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LINEAR FUNCTIONALS ON NON-LOCALLY CONVEX SPACES

by
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ABSTRACT

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A linear topological space has the Hahn-Banach Extension Property (HBEP) if every continuous linear functional on a closed subspace has a continuous linear extension to the whole space. The Hahn-Banach theorem implies that every locally convex space has the HBEP, and examples are known of non-locally convex spaces with the HBEP. In this work we study the relationship between the HBEP and local convexity. We prove in particular that a complete linear metric space with a basis must be locally convex if it has the HBEP.

We then study in detail a class of spaces of analytic functions. Let $H^{p,q}$ denote the space of functions analytic in the open unit disc with

$$\|f\|_{p,q} = \left\{ \int_0^1 M_p(f;r)^q r\,dr \right\}^{1/q} < \infty \quad .$$

We show that these spaces are not locally convex when $\min(p,q) < 1$, and characterize their duals in the cases $0 < p,q \leq 1$ and $1 < p,q < \infty$. We obtain a partial characterization when $1 < q < \infty$, $0 < p \leq 1$. We consider the subspaces consisting of all functions in $H^{p,q}$ whose Taylor coefficients vanish off a fixed lacunary sequence of non-negative integers, and determine when every continuous linear

functional on such a subspace can be extended to the whole space. When $q = \min(p,q) < 1$ we use a mapping technique to show that $H^{p,q}$ contains proper closed subspaces which are weakly dense, and to obtain information about the concavity of the unit ball in $H^{p,q}$. Finally we apply our methods to the problem of obtaining proper, closed, weakly dense subspaces in some other non-locally convex spaces of analytic functions.

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INTRODUCTION

We say a linear topological space has the Hahn-Banach Extension Property (HBEP) if every continuous linear functional on a closed subspace has a continuous linear extension to the whole space. The Hahn-Banach theorem guarantees that every locally convex space has the HBEP (Kelley-Namioka [8, page 117]).

On the other hand if E is a linear space of uncountable Hamel dimension, the strongest vector topology σ on E is not locally convex ([8, pages 42, 53]). Since σ is stronger than the weak topology induced on E by its algebraic dual, every linear functional on E is σ-continuous, and it follows easily that (E, σ) has the HBEP (this example is due to Mr. A. Shuchat).

Local convexity is thus a stronger assumption than the HBEP, and it is natural to seek restrictions on E under which the two properties will coincide. We consider this question in Chapter II, and show that the HBEP and local convexity are equivalent for complete linear metric spaces (F-spaces) with bases. In Chapter I we record some notation and terminology, and prove some results about the Mackey topology of a pseudo-metrizable space.

In Chapter III we study in detail a particular class of spaces. Let $H^{p,q}$ $(0 < p, q \leq \infty)$ be the space of functions analytic in

$|z| < 1$ such that

$$(0.1) \qquad \|f\| = \|f\|_{p,q} = \left\{ \int_0^1 \left[\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right]^{q/p} r \, dr \right\}^{1/q}$$

is finite, with appropriate modifications when p or $q = \infty$. Thus $H^{p,\infty}$ is the Hardy space H^p, while $H^{p,p}$ is the corresponding "area space." When $1 \leq p,q \leq \infty$, $H^{p,q}$ is a Banach space, with the norm defined by (0.1). Our main interest, however, is in the case $m = \min(p,q) < 1$, when $\| \ \|^m$ (rather than $\| \ \|$) obeys the triangle inequality and induces a complete metric. We observe various failures of the Hahn-Banach theorem, and prove in particular that $H^{p,q}$ $(m < 1)$ is not locally convex.

The case $H^p = H^{p,\infty}$ $(0 < p < 1)$ has been treated by Duren, Romberg, and Shields [6], and we extend some of their results to $H^{p,q}$. Specifically, if $0 < p,q \leq 1$ we find that the dual of $H^{p,q}$ is linearly homeomorphic with O^σ, the space of functions g analytic in $|z| < 1$ whose fractional derivative $g^{[1/\sigma]}$ has growth $O((1 - |z|)^{-1})$, where

$$(0.2) \qquad 1/\sigma = 1/p + 1/q \quad ,$$

$$(0.3) \qquad \|g\|_{O^\sigma} = \sup_{|z|<1} |g(z)|(1 - |z|) \quad ,$$

and the pairing between the spaces is

$$(0.4) \qquad \langle f, g \rangle = \lim_{r \to 1-} \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) g(re^{-i\theta}) \, d\theta \quad .$$

We show that $H^{p,q}$ $(0 < p,q \leq 1)$ is a dense subspace of the Banach space B^{σ} of functions f analytic in $|z| < 1$ with

$$(0.5) \qquad \|f\|_{B^{\sigma}} = \iint_{|z|<1} |f(z)|(1 - |z|)^{1/\sigma - 2}\, dx\, dy < \infty \quad ,$$

and show that (0.5) induces the Mackey topology on $H^{p,q}$. When $1 < q < \infty$, $0 < p \leq 1$ we obtain a partial description of the duality.

We next associate to each lacunary sequence $\mathbf{E}$ of non-negative integers the closed subspace $H^{p,q}(\mathbf{E})$ consisting of all $f(z) = \Sigma a_n z^n$ in $H^{p,q}$ with $a_n = 0$ when $n \notin \mathbf{E}$. When $0 < p < 1$ these subspaces have continuous linear functionals not extendable to the whole space, while if $1 \leq p < \infty$ the situation is entirely different; $H^{p,q}(\mathbf{E})$ is complemented in $H^{p,q}$.

When $q = \min(p,q) < 1$ we observe another way in which the Hahn-Banach theorem may fail: $H^{p,q}$ contains proper, closed subspaces which are dense in the weak topology. In addition, our analysis yields an estimate of the concavity of the unit ball. We conclude Chapter III by showing that the dual of $H^{p,q}$ $(1 < p,q < \infty)$ is linearly homeomorphic with $H^{p',q'}$, the prime denoting conjugate index.

In Chapter IV we apply some of the methods used in Chapter III to describe the failure of the Hahn-Banach theorem in two other classes of non-locally convex spaces of analytic functions.

CHAPTER 1

LINEAR TOPOLOGICAL SPACES

In this chapter we set out our notation and terminology concerning linear topological spaces, and prove some useful results about the Mackey topology in a pseudo-metrizable space. A basic reference will be Kelley-Namioka [8].

1. Linear topological spaces.

Let $\mathbf{E}$ be a linear topological space over the real or complex field. A local base for $\mathbf{E}$ is a base for the neighborhood system of the origin. $\mathbf{E}$ is locally convex if there is a local base consisting of convex sets. A set A in $\mathbf{E}$ is circled if $x \in A$ implies $\alpha x \in A$ whenever $|\alpha| = 1$. A locally convex space always has a local base $\mathcal{U}$ of convex, circled neighborhoods. For each U in $\mathcal{U}$ the Minkowski functional

$$P_U(x) = \inf\{\alpha > 0 : \alpha^{-1} x \in \overline{U}\} \qquad (x \in \mathbf{E})$$

is finite, subadditive, and absolutely homogeneous. A net $\{x_\alpha\}$ converges to 0 in $\mathbf{E}$ if and only if $P_U(x_\alpha) \to 0$ for each U in $\mathcal{U}$.

A p-seminorm on $\mathbf{E}$ is a non-negative functional $\| \ \|$ which is subadditive and absolutely p-homogeneous $(0 < p \leq 1)$:

$$\|\alpha x\| = |\alpha|^p \|x\| \qquad (x \in \mathbf{E}, \ \alpha \in \text{scalars}) \ .$$

If $x = 0$ whenever $\|x\| = 0$ we call $\| \ \|$ a p-norm. A 1-seminorm is simply called a seminorm; a similar convention is made for 1-norms. A space whose topology is induced by a single p-norm is called a p-normed space.

A subset A of $\mathbf{E}$ is said to be bounded if for each neighborhood $\mathbf{U}$ of 0, $\lambda \mathbf{U} \supset A$ for some scalar λ. $\mathbf{E}$ is said to be locally bounded if it has a bounded neighborhood of 0. A theorem of Kolmogorov (Kelley-Namioka [8, page 44]) states that a Hausdorff linear topological space is normable if and only if it has a bounded convex neighborhood of 0. More generally, we have

Theorem 1.1 (Rolewicz [12]). A Hausdorff linear topological space is locally bounded if and only if it is p-normed for some $0 < p \leq 1$.

2. Pairing and duality.

The space of linear functionals on a linear space $\mathbf{E}$ is called the algebraic dual of $\mathbf{E}$, and denoted by $\mathbf{E}^*$. If $\mathbf{E}$ is a linear topological space the space of continuous linear functionals on $\mathbf{E}$ is called the dual of $\mathbf{E}$, and denoted by $\mathbf{E}'$. We sometimes refer to $\mathbf{E}'$ as the τ-dual of $\mathbf{E}$, where τ denotes the vector topology on $\mathbf{E}$.

If $\mathbf{E}$ is a linear space and $\mathbf{F}$ is a subspace of $\mathbf{E}^*$, we can consider the collection of locally convex topologies τ on $\mathbf{E}$ yielding $\mathbf{F}$ as the τ-dual of $\mathbf{E}$. There is a weakest such topology, $w(\mathbf{E},\mathbf{F})$, called the weak topology of the dual pair $(\mathbf{E},\mathbf{F})$; and there is also a strongest such topology, $m(\mathbf{E},\mathbf{F})$, called the Mackey topology of $(\mathbf{E},\mathbf{F})$. Both these topologies are Hausdorff if and only if $\mathbf{F}$ separates points of $\mathbf{E}$. The topology τ of a linear topological space $\mathbf{E}$ is said to be Mackey if $\tau = m(\mathbf{E},\mathbf{E}')$ Every pseudo-metrizable locally convex topology is Mackey (Kelley-Namioka [8, page 210]). A useful consequence of this is

Theorem 1.2. If $\mathbf{E}$ is a pseudo-metrizable linear topological space then the convex hulls of the sets in a local base form a local base for $m(\mathbf{E},\mathbf{E}')$. Consequently, $m(\mathbf{E},\mathbf{E}')$ is pseudo-metrizable and at least as weak as the original topology

Proof. A vector topology is pseudo-metrizable if and only if it has a countable local base (Kelley-Namioka [8, page 48]). Let $\{U_n\}$ be a countable local base for τ_0, the original topology of $\mathbf{E}$. Let V_n be the convex hull of U_n. It is easy to check that $\{V_n\}$ is a local base for a vector topology τ on $\mathbf{E}$ which is at least as weak as τ_0. Thus every τ-continuous linear functional is τ_0-continuous. Conversely, each τ_0-continuous linear functional is bounded on some neighborhood U_n, hence on V_n, so it must

be τ_0-continuous (a linear functional is continuous on a linear topological space if and only if it is bounded on a neighborhood of 0 [8, page 37]). Therefore, the τ-dual of E is E'. Since τ has a countable local base it is pseudo-metrizable, hence Mackey. Thus $\tau = m(E,E')$. |||

The next result describes for pseudo-metrizable spaces the connection between the Mackey topology and the HBEP.

Theorem 1.3. Let E be a pseudo-metrizable space and F a subspace. The following are equivalent:

(i) Every continuous linear functional on F has a continuous linear extension to E.

(ii) $m(F,F')$ is the restriction of $m(E,E')$ to F.

Proof. Let F^e denote those continuous linear functionals on F having continuous linear extensions to E. It follows from the Hahn-Banach theorem that every $m(E,E')$-continuous linear functional on F is in F^e. Conversely, each member of F^e is $m(E,E')$-continuous, so F^e is the $m(E,E')$-dual of F. By Theorem 1.2, $m(E,E')$ is pseudo-metrizable, hence its restriction to F is pseudo-metrizable, therefore Mackey, and therefore equal to $m(F,F^e)$. If (i) holds, then $F = F^e$ and we have (ii).

Conversely, if (ii) is true, then every linear functional λ continuous on F is $m(F,F')$, hence $m(E,E')$-continuous.

By the Hahn-Banach theorem λ has an extension to $\mathbf{E}$ that is $m(\mathbf{E}, \mathbf{E}')$-continuous, hence continuous in the original topology, and we have (i). |||

This theorem is true in general linear topological spaces if "Mackey topology" is replaced by "weak topology." The difficulty is that the restriction of a Mackey topology to a subspace need not be Mackey. In fact, every locally convex space is linearly homeomorphic with a subspace of a space whose topology is Mackey [8, page 173].

3. **F-spaces**.

An **F**-space is a complete linear metric space over the real or complex field. If $\mathbf{E}$ is an **F**-space, there is a complete metric d on $\mathbf{E}$ which is translation invariant:

$$d(x,y) = d(x+z,\ y+z) \qquad (x,y,z \text{ in } \mathbf{E}) \ ,$$

circularly symmetric:

$$d(\omega x,\ \omega y) = d(x,y) \qquad (|\omega| = 1,\ x,y \text{ in } \mathbf{E}) \ ,$$

and which induces the original topology [8, Sections 5 and 6]. The functional

$$\|x\| = d(x,0) \qquad (x \text{ in } \mathbf{E})$$

is called an **F**-norm, so we may write

$$d(x,y) = \|x - y\| \ .$$

Two F-norms $\| \ \|_1$ and $\| \ \|_2$ are <u>equivalent</u> provided

$$\lim_n \|x_n\|_1 = 0 \quad \text{iff} \quad \lim_n \|x_n\|_2 = 0 \quad .$$

If $\| \ \|$ is an F-norm on E, the continuity of scalar multiplication insures that

$$\|x\|_1 = \sup \|tx\| \qquad (0 < t < 1)$$

defines an equivalent F-norm which is <u>monotone</u>:

$$\|t_1 x\|_1 \leq \|t_2 x\|_2 \qquad (t_1 \leq t_2, \quad x \in E) \quad .$$

The importance of F-spaces derives from the following:

<u>Theorem</u> 1.4. (the uniform boundedness principle) [4, page 54]. Let $\{T_n\}$ be a sequence of continuous linear maps of an F-space E into an F-space F, and let $Tx = \lim_n T_n x$ exist for each x in E. Then T is a continuous linear map of E into F.

<u>Theorem</u> 1.5. (the interior mapping principle) [4, page 55]. Under a continuous linear map of one F-space onto all of another, the image of every open set is open.

A sequence $\{e_n\}$ in an F-space E is called a <u>basis</u> if to each x in E there corresponds a unique sequence $\{\xi_n(x)\}$ of scalars such that the series $\Sigma \xi_n(x) e_n$ converges in E to x. ξ_n is called the nth coordinate functional for $\{e_n\}$. Each ξ_n

is clearly linear, and we will see shortly that it is continuous. Thus an F-space with a basis always possesses a separating family of continuous linear functionals. A sequence $\{e_n\}$ is called basic if it is a basis for the subspace it spans. The following result is well known:

Theorem 1.6. Suppose E is an F-space with a basis $\{e_n\}_{n\geq 0}$. Let $\|\ \|$ be the F-norm in E. Then

$$\|x\|_1 = \sup_{0\leq n\leq m} \|\sum_n^m \xi_k(x)e_k\|$$

defines an F-norm on E equivalent to $\|\ \|$.

Proof. Clearly, $\|\ \|_1$ is a non-negative, subadditive functional on E, and $\|x\| \leq \|x\|_1$ for each x in E. If we can show that $(E, \|\ \|_1)$ is complete, an application of the interior mapping principle to the identity map $i : (E, \|\ \|_1) \to (E, \|\ \|)$ will show that the norms are equivalent.

Suppose $\{x_N\}_{N\geq 0}$ is Cauchy in $\|\ \|_1$. For a fixed non-negative integer j,

$$\|[\xi_j(x_N) - \xi_j(x_M)]e_j\|_1 \leq \|x_N - x_M\|_1 \quad ,$$

so $\{\xi_j(x_N)\}_{N\geq 0}$ is a Cauchy sequence of scalars. Thus the limit

$$\xi_j = \lim_N \xi_j(x_N)$$

exists. $(j = 0, 1, 2, \ldots)$. We claim the sequence $\{\Sigma_0^n \xi_j e_j\}_{n \geq 0}$ is $\| \ \|_1$-Cauchy. Indeed, if $N \geq 0$,

$$\| \sum_{j=n}^{m} \xi_j e_j \|_1 \leq \| \sum_{j=n}^{m} [\xi_j - \xi_j(x_N)]e_j \|_1 + \| \sum_{j=n}^{m} \xi_j(x_N)e_j \|_1$$

$$= \lim_M \| \sum_{j=n}^{m} [\xi_j(x_M) - \xi_j(x_N)]e_j \|_1 + \| \sum_{j=n}^{m} \xi_j(x_N)e_j \|_1$$

$$\leq \overline{\lim_M} \| x_M - x_N \|_1 + \| \sum_{j=n}^{m} \xi_j(x_N)e_j \|_1 \quad .$$

Given $\varepsilon > 0$, since $\{x_N\}$ is Cauchy in $\| \ \|_1$ the first term can be made less than $\varepsilon/2$ for some N. Since $\Sigma \xi_j(x_N)e_j$ is $\| \ \|$-convergent we have for some $K > 0$,

$$\| \sum_{j=k}^{\ell} \xi_j(x_N)e_j \| < \varepsilon/2$$

when $\ell \geq k > K$. Thus when $m \geq n > K$,

$$\| \sum_{j=n}^{m} \xi_j(x_N)e_j \|_1 = \sup \{ \| \sum_{j=k}^{\ell} \xi_j(x_N)e_j \| : n \leq k \leq \ell \leq m \}$$

$$\leq \varepsilon/2 \quad .$$

Thus $\| \Sigma_n^m \xi_j e_j \|_1 < \varepsilon$ when $m \geq n > K$, and our assertion is proved. Since $\| \ \| \leq \| \ \|_1$ and $\mathbf{E}$ is complete in $\| \ \|$, the series $\Sigma_0^\infty \xi_j e_j$ converges in $\| \ \|$ to an element x in $\mathbf{E}$. We claim that $\lim_N \| x_N - x \|_1 = 0$. For

$$\|x_N - x\|_1 = \sup_{n,m} \Big\| \sum_{j=n}^{m} [\xi_j(x_N) - \xi_j]e_j \Big\|$$

$$= \sup_{n,m} \lim_{M} \Big\| \sum_{j=n}^{m} [\xi_j(x_N) - \xi_j(x_M)]e_j \Big\|$$

$$\leq \overline{\lim_{M}} \sup_{n,m} \Big\| \sum_{j=n}^{m} [\xi_j(x_N) - \xi_j(x_M)]e_j \Big\|$$

$$= \overline{\lim_{M}} \|x_N - x_M\|_1 \quad .$$

As before, the last term tends to zero. Thus $\| \ \|_1$ is complete. |||

The F-norm $\| \ \|_1$ has the following useful property:

(*) $$\Big\| \sum_{n}^{m} \xi_j(x)e_j \Big\|_1 \leq \|x\|_1 \qquad (x \text{ in } \mathbf{E}, \ 0 \leq n \leq m) \quad .$$

Moreover, if $\| \ \|$ is monotone so is $\| \ \|_1$. The following is an immediate consequence of (*):

Corollary 1.1. The coordinate functionals for a basis in an F-space are continuous.

CHAPTER 2

In this chapter we show that the HBEP and local convexity are equivalent in F-spaces with bases.

1. Mackey topology and basic sequences.

We begin by characterizing locally convex F-spaces in terms of their Mackey topologies.

Lemma 2.1. Let $\mathbf{E}$ be an F-space. The following are equivalent:

(i) $\mathbf{E}$ is not locally convex.

(ii) The topology of $\mathbf{E}$ is strictly stronger than $m(\mathbf{E},\mathbf{E}')$.

(iii) $m(\mathbf{E},\mathbf{E}')$ is not complete.

Proof. (i) $\rightarrow$ (ii): By Theorem 1.2 the original topology τ_0 on $\mathbf{E}$ is no weaker than $m(\mathbf{E},\mathbf{E}')$. Thus if τ_0 were not strictly stronger than $m(\mathbf{E},\mathbf{E}')$, the two topologies would coincide, and τ_0 would be locally convex.

(ii) $\rightarrow$ (iii): Suppose $m(\mathbf{E},\mathbf{E}')$ is complete. Since it is no stronger than τ_0, the identity map $i : (\mathbf{E}, \tau_0) \rightarrow (\mathbf{E}, m(\mathbf{E},\mathbf{E}'))$ is continuous, hence open by the interior mapping principle. Thus i^{-1} is continuous so τ_0 coincides with $m(\mathbf{E},\mathbf{E}')$.

(iii) $\rightarrow$ (i): If $m(\mathbf{E},\mathbf{E}')$ is not complete it cannot coincide with τ_0 (which is complete). By Theorem 1.2 τ_0 must be strictly stronger than $m(\mathbf{E},\mathbf{E}')$. Therefore, τ_0 is not locally convex, because $m(\mathbf{E},\mathbf{E}')$ is the strongest locally convex topology on $\mathbf{E}$ whose dual is $\mathbf{E}'$. |||

In the next lemma we isolate the crucial step in the proof of our main theorem.

Lemma 2.2. Suppose $\mathbf{E}$ is an F-space and $\{f_n\}$ is a basic sequence in $\mathbf{E}$ with

$$\inf_n \|f_n\| = \delta > 0 \quad , \tag{2.1}$$

and

$$\lim_n f_n = 0 \quad \text{in} \quad m(\mathbf{E},\mathbf{E}') \quad . \tag{2.2}$$

Then there is a continuous linear functional on the closed linear span of $\{f_n\}$ that cannot be extended to $\mathbf{E}$.

Proof. Let N be the closed linear span of $\{f_n\}$. If $f \in N$ then $f = \Sigma \phi_n(f) f_n$, where ϕ_n is the nth coordinate functional for $\{f_n\}$. We claim that $\lim \phi_n(f) = 0$. If not there is an integer $N > 0$ and a subsequence $\{n_k\}$ of non-negative integers such that

$$|\phi_{n_k}(f)| > 1/N \qquad (k = 0, 1, \ldots) \quad .$$

By Theorem 1.6 and the remark following it we can find a monotone F-norm $\| \ \|_1$ in N which is equivalent to $\| \ \|$ and has property (*) relative to the basis $\{ f_n \}$. Then whenever $n \leq n_k \leq m$,

$$\begin{aligned} \| \sum_n^m \phi_j(f) f_j \|_1 &\geq \| \phi_{n_k}(f) f_{n_k} \|_1 \\ &= \| \, | \phi_{n_k}(f) | \, f_{n_k} \|_1 \\ &\geq \| N^{-1} f_{n_k} \|_1 \\ &\geq N^{-1} \| f_{n_k} \|_1 \\ &\geq N^{-1} \delta_1 \quad , \end{aligned}$$

where $\delta_1 = \inf \| f_n \|_1 > 0$ (the next-to-last line follows from the subadditivity of $\| \ \|_1$, the line above from monotonicity). But this contradicts the convergence of $\Sigma \phi_n(f) f_n$, and our assertions is proved.

Let V_n be the convex hull of $\{ x \in \mathbf{E} : \|x\| < 1/n \}$ $(n = 1, 2, \ldots)$. By Theorem 1.2, $V_1 \supset V_2 \supset \ldots$ is a local base for $m(\mathbf{E}, \mathbf{E}')$. If p_n is the Minkowski functional of V_n $(n = 1, 2, \ldots)$ then $p_1 \leq p_2 \leq \ldots$. From (2.2) $\lim_n p_k(f_n) = 0$ $(k = 1, 2, \ldots)$, so for some subsequence $\{ n_k \}$ we have

$$\lim_k p_k(f_{n_k}) = 0 \quad .$$

Choose a sequence of scalars $\{\lambda_n\}$ in ℓ^1 with

$$\lambda_{n_k} \neq O(p_k(f_{n_k})) \quad . \tag{2.3}$$

Since $\{\phi_n(f)\}$ tends to 0 for each f in N,

$$\lambda(f) = \sum_0^\infty \lambda_n \phi_n(f) \qquad (f \text{ in } N)$$

defines a linear functional on N which is continuous by the principle of uniform boundedness.

We claim λ has no continuous linear extension to **E**. For if $\tilde{\lambda}$ were such an extension it would be $m(\mathbf{E},\mathbf{E}')$-continuous, so for some integer $N > 0$ and some constant $C > 0$ we would have

$$|\tilde{\lambda}(x)| \leq C p_N(x) \qquad (x \text{ in } \mathbf{E}) \quad .$$

But $\tilde{\lambda}(f_n) = \lambda(f_n) = \lambda_n$, which implies

$$|\lambda_n| = O(p_N(f_n)) \qquad (n \to \infty) \quad .$$

Since $p_k \geq p_N$ whenever $k \geq N$, this yields

$$|\lambda_{n_k}| = O(p_k(f_{n_k})) \quad ,$$

which contradicts (2.3). |||

2. HBEP in F-spaces with bases.

We begin with a method for constructing basic sequences from a given basis.

Lemma 2.3. If $\{e_n\}_{n\geq 0}$ is a basis for an F-space E, $\{n_k\}_{k\geq 0}$ is an increasing sequence of non-negative integers, and $\{\alpha_n\}_{n\geq 0}$ is a sequence of scalars, then

$$f_k = \sum_{n=n_k+1}^{n_{k+1}} \alpha_n e_n \qquad (k = 0, 1, 2, \ldots)$$

is a basic sequence.

Proof. We may suppose the F-norm $\|\ \|$ in E is monotone and has property (*) of Chapter 1. Suppose x is in N, the closed subspace of E spanned by $\{f_k\}_{k\geq 0}$. Then $x = \lim x_n$ where

$$x_n = \sum_{k=0}^{\infty} \beta_{n,k} f_k \qquad (n = 0, 1, 2, \ldots),$$

and for each fixed n, $\beta_{n,k} = 0$ except for finitely many k. Since $\|\ \|$ has property (*),

$$\|(\beta_{n,k} - \beta_{m,k})f_k\| \leq \|x_m - x_n\| \quad ,$$

so by the continuity of scalar multiplication $\{\beta_{n,k}\}_{n\geq 0}$ is a Cauchy sequence (k = 0, 1, 2, ...). Let

$$\beta_k = \lim_n \beta_{n,k} \qquad (k = 0, 1, 2, \ldots) \ .$$

If $n_k + 1 \leq j \leq n_{k+1}$, then for each $n \geq 0$

$$\xi_j(x_n) = \xi_j(\beta_{n,k} f_k) = \beta_{n,k} \xi_j(f_k) \ ,$$

where ξ_j is the jth coordinate functional for $\{e_n\}$. Thus

$$\sum_{j=n_k+1}^{n_{k+1}} \xi_j(x) e_j = \beta_k \sum_{j=n_k+1}^{n_{k+1}} \xi_j(f_k) e_j$$

$$= \beta_k f_k$$

$(k = 0, 1, 2, \ldots)$, so properly inserting parentheses in the convergent series $x = \Sigma \xi_j(x) e_j$ yields the convergent series $x = \Sigma \beta_k f_k$. Since $\|\beta_k f_k\| \leq \|x\|$ (from (*) and the definition of f_k) we see that the latter representation of x is unique, so $\{f_k\}$ is a basis for N. |||

We can now prove the main result of this chapter.

Theorem 2.1. If **E** is a non-locally convex **F**-space with a basis then **E** does not have the HBEP.

Proof. Let $\{e_n\}$ be the basis with coordinate functionals $\{\xi_n\}$. Suppose the **F**-norm $\| \ \|$ in **E** is monotone and obeys (*) of Chapter 1. By Lemma 2.1, $m(\mathbf{E}, \mathbf{E}')$ is not complete. Now

suppose we can find an increasing sequence $\{n_k\}$ of non-negative integers, and a scalar sequence $\{\xi_j\}$ such that

$$f_k = \sum_{j=n_k+1}^{n_{k+1}} \xi_j e_j \qquad (k = 0, 1, 2, \ldots)$$

satisfies (2.1) and (2.2). By Lemma 2.3 $\{f_k\}$ is a basic sequence, so Lemma 2.2 insures that its closed linear span has a continuous linear functional that cannot be extended to E.

We finish the proof by showing that if no such $\{\xi_j\}$ exists then $m(E,E')$ must be complete, contradicting our previous assertion. Suppose, then, that $\{\Sigma_0^k \xi_j e_j\}_{k\geq 0}$ is $\|\ \|$-Cauchy whenever it is $m(E,E')$-Cauchy. Let $\{x_n\}$ be an $m(E,E')$-Cauchy sequence in E. Since the coordinate functionals are continuous there is a sequence $\{\xi_j\}$ of scalars such that

$$\lim_n \xi_j(x_n) = \xi_j \qquad (j = 0, 1, 2, \ldots) .$$

Let V_n denote the convex hull of $\{x \in E : \|x\| < 1/n\}$, and let p_n be the Minkowski functional of V_n $(n = 1, 2, \ldots)$. We claim that each p_n inherits the property (*) possessed by $\|\ \|$. Temporarily granting this, whenever $p \in \{p_n\}$,

$$p\Big(\sum_{j=k}^{\ell} \xi_j e_j\Big) \le p\Big(\sum_{j=k}^{\ell} [\xi_j - \xi_j(x_n)]e_j\Big) + p\Big(\sum_{j=k}^{\ell} \xi_j(x_n)e_j\Big)$$

$$= \lim_m p\Big(\sum_{j=k}^{\ell} [\xi_j(x_m) - \xi_j(x_n)]e_j\Big) + p\Big(\sum_{j=k}^{\ell} \xi_j(x_n)e_j\Big)$$

$$\le \overline{\lim_m}\, p(x_m - x_n) + p\Big(\sum_{j=k}^{\ell} \xi_j(x_n)e_j\Big) \quad .$$

From Theorem 1.2 a sequence is $m(\mathbf{E},\mathbf{E}')$-Cauchy if and only if it is p-Cauchy for each $p \in \{p_n\}$. Given p and $\varepsilon > 0$ we may choose $N > 0$ such that

$$\overline{\lim_m}\, p(x_m - x_N) < \varepsilon/2 \quad .$$

Since $\Sigma \xi_j(x_N)e_j$ converges in $\mathbf{E}$ it converges in the weaker topology $m(\mathbf{E},\mathbf{E}')$, so for some $K > 0$

$$p\Big(\sum_{j=k}^{\ell} \xi_j(x_N)e_j\Big) < \varepsilon/2 \quad ,$$

hence

$$p\Big(\sum_{k}^{\ell} \xi_j\, e_j\Big) < \varepsilon$$

when $\ell \ge k > K$. Thus $\{\Sigma_0^k \xi_j\, e_j\}_{k\ge 0}$ is $m(\mathbf{E},\mathbf{E}')$-Cauchy, so $\|\ \|$-Cauchy by the hypothesis of this paragraph. Therefore, $\Sigma \xi_j e_j$ is $\|\ \|$-convergent, hence

$m(\mathbf{E}, \mathbf{E}')$-convergent to some x in $\mathbf{E}$. Moreover,

$$p(x - x_n) = \sup_{0 \le k \le \ell} p\Big(\sum_{j=k}^{\ell} [\xi_j - \xi_j(x_n)]e_j\Big)$$

$$\le \overline{\lim_m} \sup_{0 \le k \le \ell} p\Big(\sum_{j=k}^{\ell} [\xi_j(x_m) - \xi_j(x_n)]e_j\Big)$$

$$\le \overline{\lim_m} p(x_m - x_n) \quad ,$$

so $\{x_n\}$ is $m(\mathbf{E}, \mathbf{E}')$ convergent to x, and we see that $m(\mathbf{E}, \mathbf{E}')$ is complete.

It remains to show that each p in $\{p_n\}$ has property (*). Suppose p is the Minkowski functional of $V \in \{V_n\}$. If $\alpha > p(x)$ then $\alpha^{-1}x \in V$, so

$$\alpha^{-1}x = \sum_1^n \lambda_i x_i$$

where $\lambda_i \ge 0$, $\|x_i\| < 1/n$ $(i = 1, 2, \ldots, M)$ and $\sum_1^M \lambda_i = 1$. Thus

$$\alpha^{-1} \sum_{j=k}^{\ell} \xi_j(x)e_j = \sum_{i=1}^{M} \lambda_i \Big(\sum_{j=k}^{\ell} \xi_j(x_i)e_j\Big) \quad .$$

Since $\| \ \|$ obeys (*) we have

$$\Big\| \sum_{j=k}^{\ell} \xi_j(x_i)e_j \Big\| \le \|x_i\| < 1/n \qquad (i = 1, 2, \ldots, M) \quad ,$$

so $\alpha^{-1}\sum_{k}^{\ell} \xi_j(x)e_j \in V$, hence

$$p(\sum_{k}^{\ell} \xi_j(x)e_j) < \alpha \quad .$$

This holds for each $\alpha > p(x)$, so

$$p(\sum_{k}^{\ell} \xi_j(x)e_j) \leq p(x) \quad ,$$

and p obeys (*). |||

3. Related results.

We conjecture that the HBEP fails in every non-locally convex F-space. According to Theorem 1.2 such a space always contains a sequence which tends to 0 in the Mackey topology but not in the original topology. If it were true that every such sequence contained a basic subsequence, the conjecture would follow immediately from Lemma 2.2. Bessaga and Pelczynski [2] have proved a result in this direction: If **E** is locally bounded and $\{f_n\}$ is a sequence in **E** with $\inf\|f_n\| > 0$ and $\lim_n \xi_j(f_n) = 0$ $(j = 0, 1, 2, \ldots)$, then $\{f_n\}$ contains a basic subsequence.

An immediate consequence of this and Lemma 2.2 is

Theorem 2.2. Suppose **E** is an F-space embedded in a locally bounded space with a basis. If **E** is not locally convex then **E** does not have the HBEP.

We close this chapter by describing another consequence of Lemma 2.2.

Proposition 2.1. Let $\mathbf{E}$ be an F-space with basis $\{e_n\}$. Then a subsequence $\{e_{n_k}\}$ converges to 0 in $\mathbf{E}$ if and only if it converges to 0 in $m(\mathbf{E},\mathbf{E}')$.

Proof. Since the original topology on $\mathbf{E}$ is at least as strong as $m(\mathbf{E},\mathbf{E}')$, any sequence converging to 0 in $\mathbf{E}$ must converge to 0 in $m(\mathbf{E},\mathbf{E}')$. Conversely, suppose $\{e_{n_k}\}$ does not converge to 0 in $\mathbf{E}$. Passing to a subsequence if necessary, we may assume $\|e_{n_k}\| > \delta > 0$ $(k = 0, 1, 2, \ldots)$. If $e_{n_k} \to 0$ in $m(\mathbf{E},\mathbf{E}')$ then by Lemma 2.2 the closed linear span N of $\{e_{n_k}\}$ has a continuous linear functional that is not extendable to $\mathbf{E}$. But N is clearly a complemented subspace of $\mathbf{E}$, so a continuous linear functional on N may be extended to $\mathbf{E}$ by simply defining it to be zero on the closed subspace of $\mathbf{E}$ complementary to N. Thus $e_{n_k} \not\to 0$ in $m(\mathbf{E},\mathbf{E}')$. |||

CHAPTER III

MIXED-NORM SPACES OF ANALYTIC FUNCTIONS

In this chapter we study the space $H^{p,q}$ defined in the Introduction. Since we do not know whether $H^{p,q}$ has a basis when $0 < p \leq 1$, Theorem 2.1 cannot be immediately applied. In any case our main interest is the study of some concrete problems suggested in these spaces by the more general questions discussed previously.

1. Notation and preliminaries

If f is analytic in the unit disc $|z| < 1$, the angular means

$$M_p(f;r) = \frac{1}{2\pi}\left\{\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right\}^{1/p} \qquad (0 < p < \infty).$$

and

$$M_\infty(f;r) = \sup_\theta |f(re^{i\theta})|$$

are increasing functions of r $(0 \leq r < 1)$. Recall from the Introduction that $H^{p,q}$ is the class of functions f analytic in $|z| < 1$ with

$$\|f\|_{p,q} = \{\int_0^1 M_p(f;r)^q \, r\,dr\}^{1/q} < \infty$$

with appropriate modifications if p or $q = \infty$. B^p is the class of functions f analytic in $|z| < 1$ such that

$$\|f\|_{B^p} = \int_0^1 M_1(f;r)(1 - r)^{1/p-2} \, r\,dr < \infty \quad .$$

We write $H^p = H^{p,\infty}$ and $\|f\|_p = \|f\|_{p,\infty}$. The following estimate is due to Hardy and Littlewood [7, Theorem 31].

Theorem 3.1. If $0 < p < s \leq \infty$, $f \in H^p$, $\alpha = 1/p - 1/s$ and $\lambda \geq p$, then

$$\int_0^1 M_s(f;r)^\lambda (1 - r)^{\lambda\alpha - 1} dr \leq K\|f\|_p^\lambda \tag{3.1}$$

where K is independent of f.

We remark that when $\lambda = s = 1$ this theorem shows that H^p is continuously embedded in B^p.

Suppose $\beta > 0$ and $f(z) = \Sigma a_n z^n$ is analytic in $|z| < 1$. Following Duren, Romberg, and Shields [6], we define

$$f^{[\beta]}(z) = \sum_{n=0} \frac{\Gamma(n + 1 + \beta)}{\Gamma(n + 1)} a_n z^n \quad ,$$

and

$$f_{[\beta]}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1+\beta)} a_n z^n$$

to be respectively the fractional derivative and fractional integral of f of order β. These definitions differ by an unimportant factor of z^{β} from those of Hardy and Littlewood [7, pages 409, 424]. Fractional integrals and derivatives give a way of passing from one B^p space to another. More specifically we have the following theorem of Duren, Romberg, and Shields [6, Theorem 5]:

Theorem 3.2. Suppose $\alpha > -1$ and $\beta > 0$. If f is analytic in $|z| < 1$, then

$$\text{(i)} \quad \int_0^1 M_1(f^{[\beta]};r)(1-r)^{\alpha+\beta}\,dr \leq K \int_0^1 M_1(f;r)(1-r)^{\alpha}\,dr$$

and

$$\text{(ii)} \quad \int_0^1 M_1(f_{[\beta]};r)(1-r)^{\alpha}\,dr \leq K \int_0^1 M_1(f;r)(1-r)^{\alpha+\beta}\,dr \quad ,$$

where K is independent of f.

In other words if $0 < p < q < 1$ and $\beta = 1/p - 1/q$ then $f \to f^{[\beta]}$ is a linear homeomorphism of B^q onto B^p.

We will need to know the dual space of B^p $(0 < p < 1)$. In general if f and g are analytic in $|z| < 1$ we write

$$\langle f, g \rangle = \lim_{r \to 1-} \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) g(re^{-i\theta})\,d\theta \quad .$$

Theorem 3.3. (Duren, Romberg, and Shields [6, Theorem 8]) If λ is a continuous linear functional on B^p $(0 < p < 1)$ then there is a unique g in O^p (for definition see Introduction) such that

$$\lambda(f) = \langle f, g \rangle \qquad (f \in B^p) \qquad . \tag{3.2}$$

Conversely, if $g \in O^p$ then $\langle f, g \rangle$ exists for each f in B^p and (3.2) defines a continuous linear functional on B^p. The correspondence $g \leftrightarrow \lambda$ is a linear homeomorphism between $(B^p)'$ and O^p.

Actually in [6] this result is phrased in terms of Lipschitz, rather than growth conditions. The statements are, however, equivalent (see [6, Theorems B and C], also [7, Theorem 43]).

We close this section with a useful estimate.

Lemma 3.1. (Duren [5, Chapter 4]) If $\gamma > 1$ then

$$\int_0^{2\pi} |1 - re^{i\theta}|^{-\gamma} d\theta = O((1 - r)^{1-\gamma}) \qquad .$$

Proof: Since

$$|1 - re^{i\theta}|^2 \geq (1 - r)^2 + 4\pi^{-2}\theta^2 r \qquad (|\theta| \leq \pi) \quad ,$$

we have, after a change of variable,

$$\int_0^{2\pi} \frac{d\theta}{(1 - re^{i})^{\gamma}} \leq \frac{C}{(1 - r)} \int_0^{\frac{2\pi}{1-r}} \frac{du}{(1 + u^2)^{\gamma/2}} ,$$

where $r > 1/2$ and C is independent of r. The integral on the right-hand side is dominated by

$$1 + \int_1^{\infty} u^{-\gamma} du ,$$

which converges since $\gamma > 1$. |||

2. Basic properties of $H^{p,q}$.

$H^{p,q}$ may be regarded as a subspace of the corresponding space $L^{p,q}$ of equivalence classes of measurable functions. $L^{p,q}$ is a metric space in its "norm" topology, and a sequence converging in this topology has a subsequence converging pointwise a.e. We prove that $H^{p,q}$ is complete by showing that it is a closed subspace of $L^{p,q}$. In the course of our proof we need the following growth estimate:

Proposition 3.1. If $f \in H^{p,q}$ $(0 < p, q \leq \infty)$ then

$$|f(z)| \leq C\|f\|(1 - |z|)^{-1/\sigma} \qquad (|z| < 1) ,$$

where $1/\sigma = 1/p + 1/q$ and $C = C(p,q)$.

Proof: Suppose $0 < p < \infty$. Then $|f|^p$ is subharmonic. Therefore, if $z = re^{i\theta}$ and $r < R < 1$,

$$|f(z)|^p \leq \frac{1}{2\pi} \int_0^{2\pi} |f(Re^{it})|^p \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - t) + r^2} dt$$

$$\leq 2(R - r)^{-1} M_p(f;R)^p \quad .$$

If $q = \infty$ the result follows upon letting $R \to 1$. Otherwise

$$|f(z)|^q (R - r)^{q/p} \leq 2^{q/p} M_p(f;R)^q \quad ,$$

and the result follows upon integrating both sides with respect to RdR over $[r, 1)$. If $p = \infty$, the same procedure, starting with the trivial inequality $|f(z)| \leq M_\infty(f;r)$, is employed.

|||

Proposition 3.2. $H^{p,q}$ $(0 < p,q \leq \infty)$ is complete.

Proof: Suppose $\{f_n\}$ is a sequence in $H^{p,q}$ converging in the $L^{p,q}$ metric to $f \in L^{p,q}$. We claim that f is equivalent to an analytic function. By Proposition 3.1 $\{f_n\}$ is Cauchy uniformly on compact subsets of $|z| < 1$, so it converges in this manner to an analytic function g. Since $\{f_n\}$ converges to f in $L^{p,q}$, some subsequence converges pointwise a.e. to f, hence $f = g$ a.e. Thus g is in $H^{p,q}$ and $f_n \to g$ in the $L^{p,q}$ metric. $H^{p,q}$ is therefore a closed subspace of the complete metric space $L^{p,q}$.

|||

Proposition 3.3. If $f \in H^{p,q}$ $(0 < p,q < \infty)$ and $f_r(z) = f(rz)$ $(0 \le r, |z| < 1)$, then

$$\lim_{r \to 1-} \| f_r - f \|_{p,q} = 0 \quad .$$

Proof: Fix r in $[0,1)$. If $0 \le \rho < 1$, then

$$M_p(f_r - f;\rho)^q \le 2^q [M_p(f_r;\rho)^q + M_p(f;\rho)^q] \quad .$$

Since $M_p(f;\rho)$ is increasing and $M_p(f_r;\rho) = M_p(f;r\rho)$, the last inequality yields

$$M_p(f_r - f;\rho)^q \le 2^{q+1} M_p(f;\rho)^q \quad .$$

But $f_r \to f$ uniformly on the circle $|z| = \rho$ as $r \to 1-$, so the term on the left approaches 0. Since the right-hand side is integrable, our result follows from the Dominated Convergence Theorem.

|||

In particular this result shows that the functions analytic in a neighborhood of $|z| \le 1$ form a dense subset of $H^{p,q}$ $(0 < p,q < \infty)$. We next prove that $H^{p,q}$ $(0 < p,q \le 1)$ is continuously embedded in B^σ, where

$$1/\sigma = 1/p + 1/q \quad . \tag{3.3}$$

Theorem 3.4. If $f \in H^{p,q}$ $(0 < p,q \le 1)$ then $f \in B^\sigma$ and $\| f \|_{B^\sigma} \le K \| f \|_{p,q}$ where $K = K(p,q)$.

Proof: Suppose $f \in H^{p,q}$. Since $M_p(f;r)$ is increasing, we have for $0 \leq \rho < 1$,

$$M_p(f;\rho)^q \int_\rho^1 u\,du \leq \int_\rho^1 M_p(f;u)^q u\,du \quad .$$

Therefore

$$M_p(f;\rho)^q (1-\rho) \leq 2\|f\|_{p,q}^q \quad ,$$

so

$$M_p(f;\rho) \leq 2^{1/q}\|f\|_{p,q}(1-\rho)^{-1/q} \quad . \tag{3.4}$$

We consider first the case $p = 1$. Using (3.4),

$$M_1(f;\rho) \leq 2^{1-1/q}\|f\|_{1,q}^{1-q} M_1(f;\rho)^q (1-\rho)^{1-1/q} \quad ,$$

hence

$$\int_0^1 M_1(f;\rho)(1-\rho)^{1/q-1}\rho\,d\rho \leq 2^{1-1/q}\|f\|_{1,q} \quad ,$$

which was to be proven.

Next suppose $0 < p < 1$. In view of Proposition 3.3 it is enough to prove the result for functions f analytic in a neighborhood of $|z| \leq 1$. Given such an f, fix $0 \leq \rho < 1$, and apply Theorem 3.1, with $\lambda = s = 1$, to f_ρ:

$$\int_0^1 M_1(f_\rho;r)(1-r)^{1/p-2}\,dr \leq KM_p(f;\rho) \quad , \tag{3.5}$$

where K is independent of f and ρ.

We claim that by suitably altering the value of K we may replace the left side of (3.5) by

$$\int_0^1 M_1((f_\rho)_{[1/p-2]}; r)dr \quad .$$

If $1/p - 2 > 0$ this follows from Theorem 3.2 (i) upon setting $\alpha = 0$ and $\beta = 1/p - 2$. If $1/p - 2 < 0$ we let $\alpha = 1/p - 2$ and $\beta = 2 - 1/p$. Then $\alpha > -1$, $\beta > 0$, and the result follows from Theorem 3.2 (ii).

Since $(f_\rho)_{[\alpha]} = (f_{[\alpha]})_\rho$ we obtain

$$\int_0^1 M_1(f_{[1/p-2]};\rho r)dr \leq KM_p(f;\rho) \quad .$$

Set $g = f_{[1/p-2]}$. The left side of the last inequality is

$$\rho^{-1}\int_0^\rho M_1(g;u)du \quad ,$$

while from (3.4) the right side is dominated by

$$2^{1/q-1}K\|f\|_{p,q}^{1-q} M_p(f;\rho)^q(1 - \rho)^{1-1/q} \quad .$$

Thus (suitably altering K again)

$$\int_0^1\int_0^\rho M_1(g;u)du(1 - \rho)^{1/q-1}d\rho \leq K\|f\|_{p,q} \quad .$$

Integrating the left side by parts we have

$$[-q(1-\rho)^{1/q}\int_0^{\rho} M_1(g;u)du]_0^1 + \int_0^1 M_1(g;\rho)(1-\rho)^{1/q}d\rho \le K\|f\|_{p,q} .$$

Since f is analytic in a neighborhood of $|z| \le 1$, so is g, hence the first term on the left side vanishes. A final application of Theorem 3.2 ((i) with $\beta = 1/p - 2$, $\alpha = 1/q$ when $1/p - 2 > 0$; (ii) with $\beta = 2 - 1/p$, $\alpha = 1/p + 1/q - 2$ when $1/p - 2 < 0$) finishes the proof.

|||

This method of proof does not appear to work if $1 < q < \infty$, and we have only a partial result.

Theorem 3.5. If $f \in H^{p,q}$ $(0 < p \le 1, \quad 1 < q < \infty)$ then $f \in B_\tau$ for each $0 < \tau < \sigma$, and $\|f\|_{B^\tau} \le K\|f\|_{p,q}$ where $K = K(p,q,\tau)$.

Proof: If $p = 1$, (3.4) becomes

$$M_1(f;\rho)(1-\rho)^{1/q} \le 2^{1/q}\|f\|_{1,q} .$$

If $\varepsilon > 0$ we have

$$\int_0^1 M_1(f;\rho)(1-\rho)^{1/q-1+\varepsilon}\rho\, d\rho \le K\|f\|_{p,q}$$

where

$$K = 2^{1/q} \int_0^1 (1 - \rho)^{\varepsilon - 1} \rho \, d\rho < \infty \quad .$$

Suppose $0 < p < 1$. Apply (3.4) to the right-hand side of (3.5) to get

$$\int_0^1 M_1(f; r\rho)(1 - r)^{1/p - 2} dr \leq K \| f \|_{p,q} (1 - \rho)^{-1/q}$$

$(0 \leq r, \rho < 1)$ for K suitably chosen. Treating the left-hand side of this inequality as in Theorem 3.4 and multiplying both sides by $(1 - \rho)^{1/q - 1 + \varepsilon}$ $(\varepsilon > 0)$, we obtain

$$\int_0^1 \int_0^\rho M_1(g, u) du (1 - \rho)^{1/q - 1 + \varepsilon} d\rho \leq K \| f \|_{p,q}$$

where $K = K(p, q, \varepsilon)$. The proof is now finished in the same manner as that of Theorem 3.4.

|||

3. Linear functionals on $H^{p,q}$ $(0 < p \leq 1, \ \min(p,q) < 1)$.

A linear functional λ on p-normed space E $(0 < p \leq 1)$ is said to be bounded if

$$\| \lambda \| = \sup \{ |\lambda(f)| : \| f \| \leq 1 \} < \infty \quad .$$

λ is continuous if and only if it is bounded, and $\| \ \|$ is a norm under which E' becomes a Banach space. Our first result gives a

representation of the continuous linear functionals on $H^{p,q}$ $(0 < p \leq 1, \quad 0 < q < \infty)$.

Theorem 3.6. If λ is a continuous linear functional on $H^{p,q}$ $(0 < p \leq 1, \quad 0 < q < \infty)$ then there is a unique g in O^{σ} such that $\lambda(f) = \langle f, g \rangle$ for each f in $H^{p,q}$. Moreover, $\|g\|_{O^{\sigma}} \leq C\|\lambda\|$, where C is independent of λ.

Proof: Suppose $f(z) = \Sigma a_n z^n$ is in $H^{p,q}$. From the continuity of λ, and Proposition 3.3, we have

$$\lim_{r \to 1-} \lambda(f_r) = \lambda(f) \quad .$$

Fix $0 \leq r < 1$. Since the power series expansion of f_r converges uniformly in $|z| \leq 1$, it converges in $H^{p,q}$. Letting

$$b_n = \lambda(z^n) \qquad (n = 0, 1, 2, \ldots)$$

we get

$$\lambda(f) = \lim_{r \to 1-} \sum_{n=0}^{\infty} a_n b_n r^n \quad .$$

Since λ is bounded,

$$|b_n| \leq \|\lambda\| \, \|z^n\|_{p,q} = O(n^{-1/q}) \quad ,$$

so the power series

$$g(z) = \sum_{n=0}^{\infty} b_n z^n$$

defines a function analytic in $|z| < 1$. Let

$$K_z(\zeta) = (1 - \zeta z)^{-1} \qquad (|\zeta|, |z| < 1) \quad .$$

Since K_z is analytic in a neighborhood of $|\zeta| \leq 1$, it belongs to $H^{p,q}$. The same is true for $K_z^{[\beta]}$ for every real β, and a computation with power series shows that

$$\lambda(K_z^{[\beta]}) = g^{[\beta]}(z) \quad .$$

Since λ is a bounded linear functional on $H^{p,q}$,

$$|g^{[\beta]}(z)| \leq \|\lambda\| \, \|K_z^{[\beta]}\|_{p,q} \qquad (|z| < 1) \quad .$$

If $\beta > 0$ the Binomial Theorem yields

$$(1 - \zeta z)^{-(1+\beta)} = \sum_{n=0}^{\infty} \frac{\Gamma(n + \beta + 1)}{\Gamma(\beta+1)\Gamma(n+1)} (\zeta z)^n$$

$$= \Gamma(\beta + 1)^{-1} K_z^{[\beta]}(\zeta) \quad .$$

A straightforward calculation involving Lemma 3.1 then shows that whenever $\beta + 1 > 1/\sigma$,

$$\|K_z^{[\beta]}\|_{p,q} = O((1 - |z|)^{1/\sigma - (\beta+1)}) \quad .$$

In particular if $\beta = 1/\sigma$ we have

$$|g^{[1/\sigma]}(z)| \leq C\|\lambda\|(1 - |z|)^{-1} \quad ,$$

where C does not depend on λ. A simple computation shows that (3.6) is equivalent to

$$\lambda(f) = \lim_{r \to 1-} \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) g(re^{-i\theta}) d\theta = \langle f, g \rangle \quad .$$

The uniqueness of g follows from the uniqueness of the power series representation of an analytic function.

|||

We remark that g actually has a continuous extension to $|z| \leq 1$; in fact $g^{[1/\sigma - 1]}$ belongs to the Zygmund class Λ^* (see Duren [5, Chapter 5]).

We next give sufficient conditions for a function analytic in $|z| < 1$ to represent a continuous linear functional on $H^{p,q}$ $(0 < p \leq 1)$.

Theorem 3.7.

(i) If $0 < p, q \leq 1$ and $g \in O^\sigma$ then $\langle f, g \rangle$ exists for each f in $H^{p,q}$ and defines a continuous linear functional on $H^{p,q}$.

(ii) If $0 < p \leq 1 < q < \infty$ and $g \in O^\tau$ for some $\tau < \sigma$ then $\langle f, g \rangle$ exists for each f in $H^{p,q}$ and defines a continuous linear functional on $H^{p,q}$.

Proof: If $0 < p, q \leq 1$, Theorem 3.4 shows that $H^{p,q}$ is continuously embedded in B^σ, so every continuous linear

functional on B^σ gives rise to one on $H^{p,q}$ by restriction. Thus part (i) is an immediate consequence of Theorem 3.3. If $0 < p \leq 1 < q < \infty$ we use Theorem 3.4 instead of 3.3.

|||

Corollary 3.1. If $0 < p,q \leq 1$ the correspondence $\lambda \rightarrow g$ is a linear homeomorphism between $(H^{p,q})'$ and O^σ.

Proof: According to Theorem 3.6 the norm on $(H^{p,q})'$ dominates the O^σ norm when $0 < p,q \leq 1$, hence $\lambda \rightarrow g$ defines a continuous one-to-one linear mapping of $(H^{p,q})'$ onto O^σ. The inverse of this mapping is continuous by the interior mapping principle.

|||

We remark that Theorems 3.6 and 3.7 and Corollary 3.1 remain true when $0 < p < 1$, $q = \infty$, and in this case are due to Duren, Romberg, and Shields [6, Theorems 1 and 2]. The proofs we have given are adaptations of theirs.

We close this section by interpreting the foregoing results in terms of the Mackey topology.

Corollary 3.2. If $0 < p,q \leq 1$ or $0 < p < 1$, $q = \infty$, then the Mackey topology m of $H^{p,q}$ is the restriction of the B^σ topology to $H^{p,q}$. If $0 < p \leq 1 < q < \infty$ then m is stronger than the restriction of the B^τ topology for any $\tau > \sigma$.

Proof: Suppose $0 < p,q \leq 1$. The B^σ topology on $H^{p,q}$ is locally convex and metrizable, hence Mackey. Since it has the same continuous linear functionals as $H^{p,q}$ it must coincide with m.

If $0 < p \leq 1 < q < \infty$, then $H^{p,q}$ is contained in B^τ for each $\tau > \sigma$, and the B^τ topology is locally convex and weaker than that of $H^{p,q}$. From Theorem 1.2, m is a norm topology whose unit ball V can be taken to be the convex hull of the $H^{p,q}$ unit ball. Since $\{ f \in H^{p,q} : \|f\|_{B^\sigma} < 1 \}$ contains a multiple of the $H^{p,q}$ unit ball and is convex, it contains a multiple of V. |||

4. Lacunary subspaces

In this section we exhibit a class of closed subspaces in $H^{p,q}$ $(0 < p < 1)$ having continuous linear functionals which cannot be extended to the whole space. Subspaces of this type are discussed by Duren, Shields, and Romberg [6, Theorem 13] when $q = \infty$.

Let E be a set of non-negative integers and let $H^{p,q}(E)$ denote the set of functions $f(z) = \Sigma a_n z^n$ in $H^{p,q}$ with $a_n = 0$ if $n \notin E$. Since the $H^{p,q}$ topology is stronger than the topology of uniform convergence on compact subsets, the linear functional $f \to a_n$ is continuous $(n = 0, 1, 2, \ldots)$. Thus $H^{p,q}(E)$ is weakly closed, hence closed. Duren, Romberg, and Shields [6] show that if E is lacunary $(E = \{ n_k \}$ where $n_{k+1}/n_k > \delta > 1)$ then $H^p(E)$ has a continuous linear functional that cannot be extended to

the whole space. We extend this result to $0 < q < \infty$.

Lemma 3.2. (Zygmund [17, Vol. I, Chapter 5, Theorem 6.4]) Suppose $\{n_k\}$ is lacunary and $\Sigma b_k e^{in_k\theta}$ is Abel summable in a set of positive measure. Then $\Sigma |b_k|^2 < \infty$.

Corollary 3.3. If E is lacunary then $H^p(E) = H^2(E)$, and for each f in $H^p(E)$, $\|f\|_2 \leq C\|f\|_p$ where $C = C(p)$.

Proof: Since the boundary function of each f in H^p is Abel summable a.e. (i.e., has radial limits a.e.), the first part follows from Lemma 3.2. Since the identity mapping $i : H^2(E) \to H^p(E)$ is continuous, the norm inequality follows from the interior mapping principle.

Theorem 3.8. Suppose E is lacunary and $0 < p < 1$, $0 < q \leq \infty$. Then there exist linear functionals on $H^{p,q}(E)$ having no continuous linear extension to $H^{p,q}$.

Proof: If $f \in H^{p,q}(E)$ we can apply Corollary 3.3 to f_r $(0 \leq r \leq 1)$ and integrate qth powers of both sides of the resulting inequality to get

$$\|f\|_{2,q} \leq C\|f\|_{p,q} \quad , \tag{3.7}$$

where C is independent of f. Thus $H^{p,q}(E) = H^{2,q}(E)$ and the relative $H^{p,q}$ and $H^{2,q}$ topologies agree on this subspace.

We need a coefficient estimate. Suppose $f(z) = \Sigma a_n z^n \in H^{p,q}$. Then, as in (3.4),

$$M_2(f;r) \leq 2^{1/q} \|f\|_{2,q} (1 - r)^{-1/q} \quad .$$

Applying this estimate and the Hölder inequality to the representation

$$a_n r^n = \frac{}{2\pi} \int_0^{2\pi} f(r e^{i\theta}) e^{-in\theta} \, d\theta$$

$(n = 0, 1, 2, \ldots)$, we get

$$|a_n| \leq 2^{1/q} \|f\| r^{-n} (1 - r)^{-1/q} \qquad (0 < r < 1) \quad .$$

Letting $r = 1 - 1/n$ we get the required result:

$$|a_n| = O(n^{1/q}) \quad . \tag{3.8}$$

If $f(z) = \Sigma a_n z^n \in H^{p,q}(E)$ then $f \in H^{2,q}$. Thus, letting

$$u_n(z) = z^n (nq + 2)^{1/q} \quad ,$$

and

$$b_n = (nq + 2)^{-1/q} a_n \qquad (n = 0, 1, 2, \ldots) \, ,$$

we may write $f(z) = \Sigma b_n u_n(z)$, where, according to (3.8), $\{b_n\}$ is bounded.

Consequently, whenever $\{\lambda_n\} \in \ell^1$, we may define a linear functional λ on $H^{p,q}(E)$ by the formula

$$\lambda(f) = \sum_{n=0}^{\infty} \lambda_n b_n \qquad (f \in H^{p,q}(E)) \quad .$$

λ is continuous by the uniform boundedness principle. Suppose λ has a continuous linear extension $\tilde{\lambda}$ to $H^{p,q}$. By Theorems 3.3 and 3.7 every linear functional continuous on $H^{p,q}$ is continuous in the B^{σ} norm, hence

$$|\tilde{\lambda}_n| = |\tilde{\lambda}(u_n)| \leq \|\tilde{\lambda}\| \, \|u_n\|_{B^{\sigma}} \quad .$$

Now $\|u_n\|_{B^{\sigma}} = (nq + 2)^{1/q} \|z^n\|_{B^{\sigma}}$, where

$$\|z^n\|_{B^{\sigma}} = \int_0^1 r^{n+1}(1 - r)^{1/\sigma - 2} \, dr$$

$$= O(n^{1-1/\sigma}) \quad ,$$

so we have

$$|\lambda_n| = O(n^{1-1/p}) \quad .$$

Since there are absolutely summable sequences $\{\lambda_n\}$ which do not satisfy this condition, there are linear functionals continuous on $H^{p,q}(E)$ which have no continuous linear extension to the whole space.

|||

An immediate consequence of this theorem is

Corollary 3.4. If $0 < p < 1$ then $H^{p,q}$ is not locally convex.

We will show in the next section that $H^{p,q}$ is not locally convex when $1 \leq p < \infty$, $0 < q < 1$. In this case, however, the subspaces $H^{p,q}(E)$ have a drastically different character: every continuous linear functional on $H^{p,q}(E)$ can be extended to the whole space. In fact, $H^{p,q}(E)$ is a complemented subspace of $H^{p,q}$ (i.e., $H^{p,q}(E)$ has an algebraic complement that is closed in $H^{p,q}$). This is an immediate consequence of

Theorem 3.9. Let E be a lacunary sequence of non-negative integers. If $f(z) = \Sigma a_n z^n$ is in $H^{p,q}$ define

$$\pi f(z) = \sum_{n \in E} a_n z^n \quad .$$

If $1 \leq p < \infty$, $0 < q < \infty$, then π is a bounded projection from $H^{p,q}$ onto $H^{p,q}(E)$.

Proof: That π is a projection onto $H^{p,q}(E)$ will follow from its boundedness. According to a theorem of Paley (Zygmund [17, Vol. II, Chapter 12, Theorem 7.8]) if $f \in H^1$ then $\pi f \in H^2$ and

$$\|\pi f\|_2 \leq C \|f\|_1$$

where C is independent of f. On the other hand if $g(e^{i\theta})$ in L^2 has a lacunary Fourier series then $g \in L^p$ $(0 < p < \infty)$ and

$$A^{-1}\|g\|_p \leq \|g\|_2 \leq A\|g\|_p \quad ,$$

where A is independent of g (Zygmund [17, Vol. I, Chapter 5, Theorem 8.20]). Fixing f in $H^{p,q}$ and applying Paley's result to the function f_r $(0 < r \leq 1)$, we get

$$M_2(\pi f;r) \leq CM_1(f;r) \leq CM_p(f;r) \quad .$$

Applying the second result to $\pi(f_r) = (\pi f)_r$:

$$A^{-1}M_p(\pi f;r) \leq M_2(\pi f;r) \quad .$$

Combining these inequalities we get

$$M_p(\pi f;r) \leq ACM_p(f;r) \quad ;$$

hence

$$\|\pi f\|_{p,q} \leq AC\|f\|_{p,q} \quad ,$$

where A and C are independent of f. |||

5. Weakly dense subspaces

Duren, Romberg, and Shields [6, Theorem 14] exhibit a class of proper, closed subspaces of H^p $(0 < p < 1)$ which are weakly dense: no non-trivial continuous linear functional can vanish identically on the subspace. In this section we show that such subspaces are present in $H^{p,q}$ when $q = \min(p,q) < 1$. In addition,

our method yields a class of proper, closed, weakly dense subspaces in H^p quite different from those of [6]. It also provides another proof of a result of Landsberg [10, Satz 3] on the concavity of the unit ball in H^p, and an extension of this result to $H^{p,q}$ $(q = \min(p,q) < 1)$.

We require the following result concerning the sequence spaces ℓ^p $(0 < p \leq 1)$, which for $p = 1$ is found in Köthe [9, page 283].

Theorem 3.10. If E is a complete separable p-normed space $(0 < p \leq 1)$ then there is a continuous linear mapping of ℓ^p onto E.

Proof: Let $\{e_n\}_{n\geq 0}$ be a countable dense subset of $S = \{f \in E : \|f\| \leq 1\}$. Suppose $x = (\xi_n)_{n\geq 0}$ is in ℓ^p. For $n, k \geq 0$,

$$\| \sum_n^{n+k} \xi_j e_j \| \leq \sum_n^{n+k} |\xi_j|^p \quad ,$$

so the sequence $\{\Sigma_0^n \xi_j e_j\}_{n\geq 0}$ is Cauchy in E and converges to an element $\Sigma_0^\infty \xi_j e_j$ of E. Define the map $T : \ell^p \to E$ by the equation

$$Tx = \sum_0^\infty \xi_j e_j \quad .$$

Then $\|Tx\| \leq \|x\|_{\ell^p}$, so that T is a continuous linear mapping of ℓ^p into E.

To show T is onto E, suppose f is in E with $\|f\| \leq 1$. Since $\{e_n\}$ is dense in S we can choose e_{n_0} such that

$$\|f - e_{n_0}\| < 2^{-p} .$$

Since $\{\frac{1}{2} e_n\}$ is dense in $2^{-p}S$ we can choose e_{n_1} such that

$$\|f - e_{n_0} - 2^{-1} e_{n_1}\| < 4^{-p} .$$

Continuing in this manner we obtain a subsequence $\{e_{n_k}\}$ such that

$$\|f - (e_{n_0} + 2^{-1} e_{n_1} + 2^{-2} e_{n_2} + \ldots + 2^{-k} e_{n_k})\| < 2^{-(k+1)p}$$

$(k = 0, 1, 2, \ldots)$. Let x be the sequence with 2^{-k} in the n_k'th position $(k = 0, 1, 2, \ldots)$ and zeroes elsewhere. Then $x \in \ell^p$ and $Tx = f$.

|||

We next show that $H^{p,q}$ can be mapped onto ℓ^q when $q = \min(p, q) \leq 1$. We will need the following lemma, whose proof may be found in Duren [5, Theorem 9.2].

Lemma 3.3. Suppose $0 \leq |z_0| < |z_1| < \ldots \to 1$, and

$$1 - |z_{n+1}| \leq C(1 - |z_n|) \qquad (n = 0, 1, 2, \ldots), \tag{3.9}$$

where $0 < C < 1$. Then there exists $\delta > 0$ such that

$$\prod_{\substack{j=0 \\ j \neq n}}^{\infty} \left| \frac{z_n - z_j}{1 - z_j z_n} \right| > \delta \qquad (n = 0, 1, 2, \ldots).$$

Theorem 3.11. Suppose $0 \leq |z_0| < |z_1| < \ldots \to 1$ and (3.9) is satisfied. If $q = \min(p, q) \leq 1$, then

$$Tf = \left\{ (1 - |z_n|^2)^{1/\sigma} f(z_n) \right\}_{n \geq 0}$$

$(\sigma = 1/p + 1/q)$ defines a continuous linear mapping of $H^{p,q}$ onto ℓ^q.

Proof: We first show that T maps $H^{p,q}$ continuously into ℓ^q. If $0 < p < \infty$ then $|f|^p$ is subharmonic in $|z| < 1$, so when $|z_n| \leq r < 1$ we have, as in the proof of proposition 3.1,

$$|f(z_n)|^p (r - |z_n|) \leq 2M_p^p(f;r) \qquad (n = 0, 1, 2, \ldots).$$

Raising both sides of this unequality to the power q/p and integrating over $|z_n| \leq r \leq |z_{n+1}|$ we obtain

$$|f(z_n)|^q \int_{|z_n|}^{|z_{n+1}|} (r - |z_n|)^{q/p} r\,dr \le 2^{q/p} \int_{|z_n|}^{|z_{n+1}|} M_p(f;r)^q r\,dr$$

$(n = 0, 1, 2, \ldots)$. An integration by parts yields

$$\int_{|z_n|}^{|z_{n+1}|} (r - |z_n|)^{q/p} r\,dr \ge \frac{|z_1|}{q/p+1} (|z_{n+1}| - |z_n|)^{q/p+1}$$

$(n = 0, 1, 2, \ldots)$. Thus

$$|f(z_n)|^q (|z_{n+1}| - |z_n|)^{q/p+1} \le K \int_{|z_n|}^{|z_{n+1}|} M_p(f;r)^q r\,dr \tag{3.10}$$

$(n = 0, 1, 2, \ldots)$, where K is independent of f and n. It is easy to see that (3.9) is equivalent to

$$1 - |z_n| \le (1 - C)^{-1} (|z_{n+1}| - |z_n|) \qquad (n = 0, 1, 2, \ldots),$$

hence

$$1 - |z_n|^2 \le 2(1 - |z_n|) \le 2(1 - C)^{-1} (|z_{n+1}| - |z_n|)$$

$(n = 0, 1, 2, \ldots)$. Substituting this inequality into (3.10) and summing:

$$\sum_{n=0}^{\infty} (1 - |z_n|^2)^{q/} |f(z_n)|^q \le K \|f\|_{p,q}^q ,$$

where K does not depend on f. If $p = \infty$ we integrate the trivial inequality

$$|f(z_n)| \leq M_\infty(f;r) \qquad (|z_n| \leq r < 1)$$

over $|z_n| \leq r \leq |z_{n+1}|$ to obtain (3.10) (where q/∞ is interpreted as 0), and proceed as before. Thus T is a continuous linear mapping of $H^{p,q}$ into ℓ^q. We remark that the proof thus far is valid for all p and q as long as $q < \infty$.

We now show that T is onto. Let $\{w_n\} \in \ell^q$, and form the series

$$f(z) = \sum_{n=0}^{\infty} (1 - |z_n|^2)^{\beta - 1/\sigma} w_n \frac{b_n(z)}{b_n(z_n)} g_n(z) \qquad (3.11)$$

where $\beta > 1/\sigma$,

$$b_n(z) = \prod_{\substack{j=0 \\ j \neq n}}^{\infty} \frac{\overline{z}_j}{|z_j|} \frac{z_j - z}{1 - \overline{z}_j z}$$

and

$$g_n(z) = (1 - \overline{z}_n z)^{-\beta} .$$

We claim that the series (3.11) converges uniformly on compact subsets of $|z| < 1$ to a function f in $H^{p,q}$ such that $Tf = \{w_n\}$. To see this we need only show that the series (3.11) converges in the metric of $H^{p,q}$. According to Lemma 3.3,

$$|b_n(z_n)| > \delta > 0 \qquad (n = 0, 1, 2, \ldots) ,$$

and it is clear that $|b_n(z)| < 1$ when $|z| < 1$. Letting f_n denote the nth partial sum of (3.11), we have for $m > n$,

$$\|f_m - f_n\|^q \leq \delta^{-q} \sum_{j=n+1}^{m} (1 - |z_j|^2)^{\beta - 1/\sigma)q} |w_j|^q \|g_j\|^q \quad .$$

A calculation using Lemma 3.1 shows that since $\beta > 1/\sigma$

$$\|g_j\| \leq C(1 - |z_j|)^{1/\sigma - \beta} \qquad (j = 0, 1, 2, \ldots) \ .$$

Therefore,

$$\|f_m - f_n\| \leq K \sum_{n+1}^{m} |w_j|^q$$

whenever $m > n \geq 0$. Since $\{w_j\} \in \ell^q$, the sequence $\{f_n\}$ is Cauchy in $H^{p,q}$ and the convergence of (3.11) follows.

|||

We can now prove the main result of this section.

Theorem 3.12. If $q = \min(p,q) < 1$ then $H^{p,q}$ contains proper closed subspaces that are weakly dense.

Proof: Let E be an infinite dimensional complete separable q-normed space having no non-trivial continuous linear functional (for example, $E = L^q([0,1])$; see Corollary 3.7, Section 6). By Theorems 3.10 and 3.11 there are continuous linear maps S and T taking ℓ^q

onto E and $H^{p,q}$ onto ℓ^q, respectively. Thus $U = S \circ T$ is a continuous linear transformation of $H^{p,q}$ onto E. The null space N of U is a proper closed subspace of $H^{p,q}$, and we claim it is weakly dense.

Now the quotient space $H^{p,q}/N$ is complete in the q-norm

$$\|f + N\| = \inf \|f + g\| \qquad (g \in N) \quad ,$$

which induces the quotient topology. The equation

$$\tilde{U}(f + N) = Uf \qquad (f \in H^{p,q})$$

defines a linear isomorphism of $H^{p,q}$ onto E. From the definitions of $\tilde{U}$ and the quotient norm:

$$\|\tilde{U}(f + N)\| \leq \|U\| \, \|f + N\| \qquad (f \in H^{p,q}) \quad .$$

$\tilde{U}$ is therefore continuous, and the interior mapping principle shows that the same is true of its inverse. Thus $H^{p,q}/N$ is linearly homeomorphic with E, and therefore has no non-zero continuous linear functional. If N were not weakly dense there would be a point f_0 in $H^{p,q}$ not in N, and a continuous linear functional λ on $H^{p,q}$, vanishing on N, with $\lambda(f_0) \neq 0$. The equation

$$\tilde{\lambda}(f + N) = \lambda(f) \qquad (f \in N)$$

would then define a non-trivial continuous linear functional on $H^{p,q}/N$, contradicting our previous assertion.

$\|\|$

The same idea applied to the null space of $S : \ell^q \to E$ shows that ℓ^q $(0 < q < 1)$ contains proper, closed, weakly dense subspaces, improving a result of Peck [11, Section 2].

Theorem 3.12 and Corollary 3.4 immediately yield

Corollary 3.5. If $\min(p,q) < 1$ then $H^{p,q}$ is not locally convex.

Our methods are also applicable in H^p $(0 < p < 1)$. An extension due to Kabaila (see Duren [5, Chapter 9]) of a result of Shapiro and Shields characterizes all sequences $\{z_n\}$ in the unit disc for which

$$Tf = \{ (1 - |z_n|^2)^{1/p} f(z_n) \} \qquad (f \in H^p)$$

defines a continuous linear mapping of H^p onto ℓ^p. These are exactly the sequences for which the conclusion of Lemma 3.3 is true. In particular, any sequence satisfying (3.9) qualifies, and it follows as before that H^p contains proper, closed, weakly dense subspaces. In this case the subspaces we have found are quite different from the ones described by Duren, Romberg, and Shields [6, Theorem 14].

6. Concavity.

The methods of the last section provide information about the geometry of the unit ball in $H^{p,q}$ when $q = \min(p,q) < 1$. A subset S of a linear space is called r-convex $(0 < r \leq 1)$ if

$ax + by \in S$ whenever $x, y \in S$ and a, b are scalars with $|a|^r + |b|^r = 1$. It is easy to see that every r-convex set is s-convex for $0 < s \leq r$. A linear space is called r-convex if it has a local base of r-convex sets. Thus a p-normed space is p-convex $(0 < p < 1)$. On the other hand, a p-normed space may be r-convex for some $p < r \leq 1$; for example, if $\| \ \|$ is a norm then $\| \ \|^{1/2}$ is a $1/2$-norm, so a normed space can masquerade as a $1/2$-normed space. Landsberg [10] has given a method for determining the convexity of certain spaces, and has shown that H^p, ℓ^p, and L^p $(0 < p < 1)$ are not r-convex for $p < r \leq 1$. Using Theorems 3.10 and 3.11 we can give different proofs of the above quoted results for H^p and ℓ^p, and can show that $H^{p,q}$ $(\min(p,q) = q < 1)$ is not r-convex for $q < r \leq 1$. In the next chapter we shall apply this method to some other spaces of analytic functions.

We begin with a simple result about $L^p([0,1])$.

Lemma 3.4. If $p < r < 1$ then the r-convex hull of the unit ball in $L^p([0,1])$ is the whole space.

Proof: It is enough to show that every member of the L^p unit sphere is an r-convex combination of elements in arbitrarily small spheres. Suppose $f \in L^p$ and

$$\|f\| = \int_0^1 |f|^p dx = 1 \quad .$$

Fix $n > 0$ and let $I_1, \ldots, I_n$ be a partition of $[0,1]$ into disjoint intervals such that

$$\int_{I_j} |f|^p dx = 1/n \qquad (j = 1, 2, \ldots, n) \quad .$$

Let $g_j = f\chi_j$, where χ_j is the characteristic function of I_j.
Let $f_j = n^{1/r} g_j$ $(j = 1, 2, \ldots, n)$, so

$$f = n^{-1/r}(f_1 + \ldots + f_n) \quad ,$$

and $\|f_j\| = n^{p/r-1}$ $(j = 1, 2, \ldots, n)$. Since $p < r$ we can make $\|f_j\|$ as small as desired by choosing n large.

|||

Corollary 3.6. $L^p([0,1])$ is not r-convex for $p < r \leq 1$.

Proof: If L^p were r-convex for some $p < r \leq 1$ there would be a proper r-convex subset containing some multiple of the L^p unit ball. Thus the r-convex hull of the unit ball in L^p would be a proper subset of L^p, contradicting Lemma 3.4.

|||

Corollary 3.7. If $0 < p < 1$ then the only continuous linear mapping of $L^p([0,1])$ into an r-normed space $(p < r \leq 1)$ is the zero

mapping. In particular, $L^p([0,1])$ has no non-zero continuous linear functional.

Proof: Suppose the linear transformation T maps L^p continuously into an r-normed space E $(p < r \leq 1)$. If U is the unit ball in E, $T^{-1}(U)$ is an r-convex open set in L^p, so $L^p = T^{-1}(U)$ by Lemma 3.4. Thus for every $\varepsilon > 0$, $L^p = T^{-1}(\varepsilon U)$, hence $T \equiv 0$.

|||

We now prove the main result of this section.

Theorem 3.13. If $0 < q < 1$ then ℓ^q, H^q, and $H^{p,q}$ $(\min(p,q) = q)$ are not r-convex for $q < r \leq 1$.

Proof: Let E denote any of the above mentioned spaces. Theorems 3.10 and 3.11 show that there is a continuous linear transformation of E onto $L^p([0,1])$. If N is the null space of this mapping, as in Theorem 3.2, L^q is linearly homeomorphic with E/N. If E is r-normed for $p < r \leq 1$, so is E/N, hence so is L^p, which contradicts Corollary 3.6. Thus E is not r-normed. If E is r-convex there is a bounded r-convex circled neighborhood U of 0, and the r-norm

$$\|x\|_r = \inf \{ \alpha^r : \alpha > 0,\ \alpha^{-1} x \in U \}$$

induces the topology of E. Thus E is r-normable, contradicting our previous assertion.

|||

In closing we note another consequence of Lemma 3.4.

Theorem 3.14. If $0 < q < 1$ then there are proper, closed subspaces N in ℓ^q, H^q, and $H^{p,q}$ $(p \geq q)$ on which no non-trivial continuous linear mapping of the space into an r-normed space $(q < r \leq 1)$ can vanish identically.

Proof: Let E denote ℓ^q, H^q, or $H^{p,q}$ $(p \geq q)$. Then there is a closed subspace N of E for which E/N is linearly homeomorphic with $L^q([0,1])$. Suppose F is an r-normed space and T is a continuous linear mapping of E into F, which vanishes on N. If T is non-trivial then it induces a non-trivial continuous linear mapping of L^q into F, contradicting Corollary 3.7.

|||

7. Linear functionals on $H^{p,q}$ $(1 < p,q < \infty)$.

In this section we depart briefly from our study of non-locally convex spaces and consider the dual space of $H^{p,q}$ when $1 < p,q < \infty$. Recall that when $0 < p,q \leq 1$, $(H^{p,q})'$ depends only on the parameter $\sigma = pq(p+q)^{-1}$. When $1 < p,q < \infty$ the situation is different; $(H^{p,q})'$ is linearly homeomorphic with $H^{p',q'}$ $(1/p + 1/p' = 1, \quad 1/q + 1/q' = 1)$, where the spaces are paired by

$$(3.12) \qquad \langle f, g \rangle = \iint_{|z|<1} f(z)g(z)\,dx\,dy \quad .$$

This result is known for the case $p = q$ (see Zaharjuta and Judovič [16]). Our strategy is to show that

$$(3.13) \qquad Pf(z) = \frac{1}{\pi} \iint_{|\zeta|<1} \frac{f(\zeta)}{(1 - \zeta z)^2}\,du\,dv$$

($f \in L^{p,q}$, $|z| < 1$, $\zeta = u + iv$) defines a bounded projection from $L^{p,q}$ onto $H^{p,q}$. The result then follows from the fact that $(L^{p,q})'$ is isometrically isomorphic to $L^{p',q'}$ in the pairing (3.12) (see Benedek and Panzone [1, page 304]). Our main tool is a result of Schwartz generalizing the Calderon-Zygmund inequality to vector valued functions.

Before proceeding we briefly review some facts about vector integration. For more details see Dunford and Schwartz [4, Chapter 3] or Yosida [15, page 130]. Let (X, Σ, μ) be a measure space and E a Banach space. A function $f : X \to E$ is said to be <u>simple</u> if there are disjoint sets $S_1, \ldots, S_n$ in Σ having finite measure such that f is constant on S_i $(i = 1, 2, \ldots, n)$ and vanishes off their union. If $f = c_i$ on S_i ($c_i \in E$, $i = 1, 2, \ldots, n$) we write

$$\int f d\mu = \sum_{i=1}^{n} c_i \mu(s_i) \quad .$$

A function $f : X \to E$ is strongly measurable if there is a sequence $\{f_n\}$ of simple functions such that

$$\lim_n \| f_n(x) - f(x) \| = 0$$

for almost every x in X. The strongly measurable functions form a vector space, and for each such function the scalar valued function $\|f\|$ is measurable in the ordinary sense.

A strongly measurable function f is Bochner integrable if there is a sequence $\{f_n\}$ of simple functions such that

$$\lim_n \| f_n(x) - f(x) \| = 0 \qquad \text{a.e.}$$

and

$$\lim_n \int \| f_n - f \| d\mu = 0 \quad . \tag{3.14}$$

In this case the sequence $\{\int f_n d\mu\}$ converges in E, and the limit is independent of the approximating sequence $\{f_n\}$. We call this limit the Bochner integral of f with respect to μ and write

$$\int f d\mu = \lim_n \int f_n d\mu \quad .$$

The collection of strongly measurable functions $f : X \to E$ such that

$$\|f\|_p^p = \int \|f\|^p \, d\mu < \infty$$

is denoted by $L^p(\mu, E)$. If $p \geq 1$ it is a Banach space and contains the simple functions as a dense subspace.

The strongly measurable functions may be characterized in the following way. A function $f : X \to E$ is called weakly measurable if $\lambda \circ f$ is measurable for each λ in E'. A theorem of Pettis (see Yosida [15, page 131]) states that f is strongly measurable if and only if it is weakly measurable and $f(X - B_0)$ is separable for some set $B_0 \in \Sigma$ whose measure is zero. We will be concerned with separable spaces E, in which case Pettis' theorem asserts that strong and weak measurability are equivalent.

From now on we take $E = L^p = L^p([0, 1])$ and $\mu =$ planar Lebesgue measure. We write $L^q(L^p)$ instead of $L^q(\mu, E)$ $(1 \leq p, q < \infty)$. If $f \in L^q(L^p)$ we will write

$$f(z)(\theta) = f(z, \theta) \qquad (z \in C, \ 0 \leq \theta \leq 2\pi),$$

so f may be considered as a function on $C \times [0, 2\pi]$. As such, f is product measurable, for there is a sequence f_n of simple functions in $L^q(L^p)$ converging in norm to f. The f_n are clearly product measurable, so they belong to the mixed-norm

$\mathcal{L}^{p,q}$ of measurable functions f on $C \times [0, 2\pi]$ with

$$\iint_C \left[\int_0^{2\pi} |f(z, \theta)|^p d\theta \right]^{q/p} dx\, dy < \infty .$$

Since $\mathcal{L}^{p,q}$ is complete, the sequence $\{f_n\}$ converges in its topology to $g \in \mathcal{L}^{p,q}$. Thus g is product measurable, and we need only show that $g \in L^q(L^p)$; i.e., when viewed as a function from C to L^p, g is strongly measurable. In view of Pettis' theorem it suffices to show that g is weakly measurable. This is true because for each f in $L^{p'}([0, 2\pi])$,

$$\phi(z) = \int_0^{2\pi} g(z, \theta) f(\theta) d\theta \qquad (z \in C)$$

is a measurable function.

We have shown that $L^q(L^p)$ may be considered as either a space of equivalence classes of vector valued functions or a mixed norm space. In the following lemma we consider it to be a space of vector valued functions.

Lemma 3.4. If $f \in L^1(L^p)$ $(1 \leq p \leq \infty)$, then for almost all θ in $[0, 2\pi]$,

$$\left(\int f dA \right)(\theta) = \int f(z, \theta)\, dA(z) \qquad ,$$

where dA is planar Lebesgue measure.

Proof: Let $f_n : C \to L^p$ be a sequence of simple functions converging to f in the norm of $L^1(L^p)$. Then $f_n \to f$ in $L^1(L^1)$, so that

$$\lim_n \int [\int_0^{2\pi} |f_n(z,\theta) - f(z,\theta)| \, d\theta] dA(z) = 0 \quad .$$

Since f_n and f are measurable on $C \times [0, 2\pi]$, we may reverse the order of integration. Thus, for almost all θ,

$$\lim_n \int f_n(z,\theta) dA(z) = \int f(z,\theta) dA(z) \quad .$$

the limit being taken in L^p. If a sequence converges in L^p then a subsequence converges pointwise a.e. Thus there is an increasing sequence $\{n_k\}$ of non-negative integers such that for almost all θ

$$(\int f dA)(\theta) = \lim_n (\int f_{n_k} dA)(\theta)$$

$$= \lim_n \int f_{n_k}(z,\theta) dA(z)$$

$$= \int f(z,\theta) dA(z). \qquad \|$$

We will use the following form of the Calderon-Zygmund inequality for vector valued functions.

Theorem 3.15 (Schwartz [13, Theorem 7]). Suppose $f \in L^q(L^p)$. Then for each $z \in C$

$$\tilde{f}(z) = \lim_{\varepsilon \to 0} \iint_{|w-z|>\varepsilon} \frac{f(w)}{(w-z)^2}\, dA(w) \tag{3.15}$$

exists, and the mapping $f \to \tilde{f}$ is a bounded linear operator on $L^q(L^p)$.

We can now prove our main result.

Theorem 3.16. The map P defined by (3.13) is a bounded projection from $L^{p,q}$ onto $H^{p,q}$ $(1 < p, q < \infty)$.

Proof: Equation (3.13) defines a linear map P on those functions in $L^q(L^p)$ which vanish outside $|z| < 1$. If F is such a function, then according to Lemma 3.4, whenever $|z| < 1$ we have

$$\overline{PF(\bar{z}, \theta)} = \iint \frac{G(\eta, \theta)}{(\eta - z)^2}\, dA(\eta) \quad , \tag{3.16}$$

where for almost every θ,

$$G(\eta, \theta) = \begin{cases} \pi^{-1} F(1/\eta, \theta)\, \eta^2 |\eta|^{-4} & , \quad \text{if } |\eta| > 1 \\ 0 & , \quad \text{otherwise} \end{cases} .$$

We remark that since $|z| < 1$ and $G(\eta) = 0$ for all $|\eta| < 1$, the integrand in (3.16) has no singularities. From (3.16) and

Theorem 3.15 we obtain

$$\|PF\| \leq K\|G\| \quad , \tag{3.17}$$

where K is independent of F, and $\|\ \|$ is the $L^q(L^p)$ norm. Computing $\|G\|$ we have

$$\|G\|^q = \int\int \|G(\eta)\|_p^q dA(\eta)$$

$$= \int\int \|G(1/w)\|_p^q |d\eta/dw|^2 dA(w) \qquad (\eta = 1/w)$$

$$= \pi^{-q} \int\int_{|w|<1} \|F(w)\|_p^q |w|^{2(q-2)} dA(w) \quad .$$

When $q \geq 2$,

$$\|G\|^q \leq \pi^{-q} \int\int_{|w|<1} \|F(w)\|_p^q \, dA(w)$$

$$= \pi^{-q} \|F\|^q \quad ,$$

and it follows from (3.17) that

$$\|PF\| \leq K\|F\| \quad . \tag{3.18}$$

We can now show that P is a bounded operator on $L^{p,q}$ $(1 < p < \infty, \ 2 \leq q < \infty)$. If $f \in L^{p,q}$ then

$$F(z,\theta) = \begin{cases} f(z\,e^{i\theta}) & (0 \le \theta \le 2\pi, \quad |z| < 1) \\ 0 & (0 \le \theta \le 2\pi, \quad |z| \ge 1) \end{cases}$$

defines a function in $L^q(L^p)$ supported on $|z| < 1$. Therefore, (3.18) applies. An easy calculation shows that

$$PF(z,\theta) = Pf(z\,e^{i\theta}) \qquad (|z| < 1) \quad . \tag{3.19}$$

Moreover,

$$\|F\|^q = \iint_{|z|<1} \|F(z)\|_p^q \, dA(z)$$

$$= \int_0^1 \int_0^{2\pi} \left[\int_0^{2\pi} |f(r\,e^{i(\theta+t)}|^p \, dt\right]^{q/p} d\theta \, r\,dr$$

$$= \int_0^1 \int_0^{2\pi} \left[\int_0^{2\pi} |f(r\,e^{it})|^p \, dt\right]^{q/p} d\theta \, r\,dr$$

$$= 2\pi \int_0^1 \left[\int_0^{2\pi} |f(r\,e^{it})|^p \, dt\right]^{q/p} r\,dr$$

$$= (2\pi)^{q/p+1} \|f\|_{p,q}^q \quad .$$

In view of (3.19) the same calculation gives

$$\|PF\|^q = (2\pi)^{q/p+1} \|Pf\|_{p,q}^q \quad ,$$

so (3.18) yields

$$\| Pf\| \leq K\| f\| \quad .$$

Next suppose $f \in L^{p,q}$ $(1 < q < 2, \quad 1 < p < \infty)$. By the Hahn-Banach theorem there is a $g \in L^{p',q'}$ such that $\| g\|_{p',q'} = 1$ and

$$\| Pf\|_{p,q} = \langle Pf, g\rangle \quad ,$$

where $\langle \ , \ \rangle$ is defined by (3.12). Fubini's theorem shows that

$$\langle Pf, g\rangle \ = \langle f, \ Pg\rangle \quad .$$

Applying Hölder's inequality to the right side, we have

$$\| Pf\|_{p,q} \leq \| f\|_{p,q} \| Pg\|_{p',q'} \quad .$$

Now $1 < p' < \infty$, and $2 < q' < \infty$, so by the previous discussion,

$$\| Pg\|_{p',q'} \leq K\| g\|_{p',q'} = K \quad .$$

Therefore, P is a bounded operator on $L^{p,q}$. A calculation with power series shows that P is a projection onto $H^{p,q}$.

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Corollary 3.6. If λ is a continuous linear functional on $H^{p,q}$ $(1 < p,q < \infty)$ then there is a unique g in $H^{p',q'}$ with

$$\lambda (f) = \langle f, g\rangle \qquad (f \in H^{p,q}) \quad . \tag{3.20}$$

Conversely each such g defines a continuous linear functional on $H^{p,q}$ via (3.20). The correspondence between λ and f is a linear homeomorphism between $(H^{p,q})'$ and $H^{p',q'}$.

Proof: Suppose $\lambda \in (H^{p,q})'$. By the Hahn-Banach theorem, λ can be extended to $L^{p,q}$ without increasing its norm, so there exists g_1 in $L^{p',q'}$ with $\|g_1\|_{p',q'} = \|\lambda\|$ and

$$\lambda(f) = \langle f, g_1 \rangle \qquad (f \in H^{p,q}) \quad .$$

Since $Pf = f$ for each f in $H^{p,q}$,

$$\begin{aligned} \lambda(f) &= \langle Pf, g_1 \rangle \\ &= \langle f, Pg_1 \rangle \\ &= \langle f, g \rangle \quad , \end{aligned}$$

where, according to Theorem 3.16 we have $g = Pg_1 \in H^{p',q'}$ and $\|g\| \leq K\|g_1\| = K\|\lambda\|$. The uniqueness of g is evident from the fact that $(n+1)\pi^{-1}\lambda(z^n)$ is the nth Taylor coefficient of g. Therefore, the mapping $\lambda \to g$ defines a one-to-one, continuous linear transformation of $(H^{p,q})'$ onto $H^{p',q'}$. The interior mapping principle insures the continuity of its inverse.

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CHAPTER IV

FURTHER EXAMPLES

In this chapter we introduce two new classes of non-locally convex F-spaces: a class related to the H^p spaces $(0 < p < 1)$, and a class of spaces of entire functions. We apply the techniques of Chapter III to these spaces, determining the convexity of their unit balls and proving the existence of proper, closed, weakly dense subspaces.

1. The spaces $H^p(K)$ $(0 < p < 1)$.

Recall that a function f analytic in $|z| < 1$ belongs to H^p $(0 < p < \infty)$ if

$$\|f\|_p^p = \lim_{r \to 1-} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty$$

For each f in H^p the radial limit

$$f(e^{i\theta}) = \lim_{r \to 1-} f(re^{i\theta})$$

exists a.e., and the correspondence between f and its boundary function is an isometry between H^p and a subspace of L^p.

If K is a measurable subset of $[0,2\pi]$, and $0 < p < 1$, let $H^p(K)$ denote the set of functions f in H^p with

$$\|f\|_K^2 = \int_K |f(e^{i\theta})|^2 d\theta < \infty \quad .$$

We claim that the p-norm

$$\|f\|^p = \|f\|_p^p + \|f\|_K^p \tag{4.1}$$

makes $H^p(K)$ into a complete space. Note that a sequence converges in this space if and only if it converges in both $\|\ \|_p$ and $\|\ \|_K$. Now suppose that $\{f_n\}$ is $\|\ \|^p$-Cauchy. Then it is Cauchy in both H^p and $L^2(K)$, hence for some $f \in H^p$ and $g \in L^2(K)$ we have

$$\lim_n \|f_n - f\|_p^p = \lim_n \|f_n - g\|_K = 0 \quad .$$

Thus a subsequence of $\{f_n\}$ converges to f pointwise a.e., and a further subsequence converges to g a.e. on K. Thus $f = g$ a.e. on K, and it follows immediately that $f \in H^p(K)$ and $\lim_n \|f_n - f\|^p = 0$.

If $|K|$, the Lebesgue measure of K, is 0 then $H^p(K) = H^p$, while if $|K| = 2\pi$ then $H^p(K) = H^2$. In general

$$H^2 \subset H^p(K) \subset H^p$$

and the embeddings are continuous. It is natural, then, to ask how the convexity of $H^p(K)$ depends upon K. We prove in this section that when K is not dense $H^p(K)$ is not r-convex for $p < r \leq 1$, and contains proper, closed, weakly dense subspaces.

The lacunary subspaces of $H^p(K)$ $(0 < p < 1)$ are also of interest. When $|K| = 0$ they support continuous linear functionals which are not extendable to the whole space. However, when $|K| > 0$ they behave like the lacunary subspaces of $H^{p,q}$ $(1 \leq p < \infty, 0 < q < 1)$: every continuous linear functional on such a subspace has a continuous linear extension to the whole space. This is a consequence of

Lemma 4.1. (Zygmund [17, Volume I, Chapter 5, Lemma 6.5]) If $|K| > 0$ and $q > 1$ then there exists a positive integer $h_0 = h_0(K,q)$ such that if

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$$

is in H^2, with $n_{k+1}/n_k > q > 1$ and $n_0 \geq h_0$, then

$$\frac{1}{2}|K| \, \|f\|_2^2 \leq \int_K |f|^2 d\theta \leq 2|K| \, \|f\|_2^2 .$$

If E is a set of non-negative integers, let $H^p(K;E)$ denote those functions $f(z) = \Sigma a_n z^n$ in $H^p(K)$ with $a_n = 0$ when $n \notin E$. Each continuous linear functional on H^p has a continuous

restriction to $H^p(K)$, and therefore the Taylor coefficients can be considered continuous linear functionals on $H^p(K)$. Clearly, then, $H^p(K;E)$ is weakly closed, hence closed.

Theorem 4.1. Suppose $|K| > 0$. If E is lacunary then every continuous linear functional on $H^p(K;E)$ has a continuous linear extension to $H^p(K)$.

Proof: Suppose $E = \{ n_k \}_{k=0}^{\infty}$ where $n_{k+1}/n_k > q > 1$. We may assume without loss of generality that $n_0 > h_0$, where h_0 is chosen as in Lemma 4.1 (a linear functional can always be extended from a subspace of finite codimension). Suppose λ is a continuous linear functional on $H^p(K;E)$. Then for each f in $H^p(K;E)$,

$$|\lambda(f)| \leq \|\lambda\| \, \|f\| \quad ,$$

where $\|f\|$ is given by (4.1). From Lemma 4.1,

$$\|f\|_2^2 \leq 2|K|^{-1} \|f\|_K^2 \quad ,$$

so that

$$\begin{aligned} \|f\|^p &= \|f\|_p^p + \|f\|_K^p \\ &\leq \|f\|_2^p + \|f\|_K^p \\ &\leq [(2|K|^{-1})^{p/2} + 1] \|f\|_K^p \quad . \end{aligned}$$

Therefore,

$$|\lambda(f)| \leq C\|\lambda\| \, \|f\|_K$$

for each f in $H^p(K;E)$, where $C = C(p,K)$. We see from the Hahn-Banach theorem that λ has a linear extension $\tilde{\lambda}$ to $H^p(K)$ with

$$|\tilde{\lambda}(f)| \leq C\|\lambda\| \, \|f\|_K \qquad (f \in H^p(K)) \quad .$$

Clearly $\tilde{\lambda}$ is continuous on $H^p(K)$. This completes the proof of the theorem.

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According to this result the dual space of $H^p(K)$ $(|K| > 0)$ is significantly richer than that of H^p. Nevertheless, when K is thin enough near some point there are not enough continuous linear functionals to prevent the existence of proper, closed, weakly dense subspaces. This is a corollary of our next theorem.

Theorem 4.2. If

$$\int_K |\theta - \theta_0|^{-2\beta} d\theta < \infty \tag{4.2}$$

for some θ_0 in $[0, 2\pi]$, and $\beta > 1/p$, then there is a continuous linear mapping of $H^p(K)$ onto ℓ^p $(0 < p < 1)$.

Proof: We may suppose $\theta_0 = 0$. Choose r_n in $(0,1)$ with $1 - r_{n+1} \leq C(1 - r_n)$ $(n = 0, 1, 2, \ldots)$ where $0 < C < 1$. From the remarks following Corollary 3.5 we see that

$$Tf = \left\{ f(r_n)(1 - r_n^2)^{1/p} \right\}_{n=0}^{\infty} \qquad (f \text{ in } H^p)$$

defines a continuous linear mapping of H^p onto ℓ^p. Thus T maps $H^p(K)$ continuously into ℓ^p. To show that T is onto we proceed as in Theorem 3.11. Take $\{w_n\}$ in ℓ^p and form the series

$$f(z) = \sum_{n=0}^{\infty} (1 - r_n^2)^{\beta - 1/p} w_n \frac{b_n(z)}{b_n(z_n)} g_n(z) \quad , \tag{4.3}$$

with $\beta > 1/p$,

$$b_n(z) = \prod_{j \neq n} \frac{z - r_j}{1 - r_j z} \quad ,$$

and

$$g_n(z) = (1 - r_n z)^{-\beta} \qquad (n = 0, 1, 2, \ldots) .$$

By Lemma 3.3, $|b_n(r_n)| > \delta > 0$, so as in the proof of Theorem 3.11 we see that the series (4.3) converges in H^p, and $Tf = \{w_n\}$. To finish the proof we show that f belongs to $H^p(K)$, i.e., $\|f\|_K < \infty$.

For $|\theta| \leq \pi$,

$$|1 - r_n e^{i\theta}|^2 \geq (1 - r_n)^2 + 4 r_n \pi^{-2} \theta^2$$

$$\geq 4 r_0 \pi^{-2} \theta^2 \quad .$$

Since $r_0 > 0$,

$$\|g_n\|_K^2 \leq (4r_0)^{-\beta}\pi^{2\beta} \int_K |\theta|^{-2\beta} d\theta$$

$$= M^2 < \infty \quad .$$

From this we have

$$\|f\|_K \leq \delta^{-1} \sum_{n=0}^{\infty} (1 - r_n^2)^{\beta - 1/p} |w_n| \, \|g_n\|_K$$

$$\leq \delta^{-1} M \sum_{n=0}^{\infty} |w_n|$$

$$\leq \delta^{-1} M \left(\sum_{n=0}^{\infty} |w_n|^p \right)^{1/p} < \infty \quad . \qquad \|\|$$

Just as in Chapter III we have

Theorem 4.3. If K satisfies the hypotheses of Theorem 4.2 then $H^p(K)$ is not r-convex for $p < r \leq 1$ and contains proper, closed, weakly dense subspaces.

Clearly any non-dense measurable subset of $|z| = 1$ satisfies the conditions of Theorem 4.2. These conditions seem far from the best possible, however, and it would be of interest to know whether there are any sets K whose complement has positive measure for which $H^p(K)$ is locally convex or has the HBEP.

2. The spaces E_τ^p $(0 < \tau < \infty, \quad 0 < p < 1)$.

Let E_τ^p denote the space of entire functions of exponential type τ whose restrictions to the real axis belong to $L^p(-\infty, \infty)$. More specifically, f is in E_τ^p $(0 < p, \tau < \infty)$ if and only if

$$M_\infty(f;r) = O(e^{(\tau+\varepsilon)r}) \tag{4.4}$$

for each $\varepsilon > 0$, and

$$\|f\|_p^p = \int_{-\infty}^{\infty} |f(x)|^p dx < \infty \quad . \tag{4.5}$$

We will find that in the topology induced by (4.5) E_τ^p can be considered a closed subspace of $L^p(-\infty, \infty)$, and (4.4) can actually be replaced by

$$M_\infty(f;r) = O(e^{\tau r}) \tag{4.6}$$

(see Proposition 4.1). When $0 < p < 1$, (4.5) defines a p-norm on E_τ^p, and we will be able to use our mapping technique to show that E_τ^p is not r-convex for $p < r \leq 1$, and contains proper, closed, weakly dense subspaces. The facts we will need concerning entire functions can be found in Boas [3, Section 6.7]. Of crucial importance is the following inequality, due to Plancherel and Polya (see Boas [3, page 98] for proof).

Lemma 4.2. If f belongs to E^p_τ $(0 < p, \tau < \infty)$ then

$$\int_{-\infty}^{\infty} |f(x + iy)|^p dx \leq \|f\|_p^p \exp(p\tau|y|) \quad .$$

We can now obtain a growth estimate which will be used to show that E^p_τ is a closed subspace of L^p $(-\infty, \infty)$.

Proposition 4.1. If f belongs to E^p_τ $(0 < p, \tau < \infty)$ then

$$|f(z)| \leq C\|f\|_p \exp(\tau|y|) \quad ,$$

where $z = x + iy$ and $C = C(p, \tau)$.

Proof: Since $|f|^p$ is subharmonic we have for any $\delta > 0$,

$$|f(z)|^p \leq \frac{1}{\pi\delta^2} \iint_{|z-\zeta|<\delta} |f(\zeta)|^p d\xi\, d\eta, \qquad (\zeta = \xi + i\eta)$$

$$\leq \frac{1}{\pi\delta^2} \int_{y-\delta}^{y+\delta} \left[\int_{-\infty}^{\infty} |f(\xi + i\eta)|^p d\xi \right] d\eta \quad .$$

According to Lemma 4.2 the inner integral is dominated by $\|f\|_p^p \exp(p\tau|\eta|)$, so

$$|f(z)|^p \leq 2(\pi\delta)^{-1} e^{p\tau\delta} \|f\|_p^p e^{p\tau|y|} \quad .$$

Setting $\delta = (p\tau)^{-1}$ we obtain

$$|f(z)| \leq (2\pi^{-1} p\tau e)^{1/p} \|f\|_p e^{\tau|y|} \quad . \qquad \|\|$$

Corollary 4.1. The topology of E_τ^p $(0 < p, \tau < \infty)$ is stronger than the topology of uniform convergence on compact subsets. Consequently, point evaluations are continuous linear functionals and E_τ^p is a closed subspace of $L^p(-\infty, \infty)$.

In particular, E_τ^p $(0 < p < 1)$ is a complete p-normed space with a separating family of continuous linear functionals. The next result shows that E_τ^p can be mapped onto ℓ^p if $0 < p \leq 1$. The first part of the proof is simply a refinement of Proposition 4.1 (see Boas [3, page 101]).

Theorem 4.4. Suppose $0 < p \leq 1$, $0 < \tau < \infty$. Then there is an increasing sequence $\{x_n\}_0^\infty$ of real numbers such that

$$Tf = \{f(x_n)\}_0^\infty \qquad (f \text{ in } E_\tau^p)$$

defines a continuous linear mapping of E_τ^p onto ℓ^p.

Proof: First we show that T maps E_τ^p continuously into ℓ^p whenever $x_{n+1} - x_n > \delta > 0$ $(n = 0, 1, 2, \ldots)$. As in Proposition 4.1,

$$|f(x_n)|^p \leq 4\pi^{-1}\delta^{-2} \iint_{|x_n - \zeta| < \delta/2} |f(\zeta)|^p d\xi\, d\eta$$

$$\leq 4\pi^{-1}\delta^{-2} \int_{-\delta/2}^{\delta/2} \left[\int_{x_n - \delta/2}^{x_n + \delta/2} |f(\xi + i\eta)|^p d\xi \right] d\eta$$

$(n = 0, 1, 2, \ldots)$. The intervals $(x_n - \delta/2, \ x_n + \delta/2)$ are disjoint, so

$$\sum_{n=0}^{\infty} |f(x_n)|^p \leq 4\pi^{-1}\delta^{-2} \int_{-\delta/2}^{\delta/2} \int_{-\infty}^{\infty} |f(\xi + i\eta)|^p \, d\xi \, d\eta \quad .$$

Using Lemma 4.2 on the inner integral we obtain

$$\sum_{n=0}^{\infty} |f(x_n)|^p \leq C \|f\|_p^p \quad ,$$

where $C = C(p, \tau, \delta)$. Thus T is a continuous linear mapping of E_τ^p into ℓ^p.

We next show that T is onto if $x_n = n\pi/\alpha$, where $\alpha = \tau/\beta$, and β is an integer $> 1/p$. Let

$$g_n(z) = \left[\frac{\sin \alpha(z - x_n)}{\alpha(z - x_n)} \right]^\beta$$

$(n = 0, 1, 2, \ldots)$. We claim that whenever $\{w_n\}_0^\infty \in \ell^p$,

$$f(z) = \sum_{n=0}^{\infty} w_n g_n(z)$$

defines a function in E_τ^p. Since $g_n(x_m)$ is 0 when $n \neq m$ and 1 when $n = m$, we will have $Tf = \{w_n\}$. Since $x_{n+1} - x_n = \pi/\alpha > 0$, T will map E_τ^p continuously onto ℓ^p.

Let $m = \max_{|z| \leq 1} |(\alpha z)^{-1} \sin \alpha z|$. Then

$$\|g_n\|_p^p = \int_{-\infty}^{\infty} \left| \frac{\sin \alpha(x - x_n)}{\alpha(x - x_n)} \right|^{\beta p} dx$$

$$= \int_{-\infty}^{\infty} \left| \frac{\sin \alpha x}{\alpha x} \right|^{\beta p} dx$$

$$\leq 2m + 2\alpha^{-\beta p} \int_{1}^{\infty} x^{-\beta p} dx$$

$$= C^p < \infty$$

$(n = 0, 1, 2, \ldots)$, where the last integral converges because $\beta p > 1$. Since $\{w_n\}$ is in ℓ^p we have

$$\|f\|_p^p \leq \sum_{n=0}^{\infty} |w_n|^p \|g_n\|_p^p$$

$$\leq C^p \sum_{n=0}^{\infty} |w_n|^p < \infty \quad .$$

We finish the proof by showing that f is of exponential type τ. Since x_n is real $(n = 0, 1, 2, \ldots)$,

$$|\sin \alpha(z - x_n)| = |\sin \alpha z \cos \alpha x_n - \cos \alpha z \sin \alpha x_n|$$

$$\leq |\sin \alpha z| + |\cos \alpha z|$$

$$\leq 2 \exp(\alpha |z|) \quad .$$

Thus if m is given by (4.6),

$$|g_n(z)| \le \begin{cases} 2\alpha^\beta e^{\tau|z|} & (|z - x_n| \ge 1) \\ (\alpha m)^\beta & (|z - x_n| < 1) \end{cases}$$

$(n = 0, 1, 2, \ldots)$. Consequently,

$$|f(z)| \le \sum_{|z-x_n|<1} + \sum_{|z-x_n|\ge 1} |w_n| \, |g_n(z)|$$

$$\le [(\alpha m)^\beta + 2\alpha^\beta \exp(\tau|z|)] \sum_0^\infty |w_n|$$

$$= O(\exp(\tau|z|)) \quad ,$$

where the last line follows from the fact that any sequence in ℓ^p $(0 < p < 1)$ is already in ℓ^1.

Corollary 4.2. When $0 < p < 1$ and $0 < \tau < \infty$, E_τ^p is not r-convex for $p < r \le 1$, and contains proper, closed, weakly dense subspaces.

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