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PERIODIC SOLUTIONS OF NONLINEAR DIFFERENTIAL EQUATIONS: EXISTENCE AND ERROR BOUNDS

by

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of the requirements for the degree of
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I owe this thesis to four persons. Without the inspiration and encouragement of them all I do not know how I could have done it. To them I dedicate this work:

Professor Lamberto Cesari
Professor Francisco A. Lacaz Netto
My father, Amaro de Lyra Borges
Eliane

with my gratitude.

PREFACE

In 1957, I met Professor Lamberto Cesari at ITA, the "Instituto Tecnologico de Aeronautica," in S. Paulo, Brazil, while he was delivering a series of lectures on recent developments in the theory of nonlinear differential equations.

This was one of those lucky events one enjoys only a few times in a lifetime because it determined my line of studies for the next several years. I was fascinated by the subject and obtained a promise from Professor Cesari to take me as a student if I could ever come to the United States for a doctors degree. I want to express my sincere gratitude to Professor Cesari for this opportunity.

I also wish to thank my committee and extend these thanks to all my professors at The University of Michigan.

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1. INTRODUCTION

The subject of this thesis is the problem of existence and approximation of periodic solutions of nonlinear differential equations. This problem, in its various forms, including existence of quasi-periodic solutions and of invariant manifolds of solutions, became, in the last few decades, of the greatest importance in several branches of applied mathematics such as electronics, econometrics, automatic controls, etc. It has been investigated in many different directions. We just mention Poincaré-Bendixson theory and N. Levinson's recent developments, the extensions of Sturn theory to nonlinear problems, the bifurcation theory of Poincaré and its recent developments by K. O. Friedrichs and J. Cronin. It should be pointed out, however, that most of the work done so far concerns problems of the perturbation type, that is, the differential equation contains a parameter, say ϵ, such that when $\epsilon = 0$ the equation reduces to a linear one and ϵ is supposed to be "sufficiently small." No indication is given on the range of the values of ϵ for which the conclusions are valid.

Recently, methods based on functional analysis have been applied extensively to the same problems. We mention here the method of Krilov-Bogoliubov-Mitropolski[3] for problems of the perturbation type, and the method of Cesari (5, particularly Section 8.5). The latter was first applied—by Cesari, in collaboration with J. K. Hale and others—to an existence analysis and approximation of periodic solutions of nonlinear

ordinary differential equations of the perturbation type. A bibliography of this early research is given in Ref. 5, Section 8.5, and an exposition has appeared in book form.[11] A recent version of Cesari's approach[4] applies to nonlinear problems not necessarily of the perturbation type, or, as we shall say, straightforward nonlinear. Recently H. Knobloch[15] has obtained by this analytical method existence theorems which are more comprehensive than analogous ones which had been obtained by Wazewski's topological method. The same method has been applied by Cesari to boundary value problems for ordinary and partial differential equations. By the previous version[5] for problems of the perturbation type, J. K. Hale[10] had proved the existence of families of periodic solutions which are analogous to the ones well known for linear problems.

In the present work we investigate periodic solutions for second order nonlinear differential equations of the type

$$\ddot{x} + g(x,t) = 0 ,$$

where $g(x,t)$ is periodic, in t, of a given period T. When g does not depend explicitly on t the equation reduces to the equation

$$\ddot{x} + g(x) = 0$$

of nonlinear "free oscillations." The periodic solutions of this equation are well known and yield the oscillations of the true pendulum in the particular case

$$\ddot{x} + \sin x = 0 .$$

Our method of investigating the problem above is connected with Cesari's general approach[4] and with Galerkin's method.

In Section 2 we describe Galerkin's method and its relations with the better known Ritz method.

In Section 3 we give an account of Keldish's recent proof of the convergence of Galerkin's method for linear nonnecessarily self adjoint differential equations.

In Section 4 we give an account of the known results concerning the periodic solutions of the equation $\ddot{x}+g(x) = 0$.

In Section 5 we give a few lemmas to be used in Section 6 concerning the use of uniform and square norms in Fourier series.

In Section 7 we give existence theorems for periodic solutions of the equation

$$\ddot{x} + g(x,t) = 0$$

by the use of the general ideas of Cesari's approach. In these sections we introduce convenient Hilbert, or Banach spaces, of periodic functions. The existence theorems are then obtained by first showing that certain mappings on convenient subspaces map these subspaces into themselves and are contractions in the corresponding metrics of these spaces, and then prove that an exact periodic solution exists in a conveniently chosen neighborhood of the first Galerkin's approximation. The size of this neighborhood represents then an error bound for the same first Galerkin's approximation.

In Section 7 we discuss a number of examples. In 7.1 we discuss the

equation $\ddot{x}+x-\alpha^2x^3 = 0$ for $\alpha^2 = 1/6$. In 7.2 and 7.3 we deduce, from the results of Section 6, very small error bounds for the sinusoidal approximation

$$x = a \sin \omega t$$

of the periodic solutions of the true pendulum, that is, for the equation

$$\ddot{x} + \sin x = 0 .$$

In Section 7.3 we obtain from the results of Section 6 criteria of existence and error bounds for the periodic solutions of the equation

$$\ddot{x} + \alpha \sin x = \beta \sin \omega t ,$$

where α and β are given constants. This equation represents the forced oscillations of a true pendulum, as well as the working of synchronous electrical machines. The numerical example chosen for study was investigated recently by W. Bader[2] who determined numerically the first five Galerkin's approximations by means of a Telefunken electronic computer. In our work we obtain at the first approximation both a proof of the existence of the exact solution and an error bound of the sinusoidal approximation. The approximate solution obtained by Bader falls of course within the narrow limits that we obtain by the present theoretical approach.

2. THE GALERKIN METHOD

2.1. GALERKIN AND RITZ METHODS

In 1915 the Russian engineer V. G. Galerkin[9] introduced in his work on plates and beams the method that bears his name for the approximate solution of boundary value problems. The method, besides being of easier application, is more general than that of Ritz, presented in 1908 for the construction of a minimizing sequence in problems of calculus of variations. Following essentially[13] we present briefly in this section both methods and their relations.

We consider the linear ordinary differential equation

$$L(y) = 0 , \tag{2.1}$$

where L is some differential operator, and we require a solution satisfying given boundary conditions, say, at a and b. Let $\{\phi_i(x)\}$ be a system of linearly independent functions, and let $y = \Phi(x,a_1,\ldots,a_n)$ be a family of functions depending on the parameters $a_1,\ldots,a_n$, all satisfying the given boundary conditions. We shall then seek for an approximate solution of the form

$$\bar{y} = \Phi(x,c_1,\ldots,c_n), \quad a \leq x \leq b , \tag{2.2}$$

where the c_i are to be determined, if possible, by the orthogonality conditions

$$\int_a^b L(\bar{y})\phi_i(x)\,dx = 0 ,$$

or

$$\int_a^b L[\phi(x,c_1,\ldots,c_n)]\phi_i(x)\,dx = 0, \quad i=1,\ldots,n . \tag{2.3}$$

The system of finite equations (2.3) serves to the determination of the c_i. For instance, in case of homogeneous boundary conditions we can take functions $\phi_i(x)$ which satisfy these conditions, and then take for Φ the family of all linear combinations $a_1\phi_1+\ldots+a_n\phi_n$. In this situation, Eqs. (2.3) are linear if L is a linear differential operator.

In order to illustrate the Ritz method we consider the following simple problem. To find a function $y = y(x)$ satisfying the boundary condictions $y(a) = \alpha$, $y(b) = \beta$, which minimizes the functional

$$I[y] = \int_a^b F(x,y,y')\,dx . \tag{2.4}$$

Let us assume that $F(x,y,y')$ is a continuous function of x,y,y', and that an exact solution $y^*(x)$ for this problem exists which is continuous with its first derivative $y'^*(x)$. Let $m = I[y^*]$ be the value of the minimum. We try to find a sequence $\{y_n(x)\}$ of functions which satisfy the boundary conditions and such that the sequence $\{I[y_n]\}$ converges in some sense, to $m = I[y^*]$. The sequence $y_n(x)$ is called a minimizing sequence.

Ritz proposed the following device to find a sequence $\{y_n\}$. Let

$$y = \Phi(x,a_1,\ldots,a_n) \tag{2.5}$$

be a family of functions depending on the parameters $a_1,\ldots,a_n$ and satisfying the boundary conditions $y(a) = \alpha$, $y(b) = \beta$, for any value of $(a_1,\ldots,a_n)$. If we substitute Φ in (2.4) we obtain I as a function, $I = I(a_1,\ldots,a_n)$, of the n parameters $a_1,\ldots,a_n$. A necessary condition for I to have a minimum is that a_k satisfy the system of equations

$$\frac{\partial I}{\partial a_k} = 0, \quad k = 1,\ldots,n . \tag{2.6}$$

We take as $y_n(x)$ the function Φ of the family (2.5) for which $a_1,\ldots,a_n$ have the values obtained from (2.6). The sequence $\{y_n(x)\}$ so obtained is minimal provided that the following condition is satisfied.

For every $y = y(x)$ with a continuous first derivative and satisfying the boundary conditions, given $\epsilon > 0$ there is an n and a function $\bar{y}_n$ in the family (2.5) such that

$$|\bar{y}_n - y| < \epsilon , \quad \left| \frac{d\bar{y}_n}{dx} - \frac{dy}{dx} \right| < \epsilon .$$

Indeed, let us apply this condition to the function $y = y^*(x)$. There is a function y_n^* in the nth family (2.5), say

$$y_n^* = \Phi_n(x,a_1,\ldots,a_n^*) ,$$

such that

$$|y_n^* - y^*| < \epsilon , \quad \left| \frac{dy_n^*}{dx} - \frac{dy^*}{dx} \right| < \epsilon .$$

Clearly, then, by continuity of $F(x,y,y')$ the number $|I[y_n^*] - I[y^*]|$ may be made arbitrarily small. Since y_n^* is in the nth family $\Phi(x, a_1,\ldots,a_n)$

and the element y_n of the minimizing sequence $\{y_n\}$ gives $I[y]$ its smallest value in comparison with all functions of $\Phi(x,a_1,\ldots,a_n)$ we have

$$I[y^*] \leq I[y_n] \leq I[y_n^*] ,$$

and so, $|I[y^*]-I[y_n]|$ is also arbitrarily small.

2.2. A CASE OF EQUIVALENCE OF GALERKIN AND RITZ METHODS

We consider now the self adjoint differential equation

$$L(y) = \frac{d}{dx}(py') - qy - f = 0 \qquad (2.7)$$

under the homogeneous boundary conditions

$$y(o) = y(\ell) = 0 . \qquad (2.8)$$

We assume that $p = p(x) > 0$ and $q = q(x) \geq 0$ on $[o,\ell]$.

Equation (2.7), as is easily seen, is the Euler equation for the real functional

$$I[y] = \int_o^\ell [py'^2+qy^2+2fy]dx , \qquad (2.9)$$

with the same boundary conditions (2.8). We show now that the nth approximation for the solution of this problem is the same when we use Ritz's or Galerkin's method.

We may take for $\{\phi_k\}$ any system of linearly independent functions of class C^1 and satisfying the homogeneous boundary conditions (2.8). We may, for example, put

$$\phi_k = \sin \frac{k\pi x}{\ell} , \quad k = 1,2,\ldots,n .$$

Let the family $\Phi(x,a_1,\ldots,a_n)$ of the preceding section be

$$y_n = \Phi(x,a_1,\ldots,a_n) = \sum_{i=1}^{n} a_i\phi_i . \tag{2.10}$$

If we substitute (2.10) in (2.9) we get

$$\begin{aligned} I[y_n] &= \int_0^{\ell} [py_n'^2+qy_n^2+2fy_n]dx \\ &= \int_0^{\ell} \left[p\left(\sum_{i=1}^{n} a_i\phi_i'\right)^2 + q\left(\sum_{i=1}^{n} a_i\phi_i\right)^2 + 2f\sum_{i=1}^{n} a_i\phi_i \right] dx \\ &= \sum_{i,j=1}^{n} \alpha_{ij}a_ia_j + 2\sum_{i=1}^{n} \beta_i a_i , \end{aligned} \tag{2.11}$$

where

$$\begin{aligned} \alpha_{ij} = \alpha_{ji} &= \int_0^{\ell} (p\phi_i'\phi_j'+q\phi_i\phi_j)dx , \\ \beta_i &= \int_0^{\ell} f\phi_i dx . \end{aligned} \tag{2.12}$$

If we differentiate (2.11) with respect to a_j we get the system (2.6) in the forms

$$\frac{1}{2}\frac{\partial I}{\partial a_j} = \int_0^{\ell} (py_n'\phi_j'+qy_n\phi_j+f\phi_j)dx = 0 , \quad j=1,\ldots,n , \tag{2.13}$$

or

$$\sum_{i=1}^{n} \alpha_{ij}a_i + \beta_j = 0 , \quad j=1,\ldots,n . \tag{2.14}$$

We put $a_i = a_i^{(n)}$ for the solution of system (2.14) and so the nth approximate solution is

$$y_n = \sum_{i=1}^{n} a_i^{(n)} \phi_i \ .$$

This solution always exists uniquely as it is easy to show. Indeed, suppose that the homogeneous system

$$\sum_{i=1}^{n} \alpha_{ij} a_i = 0, \quad j=1,2,\ldots,n$$

has a nontrivial solution $a_i = \bar{a}_i$. Then

$$\sum_{i=1}^{n} \alpha_{ij} \bar{a}_i = 0 \ , \quad j=1,\ldots,n \ ,$$

and so, from (2.12),

$$\sum_{i,j} \int_0^{\ell} (p\phi_i'\phi_j' + q\phi_i\phi_j)\bar{a}_i dx = 0 \ ,$$

or

$$\int_0^{\ell} (p\bar{y}_n'\phi_j' + q\bar{y}_n\phi_j) dx = 0, \quad j=1,\ldots,n \ , \tag{2.15}$$

where

$$\bar{y} = \sum_{i=1}^{n} \bar{a}_i \phi_i \ .$$

If we multiply (2.15) by $\bar{a}_j$ and add we obtain

$$\int_0^{\ell} (p\bar{y}_n'^2 + q\bar{y}_n^2) dx = 0 \ .$$

This implies $\bar{y}_n = 0$, or $\sum_{1}^{n} \bar{a}_k \phi_k = 0$, which is impossible since the ϕ_k

are linearly independent.

Now, integration by parts of the first term in the integrand of (2.13) yields

$$\int_0^{\ell} (py_n'\phi_n'+qy_n\phi_j+f\phi_j)\,dx \;=\; \int_0^{\ell} \left[-\frac{d}{dx}(py_n')\phi_j+qy_n\phi_j+f\phi_j\right] dx \;=\; 0\ ,$$

that is

$$\int_0^{\ell} \left[\frac{d}{dx}(py_n') + qy_n + f\right]\phi_j \;dx \;=\; 0\ , \quad j=1,2,\ldots,n\ .$$

These are exactly the Galerkin equations for the boundary value problem.

3. THE CONVERGENCE OF GALERKIN'S METHOD

3.1 A THEOREM OF KANTOROVICH

Kantorovich[12] proves a theorem concerning linear functional equations of the type

$$Kx = y$$

in convenient function spaces, which we shall use in Section 3.2 to prove a theorem of Keldish on the convergence of Galerkin's method.

Let us consider the linear normed spaces X, X', and $\overline{X}_n$, n=1,2,..., where $\overline{X}_n$ is complete, $X' \subset X$, and there is an isomorphism $\Phi_{on}: X' \to \overline{X}_n$. We shall assume that there exists a linear operator $\Phi_n: X \to \overline{X}_n$ and that Φ_{on} is the restriction of Φ_n to X'. Let H and $\overline{H}_n$ be completely continuous linear operators, $H: X \to X$ and $\overline{H}_n: \overline{X}_n \to \overline{X}_n$, n=1,2,... . We consider the equations

$$Kx = I - \lambda Hx = y , \tag{3.1}$$

and the sequence of equations

$$\overline{K}_n \overline{x} = I_n - \lambda \overline{H}_n \overline{x} = \overline{y} = \Phi_n y , \quad n=1,2,... , \tag{3.2}$$

where $x, y \in X$, $\overline{x}, \overline{y} \in \overline{X}_n$, $K = I - \lambda H$ and $\overline{K}_n = I_n - \lambda \overline{H}_n$, with λ a number.

We state without proof (see Ref. 12) the following theorem:

Theorem 3.1.

Let us assume that the following conditions are satisfied:

1. In Eq. (3.1) λ is not an eigenvalue.

2. For each n=1,2,..., numbers ϵ_n, ϵ_{1n}, ϵ_{2n} can be found such that:

(a) $\|\Phi_n Hx' - \overline{H}_n \Phi_n x'\| \leq \epsilon_n \|x'\|$ for all $x' \epsilon X'$;

(b) given any $x \epsilon X$ there is an element $x' \epsilon X'$ such that

$$\|Hx - x'\| \leq \epsilon_{1n} \|x\| ;$$

(c) there is an element $y' \epsilon X'$ such that

$$\|y - y'\| \leq \epsilon_{2n} \|y\| ,$$

where y is the second member of (3.1).

3. The relations

$$\lim_{n\to\infty} \epsilon_n \|\Phi_n\|^2 \|\Phi_{on}^{-1}\|^2 = \lim_{n\to\infty} \epsilon_{1n} \|\Phi_n\|^2 \|\Phi_{on}^{-1}\|^2 = \lim_{n\to\infty} \epsilon_{2n} \|\Phi_n\|^2 \|\Phi_{on}^{-1}\|^2 = 0$$

hold.

Then, for n large enough, system (3.2) is solvable and the solutions $\overline{x}^{(n)}$ converge to a solution x* of Eq. (3.2) that is,

$$\lim_{n\to\infty} \|x^* - \Phi_{on}^{-1} \overline{x}^{(n)}\| = 0 .$$

3.2. A THEOREM OF KELDISH

M. V. Keldish[14] proves the convergence of Galerkin's method in several different cases. In this section we present one of his results for the case of a second order differential equation with simple boundary conditions.

Let us consider the differential operator

$$L(y) = p_2(x) y'' + p_1(x,\lambda) y' + p_0(x,\lambda) y ,$$

where p_2, p_1, and p_0 are continuous functions of x for $x\epsilon[0,1]$ and λ in some interval $[\lambda_1,\lambda_2]$, and are twice continuously differentiable with respect to x; furthermore we assume that $p_2(x) \geq \alpha > 0$, $x\epsilon[0,1]$, α a constant.

Let now f(x) be a given continuous function in [0,1]. We shall study the differential equation

$$L(y) = p_2(x)y'' + p_1(x;\lambda)y' + p_0(x,\lambda)y = f(x), \qquad (3.3)$$

with the boundary conditions

$$y(0) = y'(0) = 0. \qquad (3.4)$$

We shall write system (3.3) in the form

$$\frac{d}{dx}[p(x)y'] + \frac{d}{dx}[q(x,\lambda)y] + r(x,\lambda)y = f(x), \qquad (3.5)$$

so that,

$$p(x) = p_2(x), \quad q(x,\lambda) = p_1(x,\lambda) - p_2'(x), \quad r(x,\lambda) = p_0(x,\lambda) - p_1'(x,\lambda) + p_2''(x),$$

and, therefore, p(x) and $q(x,\lambda)$ are continuous functions of x in [0,1], p(x) of class C^2 and q of class C^1.

We know that this problem either has solutions for any f(x) or else the homogeneous problem [f(x) = 0] has nontrivial solutions.

Let $\{\phi_i, i=1,2,\ldots\}$ be a set of functions of class C^2 in [0,1] satisfying the boundary conditions (3.4). We consider an nth Galerkin's approximation, if any, to the solution of our problem in the form

$$y_n = c_1^{(n)}\phi_1(x) + \ldots + c_n^{(n)}\phi_n(x) \ , \tag{3.6}$$

where the constants $c_k^{(n)}$ are determined, if possible, by the equations

$$\int_0^1 [L(y_n)-f]\phi_i(x)\,dx = 0 \ , \quad i=1,2,\ldots,n \ ,$$

and these equations are of the form

$$\sum_{j=1}^{n} C_{ij}c_j^{(n)} - f_i = 0 \ , \quad i=1,2,\ldots,n \ . \tag{3.7}$$

We now assume that the functions ϕ_i are so chosen that the functions

$$\{1,\phi_1',\phi_2',\ldots\} \tag{3.8}$$

are linearly independent and form a complete system in the norm defined by the inner product

$$(u,v) = \int_0^1 p(x)u(x)v(x)\,dx \ , \tag{3.9}$$

where we use the weight function $p(x)$. Without any lack of generality we may assume that the system (3.8) is orthonormal with respect to the weight function $p(x)$, that is,

$$\int_0^1 p(x)\phi_i'\phi_j'dx = \delta_{ij} \ , \quad i,j=0,1,2,\ldots, \tag{3.10}$$

where we put $\phi_0' \equiv 1$.

We observe that system (3.7) is now

$$\int_0^1 \left\{\frac{d}{dx}[p(x),y_n'] + \frac{d}{dx}[q(x,\lambda)y_n] + r(x,\lambda)y_n - f\right\}\phi_i(x)\,dx = 0, \quad i=1,2,\ldots,n \ .$$

Also, we note that

$$\int_0^1 \frac{d}{dx}\,[p\phi_j']\phi_i dx \;=\; -\int_0^1 p\phi_j'\phi_i' dx \;=\; -\delta_{ij}\,, \quad i,j=1,2,\ldots,n\,,$$

since the functions ϕ_i, i=1,2,..., satisfy the boundary conditions (3.4). As a consequence we have

$$\int_0^1 \frac{d}{dx}\,[py_n']\phi_i dx \;=\; -c_i^{(n)}\,, \quad i=1,2,\ldots,n\,,$$

and system (3.7) takes the new form

$$c_i^{(n)} + \sum_{j=1}^{n} A_{ij}c_j^{(n)} - f_i \;=\; 0\,, \quad i=1,2,\ldots,n\,, \qquad (3.11)$$

where we put

$$A_{ij} = A_{ij}(\lambda) \;=\; -\int_0^1 \left\{\frac{d}{dx}\,[q(x,\lambda)\phi_j] + r(x,\lambda)\phi_j\right\}\phi_i dx, \quad i,j=1,2,\ldots,n\,,$$

$$f_i \;=\; -\int_0^1 f\phi_i dx\,, \quad i=1,2,\ldots,n\,. \qquad (3.12)$$

We shall prove the following result.

Theorem 3.2.

With the assumptions made on the system of functions (3.8), if λ is not an eigenvalue of the boundary value problem (3.3) and (3.4), a solution is given by the limit of the sequence $\{y_n\}$, with coefficients defined by (3.7).

Proof

We start by writing in (3.12)

$$A_{ij} \;=\; a_{ij} + b_{ij}\,, \quad i,j=1,2,\ldots,n\,,$$

where

$$\left.\begin{aligned} a_{ij} &= \int_0^1 q(x,\lambda)\phi_j\phi_i' dx \\ b_{ij} &= \int_0^1 r(x,\lambda)\phi_j\phi_i dx \end{aligned}\right\} \quad i,j=1,2,\ldots,n \ .$$

Now, system (3.11) is obtained by truncation from the infinite system of linear equations

$$c_i + \sum_{j=1}^{\infty} A_{ij}c_j - f_i = 0 \ , \quad i=1,2,\ldots \ , \tag{3.13}$$

where the A_{ij} are given by (3.12) with $i,j=1,2,\ldots$. We will show first that system (3.13) is equivalent to the original boundary value problem in the following sense: to a solution $y(x)$ of the latter there corresponds a solution of system (3.13),

$$c_i = \int_0^1 py\phi_i dx \ , \quad i=1,2,\ldots \ , \tag{3.14}$$

which is square summable

$$\sum_{i=1}^{\infty} |c_i^2| < +\infty \ , \tag{3.15}$$

and conversely, to each square summable solution of (3.13) there corresponds a solution of the boundary value problem.

Let $y(x)$ be a solution to the boundary value problem. We expand y' in a Fourier series with respect to the orthonormal system $\{\phi_i'\}$, $i=0,1,2,\ldots$, and, with c_m defined by (3.14) we obtain a series, convergent in the square mean with weight function $p(x)$,

$$y'(x) \approx \sum_{m=1}^{\infty} c_m \phi_m'(x) \ . \qquad (3.16)$$

Actually, from the assumption of the completeness of the system (3.8) we obtain

$$y'(x) \approx c_o + \sum_{m=1}^{\infty} c_m \phi_m'(x) \ , \qquad (3.17)$$

but we proceed to show that $c_o = 0$. Indeed, if we put $S_m(x) = c_o + \sum_{1}^{m} c_m \phi_m'(x)$ and $N = \int_o^1 p(x)\,dx$, we have

$$\left| \int_o^1 S_m(x)\,dx - \int_o^1 y'(x)\,dx \right| \le \frac{1}{\alpha} \cdot \alpha \left| \int_o^1 \{S_m(x) - y'(x)\}\,dx \right|$$

$$\le \frac{1}{\alpha} \int_o^1 p(x)\,|S_m(x) - y'(x)|\,dx$$

$$\le \frac{1}{\alpha} \left(\int_o^1 p(x)\,dx \right)^{1/2} \left(\int_o^1 p(x)\,[S_m(x) - y'(x)]^2 dx \right)^{1/2}$$

$$\le \frac{1}{\alpha} N^{1/2} \left(\int_o^1 p(x)\,[S_m(x) - y'(x)]^2 dx \right)^{1/2}$$

and the last expression tends to zero as $m \to \infty$. This shows that

$$\int_o^1 S_m(x)\,dx \to \int_o^1 y'(x)\,dx \quad \text{as} \quad m \to \infty \ .$$

We can, therefore, integrate term by term between 0 and 1 the series (3.17). We obtain successively

$$\int_o^1 y'(x)\,dx = c_o + \sum_{m=1}^{\infty} c_m \int_o^1 \phi'(x)\,dx \ ,$$

$$y(1) - y(0) = c_o + \sum_{m=1}^{\infty} c_m [\phi_m(1) - \phi_m(0)]$$

and, because of the boundary conditions, $c_o = 0$.

By the same reasoning as before we have now that, for every $x\epsilon[0,1]$

$$\left| \int_o^x S_m(t)dt - \int_o^x y'(t)dt \right| \leq \frac{1}{\alpha} N^{1/2} \left\{ \int_o^1 p(t)[S_m(t)-y'(t)]^2 dt \right\}^{1/2},$$

that is,

$$\int_o^x S_m(t)dt \to \int_o^x y'(t)dt ,$$

uniformly in $[0,1]$, as $m\to\infty$. In other words

$$\lim_{m\to\infty} \left[\int_o^x \sum_1^m c_i \phi_i'(t)dt - \int_o^x y'(t)dt \right] = 0 .$$

By the boundary conditions, we have thus

$$\lim_{m\to\infty} \left[\sum_1^m c_i \phi_i(x) - y(x) \right] = 0$$

uniformly in $[0,1]$. In other words

$$y(x) = \sum_{m=1}^{\infty} c_m \phi_m(x) \qquad (3.18)$$

where the series is uniformly convergent in $[0,1]$.

If we now multiply (3.5) by $\phi_i(x)$ and integrate from 0 to 1 we get, after integration by parts

$$\int_o^1 \{-py'\phi_i' - qy\phi_i' + ry\phi_i - f\phi_i\}dx = 0 , \quad i=1,2,\ldots . \qquad (3.19)$$

Substituting (3.16) and (3.18) in (3.19) we get (3.13). On the other hand (3.18) implies (3.14) and (3.15).

Conversely, we now assume that $\{c_k\}$ is a solution for system (3.12) which is square summable. Using the theorem of Riesz-Fischer we form the function

$$\eta(x) \approx \sum_{m=1}^{\infty} c_m \phi_m'(x) \tag{3.20}$$

and set

$$y(x) = \sum_{1}^{\infty} c_m \phi_m(x) \tag{3.21}$$

with the same conventions as before. Then $y(x)$ is absolutely continuous in $[0,1]$, series (3.21) converges uniformly as proved, and we have

$$y'(x) = \eta(x)$$

almost everywhere, where $\eta(x)$ is of class L^2. Here $y(x)$ satisfies clearly the boundary conditions $y(0) = 0$, $y(1) = 0$. We will show that it satisfies the differential equation (3.5).

From (3.21) it follows that we can write system (3.13) in the form (3.19). If we integrate by parts the two last terms of (3.19) we get

$$\int_0^1 \left\{ py'+qy+ \left[\int_0^x (ry-f)\,dx \right] \right\} \phi_i' dx = 0 , \quad i=1,2,\ldots .$$

Since the system (3.8) is complete the expression in square brackets must be a constant almost everywhere, that is,

$$py' + qy + \int_0^x (ry-f)\,dx = k, \text{ a.e.} \tag{3.22}$$

By differentiation we shall see that y satisfies equation (3.5). Indeed, q is of class C^1 and y is AC with y' of class L^2. Thus qy is AC and $(qy)' = q'y+qy'$, a.e., and this shows that $(qy)'$ is of class L^2. Thus

$$py' = -qy - \int_0^x (ry-f)dx + k, \text{ a.e.,} \tag{3.23}$$

where ry-f is of class C^0, qy is AC, $(qy)'$ is of class L^2, and hence py' coincides everywhere with a continuous function. Then y', as the derivative of its indefinite integral, is continuous and (3.23) is satisfied everywhere. Also, py' is AC and

$$(py')' = -(qy)' - (ry-f), \quad \text{a.e.,}$$

hence $(py')'$ is of class L^2. Also,

$$(py')' = -q'y - qy' - ry + f ;$$

in other words, if $py' = X(x)$, $0 \le x \le 1$, then $X'(x) = -q'y-qy'-ry+f$ is of class C^0 in [0,1] and

$$py' = X(0) + \int_0^x X'(t)dt ,$$

$$y' = \frac{1}{p}\left[X(0) + \int_0^x X'(t)dt\right] .$$

Since p is of class C^2, $p \ge \alpha > 0$, and the bracket is of class C^1, we conclude that y' is of class C^1, that is y is of class C^2. By differentiation of (3.22) we obtain (3.5) for every $x \in [0,1]$.

We have thus proved the statement that the system (3.13) and the

given boundary value problem are equivalent problems in the sense indicated.

We now apply Theorem 3.1 to our systems (3.11) and (3.13) which we write as

$$x_i + \ell \sum_{j=1}^{n} A_{ij}x_j = y_i , \quad i=1,\ldots,n \tag{3.24}$$

and

$$x_i + \ell \sum_{j=1}^{\infty} A_{ij}x_j = y_i , \quad i=1,2,\ldots , \tag{3.25}$$

where we take $\ell = 1$. The hypothesis that λ is not an eigenvalue for the given problem implies that $\ell = 1$ is not an eigenvalue for the problem (3.25).

The space X is ℓ^2, $\bar{X}_n$ is R^n, X' is the subspace of elements $(x_1,x_2,\ldots,x_n, 0,0,\ldots)$ of ℓ^2 for which $x_j = 0$ for $j > n$. The operators H and $\bar{H}_n$ are defined by the matrices $(A_{ij}, i,j=1,2,\ldots)$ and $(A_{ij}, i,j=1,2,\ldots,n)$. The isomorphism $\Phi_n: X' \to \bar{X}_n$ is naturally defined by $\phi_n(n_1,\ldots,x_n,\ldots) = (x_1,x_2,\ldots,x_n)$. In our case we have $\lambda = 1$. To show that H and $\bar{H}_n$ are completely continuous it is sufficient to show (see Ref. 1) that the matrix (A_{ij}) satisfies the Koch condition

$$\sum_{i,j} |A_{ij}|^2 < \infty . \tag{3.26}$$

In order to do this we prove that the two series $\sum_{i,j} |a_{ij}|^2$ and $\sum_{i,j} |b_{ij}|^2$ converge. Consider the first of these two series. Since we have

$$a_{ij} = \int_0^1 q\phi_j\phi_i' dx = \int_0^1 p \frac{q\phi_j}{p} \phi_i' dx = \left(\frac{q\phi_j}{p}, \phi_i'\right),$$

Bessel's inequality gives

$$\sum_{i=1}^{\infty} |a_{ij}|^2 \leq \int_0^1 [q(x)\phi_j(x)]^2 \frac{dx}{p(x)}, \quad j=1,2,\ldots . \tag{3.27}$$

From (3.27) we conclude that

$$\sum_{i,j=1}^{\infty} |a_{ij}|^2 \leq \sum_{j=1}^{\infty} \int_0^1 [q(x)\phi_j(x)]^2 \frac{dx}{p(x)} . \tag{3.28}$$

Now

$$q(x)\phi_j(x) = \int_0^x q(x)\phi_j'(t)dt = \int_0^x p(t) \frac{q(x)}{p(t)} \phi_j'(t)dt ,$$

and so, for every x, $q(x)\phi_j(x)$ is the Fourier coefficient of a function of t equal to $q(x)/p(t)$ on $0 \leq t \leq x$ and equal to zero on $x < t \leq 1$. By the inequality of Bessel we have then

$$\sum_{j=1}^{\infty} |q(x)\phi_j(x)|^2 \leq \int_0^x [q(x)]^2 \frac{dt}{p(t)}, \quad 0 \leq x \leq 1 ,$$

and from (3.28)

$$\sum_{i,j=1}^{\infty} |a_{ij}|^2 \leq \int_0^1 \frac{dx}{p(x)} \int_0^x [q(x)]^2 \frac{dt}{p(t)} = \int_0^1 [q(x)]^2 \frac{dt}{p(x)} \int_0^x \frac{dt}{p(t)} .$$

It follows that $\sum_{i,j} |a_{ij}|^2$ converges.

Now we consider the series $\sum_{i,j=1}^{\infty} |b_{ij}|^2$. By Schwartz's inequality and the definition of b_{ij} we get

$$|b_{ij}|^2 \leq \int_0^1 (\phi_i)^2 dx \int_0^1 (r\phi_j)^2 dx .$$

Hence

$$\sum_{i,j=1}^{\infty} |b_{ij}|^2 \leq \left[\sum_{i=1}^{\infty} \int_0^1 (\phi_i)^2 dx\right] \left[\sum_{j=1}^{\infty} \int_0^1 (r\phi_j)^2 dx\right] .$$

If we repeat for the sum on the right the calculation made above we get

$$\sum_{i,j=1}^{\infty} |b_{ij}|^2 \leq \int_0^1 dx \int_0^x \frac{1}{p(t)}\, dt \int_0^1 [r(x)]^2 dx \int_0^x \frac{dt}{p(t)} .$$

This shows that $\sum_{i,j} |b_{ij}|^2$ converges and so $\sum_{i,j} |A_{ij}|^2 < \infty$.

Now we verify conditions 2 of the theorem. Since,

$$\Phi_n H x' = \overline{H}_n \Phi x' = \sum_{j=1}^{n} A_{ij} x_j , \quad i=1,2,\ldots,n$$

we see that (a) is satisfied trivially with $\epsilon_n = 0$. In order to verify (b) we take for each $x \epsilon X$ an x' defined by

$$x' = \Phi_{on}^{-1} \Phi_n H x ,$$

that is, if $x = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots)$ we put

$$x' = \left(\sum_{j=1}^{\infty} A_{1j} x_j, \sum_{j=1}^{\infty} A_{2j} x_j, \ldots \sum_{j=1}^{\infty} A_{nj} x_j, 0, 0, \ldots \right) .$$

Then we have

$$\|Hx-x'\| = \left\{ \sum_{i=n+1}^{\infty} \left(\sum_{j=1}^{\infty} A_{ij}x_j \right)^2 \right\}^{1/2}$$

$$\leq \left\{ \sum_{i=n+1}^{\infty} \left(\sum_{j=1}^{\infty} A_{ij}^2 \sum_{j=1}^{\infty} x_j^2 \right) \right\}^{1/2}$$

$$= \left(\sum_{i=n+1}^{\infty} \sum_{j=1}^{\infty} A_{ij}^2 \right)^{1/2} \|x\| .$$

We take thus $\epsilon_{1n} = \left(\sum_{i=n+1}^{\infty} \sum_{j=1}^{\infty} A_{ij}^2 \right)^{1/2}$. In order to verify (c), for any given $y = (y_1, y_2, \ldots) \epsilon X$ we take $y' = (y_1', y_2', \ldots)$ with $y_i' = y_i$ for $i=1,\ldots,n$ and $y_i' = 0$ for $i > n$. Then

$$\|y-y'\| = \left\{ \sum_{i=1}^{\infty} (y_i - y_i')^2 \right\}^{1/2} = \left\{ \sum_{i=n+1}^{\infty} y_i^2 \right\}^{1/2} ,$$

and so we may take $\epsilon_{2n} = \frac{1}{\|y\|} \left(\sum_{i=n+1}^{\infty} y_i^2 \right)^{1/2}$. Clearly ϵ_n, ϵ_{1n} and ϵ_{2n} tend to zero as $n \to \infty$ thus verifying condition 3 of the theorem. We have hence proved the convergence of the Galerkin method for our boundary value problem.

4. SOME KNOWN PROPERTIES OF THE EQUATION $\ddot{x}+g(x) = 0$

We consider here the autonomous second order equation

$$\ddot{x} + g(x) = 0 , \qquad (4.1)$$

where $g(x)$ is a continuous function of x with $g(0) = 0$ and $xg(x) > 0$ for all $x \neq 0$ in a neighborhood of $x = 0$, say $x\epsilon[-A,A]$. Then Eq. (4.1), besides the trivial solution $x = 0$, is known to have a two-parameter family of periodic solutions $x = x(t+\gamma,C)$, C,γ arbitrary constants (see below), representing a one-parameter family of closed trajectories in the phase plane $x\dot{x}$, linking the origin $x = \dot{x} = 0$ (the origin $x = \dot{x} = 0$ is a center).

In Section 6 we will discuss the first Galerkin approximation of these periodic solutions by means of the method of Ref. 4. We will obtain an evaluation of how close the exact periodic solutions are from the corresponding approximate solutions. This will lead to rather stringent estimates of the relations between amplitudes and periods of these solutions.

We shall assume that $g(x)$ satisfies the conditions:

1. $g(x)$ is odd, that is, $g(-x) = -g(x)$ for $|x| \leq A$, and as already assumed, $g(x) > 0$ for $x\epsilon(0,A]$.

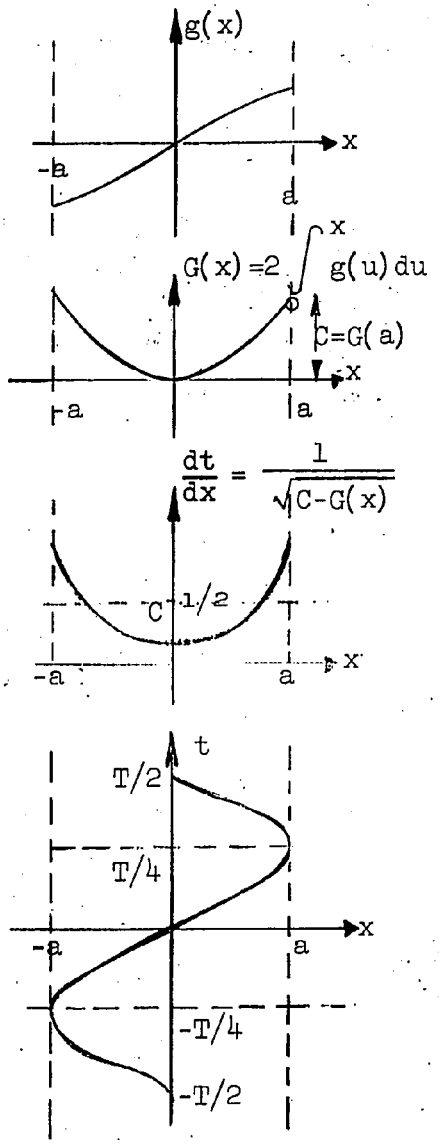
2. $g(x)$ is Lipschitzian in $[-A,A]$, that is, for some $L > 0$, $|g(x_1)-g(x_2)| \leq L|x_1-x_2|$ for $|x_1|, |x_2| \leq A$.

3. For some integer $k \geq 1$, and constants $b_0 \geq 0$, $b_1,\ldots b_{k-1}$, real, and $b_k \geq 0$ we have

$$|g(x) - (b_0x+b_1x^3 +\ldots+ b_{k-1}x^{2k-1})| \leq b_k|x|^{2k+1}$$

for $|x| \leq A$

If $G(x) = 2\int_0^x g(u)\,du$, $|x| \leq A$ then, $G(0) = 0$, by (4.1), $G(x)$ is even, that is $G(-x) = G(x)$, $|x| \leq A$, and $G(x)$ is strictly increasing in $[0,A]$. Since $2\dot{x}$ is an integrating factor for (4.1) we have successively



$$2\dot{x}\ddot{x} + 2\,g(x)\dot{x} = 0\,,$$

$$\dot{x}^2 + G(x) = C\,, \qquad (4.2)$$

$$\frac{dx}{dt} = \pm[C-G(x)]^{1/2}.$$

For $C\in[0,G(a)]$, an arc of solution $x = x(t,C)$ with $x(0,C) = 0$, $\dot{x} > 0$ is the inverse function of the function $t = t(x,C)$ defined by

$$t = t(x,C) = +\int_0^x [C-G(x)]^{-1/2}\,dx \qquad (4.3)$$

and this solution exists in the interval $(-a,a)$ where $C = G(a)$, $a\in(0,A]$.

Note that the integrand in (4.3) is continuous and finite for $x\in(-a,a)$ but approaches $+\infty$ as $x\to\pm a$. On the other hand $G(a) = C$, $G'(a) = 2g(a) > 0$ and by the continuity of g, there are constants $\mu,\delta > 0$ such that $2g(x) \geq \mu > 0$ for all $x\in[a-\delta,a]$. Hence

by the mean value theorem

$$G(a) - G(x) = (a-x)2g(\tilde{x})$$

for some $\tilde{x}\in(x,a)$,

$$C - G(x) = G(a) - G(x) \geq \mu(a-x) ,$$

$$[C-G(x)]^{-1/2} \leq [\mu(a-x)]^{-1/2}$$

for $x\in[a-\delta,a]$. This shows that the integral from 0 to a of $[C-G(x)]^{-1/2}$ is finite, and the same holds for the integral from -a to 0, because of symmetry. If

$$T = 4\int_0^a [C-G(x)]^{-1/2}\,dx , \qquad (4.4)$$

we see that $t = t(x,c)$ is a strictly increasing function of x for $x\in[a,b]$ and that $t(-a,C) = -T/4$, $t(a,C) = T/4$. Hence, the inverse function $x = x(t,C)$ is also a strictly increasing function of t with $x(-T/4,C)=-a$, $x(0,C) = 0$, $x(T/4,C) = a$,

$$x'(-T/4,C) = 0 , \quad x'(0,C) = C^{1/2} , \quad x'(T/4,C) = 0 .$$

Also $x(t,C)$ satisfies (4.2) with + sign for all $t\in[-T/4,T/4]$, hence $x(t,C)$ satisfies Eq. (4.1) in the same interval. Finally, $t(x,C)$ is odd, and so $x(t,C)$ is odd.

We may now define an analogous arc $x = x(t,C)$ of solution with $x(T/2,C) = 0$, $dx/dt < 0$, whose inverse function $t = t(x,C)$ is defined by

$$t = t(x,C) = -\int_0^x [C-G(x)]^{-1/2} + \frac{T}{2} . \qquad (4.5)$$

Then this solution exists in the interval $-a < x < a$, with the same a as

before. Actually, (4.5) defines $t = t(x,C)$ in the whole interval $-a \leq x \leq a$, $t(x,C)$ is strictly decreasing in this interval, and so is the inverse function. Also $t(-a,C) = 3T/4$, $t(a,C) = T/4$, and thus the inverse function $x(t,C)$ satisfies $x(T/4,C) = a$, $x(T/2,C) = 0$, $x(3T/4,C) = -a$, $x'(T/4,C) = 0$, $x'(T/2,C) = -C^{1/2}$, $x'(3T/4,C) = 0$. Also, $x(t,C)$ satisfies (4.2) with the minus sign for all $t \in [T/4,3T/4]$, hence $x(t,C)$ satisfies Eq. (4.1) in the same interval, and the values of $x(t,C)$, $x'(t,C)$ at $t = T/4$ are the same of those of the previous arc at $t = T/4$.

By the uniqueness theorem (which certainly holds by property 2) we have defined the unique solution of (4.1) in $[-T/4, 3T/4]$ with $x(T/4, C) = a$, $x'(T/4,C) = 0$.

If $t_1(x,C)$, $t_2(x,C)$ denote, for a moment, the two functions we have now defined, we have

$$t_1(a,C) - t_2(x,C) = \int_x^a [C-G(u)]^{-1/2} du = t_2(x,C) - t_2(a,C) ,$$

that is,

$$\frac{T}{4} - t_1(x,C) = t_2(x,C) - \frac{T}{4} .$$

Thus for the inverse functions $x_1(t,C)$, $x_2(t,C)$ we have

$$x_2\left(\frac{T}{4} + \tau\right) = x_1\left(\frac{T}{4} - \tau\right) .$$

In other words the arc $x(t,C)$ defined in $[-T/4, 3T/4]$ is even with respect to $T/4$. But we have also

$$x\left(\frac{3T}{4}\right) = x\left(-\frac{T}{4}\right) = -a\ , \quad x'\left(\frac{3T}{4}\right) = x'\left(-\frac{T}{4}\right) = 0\ .$$

Hence $x(t)$ can now be defined in $(-\infty,\infty)$ by the periodicity of period T, and this is the unique solution of (4.1) with $x(0) = 0$, $x'(0) = C^{1/2}$. Thus the period T of the so determined solution $x(t,C)$ of (4.1) is

$$T = 4\int_0^a [C-G(x)]^{-1/2}dx > 0$$

and $x(t,C)$ is odd with respect to zero, even with respect to $T/4$, and hence, also even with respect to $3T/4$ and odd with respect to $T/2$, that is, $x(-t,C) = -x(t,C)$, $x(T/2+t,C) = -x(T/2-t,C)$, $x(T/4+t,C) = x(T/4-t,C)$, $x(3T/4+t,C) = x(3T/4-t,C)$. Indeed,

$$x\left(\frac{T}{2} - t\right) = x\left(\frac{T}{2} - t - T\right) = x\left(-\frac{T}{2} - t\right) = -x\left(\frac{T}{2} + t\right) \tag{4.6}$$

and

$$x\left(\frac{3T}{4} - t\right) = x\left(\frac{T}{2} + \frac{T}{4} - t\right) = x\left(\frac{T}{2} - \left(t - \frac{T}{4}\right)\right) = x\left(t - \frac{T}{4}\right)$$

$$= x\left(t - \frac{T}{4} + T\right) = x\left(\frac{3T}{4} + t\right)\ . \tag{4.7}$$

Note that $x(t)$ has a Fourier series of the form $c_1\sin\omega t + c_3\sin 3\omega t + c_5\sin 5\omega t + \ldots$ because of the symmetries above. Indeed

$$c_{2n} = \frac{\omega}{2\pi}\int_0^T x(t)\sin 2n\omega t dt = \frac{\omega}{2\pi}\int_{-T/2}^{T/2} x(t)\sin 2n\omega t dt$$

$$= \frac{\omega}{\pi}\int_0^{T/2} x(t)\sin 2n\omega t dt = \frac{\omega}{\pi}\left[\int_0^{T/4} x(t)\sin 2n\omega t dt + \int_{T/4}^{T/2} x(t)\sin 2n\omega t dt\right]\ .$$

Now,

$$\sin 2n\omega\left(\frac{T}{4} - t\right) = \sin(n\pi - 2n\omega t) = -(-1)^n \sin 2n\omega t$$

and

$$\sin 2n\omega\left(\frac{T}{4} + t\right) = \sin(n\pi + 2n\omega t) = (-1)^n \sin 2n\omega t \ .$$

If we put $t = \frac{T}{4} - s$ in the first integral above and $t = \frac{T}{4} + s$ in the second we get

$$\frac{\pi}{\omega} c_{2n} = \int_0^{T/4} x\left(\frac{T}{4} - s\right) \sin 2n\omega\left(\frac{T}{4} - s\right) ds + \int_0^{T/4} x\left(\frac{T}{4} + s\right) \sin 2n\omega\left(\frac{T}{4} + s\right) ds = 0 \ .$$

Since (4.1) is autonomous, a two-parameter family of periodic solutions can always be written as $x = x(t+\gamma, C)$. So far, we have proved the following: under the hypotheses 1 and 2, Eq. (4.1) has a two-parameter family of nontrivial periodic solutions of the form $x = x(t+\gamma, C)$ where γ is an arbitrary constant and $C \in (0, G(A))$ is also arbitrary. For every $C \in (0, G(A))$, the function $x(t,C)$ is odd with respect to 0 and $T/2$, even with respect to $T/4$ and $3T/4$, and periodic of period

$$T = T(C) = 4 \int_0^a [C - G(u)]^{-1/2} \, du$$

with $G(a) = C$ and $a \in (0, A)$.

Since $x(t,C)$ is, at least, of class C^2 in t we conclude that $x(t,C)$ has a convergent Fourier series, and because of the symmetries, this Fourier series is of the form

$$x(t,C) = b_1 \sin \omega t + b_3 \sin 3\omega t + \ldots \text{ with } \omega = \frac{2\pi}{T} \ .$$

These results, which are well known, justify the treatment of Eq. (4.1) that we shall use later in Section 6.

5. SOME PREPARATORY REMARKS ON FOURIER SERIES

5.1. SOME LINEAR OPERATORS WITH RESPECT TO THE UNIFORM NORM

Let us consider real valued functions $x(t)$, $-\infty < t < +\infty$, which are periodic of period $T = 2\pi/\omega$, $\omega > 0$, which are L-integrable in $[o,T]$ and which are odd (with respect to the origin) and even with respect to $T/4$, that is, for which

$$x(-t) = -x(t), \quad x(\frac{T}{4} - t) = x(\frac{T}{4} + t), \quad x(T+t) = x(t). \quad (5.1)$$

As it was shown in Section 4 we have also

$$x(\frac{T}{2} - t) = -x(\frac{T}{2} + t), \quad x(\frac{3T}{4} - t) = x(\frac{3T}{4} + t),$$

and $x(t)$ has a Fourier series of the form

$$x(t) \sim c_1 \sin \omega t + c_3 \sin 3\omega t + \dots . \quad (5.2)$$

We shall consider now the space S_c of all functions $x(t)$, $-\infty < t < +\infty$, which are periodic of period $T = 2\pi/\omega$, continuous in $(-\infty,+\infty)$; and satisfy the symmetry relations (5.1). We take in S_c the uniform topology by choosing the norm

$$\|x\| = \max_t |x(t)|, \quad x \in S_c .$$

Clearly S_c is a normed linear space. Note that S_c is a subset of a separable Hilbert space of all real-valued periodic functions, of period $T = 2\pi/\omega$, L^2-integrable in $[0,T]$, and satisfying relations (5.1). This

space has ($\sqrt{2/T}$ sin ωt, $\sqrt{2/T}$ sin $3\omega t$,...) as an orthonormal sequence.

Also, for every $[\alpha,\beta]$, $-\infty < \alpha < \beta < +\infty$, every element $x(t)$ of S_c defines an element $x(t)$, $\alpha \leq t \leq \beta$, of the usual space $C[\alpha,\beta]$ of all continuous functions in $[\alpha,\beta]$.

Lemma 5.1. The space S_c is complete.

Proof. We first note that convergence in the norm means uniform convergence in $(-\infty,\infty)$. Since any Cauchy sequence $\{x_n=x_n(t)$, $-\infty < t < \infty$, $n=1,2,...\}$ in S_c defines a Cauchy sequence of elements $x_n=x_n(t)$, $\alpha \leq t \leq \beta$, $n = 1,2,...$ of the space $C[\alpha,\beta]$, for any α,β, $-\infty < \alpha < \beta < \infty$, the limit $x(t) = \lim_{n\to\infty} x_n(t)$ exists and is a continuous function in any $[\alpha,\beta]$ and so in $(-\infty,\infty)$. Now, the relations

$$x_n(\frac{2\pi}{\omega} + t) = x_n(t), \quad x_n(-t) = -x_n(t), \quad x_n(\frac{\pi}{2\omega} - t) = x_n(\frac{\pi}{2\omega} + t)$$

imply

$$x(\frac{2\pi}{\omega} + t) = x(t), \quad x(-t) = -x(t), \quad x(\frac{\pi}{2\omega} - t) = x(\frac{\pi}{2\omega} + t)$$

Thus $x(t) \in S_c$.

Definition 5.1. We define the operator P in S_c as follows. If $x(t) \in S_c$ is in the form (5.2) then

$$Px = c_1 \sin \omega t$$

or

$$Px(t) = \left(\frac{\omega}{\pi} \int_0^{2\pi/\omega} x(u) \sin \omega u du\right) \sin \omega t \ .$$

Lemma 5.2. P is a bounded, idempotent, linear operator in S_c.

Proof. Obviously P is linear and idempotent. Since

$$\|Px\| = \|\frac{\omega}{\pi}\int_0^T x(u)\sin \omega u\, du\| \leq \frac{\omega}{\pi}\int_0^T |x(u)|\,|\sin \omega u|du$$

$$\leq \frac{2\omega}{\pi}\|x(u)\|\int_0^T \sin \omega u du = \frac{4}{\pi}\|x(u)\|,$$

we have $\|P\| \leq \frac{4}{\pi}$ and P is bounded.

Definition 5.2. If $x(t)\epsilon S_c$ has the Fourier series (5.2), then we define $H:S_c\to S_c$ as

$$Hx = -\frac{c_1}{\omega^2}\sin \omega t - \frac{c_3}{9\omega^2}\sin 3\omega t - \ldots . \qquad (5.3)$$

Since the sequence $\{c_n\}$ is bounded, the series (5.3) converges absolutely and uniformly in $(-\infty,+\infty)$, and hence Hx is a continuous function of t. Also Hx obviously satifies the symmetry properties (5.1) and so is an element of S_c.

Lemma 5.3. H is a bounded linear operator in S_c and commutes with P.

Proof. Obviously H is linear in S_c. On the other hand

$$\|Hx\| = \max_t\left|-\frac{c_1}{\omega^2}\sin \omega t - \frac{c_3}{9\omega^2}\sin 3\omega t - \ldots\right|$$

$$\leq \omega^{-2}\left(|c_1| + \frac{1}{3^2}|c_3| + \frac{1}{5^2}|c^5| + \ldots\right).$$

Since

$$c_{2n+1} = \frac{\omega}{\pi}\int_0^T x(t)\sin(2n+1)\omega t\, dt$$

we get

$$|c_{2n+1}| \le \frac{\omega}{\pi}\left[\int_0^T |\sin(2n+1)\omega t|\,dt\right] \|x(t)\| = \frac{\omega\|x(t)\|}{\pi(2n+1)\omega}\int_0^{(2n+1)2\pi} |\sin u|\,du$$

$$= \frac{\omega\|x(t)\|}{\pi(2n+1)\omega}\cdot 2(2n+1)\int_0^{\pi}\sin u\,du = \frac{4}{\pi}\|x(t)\| . \qquad (5.4)$$

Then

$$\|Hx\| \le \omega^{-2}\left(\frac{|c_1|}{1^2}+\frac{|c_3|}{3^2}+\frac{|c_5|}{5^2}+\ldots\right) \le \frac{1}{\omega^2}\cdot\frac{\pi^2}{8}\cdot\frac{4}{\pi}\|x(t)\|$$

$$= \frac{\pi}{2\omega^2}\|x(t)\| .$$

Thus H is bounded and so continuous in the linear space S_c.

Finally $HPx = \omega^{-2}c_1\sin\omega t = PHx$ that is H and P commute in S_c.

Lemma 5.4. If I denotes the identity operator in S_c then $H(I-P)$ is a linear, bounded operator in S_c, and $PH(I-P) = 0$.

Proof. The first part follows immediately from the previous lemmas. Now, since P commutes with H we have

$$PH(I-P)x = P(H-HP)x = P(H-PH)x = P(I-P)(Hx)$$

$$= (P-P^2)(Hx) = (P-P)(Hx) = 0 .$$

We shall denote by k_c the norm of $H(I-P)$ or

$$k_c = \|H(I-P)\| .$$

Then k_c is the smallest number such that

$$\|H(I-P)x\| \le k_c\|x\|$$

for all $x = x(t)\in S_c$.

Let us discuss for a while the norm k_c of $H(I-P)$. An upper bound of k_c can be easily determined as follows.

If $x \epsilon S_c$ has the form of (5.2) then $H(I-P)x$ has the form

$$H(I-P)x \sim -\frac{c_3}{9\omega^2} \sin 3\omega t - \frac{c_5}{25\omega^2} \sin 5\omega t - \dots . \qquad (5.5)$$

By Eq. (5.4) we have $|c_{2n+1}| \leq \frac{4}{\pi}\|x(t)\|$. This result and Eq. (5.5) yield

$$\|H(I-P)x\| \leq \frac{4}{\pi\omega^2}\left[\frac{1}{9} + \frac{1}{25} + \dots\right]\|x\| = \frac{4}{\pi\omega^2}\left[\frac{\pi^2}{8} - 1\right]\|x\|$$

$$= 0.29755\ \omega^{-2}\|x\| . \qquad (5.6)$$

The number $0.29755\ \omega^{-2}$ is then an upper bound for k_c. This value may be used in the applications, but we shall give below (5.2) an integral representation of k_c from which we shall deduce the exact value of k_c.

Let $x(t) \epsilon S_c$ have the form (5.2) and let

$$F_1(t) = \int_{T/4}^{t} x(\beta)\, d\beta , \quad F_2(t) = \int_{0}^{t} F_1(\alpha)\, d\alpha .$$

Since $x(t)$ is odd with respect to 0 and even with respect to $T/4$ then $F_1(t)$ will be even with respect to 0 and odd with respect to $T/4$. In turn, $F_2(t)$ will be odd with respect to 0 and even with respect to $T/4$. Integration of (5.2) term by term, from $T/4$ to t, is justified by known theorems on Fourier series (see Ref. 18, p. 40), and yields

$$F_1(t) = \int_{T/4}^{t} x(t)\,dt = \int_{T/4}^{t} [c_1 \sin \omega t + c_3 \sin 3\omega t + \dots]\,dt$$

$$= -\frac{c_1}{\omega} \cos \omega t - \frac{c_3}{3\omega} \cos 3\omega t - \dots .$$

Another integration, now from 0 to t, will give the Fourier series of $F_2(t)$

$$F_2(t) = \int_{0}^{t} F_1(t)\,dt = \int_{0}^{t} [-\frac{c_1}{\omega} \cos \omega t - \frac{c_3}{3\omega} \cos 3\omega t - \dots]\,dt$$

$$= -\frac{c_1}{\omega^2} \sin \omega t - \frac{c_3}{3^2\omega^2} \sin 3\omega t - \dots .$$

By comparison with (5.3) we have

$$F_2(t) = Hx(t) = \int_{0}^{t} d\alpha \int_{T/4}^{t} x(\beta)\,d\beta . \qquad (5.7)$$

We shall use this formula later.

In order to conclude this section we have two lemmas.

Lemma 5.5. If $x(t) \in S_c$ then Hx is of class C^2 and $\frac{d^2}{dt^2} Hx = x$,

$$\frac{d^2}{dt^2} H(I-P)x = (I-P)x .$$

Proof. It is a consequence of Eq. (5.7).

Lemma 5.6. If $x(t) \in S_c$ is a function of class C^2, then

$$H(I-P)x'' = (I-P)x .$$

Proof. If $x(t) \in S_c$ is of class C^2 in $(-\infty, +\infty)$ and

$$x(t) \sim c_1 \sin \omega t + c_3 \sin 3\omega t + \dots$$

then

$$x''(t) \sim -c_1\omega^2 \sin \omega t - 9c_3\omega^2 \sin 3\omega t - \dots .$$

Indeed, by straightforward integration and integration by parts, we have

$$\frac{1}{T}\int_0^T x''(t)\,dt = \frac{2}{T}\,[x'(T) - x'(0)] = 0 ,$$

$$\frac{2}{T}\int_0^T x''(t)\sin n\omega t\,dt = -\frac{2n\omega}{T}\int_0^T x'(t)\cos n\omega t\,dt$$

$$= -\frac{2n^2\omega^2}{T}\int_0^T x(t)\sin n\omega t\,dt = -n^2\omega^2 c_n ,$$

$$\frac{2}{T}\int_0^T x''(t)\cos n\omega t\,dt = 0 .$$

Finally, $(I-P)x'' \sim -9c_3\omega^2\sin 3\omega t\ldots$ and by (5.3) also

$$H(I-P)x''(t) = c_3\sin 3\omega t + c_5\sin 5\omega t + \ldots = (I-P)x(t) .$$

5.2. DETERMINATION OF k_c

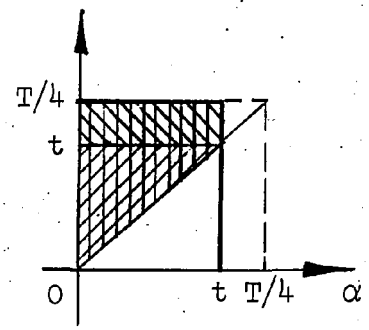
We shall now determine the exact value of k_c. We shall use the formula (5.7) obtained above which we will write in the form (see sketch, p. 39)

$$F_2(t) = \int_0^t d\alpha \int_{T/4}^{\alpha} x(\beta)\,d\beta = -\int_0^t d\alpha \int_{\alpha}^{T/4} x(\beta)\,d\beta$$

$$= -\int_0^t d\beta \int_0^{\beta} x(\beta)\,d\alpha - \int_t^{T/4} d\beta \int_0^t x(\beta)\,d\alpha \qquad (5.8)$$

$$= -\int_0^t \beta x(\beta)\,d\beta - t\int_t^{T/4} x(\beta)\,d\beta .$$

If we use the symmetries of x(t) we get



$$c_1 = \frac{\omega}{\pi} \int_0^T x(t) \sin \omega t \, dt$$

$$= \frac{\omega}{\pi} \left[\int_0^{T/2} x(t) \sin \omega t \, dt + \int_{T/2}^{T} x(t) \sin \omega t \, dt \right]$$

$$= \frac{\omega}{\pi} \left[\int_0^{T/2} x(t) \sin \omega t \, dt - \int_0^{T/2} x(\tau + \frac{T}{2}) \sin \omega\tau \, d\tau \right]$$

$$= \frac{\omega}{\pi} \left[\int_0^{T/2} x(t) \sin \omega t \, dt + \int_0^{T/2} x(\tau) \sin \omega\tau \, d\tau \right]$$

$$= \frac{2\omega}{\pi} \left[\int_0^{T/4} x(t) \sin \omega t \, dt + \int_{T/4}^{T/2} x(\tau) \sin \omega\tau \, d\tau \right]$$

$$= \frac{2\omega}{\pi} \left[\int_0^{T/4} x(t) \sin \omega t \, dt + \int_{T/4}^{0} x(\tau - \frac{T}{2}) \sin \omega\tau \, d\tau \right]$$

$$= \frac{4\omega}{\pi} \int_0^{T/4} x(t) \sin \omega t \, dt \, . \qquad (5.9)$$

Since $Px = c_1 \sin \omega t$ implies $HPx = -\frac{c_1}{\omega^2} \sin \omega t$ we have from Eqs. (5.8) and (5.9).

$$H(I-P)x = Hx - HPx = F_2(t) + \frac{c_1}{\omega^2} \sin \omega t$$

$$= -\int_0^t \beta x(\beta) \, d\beta - t \int_t^{T/4} x(\beta) \, d\beta + \frac{4}{\pi\omega} \sin \omega t \int_0^{T/4} x(\beta) \sin \omega\beta \, d\beta$$

$$= \int_0^{T/4} K(t,\beta) x(\beta) \, d\beta \, . \qquad (5.10)$$

where the kernel $K(t,\beta)$ is given by

$$K(t,\beta) = \begin{cases} -\beta + \frac{4}{\pi\omega} \sin \omega t \sin \omega\beta & \text{for} \quad 0 \le \beta \le t \le \frac{T}{4} , \\ -t + \frac{4}{\pi\omega} \sin \omega t \sin \omega\beta & \text{for} \quad 0 \le t \le \beta \le \frac{T}{4} . \end{cases} \tag{5.11}$$

Note that, by taking $\omega K = \overline{K}$, $\omega t = \xi$, $\omega\beta = \eta$, $y(t) = H(I-P)x(t)$, and

$$X(\xi) = x(\tfrac{\xi}{\omega}) , \quad Y(\xi) = y(\tfrac{\xi}{\omega}) ,$$

we have successively

$$y(\tfrac{\xi}{\omega}) = \int_0^{\pi/2} K(\tfrac{\xi}{\omega} , \tfrac{\eta}{\omega}) x(\tfrac{\eta}{\omega}) \frac{d\eta}{\omega} ,$$

$$Y(\xi) = \omega^{-2} \int_0^{\pi/2} \overline{K}(\xi,\eta) x(\eta) d\eta ,$$

where

$$\overline{K}(\xi,\eta) = \begin{cases} -\eta + \frac{4}{\pi} \sin \xi \sin \eta & \text{for} \quad 0 \le \eta \le \xi \le \frac{\pi}{2} \\ -\xi + \frac{4}{\pi} \sin \xi \sin \eta & \text{for} \quad 0 \le \xi \le \eta \le \frac{\pi}{2} . \end{cases}$$

The kernel $\overline{K}$ has obviously the properties:

(a) $\overline{K}(\xi,\eta) = \overline{K}(\eta,\xi)$,

(b) $\overline{K}(\xi,\eta)$ is continuous in the square $[0, 0, \pi/2, \pi/2]$,

(c) $\overline{K}(0,0) = \overline{K}(\xi,0) = \overline{K}(0,\eta) = 0.$

Now, we have

$$\frac{\partial \overline{K}(\xi,0)}{\partial \eta} = -1 + \frac{4}{\pi} \sin \xi$$

and this function of ξ increases from -1 to $-1 + \frac{4}{\pi}$ as ξ describes $[0,\pi/2]$. Thus, this expression is zero at the point ξ_o, $0 < \xi_o < \pi/2 = 1.5708$, with $\xi_o = \text{arc sin } \pi/4 = \text{arc sin } 0.7854 = 0.9034$. Finally, $\bar{K}(\xi,0) = 0$ for all $0 \leq \xi \leq \pi/2$ and

$$\frac{\partial \bar{K}(\xi,0)}{\partial \eta} < 0 \quad \text{for} \quad 0 \leq \xi < \xi_o , \quad \frac{\partial \bar{K}(\xi,0)}{\partial \eta} > 0 \quad \text{for} \quad \xi_o < \xi \leq \pi/2 . \tag{5.12}$$

Now, for any ξ, $0 \leq \xi \leq \pi/2$, consider the expression

$$\frac{\partial \bar{K}(\xi,\eta)}{\partial \eta} = \begin{cases} -1 + \frac{4}{\pi} \sin \xi \cos \eta & \text{for} \quad 0 \leq \eta < \xi \leq \pi/2 \\ \frac{4}{\pi} \sin \xi \cos \eta & \text{for} \quad 0 \leq \xi < \eta \leq \pi/2 , \end{cases}$$

as a function of η for $0 \leq \eta \leq \pi/2$. This derivative is positive as η describes $(\xi, \pi/2)$. The same derivative decreases from $-1 + \frac{4}{\pi} \sin \xi$ to $-1 + \frac{4}{\pi} \sin \xi \cos \xi$ as η describes $[0,\xi)$. Thus $\bar{K}(\xi,\eta)$ as a function of η alone, changes sign at most once between 0 and ξ.

Since for $0 \leq \xi < \xi_o$ we have $\partial \bar{K}(\xi,0)/\partial \eta < 0$, from (5.12), and since $\partial \bar{K}(\xi,0)/\partial \eta$ decreases for $0 \leq \eta < \xi$ the fact that $\bar{K}(\xi,\eta) = 0$ for $\eta = 0$ implies that $\bar{K}(\xi,\eta) < 0$ for $0 < \eta < \xi$, $0 \leq \xi < \xi_o$.

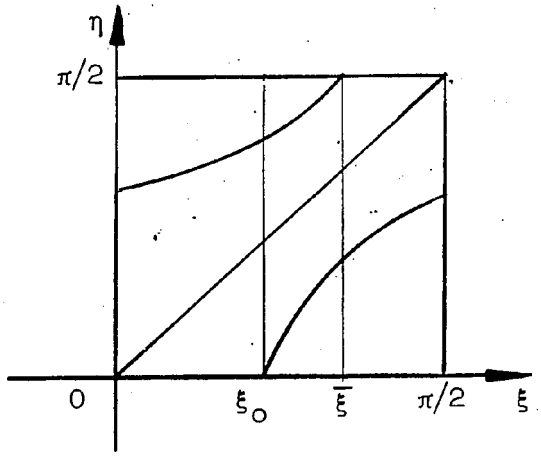
Now we consider ξ in the interval $[\xi_o,\pi/2]$. First we recall that $\bar{K}(\xi,0) = 0$ and that $\partial \bar{K}(\xi,0)/\partial \eta > 0$ for $\xi_o < \xi < \pi/2$. Then $\bar{K}(\xi,\eta)$ is positive for some region of the strip $\xi_o < \xi \leq \pi/2$. We show next that $\bar{K}(\xi,\eta)$ is negative along the line $\eta = \xi$ on $[0,0,\pi/2,\pi/2]$.

Indeed $\overline{K}(\xi,\xi) = -\xi + \frac{4}{\pi}\sin^2\xi$, $\overline{K}(0,0) = 0$ and $\overline{K}(\pi/2,\pi/2) = -\frac{\pi}{2} + \frac{4}{\pi}\sin^2\frac{\pi}{2} = \frac{4}{\pi} - \frac{\pi}{2} = \frac{8-\pi^2}{2\pi} < 0$. Now, $\frac{d}{du}\overline{K}(u,u) = -1 + \frac{4}{\pi}\sin 2u$ and this derivative is -1 at $u = 0$ and $u = \pi/2$; it is zero at the points where $\sin 2u = \pi/4$ that is at the points $\xi_1 = \xi_0/2 = 0.4517$ and $\xi_2 = \frac{\pi}{2} - \xi_1 = 1.1191$. The maximum value of $\overline{K}(u,u)$ is attained at the point $u = \xi_2$ and is $\overline{K}(\xi_2,\xi_2) = -\xi_2 + \frac{4}{\pi}\sin^2\xi_2 = -0.0885 < 0$. Thus $\overline{K}(\xi,\xi) < 0$ for all $0 \le \xi \le \pi/2$.

Now, we have

$$\overline{K}(\xi,\pi/2) = -\xi + \frac{4}{\pi}\sin\xi\,, \quad \overline{K}(0,\pi/2) = 0\,,$$

and as we have seen $\overline{K}(\pi/2,\pi/2) = (8-\pi^2)/2\pi < 0$. Also we have $\frac{\partial}{\partial\xi}\overline{K}(\xi,\pi/2) = -1 + \frac{4}{\pi}\cos\xi$. If ξ_1 is the value of ξ in $(0,\pi/2)$ for which $-1 + \frac{4}{\pi}\cos\xi = 0$ then $\xi_1 = \frac{\pi}{2} - \xi_0 = 0.6674$, and $\frac{\partial}{\partial\xi}\overline{K}(\xi,\pi/2) > 0$ for $0 \le \xi < \xi$, and $\frac{\partial}{\partial\xi}\overline{K}(\xi,\pi/2) < 0$ for $\xi < \xi \le \pi/2$. Thus $\overline{K}(\xi,\pi/2) > 0$ for $0 < \xi < \overline{\xi}$, $\overline{K}(\xi,\pi/2) < 0$ for $\overline{\xi} < \xi \le \pi/2$, where $\overline{\xi}$ is the solution of $-\xi + \frac{4}{\pi}\sin\xi = 0$ in $[0,\pi/2]$. We get $\overline{\xi} = 1.1746$.



Thus, for $0 \le \xi \le \xi_0$, $\overline{K}(\xi,\eta)$ as a function of η changes sign once; for $\xi_0 < \xi < \overline{\xi}$, $\overline{K}(\xi,\eta)$ as a function of η changes sign twice; for $\overline{\xi} \le \xi \le \pi/2$. $\overline{K}(\xi,\eta)$ as a function of η again changes sign once.

Let us denote by $\psi(\xi)$ the function

$$\psi(\xi) = \int_0^{\pi/2} |\overline{K}(\xi,\eta)|\,d\eta\,, \quad 0 \le \xi \le \pi/2\,.$$

Obviously $\psi(\xi)$ is a continuous function of ξ, and we take $k_o = \max \psi(\xi)$ in $[0,\pi/2]$, and $k_c = k_o\omega^{-2}$. Now we have, for every function $x(t)$ in S

$$y(t) = H(I-P)x(t) = \int_0^{T/4} K(t,\beta)x(\beta)\,d\beta\,,$$

$$Y(\xi) = \omega^{-2} \int_0^{\pi/2} \overline{K}(\xi,\eta)X(\eta)\,d\eta\,,$$

$$\begin{aligned} |Y(\xi)| &\le \omega^{-2} \int_0^{\pi/2} |\overline{K}(\xi,\eta)|\ |X(\eta)|\,d\eta \\ &\le \omega^{-2}\ [\max|X(\eta)|\,] \int_0^{\pi/2} |\overline{K}(\xi,\eta)|\,d\eta \\ &\le \omega^{-2}\ \|x(t)\|\psi(\xi) \\ &\le \omega^{-2}\ \|x(t)\|k_o = k_c\|x(t)\|\,. \end{aligned}$$

Thus we have

$$\|H(I-P)x\| \le k_c\|x\|\,. \tag{5.13}$$

Let us prove now that the value $k_c = k_o\omega^{-2}$ now determined is the best possible value, that is, $k_c = k_o\omega^{-2}$ is the norm of $H(I-P)$. Let ξ^* be a point where $\psi(\xi^*)$ takes its maximum value, hence

$$0 \le \xi^* \le \pi/2\,, \quad \psi(\xi^*) = \max \psi(\xi) = \int_0^{\pi/2} |\overline{K}(\xi^*,\eta)|\,d\eta\,.$$

Suppose, for instance, that for $\xi = \xi^*$, $\overline{K}(\xi^*,\eta)$ as a function of η changes sign only once, say, $\overline{K}(\xi^*,\eta) > 0$ in $(0,\overline{\eta})$, and $\overline{K}(\xi^*,\eta) < 0$ in $(\overline{\eta}, \pi/2)$. The other possible cases can be discussed analogously.

Let $\bar{x}(t)$ (not in S_c) be defined by $\bar{x}(t) = \lambda$ in $[0,\bar{t})$ and $\bar{x}(t) = -\lambda$ in $(\bar{t},\pi/2)$ where $\lambda > 0$ is arbitrary and $\bar{t} = \bar{\eta}/\omega$. Let us extend $\bar{x}(t)$ in $(-\infty,+\infty)$ by the periodicity of period T, by oddness with respect to 0, and evenness with respect to T/4. Then

$$X(\xi) = \begin{cases} \lambda & \text{for} \quad 0 < \xi < \bar{\eta} , \\ -\lambda & \text{for} \quad \bar{\eta} < \xi < \pi/2 \end{cases}$$

and

$$\bar{y}(t) = H(I-P)x = \int_0^{T/4} K(t,\beta)x(\beta)\,d\beta ,$$

$$\bar{Y}(\xi) = y\left(\frac{\xi}{\omega}\right) = \omega^{-2} \int_0^{\pi/2} \bar{K}(\xi,\eta)X(\eta)\,d\eta ,$$

$$Y(\xi^*) = \omega^{-2}\int_0^{\pi/2} \bar{K}(\xi^*,\eta)X(\eta)\,d\eta$$

$$= \lambda\omega^{-2} \int_0^{\pi/2} \bar{K}(\xi^*,\eta)\,d\eta = \lambda\omega^{-2}\psi(\xi^*)$$

$$= \lambda k_0\omega^{-2} = k_c\lambda .$$

and finally

$$|Y(\xi)| \geq k_c\lambda = \lambda\|x\| , \quad 0 \leq \xi \leq \pi/2 ,$$

$$|y(t)| \geq k_c\|x\| , \quad 0 \leq t \leq T/4 ,$$

or

$$\|H(I-P)\| \geq k_c\|x\| .$$

This and (5.13) yield

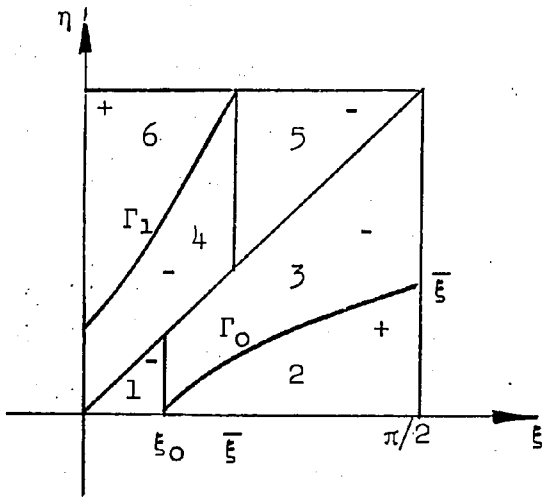
$$\|H(I-P)x\| = k_c\|x\| .$$

This would prove our claim if we had allowed discontinuous functions in the class S_c. Therefore we shall now replace $\overline{x}(t)$ by a continuous function $\overline{\overline{x}}(t)$ in S_c which has the same norm λ as $\overline{x}(t)$, and for which the norm of the corresponding function $\overline{\overline{y}}(t)$ differs as little as we want from the norm of $\overline{y}(t)$. Let us take

$$\overline{\overline{x}}(t) = \begin{cases} \lambda t/\epsilon & \text{for} \quad 0 \le t \le \epsilon \\ \lambda & \text{for} \quad \epsilon \le t \le \overline{t} - \epsilon \\ \lambda(\overline{t}-\epsilon) & \text{for} \quad \overline{t} - \epsilon \le t \le \overline{t} + \epsilon \\ -\lambda & \text{for} \quad \overline{t} + \epsilon < t < T/4 \end{cases}$$

and let us extend $\overline{\overline{x}}(t)$ in $(-\infty,+\infty)$ as before. Now, $\overline{\overline{y}}(t)$ can be made as close to $k_c\lambda$ as we want by taking $\epsilon > 0$ sufficiently small. Thus we can take $\|\overline{\overline{y}}\| \ge (k_c-\sigma)\|\overline{\overline{x}}\|$ for every fixed $\sigma > 0$ while we know that $\|\overline{\overline{y}}\| \le k_c\|\overline{\overline{x}}\|$. This choice of the function $\overline{\overline{x}}(t)$ corresponds to one of the possible cases mentioned above. We shall proceed analogously in the other cases. This shows that $k_c = k_o\omega^{-2}$, $k_o = \max \psi(\xi)$ is the best possible constant, that is, $\|H(I-P)\| = k_c = k_o\omega^{-2}$.

We now proceed to the determination of the exact numerical value of k_c, by means of the considerations above. As we will see this value is much smaller than the upper bound we obtained in (5.6). As mentioned above we have only to find the maximum of the function $\psi(\xi)$ defined by



$$\psi(\xi) = \int_0^{\pi/2} |\bar{K}(\xi,\eta)|\,d\eta\ , \tag{5.14}$$

where

$$\bar{K}(\xi,\eta) = \begin{cases} -\eta + \frac{4}{\pi}\sin\xi\,\sin\eta, & 0 \le \eta \le \xi \le \pi/2\ , \\ -\xi + \frac{4}{\pi}\sin\xi\,\sin\eta, & 0 \le \xi \le \eta \le \pi/2\ . \end{cases} \tag{5.15}$$

In order to perform the integration of (5.14) we divide the square $(0,0,\pi/2,\pi/2)$ into the regions 1...6 of the figure where Γ_0 and Γ_1 are the curves $\eta = \eta_0(\xi)$ and $\eta = \eta_1(\xi)$ defined as follows: For $\xi_0 \le \xi \le \pi/2$ the function $\eta_0(\xi)$ is defined by

$$-\eta + \frac{4}{\pi}\sin\xi\,\sin\eta = 0$$

or

$$\frac{\eta}{\sin\eta} = \frac{4}{\pi}\sin\xi\ . \tag{5.16}$$

Note that for $\xi \to \xi_0$ we have successively $\sin\xi \to \pi/4$, $\eta/\sin\eta \to 1$, $\eta \to 0$ and $\eta_0(\xi_0) = 0$. Also for $\xi = \pi/2$, $\sin\xi = 1$, $\eta/\sin\eta = 4/\pi$, $-\eta + \frac{4}{\pi}\sin\eta = 0$ and $\eta_0(\pi/2) = \bar{\xi}$.

For $0 \le \xi \le \bar{\xi}$ the function $\eta_1(\xi)$ is given by

$$-\xi + \frac{4}{\pi} \sin \xi \sin \eta = 0$$

or

$$\sin \eta = \frac{\pi}{4} \frac{\xi}{\sin \xi} .$$

Note that for $\xi \to 0$, $\sin \eta \to \pi/4$, $\eta \to \xi_o$ and $\eta_1(0) = \xi_o$. Also, for $\xi = \bar{\xi}$, $-\xi + \frac{4}{\pi} \sin \xi = 0$, $\sin \eta = 1$, $\eta = \pi/2$ and $\eta_1(\bar{\xi}) = \pi/2$. Finally we note that $\sin \xi_o = \pi/4$ and $\sin \bar{\xi} = \frac{\pi}{4} \bar{\xi}$. Now we compute integral (5.14).

For $0 \leq \xi \leq \xi_o$ we take

$$\psi_1 = \int_o^{\xi} |\bar{K}(\xi,\eta)| d\eta = \int_o^{\xi} [\eta - \frac{4}{\pi} \sin \xi \sin \eta] d\eta \tag{5.17}$$

$$= \left[\frac{\eta^2}{2} + \frac{4}{\pi} \sin \xi \cos \eta\right]_{\eta=0}^{\eta=\xi} = \frac{\xi^2}{2} + \frac{4}{\pi} \sin \xi \cos \xi - \frac{4}{\pi} \sin \xi .$$

For $\xi_o \leq \xi \leq \pi/2$ we take

$$\psi_2 = \int_o^{\eta_o(\xi)} |\bar{K}(\xi,\eta)| d\eta = \int_o^{\eta_o(\xi)} [-\eta + \frac{4}{\pi} \sin \xi \sin \eta] d\eta$$

$$= \left[-\frac{\eta^2}{2} - \frac{4}{\pi} \sin \xi \cos \eta\right]_{\eta=0}^{\eta=\eta_o(\xi)}$$

$$= -\frac{[\eta_o(\xi)]^2}{2} - \frac{4}{\pi} \sin \xi \cos \eta_o(\xi) + \frac{4}{\pi} \sin \xi . \tag{5.18}$$

For $\xi_o \leq \xi \leq \pi/2$ we take

$$\psi_3 = \int_{\eta_o(\xi)}^{\xi} |\bar{K}(\xi,\eta)|d\eta = \int_{\eta_o(\xi)}^{\xi} [\eta - \frac{4}{\pi}\sin\xi\sin\eta]d\eta$$

$$= \left[\frac{\eta^2}{2} + \frac{4}{\pi}\sin\xi\cos\eta\right]_{\eta=\eta_o(\xi)}^{\eta=\xi}$$

$$= \frac{\xi^2}{2} + \frac{4}{\pi}\sin\xi\cos\xi - \frac{[\eta_o(\xi)]^2}{2} - \frac{4}{\pi}\sin\xi\cos\eta_o(\xi) . \qquad (5.19)$$

For $0 \le \xi \le \bar{\xi}$ we take

$$\psi_4 = \int_{\xi}^{\eta_1(\xi)} |\bar{K}(\xi,\eta)|d\eta = \int_{\xi}^{\eta_1(\xi)} [\xi - \frac{4}{\pi}\sin\xi\sin\eta]d\eta$$

$$= \left[\xi\eta + \frac{4}{\pi}\sin\xi\cos\eta\right]_{\eta=\xi}^{\eta=\eta_1(\xi)}$$

$$= \xi\eta_1(\xi) + \frac{4}{\pi}\sin\xi\cos\eta_1(\xi) - \xi^2 - \frac{4}{\pi}\sin\xi\cos\xi . \qquad (5.20)$$

For $\bar{\xi} \le \xi \le \pi/2$ we take

$$\psi_5 = \int_{\xi}^{\pi/2} |\bar{K}(\xi,\eta)|d\eta = \int_{\xi}^{\pi/2} [\xi - \frac{4}{\pi}\sin\xi\sin\eta]d\eta \qquad (5.21)$$

$$= \left[\xi\eta + \frac{4}{\pi}\sin\xi\cos\eta\right]_{\eta=\xi}^{\eta=\pi/2} = \xi\frac{\pi}{2} - \xi^2 - \frac{4}{\pi}\sin\xi\cos\xi .$$

For $0 \le \xi \le \bar{\xi}$ we take

$$\psi_6 = \int_{\eta_1(\xi)}^{\pi/2} |\bar{K}(\xi,\eta)|d\eta = \int_{\eta_1(\xi)}^{\pi/2} [-\xi + \frac{4}{\pi}\sin\xi\sin\eta]d\eta \qquad (5.22)$$

$$= \left[-\xi\eta - \frac{4}{\pi}\sin\xi\cos\eta\right]_{\eta=\eta_1(\xi)}^{\eta=\pi/2} = -\xi\frac{\pi}{2} + \xi\eta_1(\xi) + \frac{4}{\pi}\sin\xi\cos\eta_1(\xi) .$$

Then, for $0 \le \xi \le \xi_o$ we have from (5.17), (5.20) and (5.22)

$$\psi(\xi) = \int_0^{\pi/2} |\bar{K}(\xi,\eta)|\,d\eta = \psi_1 + \psi_4 + \psi_6$$

$$= \frac{\xi^2}{2} + \frac{4}{\pi}\sin\xi\cos\xi - \frac{4}{\pi}\sin\xi + \xi\eta_1(\xi) + \frac{4}{\pi}\sin\xi\cos\eta_1(\xi)$$

$$- \xi^2 - \frac{4}{\pi}\sin\xi\cos\xi - \xi\frac{\pi}{2} + \xi\eta_1(\xi) + \frac{4}{\pi}\sin\xi\cos\eta_1(\xi)$$

$$= \frac{8}{\pi}\sin\xi\cos\eta_1(\xi) - \frac{4}{\pi}\sin\xi + 2\xi\eta_1(\xi) - \frac{\xi^2}{2} - \xi\frac{\pi}{2}. \qquad (5.23)$$

At the points $\xi = 0$ and $\xi = \xi_o$ we have $\psi(0) = 0$ and

$$\psi(\xi_o) = \frac{8}{\pi}\sin\xi_o\cos\eta_1(\xi_o) + \frac{4}{\pi}\sin\xi_o + 2\xi_o\eta_1(\xi_o) - \frac{\xi_o^2}{2} - \xi_o\frac{\pi}{2}$$

$$= 2\cos\eta_1(\xi_o) - 1 + 2\xi_o\eta_1(\xi_o) - \frac{\xi_o^2}{2} - \xi_o\frac{\pi}{2}. \qquad (5.24)$$

For $\xi_o \le \xi \le \bar{\xi}$ we have from (5.18) to (5.22)

$$\psi(\xi) = \psi_2 + \psi_3 + \psi_4 + \psi_6$$

$$= -\frac{[\eta_o(\xi)]^2}{2} - \frac{4}{\pi}\sin\xi\cos\eta_o(\xi) + \frac{4}{\pi}\sin\xi + \frac{\xi^2}{2} + \frac{4}{\pi}\sin\xi\cos\xi$$

$$- \frac{[\eta_o(\xi)]^2}{2} - \frac{4}{\pi}\sin\xi\cos\eta_o(\xi) + \xi\eta_1(\xi) + \frac{4}{\pi}\sin\xi\cos\eta_1(\xi)$$

$$- \xi^2 - \frac{4}{\pi}\sin\xi\cos\xi - \xi\frac{\pi}{2} + \xi\eta_1(\xi) + \frac{4}{\pi}\sin\xi\cos\eta_1(\xi)$$

$$= -\frac{8}{\pi}\sin\xi\cos\eta_o(\xi) + \frac{4}{\pi}\sin\xi + \frac{8}{\pi}\sin\xi\cos\eta_1(\xi)$$

$$- [\eta_o(\xi)]^2 - \frac{\xi^2}{2} + 2\xi\eta_1(\xi) - \xi\frac{\pi}{2}. \qquad (5.25)$$

At the points ξ_o and $\bar{\xi}$ we get from (5.25)

$$\psi(\xi_o) = -\frac{8}{\pi}\sin\xi_o\cos\eta_o(\xi_o) + \frac{4}{\pi}\sin\xi_o + \frac{8}{\pi}\sin\xi_o\cos\eta_1(\xi_o)$$

$$-\eta_o^2(\xi_o) - \frac{\xi_o^2}{2} + 2\xi_o\eta_1(\xi_o) - \xi_o\frac{\pi}{2}$$

$$= -1 + 2\cos\eta_1(\xi_o) - \frac{\xi_o^2}{2} + 2\xi_o\eta_1(\xi_o) - \xi_o\frac{\pi}{2},$$

which agrees with (5.24). Since $\sin\bar{\xi} = \frac{\pi}{4}\bar{\xi}$ and $\eta_1(\bar{\xi}) = \frac{\pi}{2}$ we get

$$\psi(\bar{\xi}) = -\frac{8}{\pi}\sin\bar{\xi}\cos\eta_o(\bar{\xi}) + \frac{4}{\pi}\sin\bar{\xi} + \frac{8}{\pi}\sin\bar{\xi}\cos\eta_1(\bar{\xi})$$

$$+ [\eta_o(\bar{\xi})]^2 - \frac{\bar{\xi}^2}{2} + 2\bar{\xi}\eta_1(\bar{\xi}) - \bar{\xi}\frac{\pi}{2}$$

$$= -2\bar{\xi}\cos\eta_o(\bar{\xi}) + \bar{\xi} + [\eta_o(\bar{\xi})]^2 - \frac{\bar{\xi}^2}{2} + \frac{\pi}{2}\bar{\xi}. \qquad (5.26)$$

For $\bar{\xi} \le \xi \le \pi/2$ we have from (5.18), (5.19) and (5.21)

$$\psi(\xi) = \psi_2 + \psi_3 + \psi_5$$

$$= -\frac{[\eta_o(\xi)]^2}{2} - \frac{4}{\pi}\sin\xi\cos\eta_o(\xi) + \frac{4}{\pi}\sin\xi$$

$$+ \frac{\xi^2}{2} + \frac{4}{\pi}\sin\xi\cos\xi - \frac{[\eta_o(\xi)]^2}{2} - \frac{4}{\pi}\sin\xi\cos\eta_o(\xi)$$

$$+ \xi\frac{\pi}{2} - \xi^2 - \frac{4}{\pi}\sin\xi\cos\xi \qquad (5.27)$$

$$= -[\eta_o(\xi)]^2 - \frac{8}{\pi}\sin\xi\cos\eta_o(\xi) + \frac{4}{\pi}\sin\xi - \frac{\xi^2}{2} + \frac{\pi}{2}\xi.$$

From (5.27) we get at $\xi = \bar{\xi}$ and $\xi = \pi/2$,

$$\psi(\bar{\xi}) = [\eta_o(\bar{\xi})]^2 - 2\bar{\xi}\cos\eta_o(\bar{\xi}) + \bar{\xi} - \frac{\bar{\xi}^2}{2} + \frac{\pi}{2}\bar{\xi}$$

as in (5.26), and since $\eta_o(\pi/2) = \bar{\xi}$,

$$\psi(\tfrac{\pi}{2}) = -\frac{8}{\pi}\cos\eta_o(\tfrac{\pi}{2}) + \frac{4}{\pi} - [\eta_o(\tfrac{\pi}{2})]^2 - \frac{1}{2}\frac{\pi^2}{4} + \frac{\pi^2}{4}$$

$$= -\frac{8}{\pi}\cos\bar{\xi} + \frac{4}{\pi} - \bar{\xi}^2 + \frac{\pi^2}{8}\,. \qquad (5.28)$$

A numerical calculation yields the following values

ξ	0	0.2	0.4	0.45	0.5	0.55	0.6	0.8
$\psi(\xi)$	0	0.0883	0.1326	0.1380	0.1401	0.1386	0.1353	0.0989

ξ	$0.9034=\xi_o$	1	1.1	$1.174602=\bar{\xi}$	1.2	1.3
$\psi(\xi)$	0.0676	0.0489	0.021	0.0606	0.0687	0.1060

ξ	1.4	1.45	1.5	1.55	$1.5708=\frac{\pi}{2}$
$\psi(\xi)$	0.1257	0.1354	0.14132	0.1443	0.1445

After computation, ψ takes its maximum $\psi_{max} = 0.14454$ at $\xi = 1.570$. Thus $k_c = 0.14454\omega^{-2}$.

5.3. SOME LINEAR OPERATORS WITH RESPECT TO THE SQUARE NORM

Let us consider now the space S_q of all real valued functions $x(t)$, $-\infty < t < +\infty$, which are periodic of period $T = 2\pi/\omega$, $\omega > 0$, which are L^2-integrable in $[0,T]$, and which satisfy the symmetry relations (5.1). We shall take in S_q the usual inner product and square norm

$$(x,y) = \frac{1}{T}\int_o^T x(t)y(t)\,dt\,, \qquad (5.29)$$

$$\mu(x) = [(x,x)]^{1/2} = \left[\frac{1}{T}\int_o^T x^2(t)\,dt\right]^{1/2}.$$

Thus S_q is a separable Hilbert space for which $(\sqrt{2/T}\sin\omega t, \sqrt{2/T}\sin 3\omega t, \ldots)$ is an orthonormal system. If $x(t)\epsilon S_q$ and

$$x(t) \sim c_1\sin\omega t + c_3\sin 3\omega t + \ldots \tag{5.30}$$

then

$$\mu(x) = \left[\frac{1}{T}\int_0^T x^2(t)\,dt\right]^{1/2} = \left\{\frac{1}{2}(c_1^2+c_3^2+\ldots)\right\}^{1/2}.$$

Thus (S_q,μ) is, in particular a normed linear space. Note that for the elements $x(t)\epsilon S_q$ which are bounded we may still define a norm $\|x\|$ by taking

$$\|x\| = \sup_t |x(t)| .$$

We will see below a property of S_q where both norms, μ and $\|\ \|$, are being used. Otherwise, when we refer to properties "in S_q" we will refer exclusively to the usual square norm μ.

Theorem 5.1. The space (S_q,μ) is complete.

Proof. We first note that each function $x(t)\epsilon S_q$ induces a function of $L_2(\alpha,\beta)$, namely, the restriction of $x(t)$ to the interval $[\alpha,\beta]$ where α and β are any two real numbers and $\alpha < \beta$.

Let $\{x_n(t)\}$, $x_n(t)\epsilon S_q$, $n=1,2,\ldots$ be a Cauchy sequence in S_q. Let $x_n^o(t)$ be the restriction of $x_n(t)$ to $[0,\pi/2\omega]$. Since $L_2[0,\pi/2\omega]$, $n=1,2,\ldots$ is complete the sequence $\{x_n^o(t)\}$, $x_n^o(t)\epsilon L_2[0,\pi/2\omega]$, $n=1,2,\ldots$ converges to a square integrable function $x^o(t)$, $t\epsilon[0,\pi/2\omega]$. We extend $x^o(t)$ to the whole real line by defining

$$x(t) = x^0(t) \text{ for } t\epsilon[0,\pi/2\omega] \text{ and } x(0) = 0 ,$$

$$x(\tfrac{\pi}{2\omega} + t) = x(\tfrac{\pi}{2\omega} - t) \text{ for } t\epsilon[0,3\pi/2\omega] ,$$

$$x(-t) = -x(t) \text{ for } t\epsilon[0,2\pi/\omega] ,$$

$$x(\tau) = x(t) \text{ for all real } \tau \equiv t \pmod{\tfrac{2\pi}{\omega}}$$

and $t\epsilon[0,2\pi/\omega]$.

Now, $x(t)$ is L-integrable with $x^2(t)$ in $[0,2\pi/\omega]$ and since

$$\int_0^{\pi/2\omega} [x_n(t) - x(t)]^2 dt = \int_0^{\pi/2\omega} [x_n^0(t) - x^0(t)]^2 dt ,$$

the norm $\mu[x_n(t)-x(t)]$ tends to zero as $n\to\infty$. Since $x_n(\frac{T}{4} - t) = x_n(\frac{T}{4} + t)$ for $t\epsilon[0,T/2]$ we have $x^0(\frac{T}{4} - t) = x^0(\frac{T}{4} + t)$ almost everywhere for $t\epsilon[0,T/2]$ and so $x(\frac{T}{4} - t) = x(\frac{T}{4} + t)$ a.e., for $t\epsilon[0,T/2]$. If $t\epsilon[T/2,T]$ we have

$$x(\tfrac{T}{4} + t) = x(\tfrac{T}{4} + t - T) = x(t - \tfrac{3T}{4}) = -x(\tfrac{3T}{4} - t)$$

$$= -x[\tfrac{T}{4} - (t - \tfrac{T}{2})] = -x[\tfrac{T}{4} + (t - \tfrac{T}{2})]$$

$$= x[-\tfrac{T}{4} - t + \tfrac{T}{2}] = x(\tfrac{T}{4} - t) \text{ a.e.}$$

If $\tau \equiv t \pmod{T}$, τ a real number and $t\epsilon[0,T]$ we have $t = \tau - kT$ for some integer k. Then

$$x(\tfrac{T}{4} - \tau) = x(\tfrac{T}{4} - t) = x(\tfrac{T}{4} + t) = x(\tfrac{T}{4} + t + kT) = x(\tfrac{T}{4} + \tau) .$$

Finally, for τ as above

$$x(-\tau) = x(-\tau + kT) = x(-t) = -x(t) = -x(\tau - kT) = -x(\tau) .$$

Hence $x(t)\epsilon S_q$.

Definition 5.3. If $x(t)\epsilon S_q$ has the Fourier series (5.30) we define the operator P by

$$Px = c_1 \sin \omega t .$$

Lemma 5.7. P is an idempotent linear bounded operator.

Proof. P is obviously idempotent and linear. It is continuous since, for any $x\epsilon S_q$,

$$\mu(Px) = \{\tfrac{1}{2}(c_1^2)\}^{1/2} \leq \{\tfrac{1}{2}(c_1^2+c_3^2+\ldots)\}^{1/2} = \mu(x) . \qquad (5.31)$$

Definition 5.4. If $x(t)\epsilon S_q$ has the Fourier series (5.30) we define $H:S_q \to S_q$ by

$$Hx = -\frac{c_1}{\omega^2}\sin \omega t - \frac{c_3}{9\omega^2}\sin 3\omega t - \ldots . \qquad (5.32)$$

Lemma 5.8. H is a bounded linear operator in S_q and commutes with P.

Proof. If $x(t) \sim c_1 \sin \omega t + c_3 \sin 3\omega t + \ldots$

then

$$Hx = -\frac{c_1}{\omega^2}\sin \omega t - \frac{c_3}{9\omega^2}\sin 3\omega t - \ldots$$

and

$$\mu(Hx) = \left\{\tfrac{1}{2}\left(\frac{c_1^2}{\omega^4} + \frac{c_3^2}{9^2\omega^4} + \ldots\right)\right\}^{1/2} \leq \frac{1}{\omega^2}\left(1 + \frac{1}{9^2} + \frac{1}{25^2} + \ldots\right)^{1/2}$$

$$\left[\tfrac{1}{2}(c_1^2+c_3^2+\ldots)\right]^{1/2} = \frac{1}{\omega^2}\frac{\pi^2}{96}\mu(x) = 1.0073\,\omega^{-2}\mu(x)$$

and so H is bounded. Also $\mu(Hx) < +\infty$ for $x\epsilon S_q$, and hence $H:S_q \to S_q$ as stated in the definition of H. A simple verification shows that P and

H commute, and H is obviously linear.

Lemma 5.9. If I denotes the identity operator in S_q then H(I-P) is a linear bounded operator in S_q.

Proof. Linearity is obvious. Now, from (5.32),

$$\mu H(I-P)x = \mu\left[-\frac{c_3}{9\omega^2}\sin 3\omega t - \frac{c_5}{25\omega^2}\sin 5\omega t - \ldots\right]$$

$$= \left\{\frac{1}{2}\left(\frac{c_3^2}{9^2\omega^4} + \frac{c_5^2}{25^2\omega^4} + \ldots\right)\right\}^{1/2} \leq \frac{1}{9\omega^2}\mu(x) ,$$

and so H(I-P) is bounded.

We denote by k_q the (square) norm of H(I-P). Thus $k_q \leq 1/9\omega^2$. On the other hand, for $x(t) = c_3 \sin 3\omega t$, we have

$$H(I-P)x(t) = -\frac{c_3}{9\omega^2}\sin 3\omega t$$

$$\mu H(I-P)x(t) = \left[\frac{1}{2}\left(\frac{c_3}{9\omega^2}\right)^2\right]^{1/2} = \frac{1}{9\omega^2}\left[\frac{1}{2}c_3^2\right]^{1/2}$$

$$= \frac{1}{9\omega^2}\mu x(t) .$$

Thus $k_q \geq \frac{1}{9\omega^2}$, and finally

$$k_q = \frac{1}{9\omega^2} = 0.111\ldots\omega^{-2} .$$

Lemma 5.10. If $x(t)\in S_q$, and $y(t) = Hx(t)$, then $y(t)\in S_q$, y(t) is A.C., y'(t) is A.C. and y"=x.

Proof. It is a consequence of (5.7).

Lemma 5.11. If $x(t)\in S_q$ is A.C. with x'(t), if x"(t) is L^2-integrable in $[0,2\pi/\omega]$ then H(I-P)x" = (I-P)x.

Proof. It is the same as in Lemma 5.6.

Lemma 5.12. If $x(t) \in S_q$ then there is a number k' such that

$$\|H(I-P)x\| \leq k'\mu(x) .$$

Proof. Indeed, if $x(t) \sim c_1 \sin \omega t + c_3 \sin 3\omega t + \ldots$ then by using Schwartz's inequality we get

$$\|H(I-P)x\| \leq \frac{1}{\omega^2} \left|\frac{c_3}{9} + \frac{c_5}{25} + \ldots\right|$$

$$\leq \frac{1}{\omega^2} \left|\frac{1}{2}(c_3^2 + c_5^2 + \ldots)\right|^{1/2} \left|\frac{1}{2}\left(\frac{1}{9^2} + \frac{1}{25^2} + \ldots\right)\right|^{1/2} .$$

Since $\left|\frac{1}{2}(c_3^2 + c_5^2 + \ldots)\right|^{1/2} \leq \left|\frac{1}{2}(c_1^2 + c_3^2 + \ldots)\right|^{1/2} = \mu(x)$, the lemma is proved if we put $k' = \frac{1}{\omega^2}\left|\frac{1}{2}\left(\frac{1}{9^2} + \frac{1}{25^2} + \ldots\right)\right|^{1/2}$. We have then

$$k' = 0.1694\, \omega^{-2} .$$

6. THE DIFFERENTIAL EQUATION $\ddot{x} + g(x,t) = 0$

6.1. THE FIRST GALERKIN'S APPROXIMATION

We shall now proceed to the study of periodic solutions of the second order differential equation

$$\ddot{x} + g(x,t) = 0. \tag{6.1}$$

We shall (a) prove the existence of periodic solutions, (b) we shall give sinusoidal approximations of these solutions and (c) we shall give error bounds for these sinusoidal approximations.

It will be assumed that $g(x,t)$ is a real function of the real variables x and t, defined for all $-A \leq x \leq A$ and $-\infty < t < +\infty$, that, for every $|x| \leq A$, $g(x,t)$ is periodic in t, of period $T = 2\pi/\omega > 0$, and integrable in $[0,T]$ for every $|x| \leq A$. Here T may not be the minimum period, but it is assumed to be independent of x.

The function $g(x,t)$ always admit of the decomposition

$$g(x,t) = g_1(x) + g_2(x,t), \tag{6.2}$$

where, for instance, $g_1(x)$ could be the average of g with respect to t, or, what is the same, the constant term in the development of $g(x,t)$, as a function of t, in Fourier series. Under these conditions we would have

$$g_1(x) = \frac{2}{T}\int_0^T g(x,t)dt, \quad \frac{2}{T}\int_0^T g_2(x,t)dt = 0.$$

If $g(x,t)$ does not depend on t, then (6.1) reduces to an antonomous equation

$$\ddot{x} + g(x) = 0, \tag{6.3}$$

and then T is undetermined. We have already seen in Section 4 that the periodic solutions $x(t)$ of (6.3) have period $T = 2\pi/\omega$ depending upon the amplitude. We also have seen that, for $g(x)$ odd in x, $x(t)$ has the following symmetries

$$x(-t) = -x(t), \quad x(\tfrac{T}{4} + t) = x(\tfrac{T}{4} - t), \quad x(T+t) = x(t). \tag{6.4}$$

As a consequence $x(t)$ has the further symmetries

$$x(\tfrac{3T}{4} + t) = x(\tfrac{3T}{4} - t), \quad x(\tfrac{T}{2} + t) = -x(\tfrac{T}{2} - t),$$

and the solutions $x(t)$ have Fourier series of the form

$$x(t) \sim c_1 \sin \omega t + c_3 \sin 3\omega t + \ldots .$$

To simplify the present discussion we shall assume that $g(x,t)$ satisfies the same symmetry conditions:

1.

$$\begin{aligned} g(-x,-t) &= -g(x,t) \\ g(x,\tfrac{T}{4} + t) &= g(x,\tfrac{T}{4} - t), \\ g(x,T+t) &= g(x,t), \end{aligned} \tag{6.5}$$

for all t and all $|x| \leq A$. We shall also assume that $g(x,t)$ has a de-

composition of the form (6.2), which we write as

$$g(x,t) = g_0(x) + R(x,t) + S(t), \tag{6.6}$$

where $S(t)$ has a Fourier series $A_1 \sin \omega t + A_3 \sin 3\omega t + \ldots$ with $A_1, A_3, \ldots$ constants. Also we assume

2.

$$g_0(x) = \sigma^2 x + \frac{\alpha_1}{3!} x^3 + \ldots + \frac{\alpha_{k-1}}{(2k-1)!} x^{2k-1}, \tag{6.7}$$

for some constants k, $\sigma \geq 0$, $\alpha_1, \ldots, \alpha_{k-1}$, and all $|x| \leq A$. In addition, if (6.1) is autonomous we assume that $g_0(x) > 0$ for all $x \in (0,A]$.

3. For some fixed number L, for all $x_1, x_2 \in [-A,A]$ and for all t we have

$$|g(x_1,t) - g(x_2,t)| \leq L|x_1 - x_2|.$$

We shall need hypotheses of continuity and measurability on g. The discussion below is divided into two parts where we use respectively a uniform norm and a square norm. We make correspondingly two different hypotheses for the two parts:

4a. The functions $g(x,t)$, $R(x,t)$, $S(t)$ are continuous in (x,t) for all (x,t), $|x| \leq A$, $-\infty < t < +\infty$,

$$|R(x,t)| \leq \frac{\alpha_\ell}{(2\ell+1)!} |x|^{2\ell+1}, \tag{6.8}$$

for all $|x| \leq A$, $-\infty < t < +\infty$, and some constant ℓ and $\alpha_\ell \geq 0$.

4b. The functions $g(x,t)$, $R(x,t)$ are measurable in (x,t) for $|x| \leq A$, $-\infty < t < +\infty$, $S(t)$ is L^2-integrable in $(-\infty,+\infty)$ and there is some function $\lambda(t)$, of t, periodic in t, with period $T = 2\pi/\omega$ and L-intergrable on $[0.T]$ such that

$$|R(x,t)| \leq \frac{\alpha_\ell}{(2\ell+1)!} |x|^{2\ell+1}\lambda(t), \quad \frac{1}{T}\int_0^T \lambda^2(t)dt = 1, \quad (6.9)$$

for some constants ℓ and $\alpha_\ell \geq 0$. Under condition 2 we do not exclude the case where the polynomial is replaced by a power series in x of the same form, convergent in $[-A,A]$, that is, the McLaurin series of the (analytic) function $g_o(x)$.

In the autonomous case, the decomposition (6.6) reduces to $g(x) = g_o(x) + R(x)$, hypothesis 1 reduces to $g(-x) = -g(x)$, hypothesis 2 remains the same, hypothesis 3 reduces to $|g(x_1)-g(x_2)| \leq L|x_1-x_2|$ for $x_1,x_2 \epsilon [-A,A]$, hypothesis 4 reduces to the single assumption

$$|R(x)| \leq \frac{\alpha_\ell}{(2\ell+1)!} |x|^{2\ell+1},$$

where $R(x)$ is assumed to be continuous under hypothesis 4a and only measurable under hypothesis 4b.

We shall consider approximate solutions of (6.1) of the form $x = a \sin \omega t$, a constant. Note that here we suppose $T = 2\pi/\omega$ as given, and hence we may be able to determine a in order to satisfy the Galerkin equation as we shall see below. The approximate solution so determined will then correspond to a harmonic solution of Eq. (6.1) or actually to a subharmonic solution of (6.1) if T is not the minimum period of

$g(x,t)$. If g does not depend on t, that is, (6.1) is autonomous, then both a and ω are undetermined, and we may be able to determine ω as a function of a in order to satisfy the Galerkin equation.

In general $a \sin \omega t$ is not an exact solution of the Eq. (6.1). If $|a| \leq A$, then the expression

$$\theta(a,t) = -a\omega^2 \sin \omega t + g(a \sin \omega t, t) \tag{6.10}$$

is defined for all $-\infty < t < +\infty$, but not necessarily identically zero for any value of $a \neq 0$, and $\omega > 0$. On the other hand, by (6.5) we have

$$\begin{aligned} \theta(a,T+t) &= \theta(a,t) \\ \theta(-a,-t) &= -\theta(a,t), \\ \theta(a, \tfrac{T}{4} + t) &= \theta(a, \tfrac{T}{4} - t), \end{aligned} \tag{6.11}$$

and $\theta(a,t)$ is obviously L-integrable in $[0,T]$. Precisely, $\theta(a,t)$ is a continuous function of a and t under hypothesis 4a and is L^2-integrable in t for every $|a| \leq A$ under hypothesis 4b. From (6.11) we have

$$\theta(a,t) \sim c_1 \sin \omega t + c_3 \sin 3\omega t + \ldots$$

where

$$c_1 = \frac{2}{T} \int_0^T \theta(a,t) \sin \omega t dt = \frac{2}{T} \int_0^T [-a\omega^2 \sin \omega t + g(a \sin \omega t, t)] \sin \omega t dt.$$

According to Galerkin's general approach we may request that the first coefficient c_1 is zero, that is, the Galerkin equation for the first approximation under study is

$$u(a) \equiv \frac{2}{T}\int_0^T [-a\omega^2 \sin \omega t + g(a \sin \omega t, t)] \sin \omega t dt = 0. \quad (6.12)$$

By using the decompositon (6.6) we have

$$u(a) \equiv \frac{2}{T}\int_0^T - a\omega^2 \sin^2 \omega t dt + \frac{2}{T}\int_0^T g_o(a \sin \omega t) \sin \omega t dt$$

$$+ \frac{2}{T}\int_0^T [A_1 \sin \omega t + A_3 \sin 3\omega t + \ldots] \sin \omega t dt$$

$$+ \frac{2}{T}\int_0^T R(a \sin \omega t, t) \sin \omega t dt,$$

and thus u(a) decomposes into

$$u(a) = - a\omega^2 + \Sigma + A_1 + \rho, \quad (6.13)$$

where

$$\Sigma = \frac{2}{T}\int_0^T g_o(a \sin \omega t) \sin \omega t dt,$$

$$\rho = \frac{2}{T}\int_0^T R(a \sin \omega t, t) \sin \omega t dt,$$

and A_1 is a constant. Thus Galerkin's equation (6.12) becomes

$$- a\omega^2 + \Sigma + A_1 + \rho = 0 \quad (6.14)$$

For the sake of simplicity we may prefer to take for a the value of a which satisfies, instead of (6.12) or (6.14), the simpler equation

$$- a\omega^2 + \Sigma + A_1 = 0, \tag{6.15}$$

or

$$- a\omega^2 + A_1 + \frac{2}{T}\int_0^T g_0(a \sin \omega t, t) \sin \omega t dt = 0.$$

If we use the expression (6.7) for g_0 we have

$$g(a \sin \omega t, t) = \sigma^2 a \sin \omega t + \frac{\alpha_1}{3!} a^3 \sin^3 \omega t + \ldots$$

$$+ \frac{\alpha_{k-1}}{(2k-1)!} a^{2k-1} \sin^{2k-1} \omega t + R_0(t) + S(t),$$

where

$$|R_0(t)| = |R(a \sin \omega t, t)| \leq \frac{\alpha_\ell}{(2\ell+1)!} |a \sin \omega t|^{2\ell+1} \leq \frac{\alpha_\ell}{(2\ell+1)!} |a|^{2\ell+1}.$$

By Moivre's inverse formulas

$$\sin^3 u = -\frac{1}{2^2}(\sin 3u - 3 \sin u),$$

$$\sin^5 u = \frac{1}{2^4}(\sin 5u - 5 \sin 3u + 10 \sin u),$$

$$\sin^7 u = -\frac{1}{2^6}(\sin 7u - 7 \sin 5u + 21 \sin 3u - 35 \sin u),$$

$$\vdots$$

$$\sin^{2k-1} u = (-1)^{k-1} \frac{1}{2^{2k-2}} \left[\sin (2k-1)u - (2k-1) \sin (2k-3)u \right.$$

$$\left. + \frac{(2k-1)(2l-2)}{2!} \sin (2k-5)u + (-1)^{k-1} \frac{(2k-1)!}{(k-1)!k!} \sin u \right],$$

we deduce

$$g(a \sin \omega t, t) = \sigma^2 a \sin \omega t + \frac{\alpha_1}{3!}\frac{a^3}{2^2}[3 \sin \omega t - \sin 3\omega t]$$

$$+ \frac{\alpha_2}{5!}\frac{a^5}{2^4}[10 \sin \omega t - 5 \sin 3\omega t + \sin 5\omega t]$$

$$+ \frac{\alpha_3}{7!}\frac{a^7}{2^6}[35 \sin \omega t - 21 \sin 3\omega t + 7 \sin 5\omega t - \sin 7\omega t] + \ldots$$

$$+ (-1)^{k-1}\frac{\alpha_{k-1}}{(2k-1)!}\frac{a^{2k-1}}{2^{2k-2}}\left[(-1)^{k-1}\frac{(2k-1)!}{k!(k-1)!}\sin \omega t\right.$$

$$+ (-1)^{k-2}\frac{(2k-1)!}{(k+1)!(k-2)!}\sin 3\omega t + \ldots$$

$$+ \frac{(2k-1)(2k-2)}{2!}\sin (2k-5)\sin \omega t - (2k-1)\sin (2k-3)\omega t$$

$$\left. + \sin (2k-1)\omega t\right] + R_0(t) + S(t).$$

From this expression we obtain

$$g(a \sin \omega t, t)$$

$$= \left\{\sigma^2 a + \frac{\alpha_1}{3!}\frac{a^3}{2^2}\frac{3!}{1!2!} + \frac{\alpha_2}{5!}\frac{a^5}{2^4}\frac{5!}{2!3!} + \ldots + \frac{\alpha_{k-1}}{(2k-1)!}\frac{a^{2k-1}}{2^{2k-2}}\frac{(2k-1)!}{k!(k-1)!}\right\}\sin \omega t$$

$$- \left\{\frac{\alpha_1}{3!}\frac{a^3}{2^2} + \frac{\alpha_2}{5!}\frac{a^5}{2^4}5 + \frac{\alpha_3}{7!}\frac{a^7}{2^6}21 + \ldots + \frac{\alpha_{k-1}}{(2k-1)!}\frac{a^{2k-1}}{2^{2k-2}}\frac{(2k-1)!}{(k+1)!(k-2)!}\right\}\sin 3\omega t$$

$$+ \ldots$$

$$+ (-1)^{k-1}\frac{\alpha_{k-1}}{(2k-1)!}\frac{a^{2k-1}}{2^{2k-2}}\sin (2k-1)\omega t + R_0(t) + S(t)$$

$$= (S_1+A_1)\sin \omega t + (S_3+A_3)\sin 3\omega t + \ldots + (S_{2k-1}+A_{2k-1})\sin (2k-1)\omega t$$

$$+ R_0(t) + \sum_{r=k}^{\infty} A_{2r+1}\sin (2r+1)\omega t, \qquad (6.16)$$

where $|R_0(t)| \leq \alpha_\ell/(2\ell+1)!\,|a|^{2\ell+1}$ and $S_1, S_3, \ldots, S_{2k-1}$ are the coefficients of the harmonics in (6.16). Note that the polynomial $S_1(a)$ has the same expression as Σ in (6.13), that is

$$S_1(a) = \Sigma = \frac{2}{T}\int_0^T g_0(a \sin \omega t) \sin \omega t dt.$$

Thus, the Galerkin equation (6.15) reduces to

$$-a\omega^2 + A_1 + \sigma^2 a + \frac{\alpha_1}{3!}\frac{a^3}{2}\frac{3!}{1!2!} + \ldots + \frac{\alpha_{k-1}}{(2k-1)!}\frac{a^{2k-1}}{2^{2k-2}}\frac{(2k-1)!}{(k-1)!k!} = 0 \qquad (6.17)$$

In the non-autonomous case, ω is fixed and this relation represents an equation in the unknown a. We shall assume that Eq. (6.12) has a root a in the interval $[-A,A]$. Then, when we use property 4a,b, for the sake of simplicity we assume as a first Galerkin approximation to the periodic solution of (6.1) the function

$$x_0 = x_0(t) = a \sin \omega t,$$

where a is any of the roots of Eq. (6.17) in $[-A,A]$.

In the autonomous case, both ω and a are undetermined, and therefore we may prefer to think of relation (6.17) as defining ω as a function of a. We shall then write (6.17) as

$$\omega^2 = \sigma^2 + \frac{\alpha_1}{1!2!}\frac{a^2}{2^2} + \frac{\alpha_2}{2!3!}\frac{a^4}{2^4} + \ldots + \frac{\alpha_{k-1}}{(k-1)!k!}\frac{a^{2k-1}}{2^{2k-2}}. \qquad (6.18)$$

Note that $A_1 = 0$ in the autonomous case.

We shall prove next that, by restricting a, if necessary, to be in an interval $[-A_0,A_0]$, with $A_0 \leq A$, the second member of (6.18) is

actually positive and thus (6.18) defines $\omega > 0$ as a single valued function of a for $-A_o \leq a \leq A_o$.

If $\sigma > 0$ the statement is obviously true.

If $\sigma = 0$ then not all real numbers $\alpha_1, \alpha_2, \ldots, \alpha_{k-1}$ are equal to zero and we denote by α_q, $1 \leq q \leq k-1$, the first one which is not zero. Then $\alpha_1 = \ldots = \alpha_{q-1} = 0$. From hypothesis 4a, (Eq. (6.8) for the autonomous case) we have

$$|g(x) - (b_q x^{2q+1} + \ldots + b_{k-1} x^{2k-1})| \leq b_\ell |x|^{2\ell+1},$$

where $b_j = \alpha_j/(2j+1)$, for all $|x| < A$ and, in particular, for $0 < x < A_o$. Now we have for $0 < x \leq A$

$$(g(x) - b_q x^{2q+1}) - (b_{q+1} x^{2k+3} + \ldots + b_{k-1} x^{2k-1}) = \theta b_\ell |x|^{2\ell+1},$$

for some θ, $-1 \leq \theta \leq 1$, or

$$|g(x) - b_q x^{2k+1}| \leq |\theta| b_\ell |x|^{2\ell+1} + |b_{q+1} x^{2q+3} + \ldots + b_{k-1} x^{2k-1}|.$$

If $m = \max [|b_{q+1}| b_q^{-1}, \ldots, |b_{k-1}| b_q^{-1}, b_\ell b_q^{-1}]$ then we have

$$|g(x) - b_q x^{2q+1}| \leq m|b_q||x|^{2q+3} + \ldots + m|b_q||x|^{2k-1} + m|b_q||x|^{2\ell+1}$$

$$\leq m|b_q|\{|x|^{2q+3} + \ldots + |x|^{2k-1} + |x|^{2\ell+1}\}$$

$$\leq m|b_q||x|^{2q+3}\{1 + A^2 + \ldots + A^{2(k-q-2)} + A^{2(\ell-q-1)}\}$$

$$\leq m|b_q| D |x|^{2q+3},$$

where D is the sum in brackets. This inequality implies that

$$b_q x^{2q+1} - m|b_q|D|x|^{2q+3} \leq g(x) \leq b_q x^{2q+1}(1+mD|x|^2 \operatorname{sign} b_q) \quad (6.19)$$

for all $x\epsilon(0,A]$. If we take any B, $0 \leq B \leq A_o$, $B < (\mu D)^{-1/2}$ then for $x\epsilon(0,B]$ we have $mD|x|^2 \leq 1$ and (6.19) says that g(x) has the same sign of $b_q x^{2q+1}$ i.e., the sign of b_q. Since $g(x) > 0$ for $x\epsilon(0,A]$ we conclude that $b_q > 0$. Thus, the first coefficient α_i which is not zero is positive and so, for some $A_o\epsilon(0,A]$ the expression (6.18) for ω^2 is positive. We shall choose a such that this is the case. Thus, in the autonomous case, when we use property 3, for the sake of simplicity, we assume as a first Galerkin approximation to the periodic solution of (6.1) the function

$$x_o = x_o(t) = a \sin \omega(a)t,$$

where $a\epsilon(0,A_o]$ and $\omega = \omega(a) > 0$ is the function of a defined by (6.18).

6.2. AN EXISTENCE THEOREM WITH THE USE OF THE UNIFORM NORM

Let us consider the space S_c, discussed in Section 5.1 of all functions x(t) which are continuous in $(-\infty,\infty)$ and satisfy the symmetry conditions (5.1), or

$$x(-t) = -x(t), \quad x(\tfrac{T}{4} + t) = x(\tfrac{T}{4} - t), \quad x(T+t) = x(t).$$

Let $\bar{S}_c$ be the subset of all elements $x(t)\epsilon S_c$ such that $|x(t)| \leq A$.

Let g denote the operator defined by taking

$$gx = g[x(t),t], \quad x\epsilon\bar{S}_c.$$

By use of (6.5) we have

$$g[x(t+T),t+T] = g[x(t),t],$$

$$g[x(-t),-t] = -g[x(t),t],$$

$$g[x(\tfrac{T}{4} - t), \tfrac{T}{4} - t] = g[x(\tfrac{T}{4} + t),t],$$

and thus gx is a function of t satisfying symmetry relations (5.1). By hypothesis 4a then gx is a continuous function of t in $(-\infty,\infty)$. Thus gx is an element of S_c for every $x(t)\epsilon\bar{S}_c$, that is, $g:\bar{S}_c \to S_c$.

Lemma 6.1. $g:\bar{S}_c \to S_c$ is a continuous operator in $\bar{S}_c$, in the sense that $\|gx^1-gx^2\| \leq L\|x^1-x^2\|$ for all $x^1 = x^1(t)$, $x^2 = x^2(t)$ in $\bar{S}_c$.

Proof. We have $\|gx^1-gx^2\| = \max_t|g(x^1(t),t)-g(x^2(t),t)| \leq \max_t L|x^1(t)-x^2(t)|$ $= L\|x^1-x^2\|$.

Definition 6.1. We define the operators f, F, and T by taking

$$\begin{aligned} fx &= (I-P)gx, \\ Fx &= -Hfx = -H(I-P)gx \qquad (6.20) \\ Tx &= Px + Fx = Px - Hfx = Px - H(I-P)gx, \end{aligned}$$

for all $x\epsilon\bar{S}_c$. Clearly f, F and T map $\bar{S}_c$ into S_c.

Lemma 6.2. The operators f, F and T are continuous operators in $\bar{S}_c$.

Proof. A consequence of Lemmas 5.2, 5.3 and Eq. (6.20).

Let us denote by b an undetermined number in a neighborhood of a. In harmony with (6.10) let us denote by $\Theta(b,t)$ the function of t that we obtain when x is replaced by $x^*(t) = b \sin \omega t$ in the expression $\ddot{x}+g(x,t)$. Then

$$\Theta(b,t) = - b\omega^2 \sin \omega t + g(b \sin \omega t,t), \tag{6.21}$$

and $\Theta(b,t) \epsilon S_c$. Also we shall denote by $\Delta(b,t)$ the function

$$\Delta(b,t) = H(I-P)\Theta(b,t). \tag{6.22}$$

Then we have

$$\ddot{x}^* + g[x^*(t),t] \equiv \Theta(b,t), \tag{6.23}$$

$$H(I-P)\ddot{x}^* + H(I-P)g[x^*(t),t] = \Delta(b,t). \tag{6.24}$$

Since, by lemma 5.6

$$H(I-P)\ddot{x}^* = (I-P)x^* = x^* - Px^*,$$

we have from (6.24)

$$x^* = Px^* - H(I-P)g[x^*(t),t] + \Delta(b,t).$$

In particular, for b = a, we have

$$x_o = Px_o - H(I-P)g[x_o(t),t] + \Delta(a,t). \tag{6.25}$$

This relation will be used later.

Let

$$\gamma_c = \|\Delta(a,t)\| = \max_t |\Delta(a,t)|,$$

where $|a| \leq A$.

Let c,d, be positive constants, $0 < c < d$, let b be any number with $|b-a| \leq c$, $|b| \leq A$.

Let V_c be the set of all elements $x^* = x^*(t)$ of S_c of the form $x^*(t) = b \sin \omega t$, with b as above. In other words

$$V_c = \{x^* | x^* \epsilon S_c, x^* = Px^*, \|P(x^*-x_o)\| \leq c\}.$$

Indeed

$$\|P(x^*-x_o)\| = \max|(b-a) \sin \omega t| = |b - a| \leq c.$$

For every $x^* \epsilon V_c$ let S^*_{co} denote the subset of S_c defined by

$$S^*_{co} = \{x | x \epsilon S, Px = Px^*, \|x-x_o\| \leq d\}.$$

We have to make sure that $x^* \epsilon S^*_{co}$. Indeed $Px^* = Px^*$ and

$$\|x^* - x_o\| = \max|(b-a) \sin \omega t| = |b - a| \leq c < d.$$

Thus $x^* \epsilon S^*_{co}$ and hence S^*_{co} is not empty. Also $S^*_{co} \subset \bar{S}_c$ since

$$\max|x(t)| = \|x\| \leq \|x_o\| + \|x - x_o\| \leq a + d$$

all we need is that $a+d \leq A_o$. Thus the constants a,c,d must be taken in such a way that $0 < c < d$, $a+d \leq A_o$, and then we may taken for b

any real number with $|b-a| \leq c$. Then, for every $x^* \epsilon V$, the set S^*_{co} is well defined, is not empty, and is a subset of $\overline{S}_c$. We shall denote by k_c the number already introduced under lemma 5.4.

Theorem 6.1. If c,d, are positive constants with $0 < c < d$, $a+d \leq A_o$, if b is any real number with $|b-a| \leq c$, and $x^* = b \sin \omega t$, if $c+\gamma_c \leq (1-k_cL)d$, $k_cL<1$ then T/S^*_{co} is a contraction and maps S^*_{co} into S^*_{co}.

Proof. If $x \epsilon S^*_{co}$ and $y = Tx$, we have

$$y = Tx = Px - H(I-P)gx$$

hence

$$Py = PPx - PH(I-P)gx$$

and by lemmas 5.2 and 5.4

$$Py = Px + 0 = Px^*$$

Also we have, by comparison with (6.25) and by previous lemmas,

$$y = Px - H(I-P)gx,$$

$$x_o = Px_o - H(I-P)gx_o + \Delta(t,a),$$

$$y-x_o = P(x-x_o) - H(I-P)(gx-gx_o) - \Delta(t,a),$$

$$\|y - x_o\| \leq \|P(x-x_o)\| + \|H(I-P)(gx-gx_o)\| + \|\Delta(t,a)\|,$$

$$\|y - x_o\| \leq \|P(x^*-x_o)\| + \|H(I-P)(gx-gx_o)\| + \|\Delta(t,a)\|,$$

$$\|y - x_o\| \leq c + k_cL\|x - x_o\| + \gamma_c,$$

$$\|y - x_o\| \leq c + k_cLd + \gamma_c \leq (1-k_cL)d + k_cLd = d.$$

Thus T/S^*_{co} maps S^*_{co} into S^*_{co}. For any two elements $x_1, x_2 \in S^*_{co}$ we have now

$$y_i = Tx_i, \quad i = 1,2,$$

and

$$y_1 = Px_1 - H(I-P)gx_1,$$

$$y_2 = Px_2 - H(I-P)gx_2,$$

where $Px_1 = Px_2 = Px^*$. By comparison we have

$$y_1 - y_2 = H(I-P)(gx_1 - gx_2),$$

$$\|y_1 - y_2\| = \|H(I-P)(gx_1 - gx_2)\|$$

$$\leq \|H(I-P)\| \|gx_1 - gx_2\|$$

$$\leq k_c L \|x_1 - x_2\|,$$

where $k_c L < 1$. This shows that T/S^*_{co} is a contraction.

Lemma 6.3. The subset S^*_{co} of S_c is a complete metric space.

Proof. Let $\{x_n\}$ be a Cauchy sequence of elements of S^*_{co}. Then the limit x exists and is an element of S_c. On the other hand, P is a continuous operator, hence $Px_n = Px^*$ implies $Px = Px^*$. Also we have $\|x - x_o\| \leq \|x - x_n\| + \|x_n - x_o\|$ where $\|x-x_n\| \to 0$ and $\|x_n - x_o\| \leq d$. Hence $\|x-x_o\| \leq d$. This proves that the limit element x is in S^*_{co} and so S^*_{co} is a complete, metric, (nonlinear) subspace of S_c.

As a consequence of the two previous results and Banach's fixed

point theorem we have the following corollary.

Corollary. Under the conditions of Theorem 6.1, $T|_{S^*_{co}}$ has a unique fixed element $y = Ty \in S^*_{co}$.

Let y be the fixed point of $T|_{S^*_{co}}$. Then

$$y = Py - H(I-P)gy.$$

Thus y is the sum of an expression b sin ωt, and of a function $\psi(t) = H(I-P)g[y(t),t]$ which is of class C^2. Then y(t) is of class C^2. By differentiation we have

$$\ddot{y} = P\ddot{y} - (I-P)gy,$$

$$\ddot{y} + g[y(t),t] = P[\ddot{y}+gy],$$

or

$$\ddot{y} + g[y(t),t] = D,$$

where D has the simple form

$$D = P[\ddot{y}+gy] = U \sin \omega t. \tag{6.26}$$

Also y is a solution of (6.1) if and only if D = 0, that is, U = 0. We denote this equation as the determining equation of (6.1). Since $x^* = b \sin \omega t$, $Px^* = Py$ and y is determined by x^* we have

$$y = b \sin \omega t + b_3 \sin 3\omega t + b_5 \sin 5\omega t + \ldots,$$

where $b_3, b_5, \ldots$ are functions of b, and also

$$g[y(t),t] = \beta_1 \sin \omega t + \beta_3 \sin 3\omega t + \ldots$$

where again $\beta_1,\beta_3,B_5,\ldots$ are functions of b. Then, from Eq. (6.26)

$$U = U(b) = \frac{2}{T}\int_0^T \{y''+g[y(t),t]\} \sin \omega t dt = -\omega^2 b+\beta_1 = -\omega^2 b+\beta_1(b) \tag{6.27}$$

We have to prove that there exists some b with $a-c \leq b \leq a+c$ such that $U(b) = 0$.

Let $g[x^*(t),t] = \beta_{1o} \sin \omega t+\beta_{3o} \sin 3\omega t+\ldots$ and note that by Eq. (6.12)

$$u = u(b) = \frac{2}{T}\int_0^T \{x^{*\prime\prime}+g[x^*(t),t]\} \sin \omega t dt = -\omega^2 b + \beta_{1o} = -\omega^2 b + \beta_{1o}(b)$$

and

$$|U(b) - u(b)| = |\beta_1 - \beta_{1o}| = \|P\{g[y(t),t] - g[x^*(t),t]\}\|,$$

then by lemma 5.2,

$$|U(b) - u(b)| \leq \frac{4}{\pi} \|g[y(t),t]\| \leq \frac{4}{\pi} L\|y(t) - x^*(t)\|. \tag{6.28}$$

We have now

$$y = Ty = Py - H(I-P)gy, \quad Py = Px^* = x^*, \qquad y(t) = x^*(t) = y - Py = - H(I-P)gy. \tag{6.29}$$

We have also from Eq. (6.25)

$$x_o = Px_o - H(I-P)g[x_o(t),t] - \Delta(t,a),$$

where

$$x_o = a \sin \omega t = Px_o.$$

Hence

$$0 = - H(I-P)gx_o - \Delta(t,a),$$

and by comparison with Eq. (6.29)

$$\begin{aligned} y(t) - x^*(t) &= - H(I-P)[gy-gx_o] + \Delta(t,a), \\ \|y(t) - x^*(t)\| &\le H(I-P)[gy-gx_o] + \Delta(t,a)\| \\ &\le \|H(I-P)\|\|gy - gx_o\| + \|\Delta(t,a)\| \\ &\le k_c L\|y - x_o\| + \gamma_c \le k_c Ld + \gamma_c. \end{aligned} \tag{6.30}$$

Thus, from Eq. (6.28)

$$|U(b) - u(b)| \le \frac{4L}{\pi}(k_c Ld+\gamma_c). \tag{6.31}$$

From (6.13) we have now

$$\begin{aligned} u(b) &= - b\omega^2 + S_1(b) + A_1 + \rho \\ &= \left[A_1 - b\omega^2 + \sigma^2 b + \frac{\alpha_1}{1!2!}\frac{b^3}{2^2} + \ldots + \frac{\alpha_{k-1}}{(k-1)!k!}\frac{b^{2k-1}}{2^{2k-2}}\right] \\ &+ \frac{2}{T}\int_0^T R(b \sin \omega t, t) \sin \omega t dt \\ &= u_o(b) + \eta(b), \end{aligned} \tag{6.32}$$

where $u_o(b)$ is the polynomial in b of degree 2k-1 above and $\eta(b)$ is the last expression. For b = a we have $u_o(a) = 0$ and so $u(a) = \eta(a)$.

Now we have

$$|R(b \sin \omega t, t)| \leq \frac{\alpha_\ell}{(2\ell+1)!} |b \sin \omega t|^{2\ell+1}$$

$$|\eta(b)| \leq \frac{2}{T} \int_0^T \frac{\alpha_\ell}{(2\ell+1)!} |b|^{2\ell+1} |\sin \omega t|^{2\ell+2} dt.$$

Hence

$$\begin{aligned} |\eta(b)| &\leq \frac{2}{T} \int_0^T \frac{\alpha_\ell}{(2k+1)!} |b|^{2k+1} |\sin \omega t|^{2k+1} dt \\ &= \frac{2}{T} \frac{\alpha_\ell}{(2\ell+1)!} |b|^{2\ell+1} \int_0^T |\sin \omega t|^{2\ell+1} dt \qquad (6.33) \\ &= \frac{4}{\pi} \frac{\alpha_\ell |b|^{2\ell+1}}{[1.3.....(2\ell+1)]^2} . \end{aligned}$$

We call λ the number $\frac{2}{\pi} \frac{\alpha_\ell |b|^{2\ell+1}}{[1.3.....(2\ell+1)]^2}$. If $R(x,t) \equiv 0$ in the decomposition (6.6) then, of course, $\lambda = 0$.

Now, from (6.32), we have

$$u_o(b) = A_1 + b(\sigma^2-\omega^2) + \frac{\alpha_1}{1!2!} \frac{b^3}{2^2} + \ldots + \frac{\alpha_{k-1}}{(k-1)!k!} \frac{b^{2k-1}}{2^{2k-2}}, \quad k = 2,3,\ldots$$

and by use of (6.17)

$$\begin{aligned} u_o(b) = A_1 - b &\left(\frac{\alpha_1}{1!2!} \frac{a^2}{2^2} + \frac{\alpha_2}{2!3!} \frac{a^4}{2^4} + \ldots + \frac{\alpha_{k-1}}{(k-1)!k!} \frac{a^{2k-2}}{2^{2k-2}} + \frac{A_1}{a} \right) \\ &+ \frac{\alpha_1}{1!2!} \frac{b^3}{2^2} + \ldots + \frac{\alpha_{k-1}}{(k-1)!k!} \frac{b^{2k-1}}{2^{2k-1}}, \quad k = 2,3,\ldots . \end{aligned}$$

For $b = a \pm c$ we have

$$u_o(a\pm c) = \mp c\left(\frac{\alpha_1}{1!2!}\frac{a^2}{2^2} + \frac{\alpha_2}{2!3!}\frac{a^4}{2^4} + \ldots + \frac{\alpha_{k-1}}{(k-1)!k!}\frac{a^{2k-2}}{2^{2k-2}} + \frac{A_1}{a}\right)$$

$$+ \frac{\alpha_1}{1!2!}\frac{(a\pm c)^3 - a^3}{2^2} + \ldots + \frac{\alpha_{k-1}}{(k-1)!k!}\frac{(a\pm c)^{2k-1} - a^{2k-1}}{2^{2k-2}}$$

$$= \mp\left[\frac{\alpha_1}{1!2!}\frac{a^2}{2^2} + \frac{\alpha_2}{2!3!}\frac{a^4}{2^4} + \ldots + \frac{\alpha_{k-1}}{(k-1)!k!}\frac{a^{2k-2}}{2^{2k-2}} + \frac{A_1}{a}\right] c$$

$$+ \left[\binom{3}{2}\frac{\alpha_1 a}{1!2!2^2} + \binom{5}{2}\frac{\alpha_2 a^3}{2!3!2^4} + \binom{7}{2}\frac{\alpha_3 a^5}{3!4!2^6} + \ldots \right.$$

$$\left. + \binom{2k-1}{2}\frac{\alpha_{k-1}a^{2k-3}}{(k-1)!k!2^{2k-2}}\right]c^2 + \left[\binom{3}{3}\frac{\alpha_1}{1!2!2^2} + \binom{5}{3}\frac{\alpha_2 a^2}{2!3!2^4}\right.$$

$$\left. + \binom{7}{3}\frac{\alpha_3 a^4}{3!4!2^6} + \ldots + \binom{2k-1}{3}\frac{\alpha_{k-1}a^{2k-4}}{(k-1)!k!2^{2k-2}}\right]c^3 + \left[\binom{5}{4}\frac{\alpha_2 a}{2!3!2^4}\right.$$

$$\left. + \binom{7}{4}\frac{\alpha_3 a^3}{3!4!2^6} + \ldots + \binom{2k-1}{4}\frac{\alpha_{k-1}a^{2k-5}}{(k-1)!k!2^{2k-2}}\right]c^4 + \ldots$$

$$+ \frac{\alpha_{k-1}}{(k-1)!k!2^{2k-2}}c^{2k-1}. \qquad (6.34)$$

We denote by u_{o1} and u_{o2} the two numbers $u_o(a\pm c)$. We shall assume that u_{o1} and u_{o2} have opposite signs and we denote by Q the minimum of $|u_{o1}|$ and $|u_{o2}|$, that is

$$Q = \min[|u_{o1}|, |u_{o2}|]. \qquad (6.35)$$

Remark. If we assume that a is not only a root of Eq. (6.17), but a simple root, that is, if we assume that the coefficient of $\pm c$ in (6.34) is not zero, that is

$$-\frac{A_1}{a} + \frac{2\alpha_1 a^2}{1!2!2^2} + \frac{4\alpha_2 a^4}{2!3!2^4} + \ldots + \frac{(2k-2)!\,\alpha_{k-1}a^{2k-2}}{(k-1)!k!2^{2k-2}} \neq 0,$$

then u_{o1}, u_{o2} will be necessarily of opposite signs for c sufficiently small.

We have now the elements to prove the following theorem.

Theorem 6.2. Let a and ω satisfy Eq. (6.17),

$$- a\omega^2 + A_1 + \sigma^2 a + \frac{\alpha_1}{3!}\frac{a^3}{2^2}\frac{3!}{1!2!} + \ldots + \frac{\alpha_{k-1}}{(2k-1)!}\frac{a^{2k-1}(2k-1)!}{2^{2k-2}(k-1)!k!} = 0.$$

Assume that c,d are positive constants, with $0 < c < d$, $a+d \leq A_o$, that $c+\gamma_c \leq (1-k_cL)d$, $k_cL < 1$, $\lambda < Q$, and

$$\frac{4}{\pi} L(k_cLd+\gamma_c) < Q - \lambda \tag{6.36}$$

where $Q = \min [|u_o(a-c)|, |u_o(a+c)|]$ and we assume $u_o(a-c)$, $u_o(a+c)$ to be of opposite signs and

$$\gamma_c = \|\Delta(t,a)\|, \quad \lambda = \frac{2\alpha_\ell |b|^{2\ell+1}}{\pi[1,3,\ldots,(2\ell+1)]^2}.$$

Then, for some real number b, $a-c < b < a+c$, and $x_o = a \sin \omega t$, $x^* = b \sin \omega t$, the fixed element $y = Ty$ of the map $T/_{S^*_{co}}$ satisfies the equation

$$\ddot{y} + g(y,t) = 0,$$

with

$$y(t) = b \sin \omega t + b_3 \sin 3\omega t + b_5 \sin 5\omega t + \ldots,$$

$$\|y(t) - x^*(t)\| = \|y - Py\| \leq \min [k_cLd+\gamma_c, c+d],$$

and

$$\|y(t) - x_o(t)\| \leq d.$$

Remark. As pointed out before, in the autonomous case we may consider ω as a function $\omega = \omega(a)$ of the parameter a defined by (6.17). In the periodic case, ω is fixed and a is assumed to be a root of Eq. (6.17).

Proof. We have only to prove that U(b) has opposite signs at the points $b = a \pm c$. Indeed, if so, there is some b, $a-c < b < a+c$, such that $U(b) = 0$, i.e., y(t) satisfies the determining equation, $U = 0$, and hence the equation $\ddot{y}+g(y) = 0$. We have, by hypothesis, either $u_{o1} < 0 < u_{o2}$ or $u_{o1} > 0 > u_{o2}$. We assume that the first case holds. The reasoning is the same in the second case. Then, by the definition of Q

$$u_{o1} < 0 < u_{o2} \tag{6.37}$$

implies

$$u_o(a-c) \leq - Q < 0 < Q \leq u_o(a+c). \tag{6.38}$$

Now, by the definition of u_o and η

$$u(a \pm c) = u_o(a \pm c) + \eta(a \pm c)$$

where $\eta(a \pm c) \leq \lambda$, and from (6.31)

$$|U(a \pm c) - u(a \pm c)| \leq \frac{4L}{\pi} (k_c L d + \gamma_c). \tag{6.39}$$

Since

$$\begin{aligned} U(a\pm c) &= u_o(a\pm c) + [U(a\pm c)-u(a\pm c)] \\ &+ [u(a\pm c)-u_o(a\pm c)-\eta(a\pm c)] + \eta(a\pm c) \end{aligned} \tag{6.40}$$

we have by using (6.33), (6.38) and (6.39)

$$\begin{aligned} U(a+c) &\geq u_o(a+c) + |U(a+c) - u(a+c)| - 0 - |\eta(a+c)| \\ &\geq Q + \frac{4L}{\pi}(k_c Ld+\gamma_c) - \lambda \geq Q - \frac{4L}{\pi}(k_c Ld+\gamma_c) - \lambda. \end{aligned}$$

By hypothesis this last expression is positive, i.e., $U(a+c) > 0$. Again from (6.40) we get

$$U(a-c) \leq u_o(a-c) + |U(a-c) - u(a-c)| + 0 + |\eta(a-c)|,$$

and by use of (6.38)

$$U(a-c) \leq - Q + \frac{4}{\pi} L(k_c Ld+\gamma_c) + \lambda$$

and so $U(a-c) < 0$. Thus the theorem is proved.

6.3. AN IMPROVED FORM OF THE EXISTENCE THEOREM

Theorem 6.2 may be improved if we find a better evaluation for $|\beta_1-\beta_{1o}|$. First we find a bound for b_3 in the expression

$$y = b \sin \omega t + b_3 \sin 3\omega t + \dots .$$

We have,

$$b_3 = \frac{2}{T}\int_0^T (y-Py)\sin 3\omega t dt = \frac{2}{T}\int_0^T [-H(I-P)gy]\sin 3\omega t dt$$

$$= \frac{2}{3\omega t}\int_0^T \frac{d}{dt}[-H(I-P)gy][-\cos 3\omega t]dt$$

$$= \frac{2}{9\omega^2 T}\int_0^T (I-P)gy \sin 3\omega t dt = \frac{2}{9\omega^2 T}\int_0^T (gy-Pgy)\sin 3\omega t dt$$

$$= \frac{2}{9\omega^2 T}\int_0^T \{(gy-gx_o)-P(gy-gx_o)+(gx_o-Pgx_o)\}\sin 3\omega t dt,$$

where we used twice integration by parts. Therefore

$$|b_3| \le \frac{2}{9\omega^2 T}\|(gy-gx_o) - P(gy-gx_o) + (gx_o-Pgx_o)\|\int_0^T |\sin 3\omega t|dt$$

$$\le \frac{2}{9\omega^2 T}\frac{2T}{\pi}\{\|gy-gx_o\|+\|P(gy-gx_o)\|+\|gx_o-Pgx_o\|\}$$

$$\le \frac{4}{9\omega^2\pi}\{\|gy-gx_o\| + \frac{4}{\pi}\|gy-gx_o\|+\|gx_o-Pgx_o\|\}$$

$$\le \frac{4}{9\omega^2\pi}\left\{\frac{\pi+4}{\pi}L\|y-x_o\|+\|gx_o-Pgx_o\|\right\},$$

$$|b_3| \le \frac{4}{9\omega^2\pi}\left\{\frac{1}{\pi}(\pi+4)Ld+h_c\right\}, \qquad (6.41)$$

where

$$h_c = \|gx_o - Pgx_o\| = \|\theta(a,t)\|$$

As we know $\beta_1-\beta_{10}$ is given by

$$\beta_1 - \beta_{10} = \frac{2}{T}\int_0^T [Pgy-gPy]\sin\omega t dt = \frac{2}{T}\int_0^T [gy-gPy]\sin\omega t dt. \qquad (6.42)$$

Now, for any real number z

$$g_o(z) = \sigma^2 z + \frac{\alpha_1}{3!}z^3 + \frac{\alpha_2}{5!}z^5 + \ldots$$

and so

$$g(Py+(y-Py)) = \sigma^2[Py+(y-Py)] + \frac{\alpha_1}{3!}[(Py)+(y-Py)]^3 + \frac{\alpha_2}{5!}[Py+(y-Py)]^5+\ldots+R_{o1}$$

and

$$y(Py) = \sigma^2 Py + \frac{\alpha_1}{3!}(Py)^3 + \frac{\alpha_2}{5!}(Py)^5 +\ldots+R_{o2},$$

where

$$R_{o1} = R[y(t),t] + S(t),$$

$$R_{o2} = R(Py,t) + S(t),$$

and hence

$$\bar{R} = R_{o1} - R_{o2} = R[y(t),t] - R[Py,t].$$

Then

$$\begin{aligned} g(y) - g(Py) &= g[Py+(y-Py)] - g(Py) \\ &= \sigma^2(y-Py) + \frac{\alpha_1}{3!}\{[Py+(y-Py)] - (Py)^3\} \\ &+ \frac{\alpha_2}{5!}\{y^5-(Py)^5\} +\ldots+ R_{o1} - R_{o2}\ , \end{aligned}$$

where

$$[Py+(y-Py)]^3 - (Py)^3 = y^3 - (Py)^3 = 3(Py)^2(y-Py) + 3(Py)(y-Py)^2 + (y-Py)^3.$$

Since $\sigma^2(y-Py)$ has no term in $\sin \omega t$ it does not affect the value of the integral (6.42). We write

$$\beta_1 - \beta_{10} = \frac{2}{T}\int_0^T [gy-gPy] \sin \omega t dt = \frac{2}{T}\int_0^T \frac{\alpha_1}{3!} 3(Py)^2(y-Py) \sin \omega t dt$$

$$+ \frac{2}{T}\int_0^T \frac{\alpha_1}{3!} 3\, Py(y-Py)^2 \sin \omega t dt + \frac{2}{T}\int_0^T \frac{\alpha_1}{3!} (y-Py)^3 \sin \omega t dt$$

$$+ \frac{2}{T}\int_0^T \sum_{n=1}^{k} \frac{\alpha_n}{(2n+1)!} \{y^{2n+1}-(Py)^{2n+1}\} \sin \omega t dt$$

$$+ \frac{2}{T}\int_0^T \frac{T}{R} \sin \omega t dt = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5. \qquad (6.43)$$

Now we evaluate each term of (6.43).

$$|\sigma_1| = \left|\frac{2}{T}\int_0^T \frac{\alpha_1}{3!} 3(Py)^2(y-Py) \sin \omega t dt\right|$$

$$\leq \left|\frac{\alpha_1}{T}\int_0^T (b^2\sin^3\omega t)(b_3\sin 3\omega t+b_5\sin 5\omega t+\ldots)dt\right|$$

$$= \left|\frac{\alpha_1 b^2}{4T}\int_0^T (3 \sin \omega t - \sin 3\omega t)(b_3\sin 3\omega t+b_5\sin 5\omega t+\ldots)dt\right|$$

$$= \frac{|b_3|b^2|\alpha_1|}{8} \leq \frac{b^2|\alpha_1|}{18\omega^2\pi}\left[\left(1+\frac{4}{\pi}\right)Ld+h_c\right]. \qquad (6.44)$$

$$|\sigma_2| = \left|\frac{2}{T}\int_0^T \frac{\alpha_1}{3!} 3(Py)(y-Py)^2 \sin \omega t dt\right|$$

$$\leq \frac{2}{T}\int_0^T \frac{|\alpha_1|}{3!} 3\|Py\|\|y - Py\|^2 |\sin \omega t| dt \qquad (6.45)$$

$$= \frac{1}{\pi}|\alpha_1|\|Py\|\|y - Py\|^2 \leq \frac{1}{\pi}|\alpha_1| b(k_c Ld+\gamma_c)^2.$$

$$|\sigma_3| = \frac{2}{T}\int_0^T \frac{\alpha_1}{3!}(y-Py)^3 \sin \omega t dt| \leq \frac{2}{T}\int_0^T \frac{|\alpha_1|}{3!}\|y - Py\|^3|\sin \omega t dt|dt$$

$$= \frac{1}{3\pi}|\alpha_1|\|y - Py\|^3 \leq \frac{1}{3\pi}|\alpha_1|(k_c Ld+\gamma_c)^3. \tag{6.46}$$

Now, we have for every $m > 1$

$$y^m - (Py)^m = [Py+(y-Py)]^m - (Py)^m$$

$$= \sum_{s=o}^{m=1} \binom{m}{s}(y-Py)^{m-s}(Py)^s$$

$$= (y-Py)\sum_{s=o}^{m-1} \binom{m}{s}(y-Py)^{m-s-1}(Py)^s,$$

and for every t we have also

$$|y^m - (Py)^m| \leq |y - Py| \sum_{s=o}^{m=o} \binom{m}{s}|y - Py|^{m-s-1}|Py|^s,$$

where

$$\binom{m}{s} = \frac{m(m-1)\ldots(m-s+1)}{1\ldots s} = \frac{m}{m-s}\frac{(m-1)(m-2)\ldots(m-s)}{1\ldots s}.$$

and $m-s \geq 1$. Therefore

$$\binom{m}{s} \leq \frac{m}{m-s}\binom{m-1}{s} \leq m\binom{m-1}{s}$$

and

$$|y^m - (Py)^m| \leq m|y - Py|\sum_{s=o}^{m-1} \binom{m-1}{s}|y - Py|^{m-s-1}|Py|^s$$

$$= m|y - Py|(|y-Py+|Py|)^{m-1}.$$

Finally we have

$$\|y^m - (Py)^m\| \leq m\|y - Py\|(\|y-Py\|+\|Py\|)^{m-1}.$$

Therefore, the term σ_4 in (6.43) can be evaluated as follows

$$|\sigma_4| = |\frac{2}{T}\int_0^T \sum_{n=2}^{k} \frac{\alpha_n}{(2n+1)!}\{y^{2n+1}-(Py)^{2n+1}\} \sin \omega t dt|$$

$$\leq \frac{2}{T}\int_0^T \sum_{n=2}^{k} \frac{|\alpha_n|}{(2n+1)!}\|y^{2n+1} - (Py)^{2n+1}\|| \sin \omega t|dt$$

$$= \frac{2}{\pi}\sum_{n=2}^{k} \frac{|\alpha_n|}{(2n)!}\|y - Py\|(\|y-Py\|+\|Py\|)^{2n}$$

$$= \frac{2}{\pi}\|y - Py\| \sum_{n=2}^{k} \frac{|\alpha_n|}{(2n)!}(\|y-Py\|+\|Py\|)^{2n}.$$

In particular, since $\|y-Py\| \leq k_c Ld+\gamma_c$, and $\|Py\| = b$, we have

$$|\sigma_4| = |\frac{2}{T}\int_0^T \sum_{n=2}^{k} \frac{\alpha_n}{(2n+1)!}\{y^{2n+1}-(Py)^{2n+1}\} \sin \omega t dt|$$

$$\leq \frac{2}{\pi}(k_c Ld+\gamma_c) \sum_{n=2}^{k} \frac{|\alpha_n|}{(2n)!}(b+k_c Ld+\gamma_c)^{2n}.$$

For the term σ_5 in (6.43) we have

$$
\begin{aligned}
|\sigma_5| &= \left|\frac{2}{T}\int_0^T \overline{R}\sin\omega t\,dt\right| \\
&= \left|\frac{2}{T}\int_0^T \{R[y(t),t]-R[Py,t]\}\sin\omega t dt\right| \\
&\le \frac{2}{T}\int_0^T \left[\frac{\alpha_\ell |y|^{2\ell+1}}{(2\ell+1)!}+\frac{\alpha_\ell |Py|^{2\ell+1}}{(2\ell+1)!}\right]|\sin\omega t|dt \\
&\le \frac{2}{T}\frac{\alpha_\ell}{(2\ell+1)!}\left[\|y\|^{2\ell+1}+\|Py\|^{2\ell+1}\right]\int_0^T |\sin\omega t|dt \\
&\le \frac{2}{\pi}\frac{\alpha_\ell}{(2\ell+1)!}\left[\|y\|^{2\ell+1}+\|Py\|^{2\ell+1}\right].
\end{aligned}
$$

Since $\|y\| \le \|y-x_o\|+\|x_o\| \le d+a$, and $\|Py\| = \|Py-x_o\|+a \le c+a$, we have

$$
|\sigma_5| = \left|\frac{2}{T}\int_0^T \overline{R}\sin\omega t dt\right| \le \frac{2}{\pi}\frac{\alpha_\ell}{(2\ell+1)!}\left[(d+a)^{2\ell+1}+(c+a)^{2\ell+1}\right].
$$

Finally we have

$$
\begin{aligned}
|\beta_1-\beta_{1o}| &\le |\sigma_1|+|\sigma_2|+|\sigma_3|+|\sigma_4|+|\sigma_5 \\
&\le \frac{b^2|\alpha_1|}{18\omega^2\pi}\left[\left(1+\frac{4}{\pi}\right)Ld+h_c\right] \\
&+ \frac{1}{\pi}|\alpha_1|b(k_cLd+\gamma_c)^2 \\
&+ \frac{1}{3\pi}|\alpha_1|(k_cLd+\gamma_c)^3 \\
&+ \frac{2}{\pi}(k_cLd+\gamma_c)\sum_{n=2}^{k}\frac{|\alpha_n|}{(2n)!}(b+k_cLd+\gamma_c)^{2n} \\
&+ \frac{2}{\pi}\frac{\alpha_\ell}{(2\ell+1)!}\left[(d+a)^{2\ell+1}+(c+a)^{2\ell+1}\right]. \qquad (6.47)
\end{aligned}
$$

We denote by B the second member of this inequality. With these results we may state the improved version of Theorem 6.2.

Theorem 6.3. Let a and ω satisfy Eq. (6.17), namely

$$- a\omega^2 + A_1 + \sigma^2 a + \frac{\alpha_1}{3!}\frac{a^3}{2^2}\frac{3!}{1!2!} + \ldots + \frac{\alpha_{k-1}}{(2k-1)!}\frac{a^{2k-1}}{2^{2k-2}}\frac{(2k-1)!}{(k-1)!k!} = 0.$$

Assume that c,d are positive constants, with $0 < c < d$, $a+d \leq A_o$, that $c+\gamma_c \leq (1-k_cL)d$, $k_cL < 1$, $\lambda < Q$, and

$$B \leq Q - \lambda, \tag{6.48}$$

where $Q = \min[|u_o(a-c)|, |u_o(a+c)|]$ and we assume $u_o(a-c)$, $u_o(a+c)$ to be of opposite signs and

$$\gamma_c = \|\Delta(t,a)\|, \quad \lambda = \frac{2\alpha_\ell |b|^{2\ell+1}}{\pi[1.3.\ldots.(2\ell+1)]^2}.$$

Then for some real number b, $a-c < b < a+c$, and $x_o = a \sin \omega t$, $x^* = b \sin \omega t$, the fixed element $y = Ty$ of the map $T/_{S^*_{co}}$ satisfies the equation

$$\ddot{y} + g(y,t) = 0,$$

with

$$y(t) = b \sin \omega t + b_3 \sin 3\omega t + b_5 \sin 5\omega t,$$

$$\|y(t) - x^*(t)\| = \|y - Py\| \leq \min[k_cLd+\gamma_c, c+d],$$

and

$$\|y(t) - x_o(t)\| \leq d.$$

The proof of Theorem 6.3 is the same as that of Theorem 6.2 with B replacing $\frac{4}{\pi} L(k_c Ld+\gamma_c)$.

6.4. AN EXISTENCE THEOREM WITH THE USE OF THE SQUARE NORM

We shall now consider, as in Section 5.3, the space S_q of all real valued functions $x(t), -\infty < t < +\infty$, which are periodic of period $T = 2\pi/\omega$, which are L^2-integrable in $[0,T]$, and satisfy the symmetry conditions (5.1). We shall take in S_q the square norm. Therefore, if $x(t) \sim c_1 \sin \omega t + c_3 \sin 3\omega t + \ldots$ we have

$$\mu(x) = \left[\frac{1}{T}\int_0^T x^2(t)dt\right]^{1/2} = [2^{-1}(c_1^2+c_3^2+\ldots]^{1/2}.$$

As it was pointed out in 5.3 the elements $x(t)$ which are bounded functions in $(-\infty,+\infty)$ admit also a norm $\|x\| = \sup_t |x(t)|$. In particular let

$$\bar{S}_q = \{x(t) \,|\, x(t)\epsilon S_q, \|x(t)\| \leq A\}.$$

Let g be the operator defined by

$$gx = g(x(t),t),\ x(t)\epsilon\bar{S}_q.$$

From condition 1, (Eqs. (6.5)), we see that $\bar{g}(t) = g(x(t),t)$ has the same symmetries of $x(t)$, that is

$$\bar{g}(-t) = -\bar{g}(t), \quad \bar{g}(\tfrac{T}{4} - t) = \bar{g}(\tfrac{T}{4} + t) \text{ and } \bar{g}(t+T) = \bar{g}(t).$$

Lemma 6.4. $g:\bar{S}_q \to S_q$ is a continuous operator in $\bar{S}_q$.

Proof.

$$\mu(gx_1-gx_2) = \left\{\frac{1}{T}\int_0^T (g[x_1(t),t]-g[x_2(t),t])^2dt\right\}^{1/2} \leq \left\{\frac{1}{T}\int_0^T L^2(x_1(t)-x_2(t))^2dt\right\}^{1/2} = L\mu(x_1-x_2). \tag{6.49}$$

We now define the operators f, F and T by

$$fx = (P-I)gx, \quad Fx = Hfx = H(P-I)gx,$$

$$Tx = Px + Fx = Px + Hfx = Px + H(P-I)gx$$

for all $x\epsilon\bar{S}_q$. Then

$$f:\bar{S}_q \to S_q, \quad F:\bar{S}_q \to S_q, \quad T:\bar{S}_q \to S_q.$$

Lemma 6.5. The operators f, F and T are continuous in $\bar{S}_q$.

Proof. An immediate consequence of preceding lemmas.

Let $x^*(t) = b \sin \omega t$ be an element of $\bar{S}_q$. Let

$$\Theta(t,b) = x^{*\prime\prime} + g(x^*) = - b\omega^2 \sin \omega t + g[b \sin \omega t, t].$$

Then $\Theta(t,b)\epsilon S_q$. Let

$$\Delta(t,b) = H(I-P)\Theta(t,b) = H(I-P)x^{*\prime\prime} + H(I-P)g[x^*(t),t].$$

By lemma 5.11

$$H(I-P)x^{*\prime\prime} = (I-P)x^* = x^* - Px^*,$$

and so

$$x^* = Px^* + H(I-P)x^{*\prime\prime} = Px^* - H(I-P)g[x^*(t),t] + \Delta(t,b).$$

For b = a we have

$$x^*(t) = x_o(t) = a \sin \omega t$$

and

$$x_o = Px_o - H(I-P)g[x_o(t),t] + \Delta(t,a). \tag{6.50}$$

We denote by γ_1 and γ_2 the numbers

$$\gamma_1 = \|\Delta(t,a)\|, \tag{6.51}$$

$$\gamma_2 = \mu(\Delta(t,a)) = \left\{\frac{1}{T}\int_o^T [\Delta(t,a)]^2 dt\right\}^{1/2}. \tag{6.52}$$

Let c,d,r,R_o be given numbers with $0 < c < d$, and $0 < r < R_o$. Let $\bar{S}_{qo}$ be the set

$$\bar{S}_{qo} = \{x | x\epsilon S_q, \mu(x-x_o) \leq d, \|x-x_o\| \leq R_o\}. \tag{6.53}$$

Let V_q be the set

$$V_q = \{x | x\epsilon S_q, \mu P(x-x_o) \leq c, \|P(x-x_o)\| \leq r\}.$$

The sets V_q and $\bar{S}_{qo}$ are not empty since x_o belongs to both.

For any real number b with

$$|b - a| \leq r, \quad \frac{1}{\sqrt{2}}|b - a| \leq c, \tag{6.54}$$

let x^* be an element of S_q of the form $x^* = b \sin \omega t$. Since

$$\|P(x^*-x_o)\| = \max|(b-a) \sin \omega t| = |b - a| \leq r$$

and

$$\mu P(x-x_o) = \left\{\frac{1}{2}(b-a)^2\right\}^{1/2} = \frac{1}{\sqrt{2}}|b - a| \leq c,$$

then x^* is an element of V_q. Conversely, if $x^* = b \sin \omega t$, then b satisfies (6.54).

Let S^*_{qo} be the subset of $\bar{S}_{qo}$ with elements x such that $Px = Px^*$, that is

$$S^*_{qo} = \{x|x\epsilon S_q, Px = Px^*, \mu(x-x_o) \leq d, \|x - x_o\| \leq R_o\}. \quad (6.55)$$

The set S^*_{qo} is not empty since $x^*\epsilon S^*_{qo}$. To assure that $S^*_{qo} \subset \bar{S}_q$ we must have R_o chosen in such a way that $R_o+a \leq A$. Indeed

$$\|x\| \leq \|x - x_o\| + \|x_o\| \leq R_o + a \leq A.$$

We conclude that, if $0 < c < d$, $a+R_o \leq A$, b is such that $|b-a| \leq r$ and $\frac{1}{\sqrt{2}}|b-a| \leq c$ then the set S^*_{qo} is defined and not empty for each $x^*\epsilon V_q$, and contained in $\bar{S}_q$. We shall denote by k_q and k' the numbers already introduced under lemmas 5.9 and 5.12.

Theorem 6.4. If c,d,r,R_o are constants such that $0 < c < d$, $0 < r < R_o$, $a+R_o \leq A$; if b is a real number such that $|b-a| \leq c$, $\frac{1}{\sqrt{2}}|b-a| \leq r$ and $x^* = b \sin \omega t$; if $c+\gamma_2 \leq (1-k_qL)d$, $r+k'Ld+\gamma_1 \leq R_o$, $k_qL < 1$ then $T|_{S^*_{qo}}$ is a contraction in the square norm and maps S^*_{qo} into S^*_{qo}.

Proof. Let

$$y = Tx = Px - H(I-P)gx,$$

with $x \in S^*_{q_0}$. Then

$$Py = PTx = PPx - PH(I-P)gx = Px + 0 = Px^*.$$

By (6.25)

$$x_o = Px_o - H(I-P)g[x_o(t)] + \Delta(t,a),$$

and so

$$y - x_o = P(x-x_o) - H(I-P)(gx-gx_o) - \Delta(t,a),$$

$$\begin{aligned}
\mu(y-x_o) &\leq \mu P(x-x_o) + \mu[H(I-P)(gx-gx_o)] + \mu\Delta(t,a) \\
&= \mu P(x^*-x_o) + \mu[H(I-P)(gx-gx_o)] + \mu\Delta(t,a) \\
&\leq \mu P(x^*-x_o) + k_q L\mu(x-x_o) + \mu\Delta(t,a) \\
&\leq c + k_q Ld + \gamma_2.
\end{aligned}$$

By the hypothesis

$$\mu(y-x_o) \leq c + k_q Ld + \gamma_2 \leq k_q Ld + (1-k_q L)d = d.$$

Also

$$\begin{aligned}
\|y - x_o\| &\leq \|P(x-x_o)\| + \|H(I-P)(gx-gx_o)\| + \|\Delta(t,a)\| \\
&\leq r + k'Ld + \gamma_1 < R_o.
\end{aligned}$$

Hence T/S^*_{qo} maps S^*_{qo} into S^*_{qo}. Now, if $x_1, x_2 \epsilon S^*_{qo}$ and $y_1 = Tx_1, y_2 = Tx_2$, we have

$$y_1 - y_2 = H(I-P)(gx_1-gx_2),$$

$$\mu(y_1-y_2) \leq \mu[H(I-P)]\mu(gx_1-gx_2) \leq k_q L\mu(x_1-x_2).$$

Since, by hypothesis $k_q L < 1$ the theorem is proved.

Lemma 6.6. The subset S^*_{qo} of S_q with the metric of S_q is a complete metric space.

Proof. We know that S_q is complete by Theorem 5.7, that is, given a Cauchy sequence $\{x_n\}$ with $x_n \epsilon S^*_{qo}$, $n = 1,2,\ldots$ there is an element $x \epsilon S_q$ such that $\mu(x_n-x)$ tends to zero as $n \to \infty$. All we have to prove is that $x \epsilon S^*_{qo}$ that is, $Px = Px^*$, $\mu(x-x_o) \leq d$ and $\|x-x_o\| \leq R_o$. Since $x_n \epsilon S^*_{qo}$ we have, for all $n = 1,2,\ldots$

$$Px_n = Px^*, \quad \mu(x_n-x_o) \leq d, \text{ and } \|x_n - x_o\| \leq R_o.$$

We first show that $Px = Px^*$. Indeed, we have

$$\mu(Px-Px^*) = \mu(Px-Px_n) = \mu P(x-x_n) \leq \mu(x-x_n).$$

Since $\mu(x-x_n) \to 0$ as $n \to \infty$ and $\mu(Px-Px^*)$ is a number, it must be zero. Hence $\mu(Px-Px^*) = 0$ and this implies $Px-Px^* = 0$. Now we claim that $\mu(x-x_o) \leq d$. Indeed,

$$\mu(x-x_o) = \mu(x-x_o+x_n-x_n) \leq \mu(x-x_n) + \mu(x_n-x_o),$$

where $\mu(x-x_n) \to 0$ as $n \to \infty$, $\mu(x_n-x_o) \leq d$ for all n and $\mu(x-x_o)$ is a fixed number. We have $\mu(x-x_o) \leq d$ as claimed. Finally we have to show that $\|x-x_o\| \leq R_o$ a.e. From integration theory we know (see Ref. 15, p. 76) that convergence in the square norm implies convergence in the mean, and convergence in the mean implies convergence in measure. This means that given any $\epsilon > 0$ there is an N such taht for $n \geq N$ the set

$$\{t\epsilon[0,T] \mid |x_u(t)-x(t)| \geq \epsilon\}$$

has measure less than ϵ. Now,

$$|x(t) - x_o(t)| \leq |x(t) - x_n(t)| + |x_n(t) - x_o(t)|,$$

for any $n \geq N$, where $|x(t)-x_n(t)| < \epsilon$ but in a set Σ_n of measure less than ϵ, and $|x_n(t)-x_o(t)| \leq R_o$ for all t. Thus $|x(t)-x_o(t)| \leq R_o+\epsilon$ for all $[0,T]$ but in a set Σ_n of measure less than ϵ, with $\epsilon > 0$ arbitrary, $n \geq N$. Thus, $|x(t)-x_o(t)| \leq R_o$ for almost all t. Since the space S_q is made up of equivalent classes of functions x(t), we may well change the values of x(t) in a set of measure zero if necessary, and assume $|x(t)| \leq R_o$ for all t. This completes the proof that the limit x(t) of a Cauchy sequence in S^*_{qo} is an element x(t) of S^*_{qo}.

As a consequence of the two previous results and Banach's fixed point theorem we have the following corollary.

Corollary. Under the conditions of Theorem 6.4 T/S^*_{qo} has a unique fixed element $y = Ty\epsilon S^*_{qo}$.

As before, y is a solution of the differential equation

$$\ddot{x} + g(x,t) = 0$$

if

$$D = P(\ddot{y}+gy) = U \sin \omega t = 0,$$

that is, if U = 0. This is equivalent to $-\omega^2 b+\beta_1(b) = 0$ where $Pgy = \beta_1(b)$. We have to show that there is a $b\epsilon[a-c,a+c]$ such that $U(b) = 0$. We put again

$$g(x^*,t) = \beta_{1o} \sin \omega t + \beta_{3o} \sin 3\omega t + \ldots \qquad (6.56)$$

$$y = b \sin \omega t + b_3 \sin 3\omega t + \ldots \qquad (6.57)$$

$$g(y,t) = \beta_1 \sin \omega t + \beta_3 \sin 3\omega t + \ldots, \qquad (6.58)$$

and so

$$U(b) = \frac{2}{T} \int_o^T (y''+g[y(t),t]) \sin \omega t dt = - \omega^2 b + \beta_1(b). \qquad (6.59)$$

Now

$$u(b) = - \omega^2 b + \beta_{1o}(b)$$

and

$$|U(b) - u(b)| = |\beta_1 - \beta_{1o}| = \sqrt{2}\left[\tfrac{1}{2}(\beta_1-\beta_{1o})^2\right]^{1/2}$$
$$= \sqrt{2}\,\mu P[g(y)-g(x^*)] \le \sqrt{2}\,\mu[g(y)-g(x^*)]$$
$$\le \sqrt{2}\,L\mu(y-x^*).$$

From Eq. (6.30) we have

$$y(t) - x^*(t) = -H(I-P)[g(y)-x(x_o)] + \Delta(t,a),$$

and hence

$$\mu(y-x^*) \le \mu H(I-P)[g(y)-g(x_o)] + \mu\Delta(t;a)$$
$$\le k_q L\mu(y-x_o) + \mu\Delta(t,a) \le k_q Ld+\gamma_2.$$

Finally

$$|U(b) - u(b)| \le \sqrt{2}\,L[k_q Ld+\gamma_2].$$

From (6.13) we have now

$$u(b) = -b\omega^2 + S_1(b) + A_1 + \rho$$
$$= \left[A_1 - b\omega^2+\sigma^2 b + \frac{\alpha_1}{1!2!}\frac{b^3}{2^2} + \ldots + \frac{\alpha_{k-1}}{(k-1)!k!}\frac{b^{2k-1}}{2^{2k-1}}\right] \qquad (6.60)$$
$$+ \frac{2}{T}\int_o^T R(b\sin\omega t,t)\sin\omega t\,dt = u_o(b) + \eta(b),$$

where $u_o(b)$ is the polynomial in b of degree 2k-1 above and $\eta(b)$ is the last expression. For b = a we have $u_o(a) = 0$ and so $u(a) = \eta(a)$.

Now we have, by hypothesis 4b, Eq. (6.9)

$$|R(b\sin\omega t,t)| \le \frac{\alpha_\ell}{(2\ell+1)!}|b\sin\omega t|^{2\ell+1}\lambda(t) \le \frac{\alpha_\ell}{(2\ell+1)!}|b|^{2\ell+1}\lambda(t),$$

with

$$\frac{1}{T}\int_0^T \lambda^2(t)dt = 1.$$

Then

$$\begin{aligned}|\eta(b)| &\le \frac{2}{T}\int_0^T \frac{\alpha_\ell}{(2\ell+1)!}|b|^{2\ell+1}\lambda(t)|\sin\omega t|^{2\ell+1}dt \\ &= \frac{2\alpha_\ell}{(2\ell+1)!}|b|^{2\ell+1}\frac{1}{T}\int_0^T \lambda(t)|\sin\omega t|^{2\ell+1}dt \\ &\le \frac{2\alpha_\ell}{(2\ell+1)!}|b|^{2\ell+1}\left[\frac{1}{T}\int_0^T \lambda^2(t)dt\right]^{1/2}\left[\frac{1}{T}\int_0^T |\sin\omega t|^{4\ell+2}dt\right]^{1/2} \\ &\le \frac{2\alpha_\ell}{(2\ell+1)!}|b|^{2\ell+1}\left\{\frac{1.3.\ldots.(4\ell-1)(4\ell+1)}{2.4.\ldots.4\ell(4\ell+2)}\right\}^{1/2}. \qquad (6.61)\end{aligned}$$

We call λ the number

$$\lambda = \frac{2\alpha_\ell|b|^{2\ell+1}}{(2\ell+1)!}\left\{\frac{1.3.\ldots.(4\ell-1)(4\ell+1)}{2.4.\ldots.4\ell(4\ell+2)}\right\}^{1/2}.$$

The formula (6.34) for $u_o(a\pm c)$ remains the same in the present situation. If $u_{o1} = u_o(a-c)$, $u_{o2} = u_o(a+c)$ we denote by Q, as in (6.35) the number $Q = \min[|u_{o1}|,|u_{o2}|]$. We state now a theorem analogous to Theorem 6.2.

Theorem 6.5. Let a and ω satisfy equation

$$-a\omega^2 + A_1 + \sigma^2 a + \frac{\alpha_1}{3!}\frac{a^3}{2^2}\frac{3!}{1!2!} + \ldots + \frac{\alpha_{k-1}}{(2k-1)!}\frac{a^{2k-1}(2k-1)!}{2^{2k-2}(k-1)!k!} = 0$$

Assume that c,d,r,R_o, are constants such that $0 < c < d$, $0 < r < R_o$, $a+R_o \le A$, $c+\gamma_2 \le (1-k_qL)d$, $r+k'L_qd+\gamma_1 \le R_o$, $k_qL < 1$, $\lambda < Q$, and

$$\sqrt{2}\,L[k_qLd+\gamma_2] < Q - \lambda, \qquad (6.62)$$

where

$$Q = \min[|u_0(a-c)|, |u_0(a+c)|]$$

and we assume $u_0(a-c)$, $u_0(a+c)$ to be of opposite signs, and

$$\gamma_1 = \|\Delta(t,a)\|, \quad \gamma_2 = \mu\Delta(t,a), \quad \lambda = \frac{2\alpha_\ell |b|^{2\ell+1}[(1.3....(4\ell-1)(4\ell+1)]^{1/2}}{(2\ell+1)![2.4....4\ell(4\ell+2)]^{1/2}}.$$

Then, for some real number b, $a-c < b < a+c$, and $x_0 = \sin \omega t$, $x^* = b \sin \omega t$, the fixed element $y = Ty$ of the map $T|_{S^*_{co}}$ satisfies equation

$$\ddot{y} + g(y,t) = 0,$$

with

$$y(t) = b \sin \omega t + b_3 \sin 3\omega t + b_5 \sin 5\omega t + \ldots,$$

$$\|y(t) - x^*(t)\| = \|y - Py\| \leq \min[k_q L d + \gamma_2, c+d],$$

$$\|y(t) - x_0(t)\| \leq d.$$

Remark. As for Theorem 6.2, we note here that in the autonomous case ω is a function $\omega = \omega(a)$ of the parameter a, defined by (6.17). In the periodic case, ω is fixed and a is assumed to be a root of Eq. (6.17). The proof of Theorem 6.5 is exactly the same as that of Theorem 6.2 and is not repeated.

As in the Section 6.3 this theorem may be improved by a better evaluation of $|\beta_1 - \beta_{10}|$. For the evaluation we first need bounds for

$$\mu(y-Py) = \mu(y-x^*)$$

and for $|y-Py|$.

Since

$$\mu(y-Py) = \left\{\frac{1}{2}\,(b_3^2+b_5^2+\ldots)\right\}^{1/2}$$

we first try to find bounds for b_j.

$$b_j = \frac{2}{T}\int_0^T (y-Py)\,\sin j\omega t dt, \quad j = 3,5,7,\ldots$$

Since $\ddot{y}-P\ddot{y} = gy-Pgy$, integration by parts gives

$$b_j = \frac{2}{j^2\omega^2 T}\int_0^T (gy-Pgy)\,\sin \omega t dt$$

$$= \frac{2}{j^2\omega^2 T}\int_0^T \phi(t)\,\sin j\omega t dt,$$

where $\phi(t) = gy-Pgy$. Now we get

$$|b_j| \le \frac{2}{j^2\omega^2}\left(\frac{1}{T}\int_0^T [\phi(t)]^2 dt\right)^{1/2}\left(\frac{1}{T}\int_0^T \sin^2 j\omega t dt\right)^{1/2}$$

$$\le \frac{2}{j^2\omega^2}\,\frac{1}{\sqrt{2}}\,\mu(\phi(t)) = \frac{\sqrt{2}}{j^2\omega^2}\,\mu[\phi(t)].$$

Now,

$$\mu\phi(t) = \mu[(I-P)(gy-gx_o)+(gx_o-Pgx_o)]$$

$$\le \mu[gy-gx_o] + \mu(gx_o-Pgx_o)$$

$$\le L\mu(y-x_o) + h_q,$$

where h_q is an upper bound for

$$\mu(gx_o - Pgx_o) = \mu\Theta(t,a).$$

Hence

$$|b_j| \le \frac{\sqrt{2}}{j^2\omega^2}\,(Ld+h_q), \qquad j = 3,5,7,\ldots, \tag{6.63}$$

and so

$$\begin{aligned}\mu(y-Py) &= \left\{\frac{1}{2}\,(b_3^2+b_5^2+\ldots)\right\}^{1/2} \\ &\le \left\{\frac{1}{2}\,(Ld+h_q)\,\frac{2}{\omega^4}\left(\frac{1}{3^4}+\frac{1}{5^4}+\ldots\right)\right\}^{1/2} \\ &= (Ld+h_q)\omega^{-2}\left(\frac{\pi^4}{96}-1\right)^{1/2} = \ell\omega^{-2}(Ld+h_q),\end{aligned} \tag{6.64}$$

with

$$\ell = \left(\frac{\pi^4}{96}-1\right)^{1/2} = 0.12115.$$

Analogously

$$\begin{aligned}|y - Py| &\le |b_3| + |b_5| + \ldots \\ &\le \frac{\sqrt{2}}{\omega^2}\,(Ld+h_q)\left(\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\right) \\ &\le \frac{\sqrt{2}}{\omega^2}\,(Ld+h_q)\,\ell',\end{aligned} \tag{6.65}$$

with

$$\ell' = 3^{-2} + 5^{-2} + \ldots = \frac{\pi^2}{8} - 1 = 0.2337.$$

Now

$$\beta_1 - \beta_{10} = \frac{2}{T}\int_0^T (gy-gPy)\sin \omega t dt$$

and as before, in Eq. (6.43) we have

$$\begin{aligned}\beta_1 - \beta_{10} &= \frac{2}{T}\int_0^T \frac{\alpha_1}{3!}\,3(Py)^2(y-Py)\sin\omega t dt \\ &+ \frac{2}{T}\int_0^T \frac{\alpha_1}{3!}\,3Py(y-Py)^2\sin\omega t dt + \frac{2}{T}\int_0^T \frac{\alpha_1}{3!}(y-Py)^3\sin\omega t dt \\ &+ \frac{2}{T}\int_0^T \sum_{n=1}^{k}\frac{\alpha_n}{(2n+1)!}\{y^{2n+1}-(Py)^{2n+1}\}\sin\omega t dt \\ &+ \frac{2}{T}\int_0^T \bar{R}\sin\omega t dt = \sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5,\end{aligned}$$

and we evaluate sucessively each term. We have as before

$$|\sigma_1| \le \frac{b^2|b_3||\alpha_1|}{8},$$

and by using (6.63) we have

$$|\sigma_1| \le \frac{b^2|\alpha_1|\sqrt{2}}{72\omega^2}[Ld+h_q]. \tag{6.66}$$

Also

$$\begin{aligned}|\sigma_2| &\le |\alpha_1|\max|Py|\{\mu(y-Py)\}^2 = |\alpha_1|b[\mu(y-x^*)]^2 \\ &\le |\alpha_1|b\ell^2\omega^{-4}\{Ld+h_q\}^2,\end{aligned} \tag{6.67}$$

$$\begin{aligned}|\sigma_3| &\le \frac{|\alpha_1|}{3}\max|y-Py|\{\mu(y-Py)\}^2 \\ &\le \frac{|\alpha_1|}{3}\sqrt{2}\,\ell'\ell^2\omega^{-6}[Ld+h_q]^3,\end{aligned} \tag{6.68}$$

Now, we have for every $m > 1$

$$\begin{aligned} y^m - (Py)^m &= [Py+(y-Py)]^m - (Py)^m \\ &= \sum_{s=0}^{m-1} \binom{m}{s} (y-Py)^{m-s}(Py)^s \\ &= \sum_{s=0}^{m-2} \binom{m}{s} (y-Py)^{m-s}(Py)^s + \binom{m}{m-1}(y-Py)(Py)^{m-1} \\ &= \frac{m(m-1)}{(m-s)(m-s-1)} (y-Py)^2 \sum_{s=0}^{m-2} \binom{m-2}{s} (y-Py)^{m-2-s}(Py)^s \\ &+ \binom{m}{m-1} (y-Py)(Py)^{m-1}, \end{aligned}$$

and for every t we have also

$$\begin{aligned} |y^m - (Py)^m| &\le \tfrac{1}{2} m(m-1)(y-Py)^2 \sum_{s=0}^{m-2} \binom{m-2}{s} |y - Py|^{m-2-s} |Py|^s \\ &+ m|y - Py||Py|^{m-1} \\ &\le \tfrac{1}{2} m(m-1)(y-Py)^2 (|y-Py|+|Py|)^{m-2} + m|y - Py||Py|^{m-1}. \end{aligned}$$

Therefore, the term σ_4 can be evaluated as follows

$$\begin{aligned} |\sigma_4| &= \left|\frac{2}{T} \int_0^T \sum_{n=2}^{k} \frac{\alpha_n}{(2n+1)!} \{y^{2n+1} - (Py)^{2n+1}\} \sin \omega t dt\right| \\ &\le \frac{2}{T} \int_0^T \sum_{n=2}^{k} \frac{|\alpha_n|}{(2n+1)!} |y^{2n+1} - (Py)^{2n+1}|| \sin \omega t| dt \\ &\le \frac{2}{T} \int_0^T \sum_{n=2}^{k} \frac{|\alpha_n|}{(2n+1)!} \left\{ \frac{1}{2} (2n+1)2n(y-Py)^2 (|y-Py|+|Py|)^{2n-1} \right. \\ &+ \left. (2n+1)|y-Py||Py|^{2n} \right\} | \sin \omega t| dt. \end{aligned}$$

If $M = \max|y-Py|$, $N = \max|Py| = b$, then

$$|\sigma_4| \leq \sum_{n=2}^{k} \frac{|\alpha_n|}{(2n-1)!}(M+N)^{2n-1}\left[\frac{1}{T}\int_0^T (y-Py)^2 dt\right]$$

$$+ \sum_{n=2}^{k} \frac{2|\alpha_n|}{(2n)!} N^{2n} \frac{1}{T}\int_0^T (y-Py)|\sin \omega t| dt$$

$$\leq \sum_{n=2}^{k} \frac{|\alpha_n|}{(2n-1)!}(M+N)^{2n-1}\left[\frac{1}{T}\int_0^T (y-Pt)^2 dt\right]$$

$$+ \sum_{n=2}^{k} \frac{2|\alpha_n|}{(2n)!} N^{2n}\left[\frac{1}{T}\int_0^T (y-Py)^2 dt\right]^{1/2}\left[\frac{1}{T}\int_0^T \sin^2 \omega t dt\right]^{1/2}.$$

Thus we have

$$|\sigma_4| \leq [\mu(y-Py)]^2 \sum_{n=2}^{k} \frac{|\alpha_n|}{(2n-1)!}(M+N)^{2n-1} + [\mu(y-Py)]\sqrt{2} \sum_{n=2}^{k} \frac{|\alpha n|}{2n!} N^{2n}$$

$$\leq \ell^2 \omega^{-4}(Ld+h_q)^2 \sum_{n=2}^{k} \frac{|\alpha n|}{(2n-1)!}(M+N)^{2n-1} + \ell\omega^{-2}(Ld+h_q)\sqrt{2} \sum_{n=2}^{k} \frac{|\alpha_n|}{2n!} N^{2n} \tag{6.69}$$

where

$$N = \max|P_y| = b, \quad M = \max|y - Py| = \ell\sqrt{2}\,\omega^{-2}(Ld+h_q).$$

Concerning the term σ_5 we have

$$|\sigma_5| = \left|\frac{2}{T}\int_0^T \bar{R} \sin \omega t dt\right| = \left|\frac{2}{T}\int_0^T \{R[y(t),t]-R[Py,t]\} \sin \omega t dt\right|$$

$$\leq \frac{2}{T}\int_0^T \left[\frac{\alpha_\ell |y|^{2\ell+1}\lambda(t)}{(2\ell+1)!} + \frac{\alpha_\ell |Py|^{2\ell+1}\lambda(t)}{(2\ell+1)!}\right] |\sin \omega t| dt.$$

We have

$$y = Py + (y-Py),$$

$$|y| \leq |Py| + |y - Py| \leq N + M,$$

$$\mu(y) \leq \mu(Py) + \mu(y-Py) \leq \frac{1}{\sqrt{2}} N + \ell\omega^{-2}(Ld+h_q),$$

amd hence

$$|\sigma_5| \leq \frac{2\alpha_\ell}{(2\ell+1)!} (M+N)^{2\ell} \left(\frac{1}{T}\int_0^T y^2 dt\right)^{1/2} \left(\frac{1}{T}\int_0^T \lambda^2 dt\right)^{1/2}$$
$$+ \frac{2\alpha_\ell}{(2\ell+1)!} N^{2\ell} \left(\frac{1}{T}\int_0^T (Py)^2 dt\right)^{1/2} \left(\frac{1}{T}\int_0^T \lambda^2 dt\right)^{1/2}.$$

Since $\mu(Py) \leq \mu(y)$ we get

$$|\sigma_5| \leq \frac{2\alpha_\ell}{(2\ell+1)!} \{(M+N)^{2\ell} + N^{2\ell}\}\mu(y),$$

$$|\sigma_5| \leq \frac{2\alpha_\ell}{(2\ell+1)!} \{(M+N)^{2\ell} + N^{2\ell}\} \left[\frac{1}{\sqrt{2}} N + \ell\omega^{-2}(Ld+h_q)\right]. \qquad (6.70)$$

From (6.66) to (6.70) we get our evaluation for $|\beta_1 - \beta_{10}|$.

$$|\beta_1 - \beta_{10}| \leq \frac{b^2|\alpha_1|\sqrt{2}}{72\omega^2} [Ld+h_q]$$
$$+ |\alpha_1| b\ell^2\omega^{-4}[Ld+h_q]^2 + \frac{|\alpha_1|}{3}\sqrt{2}\,\ell'\ell^2\omega^{-6}[Ld+h_q]^3,$$
$$+ \ell^2\omega^{-4}[Ld+h_q]^2 \sum_{n=2}^{k} \frac{|\alpha_n|}{(2n-1)!} (M+N)^{2n-1}$$
$$+ \ell\omega^{-2}(Ld+h_q)\sqrt{2} \sum_{n=2}^{k} \frac{|\alpha_n|}{2n!} N^{2n}$$
$$+ \frac{2\alpha_\ell}{(2\ell+1)!} \{(M+N)^{2\ell} + N^{2\ell}\} \left[\frac{1}{\sqrt{2}} N + \ell\omega^{-2}(Ld+h_q)\right]. \qquad (6.71)$$

We call D the second member of this inequality and state the improved version of Theorem 6.5.

Theorem 6.6. Let a and ω satisfy equation

$$-a\omega^2 + A_1 + \sigma^2 a + \frac{\alpha_1}{3!}\frac{a^3}{2^2}\frac{3!}{1!2!} + \ldots + \frac{\alpha_{k-1}}{(2k-1)!}\frac{a^{2k-1}(2k-1)!}{2^{2k-2}(k-1)!k!} = 0.$$

Assume that c,d,r,R_o, are constants such that $0 < c < d$, $0 < r < R_o$, $a+R_o \leq A$, $c+\gamma_2 \leq (1-k_qL)d$, $r+k'Ld+\gamma_1 \leq R_o$, $k_qL < 1$, $\lambda < Q$, and

$$D < Q - \lambda, \tag{6.72}$$

where

$$Q = \min[|u_o(a-c)|, |u_o(a+c)|]$$

and we assume $u_o(a-c)$, $u_o(a+c)$ to be of opposite signs, and

$$\gamma_1 = \|\Delta(t,a)\|, \quad \gamma_2 = \mu\Delta(t,a), \quad \lambda = \frac{2\alpha_\ell |b|^{2\ell+1}}{(2\ell+1)!}\left\{\frac{1.3.\ldots.(4\ell-1)(4\ell+1)}{2.4.\ldots.4\ell(4\ell+2)}\right\}^{1/2}$$

Then, for some real number b, $a-c < b < a+c$, and $x_o = a \sin \omega t$, $x^* = b \sin \omega t$, the fixed element $y = Ty$ of the map $T|_{S^*_{co}}$ satisfies equation

$$\ddot{y} + g(y,t) = 0,$$

with

$$y(t) = b \sin \omega t + b_3 \sin 3\omega t + b_5 \sin 5\omega t + \ldots,$$

$$\|y(t) - x^*(t)\| = \|y - Py\| \leq \min[k_qLd+\gamma_2, c+d],$$

$$\|y(t) - x_o(t)\| \leq d.$$

The proof of Theorem 6.6 is the same as that of Theorem 6.5 with D replacing $\sqrt{2}\, L[k_q Ld+\gamma_2]$. The better evaluating of $|\beta_1-\beta_{10}|$ permits the use of smaller values for R_o, d and c and this means that we may get smaller bounds for the exact periodic solution of equation

$$\ddot{x} + g(x,t) = 0.$$

7. EXAMPLES

7.1. THE EQUATION $\ddot{x}+x-\alpha^2x^3 = 0$

In this paragraph we shall study a few examples as applications of Theorems 6.3 and 6.6. We start with an application of Theorem 6.3 to the autonomous equation

$$\ddot{x} + x - \alpha^2x^3 = 0 , \quad \alpha > 0 . \tag{7.1}$$

The function $g(x,t)$ of Eq. (6.1) is in this case $g(x,t) = g_0(x) = x-\alpha^2x^3$ and conditions 1, 2, 3, of 6.1 are clearly satisfied by assuming $|x| \leq A$ for any fixed $A > 0$, and $R(x) \equiv 0$. On the other hand, if we take $A = \sqrt{2/3}\ 1/\alpha$, then from $g_0'(x) = 1-3\alpha^2x^2$, we obtain $1-3\alpha^2(2/3\alpha^2) \leq g_0'(x) \leq 1$, and hence $|g_0'(x) \leq 1$. Since $|g_0(x_1)-g_0(x_2)| = |x_1-x_2|g_0'(x)$ for some x between x_1 and x_2, we conclude that $|g_0(x_1)-g_0(x_2)| \leq |x_1-x_2|$ for all $x_1, x_2 \in [-A,A]$ and $A = \sqrt{2/3}\ 1/\alpha$. Thus 3 of 6.1 is satisfied with $L=1$. Also we have $S(t) \equiv 0$. In the development (6.7) of $g_0(x)$ we have in our case $\sigma^2 = 1$, $\alpha_1 = -3!\alpha^2 = -6\alpha^2$ and $\alpha_k = 0$, $k \geq 2$. We try to find solutions of the form $x_0 = a \sin \omega t$.

Galerkin's equation (6.17) is now

$$a(-\omega^2 + 1 - \tfrac{3}{4}\alpha^2a^2) = 0 ,$$

which gives, besides the trivial solution $a = 0$,

$$\omega^2 = 1 - \tfrac{3}{4}\alpha^2a^2 . \tag{7.2}$$

Note that the requirement $|x_o(t)| \leq A = \sqrt{2/3}\ 1/\alpha$, imposes on a the limitation

$$|a| \leq \sqrt{2/3}\ 1/\alpha$$

hence

$$\omega^2 = 1 - \frac{3}{4}\alpha^2a^2 \geq 1 - \frac{3}{4}\alpha^2\frac{2}{3}\frac{1}{\alpha^2} = \frac{1}{2}.$$

Since $R(x,t) = 0$ we have $\lambda = 0$. Now we have

$$\Delta(a,t) = H(I-P)[a \sin \omega t - \alpha^2(a \sin \omega t)^3]$$

$$= H(I-P)[a \sin \omega t + \alpha^2a^3(\frac{1}{4}\sin 3\omega t - \frac{3}{4}\sin \omega t)]$$

$$= H(I-P)[(a - \frac{3\alpha^2a^3}{4})\sin \omega t + \frac{1}{4}\alpha^2a^3\sin 3\omega t]$$

$$= H[\frac{1}{4}\alpha^2a^3\sin 3\omega t] = \frac{\alpha^2a^3}{36\omega^2}\sin 3\omega t .$$

Hence

$$\gamma_c = \|\Delta(a,t)\| = \frac{\alpha^2a^3}{36\omega^2}. \qquad (6.3)$$

Also

$$h_c = \|\theta(t,a)\| = \|\frac{1}{4}\alpha^2a^3 \sin 3\omega t\| = \frac{1}{4}\alpha^2a^3 .$$

Since $Q = \min[|u_o(a+c)|, |u_o(a-c)|]$ and we have

$$u_o(a\pm c) = \pm\frac{3}{4}\alpha^2a^2c - \frac{9}{4}\alpha^2ac^2 - \frac{3}{4}\alpha^2c^3 ,$$

for $\alpha > 0$, $a > 0$, $c > 0$ we put

$$Q = \frac{3}{4}\alpha^2 a^2 c + \frac{9}{4}\alpha^2 a c^2 + \frac{3}{4}\alpha^2 c^3 .$$

According to Theorem 6.2 we must verify the following inequalities:

$$0 < c < d,\ a+d < A_o,\ c+\gamma_c \le (1-k_c L)d,\ k_c L < 1,\ \lambda < Q ,$$

and $B < Q-\lambda$ where B is given by (6.47).

With $\alpha^2 = 1/6$ our constants are: $\omega^2 = 1 - \frac{1}{8}a^2$ (hence, for $a < 0.5$, $1 \ge \omega^2 \ge 0.969$), $A_o = \sqrt{2/3}\cdot\sqrt{6} = 2$, $\gamma_c = \frac{\alpha^2 a^3}{36\omega^2} = \frac{1}{216\omega^2}a^3 \le 0.005291a^3$, $h_c = \frac{1}{4}\alpha^2 a^3 = 0.04166a^3$ and $Q = 0.125a^2c+0.375ac^2+0.125c^3$. The two main inequalities are now (with $k_c = 0.16$)

$$c + 0.005291a^3 \le 0.84d \qquad (7.4)$$

and

$$\begin{aligned} B &= (0.04594d+0.0008419a^3)b^2 \\ &\quad + 0.3183(0.16d+0.005291a^3)^2 b \\ &\quad + 0.1061(0.16d+0.005291a^3)^3 \le Q \\ &= 0.125a^2c + 0.375ac^2 + 0.125c^3 . \end{aligned} \qquad (7.5)$$

If we put $c = \gamma a^3$ and $d = \delta a^3$ the linear parts of inequalities (7.4), and (7.5) are

$$\gamma + 0.005291 \le 0.84\delta$$

and

$$0.046868 + 0.0008588 \le 0.125\gamma$$

where we suppose $c < 0.01a$. That is

$$\delta \geq 1.19\gamma + 0.0063 ,$$

$$\delta \leq 2.667\gamma - 0.0183 ,$$

which are satisfied for $\gamma = 0.03$ and $\delta = 0.05$.

If we take $c = 0.03a^3$ and $d = 0.05a^3$ both inequalities (7.4) and (7.5) are satisfied as well as the remaining ones of Theorem 6.2 for every $0 < a < 0.5$. Indeed, $c < 0.01a$ and $c = 0.03a^3$ implies $a < 0.54$ and inequalities (7.4) and (7.5) become

$$0.03 + 0.0005291 \leq 0.042$$

and

$$0.003327a^5 + 0.0000727a^7 + 0.00000035a^9$$
$$\leq 0.00375a^5 + 0.000337a^7 + 0.00000337a^9 .$$

Then Eq. (7.1) has, for any $a < 0.5$, $\alpha^2 = 1/6$, a periodic solution $y = b \sin \omega t + b_3 \sin 3\omega t + \ldots$ with $b \in [a-0.03a^3, a+0.03a^3]$. The period is $T = 2\pi/\omega$ with $\omega = [1 - \frac{1}{8} a^2]^{1/2}$. Furthermore, if $x_0 = a \sin \omega t$

$$\|y-x_0\| \leq 0.05a^3 .$$

7.2. THE EQUATION $\ddot{x}+\sin x = 0$

(a) The uniform norm.—In the case to be studied in this section the function $g(x,t)$ of Eq. (6.1) is simply $g(x,t) \equiv \sin x$ and we are in the autonomous case of Eq. (6.3). Conditions 1, 2, and 3 of Section

6.1 are satisfied with $A = \pi$, $g_o(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$, $R(x) = 0$ and $L = 1$ (since $|\sin x_1 - \sin x_2| \leq |x_1 - x_2|$).

The Galerkin equation (6.18) is now

$$\omega^2 = 1 - \frac{1}{1!2!}\left(\frac{a}{2}\right)^2 + \frac{1}{2!3!}\left(\frac{a}{2}\right)^4 - \frac{1}{3!4!}\left(\frac{a}{2}\right)^6 + \ldots$$

$$= 1 - \frac{a^2}{8} + \frac{a^4}{192} - \ldots . \tag{7.6}$$

For $\omega = \bar{\omega}(a)$ given by (7.6) we want to show that in a neighborhood of the first Galerkin approximation

$$x_o = a \sin \omega(a)t ,$$

there is, for some $A_o > 0$ and all $a \in [0, A_o]$, an exact solution of

$$\ddot{x} + \sin x = 0 , \tag{7.7}$$

which is periodic, of period $T = 2\pi/\omega$. We shall find estimates for the error of the approximate solution $x_o = a \sin \omega(a)t$ by use of Theorems 6.3 and 6.6, that is, using only the uniform norm in one case and both norms in another case.

In order to apply the theorems we proceed to the determination of the constants involved. We have

$$L = 1, \quad k_c = 0.16 \quad \text{and} \quad \lambda = 0. \tag{7.8}$$

For the determination of γ_c we have

$$\gamma_c = \|\Delta(a,t)\| .$$

Now, by (6.25)

$$\Delta(a,t) = x_o - Px_o + H(I-P)g(x_o(t),t)$$
$$= H(I-P)\sin(a \sin \omega t) .$$

By (6.16)

$$\Delta(a,t) = H\{S_3 \sin 3\omega t + S_5 \sin 5\omega t + \ldots\}$$
$$= -S_3(3\omega)^{-2}\sin 3\omega t - S_5(5\omega^{-2})\sin 5\omega t - \ldots ,$$

with

$$S_3 = -\frac{a^3}{3!2^2}\left[1 - \frac{a^2}{16} + \frac{a^4}{640} - \frac{a^6}{46080} + \ldots\right] ,$$

$$S_5 = \frac{a^5}{5!2^4}\left[1 - \frac{a^2}{24} + \frac{a^4}{1344} - \ldots\right] ,$$

$$S_7 = -\frac{a^7}{7!2^6}\left[1 - \frac{a^2}{32} + \ldots\right] ,$$

$$S_9 = \frac{a^9}{2^8 9!}\left[1 - \ldots\right] .$$

Then

$$|\Delta(t,a)| \leq |S_3(3\omega)^{-2}| + |S_5(5\omega)^{-2}| + \ldots .$$

Now, if $a \in [0,1)$, we have

$$|S_3(3\omega)^{-2}| \leq \left|\frac{a^3}{3!2^2}\left(1 - \frac{a^2}{16}\right)\left(\frac{1}{3\omega}\right)^2\right| \pm \frac{a^7}{640 \times 3! \times 2^2}\left(\frac{1}{3\omega}\right)^2$$
$$\leq 0.00492\omega^{-2}a^3 \pm \frac{a^7}{640 \times 3! \times 2^2}\left(\frac{1}{3\omega}\right)^2 ,$$

$$|S_5(5\omega)^{-2}| \leq 0.0000208\omega^{-2}a^5 \pm \frac{a^7}{24x5!x2^4}\left(\frac{1}{5\omega}\right)^2 ,$$

$$|S_7(7\omega)^{-2}| \leq \frac{a^7}{7!2^6}\left(\frac{1}{7\omega}\right)^2 ,$$

$$|S_9(9\omega)^{-2}| \leq \frac{a^9}{9!2^8}\left(\frac{1}{9\omega}\right)^2 \leq \frac{a^7}{9!2^8}\left(\frac{1}{9\omega}\right)^2 ,$$

Then we have

$$|\Delta(t,a)| \leq 0.0049408\omega^{-2}a^3 \pm \left\{0.00000723\omega^2 + 0.00000087\omega^{-2} + (e-2.7180553)\left(\frac{2^{-6}}{1-2^{-6}}\right)\left(\frac{\pi^2}{8} - 1.1511\right)\omega^{-2}\right\} a^7$$

$$\leq 0.0049408\omega^{-2}a^3 \pm 0.0000084\omega^{-2}a^7$$

$$\leq 0.005\omega^{-2}a^3 .$$

We take accordingly

$$\gamma_c \leq 0.005\omega^{-2}a^3 . \qquad (7.9)$$

The determination of h_c, which we also need, is quite similar. We have $h_c = \|\theta(t,a)\|$ and since by (6.10) $\theta(t,a) = -a\omega^2\sin\omega t + g(\sin\omega t, t)$, Eq. (6.15) shows that the coefficient of $\sin\omega t$ in the Fourier series of $\theta(t,a)$ is zero. Now

$$|S_3| \leq \left|\frac{a^3}{3!2^2}\left(1 - \frac{a^2}{16}\right)\right| \pm \frac{a^7}{640x24} \leq 0.044264a^3 \pm \frac{a^7}{640x3!x2!} ,$$

$$|S_5| \leq \frac{a^5}{5!2^4} \pm \frac{a^7}{24x5!x2^4} \leq 0.0005208a^5 \pm \frac{a^7}{24x5!2^4} ,$$

$$|S_7| \leq \frac{a^7}{7!2^6} ,$$

$$|S_9| \leq \frac{a^9}{2^8 9!} \leq \frac{a^7}{2^8 9!} .$$

Hence

$$|\theta(a,t)| \leq 0.0447848a^3 \pm \left\{0.0000651 + 0.0000217 + (e-2.7180553) \times \left(\frac{2^{-6}}{1-2^{-6}}\right)\right\} a^7 ,$$

and so

$$h_c \leq 0.04487a^3 . \tag{7.10}$$

Now we find a lower bound for Q given by $Q = \min[|u_o(a+c)|, |u_o(a-c)|]$.

From (6.34) we have

$$\begin{aligned} u_o(a\pm c) = \pm & \left[-\frac{2}{1!2!2^2}a^2 + \frac{4}{2!3!2^4}a^4 - \frac{6}{3!4!2^6}a^6 + \ldots\right] c \\ & + \left[-\frac{3}{1!2!2^2}a + \frac{5\cdot4}{2!}\frac{1}{2!3!2^4}a^3 - \frac{7\cdot6}{2!}\frac{1}{3!4!2^6}a^5 + \ldots\right] c^2 \\ & + \left[-\frac{1}{1!2!2^2} + \frac{5\cdot4\cdot3}{3!}\frac{1}{2!3!2^4}a^2 - \frac{7\cdot6\cdot5}{3!}\frac{1}{3!4!2^6}a^4 + \ldots\right] c^3 \\ & + \left[\frac{5\cdot4\cdot3\cdot2}{4!}\frac{1}{2!3!2^4}a - \ldots\right] c^4 \\ & + \ldots . \end{aligned}$$

Clearly the minimum of $|u_o(a\pm c)|$ is obtained for the minus sign. If in each coefficient of c^k we stop at a negative term we get a partial sum which is less than Q. That is

$$Q \geq \left[\frac{2}{1!2!2^2}a^2 - \frac{4}{2!3!2^4}a^4\right] c - \frac{3}{1!2!2^2}ac^2 - \frac{1}{1!2!2^2}c^3 . \tag{7.11}$$

Now, with the values (7.8) to (7.11) for the constants we try to find

a,c,d satisfying the conditions of Theorem 6.3 that is,

$$0 < c < d,\ a+d \le A_0,\ c+\gamma_c \le (1-k_cL)d,\ k_cL < 1,\ \gamma < Q$$

and $B \le Q-\lambda$ with $B = \sigma_1+\sigma_2+\sigma_3+\sigma_4+\sigma_5$ given by (6.47). In this case we have $\sigma_5 = 0$. After substitution the two main ineqaulities above reduce to

$$[0.0402d+0.0007935a^3]b^2 + 0.3183[0.16d+0.005a^3]^2b + 0.1061[0.16d+0.005a^3]^3 + 0.63662[\cosh\sigma - \frac{\sigma^2}{2} - 1][0.16d+0.005a^3] \le [0.25a^2-0.0208a^4]c - 0.375ac^2 - 0.125c^3 , \qquad (7.12)$$

where

$$\sigma = b + k_cLd + \gamma_c = b + 0.16d + 0.005a^3 ,$$

and

$$c + 0.005a^3 \le 0.84d . \qquad (7.13)$$

These inequalities are satisfied for $a \le 0.3$, $c = 0.007a^3 = 0.000189$, $d = 0.016a^3 \le 0.000432$ and so, for some $b\epsilon[a-0.007a^3, a+0.007a^3]$, our equation has, for every $a \le 0.3$, a periodic solution y of period $2\pi/\omega$ with $\omega = \omega(a)$ given by (7.6) and $Py = b \sin \omega t$. Furthermore we have

$$\|y(t) - x_0(t)\| \le 0.000432.$$

(b) <u>The square norm</u>.—Now we apply Theorem 6.6 to the same equation $\ddot{x}+\sin x = 0$. We need some evaluations in addition to those of (a) From Eq. (7.6) we see that, for $|a| < 1$ we have $7/8 < \omega^2 < 1$ and so

$1 < \omega^{-2} < 8/7$. Now, from the definitions we have

$$k_q = 0.111\ldots\omega^{-2} \leq 0.127 \quad \text{and} \quad \kappa' = 0.1694\omega^{-2} \leq 0.193598 .$$

Also $\gamma_1 = \|\Delta(t,a)\| = \gamma_c = 0.00571a^3$ from 7.2. Now, $\gamma_2 = \mu\,\Delta(t,a)$ $= \mu H(I-P)\theta(t,a)$ and so we first calculate $\mu\theta(t,a) = h_q$. From the expressions of S_j in (a) we see that for $|a| < 1$

$$|S_j| \leq \frac{a^j}{j!2^{j-1}} \leq \frac{a^3}{j!2^{j-1}}, \quad j=3,5,7, \ldots .$$

Then

$$\mu\theta(t,a) = \mu(\ddot{x}_o+\sin x_o) = \left\{\frac{1}{2}\,(S_3^2+S_5^2+\ldots)\right\}^{1/2}$$

$$\leq \left\{\frac{a^6}{2}\,[(3!2^2)^{-2} + (5!2^4)^{-2} + \ldots]\right\}^{1/2}$$

$$\leq 0.0295a^3 .$$

We take

$$h_q \leq 0.0295a^3 . \tag{7.15}$$

Now, we have

$$\gamma_2 \leq \frac{1}{9\omega^2} \times 0.0295a^3 = 0.00374a^3 . \tag{7.16}$$

Again, as in (a), we have $\lambda = 0$, $L = 1$ and Q satisfying (7.11).

We have to find constants c,d,r,R_o such that for a in a convenient neighborhood of the origin we have the following inequalities of Theorem 6.6 satisfied:

$$0 < c < d,\ 0 < r < R_o,\ a+R_o \leq \pi,\ \lambda < Q,\ k_qL < 1,$$

$$c+\gamma_2 \leq (1-k_qL)d,\ r+k'Ld+\gamma_1 \leq R_o \quad \text{and} \quad D \leq Q-\lambda$$

where D is given by (6.71).

After substitutions the three last inequalities become

$$c + 0.00374a^3 \leq 0.873d\ , \tag{7.17}$$

$$r + 0.1936d + 0.00571a^3 \leq R_o\ , \tag{7.18}$$

$$0.0224478[d+0.0295a^3]b^2 + 0.0190690[d+0.0295a^3]^2b + 0.0024009[d + 0.0295a^3]^3 + 0.0190694[d+0.0295a^3]^2[\sinh(M+N)-(M+N)] + 0.1952422[d+0.0295a^3][\cosh N-1-\tfrac{1}{2}N^2] \leq Q\ , \tag{7.19}$$

where

$$M = 0.3828754[d+0.0295a^3],\quad N = b\ ,$$

and

$$Q \geq 0.25a^2c - 0.020833...a^4c - 0.375ac^2 - 0.125c^3\ .$$

Inequalities (7.9) and (7.11) are satisfied if we put $c = 0.0043a^3$, $d = 0.005a^3$ for any $a < 0.8$. We must now find r and R_o so that (7.18) holds. We have, for a and d as above

$$r + 0.0058068a^3 \leq R_o\ .$$

We may take $r = c\sqrt{2} \leq 0.006081a^3$ and $R_o \geq 0.012a^3$ satisfying $|b-a| < r$ and $a+R_o \leq \pi$. Equation (7.7) has then, for any $a < 0.8$ a periodic solu-

tion $y = b \sin \omega t + b_3 \sin 3\omega t + \ldots$ with $\omega^2 = 1 - \frac{a^2}{8} + \frac{a^4}{192} - \ldots$ and such that

$$|b-a| \leq 0.0043a^3, \quad \mu(y-x_0) \leq 0.005a^3$$

and

$$\|y-x_0\| \leq 0.012a^3 .$$

Note that, by using Theorem 6.6 rather than Theorem 6.3 we have a larger interval in which we may take a.

7.3. THE EQUATION $\ddot{x}+\sin x = \alpha \sin \omega t$

(a) The uniform norm.—In this section we consider the non-autonomous, nonlinear, second order differential equation

$$\ddot{x} + \sin x = \alpha \sin \omega t . \tag{7.20}$$

Our function $g(x,t)$ of (6.1) is now $\sin x - \alpha \sin \omega t$. In the decomposition (6.6) $g_0(x) = \sin x$ as in 7.2(a) and (b), $R(x,t)$ is zero, but $S(t) = -\alpha \sin \omega t$ rather than zero as in 7.2(a) and (b). All the conditions of 6.1 are satisfied by $g(x,t)$. In particular $|g(x_1)-g(x)| \leq L|x_1-x_2|$ with $L = 1$ as in the two preceding cases.

We shall take the values $\alpha = 0.21818$, $\omega = 0.7784$* for an approximate (harmonic) solution of (7.20) with the form $x = a \sin \omega t$, Eq. (6.17) gives

*Bader[2] studied the equation $\ddot{x}+a \sin x = b \sin \tau$ using the values $a = 1.65$ and $b = 0.36$. This equation reduces to (7.20) by a suitable change of independent variable and for the values of a and b chosen by Bader α has the value given in this example.

$$a\left\{\omega^2-\sigma^2 - \frac{\alpha_1}{1!2!}\frac{a^2}{2^2}\right\} + \alpha = 0 .$$

Since $\alpha_1 = -1$ we have

$$-0.394a + 0.125a^3 + 0.218 = 0 ,$$

$$a^3 - 3.152a + 1.744 = 0 .$$

A root of this equation is $a = 0.6342$.

The function $\theta(t,a)$ is in this case

$$\theta(t,a) = -(a\omega^2+\alpha)\sin \omega t + \sin (a \sin \omega t)$$

and $\Delta(t,a) = H(I-P)\theta(t,a)$. Both $\|\theta(a,t)\|$ and $\|\Delta(t,a)\|$ are as in the previous examples. We have then $h_c = 0.04487a^3 = 0.011445$, $\gamma_c = 0.005\omega^{-2}a^3 = 0.0021044$. Also $\lambda = 0$ and $k_c = 0.24$. As for Q we have only to add to the coefficient of c in the expression of $u_o(a\pm c)$ the term $A_1/a = \alpha/a = 0.34402$. We have then

$$Q \geq 0.4411c - 0.2378c^2 - 0.125c^3 . \qquad (7.21)$$

Now we may apply Theorem 6.3.

We must find constants c,d such that $0 < c < d$, $a+d \leq \pi$, $k_cL < 1$, $\lambda < Q$, $c+\gamma_c \leq (1-k_cL)d$ and $B \leq Q-\lambda$ with B given by (6.47)

The two last inequalities, after substitutions are

$$c + 0.0021044 \leq 0.76d , \qquad (7.22)$$

$$[0.06633d+0.0003339]b^2 + 0.3183[0.24d+0.0021044]^2 b + 0.1061[0.24d+0.0021044]^3$$
$$+ 0.63662[\cosh \sigma - \frac{\sigma^2}{2} - 1][0.24d+0.0021044]$$
$$\leq 0.4411c - 0.2378c^2 - 0.125c^3 , \qquad (7.23)$$

with

$$\sigma = b + 0.16d + 0.0021044.$$

The two inequalities (7.22) and (7.23) are satisfied for $d = 0.017$ and $c = 0.009$.

We conclude that, for $\alpha = 0.21818$, $\omega = 0.7784$, the equation $\ddot{x}+\sin x = \alpha \sin \omega t$ has a periodic solution $y = b \sin \omega t + b_3 \sin 3\omega t + \ldots$ where b lies in the interval $[a-c, a+c] = [0.6252, 0.6432]$ and the approximate solution $x_0 = a \sin \omega t$, with $a = 0.6342$, satisfies

$$\|y(t) - x_0(t)\| \leq 0.017 .$$

(b) The square norm.—We now apply Theorem 6.6 to the same Eq. (7.20). We have the additional constants:

$$\gamma_2 = \frac{1}{9\omega^2} \times 0.0295a^3 = 0.00138 ,$$

$$h_q = \mu\theta(t,a) \leq 0.0295a^3 = 0.0075248 ,$$

besides the ones already used before namely: $L = 1$, $a = 0.6342$, $\lambda = 0$, $\gamma_1 = 0.002044$, $k_q = \frac{1}{9}\omega^{-2} = 0.18333\ldots$, and $k' = 0.1694\omega^{-2} = 0.27951$. Q is as in (7.21).

We have to find constants c, d, r, R_0 such that, the following inequal-

ities are satisfied: $0 < c < d$, $0 < r < R_o$, $a+R_o \leq \pi$, $\lambda < Q$, $k_q L < 1$, $c+\gamma_2 \leq (1-k_q L)d$, $r+k'Ld+\gamma_1 \leq R_o$ and $D \leq Q-\lambda$ where D is given by (6.71). After substitutions the three last inequalities become

$$c + 0.00138 \leq 0.81667d , \tag{7.24}$$

$$r + 0.27951d + 0.002044 \leq R_o , \tag{7.25}$$

$$\begin{aligned} &0.013413[d+0.0075248] + 0.025567[d+0.0075248]^2 + 0.0216759[d \\ &+0.0075248]^3 + 0.3974[d+0.0075248]^2[\sinh(M+n)-(M+n)] \\ &+ 0.281881[d+0.0075248][\cosh N-1 - \tfrac{1}{2}N^2] \leq Q , \end{aligned} \tag{7.26}$$

where

$$N = b , \quad M = 0.54533[d+0.0075248]$$

and

$$Q \geq 0.4411c - 0.2378c^2 - 0.125c^3 .$$

Inequalities (7.24) and (7.26) are satisfied for

$$d = 0.0035, \quad c = 0.001 .$$

If we now put in inequality (7.25) $d = 0.0035$, we get

$$r + 0.003022 \leq R_o . \tag{7.27}$$

If we note that, from (6.54), r cannot be less than $c\sqrt{2}$ we may put

$$r = 0.001414 \quad \text{and} \quad R_o = 0.0444 .$$

We know, then, that for $\alpha = 0.21818$, $\omega = 0.7784$ the equation $\ddot{x}+\sin x = \alpha \sin \omega t$ has a periodic solution $y = b \sin \omega t + b_3 \sin 3\omega t + \ldots$ where b lies in the interval $[0.63278, 0.63563]$ and the approximate solution $x_o = a \sin \omega t$, with $a = 0.6342$ satisfies $\mu(y(t)-x_o(t)) \leq 0.0035$.

In Bader's paper,[2] he used the first five Galerkin's approximations and obtained for b the value $b = 0.633$ well within our result.

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