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RATE DISTORTION FUNCTIONS

FOR CONTINUOUS ALPHABET

MEMORYLESS SOURCES

by
Stephen Lynn Fix

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of the requirements for the degree of
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(Electrical Engineering)
in The University of Michigan
1977

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LIST OF ABBREVIATIONS

$R(d)$	Rate distortion function
$D(R)$	Distortion rate function
$I(\cdot\|\|\cdot)$	I-divergence
$I(\cdot;\cdot)$ or $I(\cdot)$	Mutual information
$d(x,y)$	Single letter distortion measure
P_X	Source probability measure
Q	Reproduction probability measure
$E\{\cdot\}$	Expected value
$\underline{\lim}$	Limit inferior
SLB	Shannon lower bound
iff	If and only if
nats	Natural logarithm units
$<<$	Absolutely continuous with respect to
$</<$	Not absolutely continuous with respect to
$\subseteq$	Contained in or equal to
$\forall$	For all
$\bar{A}$	Closure of the set A
A^c	Complement of the set A
ε	Is an element of
$\triangleq$	Equal by definition
$\cup$	Union

I. INTRODUCTION

Rate distortion theory as invented by C.E. Shannon ([1],[2]) was proposed to address the problem of transmitting a message from a source across a disruptive medium at a rate[1] higher than could be achieved with negligible error. This problem occurs most commonly when the source produces a continuous valued message, such as in the cases of speech and video. Since one must always sacrifice something in this situation, a distortion criteria, which may be specified by the user of the message, was introduced. With this criteria specified, the message could be modified by some means (coding for example) such that the average distortion be minimized at a given rate. Therefore, rate distortion theory is concerned with source-user pairs.

To be specific, the source will be defined here as having an alphabet of letters from which it produces one at random at regular time intervals. The randomness is controlled by a probability law. A source will be said to have memory if there is a probabilistic dependence between the outputs at different times. The user will also have an alphabet and will receive one letter from this alphabet at regular time intervals. Which letter is received is dependent on what was sent and the properties of the disruptive medium, which shall also

[1]Rate will ultimately be reduced to some physical concept such as symbols/sec or bits/sec. Since coding is not being addressed this will not be made precise. See [4] for the precise usage.

be governed by a probability law. The disruptive medium is referred to as the channel. The distortion criteria shall be a function on the source and user alphabets into the real line.

In general, one cannot control the disruptive medium and, therefore, control can only be inserted between the source and channel and between the channel and user. The processing inserted shall be referred to as coding and decoding, respectively. It is convenient to think of the coding as being separated into two processes, namely, source coding and channel coding and similarly the decoding operation into channel decoding and source decoding.

Because of Shannon's work [1] and others on the problem of transmitting a message across a disruptive medium with negligible error (the channel coding objective), the source coding problem can be viewed as the reduction of the rate of the source to one which can be reproduced with negligible error (via channel coding) at the output of the disruptive medium, while introducing the smallest possible average distortion in doing so. As one might expect, there is a lower limit to the induced distortion at a particular rate. This lower limit is the subject to be considered here. It will be referred to as the rate distortion function for the source and user specified distortion criteria. The fact that this is the lower limit to the performance (as measured by the distortion criteria) of any strategy one might invent is made precise by a source coding theorem. The fact that one can

invent a scheme to get arbitrarily close to this limit is referred to as the source coding theorem. The fact that no scheme can do better is known as the converse source coding theorem. Precise statements of source coding theorems and their converses may be found, for example, in the books by Berger [3] and Gallager [4].

The subject to be considered here is the evaluation of the rate distortion function. This problem has proven to be extremely difficult even in the seemingly simple cases. To date, the solution to only a handful of memoryless sources and distortion criteria has been found analytically and the results for even fewer sources with memory are known. Berger [3] surveys this area up to 1971.

One might sight some of the progress that has been made in this direction and that is especially relevant here as:

- The evaluation of the rate distortion function for the memoryless Gaussian source and the squared error distortion criteria by Shannon [1]. (1948)
- The evaluation of the rate distortion function for the Gaussian random process and the squared error distortion criteria by Kolmogorov (see [3]). (1956)
- The Shannon Lower Bound to the rate distortion function with difference distortion criteria by Shannon [2]. (1959)
- The variational equations are credited to Gerrish by Berger [3]. (1963)
- Bounds on the rate distortion function of sources with memory in terms of the rate distortion function of the associated memoryless source by Wyner and Ziv [7]. (1971)
- An iterative procedure for evaluating the rate distortion function by Blahut [5]. (1972)

- Evaluation of the rate distortion function for the memoryless Gaussian source and the magnitude error distortion criteria by Tan and Yao [6]. (1975)

One might note the continued interest in the continuous alphabet case after the appearance of Blahut's algorithm. A reason for this might be that Blahut's algorithm, which very nicely handles the discrete memoryless problem, is in practice subject to problems, which will be discussed later, when applied to the continuous alphabet source.

This dissertation concentrates on finding rate distortion functions for continuous alphabet memoryless sources, although many of the techniques are applicable in the case of memory. Since the Wyner-Ziv [7] bound on sources with memory is expressed in terms of the rate distortion function of a memoryless source, this work is of interest from that aspect also. The approach followed is to carry the problem analytically as far as possible and obtain analytical expressions, where possible. This approach will often not yield an analytical closed form solution but will yield a procedure which is hopefully superior to a blind use of Blahut's algorithm.

To allow a certain completeness of the presentation, many results are repeated here with reference to Berger [3] or Gallager [4]. For historic credit,one should refer to [3] or [4] for the true origin of these results. Other proofs of standard results, such as Csiszar [8], are included not only for completeness but because they seem to be the nicest way (at present) of arriving at these results.

Chapter 2 presents the basic definitions and properties

of the rate distortion function. It considers the development of the form of the solution. Csiszar's [8] recent approach to this form is utilized to develop the basic equations. This approach requires arguments that the solution exists (this will be made clear later) and results to this effect will be presented in Chapter 3. The necessity for the mathematical rigor used here seems unavoidable because of the nature of the solution. Chapter 3 should help to point out an aspect of the rate distortion function,which is not often mentioned in the literature and apparently not widely appreciated, that is,when the inf is achieved (see Chapter 2 for the definition of the rate distortion function).

Chapter 4 considers the solution of the equations developed in Chapter 2 and discusses the possible role of estimation theory in aiding the solution. It primarily sets the stage for Chapters 5 and 6, which consider the two examples of magnitude and squared error distortion, respectively.

Chapter 5 considers the magnitude error distortion criteria. In this example, integral equation techniques can be applied as illustrated by Tan and Yao [6]. In Chapter 5, the connection with estimation theory is demonstrated and used to eliminate one of the integral equations. Examples and discussion of the solution are presented.

Chapter 6 considers the squared error distortion criteria. In this case, the integral equation techniques utilized in Chapter 5 don't work. The analytic properties of the equations, however, require the solution to be of a very restric-

ted form. An algorithm is presented based upon this work to find the solution. The algorithm presented has been utilized in slightly different form by Benjamin [9] in considering a discrete source and continuous reproduction alphabet with the squared error distortion criteria. The work here subsumes Benjamin's work and shows in a broader and more interesting setting the use and utility of this thinking.

II. THE RATE DISTORTION FUNCTION AND ITS PROPERTIES

The introduction described rate distortion theory as being concerned with source-user pairs. The mathematical definitions of what this shall mean here will be given in this chapter along with the definition of the rate distortion function and some of its properties.

The source has an alphabet $\underline{\overline{X}}$ governed by a probability law on the outputs. The source emits values from $\underline{\overline{X}}$ at regular time intervals. On any N consecutive time intervals there is, therefore, an N-sequence of values $\underline{x}_N$ from this source. The statistical properties of any N-sequence of values from the source is governed by the source probability measure P_x^N on the sigma algebra $A(\underline{\overline{X}}^N)$ of sets from the product space $\underline{\overline{X}}^N$. The source will further be required to be stationary and ergodic. The source alphabet shall be real valued. In Chapter 3, further topological requirements on $\underline{\overline{X}}$ will be made, however, these need not be made at this point.

The user or reproduction alphabet $\underline{\overline{Y}}$ is also real valued and will also have further topological requirements in Chapter 3. The user will receive,at regular time intervals, a value from $\underline{\overline{Y}}$. One value from $\underline{\overline{Y}}$ shall be received for each value emitted by the source. An N-sequence of values from $\underline{\overline{Y}}$ is denoted by $\underline{y}_N$.

The statistical properties of any N-sequence of received values are governed by a probability measure P^N on the sigma algebra $A(\underline{\overline{X}}^N \times \underline{\overline{Y}}^N)$ of sets from the space $\underline{\overline{X}}^N \times \underline{\overline{Y}}^N$ with the

restriction that

$$P_x^N(A) = \int_{A \times \underline{\bar{Y}}^N} P^N(A, d\underline{y}_N) \qquad \forall\ A \in A(\underline{\bar{X}}^N)$$

P^N thus describes a probabilistic mapping between N source outputs and their corresponding reproductions. The induced probability measure on the sigma algebra $A(\underline{\bar{Y}}^N)$ of sets from $\underline{\bar{Y}}^N$ shall be denoted by Q^N. The mapping governed by P^N shall be referred to as a test channel and it will be a task of this dissertation to find the test channel which is optimum in a sense to be defined shortly.

To complete the source-user terminology, the user shall specify a class of distortion criteria $\mathcal{D}$ on N-sequences such that

$$\mathcal{D} = \{d_N(\underline{x}_N, \underline{y}_N) \qquad N=1,2,\ldots.\}$$

$d_N(\underline{x}_N, \underline{y}_N)$ will be constructed from the single letter distortion measure $d(x,y)$ such that

$$d_N(\underline{x}_N, \underline{y}_N) = \frac{1}{N} \sum_{i=1}^{N} d(x_i, y_i)$$

$d(x,y)$ shall be a mapping of values from $\underline{\bar{X}} \times \underline{\bar{Y}}$ into the positive real line $[0,\infty)$. Other restrictions on $d(x,y)$ will be presented in the following, as needed.

In the following, the average distortion attributed to a particular P^N shall be denoted by $d(P^N)$ where

$$d(P^N) \triangleq \int_{\underline{\bar{X}}^N \times \underline{\bar{Y}}^N} d_N(\underline{x}_N, \underline{y}_N) P^N(d\underline{x}_N, d\underline{y}_N)$$

The I-divergence between two probability measures P_1 and P_2 on the same space Z shall be defined as[1]

$$I(P_1||P_2) \triangleq \sup \sum_{i=1}^{\infty} P_1(A_i) \log[\frac{P_1(A_i)}{P_2(A_i)}] \tag{2.1}$$

where the sup is over all partitions of Z such that $A_i \varepsilon A(Z)$, the sigma algebra of sets on Z. The mutual information between the random N-vectors from the source and reproduction can thus be defined as

$$I(X^N;Y^N) \triangleq I(P^N) = I(P^N||P_X^N \times Q^N) \tag{2.2}$$

where $P_X^N \times Q^N$ is the product measure generated from the marginals and $\underline{X}^N \times \underline{Y}^N$ is partitioned into rectangles in the definition of I-divergence above. The partition need not be rectangular, however, as shown by Dobrushin (see Pinsker [10]).

It is well known that

$$I(P_1||P_2) \geq 0 \tag{2.3}$$

The other elementary properties of this quantity will be used throughout this dissertation without proof. The following theorem on the form of $I(P_1||P_2)$ is of importance here.

Theorem 2.1 (Gelfand-Yaglom-Perez)

If P_1 is not absolutely continuous with respect to P_2, then $I(P_1||P_2) = \infty$. If P_1 is absolutely continuous with respect to P_2 ($P_1 << P_2$), then

[1] Note all logarithms, unless specified otherwise, shall be the natural logarithm.

$$I(P_1||P_2) = \int_Z \log[a(z)]P_1(dz) \tag{2.4}$$

where $a(z)$ is the Radon-Nikodym derivative of P_1 with respect to P_2.

Proof: See Pinsker [10] p. 23-25. ///

With this terminology defined, the rate distortion function $R(d)$ may be defined as

$$R(d) \triangleq \lim_{N\to\infty} R_N(d) \tag{2.5}$$

where

$$R_N(d) \triangleq \inf_{P^N \epsilon A^N} \frac{1}{N} I(X^N;Y^N) \tag{2.6}$$

and

$$A^N = \{P^N: \int_{\underline{X}^N \times \underline{Y}^N} d_N(\underline{x}_N, \underline{y}_N) P^N(d\underline{x}_N, d\underline{y}_N) \leq d$$

$$\text{and } P^N \text{ has } P_x^N \text{ as a marginal}\}$$

The companion function or inverse function, known as the distortion rate function, shall be defined here as

$$D(R) \triangleq \lim_{N\to\infty} D_N(R) \tag{2.7}$$

where

$$D_N(R) \triangleq \inf_{P^N \epsilon B^N} E\{d_N(\underline{x}_N, \underline{y}_N)\} \tag{2.8}$$

and

$$B^N = \{P^N: \frac{1}{N} I(X^N;Y^N) \leq R \text{ and } P_x^N \text{ is given}\}$$

The limits in these definitions exist for stationary sources, and for any N

$$R_N(d) \geq R(d)$$

This fact is stated in the following lemma.

Lemma 2.1

$$\lim_{N\to\infty} R_N(d) = \inf_N R_N(d) \tag{2.9}$$

Proof: See Gallager [4] Thm. 9.8.1. The proof uses the fact that $I(X;YZ) \geq I(X;Y)$ and the definition of $R_N(d)$ to obtain

$$(L+N)R_{L+N}(d) \leq L\, R_L(d) + N\, R_N(d)$$

Gallager's [4] lemma 4A.2 is then used to prove the result. ///

If the source is memoryless then

$$R(d) = R_1(d)$$

and

$$D(R) = D_1(R)$$

Therefore, only $R_1(d)$ need be considered here although $R_N(d)$ will be considered through Chapter 3.

It should be noted that the definition of the rate distortion function is often stated in the form shown on the right hand side of equation (2.9). It should also be remem-

bered that the rate distortion function as defined here gains physical significance from a source coding theorem. For example, see Berger [3] Chapter 7.

A basic property of $R_N(d)$ is its convexity, which is stated in the following lemma.

Lemma 2.2 $R_N(d)$ is a positive nonincreasing convex U function of d.

Proof: Follow the procedure of Gallager [4] p. 445-446. ///

The convexity of $R_N(d)$ has important consequences in the properties of $R_N(d)$ to be developed. The following lemma states such a property.

Lemma 2.3 If there exists a P^N which achieves the point $R_N(d_2) \neq 0$, then $d(P^N) = d_2$.

Proof: If there is a P^N such that $d(P^N) = d_1 < d_2$ and it achieves a point on $R_N(d)$, namely $R_N(d_2)$, then it must achieve the point $R(d_1)$ as well. This implies that $R(d_1) = R(d_2)$ since the rate distortion function is nonincreasing. However, this can only happen at R=0 because the rate distortion function is convex U. ///

In developing properties of the rate distortion function it is useful to parameterize it. That is, express R and d in terms of another variable, which will be called s. It will soon become apparent that s is related to the slope of

$R_N(d)$. A point on $R_N(d)$ will often be referred to as (R_s,d_s) when that point can be associated with a particular value of s. The point (R_s,d_s) shall be associated with a straight line with slope -s,lying strictly below $R_N(d)$, and touching $R_N(d)$ at the single point (R_s,d_s). In general, however, a single point on $R_N(d)$ may have several s associated with it, as will become apparent. The following lemma will be helpful in proving results in this area.

Lemma 2.4 If a straight line lying strictly below and touching $R_N(d)$ at a single point $(R_N(d_1),d_1)$ has slope -s (not necessarily unique), then

$$R_N(d') + sd' \geq R_N(d_1) + sd_1 \qquad \forall\ d'$$

Proof: If one considers the rate axis as u and the distortion axis as v, then any point on a line touching $R_N(d)$ at d_1 with slope -s is given by

$$u+sv = R_N(d_1) + sd_1$$

For some point $v = d'$, let u be given by this equation. The convexity of $R_N(d)$ requires that $u_{d'} \leq R_N(d')$, therefore

$$R(d') + sd' \geq R(d_1) + sd_1$$

and this argument holds for all d'. ///

The relationship s has to $R_N(d)$ should now be clear. It can be seen that because of the convexity of $R_N(d)$ that s

need only be positive. Fig. 2.1 shows a typical $R_N(d)$, which might be encountered.

It will now be useful to generalize the ideas presented in Lemma 2.4 and utilize some ideas introduced by Blahut [5] and extended by Csiszar [8]. The quantity $J(P^N,Q^N,s)$ is defined as

$$J(P^N,Q^N,s) \triangleq I(P^N||P_x^N \times Q^N) + s\, d(P^N) \tag{2.10}$$

where here P^N and Q^N need not be related. That is, Q^N need not be the Y^N-marginal of P^N. However, P_x^N is still required to be the X^N-marginal of P^N. In the case when Q^N is the Y^N-marginal of P^N, the special symbol $J(P^N,s)$ shall be used. From $J(P^N,s)$ the important quantity $F_N(s)$ may be defined as

$$F_N(s) \triangleq \inf_{P^N} J(P^N,s) \tag{2.11}$$

where the inf is over all P^N with the proper X^N-marginal and s is a positive real number. The following lemma establishes the relationship between $R_N(d)$ and $F_N(s)$.

Lemma 2.5 If P_1^N achieves $F_N(s)$ then it also achieves the point on $R_N(d)$ at $d_1 = d(P_1^N)$. Further, if P_2^N achieves a point on $R_N(d)$ at $d_2 = d(P_2^N)$, then it achieves $F_N(s)$ for the s for which a straight line with slope -s and lying strictly below $R_N(d)$ is touching $R_N(d)$ at the single point $R_N(d_2)$.

Proof: For some P_3^N with $d_3 = d(P_3^N)$ and some $s>0$

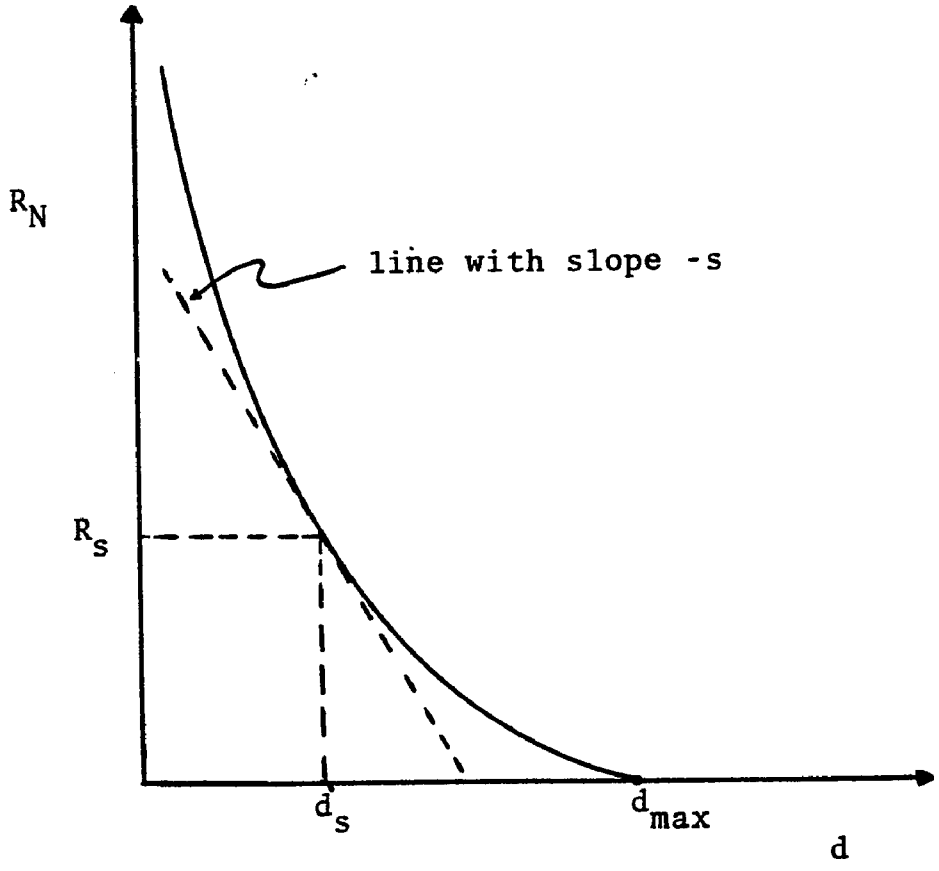


Fig. 2.1 A Typical $R_N(d)$

$$J(P_3^N,s) \geq R_N(d_3) + sd_3$$

This follows from (2.11), (2.10),and the definition of $R_N(d)$. However, Lemma 2.4 requires

$$R_N(d_3) + sd_3 \geq R_N(d_s) + sd_s$$

Therefore, for any P_3^N and any $s>0$

$$J(P_3^N,s) \geq R_N(d_s) + sd_s$$

and

$$F_N(s) \geq R_N(d_s) + sd_s$$

Now assume P_2^N achieves $R_N(d)$ at $d_2 = d(P_2^N)$, which has a line with slope -s touching $R_N(d)$ at only this point. Then

$$J(P_2^N,s) = R_N(d_s) + sd_s \geq F_N(s) \geq R_N(d_s) + sd_s$$

Therefore, P_2^N achieves $F_N(s)$.
Now assume P_1^N achieves $F_N(s)$ and let $d_1 = d(P_1^N)$ then

$$F_N(s) = J(P_1^N,s) \triangleq I(P_1^N) + sd_1$$

but this requires that

$$I(P_1^N) = \inf_{d(P^N)\leq d_1} I(P^N) \triangleq R_N(d_1)$$

thus P_1^N achieves $R_N(d)$. ///

Measures which achieve $F_N(s)$ are therefore related to

measures which achieve $R_N(d)$, however, the usefulness of $F_N(s)$ and $J(P^N,Q^N,s)$ has not yet been presented. It will be shown that

$$F_N(s) = \inf_{Q^N,P^N} J(P^N,Q^N,s)$$

and that the form which P^N must have to achieve $F_N(s)$ is very restricted. To start in this direction, define P_Q^N as the probability measure generated by a particular Q^N according to

$$P_Q^N(d\underline{x}_N,d\underline{y}_N) \triangleq \alpha(\underline{x}_N)e^{-sd_N(\underline{x}_N,\underline{y}_N)}P_x^N\times Q^N(d\underline{x}_N,d\underline{y}_N) \quad (2.12)$$

where

$$\alpha(\underline{x}_N) \triangleq [\int_{\underline{Y}^N} e^{-sd_N(\underline{x}_N,\underline{y}_N)}Q^N(d\underline{y}_N)]^{-1} \quad (2.13)$$

This is easily seen to be a probability measure. However, note that its Y^N-marginal is, in general, not Q^N. Looking ahead, one might want to keep in mind that the problem in finding a P^N which achieves $R_N(d)$ will become one of finding a Q^N which generates a P_Q^N whose Y^N-marginal is Q^N. It is also useful to introduce the notation that Q_P^N is the Y^N-marginal of P^N. For P_Q^N, this may be found by integrating the X^N dependence out of P_Q^N, ie.

$$Q_{P_Q}^N = [\int_{\underline{X}^N} \alpha(\underline{x}_N)e^{-sd_N(\underline{x}_N,\underline{y}_N)}P_x^N(d\underline{x}_N)]Q^N \quad (2.14)$$

The bracketed term is recognized as the Radon-Nikodym derivative of $Q_{P_Q}^N$ with respect to Q^N and will be denoted by $C(\underline{y}_N)$,

that is

$$C(\underline{y}_N) \triangleq \int_{\underline{X}^N} \alpha(\underline{x}_N) e^{-sd_N(\underline{x}_N, \underline{y}_N)} P_x^N(d\underline{x}_N) \tag{2.15}$$

It is easily seen that for $Q_{P_Q}^N$ to be equivalent to Q^N that $C(\underline{y}_N) = 1$ almost everywhere Q^N.

With this notation in mind and the requirement that $J(P^N, Q^N, s) < \infty$ and, of course, $s>0$, it follows by substitution and algebraic manipulation that

$$J(P^N, Q^N, s) = J(P^N, s) + I(Q_p^N || Q^N) \tag{2.16}$$

$$= J(P_Q^N, Q^N, s) + I(P^N || P_Q^N) \tag{2.17}$$

Since I-divergence is a positive quantity, it follows immediately that

$$\inf_{Q^N} J(P^N, Q^N, s) = J(P^N, s) \tag{2.18}$$

and

$$\inf_{P^N} J(P^N, Q^N, s) = J(P_Q^N, Q^N, s) = \int_{\underline{X}^N} \log[\alpha(\underline{x}_N)] P_x^N(d\underline{x}_N) \tag{2.19}$$

where the term on the right of (2.19) follows from (2.10). Further, applying the inf over P^N to (2.18) and over Q^N to (2.19) and using the definition of $F_N(s)$ gives

$$F_N(s) = \inf_{Q^N} \int_{\underline{X}^N} \log[\alpha(\underline{x}_N)] P_x^N(d\underline{x}_N) \tag{2.20}$$

It can be seen from (2.20) and (2.17) that if a measure P^N achieves $F_N(s)$, its Y^N-marginal must achieve (2.20).

The above argument also shows that for $F_N(s)$ to be achieved one must have a P^N and an associated Y^N-marginal Q^N such that

$$I(P^N||P^N_Q) = 0 \tag{2.21}$$

Further, the Y^N-marginal of P^N_Q, $Q^N_{P_Q}$, must be such that

$$I(Q^N_{P_Q}||Q^N) = 0 \tag{2.22}$$

However, a P^N such that (2.21) and (2.22) are satisfied does not guarantee that $F_N(s)$ is achieved.

One needs a necessary and sufficient condition on P^N to guarantee that $F_N(s)$ has been achieved. To consider this, note that equation (2.22) may be written with the use of Theorem 2.1 and (2.14) as

$$I(Q^N_{P_Q}||Q^N) = \int_{\underline{Y}^N} \log[C(\underline{y}_N)]C(\underline{y}_N)Q^N(d\underline{y}_N) = 0 \tag{2.23}$$

The necessary and sufficient condition which shall be developed, is that

$$C(\underline{y}_N) \leq 1 \qquad \forall\ \underline{y}_N \in \underline{Y}^N \tag{2.24}$$

Note that (2.23) requires that (See (2.14))

$$C(\underline{y}_N) = 1 \text{ almost everywhere } Q^N \tag{2.25}$$

The requirement (2.25) is equivalent to saying that $Q^N_{P_Q} = Q^N$.

Further, a Q^N which generates a $C(\underline{y}_N)$ satisfying (2.25), satisfies (2.23) and (2.22). If $Q^N_{P_Q} = Q^N$, then P^N_Q has Y^N-marginal Q^N and therefore (2.21) is satisfied. Therefore, a measure P^N_Q generated from a Q^N satisfying (2.24) will satisfy (2.21) and (2.22). The sufficiency of (2.24) shall be shown as follows.

Assume Q^N_1 satisfies (2.24) and generates $\alpha_1(\underline{x}_N)$ via (2.13). Any other Q^N generating an $\alpha(\underline{x}_N)$ is such that

$$\int_{\underline{X}^N} \log[\alpha_1(\underline{x}_N)]P^N_x(d\underline{x}_N) - \int_{\underline{X}^N} \log[\alpha(\underline{x}_N)]P^N_x(d\underline{x}_N)$$

$$= \int_{\underline{X}^N \times \underline{Y}^N} \log\left[\frac{\alpha_1(\underline{x}_N)}{\alpha(\underline{x}_N)}\right]\alpha(\underline{x}_N)e^{-sd_N(\underline{x}_N,\underline{y}_N)} \times$$

$$P^N_x \times Q^N(d\underline{x}_N, d\underline{y}_N)$$

Using $x-1 \geq \log x$,

$$\int_{\underline{X}^N} \log[\alpha_1(\underline{x}_N)]P^N_x(d\underline{x}_N) - \int_{\underline{X}^N} \log[\alpha(\underline{x}_N)]P^N_x(d\underline{x}_N)$$

$$\leq \int_{\underline{Y}^N}\left[\int_{\underline{X}^N} \alpha_1(\underline{x}_N)e^{-sd_N(\underline{x}_N,\underline{y}_N)}P^N_x(d\underline{x}_N)\right]Q^N(d\underline{y}_N) - 1$$

Because Q^N_1 was assumed to satisfy (2.24) the term in the bracket is less than one for all $\underline{y}_N \in \underline{Y}^N$, therefore

$$\int_{\underline{X}^N} \log[\alpha_1(\underline{x}_N)]P^N_x(d\underline{x}_N) \leq \int_{\underline{X}^N} \log[\alpha(\underline{x}_N)]P^N_x(d\underline{x}_N) \qquad \forall\ Q^N$$

By (2.20) the left hand side of this is $F_N(s)$ and therefore (2.24) is a sufficient condition on Q^N to achieve $F_N(s)$.

The above is summarized and the necessity of (2.24) is shown in the following theorem, Csiszar [8].

Theorem 2.2 P^N achieves $F_N(s)$ iff its Y^N-marginal Q^N is such that

$$C(\underline{y}_N) \leq 1 \qquad \forall\ \underline{y}_N \in \underline{\bar{Y}}^N \tag{2.26}$$

Further, if Q^N satisfies (2.26) then

$$P_Q^N(d\underline{x}_N, d\underline{y}_N) = \alpha(\underline{x}_N) e^{-sd_N(\underline{x}_N, \underline{y}_N)} P_x^N \times Q^N(d\underline{x}_N, d\underline{y}_N) \tag{2.27}$$

achieves $F_N(s)$ where

$$\alpha(\underline{x}_N) \triangleq \left[\int_{\underline{\bar{Y}}^N} e^{-sd_N(\underline{x}_N, \underline{y}_N)} Q^N(d\underline{y}_N)\right]^{-1} \tag{2.28}$$

Proof: It has been shown (2.20) that

$$F_N(s) = \inf_{Q^N} \int_{\underline{\bar{X}}^N} \log[\alpha(\underline{x}_N)] P_x^N(d\underline{x}_N)$$

If Q_o^N achieves $F_N(s)$ then for any other Q^N

$$\int_{\underline{\bar{X}}^N} \log[\alpha(\underline{x}_N)] P_x^N(d\underline{x}_N) \geq \int_{\underline{\bar{X}}^N} \log[\alpha_o(\underline{x}_N)] P_x^N(d\underline{x}_N) \tag{2.29}$$

Let

$$Q_1^N = (1-\beta) Q_o^N + \beta \delta_{\underline{y}_{N_o}}$$

where $\delta_{\underline{y}_{N_o}}$ is a point mass at some $\underline{y}_{N_o} \in \underline{\bar{Y}}^N$ and $0 \leq \beta < 1$. Then

$$\alpha_1(\underline{x}_N) = \alpha_o(\underline{x}_N)\left[1-\beta + \beta e^{-sd_N(\underline{x}_N, \underline{y}_{N_o})} \alpha_o(\underline{x}_N)\right]^{-1}$$

Thus, using (2.29)

$$g(\beta) \triangleq -\int_{\underline{X}^N} \log[1-\beta + \beta e^{-sd_N(\underline{x}_N, \underline{y}_{N_o})} \alpha_o(\underline{x}_N)] P_x^N(d\underline{x}_N) \geq 0 \tag{2.30}$$

Differentiating the left hand side of this with respect to β one gets

$$g'(\beta) = \int_{\underline{X}^N} \frac{1-\alpha(\underline{x}_N) e^{-sd_N(\underline{x}_N, \underline{y}_{N_o})}}{1-\beta + \beta e^{-sd_N(\underline{x}_N, \underline{y}_{N_o})} \alpha_o(\underline{x}_N)} P_x^N(d\underline{x}_N)$$

At $\beta=0$, it is recognized that

$$g'(0) = 1 - C(\underline{y}_{N_o})$$

Note that $g(0) = 0$. Therefore as one moves away from $\beta=0$, $g(\beta)$ must increase or stay the same to satisfy (2.30). Therefore

$$g'(0) \geq 0$$

or

$$1 \geq C(\underline{y}_{N_o})$$

Since $\underline{y}_{N_o}$ was arbitrary, the conclusion is that equation (2.26) is a necessary condition on a Q^N (and thus P^N) that achieves $F_N(s)$. If (2.26) is satisfied, however, then it has already been argued that the condition is sufficient. ///

A useful corollary of this theorem is:

Corollary of Theorem 2.2 If one has a Q_1^N such that $C(\underline{y}_{N_o}) > 1$ for some $\underline{y}_{N_o} \varepsilon \overline{\underline{Y}}^N$, then there exists a positive number β, $0 < \beta < 1$, such that

$$J(P_{Q_1}^N, Q_1^N, s) > J(P_{Q_2}^N, Q_2^N, s)$$

where

$$Q_2^N = (1-\beta)Q_1^N + \beta\delta_{\underline{y}_{N_o}}$$

Proof: From (2.19) $J(P_Q^N, Q^N, s) = \int_{\overline{\underline{X}}^N} \log[\alpha(\underline{x}_N)] P_x^N(d\underline{x}_N)$

then $J(P_{Q_1}^N, Q^N, s) - J(P_{Q_2}^N, Q_2^N, s) = -g(\beta)$

where g is as defined in (2.30) with α_o replaced by α_1. Since

$$g'(0) = 1 - C(\underline{y}_{N_o}) < 0$$

then

$$-g'(0) > 0$$

which implies since $g(0) = 0$ that there is a $\beta > 0$ such that $-g(\beta) > 0$ or that

$$J(P_{Q_1}^N, Q^N, s) > J(P_{Q_2}^N, Q_2^N, s) \qquad ///$$

Theorem 2.2 gives the basic form P^N must have if it is to achieve $F_N(s)$. In Chapter 3, the existence of P^N, which achieves $R_N(d)$ and therefore $F_N(s)$, will be shown for the examples to be considered in Chapters 5 and 6. $F_N(s)$ is, of course, related to $R_N(d)$ by Lemma 2.5.

A P^N which achieves $R_N(d)$ might achieve $F_N(s)$ for more than one value of s. This is the situation which arises if the slope of $R_N(d)$ is discontinuous. This will happen for the cases to be considered later at one point, namely, d_{max} (see Fig. 2.1). In what follows, d_{max} will occur at the smallest d for which $R_N(d) = 0$. If d_{max} exists, then the point ($R = 0$, $d = d_{max}$) is achieved by mapping all source outputs into a point $\underline{y}_{N_{max}}$, which solves (see Berger [3] p.90)

$$d_{max} = \inf_{\underline{y}_N \in \underline{\overline{Y}}^N} \int_{\underline{\overline{X}}^N} d_N(\underline{x}_N, \underline{y}_N) P_x^N(d\underline{x}_N) \tag{2.31}$$

The following lemma considers the occurrence of discontinuities in the slope of $R_N(d)$ for distortion values less than d_{max}. The proof generalizes that of Berger [3], Theorem 2.5.5.

Lemma 2.6 If there is an achieving measure for $R_N(d)$ for all values of $d < d_{max}$ and such that $R_N(d) < \infty$, then $R_N'(d)$ is continuous for all values of d satisfying these conditions.

Proof: If there is a P^N which achieves $R_N(d)$, there is a P^N of the form of (2.26) and (2.27) which achieves $R_N(d)$. For there to be a discontinuity in $R_N'(d)$ at a point achieved by P^N, P^N must achieve $F_N(s)$ at two values of s. This requires that $\alpha_1(\underline{x}_N)e^{-s_1 d_N(\underline{x}_N, \underline{y}_N)} = \alpha_2(\underline{x}_N)e^{-s_2 d_N(\underline{x}_N, \underline{y}_N)}$ almost everywhere $P_x^N \times Q^N$ by the uniqueness of the Radon-Nikodym derivative. Therefore,

$$\frac{\alpha_1(\underline{x}_N)}{\alpha_2(\underline{x}_N)} = e^{(s_1-s_2)d_N(\underline{x}_N,\underline{y}_N)}$$

almost everywhere $P_x^N \times Q^N$. This can only occur if $s_1 = s_2$ otherwise, it implies that

$$\alpha_1(x)e^{-s_1 d_N(\underline{x}_N,\underline{y}_N)}$$

is independent of $\underline{y}_N$ almost everywhere Q^N which implies that the mutual information is zero. This can occur only at d_{max} by definition. ///

Lemma 2.6 has pointed out that for $d < d_{max}$ and $R_N(d) < \infty$ there is a one-to-one correspondence between values of s and points on $R_N(d)$, namely (R_s,d_s), if $R_N(d)$ is achieved at every $d \leq d_{max}$. This prompts the following parametric form of (R_s,d_s) and enhances the relationship to $F_N(s)$.

Lemma 2.7 If $R_N(d)$ is achieved for all d such that $d < d_{max}$ and $R_N(d) < \infty$ then

$$R_N'(d_s) = -s \qquad \text{if } d_s < d_{max}\ , \tag{2.32}$$

$$R_s = F_N(s) - sd_s \tag{2.33}$$

and

$$d_s = \frac{d}{ds} F_N(s) \tag{2.34}$$

Proof: $R'(d)$ is continuous by Lemma 2.6 and there is a line tangent to $R_N(d)$ at (R_s,d_s) with slope -s, there-

fore (2.32). Equation (2.33) follows immediately from Lemma 2.5. Differentiating (2.33) with respect to s gives

$$\frac{dR_s}{ds} = \frac{dF_N(s)}{ds} - d_s - s\,\frac{dd_s}{ds}$$

or

$$0 = \frac{dR_s}{ds} - \frac{dR_s}{dd_s}\,\frac{dd_s}{ds} = \frac{dF_N(s)}{ds} - d_s$$

therefore,

$$\frac{dF_N(s)}{ds} = d_s \qquad ///$$

Lemma 2.7 indicates an important relationship between $R_N(d)$ and $F_N(s)$ since both R_s and d_s are expressed in terms of $F_N(s)$. It is further interesting to note that if one knows the relationship between d_s and s, one can find $F_N(s)$ by integrating.

It will be useful in considering lower bounds to $R_N(d)$ if $F_N(s)$ is expressed in a different way than (2.20). The following lemma gives this form.

Lemma 2.8[1] If $F_N(s)$ is achieved then

$$F_N(s) = \max_{\alpha(\underline{x}_N)\,\varepsilon B} \int_{\underline{X}^N} \log(\alpha(\underline{x}_N))\, P_x^N(d\underline{x}_N) \qquad (2.35)$$

[1] This lemma, along with prior reasoning given here, is the basis for an alternate definition of $R_N(d)$ such as Theorem 4.2.3 of Berger [3].

where

$$B = \{\alpha(\underline{x}_N): \alpha(\underline{x}_N) > 0;\ C(\underline{y}_N) \leq 1 \qquad \forall\ \underline{y}_N \in \underline{\overline{Y}}^N\}$$

Proof: This follows Csiszar [8]. If $F_N(s)$ is achieved, then there is an $\alpha_1(\underline{x}_N)$ in B which yields $F_N(s)$ and an associated Q_1^N. Now consider any other $\alpha(\underline{x}_N)$ from B and note that

$$\int_{\underline{\overline{X}}^N} \log[\alpha(\underline{x}_N)]P_x^N(d\underline{x}_N) - \int_{\underline{\overline{X}}^N} \log[\alpha_1(\underline{x}_N)]P_x^N(d\underline{x}_N) =$$

$$= \int_{\underline{\overline{X}}^N \times \underline{\overline{Y}}^N} \log\left[\frac{\alpha(\underline{x}_N)}{\alpha_1(\underline{x}_N)}\right] P_1^N(d\underline{x}_N, d\underline{y}_N)$$

now using $\log x \leq x - 1$ one gets

$$\int_{\underline{\overline{X}}^N} \log[\alpha(\underline{x}_N)]P_x^N(d\underline{x}_N) - F_N(s)$$

$$\leq \int_{\underline{\overline{X}}^N \times \underline{\overline{Y}}^N} \alpha(\underline{x}_N) e^{-sd_N(\underline{x}_N, \underline{y}_N)} P_x^N(d\underline{x}_N) Q_1^N(d\underline{y}_N) - 1$$

but since $\alpha(\underline{x}_N) \in B$, the right side is less than or equal to zero therefore,

$$F_N(s) \geq \int_{\underline{\overline{X}}^N} \log[\alpha(\underline{x}_N)]P_x^N(d\underline{x}_N) \qquad \forall\ \alpha(\underline{x}_N) \in B \qquad ///$$

Lemma 2.8 suggests a possible optimization in terms of $\alpha(\underline{x}_N)$ directly (avoiding Q^N), however, no success has been achieved in this direction. The lemma is useful, however, in generating lower bounds to $R_N(d)$. One such bound is the

Shannon Lower Bound (SLB) which is of interest here because it is sometimes tight and because of its link to the theory to be presented later. The following restrictions are made in illustrating this bound.

1. $P_x^N(d\underline{x}_N) = p_x^N(\underline{x}_N)d\underline{x}_N$ (P_x^N has a density function)
2. $d(x,y) = d(|x-y|)$ (Difference distortion measure)
3. $\underline{X}^N$ and $\underline{Y}^N$ are R^N (N-dimensional Euclidean space)
4. $\int_{R^N} e^{-sd_N(\underline{z}_N)}\, d\underline{z}_N < \infty \qquad \forall\ s>0$.

Following Berger [3], fix an s and choose

$$\alpha(\underline{x}_N) = \frac{K_s}{p_x^N(\underline{x}_N)} \tag{2.36}$$

where K_s is a constant to be determined. For this $\alpha(\underline{x}_N)$ to be in the set B of Lemma 2.8, K_s must be such that

$$K_s\int_{R^N} e^{-sd_N(\underline{z}_N)}\, d\underline{z}_N \leq 1$$

K_s is chosen to make this 1 and thus

$$F_N(s) \geq \log K_s - \int_{\underline{X}^N} \log[p_x^N(d\underline{x}_N)]P_x^N(d\underline{x}_N)$$

and therefore

$$R_N(d_s) \geq \log K_s - \int_{\underline{X}^N} \log(p_x^N(\underline{x}_N))P_x^N(d\underline{x}_N) - sd_s$$

Lemma 2.4 shows that

$$R_N(d) \geq \log K_s - \int_{\underline{X}^N} \log(p_x^N(\underline{x}_N)) P_x^N(d\underline{x}_N) - sd \qquad \forall\, d\ ,$$

and since this holds for every $s>0$, one may write

$$R_N(d) \geq \log K_t - \int_{\underline{X}^N} \log(p_x^N(\underline{x}_N)) P_x^N(d\underline{x}_N) - td \qquad \forall\, t>0\ . \tag{2.37}$$

With d fixed in (2.37), Berger ([3], p. 92-93) shows that the right side is maximized, thus making the bound as good as possible, by relating t to d via the equation

$$K_t \int_{R^N} d_N(\underline{z}_N) e^{-sd_N(\underline{z}_N)} d\underline{z}_N = d \tag{2.38}$$

This is Berger's [3] construction of the SLB. One should consult Berger [3] for examples and more details.

When the SLB is tight, the optimum $\alpha(\underline{x}_N)$ will be related to $p_x^N(\underline{x}_N)$ via (2.36), which requires that

$$\frac{1}{K_s}\, p_x^N(\underline{x}_N) = \int_{\underline{Y}^N} e^{-sd_N(\underline{x}_N, \underline{y}_N)} Q^N(d\underline{y}_N) \qquad \forall\, x_N \in R^N \tag{2.39}$$

Since $d_N(\underline{x}_N, \underline{y}_N)$ is a difference distortion criteria, (2.39) suggests inversion by Fourier Transform techniques (if possible) to acquire a Q^N which may be tested to see if it satisfies the requirements of Theorem 2.2. This technique is explored in Berger [3] Chapter 4 and will not be repeated here. Integral equations of the type of (2.39) will appear in later chapters and the relationship of the solutions to the SLB when it is tight will become apparent.

For most sources and distortion criteria, including those to be considered in the later chapters, the SLB will not be tight and it will be the subject of the later chapters to consider what is to be done in this case. In some cases, one may not want to seek an analytical solution to $R_N(d)$ or one may not exist. An algorithm in this case may be the only alternative. Blahut's [5] algorithm has become an important part of rate distortion theory and will be described here.

If one wants to pursue an iterative solution from the relationships which have been developed in this chapter, one might want to start with a reproduction measure Q_o^N and generate from it the associated joint measure $P_{Q_o}^N$ via (2.12). That is

$$P_{Q_o}^N = \alpha(\underline{x}_N) e^{-sd_N(\underline{x}_N, \underline{y}_N)} \; P_X^N \times Q_o^N$$

Now remember that

$$F_N(s) = \inf_{Q^N} \inf_{P^N} J(P^N, Q^N, s)$$

and that from (2.16)

$$J(P_{Q_o}^N, Q_o^N, s) \geq J(P_{Q_o}^N, Q_1^N, s)$$

where

$$Q_1^N(\cdot) = \int_{\underline{X}^N} P_{Q_o}^N(d\underline{x}_N, \cdot)$$

therefore, one has reduced J and moved closer to $F_N(s)$ by

replacing Q_o^N by Q_1^N. Generating $P_{Q_1}^N$ from Q_1^N and using (2.17) results in a further reduction of J, that is

$$J(P_{Q_o}^N, Q_1^N, s) \geq J(P_{Q_1}^N, Q_1^N, s)$$

Therefore, one may continue to decrease J and thus approach $F_N(s)$ by generating a sequence of Q^N's starting with Q_o^N and utilizing the recursive relationship

$$Q_{n+1}^N = \left[\int_{\underline{X}^N} \alpha_n(\underline{x}_N) e^{-sd_N(\underline{x}_N, \underline{y}_N)} P_x^N(d\underline{x}_N)\right] Q_n^N$$

or

$$Q_{n+1}^N = C_N(\underline{y}_N) Q_n^N$$

This iteration will proceed until, hopefully one approaches

$$C_n(\underline{y}_N) = 1 \text{ almost everywhere } Q_n^N.$$

This is Blahut's algorithm [5].

Since J is nonincreasing at every point and is bounded from below (by zero), the sequence converges. The problems are showing that it converges to $F_N(s)$ and the speed of convergence. Clearly the starting Q_o^N must be considered when answering the first question since one can see cases where the wrong Q_o^N will be such that $F_N(s)$ will not be the limit of the sequence of J's. Blahut [5] presents a theorem which shows convergence to $F_N(s)$ if $Q_o^N = q_o^N(\underline{y}_N) d\underline{y}_N$ and $q_o^N(\underline{y}_N) > 0$. Numerically, this will require a numerical integration for every $\underline{y}_N$ carried in $C_n(\underline{y}_N)$ and since $q_o^N(\underline{y}_N) > 0$

$\forall\ \underline{y}_N \in \underline{\bar{Y}}^N$ there are in any case many which must be carried to keep the accuracy up. As will become apparent in the later chapters, the true solution, more often than not, will contain discrete points, that is $Q^N \not\ll d\underline{y}_N$. Therefore, $q_n^N(\underline{y}_N)$ must be approaching delta functions as it progresses. This fact makes a blind use of Blahut's algorithm in this situation subject to possible numerical problems. One cannot simply insert δ-functions into q^N to solve this problem, for Blahut's algorithm can only adjust their weight and not their position and the algorithm will no longer converge to $F_N(s)$. More will be said about this in Chapter 6.

Blahut's algorithm emphasizes the importance of $C(\underline{y}_N)$. The importance of $C(\underline{y}_N)$ in the remaining chapters will become apparent. The following lemma establishes the uniqueness of the optimum $C(\underline{y}_N)$.

Lemma 2.9 If $R_N(d)$ is achieved at some point, then the function $C(\underline{y}_N)$ associated with that point is unique.

Proof: The convexity of mutual information implies that if two measures P_1^N and P_2^N both achieve $R_N(d)$ at the same point then their weighted sum P_λ^N must also achieve $R_N(d)$ at that point. The $C(\underline{y}_N)$ associated with P_λ^N must be equal almost everywhere to one for both Q_1^N and Q_2^N. Therefore, $\int_{\underline{\bar{X}}} \frac{\alpha_\lambda(\underline{x}_N)}{\alpha_1(\underline{x}_N)} P_x^N(d\underline{x}_N) = 1$ and since $\alpha_\lambda(\underline{x}_N)$ and $\alpha_1(\underline{x}_N)$ achieve $F_N(s)$ (see 2.20) the implication is that $\alpha(\underline{x}_N)$ must be unique with respect to P_x^N. This requires $C(\underline{y}_N)$ (see 2.15) to be unique. ///

III. THE EXISTENCE OF AN ACHIEVING MEASURE

Chapter 2 presented many properties of the rate distortion function. Of specific interest for the remainder of this dissertation is Theorem 2.2, which specifies the form the achieving measure must have if it exists. This chapter is concerned with the existence of a measure which achieves $R_N(d)$. The arguments to be given here are topological and shall require $\underline{X}^N$ and $\underline{Y}^N$ to be complete separable metric spaces. Note that R^N along with the usual metric $\sum_i^N |x_i - y_i|$ is such a space. Note also that the product space $\underline{X}^N \times \underline{Y}^N$ is also a complete separable metric space (see Friedman [11], for example). In Chapters 5 and 6, $\underline{X}^N$ and $\underline{Y}^N$ will be R^N and $d(x,y)$ will be $|x-y|$ and $(x-y)^2$, respectively, and therefore this chapter is directed toward this situation. The presentation, however, is quite general and one should become familiar with what needs to be shown and can be shown for other situations which might occur.

The procedure to be followed here in showing the conditions on which there is a P^N which achieves $R_N(d)$ (see (2.6)) is to construct a weak topology of probability measures on $\underline{X}^N \times \underline{Y}^N$ and to show that under certain conditions, the set A^N of equation (2.6) is a compact set with respect to this topology. Further, it will be necessary to show that $I(X^N;Y^N)$ is a lower semicontinuous function with respect to this topology. The following theorem can then be used to show that the inf in (2.6) is achieved.

Theorem 3.1 A lower semicontinuous real valued function on a countably compact set assumes a minimum on that set.

Proof: For the relationship among definitions of compactness see Royden [12]. Compactness implies countable compactness. Then use the same arguments as Royden [12] Prop. 10. Chapter 9 except for a lower semicontinuous function and the inf rather than an upper semicontinuous function and the sup. ///

To construct a topology of the probability measures P^N on the metric space $\underline{X}^N \times \underline{Y}^N$, the development shall follow Parthasarathy [14] (see also Billingsley [13]). This topology is referred to as the weak topology of the space of measures. $C(\underline{X}^N \times \underline{Y}^N)$ shall stand for the space of all bounded real valued continuous functions on $\underline{X}^N \times \underline{Y}^N$. Convergence in the weak topology, denoted by $P_\alpha^N \to P^N$, shall mean that

$$\int_{\underline{X}^N \times \underline{Y}^N} f \, P_\alpha^N(d\underline{x}_N, d\underline{y}_N) \to \int_{\underline{X}^N \times \underline{Y}^N} f \, P^N(d\underline{x}_N, d\underline{y}_N)$$

$$\forall \; f \in C(\underline{X}^N \times \underline{Y}^N)$$

For further properties and equivalent constructions of this topology one should refer to [13] or [14].

Before presenting the results needed here, the two following definitions will be useful.

Definition 3.1 A probability measure P on a metric space $\underline{X}$ is said to be tight if for each $\varepsilon > 0$, there

exists a compact set $K \subseteq \overline{X}$ such that $P(K) > 1-\varepsilon$.

Definition 3.2 A family Π of probability measures on a metric space $\overline{X}$ is said to be uniformly tight if for every $\varepsilon>0$, there exists a compact set $K \subseteq \overline{X}$ such that $P(K) > 1-\varepsilon \quad \forall\ P\in\Pi$.

These definitions may be found in essentially the same form in Billingsley [13] and Parthasarathy [14].

Since the considerations have been restricted here to complete separable metric spaces, the following lemma will be useful.

Lemma 3.1 If $\overline{X}$ is a complete separable metric space, then each probability measure on $\overline{X}$ is tight.

Proof: Billingsley [13] Theorem 1.4. ///

The following theorem of Prohorov expresses the fundamental tool to be used here to show the compactness of the set A^N of equation (2.6).

Theorem 3.2 Let $\overline{X}$ be a complete separable metric space and Γ some set contained in the weak topology of measures on $\overline{X}$, then a necessary and sufficient condition that the closure of Γ (i.e. $\overline{\Gamma}$) is compact is that Γ be uniformly tight.

Proof: Parthasarathy [14] Theorem 6.7. ///

To apply this theorem to

$$A^N = \{P^N : \int_{\underline{\bar{X}}^N \times \underline{\bar{Y}}^N} d_N(\underline{x}_N, \underline{y}_N) P^N(d\underline{x}_N, d\underline{y}_N) \leq d \text{ and } P_x^N \text{ given}\} \tag{3.1}$$

It will be necessary to show that A^N is closed and that A^N is uniformly tight since we have already assumed that $\underline{\bar{X}}^N \times \underline{\bar{Y}}^N$ is complete and separable. It should be noted here that, if $\underline{\bar{X}}^N \times \underline{\bar{Y}}^N$ is restricted even further (compact, for example), the conditions on A^N can be weakened. For our purposes, however, the restriction of $\underline{\bar{X}}^N \times \underline{\bar{Y}}^N$ to be a complete separable metric space and the requirements on $d_N(\underline{x}_N, \underline{y}_N)$, which will be made next, will be adequate.

Remember that

$$d_N(\underline{x}_N, \underline{y}_N) \triangleq \frac{1}{N} \sum_{i=1}^{N} d(x_i, y_i)$$

$d(x,y)$ will be said to be nondecreasing if for all $x \in \underline{\bar{X}}$, there is an associated y_o such that

$$d(x, y_o) \leq d(x,y) \qquad \forall\ y \in \underline{\bar{Y}}$$

and

$$|y_o - y_1| > |y_o - y_2|$$

implies that

$$d(x, y_1) \geq d(x, y_2)$$

The following lemma, therefore, addresses the uniform tightness of A^N.

Lemma 3.2 If $\underline{\bar{X}}^N$ and $\underline{\bar{Y}}^N$ are complete separable metric spaces equipped with the usual metric and $d(x,y)$ is a continuous nondecreasing function with the property that $d(x,y) \to \infty$ as $|y| \to \infty$ $\forall\, x \in \underline{\bar{X}}$, then

$$A^N = \{P^N : \int_{\underline{\bar{X}}^N \times \underline{\bar{Y}}^N} d_N(\underline{x}_N, \underline{Y}_N)\, P^N(d\underline{x}_N, d\underline{Y}_N) \leq d \text{ and } P_x^N \text{ given}\}$$

is uniformly tight for all $d < \infty$.

Proof: Consider the compact set $A_1^N \subseteq \underline{\bar{X}}^N$ and let B^N be the set

$$B^N = \{\underline{Y}_N : d_N(\underline{x}_N, \underline{Y}_N) \leq d_o \qquad \forall\, \underline{x}^N \in A_1^N\}$$

where d_o is some finite constant. B^N is compact because it is totally bounded with respect to the usual metric and is a closed subset of $\underline{\bar{Y}}^N$ (remember $\underline{\bar{Y}}^N$ is complete, see Friedman [11] Theorem 3.5.6 and Royden [12] p. 120). Therefore, $A_1^N \times B^N$ is a compact subset of $\underline{\bar{X}}^N \times \underline{\bar{Y}}^N$ (see Royden [12] p. 143). Now, for all measures $P^N \in A^N$

$$d \geq \int_{\underline{\bar{X}}^N \times \underline{\bar{Y}}^N} d_N(\underline{x}_N, \underline{Y}_N)\, P^N(d\underline{x}_N, d\underline{Y}_N)$$

and, therefore,

$$d \geq \int_{A_1^N \times (B^N)^c} d_N(\underline{x}_N, \underline{Y}_N)\, P^N(d\underline{x}_N, d\underline{Y}_N)$$

but

$$\int_{A_1^N \times (B^N)^c} d_N(\underline{x}_N, \underline{y}_N) P^N(d\underline{x}_N, d\underline{y}_N) > d_o \; P^N(A_1^N \times (B^N)^c)$$

therefore,

$$d/d_o > P^N(A_1^N \times (B^N)^c)$$

d_o may be made as large as one wants or

$$d/d_o > \varepsilon/2$$

and B^N will remain a compact set. Thus, since

$$P^N(A_1^N \times B^N) = 1 - P_x^N((A_1^N)^c) - P^N(A_1^N \times (B^N)^c)$$

then

$$P(A_1^N \times B^N) > 1 - P_x^N((A_1^N)^c) - \varepsilon/2$$

But $\underline{X}^N$ is complete and separable, therefore, by Lemma 3.1, P_x^N is tight and $P_x^N((A_1^N)^c) < \varepsilon/2$ since A_1^N is compact and may be chosen to satisfy this. Therefore,

$$P(A_1^N \times B^N) > 1-\varepsilon \qquad \forall \; P^N \varepsilon A^N$$

and, therefore, A^N is uniformly tight. ///

Under the restrictions of Lemma 3.2, one may conclude that the closure of A^N is compact. It remains to show that A^N is closed. The following lemma will do this.

Lemma 3.3 Under the conditions of Lemma 3.2, A^N is closed.

Proof: To show closure, one need show that every convergent sequence of measures $P_n^N \epsilon A^N$ converges to a $P^N \epsilon A^N$. If $P_n^N \to P^N$, where $P^N \epsilon \overline{A^N}$ and $P_n^N \epsilon A^N$, then

$$X^N\text{-marginal of } P_n^N \to X^N\text{-marginal of } P^N$$

(use Billingsley [13] Theorem 3.1 and Theorem 2.1 to prove this). The X^N-marginal for all $P^N \epsilon A^N$ is P_x^N, therefore, the X^N-marginal of all $P^N \epsilon \overline{A^N}$ is P_x^N. All measures in $\overline{A^N}$ thus have the correct X^N-marginals to be in A^N. Now, with $d_N(\underline{x}_N, \underline{y}_N)$ continuous and positive, then

$$\int_{\underline{X}^N \times \underline{Y}^N} d_N(\underline{x}_N, \underline{y}_N) P^N(d\underline{x}_N, d\underline{y}_N) \leq \lim_n \int_{\underline{X}^N \times \underline{Y}^N} d_N(\underline{x}_N, \underline{y}_N) P_n^N(d\underline{x}_N, d\underline{y}_N)$$

where $P_n^N \epsilon A^N$ by assumption and $P_n^N \to P^N$. This follows from Breiman [15] Corol. 8.13 p. 164. Therefore,

$$\int_{\underline{X}^N \times \underline{Y}^N} d_N(\underline{x}_N, \underline{y}_N) P^N(d\underline{x}_N, d\underline{y}_N) \leq d$$

and thus P^N is in A^N. A^N is, therefore, closed. ///

Lemma 3.3 may now be combined with Lemma 3.2 and Theorem 3.2 to give:

Lemma 3.4 If $\underline{X}^N$ and $\underline{Y}^N$ are complete separable metric spaces each equipped with the usual metric and d(x,y)

has the properties of Lemma 3.2, then A^N (3.1) is compact for all $d < \infty$.

Proof: Follows immediately from the above. ///

One half the requirements of Theorem 3.1 have been met. One need only show that mutual information is lower semicontinuous with respect to the weak topology of probability measures.

Lemma 3.5 The mutual information is a lower semicontinuous function with respect to the weak topology of probability measures.

Proof: See Csiszar [8] or Dobrushin [16] who gives credit to Kolmogorov for this. ///

With Lemma 3.5, Lemma 3.4, and Theorem 3.1, the desired result of this chapter is achieved.

Theorem 3.3 If $\underline{X}^N$ and $\underline{Y}^N$ are complete separable metric spaces equipped with the usual metric and $d(x,y)$ has the properties of Lemma 3.2, then there exists a P^N in A^N (3.1) which achieves $R_N(d)$.

The conditions of Lemma 3.2 are satisfied by $d(x,y) = |x-y|$ and $d(x,y) = (x-y)^2$ and $R^N \times R^N$ is a complete separable metric space topologized by the usual metric. Therefore, $R_N(d)$ and, therefore, $F_N(s)$ will be achieved in the cases considered in the following chapters and the results of Chapter 2 give the form the achieving P^N must have.

IV. CONSIDERATION OF SOLUTIONS

In the introduction, it was stated that memoryless sources were to be considered. Up to now, $R_N(d)$ has been considered to illustrate the generality of the relationships developed. This chapter introduces ideas and terminology useful in finding the probability measure achieving $R_N(d)$. It will simplify matters if only memoryless sources are considered from here on and, therefore, let N=1. Most of the techniques to be discussed, however, are applicable when N>1. All superscripts will be dropped to denote N=1.

In the following, it will be useful to refer to the support of a measure and this will be defined now.

> Definition 4.1 P is supported by all sets $A_i \subseteq \overline{X}$ such that $P(A_i) = 1$. The intersection of the closures of all such sets shall be called the support of P. A point on the boundary of the support of P shall be a point on the smallest radius ball containing the support of P.

Definition 4.1 says that if P has a density function p (i.e. P<<dx),then the support of P is the closure of the set of x's where p>0. If p is discrete,then the support of P is the set of positions of the discrete masses.

With the existence of an achieving measure P established (see Chapter 3), the equations such a measure must satisfy may be repeated here. One will be required to find a repro-

duction measure Q for a particular value of s>0, which generates the function $\alpha(x)$ such that

$$C(y) \leq 1 \qquad \forall\ y \in \underline{Y} \tag{4.1}$$

and

$$C(y) = 1 \quad \text{almost everywhere } Q \tag{4.2}$$

where

$$C(y) \triangleq \int_{\underline{X}} \alpha(x) e^{-sd(x,y)} P_x(dx) \tag{4.3}$$

and

$$\alpha^{-1}(x) = \int_{\underline{Y}} e^{-sd(x,y)} Q(dy) \tag{4.4}$$

In all cases to be considered here, d(x,y) will be continuous. Therefore, C(y) will be continuous. Equation (4.2) will then require that C(y) = 1 over the support of Q.

If one knew the support of Q, then (4.3) and (4.4) would become coupled nonlinear integral equations, the analytical solution of which would depend heavily on the kernel $e^{-sd(x,y)}$. If integral equations of this type can be solved, one might then guess the support of Q, solve the integral equations, and test the resulting tentative solution to see if it satisfies (4.1) and (4.2). This technique requires two things: an intelligent guess of the support of Q and the ability to solve the integral equations. If the kernel of the integral equation is not of a form which can be handled by known tech-

niques, this method has little promise. This method shall be referred to as the integral equation method. Chapter 5 considers an important example of this method's use.

When the integral equation method is not suitable, that is when the kernel ($e^{-sd(x,y)}$) cannot be handled by known techniques, the approach must change. In some cases, the properties of the kernel which cause the integral equation method to fail also cause the solution to be of a particular form which simplifies finding the solution. The analytic properties of the kernel will be used to do this in Chapter 6. To be specific, properties of the kernel will cause the support of Q to be discrete, and as in the integral equation method, knowledge of the support of Q will be of great use.

In Chapter 5 and 6, the distortion measures $d(x,y)$ are difference measures, i.e. $d(x,y) = d(|x-y|)$. They are also strictly increasing, i.e., if $z_1 > z_2$ then $d(z_1) > d(z_2)$ where $d(0) = 0$. The following lemma illustrates the impact on the support of Q which this has.

Lemma 4.1 If $d(x,y)$ is a strictly increasing continuous difference distortion measure, then the support of Q which achieves $R(d)$ must lie within the boundary of the support of P_x.

Proof: $C(y) = 1$ on the support of Q and ≤ 1 elsewhere. $e^{-sd(|x-y|)}$ decreases as y moves away from x. Therefore, if y_1 is outside the boundary of the support of P_x, then there is a y_2 on the boundary such that $C(y_2) > C(y_1)$.

However, $1 \geq C(y_2) > C(y_1)$. Therefore, y_1 cannot be in the support of Q. ///

Lemma 4.1 illustrates the effect that the properties of the kernel $e^{-sd(|x-y|)}$ and the requirements on C(y) have on the support of Q. This thinking will be expanded upon in Chapters 5 and 6.

Another useful concept relating the requirements on C(y) to the support of Q introduces the derivative of C(y). The following obvious lemma indicates this relationship.

Lemma 4.2 If C(y) is a continuous function with a continuous first derivative C'(y) then C'(y) = 0 at all points of support of Q providing Q achieves R(d).

Proof: C(y) = 1 at all points of support of the Q which achieves R(d). If C'(y) $\neq$ 0 at a point of support, this implies that there is a y such that C(y) > 1, which violates the requirement that $C(y) \leq 1 \quad \forall\, y \in \underline{Y}$. ///

The first derivative of C(y) will play a significant role in Chapters 5 and 6 and will be shown to be related to the optimum estimate of x given y with respect to the distortion measure d(x,y). The background for this relationship will be introduced at this time.

The problem in estimation theory is depicted in Fig. 4.1. In our case, the disruptive medium may be thought of as the probabilistic mapping specified by P. x and y are, therefore, as we have been using them.

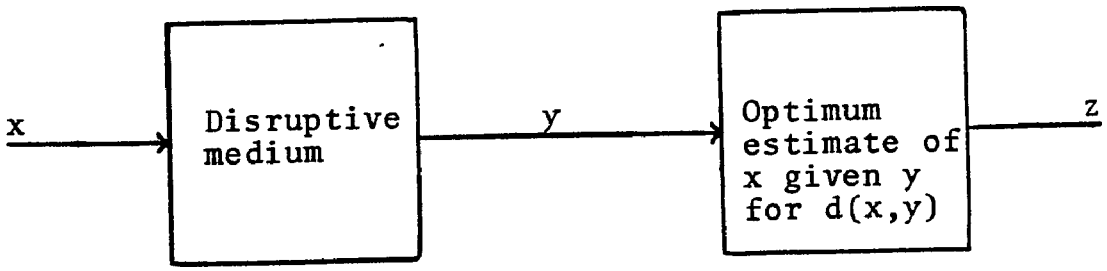


Fig. 4.1 The Estimation Problem

Suppose one constructs the optimum estimate of x given y subject to the distortion measure d(x,y) and call this z. The form of this estimator is known in certain cases, including those of the distortion measures to be used in Chapters 5 and 6. Note, however, that if we have the P which achieves a point on R(d), then our estimate z cannot improve the average distortion. The average distortion between x and z must be the same as that between x and y. Further, the estimator cannot increase the mutual information, i.e.

$$I(X;Z) \leq I(X;Y)$$

The estimator also cannot decrease the mutual information because it would be achieving a point less than R(d). Therefore,

$$I(X;Z) = I(X;Y)$$

and

$$E\{d(x,y)\} = E\{d(x,z)\}$$

These can be satisfied if the optimum estimator is an identity transformation, i.e. z = y. One might argue under what conditions the identity transformation is the only possibility and, therefore, generate another potentially useful relationship in finding Q, however, the optimum estimator's relationship to C'(y) was recognized in the cases of the distortion measures to be considered in Chapters 5 and 6. C'(y) will play an interesting role there.

V. MAGNITUDE ERROR DISTORTION

Chapter 4 presented the equations to be solved if one is to find the Q which achieves R(d). This chapter considers the particular case where $d(x,y) = |x-y|$. Equations (4.3) and (4.4) may, therefore, be written as

$$C(y) = \int_{\underline{X}} \alpha(x) e^{-s|x-y|} p_X(x)\,dx \tag{5.1}$$

and

$$\alpha^{-1}(x) = \int_{\underline{Y}} e^{-s|x-y|} Q(dy) \tag{5.2}$$

where the restriction has been made that $P_X(dx) = p_X(x)\,dx$. That is, in this chapter the source will have a density function. Further, $\underline{X}$ and $\underline{Y}$ shall be the real line.

The magnitude error distortion measure results in a kernel $e^{-s|x-y|}$, which allows one to proceed by the integral equation method. Tan and Yao [6] have utilized this technique in solving the problem for the Gaussian source and have demonstrated its usefulness for similar sources. This chapter shall further this thinking by showing that the two coupled integral equations, which result by guessing the support of Q, as discussed in Chapter 4, may be reduced to one integral equation (Section 5.1) whose form may be used to infer the support of Q. Section 5.2 considers how to pick the support of Q. A procedure for solving the problem is developed and examples are given in Section 5.3.

5.1 Reducing the Equations

If one were to follow the integral equation method as discussed in Chapter 4, one would guess the support of Q and then, since

$$C(y) = 1 \quad \text{almost everywhere } Q \tag{5.3}$$

for Q to be the solution, one would have in (5.1) and (5.2) two coupled nonlinear integral equations to solve. This is the approach of Tan and Yao [6].

Consider the function C(y), however. C(y) is a continuous function of y with a continuous derivative C'(y).

$$C'(y) = -s\int_{(-\infty,y)} \alpha(x)e^{-s|x-y|}p_x(x)dx + s\int_{(y,\infty)} \alpha(x)e^{-s|x-y|}p_x(x)dx \tag{5.4}$$

C(y) also has a second derivative C"(y).

$$C''(y) = s^2\int_{(-\infty,\infty)} \alpha(x)e^{-s|x-y|}\,p_x(x)dx - 2s\alpha(y)p_x(y) \tag{5.5}$$

Lemma 4.2 applies to this situation, and therefore, C'(y) = 0 at all points of support of the Q which achieves R(d). Therefore, from (5.4)

$$\int_{(y,\infty)} \alpha(x)e^{-s|x-y|}p_x(x)dx = \int_{(-\infty,y)} \alpha(x)e^{-s|x-y|}p_x(x)dx$$

$$\forall\ y \text{ in support of } Q \tag{5.6}$$

(5.6) may be related to estimation theory as suggested in Chapter 4 since

$$\int_{(\hat{x},\infty)} \alpha(x) e^{-s|x-y|} p_x(x)\,dx = \int_{(-\infty,\hat{x})} \alpha(x) e^{-s|x-y|} p_x(x)\,dx$$

is the requirement on $\hat{x}$ for it to be the optimum estimate of x given y for the magnitude error distortion measure. (5.6) shows that $\hat{x} = y$ is the optimum estimator as was suspected in Chapter 4.

Equation (5.6) may be combined with (5.3) to give

$$\int_{(y,\infty)} \alpha(x) e^{-s|x-y|} p_x(x)\,dx$$

$$= \int_{(-\infty,y)} \alpha(x) e^{-s|x-y|} p_x(x)\,dx = \frac{1}{2}$$

$$\forall\ y \text{ in support of } Q \qquad (5.7)$$

This may be written as

$$\int_{(y,\infty)} \alpha(x) e^{-sx} p_x(x)\,dx = \frac{1}{2} e^{-sy} \quad \forall\ y \text{ in support of } Q \qquad (5.8)$$

and

$$\int_{(-\infty,y)} \alpha(x) e^{sx} p_x(x)\,dx = \frac{1}{2} e^{sy} \quad \forall\ y \text{ in support of } Q \qquad (5.9)$$

Equations (5.8) and (5.9) will be referred to as the estimation equations. They will be very useful in establishing a relationship between the boundaries of the support of Q and

the parameter s.

If any part of the support of Q is an open interval (a,b), for example, if there is a $q(y) > 0$ $\forall$ $y\varepsilon(a,b)$ where $q(y)$ is the density of the absolutely continuous part of Q with respect to Lebesgue measure, then because $C(y)$ is continuous

$$C(y) = 1 \qquad \forall\ y\varepsilon(a,b) \quad \text{and therefore}$$

$$C'(y) = 0 \qquad \forall\ y\varepsilon(a,b),$$

and

$$C''(y) = 0 \qquad \forall\ y\varepsilon(a,b)\ .$$

Equations (5.5) and (5.3) then require that

$$\alpha(y)p_x(y) = s/2 \qquad \forall\ y\varepsilon(a,b) \tag{5.10}$$

Note that if $p_x(y)$ is piecewise continuous that (5.10) cannot hold in an interval where there is a discontinuity of $p_x(y)$ since $\alpha(y)$ is continuous. If $p_x(y)$ is continuous in (a,b) however, then (5.10) becomes

$$\frac{2}{s}\, p_x(x) = \int_{\underline{Y}} e^{-s|x-y|} Q(dy) \qquad \forall\ x\varepsilon(a,b) \tag{5.11}$$

where (a,b) is an interval in the support of Q.

(5.11) along with (5.8) and (5.9) give a set of conditions from which a solution may be constructed. One still must guess the support of Q, but one has reduced the two coupled integral equations to one integral equation (5.11) (whose solution has a known form) and the estimation equations (5.8-5.9). (Integral equations similar to (5.11) are

discussed in Van Trees [17], for example.)

Before determining how the support of Q should be picked, one should note the resemblance of (5.11) to the integral equation given in discussing the SLB, (2.39). Situations for which the SLB is tight occur when Q has interval support over the entire line and then (5.11) may be solved by Fourier transform techniques as was suggested in Chapter 2 (see Berger [3], p. 95).

5.2 The Support of Q

In determining how one should guess the support of Q, it will be useful to consider the requirements which $p_x(x)$ places on Q via equation (5.11). First, note that $\alpha(x)$ is a continuous function and (5.11) cannot hold in an interval where $p_x(x)$ has a step discontinuity. Therefore, Q has no interval support about that type of discontinuity of $p_x(y)$. Further, if (5.11) is to hold in some interval (a,b), then it follows that the derivatives of the right side of (5.11) must equal the derivatives of $p_x(x)$ almost everywhere in the interval (a,b),assuming the derivatives of $p_x(x)$ exist almost everywhere in (a,b).

If $p_x(x)$ has a continuous derivative in (a,b), then Q cannot have any discrete masses in (a,b). This can be seen by noting that $e^{-s|x|}$ has a discontinuous derivative at $x = 0$ and, therefore, a discrete mass in Q within (a,b) would require the right side of (5.11) to be discontinuous. Note, however, that a discontinuity in $p_x'(x)$ in (a,b) does not necessarily imply that Q has a discrete mass at that point

in (a,b). The Laplacian source density to be discussed later has this property.

To further consider (5.11) in some interval (a,b), one may write it as

$$\frac{2}{s}\, p_x(x) = e^{-sx}\int_{(-\infty,x)} e^{sy} Q_1(dy) + e^{sx}\int_{(x,\infty)} e^{-sy} Q_1(dy) + \sum_{i=1}^{N} q_i e^{-s|x-y_i|} \qquad \forall\, x\in(a,b) \tag{5.12}$$

where the discrete masses q_i at the positions $y_i \in \overline{Y}$ are separated out for clarity. Q_1 is what is left of Q after the discrete masses (if there are any) have been removed. Using the Lebesgue decomposition of Q_1, (5.12) may be written as

$$\frac{2}{s}\, p_x(x) = e^{-sx}\int_{(-\infty,x)} e^{sy} q(y)\,dy + e^{sx}\int_{(x,\infty)} e^{-sy} q(y)\,dy + e^{-sx} Q_S^1(x) + e^{sx} Q_S^2(x) + \sum_{i=1}^{N} q_i e^{-s|x-y_i|} \qquad \forall\, x\in(a,b) \tag{5.13}$$

where Q_S^1 and Q_S^2 are positive Borel functions whose derivatives are equal to zero almost everywhere in (a,b) (see Rudin [21], Chapter 8). q(y) is the density of the absolutely continuous part of Q with respect to Lebesgue measure.

Providing the derivatives of $p_x(x)$ exist almost everywhere in (a,b), (5.13) may be twice differentiated to give

$$\frac{2}{s}\, p_x''(x) = s^2\left[\frac{2}{s}\, p_x(x)\right] - 2s\, q(x)$$

almost everywhere in (a,b)

or

$$q(x) = p_x(x) - \frac{1}{s^2}\, p_x''(x) \text{ almost everywhere in } (a,b) \tag{5.14}$$

(5.14) is most important for if $p_x(y) - \frac{1}{s^2}\, p_x''(y)$ is negative on a subset of (a,b) of positive measure, then (a,b) cannot be an interval in the support of Q. (5.14) also gives the form which the absolutely continuous part of Q must have if two derivatives of p_x exist and Q is to have (a,b) as an interval of support. The implication that there must be an absolutely continuous part in this case if $p_x(x) - \frac{1}{s^2}\, p_x''(x) > 0$ almost everywhere in (a,b) is obvious.

Because of the occurrence of s in (5.14), it is possible that (5.14) will be positive for some values of s and not others. This suggests that the interval support of Q may change form with s. This situation will occur as a later example will demonstrate.

Fig. 5.1 shows a source density for which (5.14) is not positive in some regions. Fig. 5.1 also shows the regions (cross-hatched) for which (5.14) is positive. If one were to guess the interval support of Q for this source, one might guess it as disjoint intervals each placed in a different cross-hatched region.

To pursue this point further, assume one has n disjoint intervals (c_i,d_i) i=1,...,n of interval support of Q.

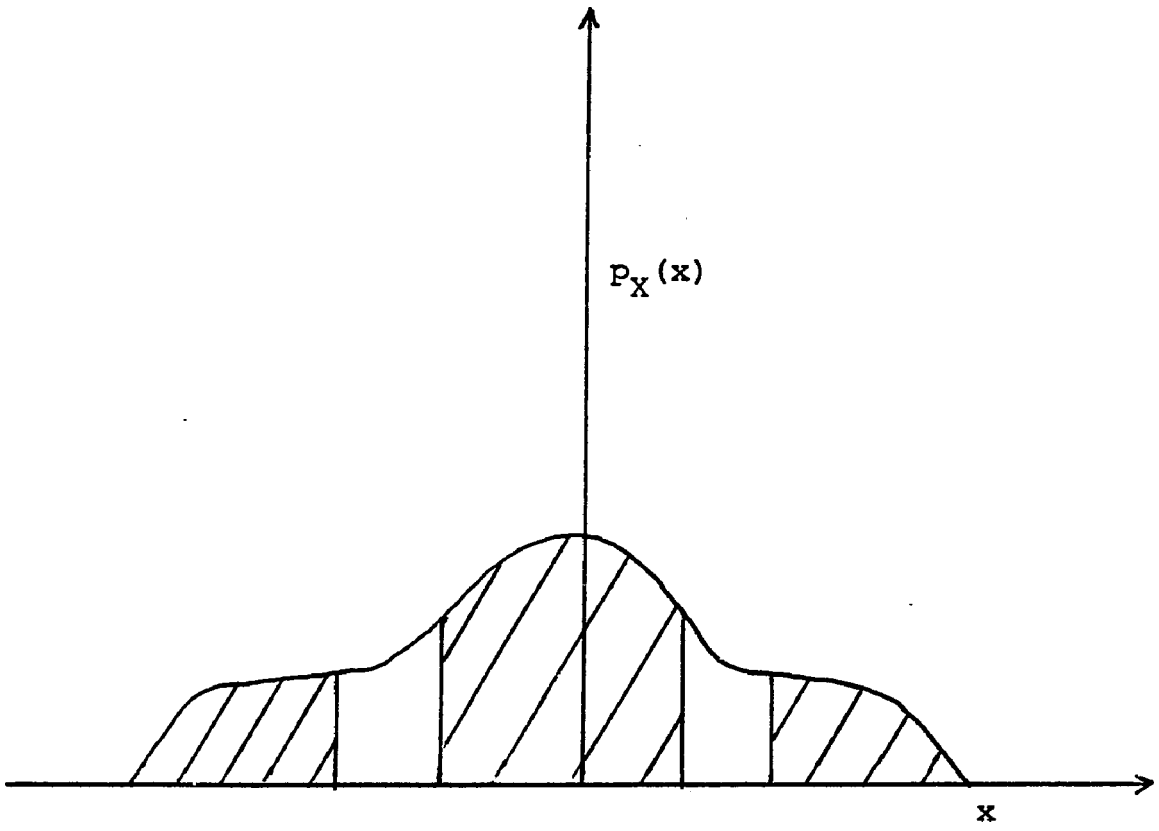


Fig. 5.1 A Possible Source Density

(5.12) may then be written as

$$\frac{2}{s}\, p_x(x) = \sum_{i=1}^{n} \int_{(c_i,d_i)} e^{-s|x-y|} q(y)\,dy$$

$$+ \sum_{i=1}^{n} \int_{(c_i,d_i)} e^{-s|x-y|} Q_s(dy) \qquad (5.15)$$

$$+ \sum_{i=1}^{N} q_i e^{-s|x-y_i|} \qquad \forall\, x \in \bigcup_{i=1}^{n} (c_i,d_i)$$

where Q_s is the continuous but singular part of Q, $q(y)$ is the absolutely continuous part, and the discrete part is represented by the masses q_i at the positions y_i. If $p_x'(y)$ and $p_x''(y)$ are finite everywhere in $\bigcup_{i=1}^{n} (c_i,d_i)$ and integrable over $\bigcup_{i=1}^{n} (c_i,d_i)$ then one may use (5.14) in (5.15) and integrate by parts (see Titchmarsh [19], p. 368 and 375). Therefore,

$$\int_{(c_i,d_i)} e^{-s|x-y|} q(y)\,dy$$

$$= \begin{cases} e^{-s(c_i-x)}\{\frac{1}{s}\,p_x(c_i) + \frac{1}{s^2}\,p_x'(c_i)\} \\ \quad - e^{-s(d_i-x)}\{\frac{1}{s}\,p_x(d_i) + \frac{1}{s^2}\,p_x'(d_i)\} \\ \qquad x \le c_i \\ \frac{2}{s}\,p_x(x) + e^{-s(x-c_i)}\{\frac{1}{s^2}\,p_x'(c_i) - \frac{1}{s}\,p_x(c_i)\} \\ \quad - e^{-s(d_i-x)}\{\frac{1}{s}\,p_x(d_i) + \frac{1}{s^2}\,p_x'(d_i)\} \\ \qquad x \varepsilon (c_i,d_i) \\ e^{-s(x-d_i)}\{\frac{1}{s}\,p_x(d_i) - \frac{1}{s^2}\,p_x'(d_i)\} \\ \quad -e^{-s(x-c_i)}\{\frac{1}{s}\,p_x(c_i) - \frac{1}{s^2}\,p_x'(c_i)\} \\ \qquad x \ge d_i \end{cases} \tag{5.16}$$

(5.16) suggests, because of the exponential terms, that a solution to (5.15) might be constructed by placing discrete masses at the points c_i, d_i $i=1,\ldots,n$. It should be noted from (5.16) that if

$$\frac{1}{s^2}\,p_x'(c_i) - \frac{1}{s}\,p_x(c_i) > 0$$

and (5.17)

$$\frac{1}{s}\,p_x(d_i) + \frac{1}{s^2}\,p_x'(d_i) < 0$$

then (c_i,d_i) is not an interval of support of Q. This occurs because (5.17) causes a contradiction in (5.15), i.e. a

strictly positive function almost everywhere cannot be also equal to zero almost everywhere.

In the following, tentative solutions (Q) will be constructed using the principles discussed above. To be specific, intervals of support shall be placed where $p_x(y) - \frac{1}{s^2} p_x''(y)$ is positive and discrete masses shall be placed at the ends of these intervals. The positions of the discrete masses must, of course, also satisfy the estimation equations (5.8-9) and (5.17). For a single interval (c_1,d_1) of support, one would, therefore, have a tentative solution

$$Q(dy) = [p_x(y) - \frac{1}{s^2} p_x''(y) + a_1\delta(y-d_1) + a_2\delta(y-c_1)]dy \qquad y\varepsilon[c_1,d_1] \qquad (5.18)$$

where a_1 and a_2 may be found by substitution of (5.18) into (5.11) to get

$$a_1 = \frac{1}{s} p_x(d_1) + \frac{1}{s^2} p_x'(d_1)$$

$$a_2 = \frac{1}{s} p_x(c_1) - \frac{1}{s^2} p_x'(c_1) \ .$$

This will be pursued further in Section 5.3.

(5.14) is not the only key one may use in guessing the support of Q. Remember that Lemma 4.1 requires that, for the magnitude error distortion measure, the support of Q cannot exceed the boundary of the support of P_x. Therefore, for a source, such as a source with a uniform density in the interval (a,b), the support of Q must lie within the interval [a,b]. A source, however, need not have a finite boundary

(see Definition 4.1) to restrict the support of Q to a finite boundary. To see this, remember that the optimum Q satisfies

$$C(y) \triangleq \int_{\underline{X}} \alpha(x) e^{-s|x-y|} p_x(x) dx \quad \begin{cases} \leq 1 \ \forall \ y \in \underline{Y} \\ = 1 \text{ almost everywhere } Q \end{cases}$$

If $p_x(x)$ dominates $\alpha(x)$ outside some finite interval (i.e. $p_x(x)\alpha(x) \to 0$ as $|x| \to \infty$, which $p_x(x)$ can do because from the definition of $\alpha(x)$, $\alpha(x)$ can grow no faster than $e^{s|x|}$) then outside some finite interval C(y) must be strictly less than one. This requires that there be no support of Q outside this interval. A source with the property that

$$p_x(x) e^{sd(|x|)} \to 0 \text{ as } |x| \to \infty$$

shall be called a strong source (with respect to that particular value of s, of course). A source, which does not have this property, shall be called a weak source. Note that this definition depends on the distortion measure.

If a source is strong, then the support of Q is restricted to lie within a finite interval, and equation (5.14) may be considered to determine what type of solution might be expected within this interval.

5.3 Solving the Problem

Section 5.1 has reduced the integral equation method to constructing a tentative solution Q, which satisfies (5.8), (5.9) and (5.11). Section 5.2 has considered how to pick the support of Q such that a tentative solution can be constructed. These ideas may be summarized in the following

procedure:

1. Determine if the source is strong for any $s > 0$, and, therefore, if consideration of the support of Q may be restricted to a finite boundary.
2. Consider the regions (c_i, d_i) where $p_x(y)$ is continuous and where $p_x(y) - \frac{1}{s^2} p_x''(y)$ is positive and finite.
3. Construct a tentative solution of the form

$$Q(dy) = \{p_x(y) - \frac{1}{s^2} p_x''(y) + \sum_i [a_i \delta(y-d_i) + b_i \delta(y-c_i)]\}dy \qquad \forall\, y \in \bigcup_i [c_i, d_i]$$

4. Evaluate the form of the a_i and b_i by substituting the tentative solution into (5.11).
5. Determine the d_i and c_i via the estimation equations (5.8-9).
6. If (5.17) is violated for some interval (c_i, d_i), drop that interval from the support and construct a new tentative solution (step 3).
7. Test the tentative solution against the necessary and sufficient condition that $C(y) \leq 1 \quad \forall\, y \in \underline{Y}$.
8. If the necessary and sufficient condition is not met, consider the region where $C(y) > 1$. Insert a discrete mass in this region, which must satisfy (5.8-9), and construct a new tentative solution. (Return to Step 3.)

No proof that this procedure will guarantee the solution is given. However, as the examples to follow indicate, one may expect the process to usually end after Step 7.

Once one has verified that a solution has been constructed, the calculation of R(d) follows. To do this, one may construct $\alpha(x)$ from Q and then use:

$$F(s) = \int_{\underline{X}} \log[\alpha(x)]\ p_x(x)dx\ , \tag{5.19}$$

$$d_s = \int_{\underline{X}}\int_{\underline{Y}} |x-y|e^{-s|x-y|}\alpha(x)\ Q(dy)p_x(x)dx = \frac{d}{ds}F(s)\ , \tag{5.20}$$

and

$$R_s = F(s) - sd_s \tag{5.21}$$

Example 5.1 Uniform Source

The uniform source is specified by

$$p_x(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & \text{elsewhere} \end{cases}$$

This is a strong source and, therefore, one is assured that the support of Q is within the interval [a,b]. $p_x''(x) = 0$ $\forall\ x \in (a,b)$ and, therefore, (5.14) is positive over (a,b). This suggests a single interval of Q support, $[c_1,d_1] \subseteq [a,b]$. (5.11) is then of the form (5.12) and a tentative Q may be specified by (5.18) as

$$Q(dy) = \begin{cases} [\frac{1}{b-a} + a_1\delta(y-d_1) + a_2\delta(y-c_1)]dy & y \in [c_1,d_1] \\ 0 & \text{elsewhere} \end{cases} \tag{5.22}$$

where a_1 and a_2 are given by (5.18) to be

$$a_1 = a_2 = \frac{1}{s}\left(\frac{1}{b-a}\right) \tag{5.23}$$

Using the tentative Q in the definition of $\alpha(x)$ gives

$$\alpha^{-1}(x) = \begin{cases} \frac{2}{es(b-a)}\, e^{-s(a-x)} & x \in [a,c_1) \\ \frac{2}{s(b-a)} & x \in [c_1,d_1] \\ \frac{2}{es(b-a)}\, e^{-s(x-b)} & x \in (d_1,b] \end{cases} \tag{5.24}$$

Using $\alpha(x)$, the estimation equations (5.8-9) may be solved to establish that

$$c_1 = a + \frac{1}{s} \tag{5.25}$$

$$d_1 = b - \frac{1}{s} \tag{5.26}$$

This illustrates the use of the estimation equations to locate the boundary of Q in terms of s.

The tentative solution is, therefore, complete. Substitution of this into C(y) verifies that the necessary and sufficient conditions on C(y) are satisfied and, therefore, the solution has been found. Substitution of (5.24) and (5.22) into the definitions of R_s and d_s ((5.19), (5.20)) gives

$$R_s = \log\left(\frac{s(b-a)}{2}\right) - 1 + \frac{2}{s(b-a)}$$
$$d_s = \frac{1}{s}\left(1 - \frac{1}{s(b-a)}\right) \qquad (5.27)$$

This holds for $R_s \geq 0$, $s \geq \frac{2}{b-a}$, or $0 < d_s \leq \frac{b-a}{4} = d_{max}$. d_{max}, of course, may be evaluated via (2.31). (5.27) is plotted along with the Laplacian source (to be presented) in Fig. 5.2 for $d_{max} = 1$. This normalization is achieved by making $b-a = 4$.

Example 5.2 One Sided Exponential Source

The one sided exponential source has density

$$p_x(x) = \begin{cases} 1/\sigma\ e^{-x/\sigma} & x \in [0,\infty) \\ 0 & \text{elsewhere} \end{cases}$$

This is not a strong source because it decays only exponentially, but it follows easily from Lemma 4.1 that there can be no Q support for $y < 0$.

(5.14) for $x > 0$ is given by

$$p_x(x) - \frac{1}{s^2} p_x''(x) = \frac{1}{\sigma} e^{-x/\sigma}\left[1 - \frac{1}{s^2\sigma^2}\right]$$

This is positive if $s \geq 1/\sigma$. If one assumes, for the moment, that $s \geq 1/\sigma$, then a single interval of support is suggested, $[c_1,d_1] \subseteq [0,\infty)$. Following example 5.1, the tentative $Q(dy)$ is given by

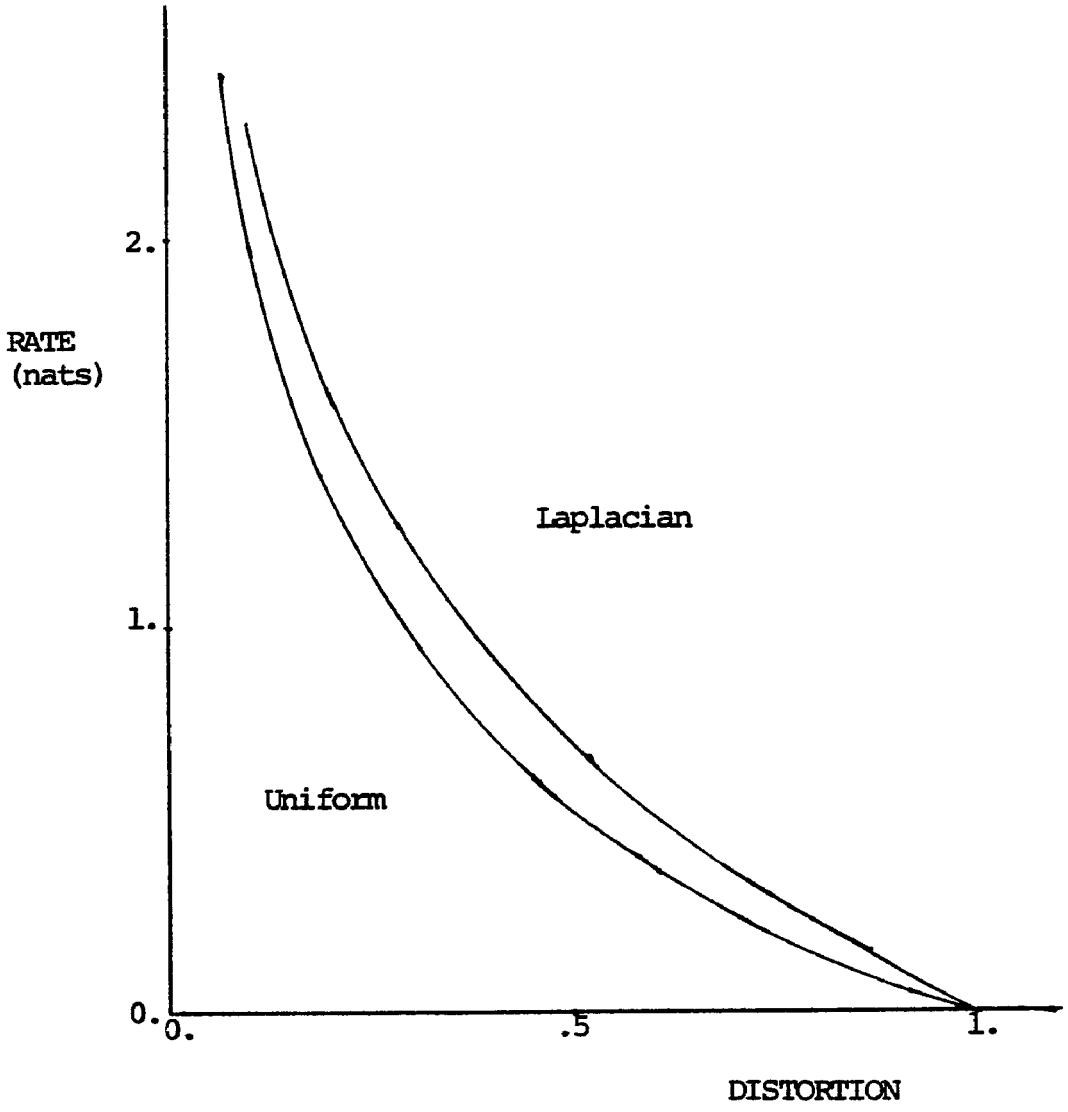


Fig. 5.2 The Rate Distortion Function for the Uniform Source and the Magnitude Error Distortion Measure for $d_{max}=1.$

$$Q(dy) = \begin{cases} [\frac{1}{\sigma}(1 - \frac{1}{s^2\sigma^2})e^{-y/\sigma} \\ \quad + a_1\delta(y-d_1) + a_2\delta(y-c_1)]dy & y \in [c_1,d_1] \\ 0 & \text{elsewhere} \end{cases} \tag{5.28}$$

where

$$a_1 = (\frac{1}{s\sigma} - \frac{1}{s^2\sigma^2})e^{-d_1/\sigma} \tag{5.29}$$

and

$$a_2 = (\frac{1}{s\sigma} + \frac{1}{s^2\sigma^2})e^{-c_1/\sigma} \tag{5.30}$$

The estimation equations (5.8-9) may be solved to find that

$$c_1 = \sigma \log(1 + \frac{1}{s\sigma}) \tag{5.31}$$

and

$$d_1 = \infty \tag{5.32}$$

The estimation equations, therefore, verify what was suggested earlier, that this source is weak. (5.32), of course, demands that $a_1 = 0$.

Q may be used in the definition of $\alpha(x)$ to give

$$\alpha^{-1}(x) = \begin{cases} \frac{2}{1+s\sigma} e^{-s\sigma\log(1+\frac{1}{s\sigma})}e^{sx} & x \in [0,c_1) \\ \frac{2}{s\sigma} e^{-x/\sigma} & x \in [c_1,\infty) \end{cases} \tag{5.33}$$

$\alpha(x)$ may be used in the definition of C(y) to verify that

this is the true solution, which it is. $\alpha(x)$ and $q(y)$ may also be used in (5.19), (5.20), and (5.21) to give

$$R_s = \log (1+s\sigma) - \log 2$$
$$d_s = \sigma \log (1+ \frac{1}{s\sigma}) \tag{5.34}$$

This holds for $R_s \geq 0$, $s \geq 1/\sigma$, and $0 < d_s \leq d_{max} = \sigma \log 2$, d_{max} being evaluated again by (2.31). Note that the original assumption that $s \geq 1/\sigma$ was sufficient to guarantee that the entire R(d) curve was generated. (5.34) is plotted in Fig. 5.3.

It is interesting to compare the one sided exponential with the two sided exponential or Laplacian density given by

$$p_x(x) = \frac{\sigma}{2} e^{-\sigma|x|} \qquad \forall\ x \in (-\infty,\infty)$$

The Laplacian is such that the SLB is tight over the entire R(d) curve as shown in Berger [3], p. 95, and is given by

$$R = - \log (\sigma d) \qquad 0 < d \leq \frac{1}{\sigma} = d_{max} \tag{5.35}$$

This is plotted in Fig. 5.3. It is, of course, noted that (5.14) is positive for all $x \in (-\infty,\infty)$ and, in fact, for the Laplacian

$$q(y) = p_x(y) - \frac{1}{s^2} p_x''(y) \qquad \forall\ y \in (-\infty,\infty)$$

Example 5.3 Rayleigh Source

The Rayleigh source is specified by its density

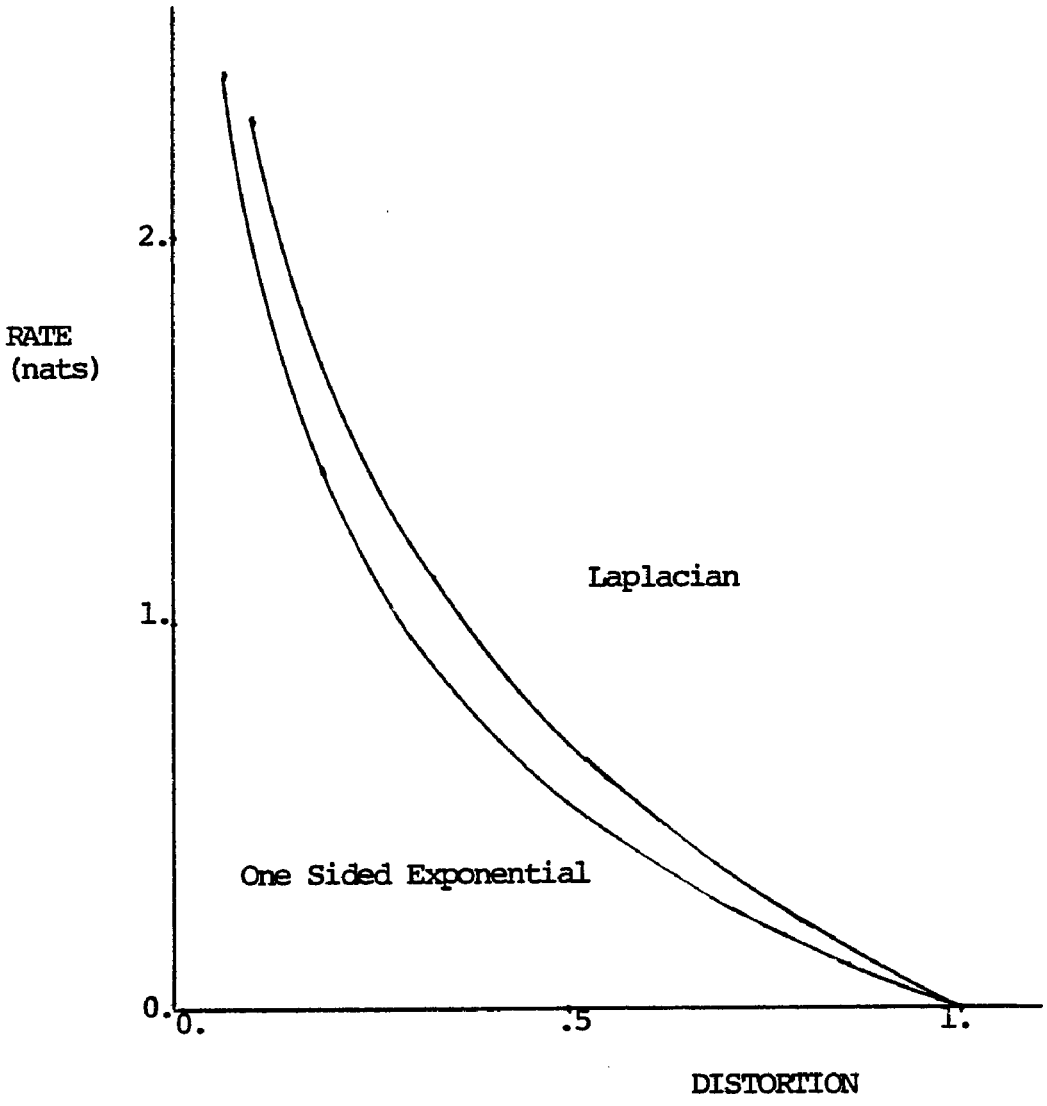


Fig. 5.3 The Rate Distortion Function for the One Sided Exponential Source and the Magnitude Error Distortion Measure for $d_{max}=1.$

$$p_x(x) = \begin{cases} \frac{x}{\sigma^2} e^{-x^2/2\sigma^2} & x \in [0,\infty) \\ 0 & \text{elsewhere} \end{cases}$$

It is easily seen that this is a strong source and, therefore, the support of Q must be contained in a finite interval, $[c_1,d_1] \subseteq [0,\infty)$. (5.14) for $x > 0$ is such that

$$p_x(x) - \frac{1}{s^2} p_x''(x) = \frac{x}{\sigma^2} e^{-x^2/2\sigma^2} [1 - \frac{1}{s^2} (\frac{x^2}{\sigma^4} - \frac{3}{\sigma^2})]$$

which is positive for

$$0 \leq x \leq \sigma \sqrt{s^2\sigma^2+3} \quad .$$

This implies a single interval $[c_1,d_1]$ of Q support such that

$$[c_1,d_1] \subseteq [0, \sigma \sqrt{s^2\sigma^2+3}\,] \quad .$$

Q(dy), therefore, follows as before from (5.18) to be

$$Q(dy) = \begin{cases} [\frac{y}{\sigma^2} e^{-y^2/2\sigma^2} (1 + \frac{3}{s^2\sigma^2} - \frac{y^2}{s^2\sigma^4}) \\ \quad + a_1\delta(y-d_1) + a_2\delta(y-c_1)]dy & y\varepsilon[c_1,d_1] \\ 0 & \text{elsewhere} \end{cases} \tag{5.36}$$

where

$$a_1 = (\frac{1}{s} d_1 + \frac{1}{s^2} - \frac{d_1^2}{s^2\sigma^2}) \frac{1}{\sigma^2} e^{-d_1^2/2\sigma^2} \tag{5.37}$$

and

$$a_2 = (\frac{1}{s} c_1 - \frac{1}{s^2} + \frac{c_1^2}{s^2\sigma^2}) \frac{1}{\sigma^2} e^{-c_1^2/2\sigma^2} \tag{5.38}$$

Of course, c_1 and d_1 must be such that a_1 and a_2 are positive for this to be a solution.

Q may be used to find $\alpha(x)$ to give

$$\alpha^{-1}(x) = \begin{cases} \frac{2c_1}{s\sigma^2} e^{-c_1^2/2\sigma^2} e^{-s|c_1-x|} & x\varepsilon[0,c_1) \\ \frac{2x}{s\sigma^2} e^{-x^2/2\sigma^2} & x\varepsilon[c_1,d_1] \\ \frac{2d_1}{s\sigma^2} e^{-d_1^2/2\sigma^2} e^{-s|x-d_1|} & x\varepsilon(d_1,\infty) \end{cases} \quad (5.39)$$

The estimation equations (5.8-9) may be used to find that

$$s = \frac{c_1}{\sigma^2} (e^{c_1^2/2\sigma^2} - 1)^{-1} \quad (5.40)$$

and

$$d_1 = \sigma^2 s \quad (5.41)$$

One, therefore, has a tentative solution which may be verified by considering C(y). This is the solution, and (5.36) and (5.39) may be used in (5.19-21) to find that

$$R_s = \log\left(\frac{s}{2}\right) + \frac{c_1^2}{2\sigma^2} - \log\left(\frac{c_1}{\sigma^2}\right) - \frac{1}{2} E_1\left(\frac{c_1^2}{2\sigma^2}\right) + \frac{1}{2} E_1\left(\frac{d_1^2}{2\sigma^2}\right)$$

$$d_s = \frac{1}{s}\left(e^{-c_1^2/2\sigma^2} - e^{-d_1^2/2\sigma^2}\right) + c - \sqrt{\frac{\pi}{2}}\,\sigma\,\mathrm{erf}\left(\frac{c}{\sqrt{2}\sigma}\right) + \sqrt{\frac{\pi}{2}}\,\sigma\,\mathrm{erfc}\left(\frac{c}{\sqrt{2}\sigma}\right) \quad (5.42)$$

where

$$\text{erf}(x) \triangleq \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

$$\text{erfc}(x) \triangleq 1 - \text{erf}(x)$$

$$E_1(x) \triangleq \int_x^\infty \frac{e^{-t}}{t} dt$$

This holds for $R_s \geq 0$, $s \geq \frac{\sqrt{2 \log 2}}{\sigma}$, or $0 < d_s \leq d_{max}$, where d_{max} may be found via (2.31) to be

$$d_{max} = \sigma \sqrt{2 \log 2} + \sqrt{2\pi}\, \sigma \left[1 - 2 \text{ erf } (\sqrt{\log 2})\right] .$$

With the aid of (5.40) and (5.41), (5.42) is plotted in Fig. 5.4.

All the examples given so far have had a single interval of Q support. The Gaussian source (see [6]) also has this property. The next example will illustrate the method of solution when this is not the case.

Example 5.4 Squared Cauchy Source

The squared Cauchy source is specified by its density

$$p_x(x) = \frac{2}{\pi} (1+x^2)^{-2} \qquad x \in (-\infty, \infty)$$

This is a weak source and (5.18) is given by

$$p_x(x) - \frac{1}{s^2} p_x''(x) = \frac{2}{\pi s^2} \left[\frac{1}{(1+x^2)^4}\right] \left[s^2(1+x^2)^2 - 24\, x^2 + 4(1+x^2)\right]$$

If $s^2 \geq 25/6$, this is positive over the entire line and

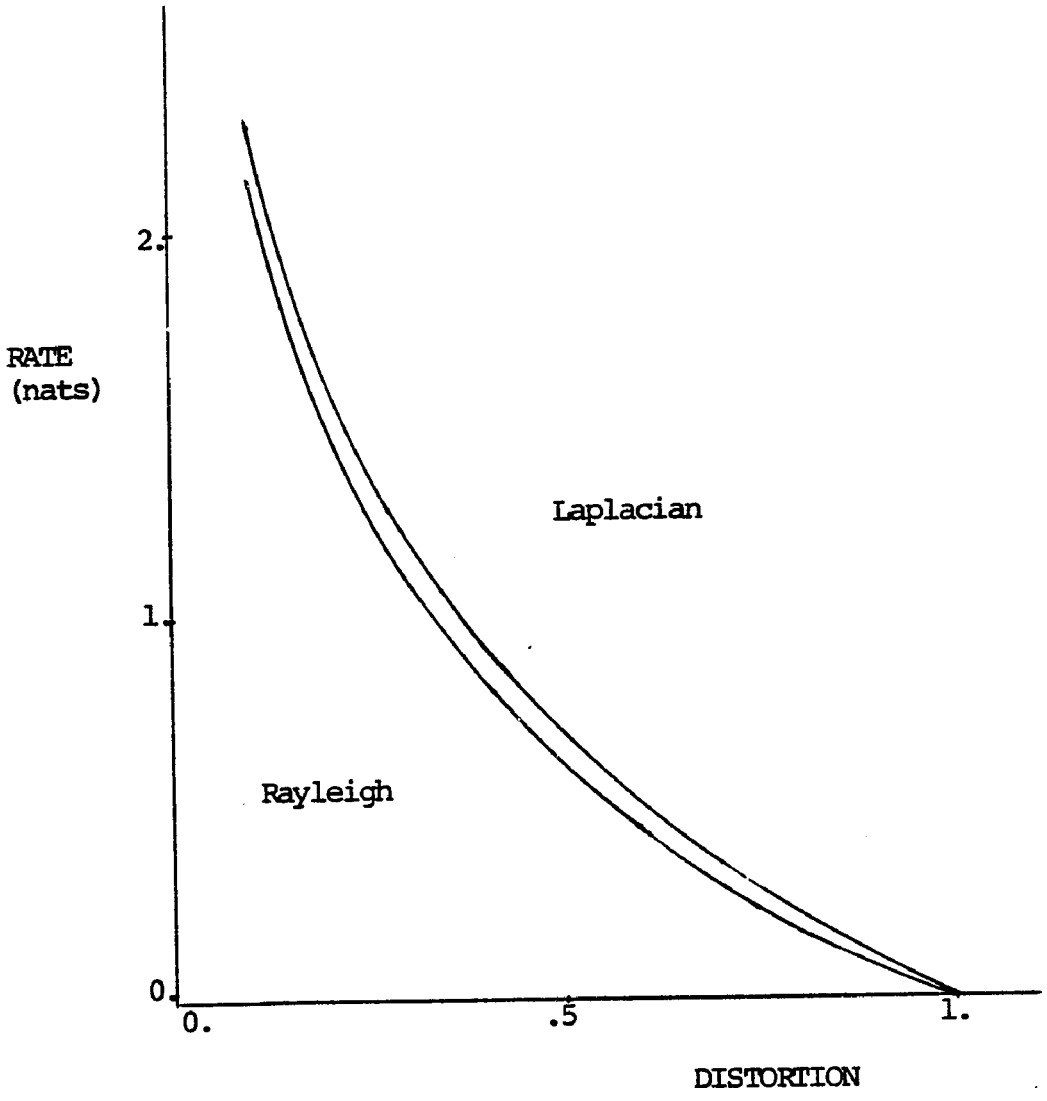


Fig. 5.4 The Rate Distortion Function for the Rayleigh Source and the Magnitude Error Distortion Measure for $d_{max}=1.$

the SLB is tight,as shown by Berger [3]. Therefore, as discussed in Berger [3], p. 96,

$$R_s = \log\left(\frac{s}{2}\right) + \log(8\pi) - 2 - sd_s \qquad s \geq 5/\sqrt{6} \quad (5.43)$$

$$d_s = \frac{1}{s}$$

d_{max} may be evaluated by equation (2.31) to find that

$$d_{max} = \frac{2}{\pi}$$

Comparing d_{max} with d_s of (5.43) indicates that (5.43) is not the entire R(d) curve. One must then consider $s^2 \leq 25/6$.

When $s^2 \leq 25/6$, $p_x(x) - \frac{1}{s^2} p_x''(x)$ is positive when

$$x^2 \geq \frac{10}{s^2} - 1 + \frac{2\sqrt{6}}{s^2}\sqrt{\frac{25}{6} - s^2}$$

and when

$$x^2 \leq \frac{10}{s^2} - 1 - \frac{2\sqrt{6}}{s^2}\sqrt{\frac{25}{6} - s^2}$$

This shows that there are three regions of the real line where (5.14) is positive if $s^2 \leq 25/6$. This suggests three intervals of Q support, $(-\infty, c_2]\ [c_1, d_1]\ [d_2, \infty)$.

From the discussion of Section 5.2, Q(dy) is given by

$$Q(dy) = \begin{cases} \{p_x(y) - \frac{1}{s^2} p_x''(y) \\ \quad + a_1\delta(y-c_2) + a_2\delta(y-c_1) + a_3\delta(y-d_1) \\ \quad + a_y\delta(y-d_2)\}dy \qquad (5.44) \\ \qquad \forall\ y \in \{(-\infty, c_2]\,[c_1, d_1]\,[d_2, \infty)\} \\ 0 \qquad \text{elsewhere} \end{cases}$$

Substituting an equation of the form (5.44) into (5.11), one may find that

$$a_1 = \frac{1}{s} p_x(c_2) + \frac{1}{s^2} p_x'(c_2) + \frac{1}{s} \frac{p_x(c_2)e^{-s(c_1-c_2)} - p_x(c_1)}{\sinh(s(c_1-c_2))} \tag{5.45}$$

$$a_2 = \frac{1}{s} p_x(c_1) - \frac{1}{s^2} p_x'(c_1) + \frac{1}{s} \frac{p_x(c_1)e^{-s(c_1-c_2)} - p_x(c_2)}{\sinh(s(c_1-c_2))} \tag{5.46}$$

$$a_3 = \frac{1}{s} p_x(d_1) + \frac{1}{s^2} p_x'(d_1) + \frac{1}{s} \frac{p_x(d_1)e^{-s(d_2-d_1)} - p_x(d_2)}{\sinh(s(d_2-d_1))} \tag{5.47}$$

$$a_4 = \frac{1}{s} p_x(d_2) - \frac{1}{s^2} p_x'(d_2) + \frac{1}{s} \frac{p_x(d_2)e^{-s(d_2-d_1)} - p_x(d_1)}{\sinh(s(d_2-d_1))} \tag{5.48}$$

(5.44) may be used in the definition of $\alpha(x)$ to give

$$\alpha^{-1}(x) = \begin{cases} \frac{2}{s} p_x(x) & x \in (-\infty, c_2] \\ \frac{2}{s}\left\{\frac{p_x(c_1)\sinh(s(x-c_2)) + p_x(c_2)\sinh(s(c_1-x))}{\sinh(s(c_1-c_2))}\right\} & x \in (c_2, c_1) \\ \frac{2}{s} p_x(x) & x \in [c_1, d_1] \\ \frac{2}{s}\left\{\frac{p_x(d_2)\sinh(s(x-d_1)) + p_x(d_1)\sinh(s(d_2-x))}{\sinh(s(d_2-d_1))}\right\} & x \in (d_1, d_2) \\ \frac{2}{s} p_x(x) & x \in [d_2, \infty) \end{cases} \tag{5.49}$$

(5.49) may be used in the estimation equations (5.8-5.9) along with some further manipulation to yield

$$s \sinh(s(c_1-c_2)) \int_{c_2}^{c_1} \frac{p_x(x)e^{sx}dx}{p_x(c_1)\sinh(s(x-c_2))+p_x(c_2)\sinh(s(c_1-x))} = e^{sc_1}-e^{sc_2} \tag{5.50}$$

$$s\int_{c_2}^{c_1} p_x(x)dx = (p_x(c_2)+p_x(c_1))\tanh[\tfrac{s}{2}(c_1-c_2)] \tag{5.51}$$

$$s \sinh(s(d_2-d_1)) \int_{d_1}^{d_2} \frac{p_x(x)e^{sx}dx}{p_x(d_2)\sinh(s(x-d_1))+p_x(d_1)\sinh(s(d_2-x))} = e^{sd_2}-e^{sd_1} \tag{5.52}$$

$$s\int_{d_1}^{d_2} p_x(x)dx = (p_x(d_2)+p_x(d_1))\tanh(\tfrac{s}{2}(d_2-d_1)) \tag{5.53}$$

Since the squared Cauchy density is symmetric about zero, one need only consider finding d_1 and d_2 because $c_1 = -d_1$ and $c_2 = -d_2$. Using this fact and substituting $\alpha(x)$ and Q into (5.19-21), one finds that

$$R_s = \log(4\pi s) - 2 + 2\int_{d_1}^{d_2} \log\left[\frac{\sinh(s(d_2-d_1))p_x(x)}{p_x(d_2)\sinh(s(x-d_1))+p_x(d_1)\sinh(s(d_2-x))}\right] p_x(x)dx - sd_s \tag{5.54}$$

$$d_s = \frac{1}{s} - \frac{2}{s}\int_{d_1}^{d_2} p_x(x)\,dx + \frac{2}{s}\,(d_2 p_x(d_2) - d_1 p_x(d_1))$$

$$+ 2\int_{d_1}^{d_2} x\left\{\frac{p_x(d_1)\cosh(s(d_2-x)) - p_x(d_2)\cosh(s(x-d_1))}{p_x(d_2)\sinh(s(x-d_1)) + p_x(d_1)\sinh(s(d_2-x))}\right\} p_x(x)\,dx$$

These may be evaluated numerically once the relationship between s, d_1, and d_2 is found.

To find the relationship between s, d_1, and d_2, equations (5.52) and (5.53) must be solved. The integration in (5.53) can be carried out to give

$$d_1 = (1+d_1^2)\left\{\frac{d_2}{(1+d_2^2)} + \arctan d_2 - \arctan d_1 - \frac{\pi}{s}\,(p_x(d_2)+p_x(d_1))\tanh\left(\frac{s}{2}(d_2-d_1)\right)\right\} \qquad (5.55)$$

(5.55) and (5.52) may be solved numerically. The relationship between s, d_1, and d_2 is shown in Fig. 5.5. Fig. 5.5 indicates that d_1 goes to zero at about s = 1.27524. It is seen that this point does not correspond to distortion d_{max}. (5.54) can, therefore, hold only in the range $1.27524 \leq s \leq 5/6$, which it does. For $s \leq 1.27524$, another solution must be constructed.

Since the three interval solution degenerates at s = 1.27524 to a solution of the form

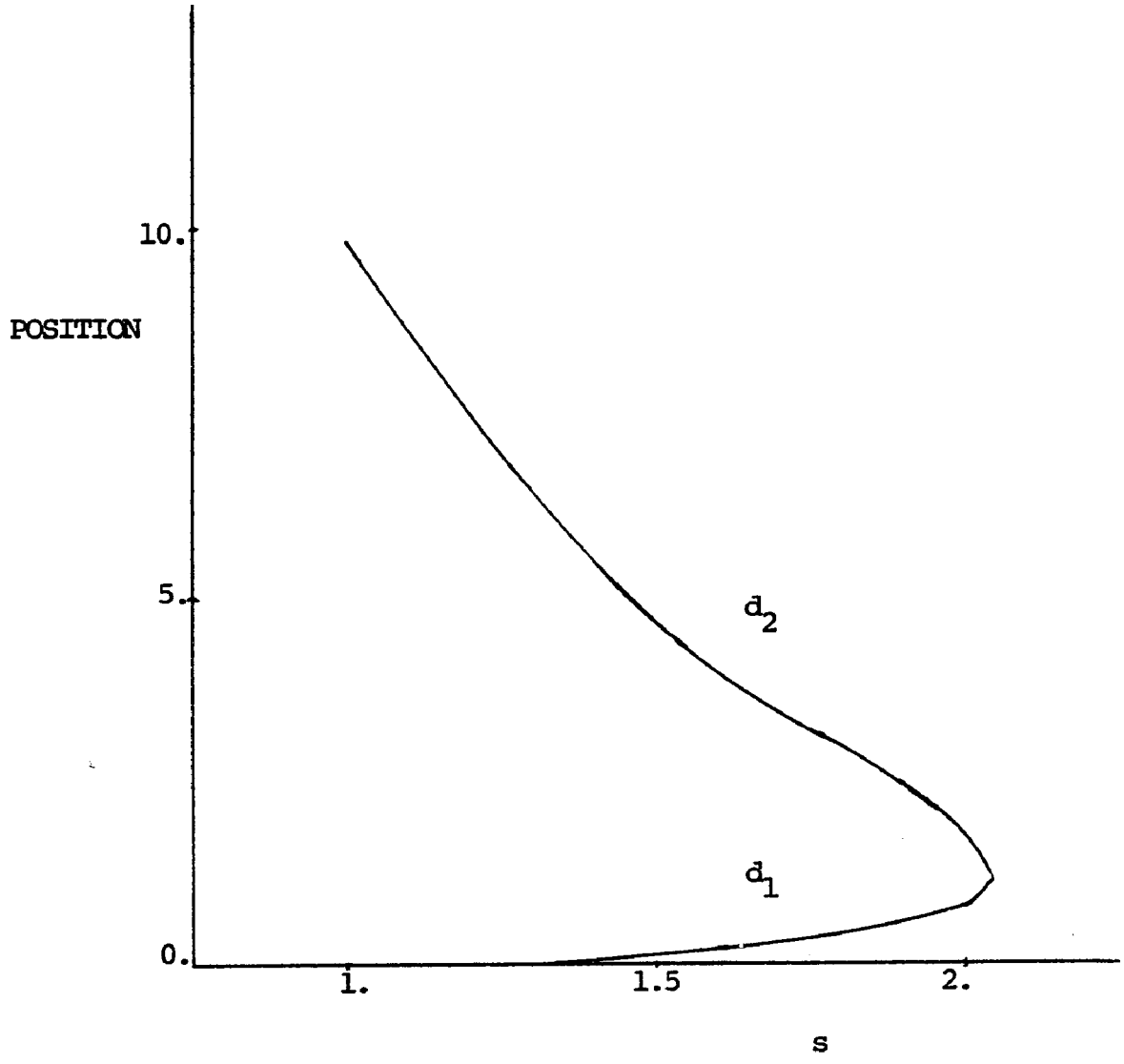


Fig. 5.5 The Positions of the Discrete Masses of the Reproduction Measure versus s for the Squared Cauchy Source and the Magnitude Error Distortion Measure.

$$Q(dy) = \begin{cases} \{p_x(y) - \frac{1}{s^2} p_x''(y) + a_5\delta(y-c_2) + a_6\delta(y-d_2)\}dy & \\ & y \in [(-\infty, c_2][d_2, \infty)] \\ a_7\delta(y) & y = [0] \\ 0 & \text{elsewhere} \end{cases} \tag{5.56}$$

one may construct a tentative solution of this form for $s \leq 1.27524$.

Again because of the symmetry of $p_x(x)$, $c_2 = -d_2$ and $a_5 = a_6$. One may find by substituting (5.56) into (5.11) that

$$a_5 = a_6 = \frac{\frac{1}{s} p_x(d_2)\sinh(sd_2) - \int_0^{d_2} p_x(x)dx}{\cosh(sd_2)-1} - \frac{1}{s^2} p_x'(d_2) \tag{5.57}$$

$$a_7 = \frac{2\cosh(sd_2)\int_0^{d_2} p_x(x)d_x - \frac{2}{s} p_x(d_2)\sinh(sd_2)}{\cosh(sd_2)-1} \tag{5.58}$$

$\alpha(x)$ is symmetric in this case and it follows from (5.56) that

$$\alpha^{-1}(x) = \begin{cases} \frac{2}{s} p_x(x) & x \in [d_2, \infty) \\ \frac{1}{s}[\theta e^{-s(d_2-x)} + \beta e^{-sx}] & x \in [0, d_2) \end{cases} \tag{5.59}$$

where

$$\theta \triangleq \frac{p_x(d_2)(e^{sd_2}-1) - s\int_0^{d_2} p_x(x)dx}{\cosh(sd_2)-1} \tag{5.60}$$

and

$$\beta \triangleq \frac{p_x(d_2)(1-e^{sd_2})+se^{sd_2}\int_0^{d_2} p_x(x)\,dx}{\cosh(sd_2)-1} \tag{5.61}$$

The estimation equation for this case may be put in the form

$$d_2 = \frac{1}{s}\log\{\,1+2s\int_0^{d_2}\frac{e^{sx}p_x(x)\,dx}{e^{-s(d_2-x)}\theta+\beta e^{-sx}}\,\} \tag{5.62}$$

(5.62) may be solved numerically and Fig. 5.5 shows the results for $s < 1.27524$. Note that $sd \to \infty$ as $s \to 0$.

R_s and d_2 may be found by using (5.59) and (5.56) in (5.19-21) yielding

$$R_s = \log(4\pi s)-2-sd_s+2\int_0^{d_2}\log\left[\frac{2p_x(x)}{e^{-s(d_2-x)}\theta+e^{-sx}\beta}\right]p_x(x)\,dx$$

$$d_s = \frac{1}{s}(1-2\int_0^{d_2}p_x(x)\,dx) + \frac{2}{s}\,d_2p_x(d_2) \tag{5.63}$$

$$+\,2\int_0^{d_2}\frac{x[e^{-sx}\beta-e^{-s(d_2-x)}\theta]}{[e^{-sx}\beta+e^{-s(d_2-x)}\theta]}\,p_x(x)\,dx$$

It may be verified by considering $C(y)$ that this is the solution for $0 \le s \le 1.27524$. Note that by letting $sd \to \infty$ and $s \to 0$ in (5.63) that $R_s \to 0$ and $d_s \to 2/\pi = d_{max}$ as they must.

The complete solution of the squared Cauchy source is, therefore, specified and is plotted in Fig. 5.6 along with

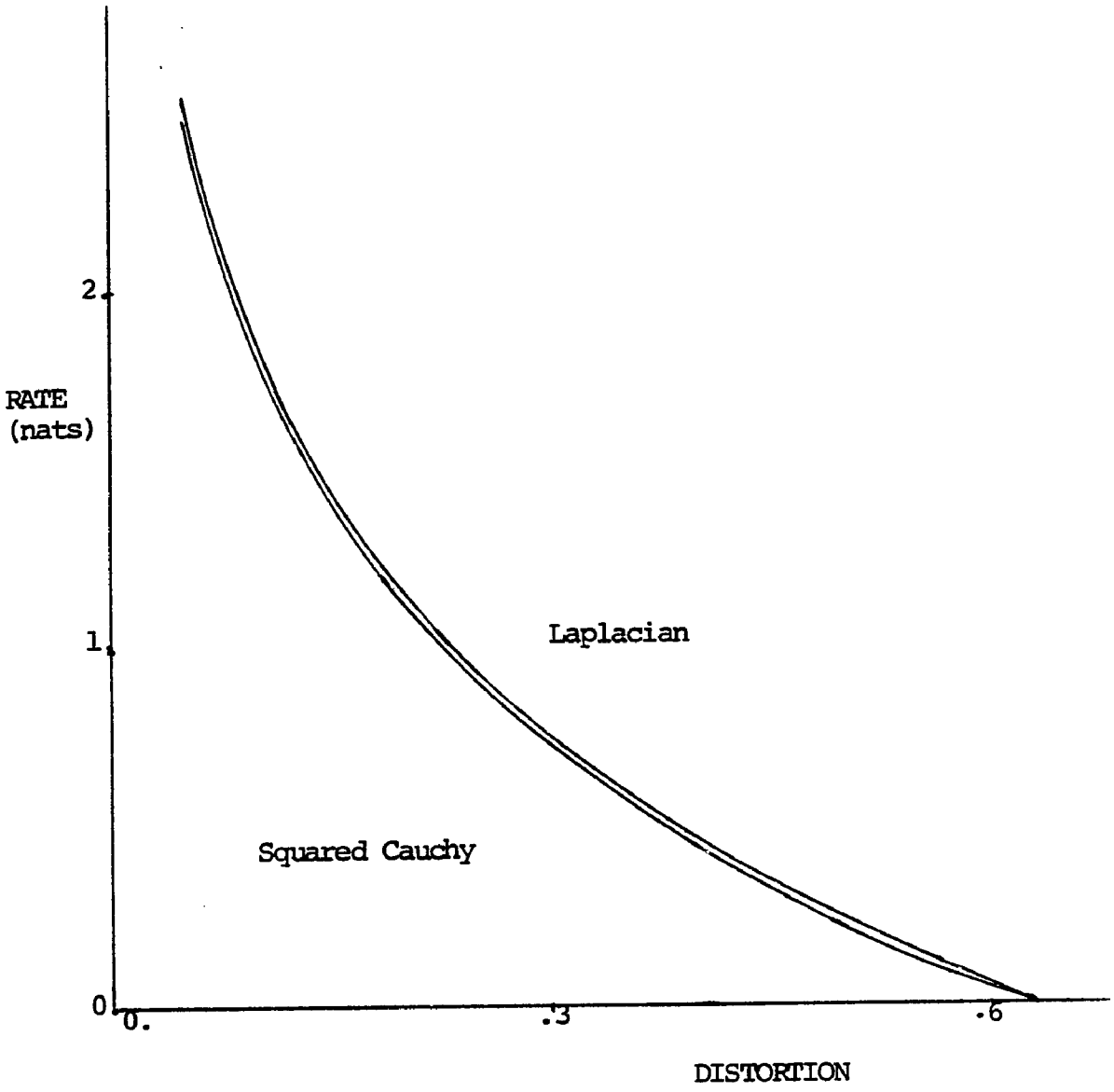


Fig. 5.6 The Rate Distortion Function for the Squared Cauchy Source and the Magnitude Error Distortion Measure for $d_{max}=2/\pi$.

R(d) for a Laplacian source with $d_{max} = 2/\pi$.

The examples demonstrate the use of the procedure for finding R(d) given in this section. The examples also demonstrate the ideas of strong and weak sources. These ideas will also be of use in Chapter 6. The examples further point out the use of the estimation equations (5.8-9) (and, therefore, C'(y)) in determining the positions of the discrete masses. This will also be the case in Chapter 6 where an algorithm based on C'(y) will be used to determine the positions of the discrete masses.

Another interesting observation is that the rate distortion function for a weak source, such as the square Cauchy, must be tangent to the distortion axis at d_{max}. This is easily seen by noting that at d_{max}

$$C(y) = \int_{\underline{X}} e^{s|x-m|} e^{-s|x-y|} p_x(x)\,dx$$

where m is the median of $p_x(x)$, and if

$$p_x(x) e^{s|x-m|} \to \infty \quad \text{as} \quad |x| \to \infty \tag{5.64}$$

($p_x(x)$ is then weak) then the only s for which $C(y) \leq 1 \;\; \forall\, y \in \underline{Y}$ is $s = 0$. An exception to this is the Laplacian source, which is weak but does not satisfy (5.64).

The Laplacian source is special in other ways. It has been plotted in the Figures of this chapter for comparison purposes. Its significance is that it has the largest rate distortion function of any source with the same d_{max}. This

property is stated in the following lemma.

Lemma 5.1 For any memoryless source with median m and $d_{max} = 1/\sigma$, the rate distortion function R(d) with respect to the magnitude error distortion measure is such that

$$R(d) \leq R_L(d) = \log \left(\frac{d_{max}}{d}\right) \tag{5.65}$$

where $R_L(d)$ is the rate distortion function of the Laplacian source with median m and $d_{max} = 1/\sigma$.

Proof: Equation (2.20) states that

$$F(s) = \inf_{Q} \int_{\underline{X}} \log\ (\alpha(x)) P_x(dx)$$

The reproduction measure which achieves the point $(R_2(d_s^L), d_s^L)$ on $R_L(d)$ generates the $\alpha(x)$

$$\alpha^{-1}(x) = \frac{2}{s}\left[\frac{\sigma}{2}\, e^{-\sigma|x-m|}\right]$$

Therefore,

$$F(s) \leq \log\ \left(\frac{s}{\sigma}\right) + \sigma \int_{\underline{X}} |x-m| P_x(dx)$$

Now

$$\int_{\underline{X}} |x-m| P_x(dx) = d_{max} = 1/\sigma$$

then

$$R(d_s) + sd_s = F(s) \leq \log\left(\frac{s}{\sigma}\right) + 1 = R_L(d_s^L) + sd_s^L \tag{5.66}$$

Applying Lemma 2.4 to the right side of (5.66) gives the desired result:

$$R(d_s) \leq R_L(d_s) \qquad ///$$

In chapter 6, the Gaussian source will play the same role for the squared error as the Laplacian plays for the magnitude error. The analogous result to Lemma 5.1 for the Gaussian source and the squared error distortion measure is given in Berger [3], Thm. 4.3.3.

VI. SQUARED ERROR DISTORTION

In this chapter the squared error distortion measure will be considered, that is $d(x,y) = (x-y)^2$. It is the most widely used distortion measure for continuous alphabet sources. The evaluation of the rate distortion function for a Gaussian source and the squared error distortion measure is a classic result in information theory. Outside of the Gaussian source, however, little is known of this problem. As may be seen from the discussion of Berger [3] p. 97-98, the SLB is usually not tight. For the SLB to be tight, a source must have a density function whose Fourier transform $F_x(w)$ when combined with $e^{w^2/4s}$ will result in a function $(F_x(w)e^{w^2/4s})$ which is positive definite.

Unlike the magnitude error distortion measure, the squared error distortion measure cannot be handled by the integral equation method. This occurs because the form of the solution for integral equations with the kernel $e^{-s(x-y)^2}$ is, in general, not known. (One may refer to Shinbrot [18], for example, for a discussion of integral equations with difference kernels.) This chapter shall demonstrate what might be done if the integral equation method fails.

In some sense, the reasons that the integral equation method fails is the reason that the results to be presented in this chapter succeed. The primary reason for this is that the kernel $e^{-s(x-y)^2}$ is an entire function. Its Fourier transform is also entire. Therefore,integral equation

methods based on Wiener-Hopf techniques (see Shinbrot [18]) do not work.

This chapter shall utilize the analytic properties of the kernel $e^{-s(x-y)^2}$ along with the requirement on C(y), to make some very strong statements about the support of the reproduction measure Q. This unfortunately will not yield closed form expressions for the rate distortion function as were found in Chapter 5. However, the results to be presented offer an improved computational procedure over Blahut's algorithm as well as being of theoretical interest.

In this chapter, Section 6.1 shall present the problem and discuss the requirements on the solution. Section 6.2 shall consider solving the problem and related issues. Section 6.3 shall present an algorithm for solving the problem.

6.1 Requirements on the Problem's Solution

From Chapter 4, the equations to be considered are:

$$\alpha^{-1}(x) = \int_{\underline{Y}} e^{-s(x-y)^2} Q(dy) \tag{6.1}$$

$$C(y) = \int_{\underline{X}} \alpha(x) e^{-s(x-y)^2} P_x(dx) \tag{6.2}$$

$$C(y) \leq 1 \qquad \forall\, y \in \overline{Y} \tag{6.3}$$

$$C(y) = 1 \qquad \text{almost everywhere } Q \tag{6.4}$$

Again in this chapter, $\underline{X} = \underline{Y} = R$ (the real line).

C(y) is obviously a continuous function of y. The following lemma establishes that at least for the measure which achieves R(d), C(y) has continuous derivatives of all orders.

Lemma 6.1 For a measure P which achieves R(d), the resulting C(y) has continuous derivatives of all orders which may be evaluated by exchanging differentiation and integration in (6.2).

Proof: $e^{-s(x-y)^2}$ is continuous for all $(x,y) \in \underline{X} \times \overline{Y}$. The derivatives of $e^{-s(x-y)^2}$ with respect to y exist for all $x \in \underline{X}$ and are continuous. There exist a finite a_i and b_i such that for any finite n and any $y \in \overline{Y}$

$$b_i e^{-s(x-y-a_i)^2} \geq (y-y)^n e^{-s(x-y)^2} \qquad \forall\, x \geq y \qquad (6.5)$$

A straight forward expansion of the exponent shows this. Note that the b_i and a_i depend on y. A similar relationship holds $\forall\, x \leq y$. That is, there exist b_i^* and a_i^* such that for any finite n

$$b_i^* e^{-s(x-y-a_i^*)^2} \geq (x-y)^n e^{-s(x-y)^2} \qquad \forall\, x \leq y$$

The nth derivative with respect to y of $e^{-s(x-y)^2}$ is a sum of terms of the form

$$(x-y)^k e^{-s(x-y)^2} \qquad \text{where } k \leq n \ .$$

One may then construct a dominating term of the form (6.5) for each term of the derivative such that

$$\sum_i^n b_i e^{-s(x-y-a_i)^2} \geq \left| \frac{d^n}{dy^n} \{e^{-s(x-y)^2}\} \right| \quad \forall\, x \geq y$$

and (6.6)

$$\sum_i^n b_i^* e^{-s(x-y-a_i^*)^2} \geq \left| \frac{d^n}{dy^n} \{e^{-s(x-y)^2}\} \right| \quad \forall\, x \leq y$$

Now form the differential coefficient

$$\frac{f(y+k)-f(y)}{k} = \int_{\underline{X}} \alpha(x) \left[\frac{A(x,y+k)-A(x,y)}{k}\right] P_x(dx)$$

where f and A will be some order of derivative with respect to y of C(y) and $e^{-s(x-y)^2}$, respectively. If the derivative of A(x,y) with respect to y is h(x,y), then because of the properties of $e^{-s(x-y)^2}$ and its derivatives

$$\frac{A(x,y+k)-A(x,y)}{k} = h(x,y+\theta_k k) \qquad 0 < \theta_k < 1$$

Therefore

$$f'(y) \triangleq \lim_{k\to 0}\left[\frac{f(y+k)-f(y)}{k}\right] = \lim_{k\to 0}\int_{\underline{X}} \alpha(x) h(x,y+\theta_k k) P_x(dx) \tag{6.7}$$

To finish the proof, the Lebesgue dominated convergence theorem will be applied to the right side of (6.7). To do this note that

$$\int_{\underline{X}} \alpha(x) |h(x,y)| P_x(dx) < \infty \qquad \forall\, y \in \underline{Y}$$

because from (6.6)

$$|h(x,y)| \leq \begin{cases} \sum_i^n b_i(y) e^{-s(x-y-a_i(y))^2} & \forall\ x \geq y \\ \sum_i^n b_i^*(y) e^{-s(x-y-a_i^*(y))^2} & \forall\ x \leq y \end{cases} \tag{6.8}$$

and using (6.3)

$$\int_{\underline{X}} \alpha(x)\,|h(x,y)|\,P_x(dx) \leq \sum_i^n b_i C(y+a_i) + \sum_i^n b_i^* C(y+a_i^*) < \infty \qquad \forall\ y \in \overline{\underline{Y}}\ .$$

(Note that in (6.8), the dependence of b_i, b_i^*, a_i, and a_i^* on y has been displayed for future use.) If one is to use the Lebesgue dominated convergence theorem, one also needs a dominating function $h^*(x,y)$ such that

$$|h(x,y+\theta_k k)| \leq h^*(x,y) \qquad \forall\ k < \text{some } \Delta$$

and

$$\int_{\underline{X}} \alpha(x)\ h^*(x,y)\ P_x(dx) < \infty.$$

Since there are finite b_i, b_i^*, a_i, and a_i^* for all y in any closed interval $[y,y+\Delta]$, there are finite $\hat{b}_i$, $\hat{b}_i^*$, $\hat{a}$, and $\hat{a}^*$ such that

$$\hat{b}_i \geq b_i(t) \qquad \forall\ t \in [y,y+\Delta],$$

$$\hat{b}_i^* \geq b_i^*(t) \qquad \forall\ t \in [y,y+\Delta],$$

$$\hat{a} \geq a_i(t) \qquad \forall\ i \text{ and } \forall\ t \in [y,y+\Delta],$$

and

$$\hat{a}^* \leq a_i^*(t) \qquad \forall\ i \text{ and } \forall\ t \in [y, y+\Delta] \ .$$

$h^*(x,y)$ may therefore be constructed to be

$$h^*(x,y) = \begin{cases} \sum_i^n \hat{b}_i e^{-s(x-y-\hat{a})^2} & \forall\ x \in (\hat{a}, \infty) \\ \sum_i^n \hat{b}_i + \sum_i^n \hat{b}_i^* & \forall\ x \in [\hat{a}^*, \hat{a}] \\ \sum_i^n \hat{b}_i^* e^{-s(x-y-\hat{a}^*)^2} & \forall\ x \in (-\infty, \hat{a}^*) \end{cases}$$

and therefore

$$\int_{\underline{X}} \alpha(x) h^*(x,y) P_x(dx) \leq \sum_i^n \hat{b}_i^* \, C(y+\hat{a}^*)$$

$$+ \left[\sum_i^n (\hat{b}_i + \hat{b}_i^*)\right] \int_{[\hat{a}^*, \hat{a}]} \alpha(x) P_x(dx)$$

$$+ \sum_i^n \hat{b}_i \, C(y+\hat{a})$$

Since $[\hat{a}^*, \hat{a}]$ is a finite interval, $\int_{[\hat{a}^*, \hat{a}]} \alpha(x) P_x(dx) < \infty$. Further, by (6.3)

$C(y+\hat{a})$ and $C(y+\hat{a}^*)$ are less than or equal to one therefore,

$$\int_{\underline{X}} \alpha(x) h^*(x,y) P_x(dx) < \infty \ .$$

The above construction is good $\forall\ y \in \overline{\underline{Y}}$, therefore there

is a dominating function $h^*(x,y)$ $\forall$ $y \in \underline{Y}$ and the Lebesgue dominated convergence theorem requires that

$$f'(y) = \int_{\underline{X}} \alpha(x) h(x,y) P_x(dx) \qquad \forall\ y \in \underline{Y} .$$

Remember that f is any order derivative of C(y) and h(x,y) is the corresponding order derivative of $e^{-s(x-y)^2}$. The continuity of any derivative of C(y) follows from the existence of the integral and the continuity of any derivative of $e^{-s(x-y)^2}$. ///

Lemma 6.1 is useful, as will soon be seen, in proving the analyticity of C(y), however, now consider the first derivative of C(y):

$$C'(y) = 2s \int_{\underline{X}} \alpha(x) e^{-s(x-y)^2} (x-y) P_x(dx) \qquad (6.9)$$

At all points in support of Q, Lemma 4.2 requires that $C'(y) = 0$ and $C(y) = 1$. Therefore, (6.9) becomes

$$y = \int_{\underline{X}} x\alpha(x) e^{-s(x-y)^2} P_x(dx) \quad \forall\ y \text{ in support of } Q \qquad (6.10)$$

The right side of (6.10) is recognized to be the conditional expectation of x given y which is known to be the optimum estimate of x given y for the squared error distortion measure. (6.10), therefore, shows that the optimum estimate $\hat{x}$ of x given y is just y. This, as in Chapter 5,

confirms the relationship between C'(y) and the results of estimation theory. (6.10) shall be referred to as the estimation equation, however, unlike in Chapter 5, the estimation equation only presents one with another integral equation.

The second derivative of C(y) is also of interest,

$$C''(y) = 2s\{2s\int_{\underline{X}}(x-y)^2\alpha(x)e^{-s(x-y)^2}P_x(dx)-C(y)\} \quad (6.11)$$

Since C(y), C'(y), and C"(y) are continuous and at the points of support of the Q which achieves R(d) C(y) = 1, C"(y) = 0, and $C(y) \leq 1 \quad \forall\, y \in \underline{Y}$, then at points of support of Q $C''(y) \leq 0$. Therefore,

$$0 \geq \int_{\underline{Y}} C''(y)Q(dy) = 2s\{2s\ d_s-1\}$$

or

$$d_s \leq \frac{1}{2s} \quad (6.12)$$

(6.12) therefore places an upper bound on the average distortion at a particular point, in terms of the slope of R(d) at that point. This result will be of interest later.

With the interest in its first two derivatives established, an important property of C(y) may be given. To do this, let z = u+jv and then recognize that $e^{-s(x-z)^2}$ is an entire function $\forall\ x \in \underline{X}$. The following theorem makes a similar statement about C(z).

Theorem 6.1 For a measure which achieves R(d), C(z) is an entire function.

Proof: C(z) may be written as

$$C(z) = f(u,v) + jh(u,v)$$

where

$$f(u,v) \triangleq e^{sv^2}\int_{\underline{X}} e^{-s(x-u)^2}\cos(2sv(x-u))\alpha(x)P_x(dx)$$

$$h(u,v) \triangleq e^{sv^2}\int_{\underline{X}} e^{-s(x-u)^2}\sin(2sv(x-u))\alpha(x)P_x(dx)$$

f(u,v) and h(u,v) exist $\forall$ v ε R because the integrals are dominated by C(u) which is constrained by (6.3) to be finite. f(u,v) and h(u,v) are obviously continuous. $\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial v}$, $\frac{\partial h}{\partial u}$, $\frac{\partial h}{\partial v}$ all exist and may be evaluated by the elementary properties of derivatives and the exchange of integration and differentiation whose proof is an identical argument to that of Lemma 6.1. Under these conditions, it is a straight forward exercise to verify that the Cauchy-Riemann conditions are met. Therefore, via a well known theorem (see Titchmarsh [19], p. 68) C(z) is analytic over the entire plane, or in other words, it is entire. ///

The analyticity of C(z) puts severe constraints on the support of Q. Since C(z) is entire, then so is C(z) - 1, and therefore, the points of support of Q are zeros of an

entire function.

The zeros of an entire function F(z) are isolated, that is if F(z) is not identically zero, there is a circle $|z-a| = \rho > 0$ about the point a such that F(z) has no zeros inside of this circle except possibly at a (see Titchmarsh [19], p. 88). The following Lemma expresses these ideas in terms of C(y).

Lemma 6.2 For a measure which achieves R(d), C(y) must be equal to one $\forall\ y \in \underline{Y}$, or it must be equal to one at at most a countable number of isolated points, but not both.

Proof: C(z) is entire. Thus by Liouville's theorem, if it is constant over any interval, it must be constant over the entire plane. Therefore, if C(y) is equal to one over any interval it must be equal to one $\forall\ y \in \underline{Y}$. If C(y) is not constant $\forall\ y \in \underline{Y}$, then since C(z)-1 is entire, the points where C(y) can be equal to one are isolated and there is at most a countable number of them (see Titchmarsh [19], p. 88). ///

If the first alternative of Lemma 6.2 is true, that is $C(y) = 1\ \forall\ y \in \underline{Y}$, then the following statements are true:

1.1 $C''(y) = 0\ \forall\ y \in \underline{Y}$ and therefore from (6.11)

$$d_s = \frac{1}{2s} .$$

1.2 $$C(y) = \int_{\underline{X}} e^{-s(x-y)^2} \alpha(x) P_x(dx) = 1 \quad \forall\ y \in \underline{Y}$$

and therefore, if the source has a density function $p_x(x)$, one may proceed by Fourier transform techniques to find that

$$\alpha(x)p_x(x) = \sqrt{\frac{s}{\pi}}$$

This is the situation when the SLB is tight (see Berger [3]).

In most situations, as will be shown, the first alternative of Lemma 6.2 will not be the case. Therefore, $C(y)$ can have at most a countable number of isolated points at which $C(y) = 1$. This restricts the support of Q to an at most countable number of discrete masses. This requires that $C''(y) < 0$ for all points in the support of Q, which requires via (6.11) that

$$d_s < \frac{1}{2s}$$

Other properties of $C(z)$ which occur if the second alternative of Lemma 6.2 is true are:

2.1 $C(z) - 1$ is a real analytic function. That is, if z is real, $C(z) - 1$ is real. (Actually true in general).

2.2 $C(z) - 1$ expanded in a Taylor series about one of its real zeros y_1 is
$C(z) - 1 = a_1 + a_2(z-y_1) + a_3(z-y_1)^2 + \ldots$
where the a_i are real.

2.3 $a_1 = 0$ because y_1 is a zero, and $a_2 = 0$ because

$C'(y_1) = 0$. $|a_3| > 0$ since $C''(y_1) < 0$. Therefore, the real zeros of $C(z) - 1$ are of multiplicity 2.

2.4 From the Taylor series expansion about a real zero y_1

$$C(z) - 1 = \frac{1}{2} C''(y_1)(z-y_1)^2 + \ldots$$

2.5 The number of zeros N of $C(z) - 1$ in a disk of finite radius r about $z = 0$ satisfies the following (Jensen's Theorem, Rudin [21], p. 330)

$$|C(0)-1| \prod_{n=1}^{N} \frac{r}{|b_n|} = \exp\{\frac{1}{2\pi} \int_{-\pi}^{\pi} \log| C(re^{j\theta})-1|d\theta\} \tag{6.13}$$

where $C(0) - 1 \neq 0$ and $b_1 \ldots b_N$ are the zeros of $C(z) - 1$ within the disk. Zeros of multiplicity k occur k times.

Property 2.5 listed above cannot be used as it stands since one does not know that $C(0) - 1 \neq 0$, or any other properties of $C(0)$. Therefore, if one wishes to bound the number of zeros of $C(z) - 1$ and therefore the number of support points of Q in any interval by using (6.13), (6.13) must be put in a different form. To do this consider the real zero of $C(z) - 1$ at y_1 and the function

$$g_1(z) \triangleq C(z+y_1) - 1 = \frac{1}{2} C''(y_1)\ z^2 + \ldots$$

Now $g_1(z)/z^2$ is a real entire function to which (6.13) may be applied giving

$$\left|\frac{1}{2} C''(y_1)\right| \prod_{n=1}^{N} \frac{r}{|b_n|} = \frac{1}{r^2} \exp\left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} \log|g_1(re^{j\theta})|d\theta\right\}$$

To handle the term $\prod_{n=1}^{N} \frac{r}{|b_n|}$, one may follow the procedure of Rudin [21], p. 332 to yield

$$\left|\frac{1}{2} C''(y_1)\right|^2 N \leq \frac{1}{4r^2} \exp\left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} \log|g_1(2re^{j\theta})|d\theta\right\} \tag{6.14}$$

Now

$$|g_1(re^{j\theta})| \leq |C(re^{j\theta}+y_1)| + 1 \leq e^{sr^2\sin^2\theta} + 1$$

therefore (6.14) becomes

$$N \leq \frac{1}{\log 2} \left\{\log\left[\frac{1}{2r^2|C''(y_1)|}\right] + \frac{1}{2\pi} \int_{-\pi}^{\pi} \log\left[e^{4sr^2\sin^2\theta}+1\right]d\theta\right\} \tag{6.15}$$

N is the total number of zeros (real and complex) in the disk of radius r and is larger than 2M-2 real zeros, where M is the number of support points contained in the disk of radius r (which have multiplicity 2) and the -2 is for the zero which was removed in generating g_1. Therefore, the number of support points M in the interval of length r on either side of y_1, is bounded by

$$M \leq 1 + \frac{1}{2 \log 2} \{\log[\frac{1}{2r^2|C''(y_1)|}] + \frac{1}{2\pi}\int_{-\pi}^{\pi} \log[e^{4sr^2\sin^2\theta}+1]d\theta\} \tag{6.16}$$

Since $e^{4sr^2\sin^2\theta}+1 \leq e^{4sr^2}+1$, (6.16) gives

$$M \leq 1 + \frac{1}{2 \log 2} \log [\frac{e^{4sr^2}+1}{2r^2|C''(y_1)|}] \tag{6.17}$$

Now $C''(y_1) < 0$ and as s increases, one expects that there will be more points of support within the disk of radius r and therefore $|C''(y_1)|$ must increase. The term $\log [\frac{1}{|C''(y_1)|}]$, therefore, is expected to decrease as s increases. Therefore, since as s increases $\log [e^{4sr^2}+1] \sim 4sr^2$, it is seen that M can grow no faster than linearly with s. This presents a useful guide to how the number of support points of Q grows with s. At d_{max}, there is one support point. Thus as one moves up the rate distortion curve, the number of support points of the achieving reproduction measure grows (from one) no faster than linearly with the slope of the rate distortion function.

The above discussion summarizes the consequences of the two alternatives of Lemma 6.2. To determine which alternative of Lemma 6.2 is in operation, the following theorem will be most important. This theorem utilizes the idea of a strong source as introduced in Chapter 5, and demonstrates

the utility of this thinking again. In this case, a strong source is one which dominates e^{sx^2} outside of some finite interval.

Theorem 6.2 The rate distortion function for any strong source and any $s > 0$ is achieved by a reproduction measure Q which is generated by a finite number of discrete masses.

Proof: A strong source is one by definition which limits the support of Q to a finite interval. Therefore, the first alternative of Lemma 6.2 is eliminated. The support of Q is, therefore, discrete. The support of Q is finite because the number of zeros of an entire function, which C(z)-1 is, in any finite circle in the complex plane is finite (see Markushovich [20], p. 43). Since the support is limited to a finite interval, it must be finite. ///

Theorem 6.2 is important in establishing the form which Q must have. In the next section, its use will be established.

6.2 Consideration of Solutions

Section 5.1 showed that there were two alternatives to the support of the Q which achieves R(d). The first alternative's solution (when $C(y) = 1 \ \forall \ y \in \underline{Y}$) was related to cases where the SLB is tight. The second alternative's solution (support of Q is discrete) gives equations to be

solved of the form:

$$y_i = \int_{\underline{X}} x\alpha(x) e^{-s(x-y_i)^2} P_x(dx) \quad i=1...N \tag{6.18}$$

$$1 = \int_{\underline{X}} \alpha(x) e^{-s(x-y_i)^2} P_x(dx) \quad i=1...N \tag{6.19}$$

where

$$\alpha^{-1}(x) = \sum_{i=1}^{N} q_i e^{-s(x-y_i)^2} \tag{6.20}$$

and q_i and y_i must be found such that

$$\sum_{i=1}^{N} q_i = 1 \tag{6.21}$$

and

$$C(y) \leq 1 \quad \forall\, y \in \overline{\underline{Y}} \tag{6.22}$$

In general, integrations of the form (6.18) and (6.19) cannot be done analytically and, therefore, numerical techniques must be employed. To perform the numerical integrations, the source will be truncated (if it is not already). In fact, in the case of either alternative of Lemma 6.2, if numerical integrations must be carried out, the source will be truncated (if it is not already). This means that the source will be replaced by another source which has no support outside of some finite interval and is approximately equal to the true source in the finite interval. Note that this must also be done when Blahut's algorithm is to be used.

However, if a source has no support outside of some finite interval, it is necessarily strong and, therefore, by Theorem 6.2 its Q support must be finite. The problem, in general, then becomes, in the case when the SLB is not tight, one of dealing with a strong source and, therefore, with a finite number of points of Q support. An algorithm for finding these points will be developed in Section 6.3, however, the remainder of this section will consider the error introduced in the rate distortion function by truncating the source. An upper bound to the error will be developed in terms of the points where the source is truncated.

To do this, the $\bar{\rho}$-distance between two random variables (X and Z) will be introduced.

$$\bar{\rho}(X,Z) \triangleq \inf E\{d(x,z)\}$$

where the inf is over all joint probability measures P with marginals P_x and P_z, and $d(x,z)$ is the distortion measure being considered. Here, of course, $d(x,z) = (x-z)^2$. The following Lemma relates the $\bar{\rho}$-distance to the distortion rate functions of the memoryless sources X and Z and the distortion measure $(x-z)^2$.

Lemma 6.3 For the squared error distortion measure

$$|D_x(R)^{1/2} - D_z(R)^{1/2}|^2 \leq \bar{\rho}(X,Z) \tag{6.23}$$

Proof: See Gray and Davisson [22]. A different proof

shall be presented here. Consider Fig. 6.1. In Fig. 6.1, P_{xz} is the joint measure between the random variables X and Z with X and Z marginals given (P_x and P_z). P_{zt} is the joint measure between the random variables Z and T with the Z-marginal (P_z) given. Let P_{xzt} be a measure with X and Z marginals P_x and P_z, respectively, and X,Z,T related as in Fig. 6.1.

Now consider the inequality

$$(x-t)^2 \leq (|x-z|+|z-t|)^2$$

and define the set A' as

$$A' = \{P_{xzt}: I(Z;T) \leq R\}$$

One may use these to give

$$\inf_{A'} E\{(X-T)^2\} \leq \inf_{A'} E\{(|X-Z|+|Z-T|)^2\} \tag{6.24}$$

and applying Minkowski's inequality to the right side of (6.24) gives

$$\inf_{A'} E\{(X-T)^2\} \leq \inf_{A'} [E\{(X-Z)^2\}^{1/2} + E\{(Z-T)^2\}^{1/2}]^2 \tag{6.25}$$

By the definition of A', (6.25) becomes

$$\inf_{A'} E\{(X-T)^2\} \leq [\bar{\rho}(X,Z)^{1/2} + D_Z(R)^{1/2}]^2 \tag{6.26}$$

where $D_z(R)$ is the distortion rate function of the source z. Since

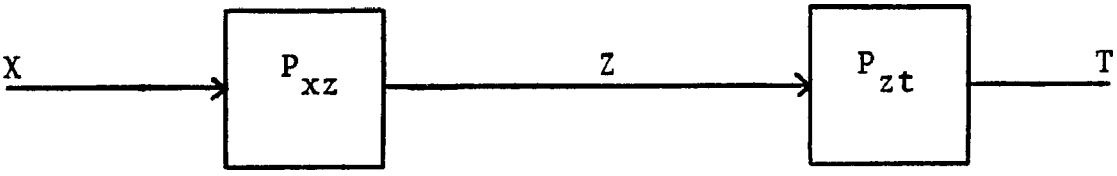


Fig. 6.1 The Relationship Among Random Variables X, Z, and T in Lemma 6.3

$$I(X;T) \leq I(Z;T) \leq R$$

then

$$\inf_{A'} E\{(X-T)^2\} \geq D_x(I(X;T)) \geq D_x(R) \tag{6.27}$$

Therefore, (6.27) may be used with (6.26) to give

$$D_x(R) \leq [\bar{\rho}(X,Z)^{1/2} + D_z(R)^{1/2}]^2$$

One may exchange the role of X and Z in the above to give the similar result that

$$D_z(R) \leq [\bar{\rho}(X,Z)^{1/2} + D_x(R)^{1/2}]^2$$

(6.23) follows immediately. ///

One would like a quantity relating the difference between $D_x(R)$ and $D_z(R)$ rather than the difference between their square roots. To accomplish this note that

$$|D_x(R) - D_z(R)| \leq \bar{\rho}(X,Z)^{1/2} \ (D_x^{1/2}(R) + D_z^{1/2}(R))$$

now recognize that d_{max} is the largest distortion for $R \geq 0$, therefore

$$|D_x(R) - D_z(R)| < \bar{\rho}(X;Z)^{1/2} \ [d_{xmax}^{1/2} + d_{zmax}^{1/2}] \tag{6.28}$$

(6.28) bounds the difference between the distortion rate functions (and, therefore, the rate distortion functions) of two memoryless sources in terms of the $\bar{\rho}$-distance between them. One now needs to be able to evaluate this

bound in order to bound the error between a source and a truncated approximation of it.

To do this choose a_1 and b_1 as the end points of the interval of support of the truncated source, and consider the mapping between X and Z such that

1. If $a_1 \leq x \leq b$, then $z = x$.
2. If $x \notin [a_1,b_1]$, then choose z at random from a uniform distribution over $[a_1,b_1]$.

The conditional distribution for this mapping is

$$P(dz|x) = \begin{cases} \delta(x-z)dz & x \in [a_1,b_1] \\ p_{un}(z)dz & x \notin [a_1,b_1] \end{cases} \tag{6.29}$$

where

$$p_{un}(z) = \begin{cases} \frac{1}{b_1-a_1} & z \in [a_1,b_1] \\ 0 & \text{elsewhere} . \end{cases}$$

It follows from (6.29) that

$$P_z(dz) = \begin{cases} P_x(dz)+[1-\int_{[a_1,b_1]} P_x(dx)]p_{un}(z)dz & \\ & z \in [a_1,b_1] \\ 0 & \text{elsewhere} \end{cases} \tag{6.30}$$

Z is, therefore, a truncated version of X, and since this mapping is in the constraint set of the definition of the

$\bar{\rho}$-distance between Z and X then

$$\bar{\rho}(X;Z) \leq \int_{\underline{X}} \int_{\underline{Y}} (x-z)^2 \, P(dz|x) P_x(dx) \tag{6.31}$$

Substituting (6.29) into (6.31) gives

$$\bar{\rho}(X,Z) \leq \frac{\Delta^2}{12} (a_1+b_1)^2 + \Delta \int_{[a_1,b_1]^c} (x - \frac{a_1+b_1}{2})^2 P_x(dx) \tag{6.32}$$

where

$$\Delta \triangleq 1 - \int_{[a_1,b_1]} P_x(dx)$$

An example will illustrate the use of (6.32).

Example 6.1 One Sided Exponential Source

The one sided exponential source is specified by its density function as

$$p_x(x) = \begin{cases} \sigma \, e^{-\sigma x} & x \in [0,\infty) \\ 0 & \text{elsewhere} \end{cases}$$

Since one side is already truncated at zero $a_1 = 0$, thus

$$\Delta \triangleq 1 - \int_0^{b_1} \sigma \, e^{-\sigma x} dx = e^{-\sigma b_1}$$

and

$$\int_{b_1}^{\infty} (x - \frac{b_1}{2})^2 \, p_x(x)\,dx = e^{-\sigma b_1}[\frac{b_1^2}{4} + \frac{b_1}{\sigma} + \frac{2}{\sigma}]$$

Therefore

$$\bar{\rho}(X,Z) \leq e^{-2\sigma b_1}[\frac{b_1^2}{3} + \frac{b_1}{\sigma} + \frac{2}{\sigma}] \tag{6.33}$$

One may, therefore, pick b_1 large enough to make (6.33) (and, therefore, the difference between D_x and D_z) as small as desired.

6.3 Evaluating R(d)

This section is concerned with evaluating R(d). From the arguments of the last two sections, one might anticipate constructing Q from a finite number of discrete masses, the number of which grows no faster than linearly with s. One also knows that at the point d_{max}, there is one discrete mass generating Q.

If Q is to be constructed from discrete masses, an iterative algorithm of the form (Blahut's iteration)

$$q_i^{k+1} = q_i^k \, C_k(y_i) \tag{6.34}$$

where q_i are the weights of the discrete masses generating Q, cannot be used by itself since it can only adjust the weights, q_i, of the masses and not their positions. One, therefore, needs a way of finding the positions as well as the weights. To do this, consider F(s) (equation (2.20))

$$F(s) = \inf_{Q} \int_{\underline{X}} \log\ [\alpha(x)]P_x(dx) \tag{6.35}$$

Since the achieving Q is made up of discrete masses, one may consider $\alpha(x)$ of the form

$$\alpha^{-1}(x) = \sum_{i=1}^{N} q_i e^{-s(x-y_i)^2}$$

where N, q_i, and y_i are unknown. Therefore,

$$F(s) \leq J \triangleq -\int_{\underline{X}} \log\ [\sum_{i=1}^{N} q_i e^{-s(x-y_i)^2}]P_x(dx) \tag{6.36}$$

J is a nonlinear functional in terms of the y_i and q_i. The minimization of functionals such as J is a topic in nonlinear programming (see Himmelblau [23], for example). In fact, one may approach the problem from here on in this context. The problem (of minimizing J and, therefore, finding F(s)) may be thought of as a minimization over two vectors $\underline{y}$ and $\underline{q}$ with $\underline{y}$ unconstrained and $\underline{q}$ linearly constrained (by (6.21)). Solutions of problems of these types are discussed in [23], for example. One is hesitant to suggest a particular nonlinear technique at this point since this is a developing field in its own right and improvements are certain to be made. However, algorithms of this type can usually be put in the form

$$\underline{y}^{k+1} = \underline{y}^{k} + \lambda^{k}\, \underline{w}^{k} \tag{6.37}$$

where λ^k is a scalar to be chosen to maximally decrease the cost function (J) for the direction vector $\underline{w}^k$. This is accomplished by a linear search varying λ^k. The direction vector $\underline{w}^k$ is chosen and updated at each iteration by a variety of schemes, many of which are discussed in [23], for example. The key element in the direction vector $\underline{w}^k$ is usually the gradient of the cost function, J. To consider this with respect to J, note that

$$\frac{\partial J}{\partial y_i} = -q_i C'(y_i) \tag{6.38}$$

and

$$\frac{\partial J}{\partial q_i} = -C(y_i) \tag{6.39}$$

(6.38) and (6.39) demonstrate the importance of $C(y_i)$ and $C'(y_i)$. (6.39) should reemphasize the importance of $C(y_i)$ in Blahut's iteration, (6.34), which has already been shown to decrease J (Chapter 2). In fact, (6.34) may be put in the form of (6.37) or

$$q_i^{k+1} = q_i^k + q_i^k[C_k(y_i)-1] \tag{6.40}$$

In (6.40), $\lambda^k=1$ and the one occurs in the right side of (6.40) to help satisfy the linear constraint (6.21).

(6.38) and (6.37) suggest an algorithm for positioning the y_is of the form

$$y_i^{k+1} = y_i^k + a_i^k\, C'(y_i^k) \tag{6.41}$$

where a_i^k is a positive gain factor (containing λ^k) to be

determined. (It should be noted that Benjamin [9] has utilized similar thinking in studying discrete sources.) The following lemma establishes (6.41).

Lemma 6.4 If $C'(y_i^k) \neq 0$ $i=1,\ldots,N$, then there exists a positive $\underline{a}^k = [a_1^k,\ldots,a_N^k]$ for which (6.41) generates a J_{k+1} via (6.36) such that

$$J_k > J_{k+1} .$$

Proof: From (6.36) and (6.41)

$$J_k - J_{k+1} = \int_{\underline{X}} \log \left[\frac{\sum_{i=1}^{N} q_i e^{-s(x-y_i^{k+1})^2}}{\sum_{i=1}^{N} q_i e^{-s(x-y_i^{k})^2}}\right] P_x(dx)$$

then

$$\frac{\partial}{\partial a_i}[J_k - J_{k+1}] = 2sq_i \int_{\underline{X}} \alpha_{k+1}(x)\,(x-y_i^{k+1})\, e^{-s(x-y_i^{k+1})^2} \frac{\partial y_i^{k+1}}{\partial a_i} P_x(dx)$$

Since $\dfrac{\partial y_i^{k+1}}{\partial a_i} = C'(y_i^k)$ this is

$$\frac{\partial}{\partial a_i}[J_k - J_{k+1}] = 2sq_i C'(y_i^{k+1}) C'(y_i^k)$$

Therefore,

$$\frac{\partial}{\partial a_i}[J_k - J_{k+1}]\Big|_{a_i=0} = 2sq_i [C'(y_i^k)]^2$$

The right side of this is greater than zero if

$C'(y_i^k) \neq 0$, therefore, because of the continuity of J_{k+1} with $\underline{a}^k$, there must be a positive $\underline{a}^k$ such that $J_k > J_{k+1}$. ///

Before developing a procedure for solving the problem of minimizing J, some specific ideas, which will be helpful, should be developed. Lemma 6.4 and Blahut's iteration (6.34) guarantee a decreasing sequence of Js. However, the sequence has not been shown to converge to F(s). One would like to specify a starting Q and be assured that the limit point of the sequence of Js is F(s). Blahut [5], as mentioned in Chapter 2, was able to do this if the starting Q was chosen to be absolutely continuous with respect to Lebesque measure and the absolutely continuous part, q(y), was strictly positive $\forall$ y ϵ $\underline{Y}$. (One now knows because of the idea of a strong source that this can be weakened such that the starting q(y) need be strictly positive only over the finite interval containing the support of the truncated source.) As has been shown, the limiting Q is discrete. Thus, although the sequence generated using Blahut's algorithm is converging to F(s) (without considering the numerical approximations), it must be attempting to approximate a discrete support as it progresses. By starting from a discrete support, as suggested here, one gives up the general starting point for which convergence can be proved but gains computational advantages.

The problem of the limit point of the algorithm is of

importance; however, in practice one has an absolute measure of the error between any J and F(s). To develop this measure, the generalization of an idea of Blahut [5] is useful. Define

$$C_{max} \triangleq \max_{y \in Y} C(y) \quad (\geq 1) \tag{6.42}$$

Since truncated sources are being considered, C(y) will be finite and attain its maximum in the finite closed interval containing the support of the truncated source. Now let

$$\alpha^*(x) \triangleq [C_{max} \int_{\underline{Y}} e^{-s(x-y)^2} Q(dy)]^{-1} = \frac{\alpha(x)}{C_{max}}$$

Clearly, $\alpha^*(x)$ is in the constraint set of Lemma 2.8 since $C_{max} \geq 1$, and, therefore,

$$F(s) \geq \int_{\underline{X}} \log[\alpha(x)] P_x(dx) - \log C_{max} \tag{6.43}$$

(6.43) implies that

$$F(s) + \log C_{max} \geq \int_{\underline{X}} \log[\alpha(x)] P_x(dx) \geq F(s) \tag{6.44}$$

(6.44) indicates the importance of C_{max} in evaluating how close any J is to F(s).

In general, evaluation of C_{max} requires a search of C(y) over the closed interval bounding the support of the source. The algorithm (6.41) is, of course, attempting to position the y_i at points of local maxima, $\{C_{im}\}$, of which

one may be C_{max}. One hopes that as the iteration progresses, C_{max} will be in the set $\{C_{im}\}$ so that the algorithm has a chance of stopping. For example, if C_{max} is a local maxima of $C(y)$, which does not have a point of Q support associated with it, then the algorithm will not converge to $F(s)$. If this situation occurs,one may, of course, insert a discrete mass at this point (at C_{max}) and proceed with the iteration. The existence of such a discrete mass, which reduces J, is assured by the corollary of Theorem 2.2.

If one is to use the nonlinear programming techniques, as suggested, the stopping conditions are usually given in terms of conditions on the gradient of J. These measures are in terms of tolerances on $|1-C(y_i)|$ and $|C'(y_i)|$ in this case. To relate tolerances on $F(s)$ to tolerances on $|1-C(y_i)|$ and $|C'(y_i)|$ one should consider that the C_{im} associated with the point y_i is such that

$$|C_{im}-1| \leq \Delta|C'(y_i)| + |C(y_i-1|$$

provided y_i is sufficiently close to the position of C_{im} so that $C''(y_i) < 0$. Δ, here, is the interval of support of the source. Therefore, if C_{max} is in the set $\{C_{im}\}$ then using (6.44)

$$J-F(s) \leq \max_i \{\Delta|C'(y_i)| + |C(y_i)-1|\} \qquad (6.45)$$

(6.45) suggests picking tolerances on the gradient of the order of the accuracy desired of $F(s)$.

One would like to find a starting Q such that after the tolerances on $|C'(y_i)|$ and $|C(y_i)-1|$ are met C_{max} is contained in $\{C_{im}\}$ and, therefore, the procedure may stop. Since one has a bound on the number of support points in the solution (for a particular s), one might wish to use this number uniformly distributed through the interval of support of the source as a starting Q. The bound on the support is, in general, larger than the actual support of the solution and, therefore, it would require carrying a larger number of points than is necessary. This can require considerably more computation time. Further, since position points are certain to merge as the iteration progresses, there is no guarantee that they might not merge to a smaller set which cannot achieve F(s). For these reasons, this approach is not the most desirable.

It seems easier to proceed from the point at d_{max} in generating the rate distortion function, since one knows that at d_{max} there is a single point of support and as one moves away from d_{max} the number of support points can grow no faster than linearly with s. One may, therefore, construct the solution by increasing s from d_{max} and using the preceeding solution as the starting point of the algorithm for the next s. If this procedure is followed, one can see that local maxima of C(y) that are not associated with a support point form as s grows until they become such that $C(y) > 1$ in their vicinity. This suggests that they then require a point of support. These ideas can best be demon-

strated by an example. Before doing this, however, the following procedure summarizes the above thinking.

1. Construct the solution at d_{max} by using

$$Q(dy) = \delta(y-\eta)dy \tag{6.46}$$

where η is the mean of the source.

2. Determine by plotting C(y) the maximum s (s_{max}) for which (6.46) is the solution.
3. Consider an s slightly larger than s_{max} and consider the regions where C(y) > 1. Place discrete masses at the maxima of these regions and, therefore, construct a new Q.
4. Pick tolerances on $|C'(y_i)|$ and $|C(y_i)-1|$ of the order of the desired tolerance of F(s) and use a nonlinear programming algorithm to iterate the y_i and q_i.
5. Iterate until the tolerances have been met and plot C(y) to determine C_{max}.
6. If C_{max} is such that the accuracy on F(s) is satisfied then stop. If the accuracy has not been met, then either continue the iteration if C_{max} is in the set $\{C_{im}\}$ or insert discrete masses to assure that $C_{max} \varepsilon \{C_{im}\}$ and continue to iterate.
7. When the tolerances are met again, determine C_{max} and repeat step 6.
8. Use the resulting Q as the starting point of the

procedure for the next value of s (increase s).

The procedure continues from step 2.

The following example and discussion should illustrate this technique.

Example 6.2 Uniform Source

The uniform source to be considered is given by

$$p_x(x) = \begin{cases} \frac{1}{2\sqrt{3}} & x \in (-\sqrt{3}, \sqrt{3}) \\ 0 & \text{elsewhere} \end{cases} \tag{6.47}$$

This has been chosen such that $d_{max} = 1$. This source has a mean value of zero and, therefore, the Q at d_{max} is given by

$$Q(dy) = \delta(y)dy \tag{6.48}$$

From (6.48)

$$C(y) = e^{-sy^2}\{ \frac{\sinh[2\sqrt{3}\ sy]}{2\sqrt{3}\ sy} \} \tag{6.49}$$

Near s = .5, (6.49) may be shown to satisfy the necessary and sufficient condition that $C(y) \leq 1 \quad \forall\ y \in \underline{Y}$ and, therefore, s_{max} is near s = .5. If one considers s = .6, Fig. 6.2 shows C(y) for y > 0 (C(y) is symmetric about y = 0). Fig. 6.2 shows that two local maxima (the one for y < 0 is not shown) occur a distance from y = 0. One may insert masses at these points and iterate an algorithm. The algorithm used to position the y_is in this example is the

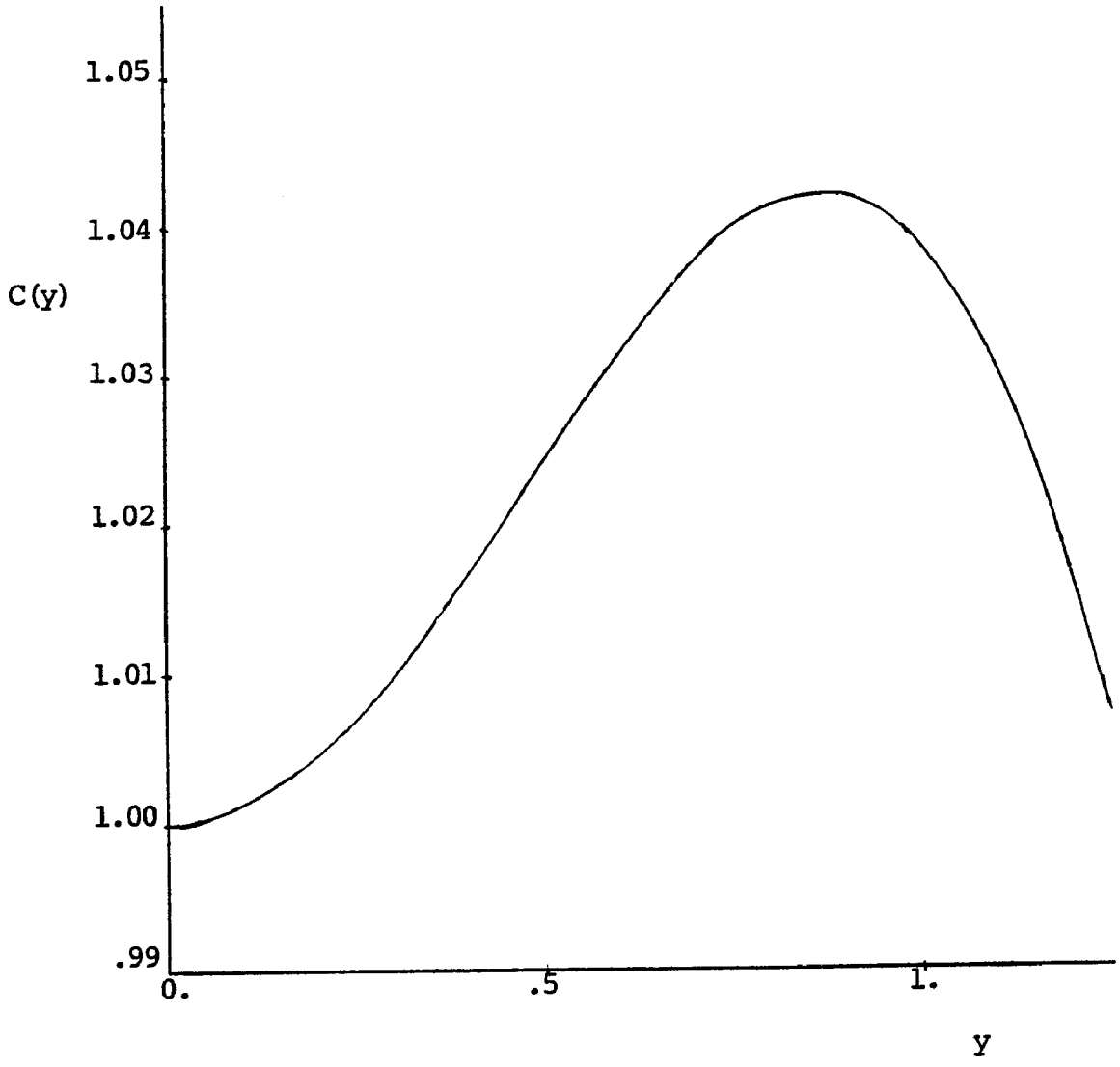


Fig. 6.2 C(y) for a Uniform Source, a Single Point of Support for Q, d_{max}=1., and s=.6 .

Davidon-Fletcher-Powell method as available in the IBM:SSP software package [24]. Table 6.1 indicates that the resulting solution for s = .6 consists of two discrete masses. Table 6.2 indicates the corresponding values of R, d, and F(s). The accuracy in Table 6.2 is to four places.

As Table 6.1 shows, the two point support solution continues to be the form of the solution until approximately s = 1.5 where a small discrete mass appears at y = 0. The three point support solution continues until approximately s = 3. where two more points of support are needed. The pattern of growth of the support points should now be clear. The growth of the support points is one of spreading and filling. New points emerge by either the splitting of old points or they emerge to fill a gap created by the separating of old points. As is illustrated in this example for N = 4, all values of N need not have solutions associated with them.

It is also interesting to note how much of the rate distortion function is mapped out by such a small number of points of support of Q. R(d) is plotted in Fig. 6.3 for the uniform source along with the rate distortion function for the Gaussian source with $d_{max} = 1$.

This example illustrates the use of the procedure presented in this section and the behavior of the solution. As was mentioned, the Davidon-Fletcher-Powell routine was used in determining the positions. The weights in this example can, of course, be determined by symmetry in the region of

Table 6.1 Positions and Weights (y_i, q_i) of the Reproduction Measure for the Uniform Source and the Squared Error Distortion Measure for $d_{max} = 1$.

s	(y_i, q_i) (positive y_i only)		
.6	(.5011, .5)		
.7	(.6306, .5)		
.8	(.6996, .5)		
.9	(.7417, .5)		
1.	(.7696, .5)		
1.1	(.7888, .5)		
1.2	(.8028, .5)		
1.3	(.8131, .5)		
1.4	(.8213, .5)		
1.5	(0., .0224)	(.8390, .4888)	
1.6	(0., .1048)	(.8914, .4476)	
1.7	(0., .1602)	(.9332, .4199)	
1.8	(0., .19)	(.9589, .405)	
1.9	(0., .221)	(.9894, .3895)	
2.	(0., .2406)	(1.01 , .3797)	
2.1	(0., .2556)	(1.028 , .3722)	
2.2	(0., .26728)	(1.0419, .36636)	
2.3	(0., .2769)	(1.0545, .36155)	
2.4	(0., .28416)	(1.0644, .35792)	
2.6	(0., .2954)	(1.0808, .3523)	
3.4	(0., .15)	(.4022, .121)	(1.1633, .304)

Table 6.2 Rate, Distortion, and F(s) for the Uniform Source and the Squared Error Distortion Measure for $d_{max} = 1$.

s	F(s)	d	R
.6	.5862	.7489	.1369
.7	.6532	.6024	.2315
.8	.7085	.5106	.3000
.9	.7563	.4499	.3515
1.	.7992	.4077	.3915
1.1	.8383	.3777	.4228
1.2	.8749	.3555	.4483
1.3	.9096	.3388	.4691
1.4	.9427	.3255	.4871
1.5	.9747	.3118	.5070
1.6	1.0047	.2887	.5428
1.7	1.0326	.2687	.5758
1.8	1.0586	.2552	.5992
1.9	1.0831	.2375	.6318
2.	1.1062	.2249	.6563
2.1	1.1281	.2141	.6786
2.2	1.1490	.2046	.6989
2.3	1.1691	.1959	.7185
2.4	1.1883	.1890	.7346
2.6	1.2249	.1770	.7648
3.4	1.3502	.1380	.8811

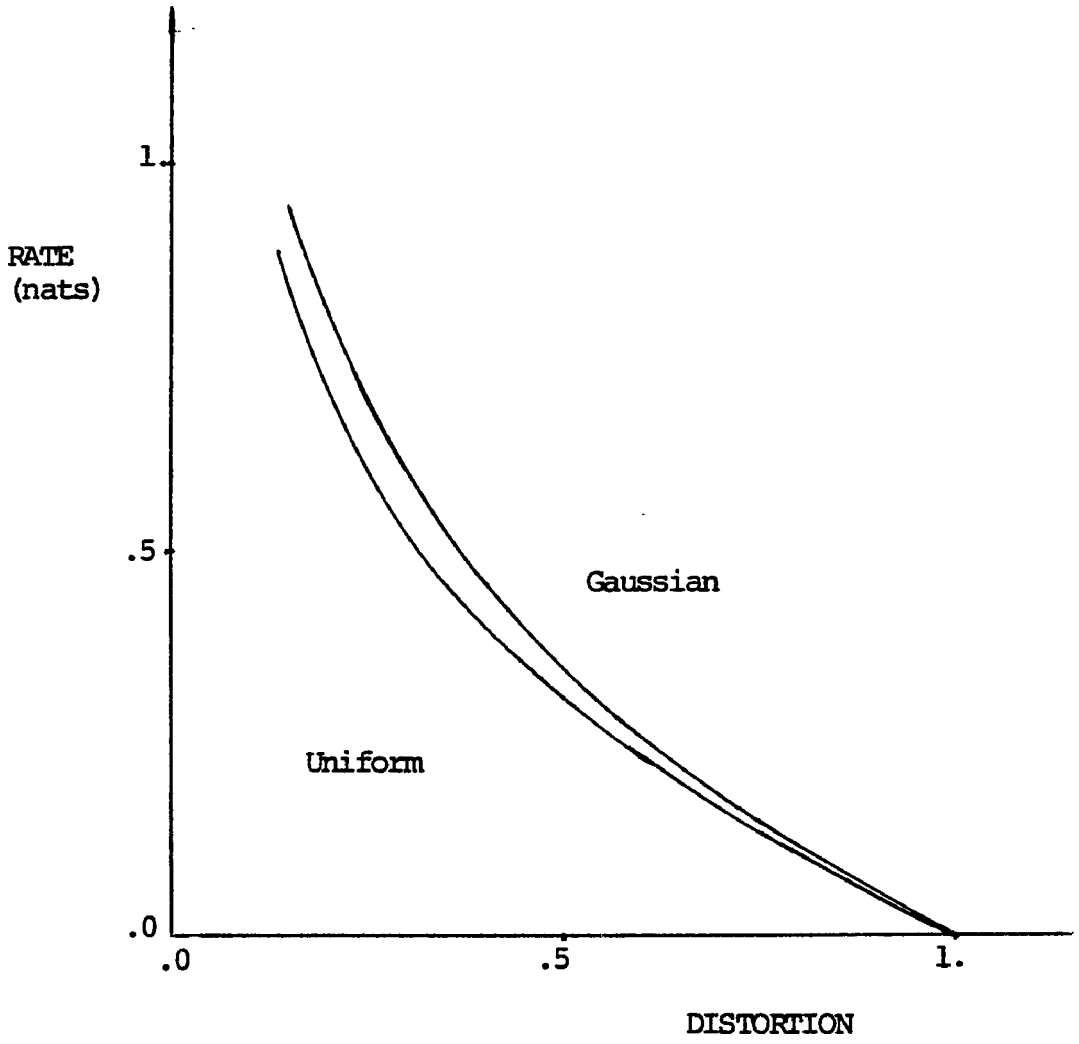


Fig. 6.3 The Rate Distortion Function for the Uniform Source and the Squared Error Distortion Measure for $d_{max}=1$.

R(d) where a two point solution holds or by Blahut's iteration in general. In utilizing these techniques one may want to alternate the iteration of the q_is and y_is or wait until after several iterations to switch. To utilize the full advantage of a nonlinear technique such as Davidon-Fletcher-Powell more than one iteration of the y_is is necessary before moving the q_is. For a small number of support points, one may find an interactive procedure quite satisfactory. That is, guess the q_i and use a nonlinear technique such as Davidon-Fletcher-Powell to give the y_i. As the above example indicates, this is a useful procedure over a large part of the curve when symmetry is also utilized since one has only a small number of variables to deal with.

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