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A THEORETICAL STUDY OF TWO BUCKLING CRITERIA FOR THIN SHELLS

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of the requirements for the degree of
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CHAPTER I

INTRODUCTION

The difficulties inherent in the theory of the buckling of thin shells are most pronounced when a cylindrical shell under axial compression is considered. This problem was first attacked in the early 1900's by Lorentz, Southwell, Timoshenko, Flugge and Donnell.[41] They neglected all terms higher than second order in the potential energy and solved the resulting linear equilibrium equations for the classical or linear buckling load (also called the Euler load).

The theoretical results of the classical theory did not, however, agree well with the early experiments of Lundquist and Donnell.[41] Significant disparities occur not only in the buckling load, which the theory predicts three to four times too high, but also in the wave form. In addition, the classical theory yields a wave pattern which envelops the entire surface of the shell and is not in accord with the observed behavior of local instabilities. Lastly, the scatter of experimental points and the definite tendency for a shell to buckle radially inward, rather than, as predicted by the theory, both inward and outward, are two additional characteristics not described by the classical theory.

Numerous attempts have been made to reconcile these discrepancies between the experiments and the classical theory. In one of the first, Flugge[16] adjusted the theoretical boundary conditions to coincide with those actually realized in the laboratory, but found only a negligible influence on the buckling loads of most cylinders. In contrast, a recent investigation by Stein[37] shows a significant drop in

the critical load when both the boundary conditions and the initial stresses were considered more precisely.

Another investigation made by Flugge, and a later attempt by Donnell,[14] lowered the theoretical buckling load by considering initial imperfections in the shell structure. Donnell used large deflection equations and assumed the initial deviations were of the same form as the buckled configuration of the shell. The application of these equations to cylindrical shells by Donnell and Wan,[15] and Nash,[35] led to the very significant result that deviations in the range of only one thickness of the shell are sufficient to lower the buckling load to about one third of the classical value. This sensitivity to imperfections in the shell structure was further examined by Koiter[26] who was able to show that buckling loads for decreasing amounts of certain types of initial imperfections approach the Euler load at infinite slope. The unfortunate aspect of this theory, however, is that in all the applications the magnitude of the initial imperfections has to be selected arbitrarily to bring the theoretical results in line with the experiments.

To circumvent this problem von Kármán and Tsien[47] performed an analysis using equations (often called the Donnell equations) similar to Donnell's but in contrast assumed no initial imperfections. This analysis pointed out a significant difference in the buckling behavior of curved shells and flat plates: it showed that a shell, even if initially perfect, can be in equilibrium at loads much lower than the classical linear buckling load. This unusual phenomenon prompted von Kármán and Tsien[49] to propose a new criterion for buckling -- "the lower buckling load". This is defined as the minimum load necessary to keep a shell in a buckled equilibrium state.

Under this new criterion when the load on the unbuckled shell reaches the "lower buckling load" the shell may "jump" into the associated buckled equilibrium state, which, however, is a state of higher potential energy. The energy necessary to make this jump, according to von Karman, Tsien and Dunn,[46] will be supplied by vibrations which occur during the loading process. Thus a lower buckling load has been defined which is independent of the amount of initial imperfections present.

This criterion, however, does not predict the observed behavior of a sudden release of energy when a shell reaches its critical load. Also, Friedrichs mentioned "it is hard to understand why the shell should leave the stable unbuckled state, jump to a state with higher potential energy and stay there."[17] To eliminate this apparent contradiction, Friedrichs proposed a new criterion--"the intermediate buckling load"--which is defined as the smallest load for which the potential energy in the buckled shell does not exceed the potential energy in the unbuckled shell. This was followed, shortly afterwards, by Tsien's[43] lowest energy criterion which not only required that the potential energies in the buckled and unbuckled states be equal but also considered the elasticity of the loading machine. As a result, different buckling loads are predicted for the same structure under different loading systems. For the dead-weight loading system, the load at which buckling occurs is the same as the Friedrichs' "intermediate load". In this dissertation, the term "energy load" will be restricted to mean the Friedrichs' "intermediate load" or, alternatively, the Tsien "dead-weight load".

Although the energy load seems to agree fairly well with the experimental data,[43] the following criticisms of the energy criterion

have been raised: 1) it is not clear why the shell "jumps" from the unbuckled state to a non-adjacent buckled equilibrium configuration; 2) the existence of the energy load has never been rigorously established; and 3) the dependence on a buckled equilibrium configuration may possibly involve deformations so large that the Donnell equations may not be accurate enough.

Nevertheless, the work of von Kármán and Tsien has received wide attention both in the Russian literature[34,45] and our own. Their analysis has been refined and extended by Kempner,[24,25] Leggett and Jones,[29,30] Michielsen,[32] Yoshimura[50] and more recently by Almroth[2] who found remarkable agreement with the experimental work of Thielemann.[40] In addition, a recent contribution by Bodner and Gjelsvik[6] points out the important significance of the energy load for certain elastic systems that exhibit snap buckling. They found that the energy load is a lower bound to the buckling loads associated with a particular class of geometrical imperfections.

As is quite often the case, the static solution may not be a sufficient description of the problem. This belief has led to investigations of the dynamic responses in a cylindrical shell due to time-dependent axial loads. In essence, this approach endeavors to simulate the actual conditions encountered in a loading machine. Some recent theoretical work of Kadashevich and Pertseur[22] and Vol'mir and Agamirov[1] indicate a load little different from that predicted by the classical theory.

A more current trend, due mainly to the rapidly advancing computer technology, is to discard energy methods and to solve the governing equations by the use of a high speed digital computer. The combined

results of Keller and Reiss, Thurston, and Budiansky and Weinitschke[8] on the buckling of spherical caps agree well with each other but not with the experiments.

The present investigation reconsiders the buckling criteria proposed by von Kármán, Tsien and Friedrichs which attempted to define a lower limit to the stability of thin shells. Alternative methods are proposed for calculating both the von Kármán "lower buckling load" and the Tsien and Friedricks "energy load". The existence of the "energy load" is then established for certain shell buckling problems.

Next, a method of approximation is developed which may enable convergence to the "energy load" from below. This is shown to be equivalent to finding a lower bound to the smallest eigenvalue in a non-linear integro-differential eigenvalue equation and is accomplished by extending an enclosure theorem of Collatz. The method is then applied to a cylindrical shell subjected to axial compression. This development is motivated by the fact that all previous solutions have been approximate and represent generally upper bounds; hence, no error estimate has been available heretofore. However, for reasons discussed later on, the present study has failed to produce a numerically significant lower bound.

CHAPTER II

DERIVATIONS OF BASIC RELATIONS

In this section we derive a functional which has as its lowest stationary value the "lower buckling load" of von Kármán and Tsien.[49] It will then be shown that this functional provides us with an alternative method for approximating the lower buckling load.

Before entering into this analysis, we first present a summary of the results of a widely used shell buckling theory which is based on Donnell-type approximations.[27] We consider a thin isotropic shell of thickness h, volume V and surface area A, which is subjected to prescribed displacements on the boundary S' and prescribed tractions on the remainder of the boundary S". The expressions for the additional, or post-buckled, strain components are, within the confines of the customary shell theory approximation,

$$\bar{E}_{\alpha\beta} = \mathcal{E}_{\alpha\beta} - z\,\mathcal{K}_{\alpha\beta} \tag{2-1}$$

in which the bar indicates validity throughout the volume; $\alpha = 1,2$ refers to the middle surface coordinates; and $\mathcal{E}_{\alpha\beta}$ and $\mathcal{K}_{\alpha\beta}$ are, respectively, the additional membrane strain components and the change in curvature components from the unbuckled to the buckled state and are functions only of the middle surface coordinates. These are defined, within the limits of the Donnell-assumptions, as

$$\mathcal{E}_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha|\beta} + u_{\beta|\alpha} - 2w b_{\alpha\beta} + w_{|\alpha} w_{|\beta}\right) \tag{2-2}$$

and

$$K_{\alpha\beta} = w|_{\alpha\beta} \tag{2-3}$$

in which u_α represents the middle surface displacement components and w is the displacement normal to the middle surface, $b_{\alpha\beta}$ is the second fundamental form, and all differentiation is covariant with respect to the first fundamental form.

The additional potential energy, i.e., the difference in the potential energies of the buckled and unbuckled states, is defined as follows:

$$V = \lambda \int_V \overset{o}{N}^{\alpha\beta} \bar{\varepsilon}_{\alpha\beta}\, dV + \frac{1}{2}\int_V \bar{N}^{\alpha\beta} \bar{\varepsilon}_{\alpha\beta}\, dV - \lambda \int_{S''} \overline{\overset{o}{T}}^{\alpha} \bar{u}_\alpha\, dS \tag{2-4}$$

in which λ is a proportional load factor, and $\overset{o}{\bar{N}}^{\alpha\beta}$ and $\overline{\overset{o}{T}}^{\alpha}$ refer to some initial or prebuckled stress components and tractions respectively and satisfy (again within Donnell's approximation)

$$\bar{N}^{\alpha\beta}|_\alpha = 0 \qquad \text{and} \qquad \bar{N}^{\alpha\beta} n_\alpha = \overline{\overset{o}{T}}^{\beta} \tag{2-5}$$

with n_α the unit normal to the boundary of the shell. In the following analysis this prebuckled state will be chosen to occur at the Euler buckling load; hence the Euler load factor λ_E may be chosen, without loss of generality, to have the value

$$\lambda_E = 1 \tag{2-6}$$

Now the stress-strain relationship is, approximately,

$$\bar{N}^{\alpha\beta} = C^{\alpha\beta\delta\delta} \varepsilon_{\gamma\delta} \tag{2-7}$$

where $C^{\alpha\beta\gamma\delta}$ (the stress-strain coefficients) are the components of a positive definite tensor with the symmetry property:

$$C^{\alpha\beta\delta\delta} = C^{\gamma\delta\alpha\beta} = C^{\beta\alpha\gamma\delta} = \ldots . \tag{2-8}$$

If we now substitute Equations (2-1), (2-2) and (2-3) in Equation (2-4), we find, after integration with respect to z and upon application of Green's theorem for surfaces, that the potential energy is now given by

$$V = U_b + U_m - \lambda W , \tag{2-9}$$

where

$$U_b = \frac{h^3}{24} \int_A C^{\alpha\beta\gamma\delta} w|_{\alpha\beta} w|_{\gamma\delta} \, dA \geq 0 , \tag{2-10}$$

$$U_m = \frac{h}{2} \int_A C^{\alpha\beta\gamma\delta} \varepsilon_{\alpha\beta} \varepsilon_{\gamma\delta} \, dA \geq 0 , \tag{2-11}$$

$$W = -\frac{h}{2} \int_A \mathring{N}^{\alpha\beta} w|_{\alpha} w|_{\beta} \, dA . \tag{2-12}$$

This reduction has involved further approximations which correspond, in essence, to Love's first approximation.(27)

We obtain the following equilibrium equations by making the potential energy stationary with respect to the displacements:

$$N^{\alpha\beta}{}_{|\alpha} = 0 \tag{2-13}$$

and, for isotropic materials,

$$\frac{D}{h} w_{|\alpha\beta}^{\alpha\beta} - \lambda N^{\circ\alpha\beta} w_{|\alpha\beta} - N^{\alpha\beta}(b_{\alpha\beta} + w_{|\alpha\beta}) = 0 \tag{2-14}$$

with

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

and E and ν represent, respectively, the elastic modulus and Poisson's ratio. These non-linear equations govern the buckled state for a given load factor λ.

We now rewrite the potential energy in Equation (2-9) after considering the membrane stress equilibrium equations (2-13) and related boundary conditions to be satisfied. This enables us to express two of the displacements in terms of the third. If we solve for u_α we obtain

$$u_\alpha(w) = \overset{1}{u}_\alpha(w) + \overset{2}{u}_\alpha(w) \tag{2-15}$$

in which $\overset{1}{u}_\alpha$ and $\overset{2}{u}_\alpha$ are, respectively, first and second order functions in w and its derivatives. Substitution of this in the strain components

(2-2) yields

$$\varepsilon_{\alpha\beta}(w) = \overset{1}{\varepsilon}_{\alpha\beta}(w) + \overset{2}{\varepsilon}_{\alpha\beta}(w) \tag{2-16}$$

in which, according to the previous notation, $\overset{1}{\varepsilon}_{\alpha\beta}$ and $\overset{2}{\varepsilon}_{\alpha\beta}$ are first and second order in w and its derivatives. Finally, substituting Equation (2-16) in the potential energy gives us the revised form, namely,

$$V = A_2(w) - \lambda W_2(w) + 2B_3(w) + C_4(w) \tag{2-17}$$

where the subscript denotes the order of w and its derivatives contained in the functionals A_2, W_2, B_3 and C_4. These functionals are defined as follows:

$$A_2(w) = U_b(w) + \frac{h}{2}\int_A C^{\alpha\beta\gamma\delta}\overset{1}{\varepsilon}_{\alpha\beta}\overset{1}{\varepsilon}_{\gamma\delta}\,dA \geq 0 \tag{2-18}$$

$$W_2(w) = W(w) \tag{2-19}$$

$$B_3(w) = \frac{h}{2}\int_A C^{\alpha\beta\gamma\delta}\overset{1}{\varepsilon}_{\alpha\beta}\overset{2}{\varepsilon}_{\gamma\delta}\,dA \tag{2-20}$$

$$C_4(w) = \frac{h}{2}\int_A C^{\alpha\beta\gamma\delta}\overset{2}{\varepsilon}_{\alpha\beta}\overset{2}{\varepsilon}_{\gamma\delta}\,dA \geq 0 \tag{2-21}$$

For the "lower buckling load" analysis we will work with the potential energy in the form found in (2-17). This notation will formally simplify our problem and considerably reduce the efforts involved. To aid us further in this respect we now introduce additional functionals which are defined by expanding the functionals in (2-17), that is,

$$A_2(a+b) = A_2(a) + A_{11}(a,b) + A_2(b)$$

$$B_3(a+b) = B_3(a) + B_{21}(a,b) + B_{12}(a,b) + B_3(b) \tag{2-22}$$

in which $B_{21}(a,b)$, for example, represents a functional which is second order in a and its derivatives and first order in b and its derivatives. Expansions for $A_2(a+b+c)$, etc., can be similarly represented.

In addition we wish to prove an identity and state several others which relate the above functionals. We first show that

$$A_{11}(a,a) = 2A_2(a) \tag{2-23}$$

Starting with $A_2(a)$ and using the expansion property (2-22) we find

$$A_2(a) = A_2\left(\frac{a}{2} + \frac{a}{2}\right) = \frac{1}{4}A_2(a) + \frac{1}{4}A_{11}(a,a) + \frac{1}{4}A_2(a) \tag{2-24}$$

from which Equation (2-23) follows immediately.

Similarly, it can be shown that:

$$W_{11}(a,a) = 2W_2(a)$$

$$B_{111}(a,b,a) = 2B_{21}(a,b) = 2B_{12}(b,a)$$

$$B_{21}(a,a) = 3B_3(a) \qquad (2\text{-}25)$$

$$C_{2111}(a,b,a) = 3C_{31}(a,b) = 3C_{13}(b,a)$$

$$C_{2111}(a,b,b) = 2C_{22}(a,b)$$

$$C_{31}(a,a) = 4C_4(a)$$

We now proceed in our discussion of the "lower buckling load" by establishing the condition of equilibrium in terms of the requirement that the potential energy be stationary in the buckled state. This corresponds to the condition that the first variation with respect to w of the potential energy (2-17) must vanish, that is,

$$\delta_w V = A_{11}(w;\eta) - \lambda W_{11}(w;\eta) + 2B_{21}(w;\eta) + C_{31}(w;\eta) = 0 \qquad (2\text{-}26)$$

for all admissible functions η.

To determine the stability or instability of these buckled states we examine the second variation, which takes the form

$$\delta_w^2 V = \frac{1}{2}\left\{A_{11}(\eta,\eta) - \lambda W_{11}(\eta\eta) + 2B_{111}(w;\eta,\eta) + C_{2111}(w;\eta,\eta)\right\} \qquad (2\text{-}27)$$

or

$$\delta_w^2 V = A_2(\eta) - \lambda W_2(\eta) + 2B_{12}(w,\eta) + C_{22}(w,\eta) \qquad (2\text{-}28)$$

For a stable equilibrium configuration the second variation must be positive definite, i.e., $\delta^2 V > 0$, while for instability we may have $\delta^2 V \gtrless 0$. The general behavior of thin shells, represented by Equations (2-17), (2-26) and (2-28), is demonstrated in Figures 1 and 2. These show the typical post buckling behavior in which the drop in the load parameter λ and the existence of unstable states of equilibrium are due to the presence of third order terms in the potential energy.

The "lower buckling load" is defined as the minimum load necessary to maintain a shell in its buckled state and is represented by the smallest value $\lambda = \lambda_K$ in Equation (2-26) for which a non-trivial solution for w exists. As can be seen from Figure 2, the lower buckling load is also that load which divides the stable equilibrium states from the unstable and can, therefore, be alternatively defined as that solution of Equation (2-26) which makes the second variation semi-definite, i.e., $\delta^2 V \geq 0$ for all nontrivial η in Equation (2-28). This means we must look for that η which minimizes the second variation and that equilibrium solution which makes this minimum equal to zero. These conditions lead to the Jacobi equations in the theory of the calculus of variations and take the form

$$\delta_\eta \delta_w^2 V = 0 \qquad \text{subject to} \quad W_2(w) = 1 \,. \qquad (2\text{-}29)$$

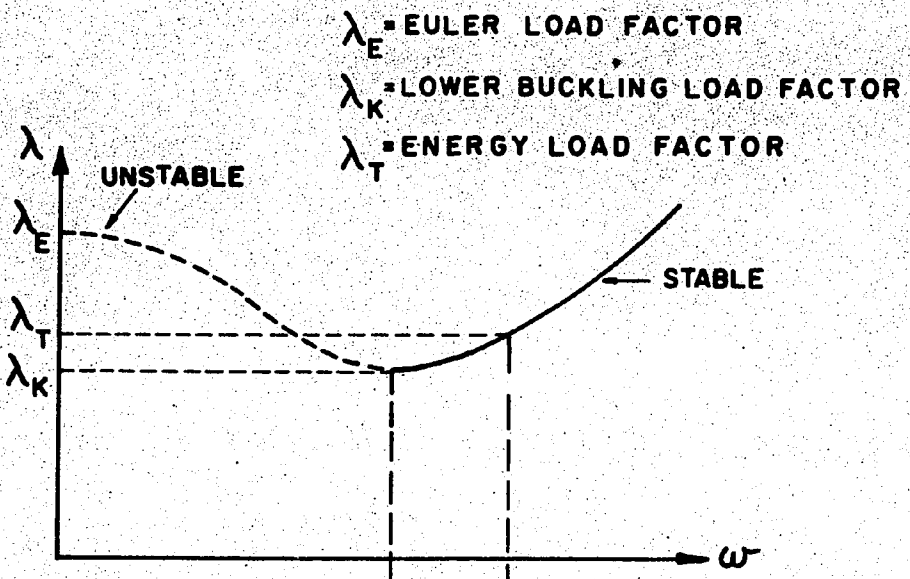


Figure 1. Load Factor vs Deflection.

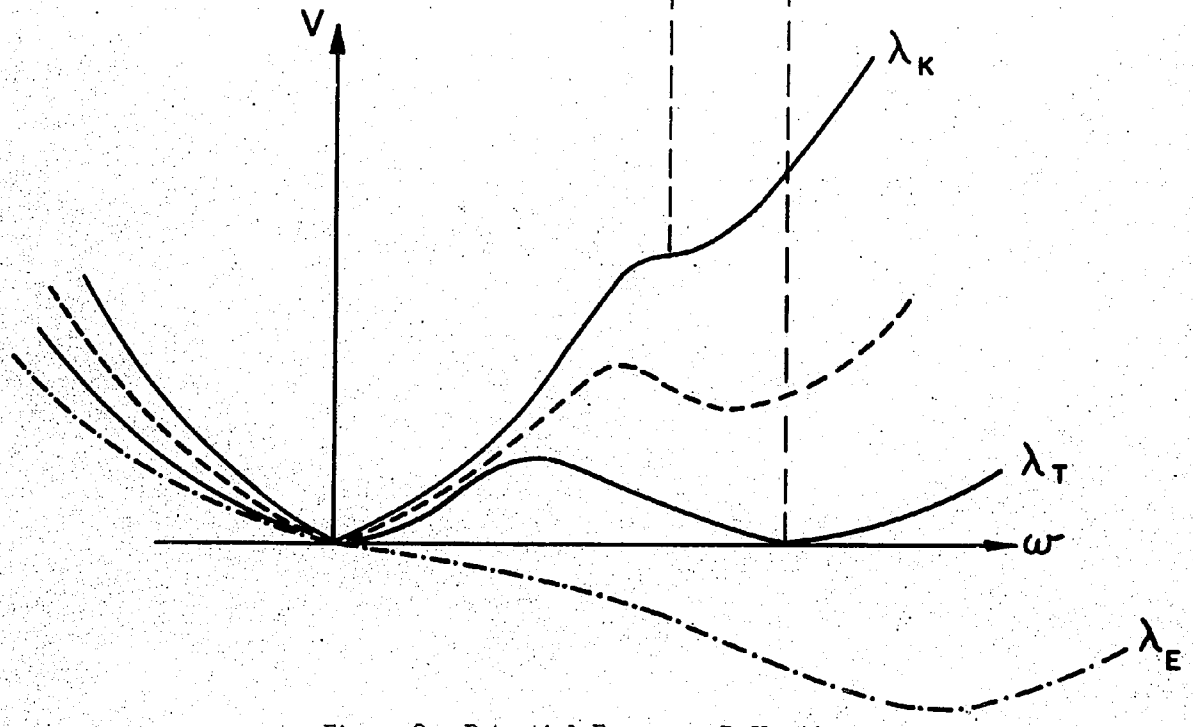


Figure 2. Potential Energy vs Deflection.

This operation is equivalent to finding the stationary equation for the functional

$$Q = \delta_w^2 V - \gamma (W_2 - 1) \tag{2-30}$$

where γ is a Lagrangian multiplier. Equation (2-29) now becomes

$$A_{11}(\eta,\beta) - \lambda W_{11}(\eta,\beta) + 2B_{111}(w,\eta,\beta) + C_{211}(w,\eta,\beta) - \gamma W_{11}(\eta,\beta) = 0 \tag{2-31}$$

for all β .

This is a linear eigenvalue equation with γ_i the eigenvalues and η_i the corresponding eigenfunctions. Letting $\beta = \eta_i$ we find that

$$\delta_w^2 V = \gamma_i \quad . \tag{2-32}$$

Since we are interested in that η_i which makes our minimum zero, we must therefore set $\gamma_i = 0$ and look for the η which satisfies the following equation:

$$A_{11}(\eta,\beta) - \lambda W_{11}(\eta,\beta) + 2B_{111}(w,\eta,\beta) + C_{211}(w,\eta,\beta) = 0 \tag{2-33}$$

for all β .

The "lower buckling load" is now represented by the smallest value of λ which permits a solution of Equation (2-26) and (2-33).

If in Equation (2-33) we let $\beta = w$ and in Equation (2-26) we restrict η to be the solution of Equation (2-33), the following system of equations results:

$$A_{11}(w;\eta) - \lambda W_{11}(w;\eta) + 4B_{21}(w;\eta) + 3C_{31}(w;\eta) = 0 \tag{2-34}$$

and

$$A_{11}(w;\eta) - \lambda W_{11}(w;\eta) + 2B_{21}(w;\eta) + C_{31}(w;\eta) = 0 \tag{2-35}$$

Subtracting the second from the first gives

$$B_{21}(w;\eta) + C_{31}(w;\eta) = 0 \tag{2-36}$$

and substituting this into either Equation (2-34) or (2-35) we arrive at the value of the "lower buckling load" factor λ_K :

$$\lambda_K = \frac{A_{11}(w;\eta) - C_{31}(w;\eta)}{W_{11}(w;\eta)} \tag{2-37}$$

in which w and η satisfy Equations (2-26), (2-33) and (2-36).

We now show that the "lower buckling load" is the lowest stationary value of the functional

$$\Lambda = \frac{A_{11}(w_1, w_2) - C_{31}(w_1, w_2)}{W_{11}(w_1, w_2)} \tag{2-38}$$

subject to

$$B_{21}(w_1, w_2) + C_{31}(w_1, w_2) = 0 \tag{2-39}$$

in which w_1 and w_2 are arbitrary functions. This requires setting equal to zero the first variation with respect to w_1 and w_2 of

$$\Pi = \Lambda - \bar{\gamma}\left[B_{21}(w_1, w_2) + C_{31}(w_1, w_2)\right] \tag{2-40}$$

in which $\bar{\gamma}$ is a Lagrangian multiplier. This yields the following two equations:

$$A_{11}(w_2, \eta_1) - \Lambda W_{11}(w_2, \eta_1) - C_{211}(w_1, w_2, \eta_1) + \bar{\gamma} W_{11}(w_1, w_2)\left[B_{111}(w_1, w_2, \eta_1) + C_{211}(w_1, w_2, \eta_1)\right] = 0 \tag{2-41}$$

for all η_1.

$$A_{11}(w_1, \eta_2) - \Lambda W_{11}(w_1, \eta_2) - C_{31}(w_1, \eta_2) + \bar{\gamma} W_{11}(w_1, w_2)\left[B_{21}(w_1, \eta_2) + C_{31}(w_1, \eta_2)\right] = 0 \tag{2-42}$$

for all η_2.

To determine $\bar{\gamma}$, let $\eta_1 = w_1$ in Equation (2-41) to get

$$A_{11}(w_1, w_2) - \Lambda W_{11}(w_1, w_2) - 3C_{31}(w_1, w_2) + \bar{\gamma} W_{11}(w_1, w_2)\left[2B_{21}(w_1, w_2) + 3C_{31}(w_1, w_2)\right] = 0, \tag{2-43}$$

and then using Equations (2-38) and (2-39) we find

$$\bar{\gamma} = \frac{2}{W_{11}(w_1, w_2)} \tag{2-44}$$

Substituting this value of $\bar{\gamma}$ into Equations (2-41) and (2-42) gives

$$A_{11}(w_2, \eta_1) - \Lambda W_{11}(w_2, \eta_1) + 2B_{111}(w_1, w_2, \eta_1) + C_{211}(w_1, w_2, \eta_1) = 0 \quad (2\text{-}45)$$

for all η_1.

and

$$A_{11}(w_1, \eta_2) - \Lambda W_{11}(w_1, \eta_2) + 2B_{21}(w_1, \eta_2) + C_{31}(w_1, \eta_2) = 0 \quad (2\text{-}46)$$

for all η_2.

We note that if Λ is identified with λ then Equations (2-45) and (2-46) are identical with Equations (2-26) and (2-33), i.e., the stationary equations of Λ are the same as the governing equations for the "lower buckling load". Therefore, the "lower buckling load" is not only the smallest value of Λ which permits a solution of Equations (2-45) and (2-46) but it is also the lowest stationary value of the functional (2-38).

This property of the functional (2-38) provides us with a new approach for approximating the "lower buckling load". When the Ritz method is applied to the potential energy (2-17) only one arbitrary function w is at our disposal; in the present approach, however, we have the freedom to select two functions w_1 and w_2. These two approaches are not unrelated since we will show that if w, w_1 and w_2 are all

chosen to be of the same form then the Ritz method applied to both the potential energy and the functional Λ yields the same approximation for the "lower buckling load".

We begin by applying the Ritz method to the potential energy (2-17) with an approximating series of the form

$$w = a_i w^i \qquad i = 1 \ldots n \qquad \text{(repeated indices denote summation)} \qquad (2\text{-}47)$$

Substituting this into the potential energy we obtain

$$V = a_i a_j [A_{ij} - \lambda W_{ij}] + 2 a_i a_j a_k B_{ijk} + a_i a_j a_k a_l C_{ijkl} \qquad (2\text{-}48)$$

with

$$A_2(a_i w^i) = a_i a_j A_{ij} \text{ , etc.} \qquad (2\text{-}49)$$

We then set the first variation of the potential energy equal to zero, that is,

$$\delta V = \frac{\partial V}{\partial a_m} \delta a_m = 0 \qquad \text{or} \qquad \frac{\partial V}{\partial a_m} = 0 \qquad (2\text{-}50)$$

to arrive at the equilibrium equations

$$a_i [A_{mi} + A_{im} - \lambda (W_{mi} + W_{im})] + 2 a_i a_j [B_{mij} + B_{imj} + B_{ijm}] \qquad (2\text{-}51)$$

$$+ a_i a_j a_k [C_{mijk} + C_{imjk} + C_{ijmk} + C_{ijkm}] = 0$$

The second governing equation for the "lower buckling load" involves minimizing the second variation, that is,

$$\delta^2 V = \frac{1}{2} \frac{\partial^2 V}{\partial a_m \partial a_n} b_m b_n \qquad (b_n = \delta a_n) \tag{2-52}$$

which results in the following equation:

$$\frac{\partial(\delta^2 V)}{\partial b_n} = 0 \qquad \text{or} \qquad \frac{\partial^2 V}{\partial a_m \partial a_n} b_m = 0 . \tag{2-53}$$

For this equation to have a non-trivial solution for the b_m the following condition must hold:

$$\text{determinant} \left(\frac{\partial^2 V}{\partial a_m \partial a_n} \right) = 0 \tag{2-54}$$

This equation plus Equation (2-51) are the governing equations for the "lower buckling load".

Now, using those a_n, b_n and λ which satisfy Equations (2-51) and (2-54), we multiply Equation (2-51) by b_m and Equation (2-53) by a_n. If we then apply Euler's homogeneity relationship, which states that if Φ is homogeneous in z of degree M then

$$\frac{\partial \Phi}{\partial z} z = M \Phi \quad , \tag{2-55}$$

we arrive at the following equations:

$$a_i b_m [A_{mi} + A_{im} - \lambda (W_{mi} + W_{im})]$$

$$+2 a_i a_j b_m [B_{mij} + B_{imj} + B_{ijm}]$$

$$+ a_i a_j a_k b_m [C_{mijk} + C_{imjk} + C_{ijmk} + C_{ijkm}] = 0 \qquad (2\text{-}56)$$

and

$$a_i b_m [A_{mi} + A_{im} - \lambda (W_{mi} + W_{im})]$$

$$+4 a_i a_j b_m [B_{mij} + B_{imj} + B_{ijm}]$$

$$+3 a_i a_j a_k b_m [C_{mijk} + C_{imjk} + C_{ijmk} + C_{ijkm}] = 0 \qquad (2\text{-}57)$$

Subtracting the first equation from the second yields an orthogonality relationship which corresponds to (2-36)

$$a_i a_j b_m [B_{mij} + B_{imj} + B_{ijm}]$$

$$+ a_i a_j a_k b_m [C_{mijk} + C_{imjk} + C_{ijmk} + C_{ijkm}] = 0 \qquad (2\text{-}58)$$

Substitution of this relationship in Equation (2-57) yields the Ritz approximation ${}_R\lambda_K$ for the "lower buckling load"

$${}_R\lambda_K = \frac{a_i b_m [A_{mi} + A_{im}] - a_i a_j a_k b_m [C_{mijk} + C_{imjk} + C_{ijmk} + C_{ijkm}]}{a_i b_m [W_{mi} + W_{im}]} \qquad (2\text{-}59)$$

We note that this expression is identical to (2-38) if we set $w_1 = a_1 w^1$ and $w_2 = b_1 w^1$. Moreover, in a similar manner we can show that the a_1 and b_1 which satisfy Equations (2-51) and (2-54) are the same coefficients which make the load factor (2-59) stationary. Thus, we can conclude that the two methods are equivalent for the particular case in which w_1 and w_2 are chosen to be of the same form.

On the other hand, the freedom to select the two functions w_1 and w_2 arbitrarily, i.e., not necessarily of the same form, may possibly increase the speed of convergence to the "lower buckling load". This numerical aid plus the simplicity of the functional form of (2-38) seems to indicate that this approach is computationally more advantageous than the previously used method for the approximation of the "lower buckling load".

CHAPTER III

THE ENERGY LOAD

This chapter is devoted to a theoretical study of the energy load proposed by Friedrichs[17] and Tsien.[43] The analysis is similar to that used in Chapter II but now a functional is derived which has the energy load as its minimum.

We begin by normalizing the deflection w_T as follows:

$$w_T = k w \qquad \text{with} \qquad W_2(w) = 1 \; . \tag{3-1}$$

On substitution of this into the potential energy (2-17) we obtain

$$V = k^2 A_2(w) - \lambda k^2 W_2(w) + 2k^3 B_3(w) + k^4 C_4(w) \; . \tag{3-2}$$

For equilibirum we again require the vanishing of the first variation of the potential energy; this leads to the following two equations:

$$\frac{\partial V}{\partial k} = 0 \tag{3-3}$$

and

$$\delta_w V = 0 \qquad \text{subject to} \quad W_2(w) = 1 \tag{3-4}$$

Performing the differentiation in Equation (3-3) we arrive at the first condition of equilibirum, namely,

$$A_2(w) - \lambda W_2(w) + 3k B_3(w) + 2k^2 C_4(w) = 0 \tag{3-5}$$

In order to utilize Equation (3-4) we introduce a Lagrangian multiplier γ and then set the first variation with respect to w of

$$\Pi = V - \gamma(W_2 - 1) \tag{3-6}$$

equal to zero. This leads to the second equilibrium condition:

$$A_{11}(w;\eta) - \gamma W_{11}(w;\eta) + 2k B_{21}(w;\eta) + k^2 C_{31}(w;\eta) = 0 \tag{3-7}$$

for all η

subject to $W_2(w) = 1$.

To describe the energy load factor λ_T, the above equilibrium equations must be supplemented by the additional condition that the potential energies of the buckled and unbuckled states are equal. For our "potential energy" function V, which is defined as the difference in the potential energies of the buckled and unbuckled states, this is equivalent to setting $V = 0$ (see Figure 2) and leads to the following equation:

$$A_2(w) - \lambda W_2(w) + 2k B_3(w) + k^2 C_4(w) = 0 \tag{3-8}$$

Our problem, then, is to find the smallestload factor λ which permits a non-trivial solution of Equations (3-5), (3-7) and (3-8). This problem can be made more compact if we solve Equations (3-5) and (3-8) simultaneously to find

$$k = -\frac{B_3(w)}{C_4(w)} \tag{3-9}$$

Substitution of this value of k into both Equations (3-7) and (3-8) reduces the number of relevant equations to the following two:

$$A_{11}(w,\eta) - \gamma W_{11}(w,\eta) - \frac{2B_3}{C_4} B_{21}(w,\eta) + \frac{B_3^2}{C_4^2} C_{31}(w,\eta) = 0 \qquad \text{for all } \eta \tag{3-10}$$

and $$\lambda = A_2(w) - \frac{B_3^2(w)}{C_4(w)} \quad \text{subject to } W_2(w) = 1 \ . \tag{3-11}$$

The energy load factor is now the smallest value of (3-11) which occurs when w satisfies Equation (3-10).

We now wish to prove two related properties of the functional in (3-11): 1) the energy load factor is the lowest stationary value of this functional and 2) this stationay value is actually the absolute minimum of this functional.

We start the proof of the first proposition by introducing a Lagrangian multiplier $\bar{\gamma}$ and setting the first variation of the functional

$$\overline{T} = A_2(w) - \frac{B_3^2(w)}{C_4(w)} - \bar{\gamma}(W_2(w) - 1) \tag{3-12}$$

equal to zero. This leads to the stationary equation for the functional (3-11):

$$A_{11}(w,\eta) - \bar{\gamma} W_{11}(w,\eta) - \frac{2B_3}{C_4} B_{21}(w,\eta) + \frac{B_3^2}{C_4^2} C_{31}(w,\eta) = 0 \qquad \text{for all } \eta \tag{3-13}$$

We determine the eigenvalue $\bar{\gamma}$ by letting $\eta = w$ in Equation (3-13) and find its value to be

$$\bar{\gamma} = A_2(w) - \frac{B_3^2(w)}{C_4(w)} \equiv \lambda \tag{3-14}$$

A comparison of Equations (3-10) and (3-13) reveals that these two equations are identical if we identify $\bar{\gamma}$ with γ. In addition, the expression (3-14) shows that the eigenvalues of the equilibrium equation (3-10) are really the stationary values of the functional (3-11). This means, therefore, that the energy load factor is the lowest stationary value of (3-11) and is equal to the smallest eigenvalue in the equilibrium equation (3-10).

To show that this stationary value is actually a minimum we must rewrite the functional (3-11) by using the definitions of the functionals A_2, B_3 and C_4 given in (2-18) through (2-21). If at the same time we incorporate the notation of an inner product by

$$(\overset{i}{\mathcal{E}}, \overset{j}{\mathcal{E}}) = \frac{h}{2}\int_A C^{\alpha\beta\rho\sigma}\, \overset{i}{\mathcal{E}}_{\alpha\beta}\, \overset{j}{\mathcal{E}}_{\rho\sigma}\, dA \tag{3-15}$$

in which the $\overset{i}{\mathcal{E}}$ are defined in (2-16), we arrive at the following expanded form for the functional (3-11):

$$\lambda = A_2 - \frac{B_3^2}{C_4} = U_b + (\overset{1}{\mathcal{E}}, \overset{1}{\mathcal{E}}) - \frac{(\overset{1}{\mathcal{E}}, \overset{2}{\mathcal{E}})^2}{(\overset{2}{\mathcal{E}}, \overset{2}{\mathcal{E}})} \tag{3-16}$$

Since $(\overset{i}{\mathcal{E}}, \overset{i}{\mathcal{E}}) > 0$, we can apply Schwarz's inequality which states that

$$(\xi', \xi^2)^2 \leq (\xi', \xi')(\xi^2, \xi^2) \quad . \tag{3-17}$$

Substitution of this in Equation (3-16) yields

$$\lambda = A_2(w) - \frac{B_3^2(w)}{C_4(w)} \geq U_b(w) > 0 \tag{3-18}$$

Therefore, the minimum of U_b with respect to w, if it exists, is a lower bound to the functional (3-11), that is,

$$\lambda \geq U_{b_{MIN}} > 0 \qquad \text{for all admissible } w \tag{3-19}$$

Now, on the basis that λ is bounded from below, we postulate the existence of a minimum for the functional (3-11). This minimum would then be described by the smallest stationary value of (3-11) which, as was shown before, is the energy load factor λ_T. Therefore, we have shown that the following inequalities hold:

$$\lambda \geq \lambda_T \geq U_{b_{MIN}} > 0 \tag{3-20}$$

If we now apply the Ritz method to the functional (3-11), we find, in an analysis similar to that at the end of Chapter II, that the resulting approximation for the energy load factor is the same as that found by applying the Ritz method to the potential energy (2-17). Moreover, because of the above indicated minimum property of this functional, this approximation will be an upper bound to the actual energy load factor λ_T.

This last result was shown previously by Langhaar and Boresi[28] but their analysis required the assumption of a stable buckled configuration at the energy load, i.e., a positive-definite second variation of the potential energy. In the present analysis, however, the stability of this state is not an assumption but rather a consequence. This can be seen most easily if we take the second variation of the functional (3-11) and evaluate it at the energy load factor and its associated mode. Carrying out the variations we obtain:

$$\delta_w^2 \lambda = A_2(\eta) - \lambda_T W_2(\eta) - \frac{2B_3}{C_4} B_{12}(w;\eta) + \frac{B_3^2}{C_4^2} C_{22}(w;\eta)$$
$$- \frac{2}{C_4}\left[B_{21}(w;\eta) - \frac{B_3}{C_4} C_{31}(w;\eta)\right]^2 > 0 \qquad \text{for all } \eta \tag{3-21}$$

This is positive-definite because the energy load is the absolute minimum of this functional. Without actually carrying out the calculations we will now note that Equation (3-21) is identical, except for a positive factor, to the second variation of the potential energy (3-2) with $k = -\frac{B_3}{C_4}$. Consequently, the second variation of the potential energy must also be positive-definite which, therefore, establishes the stability of the energy load configuration.

In the remainder of this chapter we present two examples for which the minimum of U_b exists: 1) a cylindrical shell subjected to axial compression and simply supported at its ends and 2) the same shell but with zero shear and zero slope boundary conditions.

The bending energy for a cylindrical shell under axial compression, based on the Donnell-type approximations, is given by Equation (2-10). If

we choose the middle surface coordinates the following way:



(3-22)

then the covariant derivatives become simple partial derivatives and the bending energy takes the following form:

$$U_b = \frac{h^3}{24}\int_A C^{\alpha\beta\rho\sigma} w_{,\alpha\beta}\, w_{,\rho\sigma}\, dA \tag{3-23}$$

In addition we assume all the initial stresses $\overset{\circ}{N}{}^{\alpha\beta}$ to vanish except $\overset{\circ}{N}{}^{xx}$ which is chosen to equal the negative of the Euler stress P_E, which for a cylindrical shell under axial compression equals

$$-\overset{\circ}{N}{}^{xx} = P_E = .605\,\frac{Eh}{R} \tag{3-24}$$

in which E is the elastic modulus, h the thickness and R the radius of the shell. The Euler load factor is then found by minimizing

$$A_2(w) \qquad \text{subject to} \qquad W_2(w) = 1 \tag{3-25}$$

and due to our choice for the initial stresses becomes

$$\lambda_E = 1 \tag{3-26}$$

If we compare the forms of the functionals (3-11) and (3-25), we recognize that the Euler load factor is an upper bound to the energy load factor,

that is,

$$\lambda_E = 1 > \lambda_T \tag{3-27}$$

Finally, this choice of initial stresses brings the normalizing condition, according to (2-12), into the following form:

$$W_2(w) = P_E \int_A w_{,x}^2 \, dA = 1 \tag{3-28}$$

Our problem then is to minimize the binding energy (3-23) subject to the condition (3-28); that is, we wish to minimize the functional

$$U_b(w) - \bar{\lambda}\,(W_2(w) - 1) \tag{3-29}$$

where $\bar{\lambda}$ is a Lagrangian multiplier. If we identity $U_b(w)$ with the second order functional $A_2'(w)$, that is,

$$A_2'(w) \equiv U_b(w) \tag{3-30}$$

we find the stationary equation of (3-29) to be

$$A_{11}'(w,\eta) - \bar{\lambda} W_{11}(w,\eta) = 0 \qquad \text{for all } \eta \tag{3-31}$$

By setting $\eta = w$ in Equation (3-31) we can determine the eigenvalue

$$\bar{\lambda} = A_2'(w) \equiv U_b(w) \tag{3-32}$$

The minimum of the bending energy, if it exists, is thus the smallest eigenvalue $\bar{\lambda}$ in Equation (3-31).

If we now consider (3-23) and (3-28), we can rewrite Equation (3-31) in its expanded form:

$$\frac{D}{h}\nabla^4 w + \lambda P_E w_{,xx} = 0 \tag{3-33}$$

This is immediately recognized as the equilibrium equation for a buckled rectangular plate of length L and width $2\pi R$ under the same boundary conditions as the cylindrical shell. We solve this equation for the two sets of boundary conditions mentioned before:

1) Simply supported

$$w(0) = w(L) = 0$$

$$w_{,xx}(0) = w_{,xx}(L) = 0 \tag{3-34}$$

and w periodic along the width of the plate.

2) Zero shear and sero slope

$$w_{,x}(0) = w_{,x}(L) = 0$$

$$w_{,xxx}(0) = w_{,xxx}(L) = 0 \tag{3-35}$$

and w periodic along the width of the plate.

To solve Equation (3-33) for the first set of boundary conditions, we try a solution in the form:

$$w = a \sin\frac{m_a \pi x}{L} \sin\frac{n_s S}{R} + b \sin\frac{m_a \pi x}{L} \cos\frac{n_s S}{R} \qquad \begin{matrix} 0 \le x \le L \\ 0 \le S \le 2\pi R \end{matrix} \tag{3-36}$$

which satisfies all the boundary conditions if a and b are arbitrary constants and m_1, n_1, m_2 and n_2 are integer constants. Substitution of this approximation into Equation (3-33) shows that we have an exact solution if $\bar{\lambda}$ satisfies the following equation:

$$\frac{\bar{\lambda} h R}{D} = \frac{a\left[\left(\frac{m_1\pi}{L}\right)^2+\left(\frac{n_1}{R}\right)^2\right]^2 + b\left[\left(\frac{m_2\pi}{L}\right)^2+\left(\frac{n_2}{R}\right)^2\right]^2}{\frac{\pi^2}{L^2}\left[a m_1^2 + b m_2^2\right]} \tag{3-37}$$

This is minimized when n_1 and n_2 are zero, which implies that the first term in our solution guess (3-36) does not enter. The elimination of all terms having a as a factor thus leads to

$$\frac{\bar{\lambda} h P_E}{D} = \left(\frac{m_2\pi}{L}\right)^2 \tag{3-38}$$

which is a minimum for $m_2 = 1$. The buckling stress of the plate therefore equals

$$\bar{\lambda} P_E = \frac{D\pi^2}{h L^2} \tag{3-39}$$

To determine if this value of the load factor is actually the minimum of the bending energy or simply the lowest stationary value, we note that the condition $n_2 = 0$ makes our solution independent of the circumferential direction. If this condition is imposed on Equation (3-33), the equation for the lowest eigenvalue becomes

$$\frac{D}{h} w_{,xxxx} + \bar{\lambda} P_E w_{,xx} = 0 \tag{3-40}$$

This is, of course, the equilibrium equation for the buckling of a column if $\frac{D}{h}$ is taken to be the elastic modulus. The lowest eigenvalue of this equation is known to yield a minimum to the bending energy; consequently, we have actually found a minimum to the bending energy (3-23) and the following inequalities thus hold:

$$\lambda = A_2(w) - \frac{B_3^2(w)}{C_4(w)} \geq U_b(w) \geq \frac{D\pi^2}{P_E h L^2} \tag{3-41}$$

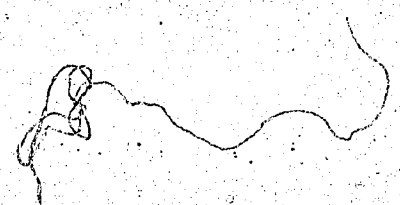
for all w which satisfy $W_2(w) = 1$.

To solve Equation (3-33) for the second set of boundary conditions we try a solution in the form:

$$w = a \cos \frac{m_1 \pi x}{L} \sin \frac{n_1 s}{R} + b \cos \frac{m_2 \pi x}{L} \cos \frac{n_2 s}{R} \tag{3-42}$$

which again satisfies all the boundary conditions if a and b are arbitrary constants and m_1, n_1, m_2 and n_2 are integer constants. If we substitute this approximation into Equation (3-33), we arrive at the same expression for the load factor (3-37) as we did for the first set of boundary conditions. Thus the remainder of the analysis would be the same as in the first case, and the conclusions reached there would apply here also. In other words, a minimum for the bending energy (3-23) exists for the two sets of boundary conditions considered.

It is noted that the lower bound (3-41) is unrealistically low and not likely to be of numerical interest (see Chapter IV). What is of interest is its existence, from which the existence of an energy load may be deduced.



CHAPTER IV

ENCLOSURE THEOREM

Since it is almost an impossible task to determine analytically the buckling load of a shell with arbitrary initial imperfections, it is necessary to resort to approximate methods and possibly intuitive approaches. This was the motivation for the energy criterion of Friedrichs [17] and Tsien. [43] They proposed that the energy load, defined as that load which makes the potential energy in the buckled and unbuckled states equal, was a lower limit to stability. Although this criterion has a weak theoretical foundation, [20] it has been made plausible by a recent publication of Bodner and Gjelsvik [6], who showed that for certain elastic systems which exhibit snap-buckling the energy load is a lower bound to the buckling loads associated with a particular class of geometrical imperfections.

If we accept the possibility that the energy load may be a lower limit of stability, then we must realize that the commonly used Ritz approximation is on the unsafe side. It would thus be desirable to obtain a lower bound to the energy load which, according to the above criterion, would then be a safe design load. In addition, a lower bound, in combination with the Ritz approximation, would enable us to estimate the error of our approximate energy load.

In the last chapter we determined, for two cases, a lower bound to the energy load P_T for a cylindrical shell under axial compression, namely,

$$P_T \equiv \lambda_T P_E \geq \frac{E h^2 \pi^2}{12(1-\nu^2)L^2} \tag{4-1}$$

If we compare this with an upper bound calculated by the Ritz method[43], that is,

$$.238\,\frac{Eh}{R} \geq P_T \tag{4-2}$$

we find that the lower bound does not appear to be a significant approximation due to the presence of h^2. In addition, its dependence on L contradicts the observed behavior that the buckling loads of a shell are not dependent on the length of the shell. This is all more apparent if, for example, we express these inequalities in terms of a typical shell with L = 4, R = 4 and h = .004; then the bounds become

$$.238 \times 10^{-3} E \geq P_T \geq .001 \times 10^{-3} E \quad , \tag{4-3}$$

which indicates the extreme variance between the two bounds.

Therefore, if our aim is to obtain "close" approximations to the energy load both from above and below, a better method for finding more realistic lower bounds must be sought. In this chapter we derive a method which may yield "close" lower bounds. This approach involves finding a lower bound to the smallest eigenvalue (i.e., to the energy load) in the equilibrium equation (3-10), which is repeated here for convenience:

$$A_{11}(w;\eta) - \lambda W_{11}(w;\eta) - \frac{2B_3}{C_4} B_{21}(w;\eta) + \frac{B_3^2}{C_4^2} C_{31}(w;\eta) = 0 \tag{4-4}$$

for all η

This equation can be rewritten in the form

$$Mw = \lambda Nw \tag{4-5}$$

in which N is the linear differential operator which has the load factor λ as a coefficient, and M is the integro-differential operator which encompasses the remainder of the equation. If we now define the Rayleigh quotient R(w) as

$$R(w) = \frac{A_2(w) - \frac{B_3^2(w)}{C_4(w)}}{W_2(w)} \geq \lambda_1 > 0 \tag{4-6}$$

which is positive by (3-19), we can establish several properties of the above operators:

1) The operator M is homogeneous of degree one and it is positive-definite, that is,

$$\int_A w M w \, dA = A_{11}(w,w) - \frac{2B_3}{C_4} B_{21}(w,w) + \frac{B_3}{C_4} C_{31}(w,w)$$
$$= 2\left(A_2 - \frac{B_3^2}{C_4}\right) > 0 \tag{4-7}$$

2) The operator N is positive-definite and self-adjoint, that is,

$$\int_A w N w \, dA = W_{11}(w,w) = 2W_2(w) > 0 \tag{4-8}$$

and

$$\int_A u N v \, dA = W_{11}(u,v) = W_{11}(v,u) = \int_A v N u \, dA \qquad (4\text{-}9)$$

We have attacked this problem by trying to extend lower bound techniques used in linear differential eigenvalue problems[21,23,39] to include the above integro-differential equation. Of the existing methods examined, however, only the enclosure theorem of Collatz[9] was found suitable for extension. One of the main difficulties which arose was that the other methods required the operator M to be self-adjoint; this is, however, not the case here, that is,

$$\int_A u M v \, dA \neq \int_A v M u \, dA \qquad (4\text{-}10)$$

as is easily seen from Equation (4-7). In fact, the proof given by Collatz for his enclosure theorem also requires the symmetry of M, but a revision of his proof is presented in this chapter which, at the expense of some of the generality of Collatz's results, does not require this property.

Before attempting to prove the revised enclosure theorem, we first note that if two shell problems have the related property:

$$W_2(w) \geq W_2^*(w) \qquad \text{for all } w \qquad (4\text{-}11)$$

or equivalently:

$$R(w) = \frac{A_2(w) - \frac{B_2^2(w)}{C_{22}(w)}}{W_2(w)} \leqslant R^*(w) = \frac{A_2(w) - \frac{B_2^2(w)}{C_{22}(w)}}{W^*(w)} , \qquad (4\text{-}12)$$

then the smallest eigenvalues satisfy

$$\lambda_T \leqslant \lambda_T^* \qquad (4\text{-}13)$$

which follows immediately by taking the minimum of both sides of the inequality (4-12). It is worthwhile to note that the inequality (4-13) would hold not only for the smallest eigenvalue but for all additional pairs of corresponding eigenvalues if M were self-adjoint because then Courant's maximum-minimum principle[11] could be invoked.

The revised proof which we now present follows closely that of Collatz.[9]

Revised Enclosure Theorem

We will work with the equation

$$Mw = \lambda N w \equiv (-1)^n \lambda [g(x)\, w^{(n)}]^{(n)} \qquad \begin{matrix} x_1 \leqslant x \leqslant x_2 \\ s_1 \leqslant s \leqslant s_2 \end{matrix} \qquad (4\text{-}14)$$

in which $w = w(x,s)$ and the n-th derivatives, denoted by (n), are taken with respect to x. This equation is accompanied by a sufficient number of homogeneous boundary conditions. Let us now assume the following:

a) M is a positive definite operator and homogeneous of degree one.

b) $g(x)$ is positive for $x_1 \leq x \leq x_2$

c) two admissible functions satisfy:

$$(-1)^n \int_A u[g v^{(n)}]^{(n)} dA = \int_A g u^{(n)} v^{(n)} dA$$

d) the Rayleigh quotient has a positive minimum:

$$R(w) = \frac{\int_A w M w dA}{\int_A w N w dA} \geq \lambda_1 > 0$$

in which λ_1 is the smallest eigenvalue in Equation (4-14).

Now define a function F_0 which satisfies:

$$M F_1 = N F_0 \tag{4-15}$$

if $F_1(x,s)$ is an arbitrary function which satisfies all of the boundary conditions.

Then, if the function:

$$\Phi(x,s) = \frac{F_0^{(n)}(x,s)}{F_1^{(n)}(x,s)} \tag{4-16}$$

does not change sign in the region and if $F_1(x,s)$ is a "sufficiently close" approximation to the first eigenfunction, the maximum and minimum of Φ enclose the smallest eigenvalue:

$$\Phi_{MIN} \leq \lambda_1 \leq \Phi_{MAX} \tag{4-17}$$

Proof

We know by Equations (4-14) and (4-15) that

$$MF_1 = NF_0 = (-1)^n [g(x) F_0^{(n)}(x,s)]^{(n)}, \tag{4-18}$$

and by Equation (4-16) that

$$F_0^{(n)}(x,s) = \Phi(x,s) F_1^{(n)}(x,s) . \tag{4-19}$$

Therefore we have

$$MF = (-1)^n [g(x) \Phi(x,s) F_1^{(n)}(x,s)]^{(n)}, \tag{4-20}$$

and, we recall, that F_1 satisfies all the boundary conditions.

We now write the following three equations for an admissible function $w(x,s)$ which satisfies all the boundary conditions:

1) $$Mw = (-1)^n \lambda' [g(x) \Phi_{MIN} w^{(n)}(x,s)]^{(n)}$$

2) $$Mw = (-1)^n \lambda'' [g(x) \Phi(x,s) w^{(n)}(x,s)]^{(n)}$$

3) $$Mw = (-1)^n \lambda''' [g(x) \Phi_{MAX} w^{(n)}(x,s)]^{(n)}$$

In cases 1 and 3 when $w = w_p(x,s)$, the eigenfunctions for (4-14), we obtain the following relationships for the associated eigenvalues:

$$\lambda_p' \Phi_{MIN} = \lambda_p \quad , \quad \lambda_p''' \Phi_{MAX} = \lambda_p \tag{4-21}$$

or

$$\lambda_p' = \frac{\lambda_p}{\Phi_{MIN}} \quad , \quad \lambda_p''' = \frac{\lambda_p}{\Phi_{MAX}} \; . \tag{4-22}$$

In case 2, $F_1(x,s)$ is an eigenfunction and its associated eigenvalue is

$$\lambda_k'' = 1 \tag{4-23}$$

which may not be the lowest eigenvalue of this equation.

These three cases can be compared by relating the denominators of their respective Rayleigh quotients. For an admissible function w we have

$$\begin{aligned} &\int_A w(-1)^n [g \Phi_{MAX} w^{(n)}]^{(n)} dA \\ &\quad = \int_A g \Phi_{MAX} [w^{(n)}]^2 dA \geq \int_A g \Phi(x,s) [w^{(n)}]^2 dA \\ &\quad \geq \int_A g \Phi_{MIN} [w^{(n)}]^2 dA \end{aligned} \tag{4-24}$$

which are valid inequalities because of the assumption (b) on g(x). The conclusions drawn from Equations (4-11) through (4-13) can now be applied here, and the following inequalities result for the smallest eigenvalues of the above equations:

$$\lambda_1''' \leq \lambda_1'' \leq \lambda_1' \tag{4-25}$$

It is important to note that λ_1'' is unknown to us unless $F_1(x,s)$ happens to be the first eigenfunction of case 2; then the smallest eigenvalue is

$$\lambda_1'' = 1 \tag{4-26}$$

by Equation (4-23). This actually occurs, for example, when F_1 is chosen to be the first eigenfunction of Equation (4-14) for then case 2 and Equation (4-14) become identical. It is plausible, then, that an F_1 which is a close approximation to the first eigenfunction of (4-14) might continue to be the first eigenfunction of case 2 so that (4-26) will still hold. This is the condition mentioned earlier that F_1 be "sufficiently close" to the first eigenfunction w_1 in Equation (4-14). Satisfaction of this condition and substitution of (4-22) into the inequalities in (4-25) leads directly to the enclosure of the smallest eigenvalue in Equation (4-14):

$$\frac{\lambda_1}{\Phi_{MAX}} \leqslant 1 \leqslant \frac{\lambda_1}{\Phi_{MIN}} \tag{4-27}$$

or

$$\Phi_{MIN} \leqslant \lambda_1 \leqslant \Phi_{MAX} . \tag{4-28}$$

This result is not as general as that obtained by Collatz, who found that the maximum and minimum of Φ enclosed at least one eigenvalue for any $F_1(x,s)$ which satisfied all the boundary contitions, that is,

$$\Phi_{MIN} \leqslant \lambda_p \leqslant \Phi_{MAX} . \tag{4-29}$$

This result, however, required that M be self-adjoint. The question now arises concerning the Collatz theorem (4-29): how do we know which eigenvalues we are enclosing? This answer is dependent on the choice of the "arbitrary" function $F_1(x,s)$. It is interesting to note that the enclosure of the smallest eigenvalue in (4-29) requires the identical restriction on F_1 indicated in our revised theorem: that F_1 be "sufficiently close" to the first eigenfunction w_1 in Equation (4-14).

Our interest, of course, lies in the determination of a lower bound to the smallest eigenvalue. The applicability of the revised enclosure theorem for finding lower bounds to the energy load is quickly realized if we note that the assumptions in the theorem are sufficient to describe the properties of the energy load equilibrium equation which are stated in Equations (4-5) through (4-9). In order to be certain that we are enclosing the energy load, however, we must check F_1 to make sure it is "sufficiently close" to w_1. For this check we propose the following test. Let the approximating function be chosen in the form of a series, such as,

$$F_1(x,s) = \sum_{i=1}^{n} a_i w^i(x,s) \tag{4-30}$$

in which the $w^i(x,s)$ satisfy all the boundary conditions and the a_i are constants which are determined by minimizing the Rayleigh quotient, namely,

$$R_n\left(\sum_{i=1}^{n} a_i w^i\right) = \frac{\int_A F_1 M F_1 \, dA}{\int_A F_1 N F_1 \, dA} \,. \tag{4-31}$$

It will then be true that

$$R_n \geq R_{n+1} \geq \cdots \geq \lambda_1 > 0 \qquad (4\text{-}32)$$

which implies that the a_i chosen in this manner make the approximating series (4-30) a continually better approximation of the first eigenfunction w_1. We now suggest that the series (4-30) satisfies the "sufficiently close" condition if the Φ_{MIN}, calculated from this series, approach the R_n from below as n increases.

CHAPTER V

APPLICATION TO A CYLINDRICAL SHELL

In this chapter we apply the Revised Enclosure Theorem, proved in the previous chapter, to the problem of finding a lower bound to the energy load for a thin circular cylindrical shell under axial compression with zero slope and zero shear boundary conditions. We showed in Chapter III that the energy load exists and is positive for this case.

If, as before, we take our coordinate system in the form (3-22) with x, s and z the axial, circumferential and radial coordinates, respectively, having the range

$$\begin{aligned} 0 &\leq x \leq L \\ 0 &\leq s \leq 2\pi R \\ -\frac{h}{2} &\leq z \leq \frac{h}{2} \end{aligned} \tag{5-1}$$

and, in addition, we introduce a stress function ϕ, satisfying

$$\begin{aligned} \phi_{,xx} &= N^{ss} \\ \phi_{,ss} &= N^{xx} \\ \phi_{,xs} &= -N^{xs} \end{aligned} \tag{5-2}$$

we arrive at the familiar form of the potential energy (2-4) for an axially compressed cylindrical shell, namely,

$$\begin{aligned} V = \frac{D}{2}\int_A [\,(w_{,xx} + w_{,ss})^2 + 2(1-\nu)(w_{,xs}^2 - w_{,xx}\, w_{,ss})\,]\, dA \\ - \frac{Ph}{2}\int_A w_{,x}^2\, dA + \frac{h}{4}\int_A [\,\phi_{,ss}\, w_{,x}^2 + \phi_{,xx}\, w_{,s}^2 - 2\phi_{,xs}\, w_{,x}\, w_{,s} \\ - 2\phi_{,xx}\,\frac{w}{R}\,]\, dA \end{aligned} \tag{5-3}$$

By minimizing the potential energy and considering the compatibility of the stress function $\phi(w)$, we arrive at the following system of equations which governs the post-buckling behavior of the shell:(47)

$$\frac{D}{h}\nabla^4 w + \lambda P_E w_{,xx} - \phi_{,xx} w_{,ss} - \phi_{,ss} w_{,xx} + 2\phi_{,xs} w_{,xs} - \frac{1}{R}\phi_{,xx} = 0 \tag{5-4}$$

and

$$\nabla^4 \phi = E\left[w_{,xs}^2 - w_{,xx} w_{,ss} - \frac{1}{R} w_{,xx}\right] \tag{5-5}$$

with P_E defined in (3-24).

We begin this analysis by approximating the radial deflection w_1 with the following series:

$$w_1 = \sum_{\alpha\beta} a_{\alpha\beta} \cos\frac{\alpha m \pi x}{L} \cos\frac{\beta n s}{R} \tag{5-6}$$

which satisfies the zero slope, zero shear boundary conditions if $a_{\alpha\beta}$ are constants and α, β, m and n are integer constants. Substituting this into the compatibility equation (5-5) and solving for the stress function ϕ on which we impose the following boundary conditions:

a) $\phi_{,sx} = 0$ at $x = 0, L$,

b) the axial displacement u equal to a constant at $x = 0, L$ subject to the condition that $\int_S \phi_{,ss}\, ds = 0$, (5-7)

we obtain

$$\Phi = E\Big\{ \sum_{\substack{\alpha\beta \\ \pm\gamma\,\pm\delta}} a_{\alpha\beta}\, a_{\gamma\delta}\, P_{\alpha\beta\gamma\delta} \cos\frac{(\alpha+\gamma)m\pi x}{L} \cos\frac{(\beta+\delta)n s}{R}$$

$$+ \sum_{\alpha\beta} a_{\alpha\beta}\, Q_{\alpha\beta} \cos\frac{\alpha m\pi x}{L} \cos\frac{\beta n s}{R} \Big\} \tag{5-8}$$

with

$$Q_{\alpha\beta} = \frac{\alpha^2\mu^2 R}{n^2\left[\alpha^2\mu^2+\beta^2\right]^2} \quad , \quad P_{\alpha\beta\gamma\delta} = \frac{\alpha\delta(\gamma\beta-\alpha\delta)\mu^2}{4\left[(\alpha+\gamma)^2\mu^2+(\beta+\delta)^2\right]^2} \tag{5-9}$$

and

$$\mu = \frac{m\pi R}{nL} \tag{5-10}$$

which is the ratio of the circumferential wavelength to the axial wavelength. The notation $\sum_{\substack{\alpha \\ \pm\gamma}}$ indicates that if α is summed over the values 0, 1, 2, then γ is summed over the values -2, -1, -0, 0, 1, 2. In addition, the subscripts on $a_{\alpha\beta}$ are always considered positive.

If we now substitute (5-6) and (5-8) into (5-3), we obtain the potential energy in the desired form (2-17), namely,

$$V = A_2(\omega) - \lambda P_E W_2(\omega) + 2B_3(\omega) + C_4(\omega) \tag{5-11}$$

with

$$A_2(\omega) = \frac{Dn^4\pi L}{4R^3} \sum_{\alpha\beta} a_{\alpha\beta}^2 \left\{ (\alpha^2\mu^2+\beta^2)^2 + \frac{E\alpha^2\mu^2 R}{Dn^2} Q_{\alpha\beta} \right\} \tag{5-12}$$

$$W_2(\omega) = \frac{n^2\pi L}{4R} \sum_{\alpha\beta} a_{\alpha\beta}^2 \alpha^2 \tag{5-13}$$

$$B_3(\omega) = \frac{E\mu^2 n^2 \pi L}{64 R^3} \sum_{\substack{\alpha \beta \\ \pm\gamma \pm\delta \\ \pm\sigma \pm\phi}} a_{\alpha\beta}\, a_{\gamma\delta}\, a_{\sigma\phi} \left\{ n^2(\beta^2\gamma\sigma - 2\alpha\beta\gamma\phi + \alpha^2\delta\phi) Q_{\alpha\beta} \right. \tag{5-14}$$

$$\left. + 8R(\alpha+\gamma)^2 P_{\alpha\beta\gamma\delta} \right\}$$

$$\text{subject to} \qquad \alpha \pm \gamma \pm \sigma = 0 \quad \text{and} \quad \beta \pm \delta \pm \phi = 0 \tag{5-15}$$

$$C_4(\omega) = \frac{E\mu^2 n^4 \pi L}{32 R^3} \sum_{\substack{\alpha \beta \\ \pm\gamma \pm\delta \\ \pm\sigma \pm\phi \\ \pm\rho \pm\theta}} a_{\alpha\beta}\, a_{\gamma\delta}\, a_{\sigma\phi}\, a_{\rho\theta} \left[(\beta+\delta)^2 \sigma\rho - 2\sigma\theta(\alpha+\gamma)(\beta+\delta) \right. \tag{5-16}$$

$$\left. + (\alpha+\gamma)^2 \phi\theta \right] P_{\alpha\beta\gamma\delta}$$

$$\text{subject to} \qquad \alpha \pm \gamma \pm \sigma \pm \rho = 0 \quad \text{and} \quad \beta \pm \delta \pm \phi \pm \theta = 0 \tag{5-17}$$

We note that the restrictions on the summation indices (5-15) and (5-17) imply that the odd coefficients a_{rs}, i.e., r + s equals an odd integer, must occur in pairs. If we represent any odd coefficient by $\mathring{a}$ and any even coefficient by $\overset{E}{a}$, we can rewrite the potential energy so that it explicitly shows all the combinations of odd and even coefficients which occur, that is,

$$V = (\mathring{a}\cdot\mathring{a}) + (\overset{E}{a}\cdot\overset{E}{a}) + (\mathring{a}\cdot\mathring{a}\cdot\overset{E}{a}) + (\overset{E}{a}\cdot\overset{E}{a}\cdot\overset{E}{a}) + (\mathring{a}\cdot\mathring{a}\cdot\overset{E}{a}\cdot\overset{E}{a}) \tag{5-18}$$

$$+ (\mathring{a}\cdot\mathring{a}\cdot\overset{E}{a}\cdot\overset{E}{a})$$

in which, for example, $(\mathring{a}\cdot\mathring{a}\cdot\overset{E}{a})$ represents the aggregate of the terms which have two odd and one even coefficient.

The stationary equations are now found by setting equal to zero the derivatives of (5-18) with respect to the coefficients, that is,

$$\frac{\partial V}{\partial \mathring{a}} = 0 \qquad \text{and} \qquad \frac{\partial V}{\partial \overset{E}{a}} = 0 \qquad (5\text{-}19)$$

Carrying out the indicated differentiation in the first part of (5-19) we find that

$$\frac{\partial V}{\partial \mathring{a}} = (\mathring{a}) + (\mathring{a}\cdot\overset{E}{a}) + (\mathring{a}\cdot\overset{E}{a}\cdot\overset{E}{a}) = 0, \qquad (5\text{-}20)$$

which is satisfied if all the odd coefficients $\mathring{a}$ are zero, independently of the values of the even coefficients $\overset{E}{a}$. Consequently, we need to consider only the even coefficients in the approximating series (5-6). This result agrees with the recent analyses of Almroth[2] and Cox.[12]

The non-zero coefficients are now chosen by minimizing the "energy load functional"

$$\frac{A_2(w) - \dfrac{B_3^2(w)}{C_4(w)}}{W_2(w)} \qquad (5\text{-}21)$$

while at the same time letting m and n take on various integer values. This analysis was carried out on an IBM 7090 digital computer using the Newton-Raphson iteration scheme for the determination of the minimizing coefficients. If these coefficients are then substituted into the

expression (5-21), we arrive at the Ritz approximation for the energy load $_{R}P_{T}$:

$$_{R}P_{T} = .0983 \frac{Eh}{R} \geq P_{T} \quad , \tag{5-22}$$

which is significantly lower than the approximate value calculated by Tsien,[43] namely,

$$_{R}P_{T} = .238 \frac{Eh}{R} \tag{5-23}$$

The present analysis was performed on a shell with dimensions $R = 4$, $L = 4$ and $h = .004$. For this case (5-21) reached a minimum when:

$$n = 12 \quad \text{and} \quad \mu = .5 \tag{5-24}$$

and the nine non-zero coefficients considered had the relative values:

a_{02}	a_{11}	a_{13}	a_{20}	a_{22}	a_{31}	a_{33}	a_{40}	a_{42}
-7.07	100.	3.09	33.	-14.8	6.70	2.88	6.28	-2.22

(5-25)

Since the end shortening is defined by

$$\epsilon = \frac{1}{L}\int_{0}^{L} u_{,x}\, dx \tag{5-26}$$

we find the end shortening in our notation to be

$$\epsilon = \frac{k^2 W_e(w)}{2\pi R L} \tag{5-27}$$

Using the coefficients in (5-25) we are able to calculate the ratio of the end shortening which occurs at the energy load, ϵ_T, to the end shortening which occurs at the classical buckling load, ϵ_{cl}, that is,

$$\frac{\epsilon_T}{\epsilon_{cl}} = 2.47 \quad . \tag{5-28}$$

In addition we are able to calculate the approximate maximum radial deflection in terms of the thickness of the shell, that is,

$$\left(\frac{w_T}{h}\right)_{MAX} = k\left(\frac{w}{h}\right)_{MAX} \simeq 15 \quad . \tag{5-29}$$

To check whether this relatively large deflection was sufficient to carry the stresses past the yield limit into the plastic range, we calculated approximately the maximum bending stress and the maximum direct stress for a steel shell, namely:

$$\text{Max. Bending Stress} = \left(N^b{}^{xx}\right)_{MAX} = \frac{Eh}{2(1-\nu^2)}\left(w_{,xx} + \nu w_{,ss}\right)_{MAX}$$

$$\simeq 25{,}000 \text{ psi.} \tag{5-30}$$

$$\text{Max. Direct Stress} = \left(N^d{}^{xx}\right)_{MAX} = \left(\Phi_{,ss}\right)_{MAX} + P_T$$

$$\simeq 37{,}000 \text{ psi.} \quad ,$$

which are within the allowable stress limit for most steels. Consequently, in the absence of initial imperfections, the shell buckles in the elastic range; this contrasts with the assumptions made by Donnell[14] in his imperfection analysis.

To determined a lower bound to the energy load we must first define a function w_o in the manner of (4-15); this results in the following equation:

$$-\frac{1}{E} w_{o,xx} = \frac{n^4}{R^4} \sum_{\alpha\beta} a_{\alpha\beta} \left\{ \frac{D}{E} [\alpha^2\mu^2 + \beta^2]^2 + \frac{\mu^2 R \alpha^2}{n^2} Q_{\alpha\beta} \right\} \cos \frac{\alpha m \pi x}{L} \cos \frac{\beta n s}{R}$$

$$- \frac{B_3 n^2 \mu^2}{4 C_4 R^4} \sum_{\substack{\pm\gamma \\ \pm\delta}} \sum_{\substack{\alpha \\ \beta}} a_{\alpha\beta} a_{\gamma\delta} \{ 4R(\alpha+\gamma)^2 P_{\alpha\beta\gamma\delta} - n^2(\alpha\delta - \beta\gamma) Q_{\alpha\beta} \} \cdot$$

$$\cdot \cos \frac{(\alpha+\gamma) m \pi x}{L} \cos \frac{(\beta+\delta) n s}{R} \qquad (5\text{-}31)$$

$$- \frac{B_3^2 n^4 \mu^2}{4 C_4^2 R^4} \sum_{\substack{\pm\gamma \\ \pm\sigma}} \sum_{\substack{\beta \\ \pm\phi}} a_{\alpha\beta} a_{\gamma\delta} a_{\sigma\phi} [(\alpha+\gamma)\phi - (\beta+\delta)\sigma] P_{\alpha\beta\gamma\delta} \cdot$$

$$\cdot \cos \frac{(\alpha+\gamma+\sigma) m \pi x}{L} \cos \frac{(\beta+\delta+\phi) n s}{R} \,.$$

After integrating this equation with respect to x, we set up the function Φ defined in (4-16), namely,

$$\Phi = \frac{w_{o,x}}{w_{i,x}} \qquad (5\text{-}32)$$

If we now substitute the results of our upper bound analysis (5-24) and (5-25) into (5-32) and minimize with respect to x and s, we should find, according to (4-17), a lower bound to the energy load. Our analysis, however, resulted in a positive Φ_{max} and a negative Φ_{min}; this not only is a meaningless lower bound but also does not satisfy the conition in the theorem that Φ must not change sign in the region. One

possible explanation for the failure of this method to yield a meaningful lower bound to the energy load is that the approximating series (5-6) is not a "close enough" approximation of the first eigenfunction of Equation (3-10). This explanation seems plausible if one notes that the last two non-zero terms considered in the series (5-6) have coefficients a_{40} and a_{42} which are of the same order of magnitude as several of the other retained coefficients. Consequently, termination of the series with a_{42} does not seem justified. Indeed, Almroth[2] found that a_{60} was also of this order of magnitude when he analyzed by the Ritz method the post-buckled range.

Moreover, it is known that the bounds calculated from the enclosure theorem of Collatz do not converge as rapidly as the upper bound calculated by the Ritz method; in fact, this is one of the reasons the enclosure theorem has been discarded in favor of other methods, for example that developed by Kato,[23] when a linear differential eigenvalue problem is considered. Consequently, a small error in the Ritz upper bound approximation may be sufficient to cause the Φ_{min} to become negative; this error may be due to the premature termination of the series (5-6).

Lastly, as we successively increased the number of non-zero terms in (5-6) for the Ritz approximation, the magnitude of k in (3-1) also increased. It ranged from .42 to .74 as the number of terms in the series went from five to nine. These successive increase in k point to the possibility that k increases indefinitely as the number of terms are increased. In other words, the functional (3-11) reaches

its minimum in the limit as k goes to infinity. If this is the case, then it is questionable that the enclosure theorem may be applied to this problem as the equality in Equation (4-6) no longer holds; this implies that λ_T is no longer an eigenvalue in Equation (4-4).

CHAPTER VI

CONCLUSIONS

A functional is derived which provides an alternative method for approximating the von Kármán "lower buckling load". It is shown that this method may yield faster convergence to the "lower buckling load" than the previously used Ritz method applied to the potential energy.

On consideration of a second functional, which has the Friedrichs and Tsien "energy load" as its lowest stationary value, we are able to establish: 1) this functional is bounded from below by the minimum of the bending energy, 2) the energy load is actually the minimum of this functional and 3) the energy load exists and is positive if the minimum of the bending energy exists.

A method of approximation, based on an enclosure theorem of Collatz, is then developed which may yield lower bounds to the smallest eigenvalue in a non-linear integro-differential equation. This method, theoretically, enables us to obtain realistic lower bounds to the energy load and, in combination with the Ritz method, provides us with an error estimate of the approximate energy load.

The application of these methods for the determination of bounds to the energy load for a cylindrical shell under axial compression was only partially successful. The Ritz method yielded an upper bound value of the energy load equal to $.0983 \frac{Eh}{R}$ in which h is the thickness and R the radius of the shell. Application of the revised enclosure theorem, however, did not produce a meaningful lower bound; the

failure of this method appears to be due to either the premature termination of the series approximation for the deflection or the possibility that the energy load is reached only in the limit as the end shortening goes to infinity. This in turn raises the question as to whether one of the fundamental assumptions of the many approximate shell analyses performed by previous investigators is based on fact. This assumption, namely that the load carried by a shell after buckling rises again after an initial drop, has never been successfully demonstrated. The present investigation has turned up no evidence which would tend to lend credence to that assumption.

REFERENCES

1. Agamirov, V. L., and Vol'mir, A. S., "Behavior of Cylindrical Shells under Dynamic Loading by Hydrostatic Pressure or by Axial Compression," Translation in American Rocket Society Journal Supplement, January, 1961.

2. Almroth, B. O., "Postbuckling Behavior of Axially Compressed Circular Cylinders," AIAA, 1, No. 3, (March, 1963), 630.

3. Appl, F. C., and Zorowski, C. F., "Upper and Lower Bounds for Special Eigenvalues," J. Appl. Mech., 26, No. 2, (June, 1959), 236.

4. Batdorf, S. B., "A Simplified Method of Elastic Stability Analysis for Thin Cylindrical Shells," I-Donnell Equations, NACA TN 1341, June, 1947, II-Modified Equilibrium Equation, NACA TN 1342, 1947.

5. Batdorf, S. B., Schildcrout, M., and Stein, M., "Critical Stress of Thin-walled Cylinders in Axial Compression," NACA TN 1343, 1947.

6. Bodner, S. R., and Gjelsvik, A., "Energy Criterion and Snap-buckling of Arches," J. Eng. Mech. Div., ASCE, 88, No. EM5, (October, 1962).

7. Bollerman, W., "The Inclusion of Eigenvalues in the Application of the Maxmin Principle (in German)," Z. angew. Math. Mech., 40, No. 8 (August, 1960).

8. Budiansky, B., and Weinitschke, H. J., "On Axisymmetrical Buckling of Clamped, Shallow, Spherical Shells," J. Aero. Sci., 27, No. 7, (July, 1960).

9. Collatz, L., Eigenwertaufgaben mit technischen Anwendungen, Leipzig, 1949, Akademische Verlagsgesellschaft.

10. Collatz, L., The Numerical Treatment of Differential Equations, 3rd ed., Berlin, Springer-Verlag, Vol. 15, 1960.

11. Courant, R., and Hilbert, D., Methods of Mathematical Physics, 1st ed., Interscience Publishers, Inc., New York, Vol. 1, 1953.

12. Cox, H. L., The Buckling of Plates and Shells, International Series of Monographs in Aeronautics and Astronautics, Pergamon Press, Mac Millan Co., New York, 1963.

13. Diaz, J. B., "Upper and Lower Bounds for Eigenvalues," Proceedings of the Eighth Symposium on Applied Mathematics, Amer. Math. Soc., (1947), 53-78.

14. Donnell, L. H., "A New Theory for the Buckling of Thin Cylinders under Axial Compression and Bending," Trans. of ASME, 56, (1934), 795.

15. Donnell, L. H., and Wan, C. C., "Effect of Imperfections on Buckling of Thin Cylinders and Columns Under Axial Compression," J. Appl. Mech., 17, No. 1, (1950), 73.

16. Flügge, W., "Die Stabilität der Kreiszylinderschale, "Ingenieur-Archiv, 3, (1932), 463-506.

17. Friedrichs, K. O., "On the Minimum Buckling Load for Spherical Shells," Theodore von Kármán Anniversary Volume, Cal. Inst. Tech., Pasadena, (1941), 258-272.

18. Fung, Y. C., and Kaplan, A., "Buckling of Low Arches and Curved Beams of Small Curvature," NACA TN 2840, 1952.

19. Fung, Y. C., and Sechler, E. E., "Buckling of Thin-walled Circular Cylinders under Axial Compression and Internal Pressure," J. Aero. Sci., 24, No. 5, (1957), 351.

20. Fung, Y. C., and Sechler, E. E., "Instability of Thin Elastic Shells," Proceedings of the First Symposium on Naval Structural Mechanics, Standford University, (August, 1958), 115-168.

21. Gould, S. H., Variational Methods for Eigenvalue Problems, University of Toronto Press, 1957.

22. Kadashevick, Iu. I., and Pertseur, A. K., "On the Loss of Stability of Cylindrical Shells under Dynamic Loading," IZV. AKAD SSSR. Odt. Tekhn. Nauk. Mech., Mashinosti, No. 3, (1960), 30-33, Translation in American Rocket Society Journal Supplement, 32, No. 1, (January, 1962), 140-143.

23. Kato, T., "Upper and Lower Bounds of Eigenvalues," J. Phys. Soc. of Japan, 4, (1949), 334.

24. Kempner, J., "Recent Results in the Theory of Large Deflections of Cylindrical Shells," Ninth Congress of Intern. Mecan. Appl., University of Bruxelles, 6, (1957), 247.

25. Kempner, J., "Post Buckling Behavior of Axially Compressed Circular Cylindrical Shells," J. Aero. Sci., 21, No. 5, (1954), 239.

26. Koiter, W. T., "Elastic Stability and Post-Buckling Behavior," presented at First All-Union Congress of Theoretical and Applied Mechanics, Moscow, 1960.

27. Koiter, W. T., "A Consistent First Approximation in the Theory of Thin Elastic Shells," Proceedings of Symposium on the Theory of Thin Elastic Shells, Technological University of Delft, Interscience Publishers, Inc., New York, 1959.

28. Langhaar, H. L., and Boresi, A. P., "Snap Through and Post Buckling Behavior of Cylindrical Shells under the Action of External Pressure," Univ. of Ill. Eng. Exp. Station, Bull. No. 443, 1957.

29. Leggett, D. M. A., and Jones, R. P. N., "The Behavior of a Cylindrical Shell under Axial Compression when the Buckling Load has been Exceeded," Rand M 2190, British Aero, Res. Comm., 1942.

30. Leggett, D. M. A., "The Buckling of Thin Cylindrical Shells under Axial Compression," Aca. Serbe Sci. Publ. Inst. Math., 7, (1954), 47-60.

31. Lo, H., Crate, H., and Schwartz, E. B., "Buckling of Thin-walled Cylinders under Axial compression and Internal Pressure, NACA TN 2021, 1950.

32. Michielsen, Herman F., "The Behavior of Thin Cylindrical Shells after Buckling under Axial Compression," J. Aero. Sci., 15, No.12, (1948), 738.

33. Mushtari, Kh. M., and Suchenkov, A. V., "Stability of Cylindrical and Conical Shells of Circular Cross-Section with Simultaneous Action of Axial Compression and External Pressure," Prikl. Mat. Mekh., 18, No. 6, 1954, Translation in NACA TM 1433, 1958.

34. Mushtari, Kh. M., and Galimov, K. Z., Non-Linear Theory of Thin Elastic Shells, Israel Program for Scientific Translations Ltd., 1961.

35. Nash, W. A., "Effect of Large Deflections and Initial Imperfections on the Buckling of Cylindrical Shells Subjected to Hydrostatic Pressure," J. Aero. Sci., 22, No. 4, (April, 1955), 264.

36. Southwell, R. V., "Some Extensions of Rayleigh's Principle," Quart. J. of Mech. and Appl. Math., 6, (1953), 257-272.

37. Stein, M., "The Effect of the Buckling of Perfect Cylinders of Prebuckling Deformations and Stresses Induced by Edge Supports," NASA TN D-1510, (December, 1962), 217.

38. Temple, G., and Bickley, W. G., Rayleigh's Principle and Its Applications to Engineering, Dover Publications, Inc., 1956.

39. Temple, G., "The Computation of Characteristic Numbers and Characteristic Functions," London Math. Soc. Proc., 29, (1928), 257.

40. Thielemann, W. G., "On the Post-Buckling Behavior of Thin Cylindrical Shells," Deutsche Forschungsanstalt fur Luft- und Raumfahrt E.V. (DFL) Braunschweig. Translated in NASA TN D-1510, (December 1962), 203.

41. Timoshenko, S., Theory of Elastic Stability, McGraw-Hill Inc., New York, 1936.

42. Tsien, H. S., "Buckling of a Column with Non-linear Lateral Supports," J. Aero. Sci., 9, No. 4, (1942).

43. Tsien, H. S., "A Theory for the Buckling of Thin Shells," J. Aero. Sci., 9, 10, (1942), 373.

44. Tsien, H. S., "Lower Buckling Load in the Non-linear Buckling Theory for Thin Shells," Quart. Appl. Math., 5, No. 2, (1947), 236-237.

45. Vol'mir, A. S., Elastic Plates and Shells, (in Russian) State Publishing House of Technical-Theoretical Literature, Moscow, 1956.

46. von Kármán, T., Dunn, L. G., and Tsien, H. S., "The Influence of Curvature on the Buckling Characteristics of Structures," J. Aero. Sci., 7, No. 7, (May, 1940), 276-289.

47. von Kármán, T., and Tsien, H. S., "The Buckling of Thin Cylindrical Shells under Axial Compression," J. Aero. Sci., 8, No. 8, (1941), 302.

48. von Kármán, T., "The Engineer Grapples with Non-Linear Problems," Bull. Amer. Math. Soc., 46, (1940), 615.

49. von Kármán, T., and Tsien, H. S., "The Buckling of Spherical Shells by External Pressure," J. Aero. Sci., 7, No. 2, (December, 1939), 43-50.

50. Yoshimura, Y., "On the Mechanism of Buckling of a Circular Cylindrical Shell under Axial Compression," NACA TM 1390, 1955.