Mixing at Double Exponential Rate and Rigidity in Analytic Dynamics

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
2026

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DEDICATION

To Ilia and Fock, the dachshund.

ACKNOWLEDGEMENTS

I would like to express my gratitude to my advisor, Prof. Ralf Spatzier, for the guidance, patience, and support throughout my doctoral studies. His mathematical insight and generosity with both time and ideas have profoundly influenced both this dissertation and my growth as a mathematician.

I am grateful to Prof. Dick Canary, Prof. Vilma Mesa, and Prof. Alex Wright for serving on my dissertation committee and their support throughout my years in Michigan.

One of the greatest privileges of graduate study has been the opportunity to engage with the broader mathematics community. I have benefited enormously from conversations, seminars, conferences, and collaborations with mathematicians from many different universities and institutes. In particular, I would like to thank Aaron Brown, Francisco Arana-Herrera, Alex Blumenthal, Keith Burns, Jon Chaika, David Fisher, Sergey Fomin, Boris Hasselblatt, Asaf Katz, Osama Khalil, Sarah Koch, Isaia Nisoli, Roland Roeder, Julia Slipantschuk, Kurt Vinhage, Amie Wilkinson, Maciej Zworski, and many others for inspiring conversations and insights that influenced this work in various ways.

I am especially thankful to the participants of various seminars, reading groups, workshops, and conferences where portions of this work were presented. Their questions, suggestions, and perspectives greatly improved both the mathematics and its exposition.

Finally, I am especially grateful to those with whom I had the privilege of learning mathematics together over the years: Karen Butt, Marisa Cantarino, Ilia Nekrasov, Bastian Nunez, Carsten Petersen, Yuping Ruan, and Truong Vu.

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ABSTRACT

This work studies mixing in dynamical systems, focusing on how fast correlations between observables decay. While most previously studied systems have polynomial or exponential rates, we investigate systems with double exponential decay.

We show that all ergodic surjective linear endomorphisms of tori mix at a double exponential rate for analytic observables. In dimension one, we provide a complete classification for finite Blaschke products on the circle: the rate of mixing (no mixing, exponential, or double exponential) is determined by the value of the derivative of the Blaschke product at its fixed point. We extend the result to free semigroup actions generated by Blaschke products.

We also show that double exponential mixing is not rigid: it is not stable under perturbations and does not imply conjugacy to linear models. In higher dimensions, we construct families of examples of nonlinear maps with and without double exponential rate of mixing, and prove that certain partially hyperbolic systems never have this property.

Our approach uses the Koopman precomposition operator acting on spaces of hyperfunctions (the dual space of analytic functions). In this setting, the operator is non-self-adjoint, compact, and quasi-nilpotent, with its spectrum reduced to zero, which can be considered an indicator of double exponential decay.

Finally, we apply mixing rates to the cohomological equation. We prove the analyticity of solutions under suitable conditions and establish existence and uniqueness results in anisotropic spaces.

Introduction

One of the central concepts in dynamics is *mixing*, which captures how correlations between observables decay under the action of a semigroup (or a group) by measure-preserving transformations. Understanding not only whether mixing occurs but also how fast it occurs—the rate of decay of correlations—is a key question. The rate depends on both the dynamical features of the action and the regularity of the observables. Typical results establish *polynomial* or *exponential* rates of mixing for *Hölder* observables [KH95].

Exponential mixing is one of the major topics in dynamics. The Bowen–Ruelle conjecture states that mixing Anosov flows are exponentially mixing. The first results were established by Dolgopyat in [Dol98]; later, Liverani extended these results to more general contact Anosov flows [Liv04]. Recently, Tsujii and Zhang proved exponential mixing for smooth 3-dimensional Anosov flows [TZ23]. This raises a natural question of whether there are systems with an even faster rate of decay.

Notably, see [Bal00], two classical linear models: the Bernoulli doubling map $x \mapsto 2x \pmod 1$ and the CAT map $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, both exhibit *double exponential decay* of correlations for *analytic* observables. We show that all surjective *ergodic* endomorphisms of tori are mixing at a double exponential rate.

Theorem A. *Let μ be the Lebesgue measure on $\mathbb{T}^d$ and $T \in \mathrm{M}(d, \mathbb{Z})$, $\det T \neq 0$, be ergodic. Then T is mixing at a double exponential rate, i.e., for any $0 < \theta < 1$, there exist $C, \beta, \gamma > 0$ such that for any analytic functions $f, g \in \mathcal{H}_\theta$ (see Definition 2.1.1) and $n \geq 0$*

$$\left| \int_{\mathbb{T}^d} (f \circ T^n)\ \overline{g}\, d\mu - \int_{\mathbb{T}^d} f\, d\mu \int_{\mathbb{T}^d} \overline{g}\, d\mu \right| \leq C e^{-\beta e^{\gamma n}} \|f\|_\theta \|g\|_\theta,$$

where T^n denotes the n-fold composition with itself.

Theorem A is proven in Chapter 2. The fact that ergodic automorphisms of tori are exponentially mixing on Hölder observables was proven by Lind in [Lin82], which is based on Katznelson's lemma [Kat71] on *Diophantine approximation*. This leads us to the following question:

Q1 (Rigidity) Are ergodic linear maps the only ones that mix at a *double exponential rate*?

Related questions have been asked in the context of *dissipation time* under stochastic perturbations in [FNW04] and [FI19]. From a physics perspective, double exponential decay of correlations is likely to be rare and easily disrupted by perturbations. In the mixing hierarchy, mixing at a double exponential rate on analytic observables implies exponential mixing on Hölder observables (see Lemma B.0.1).

In dimension one, we provide a complete answer to **Q1** in the special case where μ is the Lebesgue measure on $\mathbb{S}^1$, and $T \in \mathcal{B}$ is a finite *Blaschke product*

$$\mathcal{B} := \left\{ e^{i\psi} \prod_{i=0}^{q} \frac{z - \lambda_i}{1 - \overline{\lambda_i} z} \,:\, \psi \in [0, 2\pi),\ |\lambda_i| < 1,\ i = 0, \dots, q \ \text{ for } \ q \in \mathbb{Z}_{\geq 0} \right\}.$$

These maps extend analytically to a complex neighborhood of the unit disk and preserve $\mathbb{S}^1$. Moreover, they preserve the Lebesgue measure μ whenever $T \in \mathcal{B}^0$, where

$$\mathcal{B}^0 := \{T \in \mathcal{B} : T(0) = 0\}.$$

For $q \neq 0$, these are degree $(q+1)$ expanding maps (see [Mar83]), which are known to be exponentially mixing [Shu69]. For a fixed $q \neq 0$, McMullen studied the topological dynamics of these maps in [McM10].

Theorem B. *Let μ be the Lebesgue measure on $\mathbb{S}^1$, and $T \in \mathcal{B}^0$. Then*

- *T is not mixing if and only if $|T'(0)| = 1$,*
- *T is exponentially mixing if and only if $|T'(0)| < 1$,*
- *T is mixing at a double exponential rate if and only if $T'(0) = 0$.*

By definition, $z = 0$ is a fixed point for any $T \in \mathcal{B}^0$. In complex dynamics, $z = 0$ is called *neutral* (or *indifferent*) fixed point if $|T'(0)| = 1$, *attracting* if $|T'(0)| < 1$, and *super-attracting* if $T'(0) = 0$. Therefore, we see that mixing at a double exponential rate on the circle—a global dynamical property—can be determined by the local behavior of T at its fixed point within the unit disk.

Theorem B is proven in Chapter 1. In fact, we prove a stronger statement for the free semigroup actions on the circle generated by finite Blaschke products (see Theorem 1.1.4). The first bullet of the theorem follows from the Schwarz lemma, since $|T'(0)| = 1$ is equivalent to T being a rotation, and rotations are never mixing (see Section 1.3.1). The second bullet follows from [Shu69], since $|T'(0)| < 1$ implies that T is an analytic, uniformly expanding

map of the circle (in Section 1.4.2 we give an alternative proof based on an application of the Schwarz lemma).

In the proof of the last bullet, we use a *functional approach*, studying properties of the operator $\mathcal{L}$ on a specifically designed (dynamics-tailored) space of *hyperfunctions*, which is the dual space of analytic functions. Rather than estimating $\|\mathcal{L}\|$ and iterating the estimate (an approach typical for obtaining exponential decay), we estimate $\|\mathcal{L}^n\|$ directly. We prove that $\|\mathcal{L}^n\|$ decays double exponentially in n (see Theorem 1.3.1) and provide explicit estimates for C, β and γ. As a consequence, we show that when restricted to the subspace orthogonal to constants, $\mathcal{L}$ is a compact, *quasi-nilpotent operator* (that is, its spectral radius is equal to zero, see Proposition 1.6.1). This phenomenon is known in dynamics as *trivial Pollicott-Ruelle resonances* (see [Bal00]). Note that these operators are *non-self-adjoint*, so they are not trivial even if the spectrum is $\{0\}$.

One feature of our approach is that, when we prove that a map does *not* mix at a double exponential rate, we provide an *explicit lower bound* on the exponential decay rate (see Lemma 1.3.2). In contrast, the existing literature typically focuses only on upper bounds. We also study *the stability* of the double exponential rate of decay of correlations under C^1-perturbations. In particular, we show that mixing at a double exponential rate does not imply C^1-conjugacy to the linear model (see Example 1.5.1). This provides a negative answer to the rigidity question **Q1**, even up to a C^1-conjugacy.

Theorem B and Example 1.5.1 are easily generalized to direct sums of such systems, providing higher dimensional examples of non-linear maps with different mixing rates on $\mathbb{T}^d$. Next, we study hyperbolic examples.

Theorem C. *There exist two families of volume-preserving C^1-perturbations of a hyperbolic map $T_0 \in \mathrm{GL}(2d, \mathbb{Z})$ such that one is mixing at a double exponential rate, and the other is not. Moreover, neither is C^1-conjugate to T_0.*

Theorem C is proven in Chapter 3. In dimension two, our families fit into the framework of [PS08] and [PR10].

Next, we show that a certain family of *partially hyperbolic* nonlinear maps *never* mixes at a double exponential rate.

Theorem D. *Skew-product isometric extension of the ergodic surjective linear endomorphism $A \in \mathrm{M}(d, \mathbb{Z}_{\geq 0})$, $\det A \neq 0$,*

$$\begin{aligned} T : \mathbb{T}^d \times \mathbb{S}^1 &\to \mathbb{T}^d \times \mathbb{S}^1 \\ (z, w) &\mapsto (z^A, B(z)\, w) \end{aligned}$$

is not mixing at a double exponential rate for any finite Blaschke product

$$\begin{aligned} B : \mathbb{T}^d &\to \mathbb{T} \\ z &\mapsto B_1(z_1) \dots B_d(z_d) \end{aligned}$$

with $B_j \in \mathcal{B}$ *for each* $1 \leq j \leq d$.

Theorem D is proven in Chapter 4. In [Dol02], Dolgopyat proved that skew-product isometric extensions of *uniformly expanding* maps are exponentially mixing under a suitable cohomological condition. He also proved that, under a Diophantine condition, isometric extensions of *hyperbolic* maps are *rapidly mixing*, meaning that the correlations decay faster than any power of n. Whether exponential mixing holds in this setting remains an open question, although it is expected to follow from the techniques of Tsujii and Zhang [TZ23] (Félix Lequen, personal communication). Note that our setting includes not only uniformly expanding and hyperbolic linear maps, but also ergodic partially hyperbolic linear maps.

In the last two chapters (Chapters 5 and 6), we apply the rate of mixing to study cohomological equations. The rate of decay of correlations is used to analyze the existence (or obstructions to existence) and regularity of solutions f to the *cohomological equation*

$$f \circ T - f = g, \tag{1}$$

given g in some functional class (e.g., [KS94] and [GS14]). Classical Livšic theory [Liv72] provides an answer in the hyperbolic case within the Hölder class. In [dlL97], de la Llave considered an *analytic* version for hyperbolic maps. Wilkinson [Wil13] extended many regularity results to partially hyperbolic systems, leaving the following question.

Q2 (Cohomological equation) Is there analytic regularity of solutions to the cohomological equation for partially hyperbolic systems?

In the following theorem (proven in Chapter 5), we prove the analyticity of solutions to the cohomological equation in the case of ergodic $T \in \mathrm{GL}(d, \mathbb{Z})$ under a technical assumption on the center directions, improving Veech's C^∞ regularity result [Vee86].

Theorem E. *Let* $T \in \mathrm{GL}(d, \mathbb{Z})$ *be an ergodic automorphism of* $\mathbb{T}^d$. *Suppose that the Jordan blocks corresponding to eigenvalues of modulus one have dimension at most two. If* $g \in \mathcal{A}_\theta$ *(see Definition 5.1.1) satisfies the periodic point condition (i.e., the sums of* g *over periodic cycles for* T *vanish), then* (1) *has a solution* $f \in \mathcal{A}_\eta$, *for some* $0 < \eta < 1$.

Also, for hyperbolic $T \in \mathrm{GL}(d, \mathbb{Z})$, we prove the existence and uniqueness of solutions in *anisotropic Hilbert spaces*. This is a surprising result because no periodic point condition is needed. However, *anisotropic Hilbert spaces* are not spaces of regular functions.

Theorem F. *Let $T \in \mathrm{GL}(d, \mathbb{Z})$ be hyperbolic. If $g \in \mathcal{H}^0_{\theta_u, \theta_s}$ (see Definition 6.1.1), then* (1) *has a unique solution $f \in \mathcal{H}^0_{\theta_u, \theta_s}$.*

Theorem F is proven in Chapter 6.

We conclude the introduction with several *open questions* and directions for future research.

One of our goals is to understand nonlinear ergodic partially hyperbolic maps on tori, for which even stable ergodicity remains open in general with remarkable progress in [RH05]. Another natural setting is that of ergodic automorphisms of nilmanifolds, where exponential mixing already appears as a nontrivial phenomenon (see [GS14] and [BV24] based on [GT12]). This leads us to ask what can be said about ergodic automorphisms of more general homogeneous spaces.

It is also natural to ask whether mixing at a double exponential rate has implications for the Central Limit Theorem. Does double exponential rate of *mixing of all orders* follow automatically? More generally, what happens if the Lebesgue measure is replaced by another invariant measure, such as the harmonic measure or the measure of maximal entropy?

Another direction is the study of the regularity of solutions to cohomological equations for higher-dimensional nonlinear partially hyperbolic maps constructed from Blaschke products. We also aim to apply our results to problems of analytic conjugacy. Namely, if f and g are analytic maps and there exists a conjugacy h such that $h \circ g = f \circ h$, must h necessarily be analytic?

CHAPTER 1

Mixing for Free Semigroup Actions of Blaschke Products on the Circle

1.1 Statement of the Result

Let $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ denote the unit circle and let μ denote the Lebesgue measure on $\mathbb{S}^1$. Let

$$\phi_1, \ldots, \phi_N$$

be analytic functions on the open unit disk that admit continuous extensions to the closed unit disk and map the unit circle to itself, that is,

$$|\phi_j(z)| = 1 \quad \text{for} \quad |z| = 1, \quad 1 \leq j \leq N.$$

These maps form the space of finite *Blaschke products*

$$\mathcal{B} := \left\{ e^{i\psi} \prod_{i=0}^{q} \frac{z - \lambda_i}{1 - \overline{\lambda_i} z} \ : \ \psi \in [0, 2\pi), |\lambda_i| < 1, i = 0, \ldots, q, \ \text{ for } \ q \in \mathbb{Z}_{\geq 0} \right\}.$$

By Corollary A.0.1, each ϕ_j preserves μ if and only if $\phi_j(0) = 0$. Let

$$\mathcal{B}^0 := \{\phi \in \mathcal{B} : \phi(0) = 0\}.$$

The collection of functions $\phi_j \in \mathcal{B}^0, j = 1, \ldots, N$, generates an action of the free semigroup $\mathbb{F}_N^+$ on the circle defined by

$$\tau(a)z = \phi_{i_n} \circ \phi_{i_{n-1}} \circ \cdots \circ \phi_{i_1}(z), \quad a = i_1 i_2 \ldots i_n \in \mathbb{F}_N^+, \quad z \in \mathbb{S}^1.$$

In [Pom81], Pommerenke proved that if none of the ϕ_j are rotations, then τ is *(strongly)*

mixing; that is, for any $f, g \in L_2(\mathbb{S}^1, \mu)$, the *correlation function*

$$\mathcal{C}_{f,g}(a) := \left| \int_{\mathbb{S}^1} (f \circ \tau(a))\, \overline{g}\, d\mu - \int_{\mathbb{S}^1} f\, d\mu \int_{\mathbb{S}^1} \overline{g}\, d\mu \right| \to 0 \quad \text{as} \quad |a| \to \infty,$$

where $|a|$ denotes the word length of $a \in \mathbb{F}_N^+$.

We study the *quantitative mixing* for the action τ on the spaces of analytic functions $\mathcal{H}_\theta \subset L_2(\mathbb{S}^1, \mu)$.

Definition 1.1.1 (Analytic functions)**.** *Let $0 < \theta < 1$. The space of analytic functions $\mathcal{H}_\theta$ is defined as the closure of all linear combinations of $z^j, j \in \mathbb{Z}$, with respect to the norm*

$$\|f\|_\theta := \sqrt{\sum_{k \in \mathbb{Z}} |f_k|^2 \theta^{-2|k|}},$$

where f_k are the Fourier coefficients of f. We denote by $\mathcal{H}_\theta^0$ the subspace of $\mathcal{H}_\theta$ orthogonal to constants, that is, those with $f_0 = 0$.

Definition 1.1.2 (Mixing at a double exponential rate)**.** *An action τ is said to be mixing at a double exponential rate if for any $0 < \theta < 1$ there exist $\beta > 0$, $\gamma > 0$, and $C > 0$ such that for any $f, g \in \mathcal{H}_\theta$ and any $a \in \mathbb{F}_N^+$*

$$\mathcal{C}_{f,g}(a) \le C e^{-\beta e^{\gamma |a|}} \|f\|_\theta \|g\|_\theta.$$

Let C^α denote the space of Hölder continuous functions with $\alpha \in (0, 1]$.

Definition 1.1.3 (Exponential mixing)**.** *An action τ is said to be exponentially mixing on C^α if there exist $\gamma > 0$ and $C > 0$ such that for any $f, g \in C^\alpha$ and any $a \in \mathbb{F}_N^+$*

$$\mathcal{C}_{f,g}(a) \le C e^{-\gamma |a|} \|f\|_{C^\alpha} \|g\|_{C^\alpha}.$$

One can show that mixing at a double exponential rate on analytic observables implies exponential mixing on Hölder observables (see Lemma B.0.1). Below, we provide a complete classification of the mixing rate for the action τ distinguishing between *exponential* and *double exponential* behavior.

Theorem 1.1.4. *Let τ be as above. Then*

1. *τ is not mixing if and only if $|\phi_j'(0)| = 1$ for some $1 \le j \le N$,*
2. *τ is exponentially mixing if and only if $|\phi_j'(0)| < 1$ for all $1 \le j \le N$,*
3. *τ is mixing at a double exponential rate if and only if $\phi_j'(0) = 0$ for all $1 \le j \le N$.*

Note that, by Schwarz lemma, $|\phi_j'(0)|$ cannot be greater than 1. Thus, this theorem provides a complete classification of the mixing rates of the actions τ on $\mathcal{H}_\theta$.

The constants γ, β, and C in Definitions 1.1.2 and 1.1.3 are computed explicitly below (see equations (1.5) and (1.6)).

Theorem B is a direct corollary of Theorem 1.1.4 in the case $N = 1$. It implies that the collection of maps ϕ mixing at a double exponential rate forms a (complex) codimension one subspace within $\mathcal{B}^0$. In Proposition 1.5.2, we show that within the space of Lebesgue measure preserving Blaschke products $\mathcal{B}^0$, the ones that mix at a double exponential rate, form a nowhere dense set with respect to C^1-topology on $\mathbb{S}^1$.

The maps $\phi \in \mathcal{B}^0$ satisfying $|\phi'(0)| < 1$ are degree $q+1$, uniformly expanding analytic maps on the circle (see [Mar83]), which are known to be exponentially mixing on Hölder observables ([Shu69]). Below (see Section 1.4.2), we prove exponential mixing on a smaller space of analytic functions using a complex analytic approach.

Theorem 1.1.4 is easily generalized to direct sums of such systems, providing higher dimensional nonlinear analogs on $\mathbb{T}^d$.

Before we prove Theorem 1.1.4, let us provide some examples.

1.2 Motivating Example

Example 1.2.1. *The Bernoulli doubling map* $\phi : \mathbb{S}^1 \to \mathbb{S}^1$

$$\phi(z) = z^2$$

is mixing at a double exponential rate on $\mathcal{H}_\theta$*, i.e., for any* $f, g \in \mathcal{H}_\theta$ *and any* $n \geq 0$

$$\mathcal{C}_{f,g}(n) \leq Ce^{-\beta e^{\gamma n}} \|f\|_\theta \|g\|_\theta \tag{1.1}$$

with $C = \theta$ *,* $\beta = -\ln \theta$*, and* $\gamma = \ln 2$*. Moreover, the rate of mixing is sharp, i.e., there exist* $f, g \in \mathcal{H}_\theta$ *such that for some* $0 < C' < C$ *and* $\beta' > \beta > 0$*, and any* $n \geq 0$

$$\mathcal{C}_{f,g}(n) \geq C'e^{-\beta' e^{\gamma n}} \|f\|_\theta \|g\|_\theta.$$

Proof. In the Fourier basis $f(z) = \sum_{k\in\mathbb{Z}} f_k z^k$,

$$(f \circ \phi^n)(z) = f(z^{2^n}) = \sum_{k\in\mathbb{Z}} f_k \, z^{2^n k}.$$

Now, let us compute the correlation function explicitly. Without loss of generality, we can

assume that $f_0 = 0$, then for $f, g \in \mathcal{H}_\theta$, we compute the correlation function

$$\begin{aligned}\mathcal{C}_{f,g}(n) &= \left| \int_{\mathbb{S}^1} (f \circ \phi^n)\, \overline{g}\, d\mu \right| \\ &= \left| \sum_{0 \neq m \in \mathbb{Z}} (f \circ \phi^n)_m\, \overline{g_m} \right| \\ &= \left| \sum_{0 \neq k \in \mathbb{Z}} f_k\, \overline{g_{2^n k}} \right|.\end{aligned}$$

Next, we multiply and divide each term in the sum by $\theta^{(2^n+1)|k|}$ and use the Cauchy-Schwarz inequality

$$\begin{aligned}\mathcal{C}_{f,g}(n) &= \left| \sum_{0 \neq k \in \mathbb{Z}} \theta^{(2^n+1)|k|} f_k\, \theta^{-|k|}\, \overline{g_{2^n k}}\, \theta^{-2^n|k|} \right| \\ &\leq \sup_{0 \neq k \in \mathbb{Z}} \left(\theta^{(2^n+1)|k|} \right) \cdot \sqrt{\sum_{0 \neq k \in \mathbb{Z}} |f_k|^2 \theta^{-2|k|}} \cdot \sqrt{\sum_{0 \neq k \in \mathbb{Z}} |g_{2^n k}|^2 \theta^{-2^n|k|}} \\ &\leq \theta^{(2^n+1)} \|f\|_\theta \|g\|_\theta.\end{aligned}$$

Thus, we prove (1.1) with

$$C = \theta, \quad \beta = -\ln \theta, \quad \gamma = \ln 2.$$

Finally, we prove the sharpness of the rate. Let $f(z) = z$ and $g(z) = \sum_{0 \neq k \in \mathbb{Z}} \frac{1}{|k|} \theta^{|k|} z^k$. Note that

$$\|f\|_\theta = \theta^{-1} \quad \text{and} \quad \|g\|_\theta = \sqrt{2 \sum_{k \geq 1} \frac{1}{|k|^2}} = \frac{\pi}{\sqrt{3}}.$$

For this choice of f and g the correlation function

$$\mathcal{C}_{f,g}(n) = \overline{g_{2^n}} = \frac{\theta^{2^n}}{2^n} = e^{-\beta e^{\gamma n} - \gamma n} \geq e^{-\beta' e^{\gamma n}} = C' e^{-\beta' e^{\gamma n}} \|f\|_\theta \|g\|_\theta,$$

for some $\beta' > \beta > 0$ and $0 < C' = \theta \frac{\sqrt{3}}{\pi} < \theta = C$. □

The pullback under the Bernoulli doubling map $\phi(z) = z^2$ defines the *Koopman* precomposition operator

$$\begin{aligned}\mathcal{L} : L_2(\mathbb{S}^1, \mu) &\to L_2(\mathbb{S}^1, \mu) \\ f &\mapsto f \circ \phi.\end{aligned}$$

Next, we study the operator properties of $\mathcal{L}$ on different spaces. In particular, we will see that $\mathcal{L}$ has nice spectral properties on the space of *hyperfunctions*.

Definition 1.2.2. *Let $0 < \theta < 1$. The space of hyperfunctions $\mathcal{H}_{\theta^{-1}}$ is defined as the closure of all linear combinations of $z^j, j \in \mathbb{Z}$, with respect to the norm*

$$\|f\|_{\theta^{-1}} := \sqrt{\sum_{k\in\mathbb{Z}} |f_k|^2 \theta^{2|k|}},$$

where f_k are the Fourier coefficients of f. We denote by $\mathcal{H}^0_{\theta^{-1}} := \mathcal{H}_{\theta^{-1}} \ominus \mathbb{C}$ the subspace of $\mathcal{H}_{\theta^{-1}}$ orthogonal to constants, that is, those with $f_0 = 0$.

Note that $\mathcal{H}_{\theta^{-1}}$ can be identified with $\mathcal{H}^*_\theta$ (the space dual to analytic functions $\mathcal{H}_\theta$). Hyperfunctions are allowed to have Fourier coefficients that grow exponentially in n. The use of these types of spaces is not new in dynamics; see for example [FR06].

Proposition 1.2.3. *For $0 < \theta < 1$, the operator $\mathcal{L} : \mathcal{H}_{\theta^{-1}} \to \mathcal{H}_{\theta^{-1}}$ is trace class. Moreover, $\mathcal{L}$ is a strict contraction when restricted to the subspace orthogonal to constants $\mathcal{L} : \mathcal{H}^0_{\theta^{-1}} \to \mathcal{H}^0_{\theta^{-1}}$ with*

$$\|\mathcal{L}\|_{\mathcal{H}^0_{\theta^{-1}} \to \mathcal{H}^0_{\theta^{-1}}} = \theta.$$

Proof. We consider the orthonormal basis $e_k = \theta^{-|k|} z^k, k \in \mathbb{Z}$ of $\mathcal{H}_{\theta^{-1}}$ with respect to its norm $\|f\|_{\theta^{-1}} := \sqrt{\sum_{k\in\mathbb{Z}} |f_k|^2 \theta^{2|k|}}$.

By definition,

$$\mathcal{L} e_k = \theta^{-|k|}\, z^{2k} = \theta^{|k|} e_{2k}.$$

Therefore, the matrix of $\mathcal{L}$ in the orthonormal basis $\{e_k\}_{k\in\mathbb{Z}}$ has entries

$$\mathcal{L}_{jk} := \langle \mathcal{L} e_k, e_j \rangle_{\theta^{-1}} = \theta^{|k|} \delta_{j,2k}. \tag{1.2}$$

As we can see, this is a very sparse matrix (with many zero elements). Next, let us compute the singular values of $\mathcal{L}$, which by definition are the eigenvalues of $\sqrt{\mathcal{L}^*\mathcal{L}}$. We find

$$\begin{aligned}
\mathcal{L}^*_{jk} :&= \langle \mathcal{L}^* e_k, e_j \rangle_{\theta^{-1}} \\
&= \langle e_k, \mathcal{L} e_j \rangle_{\theta^{-1}} \\
&= \overline{\langle \mathcal{L} e_j, e_k \rangle_{\theta^{-1}}} \\
&= \overline{\mathcal{L}_{kj}} \\
&= \theta^{|j|} \delta_{k,2j}.
\end{aligned}$$

Hence,

$$\begin{aligned}(\mathcal{L}^*\mathcal{L})_{jk} &= \sum_{m\in\mathbb{Z}} \mathcal{L}^*_{jm}\mathcal{L}_{mk} \\ &= \sum_{m\in\mathbb{Z}} \theta^{|m|}\delta_{m,2j}\theta^{|m|}\delta_{m,2k} \\ &= \theta^{2|j|}\delta_{j,k}.\end{aligned}$$

Thus, $\mathcal{L}^*\mathcal{L}$ is diagonal in the basis $\{e_j\}$ with eigenvalues $\theta^{2|j|}$. Therefore, $\sqrt{\mathcal{L}^*\mathcal{L}}$ is diagonal in the basis $\{e_k\}$ with eigenvalues $\theta^{|j|}$. Hence, the singular values of $\mathcal{L}$ are

$$\sigma_j(\mathcal{L}) = \theta^{|j|}, \qquad j \in \mathbb{Z}_{\geq 0},$$

with multiplicity 2 for $j > 0$ (and multiplicity 1 for $j = 0$).

Because $0 < \theta < 1$,

$$\sum_{j\in\mathbb{Z}} \sigma_j(\mathcal{L}) = 1 + 2\sum_{j\geq 1} \theta^j = 1 + \frac{2\theta}{1-\theta} < \infty,$$

and hence $\mathcal{L}$ is trace class.

The operator norm of $\mathcal{L}$ is equal to the largest singular value

$$\|\mathcal{L}\|_{\mathcal{H}_{\theta^{-1}}\to\mathcal{H}_{\theta^{-1}}} = \max_{j\geq 0} \sigma_j(\mathcal{L}) = \sigma_0(\mathcal{L}) = 1.$$

When restricted to the subspace orthogonal to constants $\mathcal{H}^0_{\theta^{-1}}$, we have

$$\|\mathcal{L}\|_{\mathcal{H}^0_{\theta^{-1}}\to\mathcal{H}^0_{\theta^{-1}}} = \max_{j>0} \sigma_j(\mathcal{L}) = \sigma_1(\mathcal{L}) = \theta < 1.$$

Thus, $\mathcal{L}$ is a strict contraction on $\mathcal{H}^0_{\theta^{-1}}$. □

Proposition 1.2.4. *The spectrum of $\mathcal{L}$ on $\mathcal{H}_{\theta^{-1}}$*

$$\sigma(\mathcal{L}; \mathcal{H}_{\theta^{-1}}) = \{0, 1\},$$

with 1 being a simple eigenvalue corresponding to a constant eigenfunction.

The spectrum of $\mathcal{L}$ restricted to its invariant subspace $\mathcal{H}^0_{\theta^{-1}} := \mathcal{H}_{\theta^{-1}} \ominus \mathbb{C}$ is trivial

$$\sigma(\mathcal{L}; \mathcal{H}^0_{\theta^{-1}}) = \{0\}.$$

Proof. First, there is an $\mathcal{L}$-invariant decomposition

$$\mathcal{H}_{\theta^{-1}} = \mathcal{H}^+_{\theta^{-1}} \oplus \mathbb{C} \oplus \mathcal{H}^-_{\theta^{-1}},$$

where $\mathcal{H}^{\pm}_{\theta^{-1}}$ is the subspace generated by e_k with $k > 0$ ($k < 0$ respectively). Note that $\mathcal{H}^+_{\theta^{-1}} \oplus \mathcal{H}^-_{\theta^{-1}} = \mathcal{H}^0_{\theta^{-1}}$. Then

$$\sigma(\mathcal{L}; \mathcal{H}_{\theta^{-1}}) = \sigma(\mathcal{L}; \mathcal{H}^+_{\theta^{-1}}) \cup \sigma(\mathcal{L}; \mathbb{C}) \cup \sigma(\mathcal{L}; \mathcal{H}^-_{\theta^{-1}}).$$

We have $\lambda = 1$ as a simple eigenvalue with $e_0 = 1$ being the corresponding eigenfunction

$$\sigma(\mathcal{L}; \mathbb{C}) = \{1\}.$$

From (1.2), we see that $\mathcal{L}$ restricted to $\mathcal{H}^+_{\theta^{-1}}$ or $\mathcal{H}^-_{\theta^{-1}}$ has an upper triangular structure with zeros on the diagonal. Since $\mathcal{L}$ is a trace class operator, it is compact, so is its restriction to a closed invariant subspace. If an operator has an upper triangular form and is compact, then its spectrum is equal to the set of matrix entries on the main diagonal together with 0.

So,

$$\sigma(\mathcal{L}; \mathcal{H}^+_{\theta^{-1}}) = \sigma(\mathcal{L}; \mathcal{H}^-_{\theta^{-1}}) = \{0\}.$$

Thus,

$$\sigma(\mathcal{L}; \mathcal{H}_{\theta^{-1}}) = \{0, 1\} \quad \text{and} \quad \sigma(\mathcal{L}; \mathcal{H}^0_{\theta^{-1}}) = \{0\}.$$

□

Operators with zero spectral radius are called *quasi-nilpotent*.

Corollary 1.2.5 (Super exponential mixing)**.** *Assume that $\mathcal{L} : \mathcal{H}^0_{\theta^{-1}} \to \mathcal{H}^0_{\theta^{-1}}$ is quasi-nilpotent. Then, for any $\epsilon > 0$, there exists $C > 0$ such that for any $f \in \mathcal{H}^0_\theta$, $g \in \mathcal{H}_\theta$ and $n \geq 0$*

$$|\langle \mathcal{L}^n f, g \rangle_{L_2}| \leq C\epsilon^n \|f\|_\theta \|g\|_\theta.$$

Proof. From the Gelfand formula for the spectral radius

$$r(\mathcal{L}; \mathcal{H}^0_{\theta^{-1}}) = \lim_{n \to \infty} \|\mathcal{L}^n\|_{\mathcal{H}^0_{\theta^{-1}} \to \mathcal{H}^0_{\theta^{-1}}},$$

it follows that having a zero spectral radius is equivalent to super exponential decay of the norms of compositions of $\mathcal{L}$ with itself. In other words, for any $\epsilon > 0$ there exists $C > 0$ such that for any $n \geq 0$

$$\|\mathcal{L}^n\|_{\mathcal{H}^0_{\theta^{-1}} \to \mathcal{H}^0_{\theta^{-1}}} \leq C\epsilon^n.$$

Applying the Cauchy-Schwarz duality inequality, it follows that for any $f \in \mathcal{H}^0_{\theta^{-1}}$ and $g \in \mathcal{H}_\theta$

$$|\langle \mathcal{L}^n f, g\rangle_{L_2}| \le \|\mathcal{L}^n f\|_{\theta^{-1}} \|g\|_\theta \le C\epsilon^n \|f\|_{\theta^{-1}} \|g\|_\theta.$$

In particular, for any $f \in \mathcal{H}^0_\theta \subset \mathcal{H}^0_{\theta^{-1}}$, $g \in \mathcal{H}_\theta$ and any $n \ge 0$

$$|\langle \mathcal{L}^n f, g\rangle_{L_2}| \le C\epsilon^n \|f\|_\theta \|g\|_\theta.$$

□

Remark 1.2.6. *When $\theta = 1$, the norms $\|\cdot\|_\theta$ and $\|\cdot\|_{\theta^{-1}}$ degenerate to the standard $L_2(\mathbb{S}^1, \mu)$-norm. In this case, $\mathcal{L}$ is an isometry:*

$$\|\mathcal{L} f\|_{L_2} = \|f\|_{L_2} \quad \textit{for any} \quad f \in L_2(X, \mu).$$

Therefore,

$$\mathcal{L}^*\mathcal{L} = \mathrm{Id}, \qquad \sigma_j(\mathcal{L}) = 1, \qquad j \in \mathbb{Z}.$$

The condition $\theta < 1$ is therefore essential for trace class and contraction properties.

In fact, for any isometry $\mathcal{L}$, the decay of correlations on $L^0_2(X, \mu)$ can be made arbitrarily slow, as the next lemma shows. That is why, in order to gain uniform control of the decay of correlations, spaces other than $L_2(X, \mu)$ need to be considered.

Lemma 1.2.7 (No uniform quantitative mixing on L_2-space)**.** *Let $\mathcal{L} : L_2(X, \mu) \to L_2(X, \mu)$ be an isometry. Then there is no uniform correlation decay, i.e., for any sequence $\epsilon_n \to 0$ there exist $f, g \in L^0_2(X, \mu)$ such that*

$$|\langle \mathcal{L}^n f, g\rangle_{L_2}| > \epsilon_n \|f\|_{L_2} \|g\|_{L_2}.$$

Proof. We prove it by contradiction. Assume that there is a sequence $\epsilon_n \to 0$ such that

$$|\langle \mathcal{L}^n f, g)\rangle_{L_2}| \le \epsilon_n \|f\|_{L_2} \|g\|_{L_2}$$

for any $f, g \in L^0_2$. Then for any $n \ge 0$

$$\|\mathcal{L}^n\|_{L^0_2 \to L^0_2} = \sup\{|\langle \mathcal{L}^n f, g\rangle_{L_2}| : f, g \in L^0_2, \|f\|_{L_2} = \|g\|_{L_2} = 1\} \le \epsilon_n \to 0. \tag{1.3}$$

On the other hand, let us notice that if $\mathcal{L}$ is an isometry, then $\mathcal{L}^n$ is an isometry for any $n \ge 0$

$$\|\mathcal{L}^n f\|_{L_2} = \|\mathcal{L}(\mathcal{L}^{n-1} f)\|_{L_2} = \|(\mathcal{L}^{n-1} f)\|_{L_2} = \cdots = \|f\|_{L_2}.$$

So, $\|\mathcal{L}^n\|_{L_2^0 \to L_2^0} = 1$ for any $n \geq 0$, contradicting (1.3). $\square$

1.3 Proof of Theorem 1.1.4

1.3.1 Rotations

Proof of Part 1 of Theorem 1.1.4. Every μ-preserving Blaschke product ϕ_j is represented by its Fourier series, converging everywhere in some neighborhood of the unit circle

$$\phi_j(z) = \sum_{k=1}^{\infty} \frac{\phi_j^{(k)}(0)}{k!} z^k.$$

Since ϕ_j takes the unit circle into itself, we have a restriction on its coefficients

$$\sum_{k=1}^{\infty} \left| \frac{\phi_j^{(k)}(0)}{k!} \right|^2 = \int_{\mathbb{S}^1} |\phi_j(z)|^2 \, d\mu(z) = 1.$$

From here we see that the first derivative has to be bounded by 1

$$|\phi_j'(0)| \leq 1.$$

Moreover, if

$$|\phi_j'(0)| = 1,$$

then all other coefficients have to vanish. Thus, ϕ_j is a rotation, i.e.,

$$\phi_j(z) = \phi_j'(0) z = e^{i\psi} z, \quad \psi \in [0, 2\pi),$$

hence τ contains a rotation.

Circle rotations are not strongly mixing (not even weakly, see [KH95]). Thus we see that if for some $1 \leq j \leq N$, $|\phi_j'(0)| = 1$, then $\tau(a)$ is not mixing. So, having $|\phi_j'(0)| < 1$ for all $1 \leq j \leq N$ is a necessary condition for τ to be mixing. $\square$

1.3.2 Koopman Precomposition Operator

For each $a \in \mathbb{F}_N^+$, the action $\tau(a)$ induces the *Koopman* precomposition operator

$$\begin{aligned} \mathcal{L}^a : L_2(\mathbb{S}^1, \mu) &\to L_2(\mathbb{S}^1, \mu) \\ f &\mapsto f \circ \tau(a). \end{aligned}$$

Note that the constant function $f(z) = c$ is invariant under $\mathcal{L}^a$

$$\mathcal{L}^a f = f \circ \tau(a) = f.$$

Moreover, $\phi_1, \ldots, \phi_N$ are assumed to be μ-preserving, so $\mathcal{L}^a$ preserves orthogonality to the constants

$$\int_{\mathbb{S}^1} (\mathcal{L}^a f)\, d\mu = \int_{\mathbb{S}^1} (f \circ \tau(a))\, d\mu = \int_{\mathbb{S}^1} f\, d\mu,$$

thus inducing an operator

$$\mathcal{L}^a : L_2^0(\mathbb{S}^1, \mu) \to L_2^0(\mathbb{S}^1, \mu),$$

where $L_2^0(\mathbb{S}^1, \mu) := L_2(\mathbb{S}^1, \mu) \ominus \mathbb{C}$, the subspace orthogonal to constants.

As in the case of the motivating example (see Section 1.2), $\mathcal{L}^a$ has good spectral properties on $\mathcal{H}_\theta^* = \mathcal{H}_{\theta^{-1}}$ (the space dual to analytic functions $\mathcal{H}_\theta$), as the next theorem shows.

Theorem 1.3.1. *Let $\phi_1, \ldots, \phi_N \in \mathcal{B}^0$.*

1. *If $\phi_j'(0) = 0$ for all $1 \le j \le N$, then there exist $\beta, \gamma > 0$ and $C > 0$ such that for any $a \in \mathbb{F}_N^+$*
$$\|\mathcal{L}^a\|_{\mathcal{H}_{\theta^{-1}}^0 \to \mathcal{H}_{\theta^{-1}}^0} \le C e^{-\beta e^{\gamma |a|}}.$$

2. *If $|\phi_j'(0)| < 1$ for all $1 \le j \le N$, then there exist $\gamma > 0$ and $C > 0$ such that for any $a \in \mathbb{F}_N^+$*
$$\|\mathcal{L}^a\|_{\mathcal{H}_{\theta^{-1}}^0 \to \mathcal{H}_{\theta^{-1}}^0} \le C e^{-\gamma |a|}.$$

The proof of this theorem is postponed until the next section. Now we use this result to prove sufficient conditions for exponential mixing and mixing at a double exponential rate on analytic observables in Theorem 1.1.4.

Proof of Part 2 and 3 of Theorem 1.1.4. First, let us rewrite the correlation function in terms of the scalar product and the action of the precomposition operator $\mathcal{L}^a$. For any $f \in \mathcal{H}_\theta^0$ and $g \in \mathcal{H}_\theta$

$$\int_{\mathbb{S}^1} (f \circ \tau(a))\, \overline{g}\, d\mu = \langle \mathcal{L}^a f, g \rangle_{L^2(\mathbb{S}^1, \mu)}.$$

Assuming $f \in \mathcal{H}_\theta^0 \subset \mathcal{H}_{\theta^{-1}}^0$, so $\|f\|_{\theta^{-1}} \le \|f\|_\theta$, and applying the Cauchy-Schwarz inequality gives

$$\begin{aligned} |\langle \mathcal{L}^a f, g \rangle_{L^2(\mathbb{S}^1, \mu)}| &\le \|\mathcal{L}^a f\|_{\theta^{-1}} \|g\|_\theta \\ &\le \|\mathcal{L}^a\|_{\mathcal{H}_{\theta^{-1}}^0 \to \mathcal{H}_{\theta^{-1}}^0} \|f\|_{\theta^{-1}} \|g\|_\theta \\ &\le \|\mathcal{L}^a\|_{\mathcal{H}_{\theta^{-1}}^0 \to \mathcal{H}_{\theta^{-1}}^0} \|f\|_\theta \|g\|_\theta. \end{aligned}$$

The rest follows from Theorem 1.3.1. □

In the next subsection we prove that having $\phi_j'(0) = 0$ for all $1 \leq j \leq N$ is a necessary condition for τ to be mixing at a double exponential rate.

1.3.3 Lower Bound on Exponential Mixing

In the next lemma we give a lower bound on the rate of exponential mixing in the case when $\phi_j'(0) \neq 0$ for some $1 \leq j \leq N$. This way, we exclude mixing at a double exponential rate in this case and finish the proof of Theorem 1.1.4.

Lemma 1.3.2 (Lower bound on exponential mixing)**.** *Let τ be as above. If $0 \neq |\phi_j'(0)| < 1$ for some $1 \leq j \leq N$, then there exist constants $\gamma_0 > 0$, $C_0 > 0$, functions $f, g \in \mathcal{H}_\theta^0$, and a sequence $a_n \in \mathbb{F}_N^+, n \in \mathbb{N}$ with word length $|a_n| = n \to \infty$ such that*

$$\left| \int_{\mathbb{S}^1} (f \circ \tau(a))\ \overline{g}\, d\mu \right| = C_0 e^{-\gamma_0 |a|} \|f\|_\theta \|g\|_\theta. \tag{1.4}$$

Proof. Let $\phi(z) := \phi_j(z)$ and $\lambda := \phi_j'(0)$, $0 < |\lambda| < 1$. Then

$$\phi(z) = \lambda z + \sum_{k=2}^{\infty} \frac{\phi^{(k)}(0)}{k!} z^k.$$

Consider $f(z) = g(z) = z$ and the sequence $a_n = j^n, n \in \mathbb{N}$, then $|a_n| = n$ and

$$\left| \int_{\mathbb{S}^1} f(\tau(a)z)\overline{g(z)} d\mu(z) \right| = \left| \frac{1}{2\pi i} \int_{\mathbb{S}^1} \phi^n(z) z^{-1} \frac{dz}{z} \right|,$$

where $\phi^n = \phi \circ \cdots \circ \phi$ is n-fold composition of ϕ with itself. Next, we apply the Residue Theorem to the last expression, use the chain rule, and $\phi(0) = 0$, to get

$$\begin{aligned} \left| \int_{\mathbb{S}^1} f(\tau(a)z)\overline{g(z)} d\mu \right| &= |(\phi^n)'(0)| \\ &= |\phi'(\phi^{n-1}(0)) \dots \phi'(0)| \\ &= |(\phi'(0))|^n \\ &= |\lambda|^n. \end{aligned}$$

Using Definition 1.1.1, we have

$$\|f\|_\theta = \|g\|_\theta = \theta^{-1}.$$

Now take $C_0 = (\|f\|_\theta \|g\|_\theta)^{-1} = \theta^2$ and $\gamma_0 = -\ln|\lambda| > 0$ to get (1.4). □

1.4 Proof of Theorem 1.3.1

First, we estimate the operator norm by its Hilbert-Schmidt norm

$$\|\mathcal{L}^a\|^2_{\mathcal{H}^0_{\theta^{-1}}\to\mathcal{H}^0_{\theta^{-1}}} \leq \sum_{0\neq j,k\in\mathbb{Z}} |\langle \mathcal{L}^a e_k, e_j\rangle_{\mathcal{H}_\theta^{-1}}|^2$$

where $\{e_k = \theta^{-|k|}z^k\}_{k\in\mathbb{Z}}$ is an orthonormal basis of $\mathcal{H}_{\theta^{-1}}$ (since $\|z^k\|_{\theta^{-1}} = \theta^{|k|}$). Note that in the sum we take only non-zero j and k, since we restrict the operator to the subspace $\mathcal{H}^0_{\theta^{-1}}$ orthogonal to constant functions.

Lemma 1.4.1. *For $a \in \mathbb{F}_N^+$ and $j,k\in\mathbb{Z}$,*

$$|\langle \mathcal{L}^a e_k, e_j\rangle_{\mathcal{H}_{\theta^{-1}}}| = \theta^{|j|-|k|}\left|\frac{1}{2\pi i}\int_{\mathbb{S}^1}(\tau(a)z)^k z^{-j}\frac{dz}{z}\right|,$$

and

$$\left|\frac{1}{2\pi i}\int_{\mathbb{S}^1}(\tau(a)z)^k z^{-j}\frac{dz}{z}\right| \leq 1.$$

Proof. We use $e_k = \theta^{-|k|}z^k$ and the definition of the scalar product in $\mathcal{H}_{\theta^{-1}}$

$$\begin{aligned}
|\langle \mathcal{L}^a e_k, e_j\rangle_{\mathcal{H}_{\theta^{-1}}}| &= \theta^{-|k|-|j|}|\langle \mathcal{L}^a z^k, z^j\rangle_{\mathcal{H}_{\theta^{-1}}}| \\
&= \theta^{-|k|-|j|}\theta^{2|j|}|\langle \mathcal{L}^a z^k, z^j\rangle_{L_2}| \\
&= \theta^{|j|-|k|}\left|\frac{1}{2\pi i}\int_{\mathbb{S}^1}(\mathcal{L}^a z^k)z^{-j}\frac{dz}{z}\right| \\
&= \theta^{|j|-|k|}\left|\frac{1}{2\pi i}\int_{\mathbb{S}^1}(\tau(a)z)^k z^{-j}\frac{dz}{z}\right|.
\end{aligned}$$

Since $|\tau(a)z| = 1$ for $|z| = 1$,

$$\left|\frac{1}{2\pi i}\int_{\mathbb{S}^1}(\tau(a)z)^k z^{-j}\frac{dz}{z}\right| \leq 1.$$

□

Corollary 1.4.2. *For $a\in\mathbb{F}_N^+$,*

$$\|\mathcal{L}^a\|^2_{\mathcal{H}^0_{\theta^{-1}}\to\mathcal{H}^0_{\theta^{-1}}} \leq 2\sum_{\substack{k>0\\ 0\neq j\in\mathbb{Z}}} \theta^{2(|j|-|k|)}\left|\frac{1}{2\pi i}\int_{\mathbb{S}^1}(\tau(a)z)^k z^{-j}\frac{dz}{z}\right|^2.$$

Proof. First, let us notice that since $|\tau(a)z| = 1$ for $|z| = 1$,

$$\frac{1}{2\pi i}\int_{\mathbb{S}^1}(\tau(a)z)^k z^{-j}\frac{dz}{z} = \overline{\frac{1}{2\pi i}\int_{\mathbb{S}^1}(\tau(a)z)^{-k}z^j\frac{dz}{z}},$$

so by Lemma 1.4.1,

$$|\langle \mathcal{L}^a e_k, e_j\rangle_{\mathcal{H}_{\theta^{-1}}}| = |\langle \mathcal{L}^a e_{-k}, e_{-j}\rangle_{\mathcal{H}_{\theta^{-1}}}|.$$

Therefore,

$$\|\mathcal{L}^a\|^2_{\mathcal{H}^0_{\theta^{-1}}\to\mathcal{H}^0_{\theta^{-1}}} \le 2\sum_{\substack{k>0\\ 0\neq j\in\mathbb{Z}}}|\langle \mathcal{L}^a e_k, e_j\rangle_{\mathcal{H}_{\theta^{-1}}}|^2.$$

Using Lemma 1.4, we finish the proof of the corollary. □

Next, we show that many of the coefficients in the sum are zero. This allows us to obtain double exponentially decaying upper estimates from Theorem 1.3.1.

1.4.1 Double Exponential Rate

Lemma 1.4.3. *Let $\phi'_m(0) = 0$ for all $1 \le m \le N$. Denote the minimal local degree of the functions ϕ_m, $1 \le m \le N$, at $z = 0$, by p. Then for any $a \in \mathbb{F}^+_N$, $k > 0$, and $j < p^{|a|}k$*

$$\left|\frac{1}{2\pi i}\int_{\mathbb{S}^1}(\tau(a)z)^k z^{-j}\frac{dz}{z}\right| = 0.$$

Proof. Since $\phi'_m(0) = 0$, $1 \le m \le N$, it follows that $p \ge 2$ and

$$\phi_m(z) = \sum_{k=p}^{\infty}\frac{\phi_m^{(k)}(0)}{k!}z^k = O(z^p).$$

Hence, an n-fold composition of ϕ_m with itself can be estimated as

$$\phi_m^n(z) = O(z^{p^n}).$$

Similarly for any $a \in \mathbb{F}^+_N$ the action $\tau(a)$ can be estimated as

$$\tau(a)z = O(z^{p^{|a|}}).$$

Then for any $j < p^{|a|}k$

$$(\tau(a)z)^k z^{-j-1} = O(1)$$

is a function analytic in the unit disk. By Cauchy's Theorem, its integral must be zero. □

Proof of Part 1 of Theorem 1.3.1. From Lemma 1.4.1, Corollary 1.4.2, and Lemma 1.4.3 it follows that for $a \in \mathbb{F}_N^+$

$$\|\mathcal{L}^a\|^2_{\mathcal{H}^0_{\theta^{-1}} \to \mathcal{H}^0_{\theta^{-1}}} \leq 2 \sum_{\substack{k>0 \\ p^{|a|}k \leq j}} \theta^{2(|j|-|k|)}.$$

Now, applying the geometric series formula twice, we get for $|a| > 0$

$$\begin{aligned} \|\mathcal{L}^a\|^2_{\mathcal{H}^0_{\theta^{-1}} \to \mathcal{H}^0_{\theta^{-1}}} &\leq 2 \sum_{k>0} \frac{\theta^{2(p^{|a|}-1)k}}{1-\theta^2} \\ &= \frac{2\theta^{2(p^{|a|}-1)}}{(1-\theta^2)(1-\theta^{2(p^{|a|}-1)})} \\ &\leq \frac{2\theta^{2(p^{|a|}-1)}}{(1-\theta^2)^2}. \end{aligned}$$

Therefore

$$\|\mathcal{L}^a\|_{\mathcal{H}^0_{\theta^{-1}} \to \mathcal{H}^0_{\theta^{-1}}} \leq Ce^{-\beta e^{\gamma|a|}},$$

where

$$C = \frac{\sqrt{2}}{\theta(1-\theta^2)} > 0, \quad \beta = -\ln\theta > 0, \quad \gamma = \ln p \geq \ln 2, \tag{1.5}$$

with p as in Lemma 1.4.3. □

1.4.2 Exponential Rate

The next lemma is based on the application of the Schwarz lemma to the case of free semigroup action $\mathbb{F}_N^+$ (cf. [ABCC22], Lemma 3.3).

Lemma 1.4.4. *Let $|\phi_m'(0)| < 1$ for all $1 \leq m \leq N$. Then, for any $0 < r < 1$, there exists $\lambda < 1$ such that for any $a \in \mathbb{F}_N^+$*

$$|\tau(a)z| \leq \lambda^{|a|} r, \quad |z| = r.$$

Proof. For each $1 \leq m \leq N$, let $M_m(r)$ be the maximum of $|\phi_m(z)|$ for $|z| \leq r$. Since $|\phi_m'(0)| < 1$, by the Schwarz lemma,

$$\lambda_m(r) := \frac{M_m(r)}{r} < 1.$$

Then, we apply the Schwarz lemma again to the function

$$\psi_m(z) = \frac{\phi_m(rz)}{M_m(r)}, \qquad |z| \leq 1,$$

and we get

$$|\psi_m(z)| \leq |z| \quad \text{for} \quad |z| \leq 1.$$

It follows that

$$|\phi_m(z)| \leq \frac{M_m(r)}{r}|z| = \lambda_m|z| \quad \text{for} \quad |z| \leq r.$$

Now let

$$\lambda := \max_{1\leq m\leq N} \lambda_m < 1,$$

so

$$|\tau(a)z| \leq \lambda|\tau(\dot{a})z| \leq \lambda^2|\tau(\ddot{a})z| \leq \cdots \leq \lambda^{|a|}|z|, \quad \text{for} \quad |z| \leq r,$$

where $\dot{a}$ is the word of length $|a|-1$ that we obtain from a by deleting its last letter, and $\ddot{a}$ is the word of length $|a|-2$ that we obtain from a by deleting its last two letters. Finally, let $|z| = r$. □

Corollary 1.4.5. *Let $|\phi_m'(0)| < 1$ for all $1 \leq m \leq N$. Then, for any $0 < r < 1$, there exists $\lambda < 1$ such that for any $a \in \mathbb{F}_N^+$ and $k > 0$, $j \in \mathbb{Z}$*

$$\left|\frac{1}{2\pi i}\int_{\mathbb{S}^1} (\tau(a)z)^k z^{-j}\frac{dz}{z}\right| \leq \lambda^{|a|k} r^{k-j}.$$

Proof. We move the contour of integration from the unit circle to $|z| = r$

$$\left|\frac{1}{2\pi i}\int_{\mathbb{S}^1} (\tau(a)z)^k z^{-j}\frac{dz}{z}\right| = \left|\frac{1}{2\pi i}\int_{|z|=r} (\tau(a)z)^k z^{-j}\frac{dz}{z}\right| \leq \lambda^{|a|k} r^{k-j}.$$

□

Proof of Part 2 of Theorem 1.3.1. Now we are going to finish the proof of the second bullet in Theorem 1.3.1. From Corollary 1.4.2 and Corollary 1.4.5, it follows that

$$\begin{aligned}\|\mathcal{L}^a\|^2_{\mathcal{H}^0_{\theta^{-1}}\to\mathcal{H}^0_{\theta^{-1}}} &\leq 2\sum_{k>0,j\neq 0} \theta^{2(-|k|+|j|)}\lambda^{2|a|k} r^{2(k-j)} \\ &\leq 2\sum_{k>0} \theta^{-2k} r^{2k}\lambda^{2|a|k} \cdot \sum_{j\neq 0} \theta^{2|j|} r^{-2j}.\end{aligned}$$

Now choose $r = \sqrt{\theta}$ and use geometric series formula to get

$$\begin{aligned}\|\mathcal{L}^a\|^2_{\mathcal{H}^0_{\theta^{-1}} \to \mathcal{H}^0_{\theta^{-1}}} &\leq 2 \sum_{k>0} \theta^{-2k} \theta^k \lambda^{2|a|k} \sum_{j \neq 0} \theta^{2|j|} \theta^{-j} \\ &\leq 2 \frac{\theta^{-1}\lambda^{2|a|}}{1 - \theta^{-1}\lambda^{2|a|}} \frac{2\theta}{1-\theta} \\ &\leq 4 \frac{\lambda^{2|a|}}{(1-\theta)^2},\end{aligned}$$

where in the last two lines we use that $\theta^{-1}\lambda^{2|a|} < \theta < 1$, starting with some $|a| \geq C(\theta, \lambda)$. Therefore

$$\|\mathcal{L}^a\|_{\mathcal{H}^0_{\theta^{-1}} \to \mathcal{H}^0_{\theta^{-1}}} \leq Ce^{-\gamma|a|},$$

where

$$C = 2/(1-\theta) > 0, \quad \gamma = -\ln \lambda > 0, \tag{1.6}$$

with $\lambda < 1$ as in Lemma 1.4.4. □

1.5 C^1-stability of Mixing at Double Exponential Rate

Example 1.5.1. *For $0 < |\lambda| < 1$, let*

$$\phi_1(z) := z^2 \frac{z - \lambda}{1 - \overline{\lambda} z} \quad \textit{and} \quad \phi_2(z) := z \left(\frac{z - \lambda}{1 - \overline{\lambda} z} \right)^2.$$

Then ϕ_1 is mixing at a double exponential rate, but ϕ_2 is not. For small λ, both ϕ_1 and ϕ_2 are C^1-perturbations of

$$\phi_0 : z \mapsto z^3;$$

that is, $d_{C^1}(\phi_i, \phi_0) \leq C|\lambda|$ for any $i = 1, 2$. However, neither ϕ_1 nor ϕ_2 is C^1-conjugate to ϕ_0.

This example shows that mixing at a double exponential rate is *not a C^1-stable* property, since any C^1-neighborhood of z^3 contains maps of both types: those that are mixing at a double exponential rate and those that are not. Due to the structural stability (see [Shu87]), there is a C^1-neighborhood of z^3 in which every Lebesgue measure preserving map is exponentially mixing again.

Moreover, this example shows that there exist *nonlinear* maps that mix at a double exponential rate and are not C^1-conjugate to linear maps. In this way, we answer the Rigidity Question 1 (see the Introduction) negatively, even up to C^1-conjugacy.

Proof. By Theorem B, for non-zero λ, ϕ_1 is mixing at a double exponential rate but ϕ_2 is not. Let us note that both are exponentially mixing on Hölder observables since they are uniformly expanding maps.

By taking $\lambda \to 0$, we can see that both ϕ_1 and ϕ_2 are one-parameter deformations of the circle endomorphism

$$\phi_0 : z \mapsto z^3,$$

which is mixing at a double exponential rate.

In fact, for small $|\lambda|$, ϕ_1 and ϕ_2 can be viewed as C^1-perturbations of ϕ_0. We will show that there exists $C > 0$ such that

$$d_{C^1}(\phi_i, \phi_0) := \sup_{z\in\mathbb{S}^1} |\phi_i(z) - \phi_0(z)| + \sup_{z\in\mathbb{S}^1} |\phi_i'(z) - \phi_0'(z)| \le C|\lambda| \tag{1.7}$$

for any $i = 1, 2$. Since

$$\begin{aligned}\phi_1(z) &= z^2(z-\lambda)\left(1 + \overline{\lambda}z + O(|\lambda|^2)\right)\\ &= z^3 - \lambda z^2 + \overline{\lambda}z^4 + O(|\lambda|^2),\end{aligned}$$

then

$$\sup_{z\in\mathbb{S}^1} |\phi_1(z) - \phi_0(z)| = \sup_{z\in\mathbb{S}^1} | - \lambda z^2 + \overline{\lambda}z^4 + O(|\lambda|^2)| \le C_0|\lambda|$$

Next, we compute the derivative

$$\begin{aligned}\phi_1'(z) &= 2z\frac{z-\lambda}{1-\overline{\lambda}z} + z^2\frac{1-|\lambda|^2}{(1-\overline{\lambda}z)^2}\\ &= 2z(z-\lambda)(1+\overline{\lambda}z + O(|\lambda|^2)) + z^2(1-|\lambda|^2)(1+2\overline{\lambda}z + O(|\lambda|^2))\\ &= 3z^2 - 2\lambda z + 4\bar{\lambda}z^3 + O(|\lambda|^2),\end{aligned}$$

so

$$\sup_{z\in\mathbb{S}^1} |\phi_1'(z) - \phi_0'(z)| = \sup_{z\in\mathbb{S}^1} | - 2\lambda z + 4\overline{\lambda}z^3 + O(|\lambda|^2)| \le C_1|\lambda|.$$

One can get a similar estimate for ϕ_2, thus proving (1.7).

Now we are going to prove that ϕ_1 and ϕ_2 are not C^1-conjugate to z^3. In fact, we are proving more: they are not even absolutely continuously conjugate. Note that ϕ_1 and ϕ_2 are topologically conjugate to ϕ_0 (see [Shu69]); that is, there exist homeomorphisms $h_i : \mathbb{S}^1 \to \mathbb{S}^1$ such that

$$\phi_i \circ h_i = h_i \circ \phi_0$$

for any $i = 1, 2$. From the work of Shub and Sullivan [SS85], it follows that if two finite Blaschke products are absolutely continuously conjugate, then they are conjugate by Möbius transformation

$$h_i(z) = e^{i\theta_i} \frac{z - a_i}{1 - \overline{a_i} z}$$

for some $|a_i| < 1$ and $\theta_i \in [0, 2\pi), i = 1, 2$. It is a conformal map, and composition with a conformal map preserves local degree. Note that local degrees of ϕ_0, ϕ_1, ϕ_2 at $z = 0$ are 3, 2, and 1 respectively. Thus, neither ϕ_1 nor ϕ_2 is C^1-conjugate to ϕ_0. □

This example can be immediately generalized to the following statement.

Proposition 1.5.2. *Within the space of Lebesgue measure preserving Blaschke products $\mathcal{B}^0$, the ones that are mixing at a double exponential rate form a nowhere dense set with respect to C^1-topology on $\mathbb{S}^1$.*

Proof. First, within the space of Lebesgue measure preserving Blaschke products $\mathcal{B}^0$, the ones that are mixing at a double exponential rate form a closed set, let us denote it by $\mathcal{D}$. It follows from the fact that $\mathcal{D}$ is a kernel of a continuous linear functional, which can be seen from the Cauchy formula

$$\phi'(0) = \frac{1}{2\pi i} \int_{\mathbb{S}^1} \phi(z) \frac{dz}{z^2} = \int_{\mathbb{S}^1} \phi(z) \overline{z} d\mu = \langle \phi, e_1 \rangle_{L_2},$$

where $e_1(z) = z$. Next, we show that for any $\phi \in \mathcal{D}$, any of its C^1-neighborhoods will contain a map from $\mathcal{B}^0 \setminus \mathcal{D}$. Any map $\phi \in \mathcal{D}$ mixing at a double exponential rate has the form

$$\phi(z) = e^{i\psi} z \prod_{i=1}^{q} \frac{z - \lambda_i}{1 - \overline{\lambda_i} z},$$

where $\psi \in [0, 2\pi), \lambda_1 = 0, |\lambda_i| < 1, i = 1, \ldots, q$, $q \in \mathbb{Z}_{\geq 1}$. Now, deform each factor of z into a non-trivial Blaschke factor $\frac{z-\lambda}{1-\overline{\lambda} z}$, except for one. Thus, the new map $\phi_\lambda(0) = 0$ still preserves the Lebesgue measure. On the other hand, $\phi'_\lambda(0) \neq 0$, so it is not mixing at a double exponential rate anymore. By taking λ small enough, we make sure that ϕ_λ is in the C^1-neighborhood of ϕ. □

In [BN19], Bandlow and Naud use Blaschke products to prove a stronger result: there exists a dense set of analytic expanding maps of the circle with a nontrivial spectrum, which implies that these maps are not mixing at a double exponential rate.

1.6 Spectrum, Quasi-Nilpotency, and Resonances

Proposition 1.6.1 (Spectral radius)**.** *Let $N = 1$ and $\phi \in \mathcal{B}^0$ such that $|\phi'(0)| < 1$. Then, the spectral radius of the corresponding Koopman operator $\mathcal{L} : \mathcal{H}^0_{\theta^{-1}} \to \mathcal{H}^0_{\theta^{-1}}$*

$$r(\mathcal{L}) < 1.$$

Moreover, if $\phi'(0) = 0$, then

$$r(\mathcal{L}) = 0.$$

Proof. By the Gelfand formula, the spectral radius of $\mathcal{L} : \mathcal{H}^0_{\theta^{-1}} \to \mathcal{H}^0_{\theta^{-1}}$ is

$$r(\mathcal{L}) = \lim_{n\to\infty} \|\mathcal{L}^n\|^{1/n}_{\mathcal{H}^0_{\theta^{-1}} \to \mathcal{H}^0_{\theta^{-1}}}.$$

If $|\phi'(0)| < 1$, then from Theorem 1.3.1 it follows that

$$\begin{aligned} r(\mathcal{L}) &\leq \lim_{n\to\infty} C^{1/n} e^{-\frac{\gamma n}{n}} \\ &= e^{-\gamma} \\ &= \lambda < 1. \end{aligned}$$

If $\phi'(0) = 0$, then applying Theorem 1.3.1, we get

$$r(\mathcal{L}) \leq \lim_{n\to\infty} C^{1/n} e^{-\frac{\beta}{n} e^{\gamma n}} = 0.$$

□

Operators with zero spectral radius are called *quasi-nilpotent*. Equivalently, the spectrum of $\mathcal{L} : \mathcal{H}^0_{\theta^{-1}} \to \mathcal{H}^0_{\theta^{-1}}$ is

$$\sigma(\mathcal{L}) = \{0\}.$$

Such operators are *non-self-adjoint*, so they are not trivial even if the spectrum is $\{0\}$. Let us remember another trivial eigenvalue of $\mathcal{L}$, which is 1 corresponding to the constant eigenfunction of $\mathcal{L}$. The spectrum of $\mathcal{L} : \mathcal{H}^0_{\theta^{-1}} \to \mathcal{H}^0_{\theta^{-1}}$ is $\{0, 1\}$. This phenomenon is known in dynamics as *trivial Pollicott-Ruelle resonances* – 0 and 1 (see [Bal00]). In the case $\phi'(0) \neq 0$, the non-trivial resonances were computed in [BJS17].

In our setting, we have that mixing at a double exponential rate is equivalent to the quasi-nilpotency of the corresponding Koopman precomposition operator, or equivalently, the trivial Pollicott-Ruelle resonances. In general, trivial spectrum is equivalent to *super exponential mixing* (i.e., mixing faster than any exponential) but mixing at a double exponential

rate is not guaranteed (as the classical Volterra operators turn out to be quasi-nilpotent with decay of correlations at the speed of order $e^{-n \ln n}$, which is faster than any exponential but *not* double exponential).

Proposition 1.6.2. *Let $N > 1$ and $\phi_j \in \mathcal{B}^0$ such that $\phi_j'(0) = 0$ for all $j = 1, \ldots, N$. Then, the Koopman operators $\mathcal{L}_1, \ldots, \mathcal{L}_N$ corresponding to $\phi_1, \ldots, \phi_N$, and all finite noncommutative polynomials $p(\phi_1, \ldots, \phi_N)$, are quasi-nilpotent. In other words, $\{\mathcal{L}_i\}_{i=1}^N$ generate a quasi-nilpotent non-unital subalgebra.*

Proof. Theorem 1.3.1 implies that for any $a \in \mathbb{F}_N^+$,

$$\begin{aligned} r(\mathcal{L}^a) &= \lim_{n\to\infty} \|(\mathcal{L}^a)^n\|^{1/n}_{\mathcal{H}^0_{\theta^{-1}} \to \mathcal{H}^0_{\theta^{-1}}} \\ &\leq \lim_{n\to\infty} C^{1/n} e^{-\frac{\beta}{n} e^{\gamma |a|_n}} \\ &= 0. \end{aligned}$$

Let the noncommutative polynomial

$$p(\mathcal{L}_1, \ldots, \mathcal{L}_N) = \sum_{j=1}^m C_j \mathcal{L}^{a_j},$$

be such that m is the number of monomials. Let

$$C_p := \sup_{j=1,\ldots,m} |C_j| \quad \text{and} \quad a_p := \inf_{j=1,\ldots,m} |a_j|.$$

Then, the number of terms in $(p(\mathcal{L}_1, \ldots, \mathcal{L}_N))^n$ is bounded by m^n and

$$\|(p(\mathcal{L}_1, \ldots, \mathcal{L}_N))^n\| \leq m^n C_p^n \sup_{|a| \geq n a_p} \|\mathcal{L}^a\|.$$

From here, we have

$$\begin{aligned} r(p(\mathcal{L}_1, \ldots, \mathcal{L}_N)) &= \lim_{n\to\infty} \|(p(\mathcal{L}_1, \ldots, \mathcal{L}_N))^n\|^{1/n}_{\mathcal{H}^0_{\theta^{-1}} \to \mathcal{H}^0_{\theta^{-1}}} \\ &\leq m C_p \lim_{n\to\infty} C^{1/n} \sup_{|a| \geq n a_p} e^{-\frac{\beta}{n} e^{\gamma |a|}} \\ &= 0. \end{aligned}$$

□

Note that, a priori, a composition of non-commuting quasi-nilpotent operators does not necessarily have to be quasi-nilpotent. Thus, this dynamical system provides a naturally

occurring example of a quasi-nilpotent subalgebra of not necessarily commuting operators.

1.7 Rigidity for Fixed Degree Maps

Let $N = 1$. The degree two maps $\phi \in \mathcal{B}^0$ are of the form

$$\phi(z) = e^{i\theta} z \frac{z - \lambda}{1 - \bar{\lambda} z},$$

with $|\lambda| < 1$ and $\theta \in [0, 2\pi)$. Such a map is mixing at a double exponential rate if and only if $\lambda = 0$, i.e., it is an *affine* map $\phi(z) = e^{i\theta} z^2$. This way we have a rigidity for mixing at a double exponential rate in degree two.

As we have seen, in higher degrees there are many non-affine maps that mix at a double exponential rate. This is a non-rigidity type result. Next, we show that if we fix the degree of a map and the rate of double exponential decay, then we get rigidity. Let q be the degree of $\phi \in \mathcal{B}^0$ and assume that $\phi'(0) = 0$. We have proven that if ϕ has a local degree $p \geq 2$ at $z = 0$, then ϕ is mixing at a double exponential rate with $e^\gamma = p$. Clearly, $p \leq q$.

Proposition 1.7.1 (Rigidity)**.** *Assume that $\phi \in \mathcal{B}^0$ is a degree $q \geq 2$ map and it is mixing with double exponential rate $e^\gamma = q$, then $\phi(z) = e^{i\theta} z^q$.*

Proof. If the local degree p at $z = 0$ is strictly smaller than q, then $\phi^{(p)}(0) \neq 0$ and

$$\phi(z) = \sum_{k=p}^{\infty} \frac{\phi^{(k)}(0)}{k!} z^k,$$

then its n-fold composition with itself

$$\begin{aligned} \phi^n(z) &= \left(\frac{\phi^{(p)}(0)}{p!} \right)^{1+p+p^2+\cdots+p^{n-1}} z^{p^n} + O(z^{p^n+1}) \\ &= \left(\frac{\phi^{(p)}(0)}{p!} \right)^{\frac{p^n-1}{p-1}} z^{p^n} + O(z^{p^n+1}). \end{aligned}$$

Let $f(z) = \theta z$ and $g(z) = \theta^{p^n} z^{-p^n}$, note that

$$\|f\|_\theta = \|g\|_\theta = 1.$$

Then, for any $n > 0$,

$$\left|\int_{\mathbb{S}^1} (f \circ \phi^n)\, \overline{g}\, d\mu\right| = \theta^{p^n+1} \left|\frac{1}{2\pi i} \int_{\mathbb{S}^1} \phi^n(z) z^{-p^n} \frac{dz}{z}\right|$$
$$= \left(\frac{\phi^{(p)}(0)}{p!}\right)^{\frac{p^n-1}{p-1}} \theta^{p^n+1}.$$

As $n \to \infty$, this value will become larger than $Ce^{-\beta q^n}$ for any fixed $C, \beta > 0$, thus contradicting our assumption on the double exponential rate of mixing. Thus, the local degree at zero p has to coincide with the topological degree q, so $\phi(z) = e^{i\psi} z^q$ for some $\psi \in [0, 2\pi)$. $\square$

CHAPTER 2

Mixing Rate for Ergodic Toral Endomorphisms

2.1 Statement of the Result

Let $T : \mathbb{T}^d \to \mathbb{T}^d$ be a transformation preserving the Lebesgue measure μ^d. Then T is said to be *(strongly) mixing* if, for any $f, g \in L_2(\mathbb{T}^d, \mu^d)$, the *correlation function*

$$\mathcal{C}_{f,g}(n) := \left| \int_{\mathbb{T}^d} (f \circ T^n)\, \overline{g}\, d\mu^d - \int_{\mathbb{T}^d} f\, d\mu^d \int_{\mathbb{T}^d} \overline{g}\, d\mu^d \right| \to 0 \quad \text{as} \quad n \to \infty,$$

where T^n denotes the n-fold composition with itself. We study the *quantitative mixing* for T on the spaces of analytic functions $\mathcal{H}_\theta \subset L_2(\mathbb{T}^d, \mu^d)$.

Definition 2.1.1 (Analytic functions)**.** *Let $0 < \theta < 1$. The space of analytic functions $\mathcal{H}_\theta$ is defined as the closure of all linear combinations of $z^j, j \in \mathbb{Z}^d$, with respect to the norm*

$$\|f\|_\theta := \sqrt{\sum_{k \in \mathbb{Z}^d} |f_k|^2 \theta^{-2|k|}},$$

where f_k are the Fourier coefficients of f, and $|k| = |k|_1 := \sum_{1 \le j \le d} |k_j|$. We denote by $\mathcal{H}_\theta^0$ the subspace of $\mathcal{H}_\theta$ orthogonal to constants, that is, those elements f with $f_0 = 0$.

Definition 2.1.2 (Mixing at a double exponential rate)**.** *A map T is said to be mixing at a double exponential rate if for any $0 < \theta < 1$ there exist $\beta > 0$, $\gamma > 0$, and $C > 0$ such that for any $f, g \in \mathcal{H}_\theta$ and any $n \ge 0$*

$$\mathcal{C}_{f,g}(n) \le C e^{-\beta e^{\gamma n}} \|f\|_\theta \|g\|_\theta.$$

Theorem 2.1.3. *Let $T \in \mathrm{M}(d, \mathbb{Z})$, $\det T \neq 0$, be ergodic. Then T is mixing at a double exponential rate.*

The constants C, β, and γ are computed explicitly in (2.8). Theorem A from the introduction is a reformulated version of Theorem 2.1.3.

Lind [Lin82] proved that ergodic automorphisms of tori ($\det T = \pm 1$) are exponentially mixing on Hölder observables, which is based on Katznelson's lemma [Kat71] on *Diophantine approximation*. In [BS01], superexponential mixing is shown for hyperbolic endomorphisms and purely expanding endomorphisms.

2.2 Koopman Precomposition Operator

The proof of Theorem 2.1.3 is accomplished by using operator theory techniques. The main idea is to use the regularity of observables f and g in the correlation function $\mathcal{C}_{f,g}$. It can be achieved by considering the precomposition operators as operators from the spaces of analytic functions to their dual spaces, thus obtaining double exponential decay as we explain below.

The map $T : \mathbb{T}^d \to \mathbb{T}^d$ induces the *Koopman* precomposition operator

$$\begin{aligned} \mathcal{L} : L_2(\mathbb{T}^d, \mu^d) &\to L_2(\mathbb{T}^d, \mu^d) \\ f &\mapsto f \circ T. \end{aligned}$$

The constant function $f(z) = c$ is invariant under $\mathcal{L}$

$$\mathcal{L}f = f \circ T = f.$$

Moreover, since T is μ^d-preserving, we have

$$\int_{\mathbb{T}^d} (\mathcal{L}f)\ d\mu^d = \int_{\mathbb{T}^d} (f \circ T)\ d\mu^d = \int_{\mathbb{T}^d} f\, d\mu^d.$$

Therefore, if f is orthogonal to constants, then $\mathcal{L}f$ is orthogonal to constants. In other words, $\mathcal{L}$ preserves orthogonality to the constants, thus inducing an operator

$$\mathcal{L} : L_2^0(\mathbb{T}^d, \mu^d) \to L_2^0(\mathbb{T}^d, \mu^d),$$

where

$$L_2^0(\mathbb{T}^d, \mu^d) := L_2(\mathbb{T}^d, \mu^d) \ominus \mathbb{C},$$

is the orthogonal complement to constants, i.e., functions with $f_0 = 0$.

We have

$$\mathcal{H}_\theta^0 \hookrightarrow L_2^0 \hookrightarrow \mathcal{H}_{\theta^{-1}},$$

where $\mathcal{H}_{\theta^{-1}} = \mathcal{H}_\theta^*$, the space dual to analytic functions $\mathcal{H}_\theta$. Then, for any $n \geq 0$, we can

restrict $\mathcal{L}^n$ to $\mathcal{H}^0_\theta$ and consider the operator

$$\mathcal{L}^n : \mathcal{H}^0_\theta \to \mathcal{H}_{\theta^{-1}}.$$

Theorem 2.2.1. *Let $T \in \mathrm{M}(d,\mathbb{Z})$, $\det T \neq 0$, be ergodic. Then, for any $0 < \theta < 1$, there exist $\beta, \gamma > 0$ and $C > 0$ such that for any $n > 0$*

$$\|\mathcal{L}^n\|_{\mathcal{H}^0_\theta \to \mathcal{H}_{\theta^{-1}}} \leq Ce^{-\beta e^{\gamma n}}.$$

We postpone the proof of this theorem until the next section. Now, we use the result to prove Theorem 2.1.3.

Proof of Theorem 2.1.3. First, we rewrite the correlation function in terms of the $L^2(\mathbb{T}^d, \mu^d)$-scalar product and the action of the precomposition operator $\mathcal{L}^n$: for any $f \in \mathcal{H}^0_\theta$ and $g \in \mathcal{H}_\theta$

$$\mathcal{C}_{f,g}(n) = \int_{\mathbb{T}^d} (f \circ T^n)\, \overline{g}\, d\mu^d = \langle \mathcal{L}^n f, g \rangle_{L^2(\mathbb{T}^d,\mu^d)}.$$

Applying the Cauchy–Schwarz inequality gives

$$\begin{aligned} |\langle \mathcal{L}^n f, g \rangle_{L^2(\mathbb{T}^d,\mu^d)}| &\leq \|\mathcal{L}^n f\|_{\theta^{-1}} \|g\|_\theta \\ &\leq \|\mathcal{L}^n\|_{\mathcal{H}^0_\theta \to \mathcal{H}_{\theta^{-1}}} \|f\|_\theta \|g\|_\theta. \end{aligned}$$

The rest follows from Theorem 2.2.1. □

2.3 Lind's Argument

Before proving Theorem 2.2.1, we provide a variation of Lind's argument.

For $T \in \mathrm{M}(d,\mathbb{Z})$ and $z = (z_1, \ldots, z_d) \in \mathbb{T}^d$, we have

$$Tz := z^T = \left(z_1^{T_{11}} z_2^{T_{12}} \cdots z_d^{T_{1d}},\ \ldots,\ z_1^{T_{d1}} z_2^{T_{d2}} \cdots z_d^{T_{dd}} \right) \in \mathbb{T}^d.$$

For a vector $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$, the scalar monomial

$$z^k := z_1^{k_1} \cdots z_d^{k_d} \in \mathbb{T}.$$

Thus, we have

$$(Tz)^k = (z^T)^k = z^{kT},$$

and for any $n \geq 0$

$$(T^n z)^k = (z^{T^n})^k = z^{kT^n}.$$

Lemma 2.3.1 ([Lin82])**.** *Let $T \in \mathrm{M}(d, \mathbb{Z})$, $\det T \neq 0$, be an ergodic endomorphism of $\mathbb{T}^d$. Then there exists $a = a(T) > 0$ and $\nu = \nu(T) > 1$, such that for any $n > 0$ and $k, j \in \mathbb{Z}^d$ with $0 < |k|, |j| < a\nu^n$*

$$((T^n z)^k)_j = 0,$$

where $((T^n z)^k)_j$ is a j-th Fourier coefficient of $(T^n z)^k$, n-fold composition of T with itself applied to z and taken to the power k.

Proof. For $k, j \in \mathbb{Z}^d$, we have

$$((T^n z)^k)_j = \delta_{kT^n = j}. \tag{2.1}$$

Structural Lemma C.0.1 implies that there exists a T-invariant splitting

$$\mathbb{R}^d = E_s \oplus E_c \oplus E_u,$$

and any $k \in \mathbb{Z}^d$ has the decomposition

$$k = k_s + k_c + k_u.$$

We choose the norm on $\mathbb{Z}^d$ associated with this decomposition

$$|k|_T := |k_u| + |k_s| + |k_c|.$$

Since any two norms on a finite-dimensional space are equivalent, there exist constants $0 < C_1(T) < C_2(T)$ such that

$$C_1 |k|_T \leq |k| \leq C_2 |k|_T. \tag{2.2}$$

From (2.2) and Structural Lemma C.0.1 it follows that for any $n > 0$

$$\begin{aligned} |kT^n| &\geq C_1 |kT^n|_T \\ &\geq C_1 |(kT^n)_u| \\ &= C_1 |k_u T^n| \\ &\geq C_1 C_u \lambda_u^n |k_u|. \end{aligned}$$

Katznelson's Lemma C.0.2 implies that there exist $\gamma > 0$ such that for any $n \geq 0$ and

$0 < |k| < a\nu^n$

$$\begin{aligned} |kT^n| &\geq C_1 C_u \lambda_u^n \frac{\gamma}{|k|^{d_{cs}}} \\ &> C_1 C_u \lambda_u^n \frac{\gamma}{(a\nu^n)^{d_{cs}}}. \end{aligned}$$

Now, choose

$$\nu := \lambda_u{}^{\frac{1}{d_{cs}+1}} \quad \text{and} \quad a := (C_1 C_u \gamma)^{\frac{1}{d_{cs}+1}},$$

to get

$$|kT^n| > a\nu^n.$$

Thus, by (2.1), we conclude that for any $n > 0$ and $0 < |k|, |j| < a\nu^n$

$$((T^n z)^k)_j = \delta_{kT^n = j} = 0,$$

since $|kT^n| > a\nu^n > |j|$. □

2.4 Proof of Theorem 2.2.1

Applying the Riesz Representation Theorem E.0.1 to $\mathcal{H} := \mathcal{H}_\theta$ and using $\mathcal{H}^* = \mathcal{H}_{\theta^{-1}}$, we get

$$\begin{aligned} \|\mathcal{L}^n\|_{\mathcal{H}_\theta^0 \to \mathcal{H}_{\theta^{-1}}} &= \sup_{\substack{f \in \mathcal{H}_\theta^0 \\ \|f\|_{\mathcal{H}_\theta} = 1}} \|\mathcal{L}^n f\|_{\mathcal{H}_{\theta^{-1}}} \\ &= \sup_{\substack{f \in \mathcal{H}_\theta^0 \\ \|f\|_{\mathcal{H}_\theta} = 1}} \|R\mathcal{L}^n f\|_{\mathcal{H}_\theta} \\ &= \|R\mathcal{L}^n\|_{\mathcal{H}_\theta^0 \to \mathcal{H}_\theta^0}, \end{aligned} \tag{2.3}$$

where in the last equality we use that the Riesz isomorphism R and $\mathcal{L}^n$ preserve the decomposition $\mathcal{H}_\theta = \mathcal{H}_\theta^0 \oplus \mathbb{C}$.

Next, we estimate the operator norm by its Hilbert-Schmidt norm

$$\|R\mathcal{L}^n\|^2_{\mathcal{H}_\theta^0 \to \mathcal{H}_\theta^0} \leq \sum_{\mathbf{0} \neq j,k \in \mathbb{Z}^d} |\langle R\mathcal{L}^n e_k, e_j \rangle_{\mathcal{H}_\theta}|^2. \tag{2.4}$$

where $\{e_k = \theta^{|k|} z^k\}_{k \in \mathbb{Z}^d}$ is an orthonormal basis of $\mathcal{H}_\theta$. Note that in the sum we take only non-zero vectors j and k, since we restrict the operator to $\mathcal{H}_\theta^0 = \mathcal{H}_\theta \ominus \mathbb{C}$.

Lemma 2.4.1. *For $\mathbf{0} \neq j, k \in \mathbb{Z}^d$, we have*

$$|\langle R\mathcal{L}^n e_k, e_j\rangle_{\mathcal{H}_\theta}| = \theta^{|j|+|k|}\left|((T^n z)^k)_j\right|,$$

where $((T^n z)^k)_j$ is a j-th Fourier coefficient of $(T^n z)^k$, n-fold composition of T applied to z and taken to the power k.

Proof. We use the definition of $e_k = \theta^{|k|} z^k$ and Riesz Representation Theorem E.0.1 to get

$$\begin{aligned}
|\langle R\mathcal{L}^n e_k, e_j\rangle_{\mathcal{H}_\theta}| &= \theta^{|k|+|j|}|\langle R(\mathcal{L}^n z)^k, z^j\rangle_{\mathcal{H}_\theta}| \\
&= \theta^{|k|+|j|}|(\mathcal{L}^n z^k)_j| \\
&= \theta^{|k|+|j|}|((T^n z)^k)_j|,
\end{aligned}$$

where $((T^n z)^k)_j$ is a j-th Fourier coefficient of $(T^n z)^k$, n-fold composition of T applied to z and taken to the power k. □

Note that since $T^n : \mathbb{T}^d \to \mathbb{T}^d$ for any $n \geq 0$, it follows that for any $j, k \in \mathbb{Z}^d$

$$\begin{aligned}
|((T^n z)^k)_j| &= \left|\int_{\mathbb{T}^d} (T^n z)^k \, z^{-j} \, d\mu^d(z)\right| \\
&\leq \int_{\mathbb{T}^d} |(T^n z)^k| \cdot |\, d\mu^d(z)| \\
&= \int_{\mathbb{T}^d} |\, d\mu^d(z)| \\
&= 1.
\end{aligned} \tag{2.5}$$

So, (2.3)–(2.5), and Lemma 2.4.1 give the rough estimate

$$\|\mathcal{L}^n\|_{\mathcal{H}_\theta^0 \to \mathcal{H}_{\theta^{-1}}} \leq \sqrt{\sum_{\mathbf{0}\neq j,k\in\mathbb{Z}^d} \theta^{2(|j|+|k|)}} = \sum_{\mathbf{0}\neq j\in\mathbb{Z}^d} \theta^{2|j|} =: C_{\theta^2,d} < \infty,$$

where $C_{\theta^2,d}$ is computed explicitly in Lemma D.0.1 for $|k| = |k|_1 := \sum_{j=1}^d |k_j|$.

Therefore, the sum of weighted Fourier coefficients is always finite. Next, we show that many of these Fourier coefficients are zero. This allows us to get double exponential upper estimates from Theorem 2.2.1.

Proof of Theorem 2.2.1. By (2.3), (2.4) and Lemma 2.4.1, we get

$$\|\mathcal{L}^n\|^2_{\mathcal{H}_\theta^0 \to \mathcal{H}_{\theta^{-1}}} = \|R\mathcal{L}^n\|^2_{\mathcal{H}_\theta^0 \to \mathcal{H}_\theta^0} \leq \sum_{\mathbf{0}\neq j,k\in\mathbb{Z}^d} \theta^{2(|j|+|k|)}|((T^n z)^k)_j|^2. \tag{2.6}$$

For $a(T) > 0$ and $\nu(T) > 1$ chosen as in Lemma 2.3.1, we decompose the sum from (2.6) into three parts

$$\|\mathcal{L}^n\|^2_{\mathcal{H}^0_\theta \to \mathcal{H}_{\theta^{-1}}} \leq \left(\sum_{0<|j|,|k|<a\nu^n} + \sum_{|j|>0,|k|\geq a\nu^n} + \sum_{|k|>0,|j|\geq a\nu^n} \right) \theta^{2(|j|+|k|)} |((T^n z)^k)_j|^2. \tag{2.7}$$

By Lemma 2.3.1, the first sum in (2.7) is zero for any $n \geq 0$.

Using (2.5), (D.1), and (D.2), we estimate the second and third sums by

$$\sum_{|k|>0} \theta^{2|k|} \sum_{|j|\geq a\nu^n} \theta^{2|j|} \leq C_{\theta^2,d}\, C'_{\theta^2,d}\, \theta^{2a\nu^n/d},$$

where C_{d,θ^2} and $C'_{\theta^2,d}$ are defined in (D.1) and (D.2). Combining the estimates, we get

$$\|\mathcal{L}^n\|_{\mathcal{H}^0_\theta \to \mathcal{H}_{\theta^{-1}}} \leq Ce^{-\beta e^{\gamma n}},$$

where

$$C := \sqrt{2C_{\theta^2,d}\, C'_{\theta^2,d}} > 0, \quad \beta = -\frac{a}{d} \ln \theta > 0, \quad \gamma = \ln \nu > 0 \tag{2.8}$$

with $a = a(T) > 0$ and $\nu = \nu(T) > 1$ defined in Lemma 2.3.1, and $C_{\theta^2,d}$ and $C'_{\theta^2,d}$ are constants defined in (D.1) and (D.2). □

CHAPTER 3

Mixing Rate for 2d Blaschke Products

3.1 Statement of the Result

We start with the hyperbolic matrix

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

acting on a 2-torus

$$\mathbb{T}^2 = \{z = (z_1, z_2) \in \mathbb{C}^2 : |z_1| = |z_2| = 1\}$$

by the formula

$$\begin{aligned} T^{(0)} : \mathbb{T}^2 &\to \mathbb{T}^2 \\ (z_1, z_2) &\mapsto (z_1^5 z_2^2, z_1^2 z_2^1). \end{aligned}$$

For $\lambda < 1$, we consider two deformations of $T^{(0)}$ given by

$$\begin{aligned} T^{(i)} : \quad & \mathbb{T}^2 \to \mathbb{T}^2, \qquad i = 1, 2, \\ T^{(1)} : \quad & (z_1, z_2) \mapsto (z_1^3 (B_\lambda(z_1))^2 z_2^2,\ z_1 B_\lambda(z_1) z_2), \\ T^{(2)} : \quad & (z_1, z_2) \mapsto (z_1 (B_\lambda(z_1))^4 z_2^2,\ (B_\lambda(z_1))^2 z_2), \end{aligned}$$

where B_λ is a Blaschke factor

$$B_\lambda(z) := \frac{z - \lambda}{1 - \lambda z}.$$

Note that $T^{(1)}$ and $T^{(2)}$ are two families of maps depending on the parameter λ. We omit this dependence to simplify the notation. When $\lambda = 0$, Blaschke factor B_λ degenerates to

$$B_0(z) = z,$$

and therefore

$$T^{(i)}|_{\lambda=0} = T^{(0)}, \qquad i = 1, 2,$$

are indeed deformations of $T^{(0)}$.

Theorem 3.1.1. *Let $T^{(0)}$, $T^{(1)}$, and $T^{(2)}$ be defined as above. Then*

1. *$T^{(0)}$, $T^{(1)}$, and $T^{(2)}$ are Lebesgue measure preserving diffeomorphisms of $\mathbb{T}^2$,*
2. *$T^{(0)}$ and $T^{(1)}$ are mixing at a double exponential rate,*
3. *For $\lambda \neq 0$, $T^{(2)}$ is not mixing at a double exponential rate,*
4. *For small λ, $T^{(1)}$ and $T^{(2)}$ are C^1-perturbations of $T^{(0)}$,*
5. *For $\lambda \neq 0$, neither $T^{(1)}$ nor $T^{(2)}$ is C^1-conjugate to $T^{(0)}$.*

In fact, $T^{(1)}$ and $T^{(2)}$ are C^k-perturbations of $T^{(0)}$ for any k, C^∞-perturbations, and C^ω-perturbations.

From the structural stability theorem, see for example [KH95], it follows that both $T^{(1)}$ and $T^{(2)}$ are Anosov maps for small λ. Hence, for small λ, $T^{(1)}$ and $T^{(2)}$ are exponentially mixing.

Our families of maps fit into the framework of [PS08] and [PR10]. The *Pollicott-Ruelle resonances* for these families are computed in [PS23] and [SBJ17]. See also [Ada17].

Theorem C from the Introduction is a direct corollary of Theorem 3.1.1.

3.2 Invariance of the Lebesgue Measure

Proof of Part 1 of Theorem 3.1.1. The inverse maps are given by

$$\begin{aligned}
\left(T^{(i)}\right)^{-1} &: \mathbb{T}^2 \to \mathbb{T}^2, \quad i = 0, 1, 2, \\
\left(T^{(0)}\right)^{-1} &: \quad (z_1, z_2) \mapsto (z_1 z_2^{-2}, z_1^{-2} z_2^5)\,, \\
\left(T^{(1)}\right)^{-1} &: \quad (z_1, z_2) \mapsto (z_1 z_2^{-2}, z_1^{-1} z_2^3 (B_\lambda(z_1 z_2^{-2}))^{-1})\,, \\
\left(T^{(2)}\right)^{-1} &: \quad (z_1, z_2) \mapsto (z_1 z_2^{-2}, z_2 (B_\lambda(z_1 z_2^{-2}))^{-2})\,.
\end{aligned}$$

Being an ergodic automorphism of $\mathbb{T}^2$ ($\det(A) = 1$), the map $T^{(0)}$ preserves the Lebesgue measure. To prove the same for the deformations, by Lemma A.0.1, we need to check that for $i = 1, 2$

$$\int_{\mathbb{T}^2} (T^{(i)}(z_1, z_2))^k \, d\mu^2(z_1, z_2) = \delta_k$$

for any $k \in \mathbb{Z}^2$. Indeed, a direct computation shows

$$\begin{aligned}\int_{\mathbb{T}^2} (T^{(1)}(z_1,z_2))^k \, d\mu^2(z_1,z_2) &= \int_{\mathbb{S}^1} z_1^{3k_1+k_2} (B_\lambda(z_1))^{2k_1+k_2} \, d\mu(z_1) \int_{\mathbb{S}^1} z_2^{2k_1+k_2} \, d\mu(z_2) \\ &= \int_{\mathbb{S}^1} z_1^{3k_1+k_2} \, d\mu(z_1) \, \delta_{2k_1+k_2} \\ &= \delta_{3k_1+k_2} \delta_{2k_1+k_2} \\ &= \delta_k.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{\mathbb{T}^2} (T^{(2)}(z_1,z_2))^k \, d\mu^2(z_1,z_2) &= \int_{\mathbb{S}^1} z_1^{k_1} (B_\lambda(z_1))^{4k_1+2k_2} \, d\mu(z_1) \int_{\mathbb{S}^1} z_2^{2k_1+k_2} \, d\mu(z_2) \\ &= \int_{\mathbb{S}^1} z_1^{k_1} \, d\mu(z_1) \, \delta_{2k_1+k_2} \\ &= \delta_{k_1} \delta_{2k_1+k_2} \\ &= \delta_k.\end{aligned}$$

□

3.3 C^1-perturbations of the Linear Hyperbolic Map

Proof of Part 4 of Theorem 3.1.1. We want to show that for small λ, both deformations $T^{(1)}$ and $T^{(2)}$ are C^1-perturbations of $T^{(0)}$. Recall that the C^1-metric is given by

$$d_{C^1}\left(T^{(0)}, T^{(i)}\right) := \sup_{z \in \mathbb{T}^2} \|T^{(i)}(z) - T^{(0)}(z)\| + \sup_{z \in \mathbb{T}^2} \|DT^{(i)}(z) - DT^{(0)}(z)\|,$$

where for $i = 1, 2$

$$DT^{(i)}(z) = \begin{pmatrix} \partial_{z_1} T_1^{(i)} & \partial_{z_2} T_1^{(i)} \\ \partial_{z_1} T_2^{(i)} & \partial_{z_2} T_2^{(i)} \end{pmatrix}$$

is the Jacobi matrix. Therefore, it is enough to show

$$d_{C^1}\left(T^{(0)}, T^{(i)}\right) = O(\lambda)$$

for $i = 1, 2$.

Let

$$B := B_\lambda(z_1) \quad \text{and} \quad B' := B'_\lambda(z_1).$$

For $|z_1| = 1$, we have

$$B = z_1 + O(\lambda) \quad \text{and} \quad B' = 1 + O(\lambda).$$

Therefore, for $|z_1| = |z_2| = 1$, we get

$$\begin{aligned} T^{(1)}(z) &= \left(z_1^3 B^2 z_2^2 \,,\, z_1 B z_2\right) \\ &= \left(z_1^5 z_2^2 + O(\lambda) \,,\, z_1^2 z_2 + O(\lambda)\right) \end{aligned}$$

which implies that

$$\sup_{z \in \mathbb{T}^2} \|T^{(1)}(z) - T^{(0)}(z)\| = O(\lambda). \tag{3.1}$$

Next, we compute

$$DT^{(0)}(z) = \begin{pmatrix} 5z_1^4 z_2^2 & 2z_1^5 z_2 \\ 2z_1 z_2 & z_1^2 \end{pmatrix}$$

and

$$\begin{aligned} DT^{(1)}(z) &= \begin{pmatrix} z_1^2 z_2^2 \,(3B^2 + 2z_1 B B') & 2z_1^3 B^2 z_2 \\ z_2(B + z_1 B') & z_1 B \end{pmatrix} \\ &= \begin{pmatrix} z_1^2 z_2^2 \,(3z_1^2 + 2z_1^2 + O(\lambda)) & 2z_1^3 (z_1^2 + O(\lambda)) z_2 \\ z_2(z_1 + z_1 + O(\lambda)) & z_1^2 + O(\lambda) \end{pmatrix} \\ &= \begin{pmatrix} 5z_1^4 z_2^2 + O(\lambda) & 2z_1^5 z_2 + O(\lambda) \\ 2z_1 z_2 + O(\lambda) & z_1^2 + O(\lambda) \end{pmatrix}. \end{aligned}$$

Thus, we have

$$\sup_{z \in \mathbb{T}^2} \|DT^{(1)}(z) - DT^{(0)}(z)\| = O(\lambda). \tag{3.2}$$

Combining (3.1) and (3.2), we get

$$d_{C^1}\left(T^{(0)}, T^{(1)}\right) = O(\lambda).$$

The case of $T^{(2)}$ follows from a similar computation.

□

3.4 C^1-conjugacy to the Linear Hyperbolic Map

Proof of Part 5 of Theorem 3.1.1. First, we compute that all three maps $T^{(i)}$, $i = 0, 1, 2$, have the same fixed points on $\mathbb{T}^2$

$$(1, 1), \quad (1, -1), \quad (-1, 1), \quad (-1, -1).$$

To check if $T^{(i)}$, $i = 1, 2$, are C^1-conjugate to $T^{(0)}$, we need to compare the derivatives at corresponding fixed points. If two maps are C^1-conjugate, the Jacobians at corresponding fixed points must be similar matrices, hence have the same eigenvalues.

For $(z_1, z_2) = (\pm 1, \pm 1)$, we have $z_1^2 = z_2^2 = 1$, so we compute

$$DT^{(0)}(\pm 1, \pm 1) = \begin{pmatrix} 5 & 2s \\ 2s & 1 \end{pmatrix},$$

where $s = z_1 z_2 = \pm 1$. The characteristic polynomial is independent of the sign s and is equal to

$$x^2 - 6x + 1.$$

Hence, the eigenvalues of $DT^{(0)}$ are

$$x_\pm = 3 \pm 2\sqrt{2}.$$

Next, for $z_1 = \pm 1$ and $B = B_\lambda$,

$$B(z_1) = z_1, \qquad z_1 B(z_1) = 1, \qquad B'(z_1) = \left(\frac{1+\lambda}{1-\lambda}\right)^{z_1}.$$

Hence, for $(z_1, z_2) = (\pm 1, \pm 1)$,

$$\begin{aligned} DT^{(1)}(z_1, z_2) &= \begin{pmatrix} z_1^2 z_2^2 \left(3B^2 + 2z_1 B B'\right) & 2z_1^3 B^2 z_2 \\ z_2(B + z_1 B') & z_1 B \end{pmatrix} \\ &= \begin{pmatrix} 3 + 2a & 2s \\ s\,(1 + a) & 1 \end{pmatrix}, \end{aligned}$$

where $s = z_1 z_2$ and

$$a = \left(\frac{1+\lambda}{1-\lambda}\right)^{z_1} \neq 1 \quad \text{for} \quad \lambda \neq 0.$$

The characteristic polynomial is independent of the sign s

$$x^2 - (4 + 2a)\, x + 1 = 0,$$

Thus, the eigenvalues of $DT^{(i)}(z_1, z_2)$ are of the form

$$x_\pm = 2 + a \pm \sqrt{(2+a)^2 - 1} \neq 3 \pm 2\sqrt{2} \quad \text{for} \quad \lambda \neq 0.$$

Similarly, we check that for $\lambda \neq 0$, $T^{(2)}$ is not C^1-conjugate to $T^{(0)}$. □

3.5 Lower Bound on Exponential Mixing

Proof of Part 3 of Theorem 3.1.1. For $\lambda \neq 0$, let

$$\begin{aligned} T := T^{(2)} : \quad & \mathbb{T}^2 \to \mathbb{T}^2 \\ & (z_1, z_2) \mapsto (z_1 (B_\lambda(z_1))^4 z_2^2,\ (B_\lambda(z_1))^2 z_2). \end{aligned}$$

We follow the same strategy as in Lemma 1.3.2: we find two observables f and g such that the correlation decay is exactly exponential in n, thus proving that it cannot decay at a double exponential rate.

For $n \geq 0$, the second coordinate of $T^n z$, has the form

$$(T^n z)_2 = \prod_{j=0}^{n-1} \left(B_\lambda \left((T^j z)_1 \right) \right)^2 z_2.$$

We choose $f(z_1, z_2) = g(z_1, z_2) = z_2$ and compute the correlation function

$$\begin{aligned} \mathcal{C}_{f,g}(n) &= \left| \int_{\mathbb{T}^2} (f \circ T) \overline{g}\, d\mu^2 \right| \\ &= \left| \int_{\mathbb{T}^2} \prod_{j=0}^{n-1} \left(B_\lambda \left((T^j z)_1 \right) \right)^2 z_2\, z_2^{-1}\, d\mu^2(z) \right| \\ &= \left| \int_{\mathbb{T}^2} \prod_{j=0}^{n-1} \left(B_\lambda \left((T^j z)_1 \right) \right)^2 d\mu(z_1)\, d\mu(z_2) \right|. \end{aligned}$$

For $n \geq 0$, the integrand

$$\prod_{j=0}^{n-1} \left(B_\lambda \left((T^j z)_1 \right) \right)^2$$

is analytic in the polydisc $\{z \in \mathbb{C}^2 : |z_1| \leq 1, |z_2| \leq 1\}$. Thus, by applying multi-variable

Cauchy integral formula and using that $T(\mathbf{0}) = \mathbf{0}$, we get

$$\begin{aligned}
\mathcal{C}_{f,g}(n) &= \left| \prod_{j=0}^{n-1} \left(B_\lambda \left((T^j \mathbf{0})_1 \right) \right)^2 \right| \\
&= \left| \prod_{j=0}^{n-1} \left(B_\lambda \left(0 \right) \right)^2 \right| \\
&= |B_\lambda(0)|^{2n} \\
&= |-\lambda|^{2n} \\
&= |\lambda|^{2n}.
\end{aligned}$$

□

3.6 Double Exponential Decay

Proof of Part 2 of Theorem 3.1.1. The case of the linear map $T^{(0)}$ follows from Theorem 2.1.3.

The argument for $T^{(1)}$ proceeds exactly as in the proof of Theorem 2.1.3 presented in Section 2.2, once one replaces Theorem 2.2.1 with its nonlinear counterpart, stated below. □

Theorem 3.6.1. *For $\lambda < 1$, let*

$$\begin{aligned}
T := T^{(1)} : \quad & \mathbb{T}^2 \to \mathbb{T}^2 \\
& (z_1, z_2) \mapsto (z_1^3 (B_\lambda(z_1))^2 z_2^2,\ z_1 B_\lambda(z_1) z_2).
\end{aligned}$$

Then, for any $0 < \theta < 1$, there exist $\beta, \gamma > 0$ and $C > 0$ such that for any $n > 0$

$$\|\mathcal{L}^n\|_{\mathcal{H}^0_\theta \to \mathcal{H}_{\theta^{-1}}} \leq C e^{-\beta e^{\gamma n}}.$$

It is important to emphasize that our approach does not rely on perturbation theory in the regime of small λ.

Proof of Theorem 3.6.1. The argument for $T^{(1)}$ proceeds exactly as in the proof of Theorem 2.2.1 presented in Section 2.4, once one replaces Lemma 2.3.1 with the nonlinear analog of Lind's argument, stated below. □

3.7 Nonlinear Analog of Lind's Argument

Throughout this section, we assume $T := T^{(1)}$.

Lemma 3.7.1. *Let T be as above. Then there exist $a = a(T) > 0$ and $\nu = \nu(T) > 1$ such that for any $n > 0$ and $k, j \in \mathbb{Z}^2$ with $0 < |k|, |j| < a\nu^n$*

$$((T^n z)^k)_j = 0,$$

where $((T^n z)^k)_j$ is a j-th Fourier coefficient of $(T^n z)^k$, n-fold composition of T applied to z and taken to the power k.

Let

$$A := \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix},$$

then, for $k \in \mathbb{Z}^2$, we have

$$(Tz)^k = z^{kA} B_\lambda(z_1)^{k_2},$$

where

$$kA = (k_1, k_2) \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = (3k_1 + k_2, 2k_1 + k_2).$$

Define the unstable cone

$$U := \{k \in \mathbb{Z}^2 \setminus \{\mathbf{0}\} : \ 3k_1 + k_2 \geq 0, \ 2k_1 + k_2 \geq 0\}.$$

Note that U contains the unstable eigenvector for A.

Lemma 3.7.2. *Let T be as above. Then there exist constants $a(A) > 0$ and $\nu(A) > 1$ such that, for every $n > 0$, $k \in U$ and $j \in \mathbb{Z}^2$ satisfying*

$$0 < |j| < a\nu^n,$$

we have

$$((T^n z)^k)_j = 0.$$

Proof. For $n > 0$, we compute by induction

$$T^n z = \left(z_1^{(A^n)_{11}} z_2^{(A^n)_{12}} \prod_{r=0}^{n-1} (B_\lambda((T^r z)_1))^{(A^{n-r})_{12}} \ , \ z_1^{(A^n)_{21}} z_2^{(A^n)_{22}} \prod_{r=0}^{n-1} (B_\lambda((T^r z)_1))^{(A^{n-r})_{22}} \right).$$

Hence, for $n > 0$ and $k \in \mathbb{Z}^2$, we have

$$(T^n z)^k = z^{kA^n} \prod_{r=0}^{n-1} \left(B_\lambda((T^r z)_1)\right)^{(kA^{n-r})_2}.$$

If $k \in U$, then, since A has positive entries, we have

$$(kA^n)_1 \geq 0 \quad \text{and} \quad (kA^n)_2 \geq 0,$$

for all $n > 0$.

Therefore, for $n > 0$ and $k \in U$, the function $(T^n z)^k$ is analytic in the bidisc $\{z \in \mathbb{C}^2 : |z_1| \leq 1, |z_2| \leq 1\}$. Hence, by Cauchy Theorem, its Fourier coefficients vanish below the lowest occurring monomial degree. More precisely,

$$\begin{aligned}((T^n z)^k)_j &= \int_{\mathbb{T}^2} (T^n z)^k \, z^{-j} \, d\mu^2(z) \\ &= \int_{\mathbb{T}^2} z^{kA^n - j} \prod_{r=0}^{n-1} \left(B_\lambda((T^r z)_1)\right)^{(kA^{n-r})_2} \, d\mu^2(z) \\ &= 0\end{aligned}$$

whenever $j_1 < (kA^n)_1$ or $j_2 < (kA^n)_2$.

Next, suppose that

$$|j| := |j_1| + |j_2| < |(kA^n)_1| + |(kA^n)_2|.$$

Since $k \in U$ implies that both coordinates of kA^n are non-negative, we get

$$|j_1| + |j_2| < (kA^n)_1 + (kA^n)_2.$$

Consequently, at least one of the inequalities

$$|j_1| < (kA^n)_1 \quad \text{or} \quad |j_2| < (kA)_2$$

must hold.

Hence, for $k \in U$,

$$((T^n z)^k)_j = 0$$

whenever $|j| < |kA^n|$.

Finally, since U is an unstable cone for A, there exists $C > 0$ such that for any $k \in U$, its

unstable component (see Lemma C.0.1) satisfies

$$|k_u| \geq C|k|.$$

Thus, by performing a sequence of inequalities similar to those in the proof of 2.3.1, we get

$$\begin{aligned} |kA^n| &\geq C_1|kA^n|_A \\ &\geq C_1|(kA^n)_u| \\ &= C_1C_u\lambda_u^n|k_u| \\ &\geq C_1C_u\lambda_u^nC|k| \\ &\geq a\nu^n, \end{aligned}$$

with

$$a(A) := C_1C_uC \quad \text{and} \quad \nu(A) := \lambda_u,$$

where C_u and λ_u are the constants from Lemma C.0.1, constant C_1 comes from the equivalence of norms (2.2), and and the final inequality uses the fact that $|k| \geq 1$.

Therefore, there exist constants $a(A) > 0$ and $\nu(A) > 1$ such that for every $n > 0$ and every $k \in U$,

$$|kA^n| \geq a\nu^n.$$

Thus, for $k \in U$,

$$((T^nz)^k)_j = 0 \quad \text{whenever} \quad |j| < a\nu^n.$$

□

Note that since $T^n : \mathbb{T}^2 \to \mathbb{T}^2$, we have

$$\begin{aligned} ((T^nz)^k)_j &= \int_{\mathbb{T}^2} (T^nz)^k z^{-j} d\mu^2(z) \\ &= \overline{\int_{\mathbb{T}^2} (T^nz)^{-k} z^j d\mu^2(z)} \\ &= \overline{((T^nz)^{-k})_{-j}}. \end{aligned} \tag{3.3}$$

Thus, Lemma 3.7.2 also applies to $k \in \bar{U}$, where

$$\bar{U} := \{k \in \mathbb{Z}^2 \setminus \{\mathbf{0}\} : \ 3k_1 + k_2 \leq 0, \ 2k_1 + k_2 \leq 0\}.$$

Next, define the stable cone

$$S := \{k \in \mathbb{Z}^2 \setminus \{\mathbf{0}\} : \ 3k_1 + k_2 \geq 0, \ 2k_1 + k_2 \leq 0\}$$

and its counterpart

$$\bar{S} := \{k \in \mathbb{Z}^2 \setminus \{\mathbf{0}\} : \ 3k_1 + k_2 \leq 0, \ 2k_1 + k_2 \geq 0\}$$

Note that S contains the stable eigenvector for A. The next two lemmas show that the image of $k \in S$ leaves S exponentially fast and never enters $\bar{S}$.

Lemma 3.7.3. *Let T be as above. Then, for every $n > 0$, $k \in S$, and $j \in \bar{S}$, we have*

$$((T^n z)^k)_j = 0.$$

Proof. First, since T is invertible and preserves the Lebesgue measure on $\mathbb{T}^2$, we have

$$((T^n z)^k)_j = \langle (T^n z)^k, z^j \rangle_{L_2} = \langle z^k, (T^{-n} z)^j \rangle_{L_2} = \overline{((T^{-n} z)^j)_k},$$

where $T^{-n} = (T^{-1})^n$ denotes the n-fold composition of T^{-1}, and

$$\begin{aligned} T^{-1} : \mathbb{T}^2 &\to \mathbb{T}^2 \\ (z_1, z_2) &\mapsto (z_1 z_2^{-2}, \ z_1^{-1} z_2^3 \, B_\lambda(z_1^{-1} z_2^2)) \, . \end{aligned}$$

For every $n > 0$, we compute by induction

$$T^{-n} z = \left(z_1^{(A^{-n})_{11}} z_2^{(A^{-n})_{12}} \prod_{r=0}^{n-1} \left(B_\lambda \left(z^{(r)} \right) \right)^{(A^{-n+r+1})_{12}}, z_1^{(A^{-n})_{21}} z_2^{(A^{-n})_{22}} \prod_{r=0}^{n-1} \left(B_\lambda \left(z^{(r)} \right) \right)^{(A^{-n+r+1})_{22}} \right),$$

where $z^{(r)} := ((T^{-r} z)_1)^{-1} ((T^{-r} z)_2)^2$.

Hence, for $n > 0$ and $j \in \mathbb{Z}^2$, we have

$$(T^{-n} z)^j = z^{jA^{-n}} \prod_{r=0}^{n-1} B_\lambda \left(z^{(r)} \right)^{(jA^{-n+r+1})_2} .$$

Next, we compute

$$jA^{-1} = (j_1, j_2) \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} = (j_1 - j_2, -2j_1 + 3j_2).$$

Therefore, since for $j \in \bar{S}$ we have $j \in \bar{C}$ with

$$\bar{C} := \{j \in \mathbb{Z}^2 : \ j_1 \leq 0, \ j_2 \geq 0\},$$

then we have

$$jA^{-n} \in \bar{C}$$

for all $n \geq 0$. Moreover, by induction on r, for $r = 0, \ldots, n-1$,

$$z^{(r)} = \sum_{m \in \bar{C}} c_{r,m} z^m$$

and

$$(T^{-r}z)_2 = \sum_{m \in \bar{C}} t_{r,m} z^m.$$

The base case $r = 0$ is immediate, since

$$z^{(0)} = z_1^{-1} z_2^2$$

and

$$(T^0 z)_2 = z_2.$$

Now assume the claim holds for some r. Then

$$B_\lambda \left(z^{(r)}\right) = \sum_{m \in \bar{C}} b_{r,m} z^m.$$

Since,

$$(T^{-(r+1)}z)_2 = z^{(r)} B_\lambda \left(z^{(r)}\right) (T^{-r}z)_2$$

and

$$\begin{aligned} z^{(r+1)} &= ((T^{-r-1}z)_1)^{-1}((T^{-r-1}z)_2)^2 \\ &= ((T^{-r}z)_1)^{-1}((T^{-r}z)_2)^2((T^{-r}z)_1)^{-2}(T^{-r}z)_2)^6(B_\lambda(z^{(r)}))^2 \\ &= (z^{(r)})^3((T^{-r}z)_2)^2(B_\lambda \left(z^{(r)}\right))^2 \end{aligned}$$

the induction step follows.

Therefore, for $j \in \bar{S}$,

$$\begin{aligned}(T^{-n}z^j)_k &= \langle z^{jA^{-n}} \prod_{r=0}^{n-1} B_\lambda(z^{(r)})^{(jA^{-n+r+1})_2},\ z^k\rangle_{L_2} \\ &= \langle z^{jA^{-n}} \sum_{m\in\bar{C}} b_{\lambda,m} z^m, z^k\rangle_{L_2} \\ &= 0\end{aligned}$$

whenever $k_1 > (jA^{-n})_1$ or $k_2 < (jA^{-n})_2$.

Since $k \in S \subset C$, where

$$C := \{k \in \mathbb{Z}^2 :\ k_1 \geq 0,\ k_2 \leq 0\},$$

we have

$$k_1 \geq 0 \qquad \text{and} \qquad k_2 \leq 0.$$

On the other hand, since $j \in \bar{S}$, the vector jA^{-n} satisfies

$$(jA^{-n})_1 < 0 \qquad \text{and} \qquad (jA^{-n})_2 > 0.$$

Hence,

$$k_1 > (jA^{-n})_1 \qquad \text{and} \qquad k_2 < (jA^{-n})_2,$$

and therefore

$$(T^{-n}z^j)_k = 0.$$

□

Lemma 3.7.4. *Let T be as above. Then there exist constants $\tilde{a}(A) > 0$ and $\tilde{\nu}(A) > 1$ such that, for every $n > 0$, $j \in S$ and $k \in S$ satisfying*

$$0 < |k| < \tilde{a}\tilde{\nu}^n,$$

we have

$$((T^n z)^k)_j = 0.$$

Proof. Similarly to the previous lemma, if $j \in S \subset C$, then

$$(jA^{-n})_1 \geq 0 \quad \text{and} \quad (jA^{-n})_2 \leq 0,$$

for all $n \geq 0$. Therefore,

$$(T^{-n}z^j)_k = 0$$

whenever $k_1 < (jA^{-n})_1$ or $k_2 > (jA^{-n})_2$.

Next, suppose that $j \in S$ and $k \in S$ such that

$$|k| := |k_1| + |k_2| < |(jA^{-n})_1| + |(jA^{-n})_2|.$$

Since $k \in S$ and $j \in S$ imply that

$$k_1 \geq 0 \quad \text{and} \quad k_2 \leq 0,$$

and for any $n \geq 0$

$$(jA^{-n})_1 \geq 0 \quad \text{and} \quad (jA^{-n})_2 \leq 0,$$

we get

$$k_1 - k_2 < (jA^{-n})_1 - (jA^{-n})_2.$$

Consequently, at least one of the inequalities

$$k_1 < (jA^{-n})_1 \quad \text{or} \quad -k_2 < -(jA^{-n})_2$$

must hold.

Hence, for $j \in S$ and $k \in S$,

$$((T^n z)^k)_j = 0$$

whenever $|k| < |jA^{-n}|$.

Finally, since S is a stable cone for A, it is an unstable cone for A^{-1}. Therefore, there exist constants $\tilde{a} = a(A^{-1})$ and $\tilde{\nu} = \nu(A^{-1})$ such that, for any $n > 0$ and $j \in S$,

$$|jA^{-n}| \geq \tilde{a}\tilde{\nu}^n.$$

Consequently, for $j \in S$ and $k \in S$,

$$((T^n z)^k)_j = 0$$

whenever $|k| < \tilde{a}\tilde{\nu}^n$. □

By (3.3), Lemmas 3.7.3 and 3.7.4 remain valid after interchanging S and $\bar{S}$.

Proof of Lemma 3.7.1. If $k \in U \cup \bar{U}$, then the claim follows from Lemma 3.7.2.

Now suppose that $k \in S$. By invariance of the Lebesgue measure,

$$((T^n z)^k)_j = \langle (T^n z)^k, z^j \rangle_{L_2} = \langle \mathcal{L}^{\lfloor n/2 \rfloor} \mathcal{L}^{\lceil n/2 \rceil} z^k, z^j \rangle_{L_2},$$

where $\mathcal{L} f(z) = (f \circ T)(z)$.

We will show that, by Lemma 3.7.4, after $\lceil n/2 \rceil$ iterations the monomial z^k leaves the stable cone S. We may then apply Lemma 3.7.2 to the remaining $\lfloor n/2 \rfloor$ iterations to conclude that

$$((T^n z)^k)_j = 0.$$

By Lemmas 3.7.3 and 3.7.4, for every $n > 0$ and $0 < |k| < \tilde{a}\tilde{\nu}^{\lceil n/2 \rceil}$,

$$\mathcal{L}^{\lceil n/2 \rceil} z^k = (T^{\lceil n/2 \rceil} z)^k = \sum_{m \in U \cup \bar{U}} c_{m,k,n} z^m,$$

and therefore

$$((T^n z)^k)_j = \sum_{m \in U \cup \bar{U}} c_{m,k,n} \langle \mathcal{L}^{\lfloor n/2 \rfloor} z^m, z^j \rangle_{L_2}.$$

Now, using Lemma 3.7.2, we obtain that for $0 < |j| < a\nu^{\lfloor n/2 \rfloor}$,

$$\langle \mathcal{L}^{\lfloor n/2 \rfloor} z^m, z^j \rangle_{L_2} = \langle T^{\lfloor n/2 \rfloor} z^m, z^j \rangle_{L_2} = 0.$$

Hence, for $0 < |k| < \tilde{a}\tilde{\nu}^{\lceil n/2 \rceil}$ and $0 < |j| < a\nu^{\lfloor n/2 \rfloor}$,

$$\langle T^n z^k, z^j \rangle_{L_2} = 0.$$

The remaining case of $k \in \bar{S}$ follows from (3.3), completing the proof. □

CHAPTER 4

Mixing Rate for Isometric Skew-product Extensions

In this chapter we show that certain families of *partially hyperbolic* nonlinear maps *never* mix at a double exponential rate.

4.1 Isometric Extensions of Expanding Circle Endomorphisms

For $p \geq 2$, we consider the isometric skew-product extension of the form

$$\begin{aligned} T : \mathbb{T}^2 &\to \mathbb{T}^2 \\ (z, w) &\mapsto (z^p, B(z)w), \end{aligned}$$

where $B \in \mathcal{B}$ is a finite Blaschke product. Note that here we do not require $B(0) = 0$ (cf. $\mathcal{B}^0$ defined in Section 1.1).

First, let us notice that T preserves the Lebesgue measure on $\mathbb{T}^2$ defined by

$$d\mu^2(z, w) := d\mu(z)d\mu(w).$$

We check this by using Lemma A.0.1 and computing for any $k \in \mathbb{Z}^2$

$$\begin{aligned} \int_{\mathbb{T}^2} (T(z,w))^k d\mu^2(z,w) &= \int_{\mathbb{S}^1} z^{pk_1} (B(z))^{k_2} d\mu(z) \cdot \int_{\mathbb{S}^1} w^{k_2} d\mu(w) \\ &= \int_{\mathbb{S}^1} z^{pk_1} (B(z))^{k_2} d\mu(z) \cdot \delta_{k_2} \\ &= \int_{\mathbb{S}^1} z^{pk_1} d\mu(z) \cdot \delta_{k_2} \\ &= \delta_{k_1} \cdot \delta_{k_2} \\ &= \delta_k. \end{aligned}$$

In [Dol02], Dolgopyat proved that these maps are exponentially mixing under a cohomological condition.

Theorem 4.1.1. *Isometric skew-product extension of the expanding map $z \mapsto z^p$, $p \in \mathbb{Z}_{\geq 2}$,*

$$\begin{aligned} T : \mathbb{T}^2 &\to \mathbb{T}^2 \\ (z, w) &\mapsto (z^p, B(z)w), \end{aligned}$$

is not mixing at a double exponential rate for any finite Blaschke product $B \in \mathcal{B}$.

Proof. We follow the same strategy as in Lemma 1.3.2: we find two observables f and g such that the correlation decay is exactly exponential in n, thus proving that it cannot decay at a double exponential rate.

For $n \geq 0$, the n-fold composition of T with itself is given by the formula

$$T^n(z, w) = (\, z^{p^n} \, , \, \prod_{j=0}^{n-1} B(z^{p^j})w \,).$$

First, we assume that $B(0) \neq 0$. Then for $f(z, w) = g(z, w) = w$, we get

$$\begin{aligned} \mathcal{C}_{f,g}(n) &= \int_{\mathbb{T}^2} (f \circ T^n) \overline{g} \, d\mu^2 \\ &= \int_{\mathbb{T}^2} \prod_{j=0}^{n-1} B\left(z^{p^j}\right) w \, w^{-1} \, d\mu(z) \, d\mu(w) \\ &= \frac{1}{2\pi i} \int_{\mathbb{S}^1} \prod_{j=0}^{n-1} B\left(z^{p^j}\right) \frac{dz}{z} \int_{\mathbb{S}^1} d\mu(w) \\ &= \prod_{j=0}^{n-1} B(0) \\ &= (B(0))^n, \end{aligned}$$

where we used the analyticity of B in the unit disk and the Cauchy theorem.

If $B(0) = 0$, write

$$B(z) = z^q \tilde{B}(z)$$

with $\tilde{B}(0) \neq 0$ and $q \in \mathbb{Z}_{>0}$. In this case, the formula for T^n takes the form

$$T^n(z, w) = (\, z^{p^n} \, , \, z^{q \frac{p^n - 1}{p - 1}} \prod_{j=0}^{n-1} \tilde{B}(z^{p^j})w \,).$$

Then for $f(z,w) = g(z,w) = z^{-q}w^{p-1}$, we find

$$\begin{aligned}\int_{\mathbb{T}^2} (f \circ T^n)\, \overline{g}\, d\mu^2 &= \int_{\mathbb{T}^2} z^{-qp^n} z^{(p-1)q\frac{p^n-1}{p-1}} \left(\prod_{j=0}^{n-1} \tilde{B}\left(z^{p^j}\right)\right)^{p-1} w^{p-1} z^q\, w^{-(p-1)}\, d\mu(z)\, d\mu(w) \\ &= \frac{1}{2\pi i}\int_{\mathbb{S}^1} \left(\prod_{j=0}^{n-1} \tilde{B}\left(z^{p^j}\right)\right)^{p-1} \frac{dz}{z} \int_{\mathbb{S}^1} d\mu(w) \\ &= \left(\prod_{j=0}^{n-1} \tilde{B}(0)\right)^{p-1} \\ &= (\tilde{B}(0))^{(p-1)n}.\end{aligned}$$

Therefore, T has exactly exponential rate of mixing. □

4.2 Isometric Extensions of Ergodic Toral Endomorphisms

For $A \in \mathrm{M}(d, \mathbb{Z}_{\geq 0})$ with $\det A \neq 0$ and $z = (z_1, \ldots, z_d) \in \mathbb{T}^d$, we write

$$z^A := \left(z_1^{A_{11}} z_2^{A_{12}} \cdots z_d^{A_{1d}},\ \ldots,\ z_1^{A_{d1}} z_2^{A_{d2}} \cdots z_d^{A_{dd}}\right) \in \mathbb{T}^d.$$

For a vector $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$, we write the scalar monomial

$$z^k := z_1^{k_1} \cdots z_d^{k_d} \in \mathbb{T}.$$

By definition, we have

$$(z^A)^k = z^{kA}.$$

Theorem 4.2.1. *Skew-product isometric extension of the ergodic surjective linear endomorphism $A \in \mathrm{M}(d, \mathbb{Z}_{\geq 0})$, $\det A \neq 0$,*

$$\begin{aligned} T : \mathbb{T}^d \times \mathbb{S}^1 &\to \mathbb{T}^d \times \mathbb{S}^1 \\ (z, w) &\mapsto (z^A, B(z)\, w)\end{aligned}$$

is not mixing at a double exponential rate for any finite Blaschke product

$$\begin{aligned} B : \mathbb{T}^d &\to \mathbb{T} \\ z &\mapsto B_1(z_1) \ldots B_d(z_d)\end{aligned}$$

with $B_j \in \mathcal{B}$ for each $1 \leq j \leq d$.

By Lemma A.0.1, these maps preserve the Lebesgue measure on $\mathbb{T}^d \times \mathbb{S}^1$ defined by

$$d\mu^{d+1}(z,w) := d\mu^d(z)d\mu(w).$$

Under a Diophantine condition, Dolgopyat [Dol02] showed that isometric extensions of hyperbolic maps are *rapidly mixing*, meaning the correlations decay faster than any power of n. Exponential mixing remains an open question, although it is expected to follow from the techniques of Tsujii and Zhang [TZ23] (private communication from Félix Lequen).

Proof. For any $n \geq 0$, the n-fold composition of T with itself has the form

$$T^n(z,w) = (\ z^{A^n}\ ,\ \prod_{j=0}^{n-1} B(z^{A^j})\ w\ ).$$

Note that A has non-negative entries; thus, $B(z^{A^j})$ is a function analytic in the polydisc $\{z \in \mathbb{C}^d : |z_1| \leq 1, \ldots, |z_d| \leq 1\}$.

First, assume that $B(\mathbf{0}) \neq 0$. Then for $f(z,w) = g(z,w) = w$, we use Cauchy theorem to compute

$$\begin{aligned}
\mathcal{C}_{f,g}(n) &= \int_{\mathbb{T}^{d+1}} (f \circ T^n)\, \overline{g}\, d\mu^{d+1} \\
&= \int_{\mathbb{T}^{d+1}} \left(\prod_{j=0}^{n-1} B\left(z^{A^j}\right) \right) w\, w^{-1}\, d\mu^d(z)\, d\mu(w) \\
&= \int_{\mathbb{T}^d} \prod_{j=0}^{n-1} B(z^{A^j}) d\mu^d(z) \int_{\mathbb{S}^1} d\mu(w) \\
&= \prod_{j=0}^{n-1} B(\mathbf{0}) \\
&= (B(\mathbf{0}))^n.
\end{aligned}$$

If $B(\mathbf{0}) = 0$, then, using the product structure of $B(z) := B_1(z_1) \ldots B_d(z_d)$, we write

$$B(z) = z^q \tilde{B}(z),$$

with $\mathbf{0} \neq q \in \mathbb{Z}^d_{\geq 0}$ and $\tilde{B}(\mathbf{0}) \neq 0$. In this case, the composition formula takes the form

$$T^n(z,w) = (\ z^{A^n}\ ,\ z^{q\,(A-\mathrm{I_d})^{-1}(A^n-\mathrm{I_d})} \prod_{j=0}^{n-1} \tilde{B}(z^{A^j})\, w\ ),$$

where $\mathrm{I_d}$ is the $d \times d$ identity matrix. Note that $(A - \mathrm{I_d})$ is invertible, since A is ergodic, and

ergodic endomorphisms of tori do not have eigenvalues that are roots of unity.

As $(A - \mathrm{I_d})^{-1}$ has rational entries, let $k \in \mathbb{Z}$ be the least common multiple of all denominators in the entries of $(A - \mathrm{I_d})^{-1}$. Then the vector $-kq(A - \mathrm{I_d})^{-1} \in \mathbb{Z}^d$ is integral.

Choose $f(z, w) = g(z, w) = z^{-kq(A-\mathrm{I_d})^{-1}} w^k$and compute

$$\begin{aligned}
\mathcal{C}_{f,g}(n) &= \int_{\mathbb{T}^{d+1}} (f \circ T^n)\, \overline{g}\, d\mu^{d+1} \\
&= \int_{\mathbb{T}^d} z^{-kq(A-\mathrm{I_d})^{-1}A^n} z^{kq(A-\mathrm{I_d})^{-1}(A^n-\mathrm{I_d})} \left(\prod_{j=0}^{n-1} \tilde{B}\left(z^{A^j}\right) \right)^k w^k z^{kq(A-\mathrm{I_d})^{-1}} w^{-k}\, d\mu^d(z)\, d\mu(w) \\
&= \int_{\mathbb{T}^d} \left(\prod_{j=0}^{n-1} \tilde{B}\left(z^{A^j}\right) \right)^k d\mu^d(z) \cdot \int_{\mathbb{S}^1} d\mu(w) \\
&= \left(\prod_{j=0}^{n-1} \tilde{B}(\mathbf{0}) \right)^k \\
&= (\tilde{B}(\mathbf{0}))^{kn}.
\end{aligned}$$

Once again, we used that A has non-negative entries; thus, $B(z^{A^j})$ is a function analytic in the polydisc $\{z \in \mathbb{C}^d : |z_1| \leq 1, \ldots, |z_d| \leq 1\}$. □

The case of hyperbolic base map A follows from the work of Naud [Nau19], where lower bounds of the exponential decay of correlations for real analytic circle extensions of Anosov maps were computed.

CHAPTER 5

Analytic Regularity for Cohomological Equation

5.1 Statement of the Result

Let $T : \mathbb{T}^d \to \mathbb{T}^d$ be a Lebesgue measure preserving transformation. We study the existence (or obstructions to existence) and regularity of solutions f to the *cohomological equation*

$$f \circ T - f = g, \tag{5.1}$$

given g in some functional class (e.g., [KS94] and [GS14]). Classical Livšic theory [Liv72] provides an answer in the hyperbolic case within the Hölder class. In [dlL97], de la Llave considered an *analytic* version for hyperbolic maps. Wilkinson [Wil13] extended many regularity results to partially hyperbolic systems, leaving open the analytic regularity.

Assume $T \in \mathrm{GL}(d, \mathbb{Z})$ is ergodic and g satisfies the *periodic point condition* (i.e., the sums of g over periodic cycles for T vanish). In [Vee86], Veech shows that if $g \in H^r(\mathbb{T}^d)$ then cohomological equation (5.1) admits a solution $f \in H^{r-\epsilon}(\mathbb{T}^d)$ for any $\epsilon > 0$. As a consequence, he proves that if $g \in C^\infty$ then $f \in C^\infty$. Furthermore, if $E_c \neq \mathbf{0}$, there exists $g \in C^1(\mathbb{T}^d)$ for which the corresponding solution f fails to belong to $C^1(\mathbb{T}^d)$.

We consider the analytic version of these statements.

Definition 5.1.1. *Let $0 < \theta < 1$. The space of analytic functions $\mathcal{A}_\theta$ is defined as the closure of all linear combinations of $z^j, j \in \mathbb{Z}^d$ with respect to the norm*

$$\|f\|_{\mathcal{A}_\theta} := \sup_{k \in \mathbb{Z}^d} |f_k| \theta^{-|k|},$$

where f_k are the Fourier coefficients of f, and $|k| = |k|_1 := \sum_{1 \le j \le d} |k_j|$. We denote by $\mathcal{A}_\theta^0$ the subspace of $\mathcal{A}_\theta$ orthogonal to constants, that is, those with $f_0 = 0$.

The space $\mathcal{A}_\theta$ can be viewed as the analog of $\mathcal{H}_\theta$, in the same way that l_∞ is related to

l_2. Note that

$$\mathcal{A}_\theta \subset \mathcal{H}_{\theta'} \subset \mathcal{A}_{\theta'} \quad \text{for} \quad \theta' < \theta.$$

Definition 5.1.2. *Define the space of all analytic functions*

$$C^\omega(\mathbb{T}^d) := \bigcup_{0<\theta<1} \mathcal{A}_\theta(\mathbb{T}^d).$$

Theorem 5.1.3. *Let $0 < \theta < 1$ and $T \in \mathrm{GL}(d, \mathbb{Z})$ be an ergodic automorphism of $\mathbb{T}^d$. Suppose that the Jordan blocks of T, corresponding to eigenvalues of modulus one, have dimension at most two. If $g \in \mathcal{A}_\theta$ satisfies the periodic point condition (i.e., the sums of g over periodic cycles for T vanish), then (5.1) has a solution $f \in \mathcal{A}_\eta$ for some $0 < \eta(\theta, T) < 1$.*

Corollary 5.1.4. *Under the assumptions of the previous theorem, if $g \in C^\omega(\mathbb{T}^d)$, then (5.1) admits a solution $g \in C^\omega(\mathbb{T}^d)$.*

The assumption on the center directions of T is automatically satisfied if T is irreducible over $\mathbb{Q}$, since in that case T is diagonalizable over $\mathbb{C}$,and hence all the Jordan blocks are trivial. Since the characteristic polynomial of T has integer coefficients (in fact, real coefficients would suffice), all complex eigenvalues occur in conjugate pairs. We impose no restrictions on the dimension of Jordan blocks corresponding to eigenvalues whose modulus is not equal to one. Eigenvalues of modulus one also appear in complex conjugate pairs (as they are not roots of unity). Under our assumption, the only nontrivial real Jordan blocks that may arise are of the form

$$\begin{pmatrix} R_\alpha & I \\ 0 & R_\alpha \end{pmatrix},$$

where R_α is the 2×2 rotation matrix associated with the pair of eigenvalues $e^{2\pi i\alpha}$ and $e^{-2\pi i\alpha}$, and I denotes the 2×2 identity matrix.

5.2 Fourier Coefficients

Lemma 5.2.1 ([Vee86])**.** *Let $A = T^{-t}$. Then, the Fourier coefficients of a solution f of (5.1) are given by*

$$f_k = -\sum_{n\geq 0}^{\infty} g_{A^n k}, \quad k \neq 0;$$
$$f_k = \sum_{n<0}^{\infty} g_{A^n k}, \quad k \neq 0;$$
$$f_0 = \text{const.}$$

Proof. A solution of (5.1) admits two formal representations

$$f(x) = -\sum_{n\geq 0}^{\infty} g(T^n x); \tag{5.2}$$

$$f(x) = \sum_{n<0}^{\infty} g(T^n x). \tag{5.3}$$

We compute the Fourier expansion of $g(T^n x)$

$$g(T^n x) = \sum_{m\in\mathbb{Z}^d} g_m e^{2\pi i\langle m, T^n x\rangle} = \sum_{m\in\mathbb{Z}^d} g_m e^{2\pi i\langle T^{tn} m, x\rangle}.$$

Now set $k = T^{tn}m$ and $A = T^{-t}$, then $m = A^n k$ and

$$g(T^n x) = \sum_{k\in\mathbb{Z}^d} g_{A^n k} e^{2\pi i\langle k, x\rangle}.$$

Hence, the k-th Fourier coefficient of $g(T^n x)$ is $g_{A^n k}$. Substituting this into (5.2) and (5.3) yields

$$\begin{aligned} f_k &= -\sum_{n\geq 0}^{\infty} g_{A^n k}, \quad k \neq 0; \\ f_k &= \sum_{n<0}^{\infty} g_{A^n k}, \quad k \neq 0; \\ f_0 &= \text{const.} \end{aligned}$$

□

According to the Structural Lemma C.0.1, every $k \in \mathbb{Z}^d$ admits a decomposition

$$k = k_s + k_c + k_u,$$

where $k_{s,c,u} \in E_{s,c,u}$. We introduce the adapted norm

$$|k|_T := |k_s| + |k_c| + |k_u|. \tag{5.4}$$

Since all norms on the finite-dimensional space are equivalent, there exist constants $0 < b < B$ such that

$$b|k|_T \leq |k| \leq B|k|_T. \tag{5.5}$$

For $g \in \mathcal{A}_\theta$, we have

$$|g_k| \leq \|g\|_{\mathcal{A}_\theta} \theta^{|k|}. \tag{5.6}$$

Our goal is to show that there exist $0 < \eta < 1$ and C such that

$$\|f\|_{\mathcal{A}_\eta} = \sup_{k \in \mathbb{Z}^d} |f_k| \eta^{-|k|} \leq C,$$

thus implying that $f \in \mathcal{A}_\eta$.

We now distinguish three cases depending on k. By (5.4), at least one of the components satisfies

$$|k_{s,c,u}| \geq \frac{|k|_T}{3}.$$

5.3 Case of the Largest Unstable

In this section we assume that $k \neq \mathbf{0}$ and

$$|k_u| \geq \frac{|k|_T}{3}.$$

Throughout, we work with the representation (5.2).

By the invariance of E_u, the Structural Lemma C.0.1 and the norm equivalence (5.5), we obtain for any $n \geq 1$

$$\begin{aligned} |A^n k| &\geq b|A^n k|_T \\ &\geq b|A^n k_u| \\ &\geq bC_u \lambda_u^n |k_u| \\ &\geq \frac{bC_u}{3} \lambda_u^n |k|_T \\ &\geq \frac{bC_u}{3B} \lambda_u^n |k|. \end{aligned}$$

Next, choose $c_u = c(\lambda_u) > 0$ such that $\lambda_u^n \geq c_u n$ for all $n \geq 1$. Then

$$|A^n k| \geq M_u n |k|, \tag{5.7}$$

where

$$M_u := \frac{bc_u C_u}{3B}$$

and note that M_u depends only on A.

Using Lemma 5.2.1 and (5.6), we get

$$|f_k| \leq \sum_{n=0}^{\infty} |g_{A^n k}| = |g_k| + \sum_{n=1}^{\infty} |g_{A^n k}| \\ \leq \|g\|_{\mathcal{A}_\theta} \left(\theta^{|k|} + \sum_{n=1}^{\infty} \theta^{|A^n k|} \right).$$

Using (5.7) and geometric series formula, we further estimate

$$|f_k| \leq \|g\|_{\mathcal{A}_\theta} \left(\theta^{|k|} + \sum_{n=1}^{\infty} \theta^{M_u n |k|} \right) \\ \leq \|g\|_{\mathcal{A}_\theta} \left(\theta^{|k|} + \frac{\theta^{M_u |k|}}{1 - \theta^{M_u |k|}} \right).$$

Since $k \neq 0$, then $|k| \geq 1$, and hence

$$\frac{1}{1 - \theta^{M_u |k|}} \leq \frac{1}{1 - \theta^{M_u}}.$$

It follows that

$$|f_k| \leq \|g\|_{\mathcal{A}_\theta} \left(\theta^{|k|} + \frac{\theta^{M_u |k|}}{1 - \theta^{M_u}} \right).$$

Now define

$$D_u(\theta, A) := \frac{2}{1 - \theta^{M_u}} \quad \text{and} \quad \eta_u(\theta, A) := \theta^{\min(1, M_u)},$$

then we have

$$|f_k| \leq D_u \|g\|_{\mathcal{A}_\theta} \eta_u^{|k|}. \tag{5.8}$$

5.4 Case of the Largest Stable

In this section, we assume that $k \neq 0$ and

$$|k_s| \geq \frac{|k|_T}{3}.$$

Throughout, we use the representation (5.3). Proceeding as in the previous case, but now for A^n with $n < 0$, we obtain in place of (5.7) that for all $n < 0$,

$$|A^n k| \geq M_s |n| \cdot |k|,$$

where

$$M_s := \frac{bc_s C_u}{3B},$$

and c_s is chosen so that $\lambda_s^{|n|} \geq c_s|n|$. Note that M_s depends only on A. Arguing exactly as in the unstable case, we arrive at the estimate (5.8), with constants

$$D_s(\theta, A) =: \frac{2}{1-\theta^{M_s}} \quad \text{and} \quad \eta_s(\theta, A) := \theta^{\min(1,M_s)}.$$

5.5 Case of the Largest Center

In this section, we assume that $k \neq 0$ and

$$|k_c| \geq \frac{|k|_T}{3}. \tag{5.9}$$

By our assumption on the Jordan normal form of T, for each $k_c \in E_c$, exactly one of the following holds

$$|A^n k_c| \geq |k_c|, \quad n \geq 0; \tag{5.10}$$

$$|A^n k_c| \geq |k_c|, \quad n < 0. \tag{5.11}$$

In case (5.10), we use the representation (5.2), while in case (5.11), we use the representation (5.3).

From now we, consider the first case. Using (5.10) and (5.9), we get

$$\begin{aligned} |A^n k| &\geq b|A^n k|_T \\ &\geq b|A^n k_c| \\ &\geq b|k_c| \\ &\geq b\frac{|k|_T}{3} \\ &\geq \frac{b}{3B}|k|. \end{aligned} \tag{5.12}$$

Now, we decompose the sum from Lemma 5.2.1 into three parts

$$\begin{aligned}|f_k| &\leq \sum_{n=0}^{\infty} |g_{A^n k}| \\ &= |g_k| + \sum_{n=1}^{N_u} |g_{A^n k}| + \sum_{n=N_u+1}^{\infty} |g_{A^n k}| \\ &= |g_k| + \mathrm{I} + \mathrm{II}. \end{aligned} \tag{5.13}$$

where

$$N_u := \left\lceil \frac{(d_{cs}+1)\ln|k|}{\ln \lambda_u} \right\rceil.$$

The choice of N_u is motivated by the estimate of the second sum. For the moment, using (5.12) and (5.6), we estimate the first sum I as follows

$$\begin{aligned}\mathrm{I} &\leq \|g\|_{\mathcal{A}_\theta} \sum_{n=1}^{N_u} \theta^{|A^n k|} \\ &\leq \|g\|_{\mathcal{A}_\theta} \sum_{n=1}^{N_u} \theta^{\,b|k|/3B} \\ &\leq \|g\|_{\mathcal{A}_\theta} N_u \theta^{\,b|k|/3B}. \end{aligned}$$

Substituting the definition for N_u, we get

$$\mathrm{I} \leq \|g\|_{\mathcal{A}_\theta} \left\lceil \frac{(d_{cs}+1)\ln|k|}{\ln \lambda_u} \right\rceil \theta^{b|k|/3B}. \tag{5.14}$$

Next, we will explain the choice of N_u and estimate the second sum II. First, by the invariance of the splitting (5.4), Structural lemma (C.0.1), and Katznelson's lemma C.0.2, for any $n \geq 1$, we have

$$\begin{aligned}|A^n k| &\geq b|A^n k|_T \\ &\geq b|A^n k_u| \\ &\geq bC_u \lambda_u^n |k_u| \\ &\geq bC_u \lambda_u^n \frac{\gamma}{|k|^{d_{cs}}}. \end{aligned} \tag{5.15}$$

Then, for $n > N_u$, we have

$$\frac{\lambda_u^n}{|k|^{d_{cs}}} = \lambda_u^{n-N_u} |k| \frac{\lambda_u^{N_u}}{|k|^{(d_{cs}+1)}} \geq \lambda_u^{n-N_u} |k|. \tag{5.16}$$

Combining (5.15) with (5.16), we obtain for $n > N_u$

$$|A^n k| \geq b\gamma C_u \lambda_u^{n-N_u} |k|.$$

Choose $c_u = c(\lambda_u) > 0$ such that $\lambda_u^n \geq c_u n$ for all $n \geq 1$. Then, for $n > N_u$, we have

$$|A^n k| \geq m_u (n - N_u)|k|, \tag{5.17}$$

where

$$m_u = b\gamma C_u c_u.$$

Using (5.17) and (5.6), we estimate the second sum II

$$\begin{aligned} \mathrm{II} &\leq \|g\|_{\mathcal{A}_\theta} \sum_{n=N_u+1}^{\infty} \theta^{|A^n k|} \\ &\leq \|g\|_{\mathcal{A}_\theta} \sum_{n=N_u+1}^{\infty} \theta^{m_u (n-N_u)|k|} \\ &\leq \|g\|_{\mathcal{A}_\theta} \frac{\theta^{m_u |k|}}{1 - \theta^{m_u |k|}} \\ &\leq \|g\|_{\mathcal{A}_\theta} \frac{\theta^{m_u |k|}}{1 - \theta^{m_u}}. \end{aligned} \tag{5.18}$$

Next, we put (5.13),(5.14), and (5.18) together

$$|f|_k \leq \|g\|_{\mathcal{A}} \left(\theta^{|k|} + \frac{\theta^{m_u |k|}}{1 - \theta^{m_u}} + \left\lceil \frac{(d_{cs} + 1) \ln(|k|)}{\ln \lambda_u} \right\rceil \theta^{b|k|/3B} \right). \tag{5.19}$$

Choose $0 < \theta < \tilde{\theta} < 1$ such that for all $k \in \mathbb{Z}^d, k \neq 0$.

$$\ln(|k|) \theta^{b|k|/3B} \leq \tilde{\theta}^{|k|}.$$

Finally, define

$$D_{cu}(\theta, A) := 1 + \frac{1}{1 - \theta^{m_u}} + \frac{d_{sc} + 1}{\ln \lambda_u} + 1 \quad \text{and} \quad \eta_{cu}(\theta, A) := \max(\theta, \theta^{m_u}, \theta^{b/3B}, \tilde{\theta}),$$

then we have

$$|f_k| \leq D_{cu} \|g\|_{\mathcal{A}_\theta} \eta_{cu}^{|k|}. \tag{5.20}$$

Arguing in the same way for the second case (5.11), we obtain the corresponding estimate (5.20) with constants $D_{cs}(\theta, A)$ and $\eta_{cs}(\theta, A)$.

5.6 Finishing the Proof of Theorem 5.1.3

Using $A = T^t$, define

$$\eta(\theta, T) := \max(\eta_u, \eta_s, \eta_{cu}, \eta_{cs})$$

and

$$C(f, \theta, T) := \|f\|_{\mathcal{A}_\theta} \max(D_u, D_s, D_{cu}, D_{cs}).$$

Sections 5.3, 5.4, and 5.5 imply that for all $k \in \mathbb{Z}^2$

$$|f_k| \leq C\eta^{|k|}.$$

By definition 5.1.1,

$$\|f\|_{\mathcal{A}_\eta} < \infty \quad \text{and} \quad f \in \mathcal{A}_\eta.$$

CHAPTER 6

Cohomological Equation and Anisotropic Spaces

6.1 Statement of the Result

Let $T : \mathbb{T}^d \to \mathbb{T}^d$ be a hyperbolic toral endomorphism. We consider the cohomological equation

$$f \circ T - f = g. \tag{6.1}$$

The existence and regularity of solutions f depend on the properties of g. A central theme is to identify natural functional spaces in which (6.1) admits a unique solution within the same space.

Let $A = T^t$. Since T is hyperbolic, by C.0.1, there exists an A-invariant splitting

$$\mathbb{R}^d = E_u \oplus E_s,$$

where E_u and E_s are the unstable and stable subspaces, respectively.

Every $k \in \mathbb{R}^d$ admits a unique decomposition

$$k = k_u + k_s, \quad k_u \in E_u, \quad k_s \in E_s.$$

We fix the norm on $\mathbb{R}^d$ adapted to this splitting and define

$$|k|_T := |k_u| + |k_s|.$$

Then there exist constants $\lambda_u > 1$ and $\lambda_s < 1$ such that for any for any $n \geq 0$ and $k \in \mathbb{Z}^d$

$$|A^n k_u| \geq \lambda_u^n |k_u| \quad \text{and} \quad |A^n k_s| \leq \lambda_s^n |k_s|. \tag{6.2}$$

Next, we construct spaces well adapted to the hyperbolic dynamics of the map T.

Definition 6.1.1. *Let $\theta_u > 1$ and $\theta_s < 1$. The space of anisotropic hyperfunctions $\mathcal{H}_{\theta_u,\theta_s}$ is defined as the closure of all linear combinations of $z^j, j \in \mathbb{Z}^d$, with respect to the norm*

$$\|f\|_{\theta_u,\theta_s} := \sqrt{\sum_{k\in\mathbb{Z}^d} |f_k|^2 \theta_u^{-2|k_u|}\theta_s^{-2|k_s|}},$$

where f_k are the Fourier coefficients of f, and $k_{u(s)}$ are unstable (stable) components of k with respect to $A = T^t$. We denote by $\mathcal{H}^0_{\theta_u,\theta_s}$ the subspace of $\mathcal{H}_{\theta_u,\theta_s}$ orthogonal to constants, that is, those with $f_0 = 0$.

These spaces contain elements whose Fourier coefficients decay exponentially in one direction while being allowed to grow exponentially in another. In other words, elements of these spaces exhibit analytic regularity along the stable direction, whereas they behave in a dual (distributional) sense along the unstable direction. For this reason, such spaces are called *anisotropic spaces*. Anisotropic variants of Sobolev spaces play a central role in hyperbolic dynamics and in the study of transfer operators. The spaces considered here can be viewed as their analytic counterparts, where Sobolev-type polynomial weights are replaced by exponential Fourier weights. This allows one to capture analytic regularity with directional dependence aligned with the stable–unstable decomposition. These types of spaces are used in [FR06], [SBJ17], [PS23], and [J́24].

Theorem 6.1.2 (Existence and Uniqueness)**.** *Let $T \in \mathrm{GL}(d,\mathbb{Z})$ be hyperbolic. If $g \in \mathcal{H}^0_{\theta_u,\theta_s}$, then* (6.1) *has a unique solution $f \in \mathcal{H}^0_{\theta_u,\theta_s}$.*

In the space of anisotropic hyperfunctions $\mathcal{H}^0_{\theta_u,\theta_s}$, the cohomological equation (6.1) is understood in a weak (distributional) sense.

Moreover, in $\mathcal{H}_{\theta_u,\theta_s}$ there are no obstructions to the existence of solutions to (6.1).

6.2 Spectral Properties of Koopman Operator

The pullback under T induces the Koopman (precomposition) operator

$$\begin{aligned}\mathcal{L}:\ L_2(\mathbb{T}^d) &\to L_2(\mathbb{T}^d)\\ f &\mapsto f\circ T.\end{aligned}$$

Theorem 6.2.1. *Let $T \in \mathrm{GL}(d,\mathbb{Z})$. Then for any $\theta_u > 1$ and $\theta_s < 1$,*

$$\mathcal{L} : \mathcal{H}^0_{\theta_u,\theta_s} \to \mathcal{H}^0_{\theta_u,\theta_s}$$

is a strict contraction, i.e., there exist $\theta := \theta(\theta_u, \theta_s, T) < 1$ *such that*

$$\|\mathcal{L}\|_{\mathcal{H}^0_{\theta_u,\theta_s} \to \mathcal{H}^0_{\theta_u,\theta_s}} < \theta.$$

First, we use this theorem to prove Theorem 6.1.2.

Proof of Theorem 6.1.2. In $\mathcal{H}^0_{\theta_u,\theta_s}$, the cohomological equation (6.1) can be written in operator form as

$$(I - \mathcal{L})f = -g. \tag{6.3}$$

Lemma 6.2.1 implies that $I - \mathcal{L}$ is invertible on $\mathcal{H}^0_{\theta_u,\theta_s}$, with bounded inverse given by the Neumann series

$$(I - \mathcal{L})^{-1} = \sum_{k=0}^{\infty} \mathcal{L}^k.$$

The condition $\|\mathcal{L}\|_{\mathcal{H}^0_{\theta_u,\theta_s} \to \mathcal{H}^0_{\theta_u,\theta_s}} < \theta < 1$ ensures that this series converges in operator norm. Therefore, equation (6.3) admits a unique solution $f \in \mathcal{H}^0_{\theta_u,\theta_s}$. □

Proof of Theorem 6.2.1. We argue similarly to Proposition 1.2.3 and compute the matrix representation of $\mathcal{L}$ in the orthonormal basis

$$e_k = \theta_u^{|k_u|}\theta_s^{|k_s|} z^k.$$

A direct computation gives

$$\begin{aligned}
\mathcal{L}_{j,k} &= \langle \mathcal{L}e_k, e_j \rangle_{\mathcal{H}_{\theta_u,\theta_s}} \\
&= \theta_u^{|k_u|+|j_u|}\theta_s^{|k_s|+|j_s|} \langle \mathcal{L}z^k, z^j \rangle_{\mathcal{H}_{\theta_u,\theta_s}} \\
&= \theta_u^{|k_u|-|j_u|}\theta_s^{|k_s|-|j_s|} \langle \mathcal{L}z^k, z^j \rangle_{L_2} \\
&= \theta_u^{|k_u|-|j_u|}\theta_s^{|k_s|-|j_s|} \langle z^{Ak}, z^j \rangle_{L_2} \\
&= \theta_u^{|k_u|-|Ak_u|}\theta_s^{|k_s|-|Ak_s|} \delta_{Ak=j}.
\end{aligned}$$

Next, we compute the matrix representation of $\mathcal{L}^*\mathcal{L}$

$$\begin{aligned}
(\mathcal{L}^*\mathcal{L})_{j,k} &= \sum_m \mathcal{L}^*_{j,m}\mathcal{L}_{m,k} \\
&= \sum_m \overline{\mathcal{L}_{m,j}}\mathcal{L}_{m,k} \\
&= \theta_u^{2(-|Ak_u|+|k_u|)}\theta_s^{2(-|Ak_s|+|k_s|)}\delta_{j=k}.
\end{aligned}$$

In particular, $(\mathcal{L}^*\mathcal{L})_{j,k} = 0$ if $j \neq k$. Otherwise $(\mathcal{L}^*\mathcal{L})_{j,j} > 0$. Hence it is a diagonal operator with nonnegative entries. It follows that its square root $\sqrt{\mathcal{L}^*\mathcal{L}}$ is well defined.

The singular values of $\mathcal{L}$ are defined as the eigenvalues of $\sqrt{\mathcal{L}^*\mathcal{L}}$

$$\sigma_k(\mathcal{L}) := \lambda_k(\sqrt{\mathcal{L}^*\mathcal{L}}) = \theta_u^{(-|Ak_u|+|k_u|)}\theta_s^{(-|Ak_s|+|k_s|)},$$

where $k \in \mathbb{Z}^d \setminus \mathbf{0}$. Using the hyperbolicity condition (6.2), $\theta_u > 1$, and $\theta_s < 1$, we estimate

$$\sigma_k(\mathcal{L}) \leq \theta_u^{(1-\lambda_u)|k_u|}\theta_s^{(1-\lambda_s)|k_s|}, \quad k \in \mathbb{Z}^d \setminus \mathbf{0}. \tag{6.4}$$

Since $\lambda_u - 1 > 0$ and $\lambda_s - 1 < 0$, it follows that

$$\theta := \max\{\theta_u^{1-\lambda_u}, \theta_s^{1-\lambda_s}\} < 1. \tag{6.5}$$

Consequently, the singular values decay exponentially in $|k|_T = |k_u| + |k_s|$

$$\sigma_k \leq \theta^{|k|_T}.$$

Since the operator norm of $\mathcal{L}$ coincides with its largest singular value, we obtain

$$\|\mathcal{L}\|_{\mathcal{H}^0_{\theta_u,\theta_s}} = \sup_{k\neq\mathbf{0}} \sigma_k(\mathcal{L}) \leq \theta^B < 1,$$

where in the last step we use the equivalence of norms on $\mathbb{Z}^d$, namely that $|k|_T \geq B\,|k|_1$. In particular, for $k \neq \mathbf{0}$ we have $|k|_1 \geq 1$, and hence $|k|_T \geq B$.

□

APPENDIX A

Invariance of the Lebesgue Measure

Let μ^d be the Lebesgue measure on $\mathbb{T}^d = \{z \in \mathbb{C}^d : |z_1| = \cdots = |z_d| = 1\}$ given by

$$d\mu^d(z) := d\mu(z_1) \dots d\mu(z_d),$$

where each factor is the standard normalized Lebesgue measure on $\mathbb{S}^1$ given in the complex coordinate by

$$d\mu(z_j) := \frac{1}{2\pi i}\frac{dz_j}{z_j}, \quad 1 \le j \le d.$$

Lemma A.0.1. *Let $T : \mathbb{T}^d \to \mathbb{T}^d$ be a measurable map. Then T preserves μ^d if and only if*

$$\int_{\mathbb{T}^d} (Tz)^k \, d\mu^d(z) = \delta_k \qquad \text{for every} \quad k \in \mathbb{Z}^d,$$

where $z^k = z_1^{k_1} \cdots z_d^{k_d}$ and

$$\delta_k = \begin{cases} 1, & k = \mathbf{0}, \\ 0, & k \neq \mathbf{0}. \end{cases}$$

Proof. The map T preserves μ^d if, for every $f \in L_1(\mathbb{T}^d, \mu^d)$,

$$\int_{\mathbb{T}^d} (f \circ T) \, d\mu^d = \int_{\mathbb{T}^d} f \, d\mu^d.$$

Since trigonometric polynomials (linear combinations of z^k) are dense in $L_1(\mathbb{T}^d, \mu^d)$, therefore it is enough to check the above equality for $f(z) = z^k, k \in \mathbb{Z}^d$, only.

We have

$$\int_{\mathbb{T}^d} z^k \, d\mu^d(z) = \delta_k \qquad \text{for} \quad k \in \mathbb{Z}^d.$$

Therefore,

$$\int_{\mathbb{T}^d} (Tz)^k \, d\mu^d(z) = \delta_k \qquad \text{for every} \quad k \in \mathbb{Z}^d,$$

is equivalent to μ^d being invariant under T. □

Corollary A.0.2. *Let $T \in \mathcal{B}$ be a finite Blaschke product. Then T preserves the Lebesgue measure μ on $\mathbb{S}^1$ if and only if*

$$T(0) = 0.$$

Proof. For integer $k \geq 0$, we apply the Cauchy Theorem to $(Tz)^k$, a function that is analytic in the unit disk, and get

$$\begin{aligned}\int_{\mathbb{S}^1} (Tz)^k \, d\mu(z) &= \frac{1}{2\pi i} \int_{\mathbb{S}^1} (Tz)^k \, \frac{dz}{z} \\ &= (T(0))^k .\end{aligned}$$

For integer $k < 0$, we use $|T(z)| = 1$ for $|z| = 1$ to get

$$\begin{aligned}\int_{\mathbb{S}^1} (Tz)^k \, d\mu(z) &= \int_{\mathbb{S}^1} (Tz)^{-|k|} \, d\mu(z) \\ &= \int_{\mathbb{S}^1} (\overline{Tz})^{|k|} \, d\mu(z) \\ &= \overline{\int_{\mathbb{S}^1} (Tz)^{|k|} \, d\mu(z)} \\ &= \left(\overline{T(0)}\right)^{|k|} .\end{aligned}$$

In either case Lemma A.0.1 implies that $T(0) = 0$ is equivalent to T-invariance of μ. $\square$

APPENDIX B

Mixing Hierarchy

Let $T : \mathbb{T}^d \to \mathbb{T}^d$ be a Lebesgue measure preserving transformation.

Lemma B.0.1 (Mixing Hierarchy)**.** *If T is mixing at a double exponential rate on $\mathcal{H}_\theta$ for some $0 < \theta < 1$, then T is exponentially mixing on C^α for every $0 < \alpha < 1$.*

Note that the converse does not hold, as shown by examples in Section 1.5. In the proof, we use an approximation argument similar to Lemma 2.3 in [GS14].

Proof. Without loss of generality, assume that f has an average of zero. Denote by $\sigma_N f$ the N-th Fejér mean of the Fourier series of f. Then there exist $C' > 0$ and $\alpha > 0$ such that for any $f \in C^\alpha$ and $f_N = \sigma_N f$,

$$\|f_N - f\|_\infty \ \leq \ C' \, N^{-\alpha} \|f\|_{C^\alpha}.$$

We have

$$\langle \mathcal{L}^n f, g \rangle_{L_2} = \langle \mathcal{L}^n f, g - g_N \rangle_{L_2} + \langle \mathcal{L}^n (f - f_N), g_N \rangle_{L_2} + \langle \mathcal{L}^n f_N, g_N \rangle_{L_2}. \tag{B.1}$$

Now, being finite linear combinations of basis vectors, f_N and g_N belong to any $\mathcal{H}_\theta$, so there are $C > 0$, $\beta > 0$, and $\gamma > 0$ such that

$$\begin{aligned} |\langle \mathcal{L}^n f_N, g_N \rangle_{L_2}| &\leq C e^{-\beta e^{\gamma n}} \|f_N\|_\theta \|g_N\|_\theta \\ &\leq C e^{-\beta e^{\gamma n}} \theta^{-2N} \|f_N\|_{L_2} \|g_N\|_{L_2} \\ &\leq C e^{-\beta e^{\gamma n}} \theta^{-2N} \|f\|_{L_2} \|g\|_{L_2} \\ &\leq C e^{-\beta e^{\gamma n}} \theta^{-2N} \|f\|_{C^\alpha} \|g\|_{C^\alpha}. \end{aligned} \tag{B.2}$$

Next,

$$|\langle \mathcal{L}^n f, g - g_N \rangle_{L_2}| \leq \|g - g_N\|_\infty \|\mathcal{L}^n f\|_{L_2} \leq C' N^{-\alpha} \|f\|_{C^\alpha} \|g\|_{C^\alpha}, \tag{B.3}$$

and similarly,

$$|\langle \mathcal{L}^n(f - f_N), g_N\rangle_{L_2}| \leq C' N^{-\alpha} \|f\|_{C^\alpha} \|g\|_{C^\alpha}. \tag{B.4}$$

Combining the estimates (B.1)-(B.4), we get

$$|\langle \mathcal{L}^n f, g\rangle_{L^2}| \leq \left(2C' N^{-\alpha} + Ce^{-\beta e^{\gamma n}} \theta^{-2N}\right) \|f\|_{C^\alpha} \|g\|_{C^\alpha}. \tag{B.5}$$

There are two competing terms that we need to balance by choosing N appropriately as a function of n

$$N(n) := \lceil Ke^{\gamma n} \rceil, \quad K = \frac{-\beta}{4 \log \theta} \geq 0.$$

Then

$$e^{-\beta e^{\gamma n}} \theta^{-2N(n)} \leq e^{-\beta e^{\gamma n}} e^{\frac{\beta}{2} e^{\gamma n}} \theta^{-2} = \theta^{-2} e^{-\frac{\beta}{2} e^{\gamma n}}. \tag{B.6}$$

Find $C'' > 0$ such that

$$C\theta^{-2} e^{-\frac{\beta}{2} e^{\gamma n}} \leq C'' N(n)^{-\alpha} \leq C'' \left(Ke^{\gamma n}\right)^{-\alpha} \tag{B.7}$$

for all $n \geq 0$.

Combining the estimates (B.5)-(B.7), we obtain

$$|\langle \mathcal{L}^n f, g\rangle_{L^2}| \leq \hat{C} e^{-bn} \|f\|_{C^\alpha} \|g\|_{C^\alpha},$$

where $\hat{C} = (C'' + 2C')K^{-\alpha}$ and $b = \alpha\gamma > 0$. □

APPENDIX C

Structural Lemma for Ergodic Toral Automorphisms and Katznelson's Lemma

Lemma C.0.1 (Structural Lemma)**.** *Assume* $A \in \mathrm{M}(d, \mathbb{Z})$ *and* $\det A \neq 0$*. Then there exists an* A*-invariant splitting of* $\mathbb{R}^d = E_s \oplus E_c \oplus E_u$ *such that*

1. *there exist* $\lambda_u > 1$ *and* $C_u > 0$ *such that for any* $v \in E_u$

$$|A^n v| \geq C_u \lambda_u^n |v|, \quad n > 0,$$

2. *there exist* $\lambda_s > 1$ *and* $C_s > 0$ *such that for any* $v \in E_s$

$$|A^n v| \geq C_s \lambda_s^{|n|} |v|, \quad n < 0,$$

3. *there exist* $0 \leq d' < d$ *and* $C_c > 0$ *such that for any* $v \in E_c$

$$|A^n v| \geq C_c \frac{|v|}{|n|^{d'}}, \quad n \in \mathbb{Z}.$$

This decomposition is the stable–center–unstable splitting for a surjective endomorphism of the torus.

The proof is based on the Jordan normal form of A.

Now assume that A is ergodic, equivalently, A has no roots of unity as eigenvalues (see, for example, Theorem 0.14 in [Wal82]). In this case, is possible to have $E_c \oplus E_s = \mathbf{0}$, and then A is called *uniformly expanding* toral endomorphism. Also, it is possible to have $E_c = \mathbf{0}$, then A is called *hyperbolic*, or *Anosov*. However, $E_u \neq \mathbf{0}$ is necessarily nontrivial. Indeed, if $E_u = \mathbf{0}$, then modulus of every eigenvalue of A is not greater than 1. By Kronecker theorem [Kro57], all such eigenvalues must be roots of unity, contradicting the ergodicity of A. In fact, the unstable component of each $k \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ cannot be arbitrarily small, as the following Diophantine-type result shows.

Lemma C.0.2 ([Kat71]). *There exists $\gamma = \gamma(A) > 0$ such that for any $\mathbf{0} \neq k \in \mathbb{Z}^d$*

$$|k_u| \geq \frac{\gamma}{|k|^{d_{sc}}},$$

where $d_{cs} \geq 0$ is the dimension of $E_c \oplus E_s$.

APPENDIX D

Higher-dimensional Geometric Series

For $k \in \mathbb{Z}^d$, consider the norms

$$|k|_1 := \sum_{j=1}^{d} |k_j| \quad \text{and} \quad |k|_\infty := \sup_{1 \le j \le d} |k_j|.$$

Lemma D.0.1 (d-dimensional Geometric Series Formula)**.** *Assume* $0 < \theta < 1$. *Then*

$$\sum_{k \in \mathbb{Z}^d \setminus \{\mathbf{0}\}} \theta^{|k|_1} = C_{\theta,d} := \left(\frac{1+\theta}{1-\theta}\right)^d - 1. \tag{D.1}$$

Proof. The formula follows from the following computation

$$\begin{aligned}
\sum_{k \in \mathbb{Z}^d} \theta^{|k|_1} &= \sum_{k \in \mathbb{Z}^d} \theta^{\left(\sum_{j=1}^d |k_j|\right)} \\
&= \prod_{j=1}^{d} \sum_{k_j \in \mathbb{Z}} \theta^{|k_j|} \\
&= \left(\frac{1+\theta}{1-\theta}\right)^d.
\end{aligned}$$

□

Lemma D.0.2. *Assume* $0 < \theta < 1$ *and* $A > 0$. *Then*

$$\sum_{\substack{k \in \mathbb{Z}^d \\ |k|_1 > A}} \theta^{|k|_1} \le C'_{\theta,d}\, \theta^{A/d},$$

where

$$C'_{\theta,d} := 2^d \left(\frac{1+\theta}{1-\theta}\right)^d \frac{1}{1+\theta}. \tag{D.2}$$

Proof. Since

$$|k|_\infty \geq \frac{|k|_1}{d},$$

we have

$$\{k \in \mathbb{Z}^d : |k|_1 \leq A\} \supseteq \{k \in \mathbb{Z}^d : |k|_\infty \leq A/d\}$$

or

$$\{k \in \mathbb{Z}^d : |k|_1 > A\} \subseteq \{k \in \mathbb{Z}^d : |k|_\infty > A/d\}.$$

Therefore,

$$\sum_{\substack{k\in\mathbb{Z}^d \\ |k|_1>A}} \theta^{|k|_1} \leq \sum_{\substack{k\in\mathbb{Z}^d \\ |k|_\infty>\frac{A}{d}}} \theta^{|k|_1}.$$

The set $\{k \in \mathbb{Z}^d : |k|_\infty > A/d\}$ is contained in the union of 2^d half-spaces where one coordinate exceeds A/d in absolute value. Hence

$$\sum_{\substack{k\in\mathbb{Z}^d \\ |k|_\infty>\frac{A}{d}}} \theta^{|k|_1} \leq 2^d \sum_{\substack{k\in\mathbb{Z}^d \\ k_1>\frac{A}{d}}} \theta^{|k|_1}$$

Splitting the sum gives

$$\begin{aligned}
\sum_{\substack{k\in\mathbb{Z}^d \\ k_1>\frac{A}{d}}} \theta^{|k|_1} &= \sum_{k_1>\frac{A}{d}} \theta^{|k_1|} \sum_{(k_2,\ldots,k_d)\in\mathbb{Z}^{d-1}} \theta^{|(k_2,\ldots,k_d)|_1} \\
&\leq \frac{\theta^{\frac{A}{d}}}{1-\theta} \left(\frac{1+\theta}{1-\theta}\right)^{d-1} \\
&= \left(\frac{1+\theta}{1-\theta}\right)^{d} \frac{\theta^{A/d}}{1+\theta}
\end{aligned}$$

Combining the estimates yields

$$\sum_{\substack{k\in\mathbb{Z}^d \\ |k|_1>A}} \theta^{|k|_1} \leq 2^d \left(\frac{1+\theta}{1-\theta}\right)^{d} \frac{\theta^{A/d}}{1+\theta}$$

which proves the claim. □

APPENDIX E

Riesz Representation Theorem

Theorem E.0.1 (Riesz Representation Theorem,[DS58]). *There exists a Riesz isomorphism between the Hilbert space $\mathcal{H}$ and its (conjugate) dual*

$$R : \mathcal{H}^* \to \mathcal{H}$$

such that for any $g \in \mathcal{H}$ and $f \in \mathcal{H}^*$

$$f(g) = \langle g, Rf \rangle_{\mathcal{H}}.$$

Moreover, R is an isometry

$$\|Rf\|_{\mathcal{H}} = \|f\|_{\mathcal{H}^*}.$$

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