered as a combinatorial homotopy axiom.

(4.7) Corollary 2. The chain complex of a cylinder is isomorphic to the chain complex of its bases. □

(4.8) Corollary 3. The faces of a cube are acyclic.

Proof (of Corollary 3): The proof will proceed by induction. Let k be the dimension of F_β. When $k=0$ then F_β is a point and our assertion was proved in (3.3). If $k > 0$ then F_β is a cylinder over a $(k-1)$-dimensional face F'_β. By the induction hypothesis F'_β is acyclic. Hence from (4.7) F_β is acyclic.

Proof of the Theorem (4.5):

We will prove the theorem for the case $\varepsilon = +1$ only, the proof for $\varepsilon = -1$ is similar. We prove a) first.

$$(C(q) \circ i_{+1})(F_\beta) = C(q)([\text{or}\,(F_\beta) : \text{or}\,(F_{\beta_{+1}})] \cdot F_{\beta_{+1}}) =$$

$$= [\text{or}\,(F_\beta) : \text{or}\,(F_{\beta_{+1}})] \cdot C(q)(F_{\beta_{+1}}) =$$

$$= [\text{or}\,(F_\beta) : \text{or}\,(F_{\beta_{+1}})] \cdot [\text{or}\,(F_{\beta_{+1}}) : \text{or}\, f_q(F_{\beta_{+1}}))] \cdot f_q(F_{\beta_{+1}}).$$

Note that pointwise $f_q(F_{\beta_{+1}}) = F_\beta$. Hence:

$$(C(q) \circ i_{+1})(F) =$$

$$= [\text{or}\,(F_\beta : \text{or}\,(F_{\beta_{+1}})] \cdot [\text{or}\,(F_{\beta_{+1}}) : \text{or}\,(F_\beta)] \cdot F_\beta =$$

$$= ([\text{or}\,(F_\beta) : \text{or}\,(F_{\beta_{+1}})])^2 \cdot F_\beta = F_\beta .$$

We now prove b).

Assume that $F_\beta \subset I^A$ is a face such that $a \in B$. We define F^a_β to be face of I^A given by $\beta | B \setminus \{a\} \to \{-1, 1\}$. We define the chain homomorphism Δ_k as follows:

$$\Delta_k : C_k(I_a(V)) \to (I_a(V))$$

$$\Delta_k(F_\beta) = \begin{cases} 0 & \text{if } a \bar{\in} B \text{ or } \beta(a) = 1 \\ -[\text{or } (F^a_\beta) : \text{or } (F_\beta)] \cdot F^a_\beta \end{cases}.$$

We show that

$$(4.9) \qquad \delta \circ \Delta + \Delta \circ \delta = i_{+1} \circ C(q) - \mathrm{Id}_{C(I_a(V, V_0))}$$

$$i_{+1} C(q)(F_\beta) = \begin{cases} 0 & \text{if } a \bar{\in} B \\ F_\beta & \text{if } \beta(a) = 1 \\ -[\text{or } (F_\beta) : \text{or } (F_{\beta_{+1}})] \cdot F_{\beta_{+1}} & \text{if } \beta(a) = -1 \end{cases}$$

$$(i_{+1} \circ C(q) - \mathrm{Id}_{C(I_a(V,V_0))})(F_\beta) = \begin{cases} -F_\beta & \text{if } a \bar{\in} B \\ 0 & \text{if } \beta(a) = 1 \\ -[\text{or } (F_\beta) : \text{or } (F_{\beta_{+1}})] \cdot F_{\beta_{+1}} - F_\beta & \text{if } \beta(a) = -1 \end{cases}$$

i) assume that $a \notin B$.

$$(\delta\Delta)(F_\beta) = 0 \quad .$$

$$(\Delta\delta)(F_\beta) = \Delta([\text{or } (F_\beta) : \text{or } (F_{\beta_{+1}})] \cdot F_{\beta_{+1}} +$$

$$+ \Delta([\text{or } (F_\beta) : \text{or } (F_{\beta_{-1}})]) \cdot F_{\beta_{-1}} + \Delta\Big(\sum_{\substack{F'_\beta \subset F_\beta \\ \beta'(a) \neq \pm 1}} [F_\beta : F'_\beta] \cdot F'_\beta \Big)$$

$$= -[\text{or } (F_\beta) : \text{or } (F_{\beta_{-1}})] \cdot [\text{or } (F^a_{\beta_{-1}}) : \text{or } (F_{\beta_{-1}})] \cdot F^a_{\beta_{-1}} \quad .$$

Notice that pointwise $F_\beta = F^a_{\beta_{-1}}$. Hence

$$(\Delta\delta)(F_\beta) = -F_\beta \quad .$$

Thus (4.9) holds in this case

ii) assume that $\beta(a) = 1$.

$$\delta\Delta(F_\beta) = 0$$

$$(\Delta\delta)(F_\beta) = 0 \quad .$$

Hence (4.9) holds in this case

iii) assume that $\beta(a) = -1$.

$$\delta\Delta(F_\beta) = \delta(-[\text{or } (F_\beta^a) : \text{or } (F_\beta)] \cdot F_\beta^a) =$$

$$= -[F_\beta^a : F_\beta] \cdot \sum_{F_{\beta'} \subset F_\beta^a} [F_\beta^a : F_{\beta'}] \cdot F_{\beta'} \quad .$$

$$\Delta\delta(F_\beta) = \Delta\Big(\sum_{F_{\beta''} \subset F_\beta} [F_\beta : F_{\beta''}] \cdot F_{\beta''}\Big) =$$

$$\sum_{F_{\beta''} \subset F_\beta} -[F_\beta : F_{\beta''}] \cdot [F_{\beta''}^a : F_{\beta''}] \cdot F_{\beta''}^a \quad .$$

$$(\delta\Delta + \Delta\delta)(F_\beta) = -[F_\beta^a : F_\beta] \cdot [F_\beta^a : F_{\beta_{+1}}] \cdot F_{\beta_{+1}}$$

$$- [F_\beta^a : F_\beta] \cdot [F_\beta^a : F_{\beta_{-1}}] \cdot F_{\beta_{-1}} +$$

$$+ \sum_{\substack{F_{\beta'} \subset F_\beta^a \\ \beta'(a) \neq \pm 1}} - [F_\beta^a : F_\beta] \cdot [F_\beta^a : F_{\beta'}] \cdot F_{\beta'} +$$

$$+ \sum_{F_{\beta''} \subset F_\beta} -[F_\beta : F_{\beta''}] \cdot [F_{\beta''}^a : F_{\beta''}] \cdot F_{\beta''}^a \quad .$$

First note that

$$[F^a_\beta : F_\beta] \cdot [F^a_\beta : F_{\beta_{+1}}] = [\text{or}\,(F_\beta) : \text{or}\,(F_{\beta_{+1}})]$$

and that

$$F_{\beta_{-1}} = F_\beta \quad \text{(pointwise)} .$$

Hence

$$(\delta\Delta + \Delta\delta)(F_\beta) = -[\text{or}\, F_\beta : \text{or}\, F_{\beta_{+1}}] \cdot F_{\beta_{+1}} - F_\beta$$

$$+ \sum_{\substack{F_{\beta''} \subset F_\beta \\ \beta'(a) \neq \pm 1}} [F_\beta : F_{\beta''}] \cdot [F^a_{\beta''} : F_{\beta''}] \cdot F^a_{\beta''} =$$

$$+ \sum_{F_{\beta''} \subset F_\beta =} [F_\beta : F_{\beta''}] \cdot [F^a_{\beta''} : F_{\beta''}] \cdot F^a_{\beta''} =$$

$$= (i_{+1} \circ C(q) - \mathrm{Id}_{C(I_a(V,V_0))})(F_\beta) +$$

$$= \sum_{\substack{F_{\beta'} \subset F^a_\beta \\ \beta'(a) \neq \pm 1}} - [F^a_\beta : F_\beta] \cdot [F^a_\beta : F_{\beta'}] \cdot F_{\beta'} +$$

$$+ \sum_{F_{\beta''} \subset F_\beta} [F_\beta : F_{\beta''}] \cdot [F^a_{\beta''} : F_{\beta''}] \cdot F^a_{\beta''}$$

To prove (4.9) we have to show

$$(4.10)\qquad \sum_{\substack{F_{\beta'} \subset F^a_\beta \\ \beta'(a)\neq \pm 1}} [F^a_\beta : F_\beta] \cdot [F^a_\beta : F_{\beta'}] \cdot F_{\beta'} +$$

$$+ \sum_{F_{\beta''} \subset F_\beta} [F_\beta : F_{\beta''}] \cdot [F^a_{\beta''} : F_{\beta''}] \cdot F^a_{\beta''} = 0 \ .$$

Let $F_{\beta'}$ be a face of F^a_β such that $\beta'(a)\neq\pm 1$. Then $F_{\beta'} = F^a_{\beta'_{-1}}$. Let F_β be a face of F_β. Then $F^a_{\beta''}$ is a face of F^a_β given by $\beta''|B\setminus\{a\} \to \{-1, 1\}$. Note also that the above correspondence is 1-1. Thus $F_{\beta'}$ occurs in our sum with the following coefficients:

$$-[F^a_\beta : F_\beta] \cdot [F^a_\beta : F_{\beta'}] \cdot F_{\beta'} - [F_\beta : F_{\beta'_{-1}}] \cdot [F^a_{\beta'} : F_{\beta'_{-1}}] \cdot F_{\beta'}$$

$$= -([F^a_\beta : F_\beta] \cdot [F^a_\beta : F_{\beta'}] + [F_\beta : F_{\beta'_{-1}}] \cdot [F_{\beta'} : F_{\beta'_{-1}}]) \cdot F_{\beta'}$$

Proving lemma (2.2) we showed:

$$[F^a_\beta : F_\beta] \cdot [F_\beta : F_{\beta'_{-1}}] + [F^a_\beta : F_{\beta'}] \cdot [F_{\beta'} : F_{\beta'_{-1}}] = 0 \ .$$

Our result is a consequence of the following simple lemma:

(4.11) Lemma: Let $|A| = |B| = |C| = |D| = 1$ and $A \cdot B + C \cdot D = 0$.

Then $A \cdot C + B \cdot D = 0$. □

(4.12) Remark: All of the chain homomorphisms constructed above commute with augmentations. Hence an augmented complex of a face is also acyclic (look (3.4)).

§5. Cubical carriers. Category Ch.

We will state and prove the cubical carrier theorem based on (4.8)*) . Let $G = \{g_t: (V, V_0) \to (W, W_0);\ t \in T\}$ be a family of cubical morphisms admitting a common carrier [look II. 1.1]. We define the acyclic cubical carrier $E[G]$ of G as follows:

(5.1) Let F_β be a face of V .

$$E(G)(F_\beta) \overset{dt}{=} C(carr(\bigcup_t f_{q_t}(F_\beta)) \quad .$$

*) For the acyclic carrier theorem in full generality look at Hilton-Wylie [1] and Mosher-Tangora [1] .

[Note that by (4.8) $\operatorname{carr}(\bigcup_t f_{q_t}(F_\beta))$ is acyclic and that $E(G)(F_\beta) \subset C(W_0)$ for $F_\beta \subset V_0$.]

(5.2) Definition: A chain homomorphism $h: C(V) \to C(W)$ is said to be carried by $E(G)$ iff $h(F_\beta) \in E(G)(F_\beta)$ for each $F_\beta \subset V$.

[Note that $h(C(V_0)) \subset C(W_0)$].

(5.3) Theorem: Let $G = \{g_t: (V, V_0) \to (W, W_0);\ t \in T\}$ be a family of cubical morphisms admitting a common cubical carrier.

Then there exists a chain homomorphisms $h: C(V) \to C(W)$ carried by $E(G)$.

Proof. We define $h_{-1} = Id_Z: Z \to Z$. The proof proceeds by induction. Suppose that h has been extended over all of $C_k(V)$. Assume that $F_\beta \in C_{k+1}(V)$. $h(\delta\gamma)$ is a cycle in $C_k(E(G)(F_\beta))$. Thus, there exists $\tilde{\gamma} \in C_{k+1}(E(G)(F_\beta))$ such that $\delta\tilde{\gamma} = \gamma$. We define $h(F_\beta) = \tilde{\gamma}$. □

(5.4) Theorem: The map h constructed in the proof of (5.3) is unique up to a chain homotopy carried by $E(G)$.

Proof: Suppose that h^1 and h^2 are two chain homomorphisms carried by $E(G)$. Let $h=h^1-h^2$. We will show that h is chain homotopic to zero. Note that

$$h_{-1} = h^1_{-1} - h^2_{-1} = 0 \quad .$$

We will inductively construct a chain homotopy Δ carried by $E(G)$. We define $\Delta_0=0$. Suppose that Δ has been extended over all of $C_k(V)$ and that Δ is carried by $E(G)$. Let $F_\beta \in C_{k+1}(V)$.

$\gamma = (\Delta\delta)(F_\beta) - h(F_\beta)$ is a cycle in

$C_{k+1}(E(G)(F_\beta))$, for:

$$\delta\gamma = \delta\Delta\delta(F_\beta) - \delta h(F_\beta) =$$

$$(h-\Delta\delta)\delta(F_\beta) - \delta h(F_\beta) = h\delta(F_\beta) - \delta h(F_\beta) = 0 \ .$$

Hence there exists $\tilde{\gamma} \in C_{k+2}(E(G)(F_\beta))$ such that $\delta\tilde{\gamma} = \gamma$. We define $\Delta(F_\beta) = \tilde{\gamma}$. Thus concludes our construction. □

As an immediate corollary we have the following:

(5.5) Theorem: Let $q:A_1 \to A$ be a cubical morphism of $(V,V_0) \to (W,W_0)$. Then there exists a chain map $h:C(V,V_0) \to C(W,W_0)$ such that $h(F_\beta) \subset C(\text{carr } f_q(F_\beta))$ and and h is uniquely determined up to chain homotopy.

Proof: Apply (5.3) and (5.4) to $G=\{q\}$. □

We will denote the chain homotopy class of h by $C(q)$.

(5.6) Corollary: If $G=\{g_t:(V,V_0) \to (W,W_0)\}$ is a family of cubical morphisms admitting a common carrier. Then $C(q_{t_1})=C(q_{t_2})$ for every $t_1, t_2 \in T$.

Proof: Apply (5.4) to $C(q_{t_1})$ and $C(q_{t_2})$.

(5.7) Corollary: If cubical morphisms $p,q:(V,V_0) \to (W,W_0)$ are contiguous then $C(p)=C(q)$. □

(5.8) Corollary: Let $q:(V,V_0) \to (W,W_0)$ and $q_1:(W,W_0) \to (U,U_0)$ be cubical morphisms. Then:

a) $C(q_1 \circ q) = C(q_1) \circ C(q)$

b) if $q = Id_A:A \to A$ then $C(q:(V,V_0) \to (V,V_0)=Id_{C(V,V_0)}$. □

Let Ch denote the category of chain complexes and

chain homotopy classes of chain maps. Thus we have obtained a functor C from the category QP_0 of finite cubical pairs into Ch :

$$(5.9) \qquad (V,V_0) \longmapsto C(V,V_0)$$

$$[q:(V,V_0) \longrightarrow (W,W_0)] \longmapsto C(q) \quad .$$

By (5.7) there is a factorization of the functor C :

$$(5.10) \qquad C = \tilde{C} \circ Cg \quad .$$

The functor $\tilde{C}:CQ_0 \to Ch$ is unique.

[CQ_0 denotes here contiguous category restricted to finite polyhedra.]

(5.10) Remark: We will show that up to isomorphism $C(V,V_0)$ does not depend on the orientation. Let or' be another orientation of the cube I^A . We define a natural transformation

$$\Gamma:C(V,V_0;\ or) \longrightarrow C(V,V_0;\ or')$$

as follows:

$$\Gamma([F_\beta, \text{or } F_\beta]) = [\text{or } F_\beta : \text{or}' \cdot F_\beta] \cdot [F_\beta, \text{or } F_\beta].$$

It is easy to notice that

$$\Gamma' : C(V, V_0; \text{or}') \longrightarrow C(V, V_0; \text{or})$$

given by

$$\Gamma([F_\beta, \text{or}' F_\beta]) = [\text{or}' F_\beta : \text{or } F_\beta] \cdot [F_\beta, \text{or } F_\beta]$$

is the inverse of Γ . Thus up to the natural equivalence C does not depend on the orientation.

CHAPTER IV.

§1. Combinatorial homology of cubical polyhedra.

We are now in a position to define homology for cubical polyhedra.

(1.1) Definition: Let G be a group.

$$H_k(V,V_0\ ;G) \overset{df}{=} H_k(C(V,V_0);G)$$

$$H^k(V,V_0\ ;G) \overset{df}{=} H^k(C(V,V_0);G) \quad .$$

$H_k(V,V_0)$ will be used to denote $H_k(V,V_0;Z)$.

We will now show that our homology satisfies a set of axioms that may be considered as the combinatorial equivalent of the Eilenberg-Steenrod's axioms for homology.

(1.2) Lemma: Let $V_0 \subset V \subset I^A$ be a pair of cubical polyhedra. Let $i:C(V_0) \to C(V)$ be the canonical injection. Then

$$C(Id_A : V_0 \to V) = [i] \quad .$$

Proof: Put $G=\{Id_A : V_0 \to V\}$. i is carried by $E(G)$. Hence by (III. 5.4) $[i]=C(Id_A)$.

(1.3) Lemma: Let $V_0 \subset V \subset I^A$ be a pair of cubical polyhedra. Let $j:C(V) \to C(V,V_0)$ be the canonical projection. Then $C(Id_A : (V,\emptyset) \to (V,V_0)) = [j]$.

The meaning of the above two lemmas is that the algebraic homomorphisms i and j are induced by the respective cubical morphisms. From the definition of $C(V,V_0)$ we have an exact sequence

$$(1.4) \qquad 0 \longrightarrow C(V_0) \xrightarrow{i} C(V) \xrightarrow{j} C(V,V_0) \longrightarrow 0$$

for every pair of cubical polyhedra (V,V_0) .

From (1.2) and (1.3) we have $H([i])=H(C(Id_A : V_0 \to V))$ and $H([j])=H(Id_A : (V,\emptyset) \to (V,V_0))$. It is a standard conclusion in homological algebra that (1.4) gives us an exact sequence:

$$\longrightarrow H_k(V_0) \longrightarrow H_k(V) \longrightarrow H_k(V,V_0) \xrightarrow{\delta} H_{k-1}(V_0) \longrightarrow \ldots$$

[For example look at Godement "Topologie algebrique et theorie

des faisceaux" Theorem 2.1.1] .

We now establish an excision axiom. Let $(V,V_0) \subset I^A$ be a pair of cubical polyhedra. Notice that $V \setminus \mathrm{Int}_V V_0 \subset I^A$ is a polyhedron (if $x \in V \setminus \mathrm{Int}_V V_0$ then $\mathrm{carr}(x)$ is a face in $V \setminus \mathrm{Int}_V V_0$). By considering generators, we obtain the canonical chain isomorphism

$$C(V \setminus \mathrm{Int}_V V_0 \; ; \; V_0 \setminus \mathrm{Int}_V V_0) \simeq C(V,V_0) \quad .$$

Hence, we have

(1.5) If (V,V_0) is a polyhedral pair in I^A then

$$H(V \setminus \mathrm{Int}_V V_0 \; ; V_0 \setminus \mathrm{Int}_V V_0) \simeq H_k \; V, \; V_0) \; ,$$

and this isomorphism is induced by the inclusion

$$(V \setminus \mathrm{Int}_V V_0 \; , \; V_0 \setminus \mathrm{Int}_V V_0) \longrightarrow (V,V_0) \quad .$$

§2. Eilenberg-Steenrod axioms for finite cubical complexes.

There is a sequence of functors $\{H_K\}_{k=0}^{\infty}$ from the category of finite cubical pairs into the category of groups and a function $\delta = \delta(k,V,V_0) : H_k(V,V_0) : H_k(V,V_0) \to H_{k-1}(V_0)$

such that

1) If $g:A_1 \to A$ is a cubical morphism of (V,V_0) into (W,W_0) then the following diagram commutes:

$$\begin{array}{ccc} H_k(V,V_0) & \xrightarrow{H_k(g)} & H_k(W,W_0) \\ \downarrow \delta & & \downarrow \delta \\ H_{k-1}(V_0) & \xrightarrow{H_{k-1}(g)} & H_{k-1}(W_0) \end{array}$$

2) Strong excision axiom.

If $(V,V_0) \subset I^A$ is a cubical pair and $Id_A:A \to A$ is an inclusion morphism of $(V \setminus Int_V V_0 , V_0 \setminus Int_V V_0) \to (V,V_0)$, then

$$H(Id_A): H(V \setminus Int_V V_0 , V_0 \setminus Int_V V_0) \longrightarrow H(V,V_0)$$

is an isomorphism.

3) Exactness axiom.

If $(V,V_0) \subset I^A$ is a cubical pair and $Id_A:A \to A$ is a morphism of V_0 into V, and of V into (V,V_0) then the sequence

$$\ldots \longrightarrow H_k(V_0) \xrightarrow{H_k(Id_A)} H_k(V) \xrightarrow{H_k(Id_A)} H_k(V,V_0) \xrightarrow{\delta} H_{k-1}(V_0) \longrightarrow$$

is exact.

4) Homotopy axiom.

If two cubical morphisms $p,q:(V,V_0) \to (W,W_0)$ are contiguous then $H(p)=H(q):H(V,V_0) \to H(W,W_0)$.

5) Dimension axiom.

If $P=F_\beta \subset I^A$ is a vertex of I^A then

$$H_k(P) = 0 \quad \text{for} \quad k \neq 0 \quad .$$

CHAPTER V.

§1. Cubical homology of compact pairs.

Using results and definitions of preceeding sections we will define the homology of a pair of compact topological spaces. Throughout this chapter all spaces will be compact. Let us recall (I. 3.8) that

$$Q(X,X_0) = \varprojlim(N_B^A(X,X_0),\ i_B^A : B \subset A \in \mathrm{Fin}\ A(X)) .$$

We define the cubical homology of a compact pair.

(1.1) Definition: Let G be a group.

$$H_k(X,X_0;G) \overset{df}{=} \varprojlim(H_k(N_B^A(X,X_0(;G), H_k(i_B^A) : B \subset A \in \mathrm{Fin}\ A(X))$$

$$H^k(X,X_0;G) \overset{df}{=} \varinjlim(H^k(N_B^A(X,X_0);G), H^k(i_B^A) : B \subset A \in \mathrm{Fin}\ A(X))$$

$H_k(X,X_0)$ will be used to denote $H_k(V,V_0;Z)$.

Let $g:(X,X_0) \to (Y,Y_0)$ be a continuous map. Let us recall (I. 3.9) that:

$$Q(g) = \varprojlim(N_B^A g;\ g^{-1}(B) \subset A \in \mathrm{FinA}(X);\ B \in \mathrm{FinA}(Y)) .$$

(1.2) Definition:

$$H(g) \overset{df}{=} \varprojlim(H(N_B^A g);\ g^{-1}(B) \subset A \in \mathrm{FinA}(X);\ B \in \mathrm{FinA}(Y))$$

(in cohomological case there should be direct limit). Note that H is a functor (I. 3.4).

§2. Boundary operator, exactness, dimension axiom.

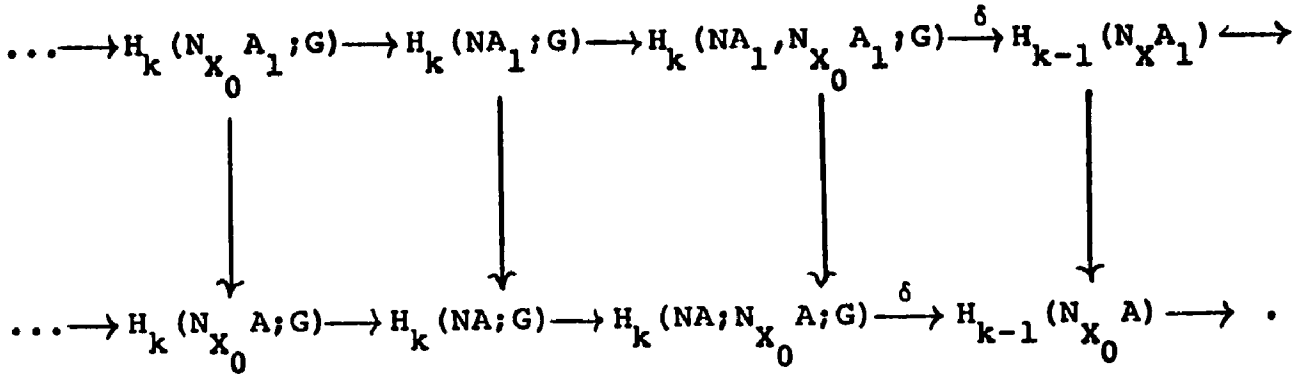
Let us now consider the boundary operator and exactness. Assume that (X,X_0) is a compact pair. Let $A \in \mathrm{FinA}(X)$. Let $i:X_0 \to X$ and $j:X \to (X,X_0)$ be the canonical inclusion. Let $A \in \mathrm{FinA}(X)$.

From the exactness of the combinatorial cubical homology we have an exact sequence:

$$\ldots \longrightarrow H_k(N_{X_0} A;G) \longrightarrow H_k(NA;G) \longrightarrow H_k(NA,N_{X_0} A;G) \longrightarrow H_{k-1}(NA;G) \longrightarrow$$

Assume that $A_1 \in \mathrm{FinA}(X)$ and that $A \subset A_1$.

Then the following diagram commutes



Passing to the limit we obtain the sequence:

(2.1)

$$\ldots \longrightarrow H_k(X_0;G) \xrightarrow{H_k(i)} H_k(X;G) \xrightarrow{H_k(j)} H_k(X,X_o;G) \xrightarrow{\delta} H_{k-1}(X_0;G) \longrightarrow$$

which is in general half-exact, in the sense that the composition of any two maps is zero. But if G is a compact topological group, or a vector space over a field then (2.1) is exact. [Look Eilenberg-Steenrod, "Foundations of algebraic topology" VIII. 5.6].

The cohomological analogue of (2.1) is always exact.

Let $f:(X,X_0) \to (Y,Y_0)$ be a continuous map. Let $A \in \mathrm{FinA}(X)$, $B \in \mathrm{FinA}(Y)$ and $f^{-1}(B) \subset A$. Axiom 3 of the combinatorial theory shows that the diagram

$$\begin{array}{ccc} H\,(NA(X,X_0)) & \xrightarrow{H(N_B^A f)} & H\,(NB(Y,Y_0)) \\ \Big\downarrow \delta & & \Big\downarrow \delta \\ H_{k-1}(N_{X_0}A) & \xrightarrow{H(N_B^A f)} & H_{k-1}(N_{Y_0}B) \end{array}$$

commutes.

Passing to the limit we have the commutative diagram:

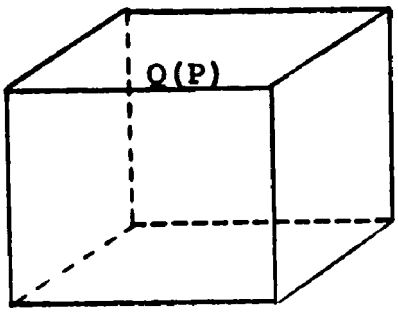
$$\begin{array}{ccc} H_k(X,X_0) & \xrightarrow{H_k(f)} & H_k(Y,Y_0) \\ \Big\downarrow \delta_k & & \Big\downarrow \delta_k \\ H_{k-1}(X_0) & \xrightarrow{H_{k-1}(f)} & H_{k-1}(Y_0) \end{array} \qquad (2.2)$$

(2.2) is the third Eilenberg-Steenrod axiom.

We will now discuss the dimension axiom. Let P be a point. Then A(P) consists of three elements:

$A_1 = (P,P)$; $A_2 = (P,\emptyset)$ and $A_3 = (\emptyset,P)$.

$Q(P) = NA(P) \subset I^3$



NA(P) has one one-dimensional face given by:

$$\beta : \{A_2 , A_3\} \to \{-1, 1\}$$

$$\beta(A_3) = -1$$

$$\beta(A_3) = 1$$

and two vertices: (-1,-1, 1) and (1,-1, 1) . By (III.4.8)

(2.3) $\quad H_k(P) = 0 \quad$ for $\quad k \neq 0$.

(2.3) is the Eilenberg-Steenrods dimension axiom.

§3. Homotopy axiom. Excision axiom.

(3.1) Lemma: If $f,g:(X,X_0) \to (Y,Y_0)$ are continuous maps such that $Q(f)$ and $Q(g)$ are contiguous then $H(f)=H(g)$.

Proof: It is a consequence of (II. 1.2') . □

(3.2) Theorem: (Homotopy axiom)

Let f and g be a pair of homotopic maps from $(X,X_0) \to (Y,Y_0)$. Then $H(f)=H(g):H(X,X_0) \to H(Y,Y_0)$.

Proof: By (II. 3.6) $Q(f)$ and $Q(g)$ are contiguous. □

Let (X,X_0) be a compact pair. Let $U \subset X$ be an open subset of X such that $\bar{U} \subset \operatorname{int} X_0$. We have the canonical inclusion map $i:(X \setminus U,\ X_0 \setminus U) \to (X,X_0)$. Notice that there exists $G=(G_{-1}, G_1) \in A(X)$ such that $G_{-1} \subseteq \operatorname{Int} X_0$ and $G_1 \subseteq X \setminus \bar{U}$. Note that finite subsets of $A(X)$ containing G are cofinal in $\operatorname{Fin} A(X)$. Furthermore, the family of all subsets of $A(X \setminus U)$ of the form $i^{-1}(A)$, where $G \in A \in \operatorname{Fin} A(X)$ are cofinal in $\operatorname{Fin} A(X \setminus U)$. Thus to show that

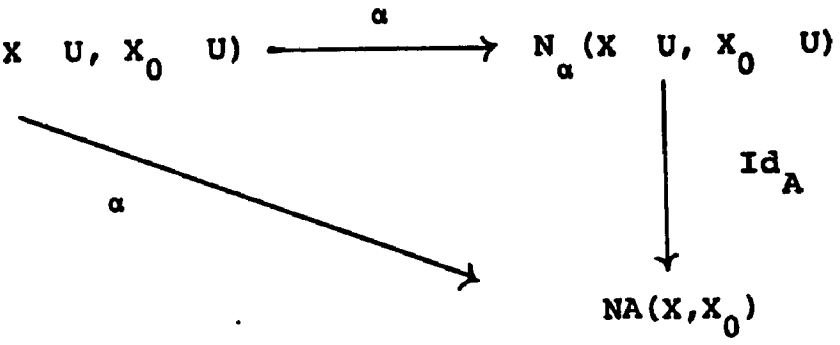
$$H(i):H(X \setminus U,\ X_0 \setminus U) \to H(X,X_0)$$

is an isomorphism we need only to show that:

$$(3.3) \quad C_g(N_A^{i^{-1}(A)} i):N_i^{-1}(A)(X \setminus U,\ X_0 \setminus U) \to NA(X,X_0)$$

is an isomorphism for finite subsets A of A(X) containing G . Let $\alpha: A \to i^{-1}(A)$ be the canonical map. Let us recall (II. 2.6) that the following diagram commutes

(3.4)



(3.5) Lemma: If G is an element of A then

$$C(Id_A): C(N_\alpha(X \setminus U, \ X_0 \setminus U)) \to C(NA(X,X_0))$$

is an isomorphism in the category Ch .

Proof: Let $h \in C(Id_A)$

i) Clearly h is an injection.

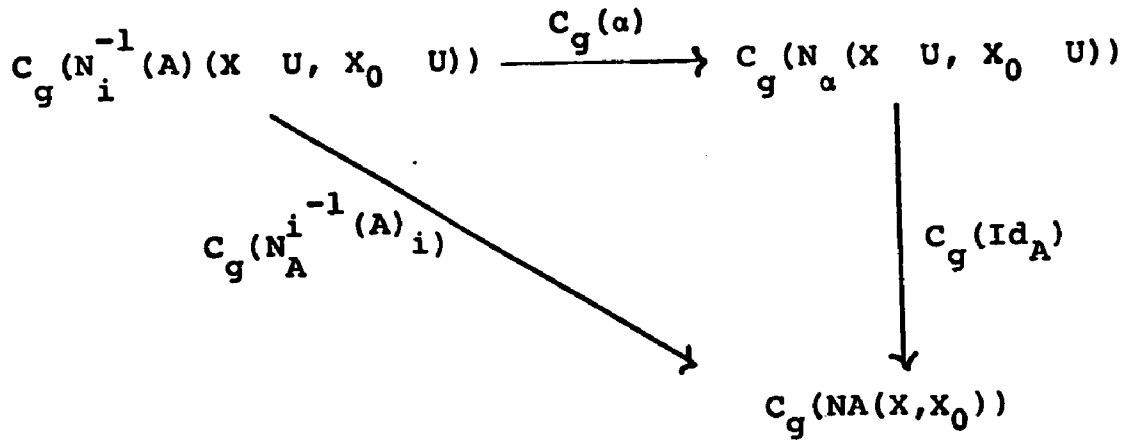
ii) Let F_β be a face of $NA(X,X_0)$ such that $F_\beta \neq 0$ $C(NA(X,X_0))$.

Let $B \subset A$ be the domain of B . Note that $G \in B$ and $\beta(G)=1$. [Otherwise, F_β would be zero in $C(NA(X,X_0))$. because $G_{-1} \cap G_1 \subset G_{-1} \subset X_0$]. For similar reasons

$$\bigcap\{G_{(a,\varepsilon)} : \varepsilon = \beta(a) \quad \text{or} \quad a \in A \setminus B\} \cap X_0 \neq \emptyset \ .$$

Hence F_β is also a non-zero chain in $C(N_\alpha(X \setminus U, X_0 \setminus U))$. □

Passing to the contiguous level we obtain the commutative diagram:



By (II. 2.5) $Cg(\alpha)$ is an isomorphism. By (3.6) $Cg(Id_A)$. is an isomorphism. Hence $Cg(N_A^{i^{-1}(A)}{}_i)$ is an isomorphism. This concludes the proof of (3.4). Thus we have:

(3.6) Theorem: (Excision axiom)

Let (X,X_0) be a compact pair and let U be an open subset of X such that $\bar{U} \subset \text{int}X_0$. Let $i:(X \setminus U, X_0 \setminus U) \to (X,X_0)$ be the canonical inclusion. Then $H(i):H(X \setminus U, X_0 \setminus U) \to H(X,X_0)$ is an isomorphism. □

§4. Continuity of cubical homology.

Throughout this section $(X, p_t; t \in T)$ is a representation of a compact space X as limit of an inverse system $(X_t, p_t^s: t < S \in T)$ of compact spaces X_t, $t \in T$.

The proofs of the following lemmas are standard in the point set topology.

(4.1) Lemma: For every finite $A \subseteq A(X)$ there exists $t \in T$ and a function $f:A \to A(X_t)$ such that $p_t^{-1}(f(G)) \subseteq G$ for every $G \in A$. □

(4.2) Lemma: For every finite $A \subseteq A(X_t)$ there exists $s > t$ and a function $f:A \to A(X_t)$ such that

(i) $f(G) \subseteq (p_t^s)^{-1}(G)$

(ii) $\bigcap\{\pi_\varepsilon(G): (G,\varepsilon) \in Z\} = \emptyset$ iff $\bigcap\{\pi_\varepsilon(p_t^{-1}(G)): (G,\varepsilon) \in Z\} \neq \emptyset$

for every $Z \subseteq f(A)\times\{-1, 1\}$. □

(4.3) Remark: (ii) shows that p_s induces cubical isomorphism

$$N^{p_s^{-1}(f(A))}_{f(A)}(p_s): Np_s^{-1}(f(A)) \to Nf(A).$$

Assertions (4.1), (4.2), (4.3) and (II. 3.1) imply:

(4.4) Corollary: For every finite $A' \subseteq A(X)$ there exists

$t \in T$, $A \subseteq A(X)$ and a function $f:A' \xrightarrow{\text{onto}} A$ such that:

(i) $p_t^{-1}(f(G)) \subseteq G$ for every $G \in A$.

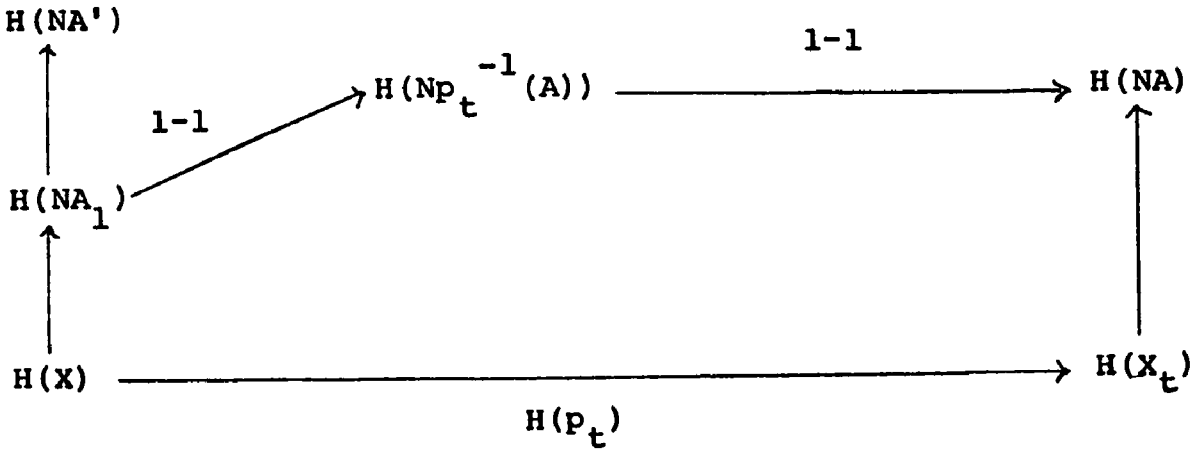
(ii) Morphism

$N^{p_t^{-1}(A)}_{A}(p_t):Np_t^{-1}(A) \to NA$ is a contiguous equivalence.

(iii) For $A_1 = A' \cup p_t^{-1}(A)$ a morphism

$i^{p_t^{-1}(A)}_{A_1}:NA_1 \to Np_t^{-1}(A)$ is a contiguous equivalence. □

Thus we have the following diagram.



$H(NA')$, $H(NA)$, $H(NA_1)$ and $H(Np_t^{-1}(A))$ denote combinatorial homology of respective polyhedra.

As a corollary we have:

(4.5) Theorem: (continuity) Cubical homology (and cohomology) theory is continuous. □

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