

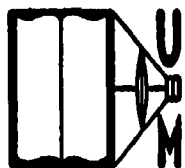
DOCTORAL DISSERTATION SERIES

TITLE EXPERIMENTAL INVESTIGATION OF
COLUMN ACTION AFTER BUCKLING

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EXPERIMENTAL INVESTIGATION OF
COLUMN ACTION AFTER BUCKLING

by

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of the requirements for the degree of
Doctor of Philosophy in the
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PREFACE

The author wishes to express his sincere gratitude to two persons: first, to Professor J. A. Van den Broek for his suggestion of the problem and the benefit of his many years of interest in columns, and secondly to the author's wife, Gloria, for her never failing help and constantly inspiring interest in the work.

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HISTORY AND BACKGROUND OF THE PROBLEM

Problems involving elastic stability were all but unknown to the ancients. In 1736 Van Musschenbroek reported that the strength of a wood column varied inversely as the square of its length, and directly as the width and depth squared. It was not until 1757, when Euler did his classic work on columns, that the first solution to an elastic stability problem was formulated. Even at that time Euler's solution was only of academic interest since there were almost no structures being built which used materials other than heavy pieces of masonry or wood, and machines were still built by rule of thumb.

For almost 100 years after Euler's paper little attention was paid to it. In fact, fifty years passed before Young's modulus was recognized in 1807, and 101 years went by before Scheffler published the secant formula. It was not until the development of commercial cast iron about 1845 that metal members began to be used extensively to withstand axial compression. Before cast iron became plentiful the wood or masonry columns in use were of such massive cross section contrasted to their height that no stability problems arose. Cast iron, being able to stand a higher load per unit of area permitted smaller cross

sections to be used and gradually brought problems of stability into the engineering picture. Even Professor Rankine, in his works as late as the 1870's, pays little attention to the column problem.

The development of commercial wrought iron and later the production of open hearth steel about 1868 both raised the allowable unit loads still further, thus making the columns even more slender and the stability problem more important.¹ The Euler formula was rediscovered about 1880 and put into more modern form by using newer mathematical notation. Since that time several hundreds of publications have appeared on the subject of columns. A complete list up to 1921 was compiled by Salmon in England² and recently Jakkula has brought the list up to 1947 for steel columns.³ It is safe to say that no other single engineering problem has received such attention.

Many hundreds of column design formulas have been proposed for finding the working loads. Some of these are based on the results of many experiments with a given material, others on purely mathematical analysis. The experimental formulas are frequently developed to be used with new materials as they are brought out, and thus fit

1. White, A. H., Engineering Materials McGraw-Hill, New York, 1931 Chapter VI.
2. Salmon, E. H., Columns Henry Froude and Hodder and Straughton, London, 1921.
3. Jakkula, A. A. and Stephenson, H. K., Steel Columns, A Survey and Appraisal of Past Works. Texas Engineering Experiment Station Bulletin 91, 1947.

only specific cases, but almost all of the formulas fall into one of four general types. These are: 1) Straight Line, 2) Rankine-Gordon, 3) Johnson Parabolic, and 4) Plastic Model.

Each of these types represents an attempt to reduce the theoretical Euler values to workable practice. It should be noted that the Euler load is the theoretical upper limit for axial compression, but in practice this theoretical load is seldom reached. Even in the laboratory, under conditions much more ideal than can possibly be realized in actual structures, the Euler load is very difficult to obtain for certain slenderness ratios. For this reason it was not until about 150 years after Euler's paper, and about 20 years after its rediscovery, that Euler's theory was ever satisfactorily verified experimentally, although many attempts were made.

All of the usual column formulas are directed at finding the maximum safe axial load which the member will support. Some safety factor based on experience or judgment may be inserted in the relationship even though the problem is non-linear, and frequently combinations of two formulas are used to cover the working range of slenderness ratios. Little attempt is made to examine the action of the column after the maximum load has been reached. It is assumed that once the maximum load has been attained, the slightest increase in load will produce disastrous deflections and the result is said to be a failure.

This assumption is true in the case of gravity loads on statically determinate structures, but is not necessarily true in the case of statically indeterminate or redundant structures where more than one supporting member may be used to carry the load. In a redundant structure the failure of a member by buckling means that some, but not all of the load formerly carried by the buckled member must be shifted to some other member or members. This process of shifting the loads continues until enough members have either buckled or plastically bent or twisted so that the failure of one more will result in a structure which is no longer able to maintain its geometry. The failure of this last member calls for a shift in load, but there is now no member able to assume this shifted load, and therefore the structure collapses.

This method of analysis of the failure of indeterminate structures is in accord with the principles of limit design as laid down by Van den Broek. It should be noted that the primary emphasis is upon deformations, rather than upon stresses in the designation of what constitutes a failure.

One of the most important questions in the analysis is what happens after a member fails by buckling. The usual column formulas give no information as to this, but they limit themselves to design procedures to permit selection of a member to withstand a certain design load.

They ignore the load shifting after buckling and thus result in some overdesign.

The critical factor in the load shifting after buckling is the amount of load to be shifted. It has been the common practice to assume that the entire load was to be transferred, none of it to be held by the buckled member. It was acknowledged that this was in error, but that the error was on the safe side. In more recent years the practice has changed to the assumption that a member buckles and continues to deflect at some constant load, that is, the limit load. This provides a convenient manner of apportioning the loads in structural members but neglects the fact that the supported load is not really a constant but is a constantly decreasing quantity as the deflection increases. This error tends to be on the unsafe side in the useful column range.

For certain structural members having very large slenderness ratios (of the order of 200 or larger for steel) there is a range of deflection of about 0.10 per cent where the supported load is actually very nearly constant, but such slender members are outside the useful range and the 0.10 per cent range is really quite small. For instance, it is only one half of the range commonly used for Johnson's apparent elastic limit for materials which do not have a sharp yield point. For columns in the useful range, the assumption of deflection at constant load is not justified.

It is the purpose of this investigation to determine experimentally the action of mild steel columns after they have buckled, in order to determine how much load they will actually support, and thus how much load must be shifted to other members. The range of slenderness ratios has been chosen from 20 to 160 to cover the usual useful range for steel. In addition, since one of the common methods of analyzing frameworks is by the energy method, it is intended to investigate the amount of energy dissipated during the large buckling, and also to examine the action of the column as the load is removed. Finally, it is proposed to verify a published column formula which is based on the action of a prismatic bar after buckling.

While it is beyond the scope of the present investigation, it should be pointed out that in addition to the amount of load to be shifted, the question of which member assumes the shifted load is of equal importance. Such a determination would depend upon the specific structure under consideration, and it leads to the very important question of the sequence in which the redundant members fail. In an n -fold redundant structure each permutation of n bars, n at a time, over all the bars of the structure, must be considered in order to find the actual sequence of failure. This analysis would also indicate where to put additional strength to the best advantage, since the weakest members would show up automatically in the sequence analysis.

There has been one unique column design formula worked out by Van den Broek which is based on the load carrying capacity of the member immediately after it has buckled. This load capacity is considered to be the working limit for the member and puts the formula in the plastic model class. This formula is

$$\frac{P}{A} = S_2 \left[\sqrt{1 + \left(\frac{S_2}{2 S_{cr}} \right)^2} - \left(\frac{S_2}{2 S_{cr}} \right)^2 \right]$$

where S_2 is the lower yield stress for a ductile material and S_{cr} is the Euler load stress.¹ This is a ductile equilibrium column formula for use with limit design.

It should be noted that the equilibrium which is called ductile equilibrium is really a state of quasi-equilibrium. This comes about because of the fact that, after buckling occurs and some of the material has been strained into the plastic range, the load supported by the column decreases regardless of whether the axial deflection of the bar increases or decreases.

This formula is further based on the consideration that the equilibrium load just after buckling is developed without any increase in axial strain. This situation is theoretically conceivable, but is unattainable experimentally because of the elasticity of the parts which serve to support the ends of the member. As soon as the bar buckles and its load carrying capacity decreases, the

1. Van den Broek, J. A., Theory of Limit Design
J. Wiley, New York, 1948. Pp. 97-103.

supporting members are also under a reduced load. They give up elastic strain energy which was stored in them. They act like a spring, forcing the ends of the buckled member to develop additional axial shortening beyond what it would develop by itself during the buckling.

Another limitation of the formula is that it is derived for a rectangular cross section whose sides are not greatly different from each other. It is not clear at present whether long, narrow rectangular cross sections or sections with re-entrant corners would fall within the analysis given or not. Sections with thin portions which fail by local buckling are also subject to doubt.

A relationship very similar to Van den Broek's was derived in 1936 by D. H. Young.¹ It was to be used with columns with some initial eccentricity, and was given as

$$S_y = \frac{f_y}{1 + \frac{\Delta}{K} \left(\frac{1}{1 - \frac{S_y}{S_c}} \right)}$$

where S_y is average yield stress for column section

f_y is yield stress obtained from test sample

Δ is initial deflection or wow

K is section core radius or the quotient of the section modulus by the area of the section

S_c is the Euler stress.

This formula plots very closely to Van den Broek's formula IX if Δ of $1/700$ is used for a square section.

1. Young, D. H., "Rational Design of Steel Columns," ASCE Transactions, 101 1936, Pp. 422-500.

Young's formula allows for different shapes of sections by using the core radius, K , and he also plotted curves for H sections. For design purposes Young's formula would have to be drawn up in a series of curves to make its use practical, whereas formula IX can be used directly.

PROCEDURE AND SPECIMEN PREPARATION

As in most experimental work, the actual method of the experiment was quite detailed and became rather involved with elusive error producing factors. Many hours were spent in fruitless search for things which prevented successful duplication of results. However, after many attempts, a system was evolved which gave answers which could be duplicated within 0.0001 inch of total deformation. Since this was the smallest amount of deflection which could be read with the available equipment, this limit was considered satisfactory.

The basic attack to the problem was simply to measure the axial loads on a column and the corresponding axial deformation. Measurement of the load was easily accomplished by use of a Riehle universal testing machine whose accuracy was within one-half of one per cent. Deformation measurements were very much more difficult, however, due both to the elasticity of the supporting mechanism and the changes of shape of the specimen after buckling occurred.

The method which finally proved reliable made use of a Starrett dial gauge which indicated the motion of the movable crosshead of the testing machine and a correction

curve which accounted for all other deflections except those of the specimen. When a certain crosshead deflection was indicated, the corresponding axial load was read, and the correction in terms of the load was used in a simple subtraction to give the true specimen deformation.

The correction curve was obtained by using a short (4 to 5 inch) piece of one inch diameter bar stock, cut from the same lot as the test pieces, and cementing a pair of SR-4 strain gauges to it. By choosing a short piece no buckling was encountered at the loads used. By placing the gauges diametrically opposite each other any bending effects were eliminated by cancellation. The piece was then mounted in exactly the same manner as the test specimens and loads applied. The movement of the crosshead was indicated by the dial gauge, and included the bending of the crosshead, deflection of the specimen holders at each end, deflection of the bed of the machine, and the deflection of the test piece. The true specimen deformation was given by the SR-4 gauges. Subtraction of the SR-4 readings from the indicator dial readings gave the amount of deformation in all of the other parts of the system for the various loads.

While this method sounds very simple, the actual development of the error curve was a very tedious and exasperating process. It was found that any crosshead deflection readings for loads under 50 pounds were all but

impossible to reproduce within the desired range of 0.0001 inch, and for loads under 20 pounds the variations in the errors were often larger than what was being measured. It was seen that this situation could be avoided by putting on an initial load of 100 pounds. Such a load was large enough to hold the specimen securely, gave reliable readings and yet was so small as to be negligible in contrast to the thousands of pounds used as buckling loads. By using the initial loading technique, an error curve was obtained for the entire load range up to 30,000 pounds. This curve showed the amount of deflection which occurred in everything except the test piece itself, and values picked from the curve were the amounts to be subtracted from the indicated crosshead deflections.

In attempting to verify the error curve on successive days, considerable trouble was experienced on the higher load ranges, rather than at low loads. After several trials it was noticed that when the proposed error curve was applied to a test piece, the calculated modulus of elasticity came out to be about 18×10^6 psi. This was badly in error, so another complete error curve was built up. This new curve gave E as about 45×10^6 psi. for the same piece of material and indicated that a thorough check of the entire method was needed.

Time effects were considered by applying a load of 20,000 pounds for periods of from thirty seconds up to ten minutes and then noting any residual displacement of the

crosshead after the load was removed. The first few brief loadings showed considerable residual displacement, but successively longer loadings showed very little, if any, such displacement. It was concluded that the test piece had been cold-worked somewhat, so a fresh one was made up immediately. It showed no time effects at all under identical loading conditions, either for brief or for long time loads. It was then concluded that the material was not responsible for the action observed. Something in the machine was therefore suspected of being time-sensitive. This was found to account for the residual deformation of the first specimen after its first few brief loadings. After six or so cycles of loading and unloading as a "warm up", the machine became stable and gave reproducible results.

With the "warm up" and the 100 pound initial load technique, the error curve was measured again and was found to be reproducible for the whole range of loads to within the desired 0.0001 inch. When applied to test specimens it consistently gave elastic moduli of about 30 to 30.5 million, which was a close check on the value obtained from the use of the SR-4 gauges.

It should be noted that at low loads, under 3000 pounds, the correction curve proved to be less reliable than at the higher loads. In many cases, the correction curve value was from five to ten times the net reading. Yet in spite of this apparent difficulty, good results were obtained

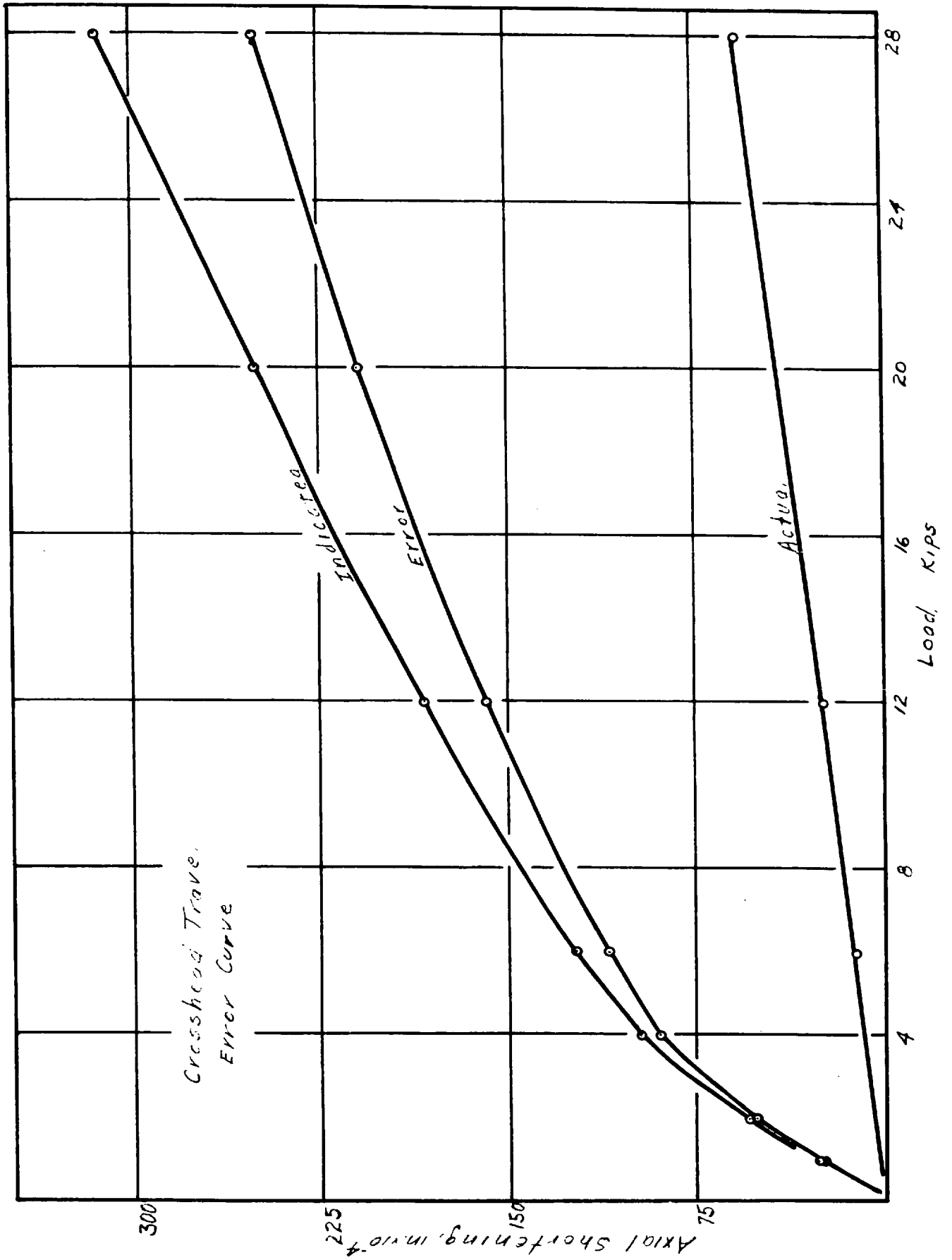


Fig. 1

consistently. The correction curve is shown in Fig. 1.

The method of holding the specimens has been developed and described by Van den Broek.¹ At each end of the specimen one half of a ball bearing is placed with the flat, or cut, side against the end of the specimen. Centering is achieved by the insertion of a template. The round part of the ball bears against a carboloy flat, and the flat, in turn, bears against the testing machine. A nearly perfect pinned end condition is thus provided by the unique properties of a sphere. The mounting parts are shown in Fig. 2. Arrangement of the components prior to a test is shown in Fig. 3.

In placing the carboloy flats against the machine, a double-ended aligning bar was used to insure that the flats were parallel, level, and vertically in line with each other. After centering, the flats were then secured to the machine by wooden jigs and clamps, and left in place for the duration of the tests. The flats, and especially the half-spheres, were kept scrupulously clean as it was found that even a small fleck of mill scale on the surfaces would affect the results adversely.

Two prestwood spacers were made up to simplify the mounting of the specimen in the machine. The spacers were designed to fit very snugly over the carboloy flats, and they had a relieved hole in the center which allowed

1. Van den Broek, J. A., "Rational Column Analysis," The Engineering Journal, Dec. 1941 P. 579

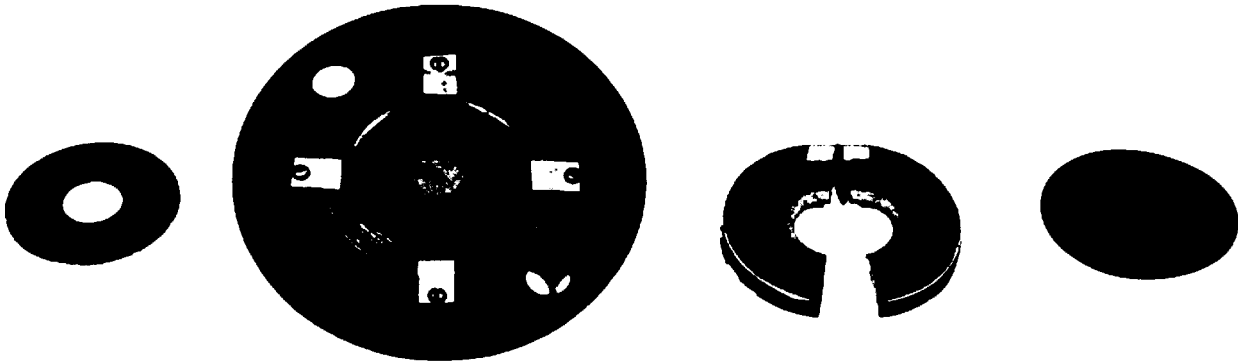


Fig. 2 Parts Used to Obtain Pin End Conditions

Left to right (order of assembly): Thin brass template, hemispherical ball, spacer block, and Carboloy bearing block. The spacer block is used to center the ball on the Carboloy bearing block before loading, but the spacer is removed after centering is achieved.

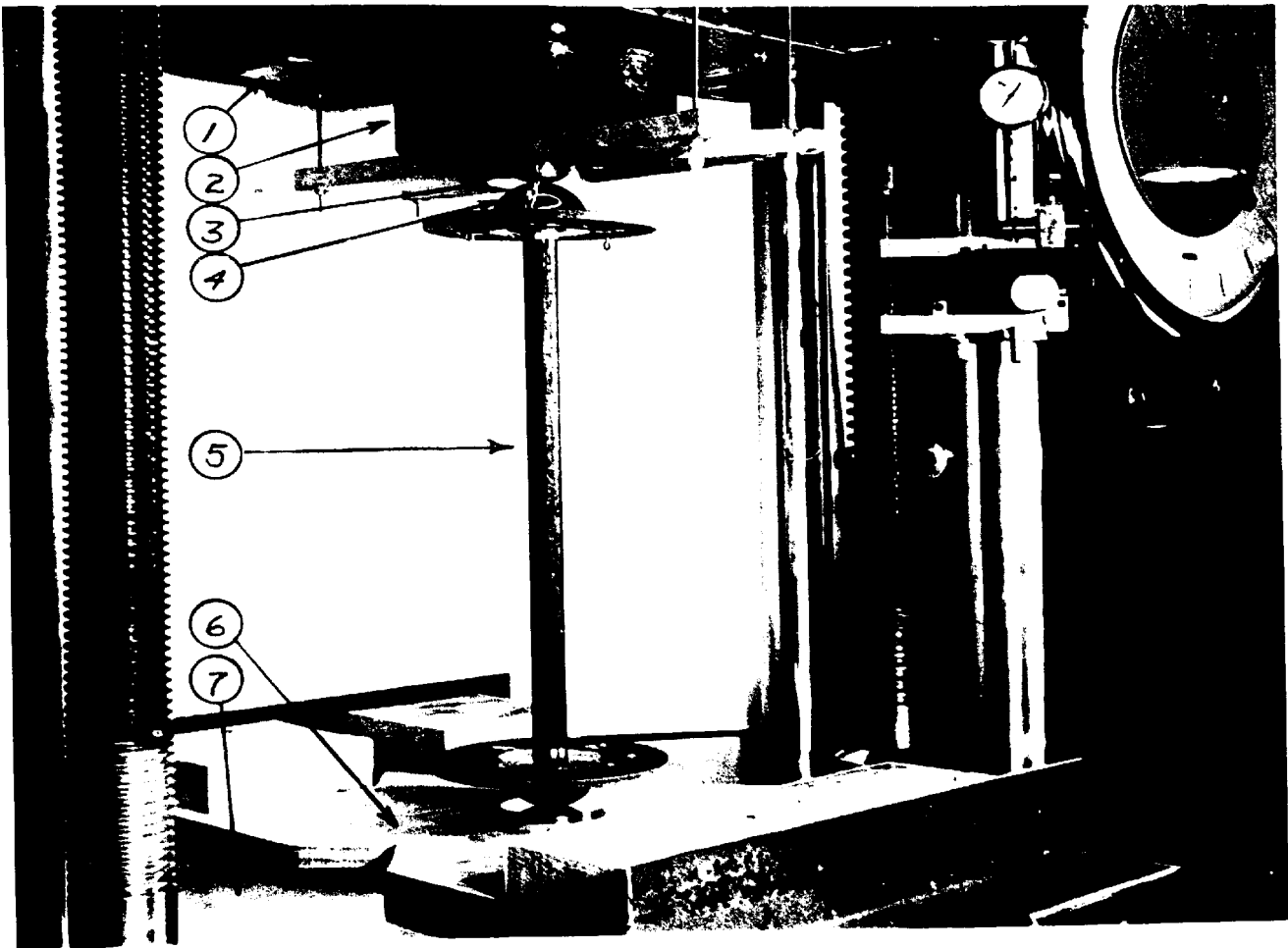


Fig. 3 Arrangement of Test Specimens

The parts used are as follows:

1. Movable crosshead
2. Steel crosshead block
3. Carboloy bearing block
4. Hardened steel hemisphere
5. Test specimen
6. Wood jig
7. Bed of machine

Dials to indicate motion of the crosshead and load on the bed are at upper right.

the curved sphere surface to seat itself smoothly. With the spacers in place, the movable crosshead was brought down slowly and the upper half sphere slid into its spacer so as to have the specimen very nearly vertical and lightly restrained against sidewise movement. A spirit level was then used to get the specimen as nearly vertical as possible.

After alignment was completed, a load of 500 pounds for the smaller bars and up to 1000 pounds for the larger ones was applied to "set" them and work out any slack in the arrangement. The spacer blocks were then removed and the load taken off down to about 50 pounds. Then the initial load of 100 pounds was brought to bear, always by increasing the load.

The dial gauge which indicated the motion of the machine crosshead was very firmly mounted on a height gauge base. This base was clamped to an arm, which was in turn, fastened to one of the vertical pillars of the machine. This mounting was made as stiff as possible, even though the only variable force acting on it was that due to the comparatively weak dial gauge return spring. The mounting arm was moved upwards along the vertical pillar to accommodate the longer specimens.

A fine adjustment was built into the base of the height gauge, permitting accurate zeroing of the dial. Since the range of the dial gauge was about 0.2100 inch, it would not read the total deflection of the longer

specimens without resetting. Resetting was accomplished very rapidly by use of the fine adjustment while the machine crosshead was momentarily stopped. No resetting ever occurred before the test was well beyond the buckling point and into the plastic range, where the load was changing very slowly. It is believed that any creep errors introduced by the resetting are strictly of a minor nature.

With the specimen accurately aligned, the dial gauge zeroed, and 100 pounds initial load on the specimen, the constant strain rate control was set at 0.005 inches per minute and the test begun. Readings of load and deflection were taken at appropriate intervals up to the buckling load.

For long specimens the buckling was slow and easy, so the readings were easily obtainable at all times. In the intermediate range, (slenderness ratio between 70 and 110), the buckling action was quite sharp and the drop in load carrying capacity quite large and rapid. This sudden change could not be read accurately due to personal reflex times and also because of the dash-pot damper in the machine. However, as soon as the dials slowed down enough to be read reasonably, readings were resumed. This lag usually allowed for a unit strain of about 0.0005 to 0.0010 inches per inch to take place. Such a lag is small contrasted to the total deformation of 0.0200 and gave smooth curves when plotted.

After the piece had buckled and was well into the plastic range the speed of straining was doubled to 0.010 inches per minute to reduce the time required to reach two per cent total strain. Even with this, the testing time for some of the longer bars was about 90 minutes.

After two per cent total strain was reached, the load was removed and readings taken during unloading down to the initial load of 100 pounds. With the load removed, the specimen was taken out of the machine. For the shorter specimens the ends were carefully examined under a magnifying glass. This was done to make certain that the ends had borne against the half spheres in a uniform manner. Any high spots were crushed down during the tests, and showed themselves as shiny spots under the glass. Several tests were re-run because of faulty ends which showed up as a poor modulus check.

All of the specimens were of ordinary mild hot rolled structural steel. No attempt was made to select special bars, but they were taken as received just as a contractor would get them. Pieces were cut off approximately to length in a saw, turned closely with a lathe, and then ground off square to length shortly before testing. Any burrs which would interfere with fitting the end of the piece into its holding and centering template were removed with a smooth file. In each case, the diameter near the end was filed down enough to allow a free slip fit for the template. This avoided the possibility that the template

would take some of the load that should go on the bar during the test.

No special pains were taken to see that the bars were perfectly straight. Some had eccentricities which were easily visible when the bars were laid on a flat, but all were used "as is." The lack of straightness undoubtedly contributed to the low buckling values which were observed in several cases, but on the average the results were quite consistent in spite of these defects.

Four sets of bars were used. One set was of $5/8$ inch square cross section; the other three were round and of $1/2$ inch, $3/4$ inch and one inch diameter. Fifteen bars were in each set, cut to lengths to give values of slenderness ratio from twenty to 160 by steps of ten.

DATA AND COMPUTATIONS

Data are presented for the testing of sixty bars in four sets as described previously. The raw data consists of the indicated crosshead deflections in units of 10^{-4} inches and the applied load in pounds. From the correction curve, the error in crosshead deflection for any given load may be determined, also in units of 10^{-4} inches. A simple subtraction of the error from the indicated deflection yields the true shortening of the specimen. By dividing the total true shortening by the original length of the bar, the unit shortening or unit compressive strain is obtained. Division of the indicated load by the original area gives the average unit compressive load. As the tests were being run, a simple check of the accuracy was made by dividing the unit stress by the unit strain to gauge the elastic modulus.

Each data point was plotted on a large sized graph and a smooth curve drawn for each bar showing the average load-unit axial deformation curve for deflections up to two per cent. The initial slopes of these curves served as a further check on the value of E and the accuracy of the work. From these curves, the following quantities were obtained:

1. The unrecovered work done on the bars.

2. The apparent modulus of elasticity during unloading.
3. The buckling values.

The unrecovered work is given by the expression

$$W = \int F \cdot ds$$

and is represented on the plot by the area enclosed by the curve. This area was measured with a planimeter, and gives the work lost per unit volume of material in inch pounds per cubic inch, which is the specific energy loss. The average specific energy loss for the four series of tests is plotted in Fig. 4 on a uniform scale, and again in Fig. 5 on a log-log scale. It may be seen that the log-log plot is nearly a straight line, and the usual operations show that the relationship may be represented by the equation

$$\epsilon = 1800 \left(\frac{L}{i} \right)^{-1.27}$$

The apparent modulus of elasticity during the unloading is simply the slope of the unloading curve after the two per cent unit strain has been reached. The unloading curve can be represented closely by a straight line in almost every case. The average values of unloading modulus for the four series are shown in a uniform plot in Fig. 6 and in a log-log plot in Fig. 7. The log-log plot shows that the relationship is made up of two parts,

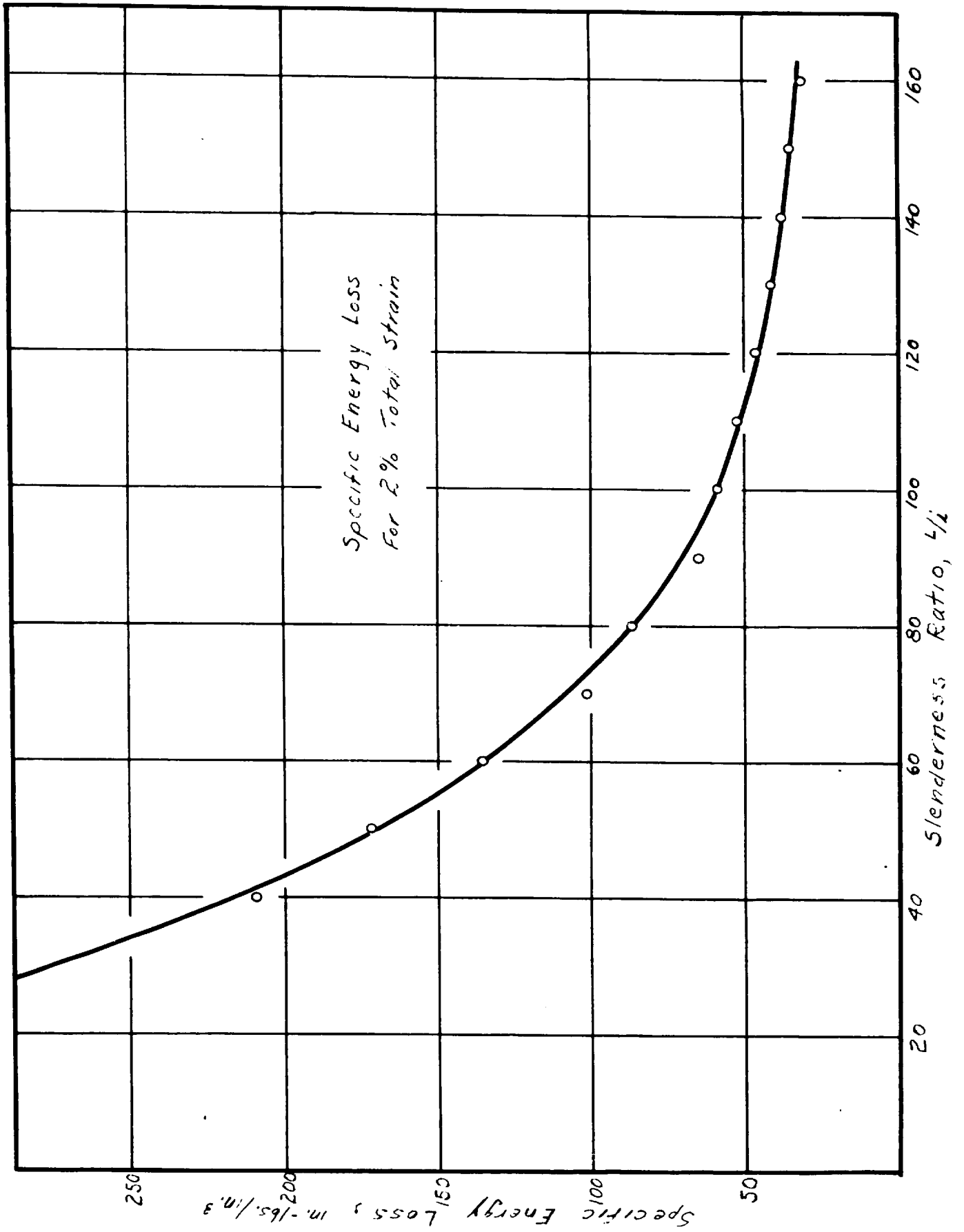


Fig. 4

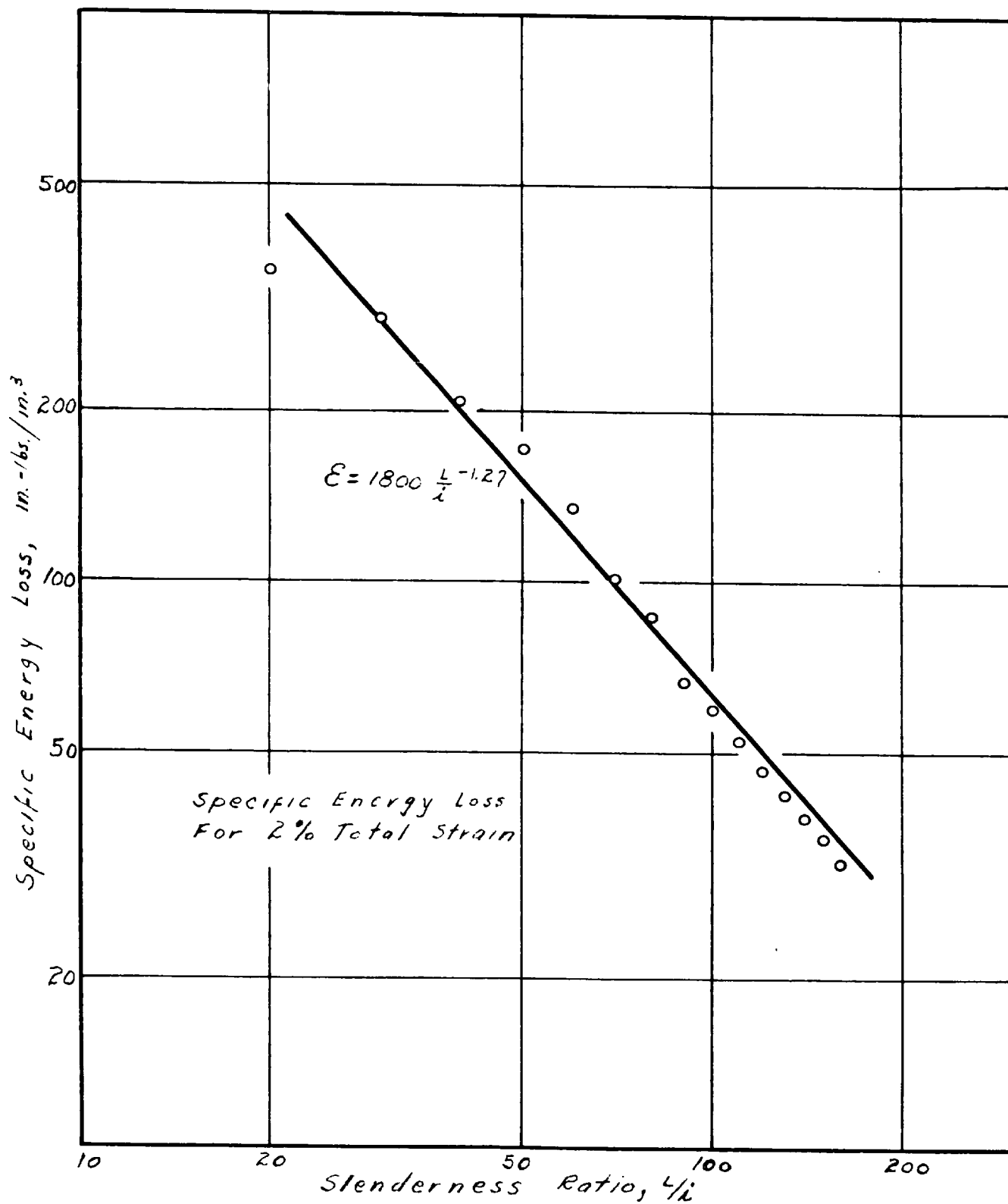


Fig. 5

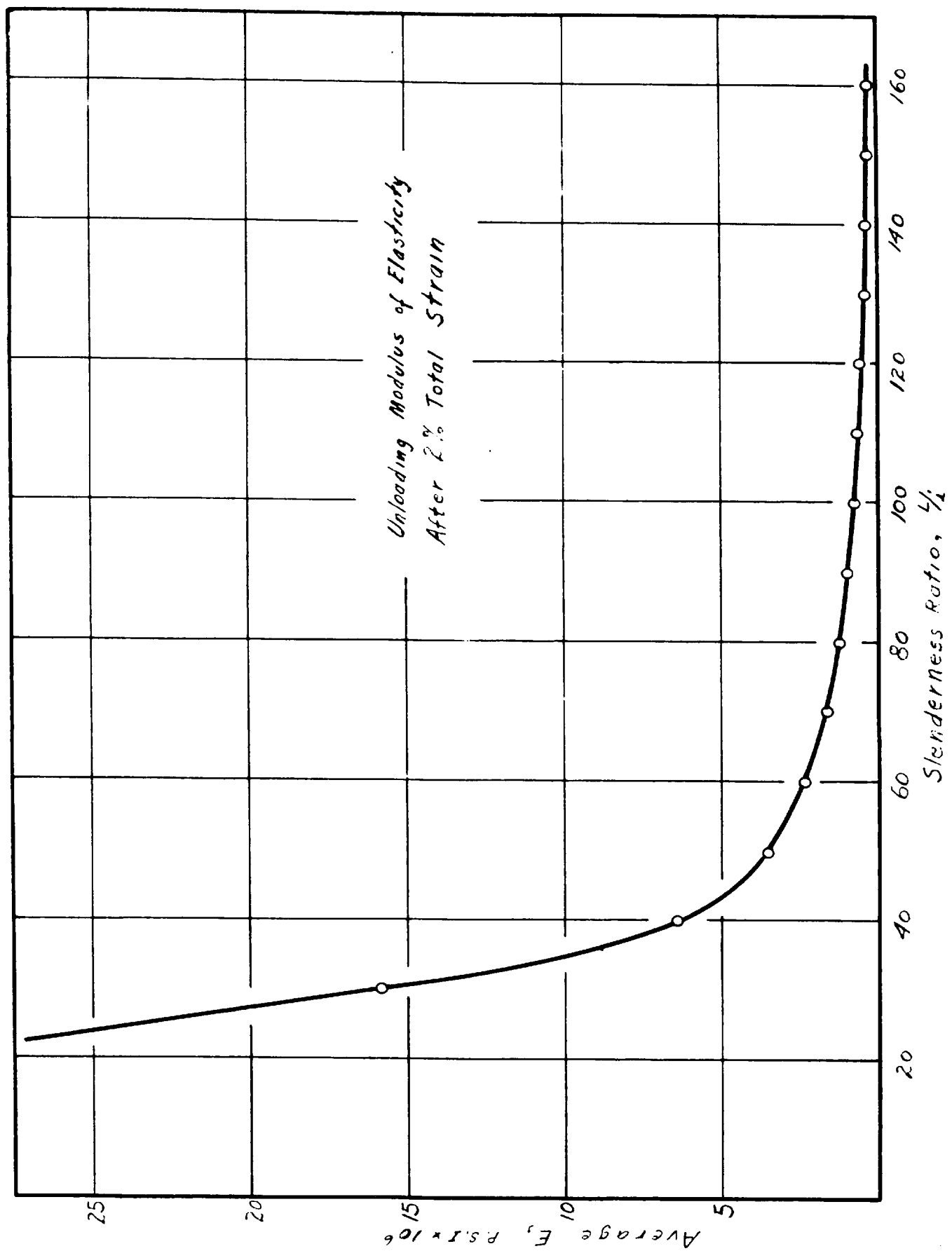


Fig. 6

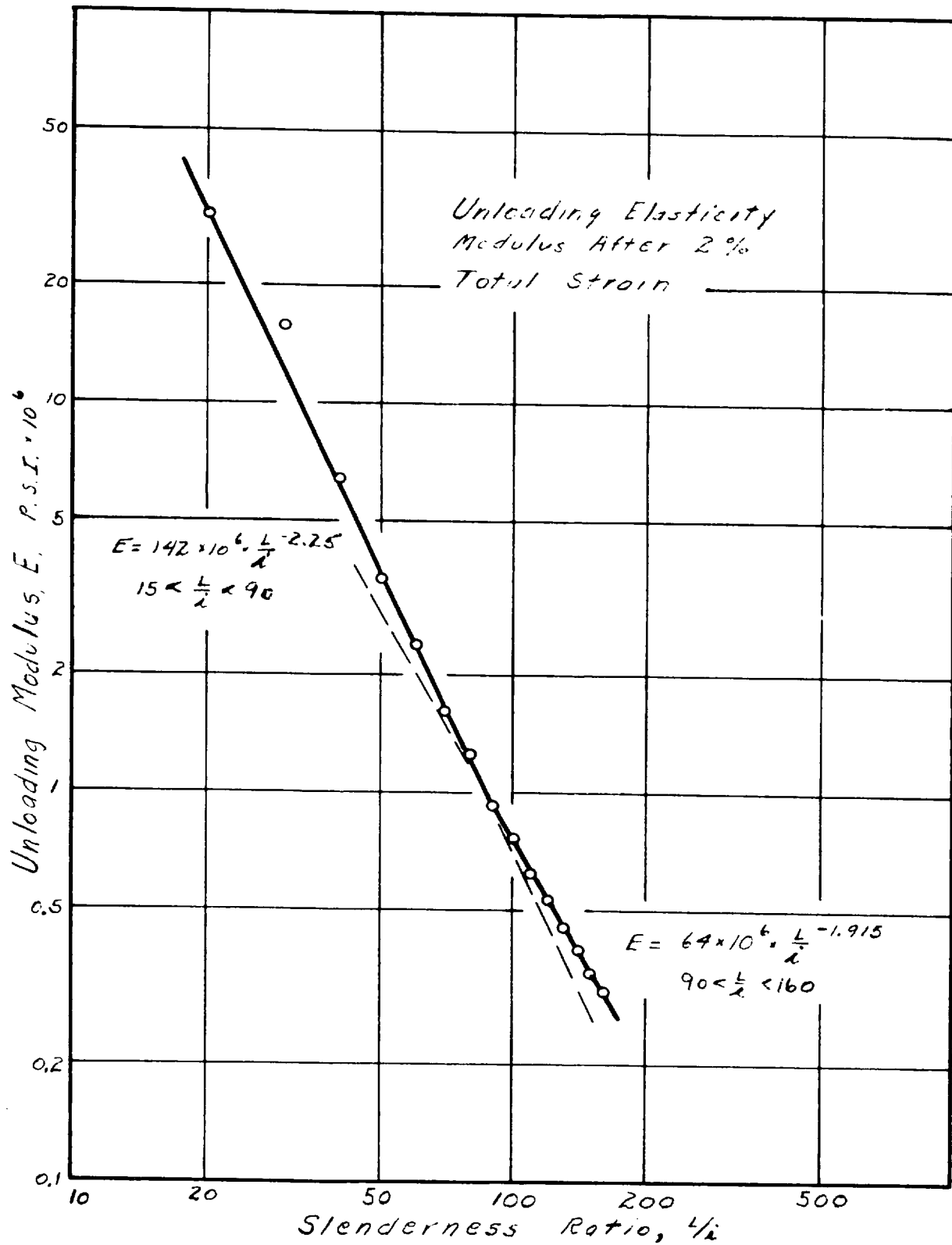


Fig. 7

$$E_U = 142 \times 10^6 \cdot \left(\frac{L}{i}\right)^{-2.25} \quad 15 < \frac{L}{i} < 90$$

and

$$E_U = 64 \times 10^6 \cdot \left(\frac{L}{i}\right)^{-1.915} \quad 90 < \frac{L}{i} < 160$$

If one equation which is reasonably close over the entire range is desired, the value of the approximation

$$E_U = 100 \times 10^6 \cdot \left(\frac{L}{i}\right)^{-2} \quad 15 < \frac{L}{i} < 160$$

may be used. With this relationship, the dimensionless form in terms of the original loading modulus becomes

$$\frac{E_U}{E} = \frac{\pi}{10^4} \left[\frac{\left(\frac{L}{i}\right)_{cr}}{\left(\frac{L}{i}\right)} \right]^2$$

where $\left(\frac{L}{i}\right)_{cr}$ is the transition slenderness ratio for steel, in this case about 100.

The buckling values are the maximum unit load values attained before the resisting ability of the bar suddenly decreases. Buckling values were obtainable for all specimens except those with slenderness ratio of twenty. In that case, of the four bars tested, three developed some rapid drop in load, hence the maximum load preceding this drop is called the buckling value. In the fourth case, for the 1/2 inch diameter bar, there was no buckling, only a shortening and concurrent swelling of the piece.

The average buckling values are plotted in Fig. 8, and it may be noted that the curve resembles rather closely the well known secant formula

$$S = \frac{P}{A} \left(1 + \frac{e c}{i^2} \sec \frac{L}{2} \sqrt{\frac{P}{E I}} \right)$$

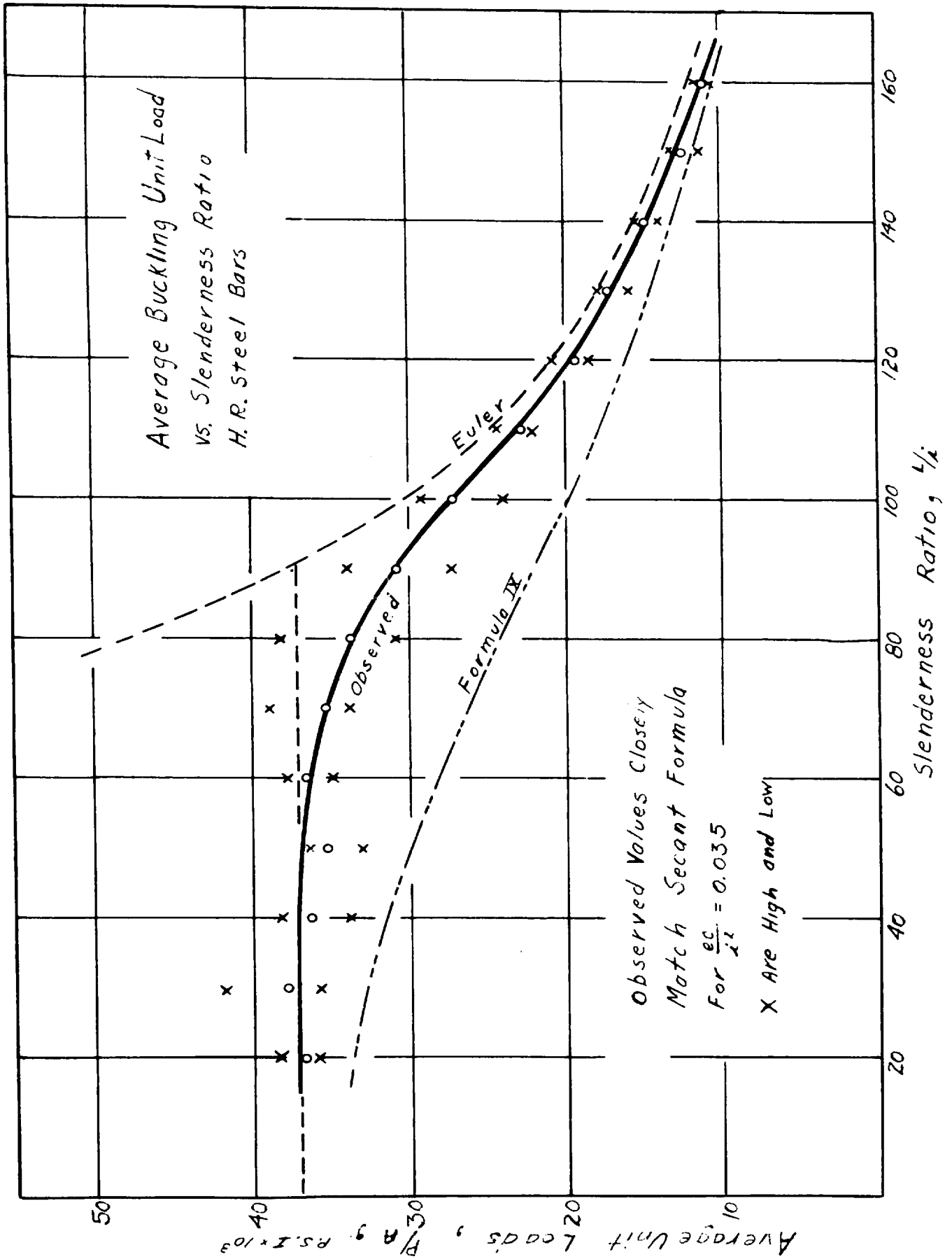


Fig. 8

Choosing S as 37,000 psi. and using P/A values from the plotted curve with lengths from the one inch diameter series, it was found that when ec/i^2 was about 0.035 the curve fit quite well for slenderness ratio from 20 to 130. The closeness of the agreement is considered as one verification of the experimental technique.

The curves in Figs. 9 and 10 are plotted to show the deformation curves of representative samples. Only a few of the many data points taken are shown on each curve, and they also show only even multiples of $10 L/i$. They do show the general tendencies of the problem, and all four sets of curves were similar to those shown in all respects. In Fig. 11 are shown all of the even multiple curves for the first 0.25 per cent of deformation. This plot magnifies the horizontal scale of the plot in Fig. 9 and 10 by a factor of about twelve and shows the action in the vicinity of the buckling values. Dashed lines are used on three curves to indicate that they are not substantiated by observations.

Average stresses at various values of deformation are shown in Fig. 12. These values are picked directly from the original data plots and average over the four series of tests. All of the curves appear to belong to the same family, and the three curves plotted serve to show how the transition progresses.

The data also show that the bending of most of the bars in the medium column range did occur before the

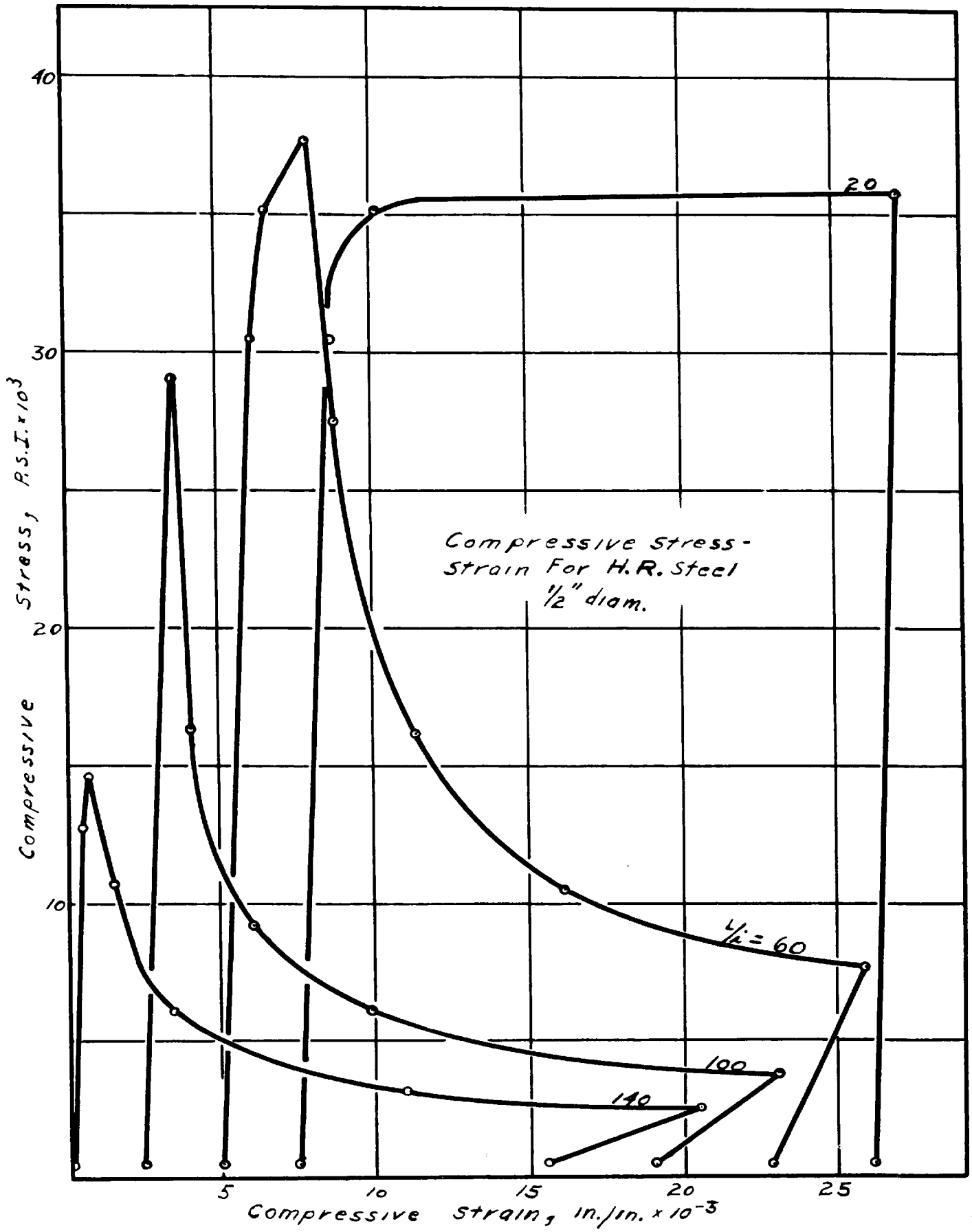


Fig. 9

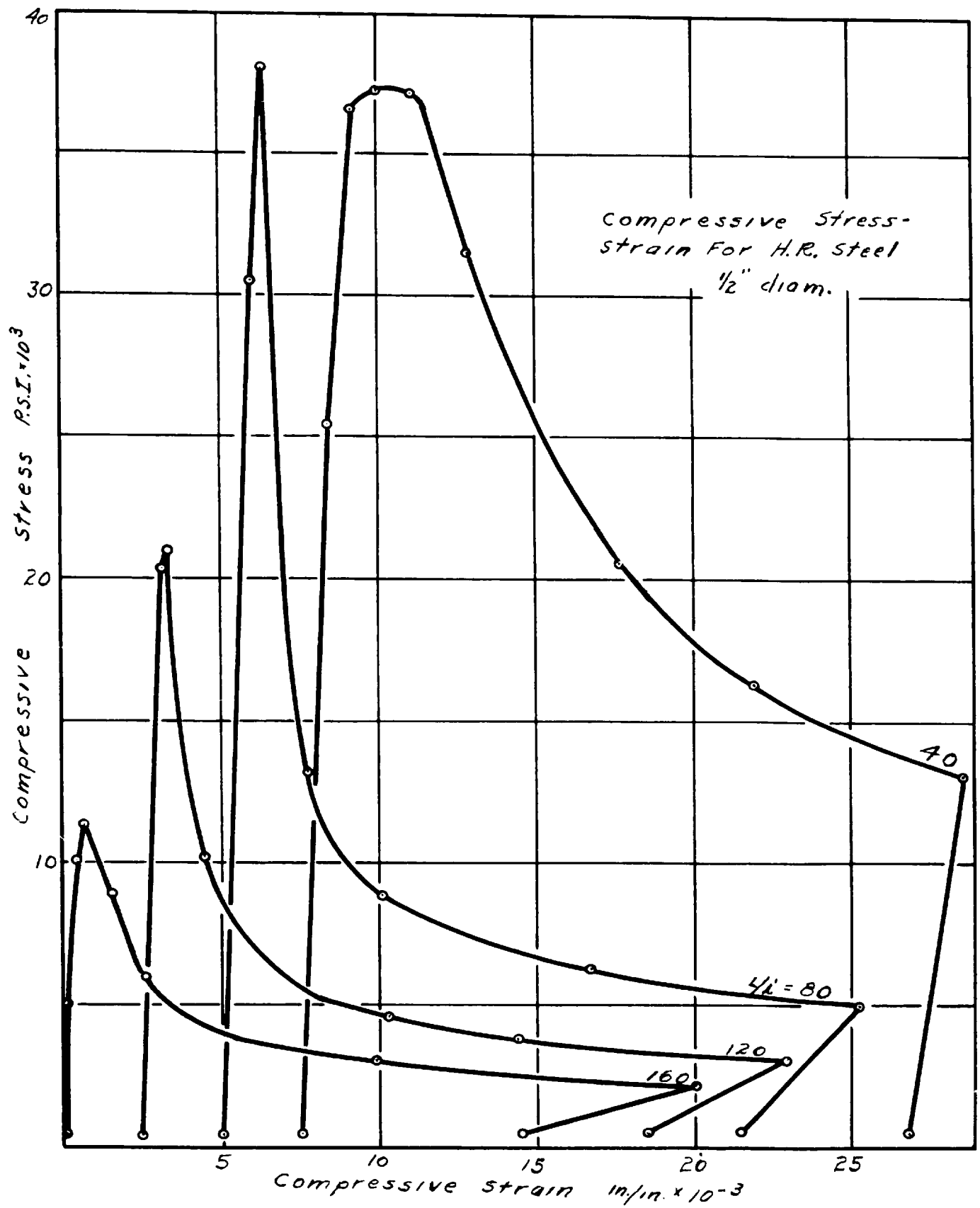


Fig. 10

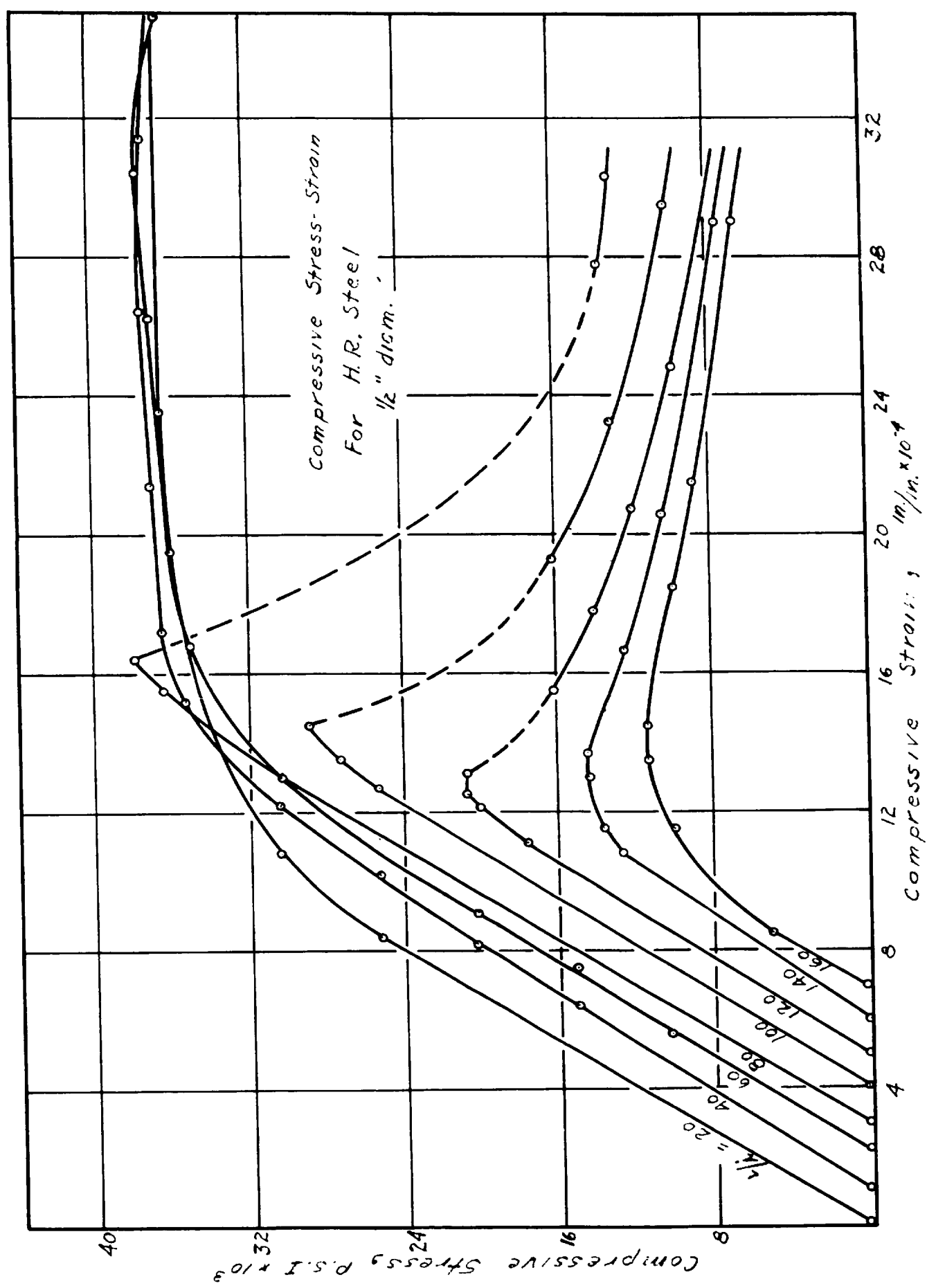


Fig. 11

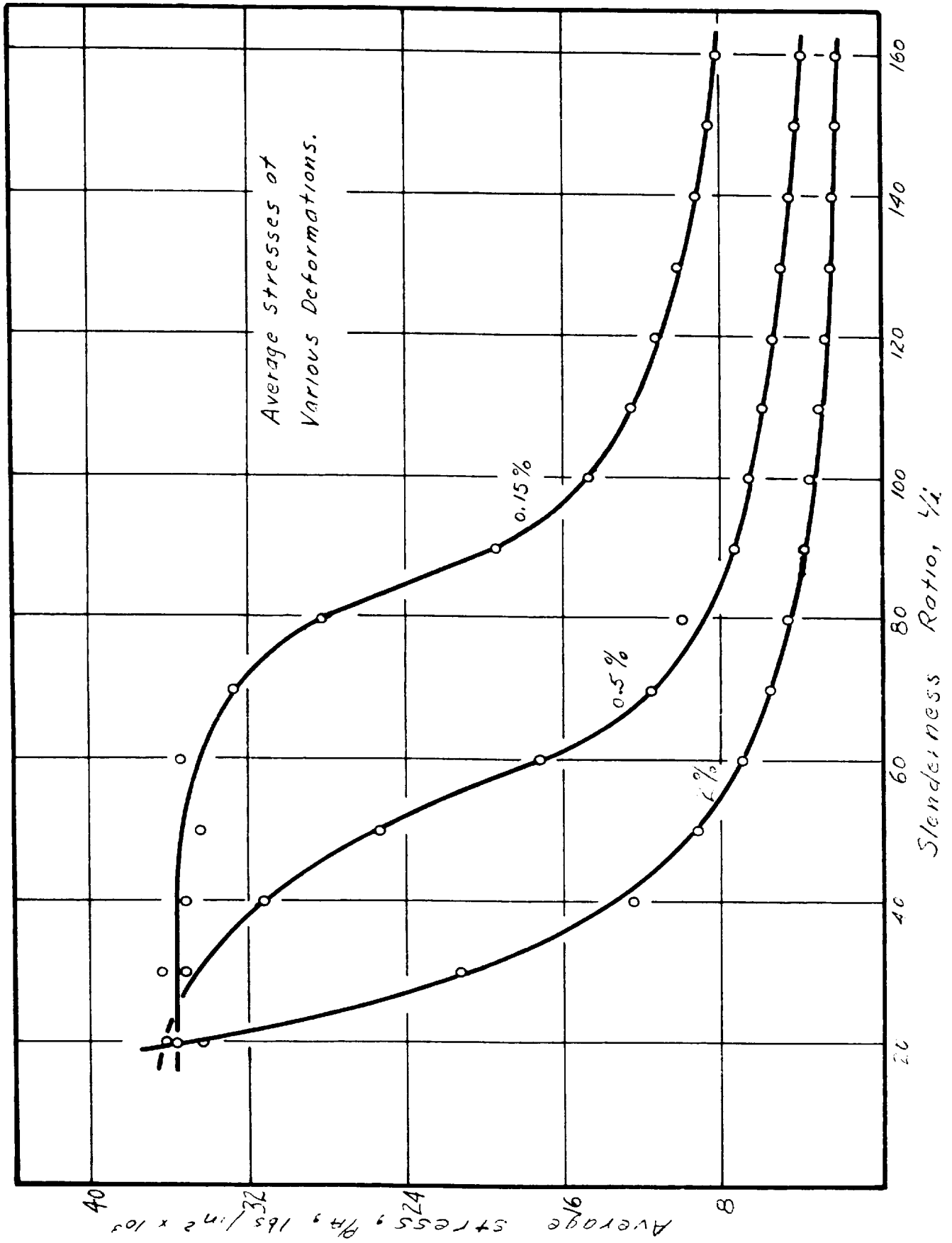


Fig. 12

buckling took place. This appears as a gradual rounding of the top portion of the deflection curve, and is particularly noticeable in the bars from L/i of 140 to 160. During the conduct of the tests, the bending prior to buckling was often easily noticeable to the eye. This action is in accordance with the predictions by Shanley and calculations by others.¹

As an experiment with the data, all of the curves were traced from the original large plot sheets onto another sheet in such a way as to give all curves the same origin. They had previously been plotted with different origins in order to avoid excessive overlapping and to render each curve distinct. When traced with a common origin, the curves formed a family which resembled a set of lines of constant elevation cut on a curved surface. There was no overlapping except on the portion of the curves representing the initial loading.

Making use of this pattern, it should be possible to construct a model surface which would show the action of any length column of any given material. Such surfaces are used in thermodynamics and metallurgy to show phases and compositions. In the present case the surface would have the length dimension to indicate strain, the width to indicate stress, and the height to indicate the slenderness ratio.

1. Shanley, F. R. "Inelastic Column Theory", Journal Aeronautical Science 14, No. 5 May 1947, Pp. 261-268.

Because all of the stress-strain curves appeared to be so closely related, another experiment was tried to see if the curves could be shifted about until considerable portions of the curves were coincident. It was found that this could be done for large parts of the curves for values of slenderness ratio between 80 and 160. All of these curves fitted into one common curve for the portion just after buckling, and there was rather small divergence from each other even as the deformations became of the order of 1.5 per cent. The common curve is not a true envelope since the origins of the component curves are in different places. The common curve does show, however, that the post-buckling action of all the bars in the range stated is closely related in a remarkable fashion.

By direct substitution, it can be shown that the curves can be represented to a reasonable degree of accuracy by a relationship of the form

$$\frac{P}{P_{cr}} = K \frac{\epsilon_{cr}}{\epsilon} \sqrt{\frac{\epsilon}{\epsilon_{cr}} - 1}$$

where P is the axial load, P_{cr} is the Euler load, ϵ is the axial strain, and ϵ_{cr} is the strain at the Euler load. The constant, K , must be determined for each case. For deformation in the range from 0.002 to 0.020 the agreement is quite good.

Another possible relationship is given by Running.¹

1. Running, T. R., Empirical Formulas, John Wiley, New York, 1917. P. 137 and Fig. XI.

However, it has the disadvantage of not being correct dimensionally, and would be clumsy to use. It is given as

$$\frac{P}{A} = S = \frac{\epsilon}{\alpha - \beta\epsilon + \gamma\epsilon^2}$$

Running has a set of curves of this function with various values in them, and the family resembles the observed curves strongly.

Attempts to fit the exponential curves of the form

$$y = ae^{\pm bx}$$

or

$$y = ae^{\pm bx \pm c}$$

proved to be unsatisfactory.

RESULTS AND CONCLUSIONS

The portion of the investigation which was directed towards the determination of the ductile equilibrium column values, as proposed by Van den Broek's formula IX proved to be futile. The formula is based entirely on the material and the shape of the bar, and does not include the elastic properties of the supporting structure. However, the elastic strain energy which is released by the supporting structure immediately following the buckling of a given member plays a large part in the subsequent action of that member. Once buckling has taken place, the deformation will not suddenly assume some fixed value and the load will not come to some stable value which depends only on the material and its shape. This occurs because, for medium length columns, the slope of the load-deformation curve after buckling is always downwards for increasing deflection, never horizontal or upwards. Release of strain energy will increase the deformation and decrease the supported load of the member arbitrarily.

Ductile equilibrium implies two things; first, that the material has been strained beyond the elastic limit, and second, that equilibrium prevails. It should be

noted that this equilibrium is unstable. For stability, the slightest change in the equilibrium configuration must not result in a non-recoverable deformation. This does not imply that the testing machine merely has to come to rest with the scale indicating a certain value after the test piece has buckled, for such a condition is not stable equilibrium. This is shown by the observed results, for if, after buckling, the "equilibrium" is disturbed by raising the crosshead of the machine by a small amount, the indicated load will decrease in an almost elastic manner. Running the crosshead down again by the same amount again gives the same load indication as existed before the disturbance was initiated. For such a disturbance the equilibrium is, indeed, stable. However, if the crosshead of the machine were to be lowered by a small amount, such a disturbance will also produce a decrease in the load, and in a non-elastic manner. Raising the crosshead again to the original level does not return the load indication to its former place, but decreases it still further. This is a non-recoverable disturbance, hence the equilibrium is not stable.

Under such conditions, any slight disturbance would materially affect laboratory measurements of ductile equilibrium values beyond the bounds of reproducibility. In practice the effects of creep, velocity of buckling, or variations in the load would render the theoretical values unattainable.

In attempting to evaluate the data for ductile equilibrium verification, three criteria can be used.

1. The value of the increase of deformation from elastic to plastic conditions at the formula IX stress value.
2. The amount of decrease of load from maximum observed to the formula IX value.
3. The slope of the stress-strain curve at the formula IX value after buckling has occurred.

None of these bases gave any recognizable relationship when plotted on Cartesian, semi-log or log-log paper. Other arbitrary values of slope and increase of deformation were tried with no success.

In analysing formula IX, one of the natural steps was to plot it for certain assumed values. The plot showed only small variations from a straight line, and suggested an attack to linearize and thus simplify the formula. A careful choice of line showed that one represented by

$$\frac{P}{A} = S_2 \left[1.1 - \frac{1}{200} \frac{L}{r} \right] \quad 20 < \frac{L}{r} < 160$$

was reasonably close. A comparison of the two curves for assumed values of S_2 as 34,500 psi. and E as 30.2×10^6 psi. gave values which differed by a maximum of 6.63 per cent, and an average of 1.96 per cent. These limits are about the same as the accuracy with which the value of E is determined for use to calculate S_{cr} in formula IX. The

linear equation has the advantage of not needing the calculation of S_{cr} and can be used for values of slenderness ratio for the whole medium length range from 20 to 160.

In the process of testing the 5/8 inch square bars an unexpected directional buckling effect became apparent. As shown in Fig. 13 the long members of the series all buckled across a section diagonal, while the shorter members buckled about axes which were inclined progressively less and less to the sides of the square.

During the testing of the long bars it was noted that when buckling was imminent, the bowing was in a more or less random direction, but as soon as buckling had taken place and some of the material was forced into the plastic range, the bar whipped around and buckled across the diagonal.

This action may be explained by the following analysis. Consider an arbitrary simply connected area, A . It will possess a centroid, G , which may or may not lie within the area, A . Through G will pass an axis in the plane of A about which the moment of inertia, I , is smaller than for any other axis in the plane through A . Call this axis the I_{min} axis. If a long prismatic bar having A as its cross section is loaded axially in compression, the bar will buckle about this I_{min} axis. As soon as buckling has occurred, a bending moment Py is developed, where P is the axial load and y is the lateral deflection of any section from the load axis.

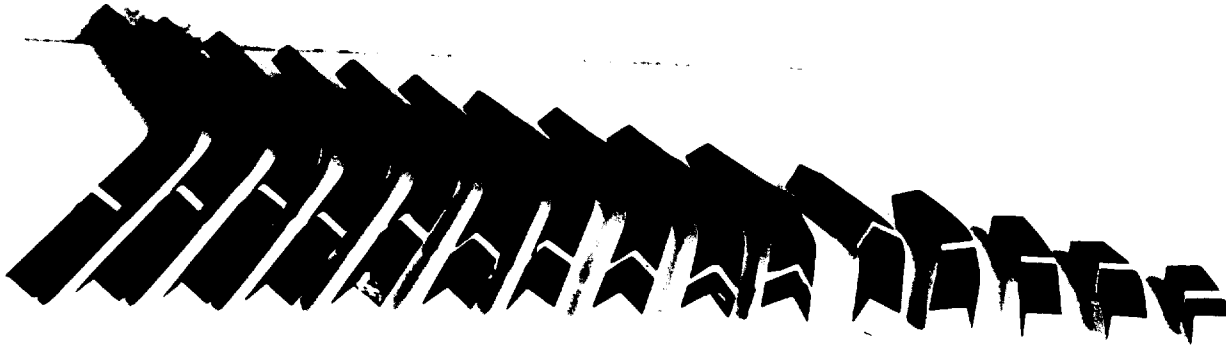


Fig. 13 5/8 inch Square Bars After Test

Slenderness ratios from 160 to 20 in steps of ten. Note progressive rotation of end cross section and bending axis as L/i increases. All specimens show some bending.

It should be noted that the action of bending is separate and distinct from the buckling action. Buckling must occur first, then the bending takes place. The buckling takes place about the axis of I_{min} , but the bending will occur about the axis with the minimum section modulus, Z_{min} . This axis may or may not coincide with the axis of I_{min} . For example, for a square section such as used in the tests, the I_{min} axis may be any axis through the centroid, G, because of the symmetry. The Z_{min} axis is the one with the maximum value for C, the distance to the extreme fiber, and this occurs across the diagonal. Hence, while buckling has no preferred direction, bending does. This requires the observed rapid swing to bending across the diagonal axis.

The rotation question leads to the problem of what kinds of rectangular shapes will develop the phenomenon, and which can not do so. In other words, what is the largest variation from a square that will permit rotation? Mohr's circle for moment of inertia is an obvious tool, with simple maxima and minima calculations for limits.

If the values of the two principal moments of inertia are calculated and plotted, the center of Mohr's circle, M, and its radius, R, can be found directly. Then by direct substitution the expression

$$\frac{d^2 I}{d\theta^2} = -(R+M) \cos \theta + 6R \cos^3 \theta - 12R \sin^2 \theta \cos \theta$$

may be obtained. If this second derivative is negative, the function has a maximum at the chosen angle. Using

the minor axis as the reference direction, a negative second derivative indicates that the bar will buckle in a direction perpendicular to the minor axis and then bend about some other axis which has the minimum value for Z. This will happen only when $M \geq 5R$ since $\cos \theta = 1$ and $\sin \theta = 0$.

Taking the case where $M = 5R$, direct substitution shows that where $\frac{I_{max}}{I_{min}} \leq 1.5$ the bar can possibly turn. This gives a ratio for the long side to the short side of $\frac{b}{h} \leq 1.225$. With these proportions the bar will buckle about the minor axis and then turn to bend about an axis inclined at about $50^\circ 40'$ counterclockwise from the major axis.

It should be noted that for $1.225 < \frac{b}{h} < 1.455$ the value of Z for the minor axis is a minimum, but that there is another lower minimum about another axis. If forced away from the minor axis, the bar might bend about this other weaker axis. For $\frac{b}{h} > 1.455$ the minor axis is the weakest axis of all, and no rotation is expected.

For short bars, the initial buckling was always in a direction perpendicular to one of the sides. After buckling occurred, the bars tried to rotate to bend about the diagonal axis, but were prevented from doing so by friction in the ball supports. The smallest bars took the largest load after buckling, hence they had the largest amount of friction developed and could turn with difficulty only through the smallest angle.

In addition to the preferential direction for the buckling of the square bars, another phenomenon easily observable during the tests of all four sets of bars was that the shorter bars usually did not buckle and bend symmetrically. In structural theory it is common to assume that the angular deflections of the ends of the bars are the same during buckling, and the maximum deflection occurs at the middle. This was not observed in these experiments for small deflections. After the deflection had been increased to about one per cent, the symmetry was closely approached, but for smaller deflections it was easily visible that symmetry did not prevail. Fig. 14 shows a one inch diameter bar which is not symmetrical even at two per cent strain.

Verifications of several earlier works were achieved. The average buckling values for the four series of bars fitted the secant formula for $\frac{P_C}{72} = 0.035$ quite closely for slenderness ratios up to 130. The general shapes of the deformation curves as they approached the buckling point also verified the theory proposed by Shanley and verified by Lin. The curves for the large slenderness ratios also verified Van den Broek's curves.

Values for the specific energy losses and for the value of the modulus of elasticity during unloading were both determined on an empirical basis for two per cent total strain. The average energy loss curve may be reasonably represented by the expression

$$\mathcal{E} = 1800 \left(\frac{L}{i}\right)^{-1.27}$$

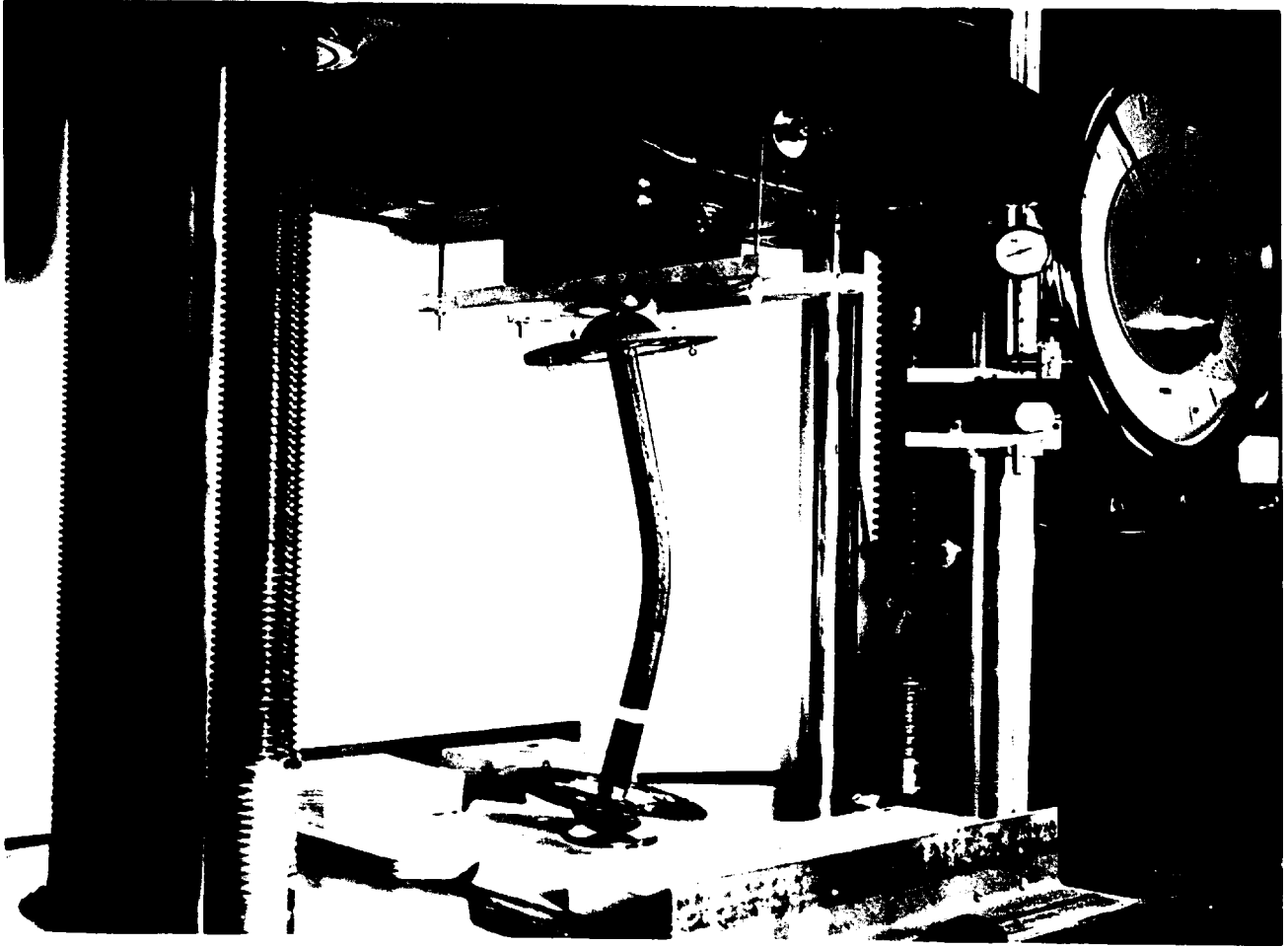


Fig. 14 Specimen After Shortening Two Per Cent

Note that the maximum lateral deflection does not occur at the middle of the specimen and that the angular deflection is different at each end. Slenderness ratio is 60.

It should be noted that all of the values for $L/i \geq 90$ plotted very nicely, but there was some spread among the shorter ones.

Values for the unloading modulus appear to be best represented by two portions of a curve. Letting E_u be the unloading modulus of elasticity, then

$$E_u = 142 \times 10^6 \cdot \left(\frac{L}{i}\right)^{-2.25} \quad 15 < \frac{L}{i} < 90$$

and

$$E_u = 64 \times 10^6 \cdot \left(\frac{L}{i}\right)^{-1.915} \quad 90 < \frac{L}{i} < 160$$

If some reasonable approximating is done, then

$$E_u = 100 \times 10^6 \left(\frac{L}{i}\right)^{-2} \quad 15 < \frac{L}{i} < 160$$

may be used. This may be arranged in dimensionless form by letting $\left(\frac{L}{i}\right)_{cr}$ be the slenderness ratio for the transition length, in this case about 100. Then

$$\frac{E_u}{E} = \frac{\pi}{10^4} \left[\frac{\left(\frac{L}{i}\right)_{cr}}{\left(\frac{L}{i}\right)} \right]^2 \quad 15 < \frac{L}{i} < 160$$

Values of stresses at various strain values are shown in graphical form in Fig. 12. The intervals between strain values are chosen for about a factor of four, and allow easy interpolation to be used.

Like many other researches, this one pointed the way to several other things which could profitably be investigated. For example, the action of a column after buckling but with successive loadings and unloadings to produce

additional finite steps of deformations would show both the variation of energy loss and unloading modulus with deformation. Work also needs to be done on the angular changes at the ends of columns during buckling and thereafter. It is surmised that one of the easiest ways of recognizing incipient buckling would be to measure the tilting of the ends of the bar.

Variations in shape may also bear looking into. All of the present tests were run on round or square sections, but it should be possible to vary the length of the bar and the radius of gyration of the cross section and thus investigate for the most efficient sections for a given area or weight per foot.

Conclusions

1. Except for very large values of slenderness ratio, no ductile equilibrium values are experimentally obtainable.
2. Ductile equilibrium, as defined by Van den Broek, is not a stable equilibrium for steel in the slenderness ratio range below 160.
3. For steel columns in the slenderness ratio range between 20 and 160, Van den Broek's formula IX may be approximated by a straight line relationship within the usual limits of accuracy on the values of E .
4. Specific energy losses during large deformations may be represented by simple empirical relationships.

5. Columns subjected to large deformations past the buckling point relax in a linear fashion as the load is removed.

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