REGULAR CURVES ON RIEMANNIAN MANIFOLDS

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in the
University of Michigan
1956

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A regular curve on a Riemannian manifold is a curve with a continuously turning non-trivial tangent vector. A regular homotopy is a homotopy which at every stage is a regular curve and keeps end points and directions fixed. A regular curve is closed if its initial point and tangent coincide with its end point and tangent. In 1937 Hassler Whitney classified the closed regular curves in the plane according to equivalence under regular homotopy. The main goal of this work is to extend this result to regular curves on Riemannian manifolds.

Let x_0 be a point of the unit tangent bundle T of a Riemannian manifold M. The space of all regular curves on M starting at the point and direction determined by x_0 is denoted by E. A map p from E onto T is defined by sending a curve into the tangent of its endpoint at its endpoint. The main theorem of this work is that the triple (E,p,T) has the covering homotopy property for polyhedra. Let Γ be the fiber over x_0 of (E,p,T) and let Ω be the ordinary loop space of T at x_0. A map $\varphi:\Gamma \longrightarrow \Omega$ is defined as follows: If $f \in \Gamma$ let φf at t be the vector of T whose base point is f(t) and whose direction is defined by f'(t), the derivative of f at t. From the main theorem it follows that φ induces an isomorphism between the homotopy groups of Γ and Ω. Then it is shown that φ induces a 1-1 correspondence between the set of classes (under regular homotopy) of regular curves on M which start and end at the point and direction determined by x_0 and $\pi_1(T,x_0)$. If f is a closed

regular curve in the plane then its rotation number $\gamma(f)$ is the total angle which $f'(t)$ turns as t traverses the unit interval. The Whitney–Graustein Theorem says that two closed regular curves on the plane are regularly homotopic if and only if they have the same rotation number. Using the fact that the unit tangent bundle of the plane is the product of the plane and the circle, this theorem follows from the previous work.

If f is a regular curve on M, φf is a curve, as we shall say, a lifted curve, on T. Clearly, not all curves on T are lifted curves. In particular, every lifted curve must be an integral curve of a canonical 1–form ω_o on T. If the integral curves of ω_o were exactly the lifted curves, the previous results could be obtained by proving theorems on integral curves. Unfortunately, however, ω_o admits as integral curves some non–lifted curves. Nevertheless, these considerations raise questions concerning the loop space of integral curves of ω_o on T. Let ω be a 1–form of a certain category (including ω_o) on a three dimensional manifold M and let $x_o \in M$. Denote by Ω_ω the loop space at x_o of piecewise regular curves on M which are integral curves of ω and by Ω the ordinary loop space of M at x_o. Then it is proved that the inclusion $i: \Omega_\omega \longrightarrow \Omega$ induces an isomorphism between the homotopy groups of Ω_ω and Ω.

ACKNOWLEDGMENTS

I would like to express my appreciation to my adviser, Professor Raoul Bott, for the encouragement and advice which he gave throughout the preparation of this thesis.

I would also like to thank Professor Hans Samelson for reading the manuscript and for suggesting several corrections.

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INTRODUCTION

A regular curve on a Riemannian manifold is a curve with a continuously turning non-trivial tangent vector.[1] A regular homotopy is a homotopy which at every stage is a regular curve and keeps end points and directions fixed. A regular curve is closed if its initial point and tangent coincides with its end point and tangent. In 1937 Hassler Whitney [17] classified the closed regular curves in the plane according to equivalence under regular homotopy. The main goal of this work is to extend this result to regular curves on Riemannian manifolds.

Theorem A. Let x_o be a point of the unit tangent bundle T of a Riemannian manifold M. Then there is a 1-1 correspondence between the set Π_o of classes (under regular homotopy) of regular curves on M which start and end at the point and direction determined by x_o and $\Pi_1(T,x_o)$.

This correspondence may be described as follows. If $\bar{f} \in \Pi_o$ let f be a representative of $\bar{f}$ and let $\bar{\varphi}f$ at t be the vector of T whose base point is f(t) and whose direction is defined by f'(t), the derivative of f at t. Then $\bar{\varphi}f$ is a curve on T which represents an element of $\Pi_1(T,x_o)$. The correspondence of Theorem A is that induced by $\bar{\varphi}$.

[1] These definitions will be made precise in the body of the work. Also, the theorems stated here will be proved later. The numbers in brackets refer to the Bibliography.

If f is a closed regular curve in the plane then its rotation number $\gamma(f)$ is the total angle which $f'(t)$ turns as t traverses I. The Whitney–Graustein Theorem says that two closed regular curves on the plane are regularly homotopic if and only if they have the same rotation number. Using the fact that the unit tangent bundle of the plane is $E^2 \times S^1$, this theorem follows from Theorem A.

Let x_0 be a point of the unit tangent bundle T of a Riemannian manifold M. The space of all regular curves on M starting at the point and direction determined by x_0 is denoted by E. A map π from E onto T is defined by sending a curve into the tangent of its endpoint at its endpoint. The following can be considered as the fundamental theorem of this work.

Theorem B. The triple (E, π, T) has the covering homotopy property for polyhedra.

Let Γ be the fiber over x_0 of (E, π, T) and let Ω be the ordinary loop space of T at x_0.

Theorem C. The map $\overline{\varphi}$ is a weak homotopy equivalence between Γ and Ω.

Theorem B is used to obtain Theorem C and, in turn, Theorem A follows from Theorem C.

If f is a regular curve on M, $\overline{\varphi}f$ is a curve, as we shall say, a lifted curve, on T=T(M). Clearly, not all curves on T are lifted curves. In particular, every lifted curve must be an integral curve of a canonical 1-form ω_0 on T. If the integral curves of ω_0 were

exactly the lifted curves, Theorems A, B, and C could be proved by proving theorems on integral curves. Unfortunately, however, ω_0 admits as integral curves some non-lifted curves. Nevertheless, these considerations raise questions concerning the loop space of integral curves of ω_0 on T.

Theorem D. Let ω be a 1-form of Class A on a three dimensional manifold M and let $x_0 \in M$. Denote by Ω_ω the loop space at x_0 of piecewise regular curves on M which are integral curves of ω and by Ω the ordinary loop space of M at x_0. Then the inclusion $i: \Omega_\omega \longrightarrow \Omega$ is a weak homotopy equivalence.

Class A (to be defined in Section 9) includes ω_0 but excludes the completely integrable 1-forms.

SECTION 1

FIBER SPACES

By a space is meant a topological space (not necessarily Hausdorff). By I is meant the closed unit interval and, as usual, a homotopy of a space X into a space Y is a map (continuous transformation) $H:X \times I \longrightarrow Y$. Often we will set $H(p,v)=h_v(p)$ and write $h_v:X \longrightarrow Y$. A homotopy h_v is called a homotopy of a map h if $h_o=h$. A triple (E,p,B) will be considered to consist of two arcwise connected spaces E, B and a map p from E into B. The map p is called the projection. If (E,p,B) is a triple, and X any space, a map $\overline{f}:X \longrightarrow E$ covers another map $f:X \longrightarrow B$ if $p\overline{f}=f$. A triple (E,p,B) will be said to have the covering homotopy property for polyhedra or simply the CHP if the following condition is satisfied:

Condition A. Let be given a polyhedron P, a homotopy $f_v:P \longrightarrow B$, and a map $\overline{f}:P \longrightarrow E$ covering f_o. Then there exists a homotopy $\overline{f}_v:P \longrightarrow E$ of $\overline{f}$ covering f_v.

Naturally enough, $\overline{f}_v$ is called a covering homotopy. If the triple (E,p,B) has the CHP then E is called a fiber space in the sense of Serre over B or simply a fiber space. The space $F=p^{-1}(y)$ for each $y \in B$ is called the fiber over y or simply the fiber. Note that if (E,p,B) has the CHP, then p must be onto.

If g is a map from a space X into a space Y, then the restriction of g to a subset A of X will be denoted by $g_{/A}$ or sometimes just g. The following proposition may be

found in [11].

Proposition 1.1. Let a triple (E,p,B) have the CHP, $y \in B$ and $F=p^{-1}(y)$. Then the following sequence (the exact homotopy sequence of E) can be defined and is exact:

$$\cdots \longrightarrow \pi_k(F) \xrightarrow{j_\#} \pi_k(E) \xrightarrow{p_\#} \pi_k(B) \xrightarrow{\Delta} \pi_{k-1}(F) \longrightarrow \cdots$$

$$\cdots \longrightarrow \pi_1(B) \xrightarrow{\Delta} \pi_0(F) \longrightarrow 0$$

Here $\pi_k(X)$ is the k^{th} homotopy group of X (with the base point suppressed) and $\pi_0(F)$ is the set of arcwise connected components of F with the component containing the base point of $\pi_1(E)$ designated as the neutral one. The homomorphisms $j_\#$ and $p_\#$ are those induced by the inclusion of $j:F \longrightarrow E$, and the projection $p:E \longrightarrow B$ respectively. The "boundary" homomorphism Δ is defined as follows. The map $p:(E,F) \longrightarrow (B,y)$ induces an isomorphism $\bar{p}_\# : \pi_k(E,F) \longrightarrow \pi_k(B)$. If $\partial: \pi_k(E,F) \longrightarrow \pi_{k-1}(F)$ is the boundary of the exact homotopy sequence of the pair (E,F) then $\Delta = \partial \bar{p}_\#^{-1}$.

In this work P will always denote a polyhedron. The set $\{t \text{ real} \mid a \leq t \leq b\}$ is denoted by [a,b]. A cube I^k is the Cartesian product of k copies of I. A triple (E,p,B) has the covering homotopy property for cubes if Condition A is true with "polyhedron" replaced by "cube". The following proposition is well-known. It is a special case of a theorem proved in [6, p. 136].

Proposition 1.2. Let (E,p,B) be a triple which has

the covering homotopy property for cubes. Then it also has the CHP.

Lemma 1.3. Let a triple (E,p,B) have the CHP. Let σ be a simplex of some dimension n and let $g:\sigma \times I \longrightarrow B$ be given. Suppose $\dot{\sigma}$ is the (pointset) boundary of σ, $A = \dot{\sigma} \times I \cup \sigma \times 0 \subset \sigma \times I$ and $f:A \longrightarrow E$ covers $g_{/A}$. Then there exists an extension F of f to all of $\sigma \times I$ covering g.

The proof is immediate. There is a homeomorphism from $\sigma \times I$ onto $I^n \times I$ which sends A homeomorphically onto $I^n \times 0$. Then the application of the CHP yields the desired map F.

The statement of the following theorem is quite similar to what Hurewics calls the "Uniformization Theorem" in [5]. It will be found useful in proving the CHP for a triple whose base space is a manifold.

Proposition 1.4. Suppose a triple (E,p,B) has the CHP locally; that is, for each point $x \in B$, there exists a neighborhood V of x such that $(p^{-1}(V),p,V)$ has the CHP. Then (E,p,B) has the CHP.

Proof. Let $H:P \times I \longrightarrow B$ be a given homotopy and $h:P \times 0 \longrightarrow E$ a covering of $H_{/P \times 0}$. We will define a covering homotopy $\overline{H}:P \times I \longrightarrow E$. For each $y \in H(P \times I)$ let V_y be a neighborhood of y so that $(p^{-1}(V_y),p,V_y)$ has the CHP. Assume $P \times I$ has been given some definite metric. Denote by δ the Lebesgue number of the covering $\{U_y = H^{-1}(V_y) \,/\, y \in H(P \times I)\}$ of $P \times I$. Put $I_o = [0, \delta/3]$. It is sufficient to define $\overline{H}$ on $P \times I_o$ for then iteration

will yield a full covering homotopy.

Take a simplicial complex K such that $|K|=P$. Let Sd(K) be a subdivision of K such that the diameter of any simplex of Sd(K) is less than $\delta/3$, and let $Sd(K)^r$ be the r-skeleton of Sd(K). If v is a vertex of Sd(K) the choice of the I_o yields that $H(v \times I_o)$ is contained in some neighborhood V where $(p^{-1}(V),p,V)$ has the CHP. This fact immediately gives a definition of $\overline{H}$ on $|Sd(K)^o| \times I_o$. Proceeding by induction suppose $\overline{H}$ has been defined on $|Sd(K)^{r-1}| \times I_o$, and σ^r is an r-simplex of Sd(K). From the choices of Sd(K) and I_o it follows that $\sigma^r \times I_o$ has diameter less than δ. Then $H(\sigma^r \times I_o)$ is contained in some V such that $(p^{-1}(V),p,V)$ has the CHP. Already $\overline{H}$ has been defined on $\dot{\sigma}^r \times I_o \cup \sigma^r \times 0$. Then application of Lemma 1.3 yields $\overline{H}$ on $\sigma^r \times I_o$. In this manner $\overline{H}$ is defined on all of $|Sd(K)^r| \times I_o$ and then by induction on all of $|Sd(K)| \times I_o$. This proves 1.4.

SECTION 2

REGULAR CURVES

By a manifold we shall mean a connected Riemannian manifold of class 3 and dimension greater than 1. There are no assumptions such as completeness or compactness. If M is a manifold, $T_0(M)$ or often just T_0 will be the space (or bundle) of all tangent vectors of M. By T(M) or T is meant the sub-bundle of T_0 which consists of the unit tangent vectors of M. These spaces are well-known manifolds; for example see [1] or [13].

As usual, a curve on a space is a map of I into the space. Let M be a manifold and let f be a curve on M whose derivative [1, p. 46] exists at the value $t_0 \in I$. This derivative is an element $v(t_0)$ of $M_{f(t_0)}$, the tangent vector space of M at $f(t_0)$. By convention in this work the derivative $f'(t_0)$ of f at t_0 will be the pair $(f(t_0), v(t_0))$. Thus, $f'(t_0)$ will be an element of $T_0(M)$. By the magnitude of $f'(t_0)$ or $|f'(t_0)|$ we mean the magnitude of $v(t_0)$.

A parametrized regular curve on a manifold M is a curve f on M such that $f'(t)$ exists, is continuous and has positive magnitude for each $t \in I$. Two parametrized regular curves f and g will be called equivalent if there exists a homeomorphism h of I onto itself such that for all $t \in I$, $h'(t)$ exists, is continuous and positive, and $f(t)=g(h(t))$. It is an easy matter to check that this is a true equivalence. A regular curve is an equivalence

class of parametrized regular curves.

The arc-length of a regular curve g, defined in the usual way, exists, and is independent of its representative. It will be denoted by L(g). Implicit use of the following proposition will be made throughout this work.

Proposition 2.1. If g is a regular curve on a manifold then there exists a unique representative of g still denoted by g such that $|g'(t)| = L(g)$ for all $t \in I$. L(g) is the only possible constant value here.

The proof for the plane is in [17]. The same proof holds for the general case of a manifold. The representative of a regular curve given by 2.1 will be called distinguished. A distinguished representative is just a parametrization proportional to arc-length. Unless we note otherwise, a regular curve will be identified with its distinguished representative.

Let M be a manifold and x_o a fixed point of T(M). We denote by E(M) or sometimes simply E, the space of regular curves on M whose normalized initial tangents are x_o; in other words,

$$E(M) = \left\{ g \text{ is a regular curve on } M \;\middle|\; \frac{g'(0)}{|g'(0)|} = x_o \right\}$$

Let $\bar{d}: T_o \times T_o \longrightarrow R^+$ be any metric on T_o (R^+ is the space of non-negative real numbers). Then for f and $g \in E$, let

$$d(f,g) = \max \left\{ \bar{d}[f'(t), g'(t)] \;\middle|\; t \in I \right\}$$

From the fact that $\bar{d}$ is a metric, it follows easily that d is a metric on E. We will suppose E to have the topology induced by d.

Let f_i be a sequence of points of E converging to a point f of E. Then for each $t \in I$, $f_i'(t)$ converges to $f'(t)$. If a sequence x_n of points of T_o converges to x_o then from the topology of T_o it follows that the base points of x_n in M converge to the base point of x_o. Thus $f_i(t)$ converges to $f(t)$ for each $t \in I$.

The map $\pi: E \longrightarrow T$ of Theorem B may be defined by $\pi(f) = f'(1)/|f'(1)|$. To speak of (E, π, T) as a triple, E must be arcwise connected. This is proved later (Lemma 6.2).

SECTION 3

THE REDUCTION OF THE PROOF OF THEOREM B TO 3.1 AND 3.2

The proof of Theorem B depends essentially on Propositions 3.1 and 3.2.

Proposition 3.1. Let M be a manifold and $\overline{p}:E(M)\longrightarrow M$ be the map $\overline{p}(g)=g(1)$. Then $(E,\overline{p},M)$ has the CHP.

Suppose M is a manifold. Let $\pi_1:T_o(M)\longrightarrow M$ be the be the map which sends a tangent vector into its base point. A homotopy $f_v:P\longrightarrow T_o(M)$ will be called vertical if for all $p\in P$ and $v\in I$, $\pi_1 f_v(p)=\pi_1 f_o(p)$. A homotopy $f_v:P\longrightarrow X$ (X any space) is said to be stationary on a subpolyhedron A of P if $f_v(p)=f_o(p)$ for all $p\in A$ and $v\in I$.

Proposition 3.2. Let M be a manifold and $f_v:P\longrightarrow T(M)$ be a given vertical homotopy with $\overline{f}:P\longrightarrow E(M)$ covering f_o. Then there exists a covering homotopy $\overline{f}_v:P\longrightarrow E$. Furthermore if f_v is stationary on a subpolyhedron A of P then $\overline{f}_v$ will also be.

Propositions 3.1 and 3.2 will be proved in the following sections. Now we will show how Theorem B follows from 3.1 and 3.2.

Lemma 3.3. Let P be a polyhedron and A be a subpolyhedron which is a strong deformation retract [2] of P. Let g and h be maps of P into a space X which agree on A. Then there exists a homotopy $H:P\times I\longrightarrow X$ between g and h which is stationary on A.

Proof. Since A is a strong deformation retract of P there is a homotopy $K:P\times I\longrightarrow P$ such that $K(p,0)=p$,

$K(p,1)\in A$, and if $p\in A$, $K(p,t)=p$. The desired homotopy $H:P\times I\longrightarrow X$ may be defined as follows:

$$H(p,t)=gK(p,2t) \qquad 0\leq t\leq 1/2$$

$$H(p,t)=hK(p,2-2t) \qquad 1/2\leq t\leq 1$$

The k-sphere is denoted by S^k.

Lemma 3.4. Let P and A be as above with P contractible, and M be an n dimensional manifold. Let $F:P\longrightarrow M$ be given and $g:P\longrightarrow T=T(M)$, $h:P\longrightarrow T$ be two covering maps of F which agree on A. Then there exists a homotopy $h_v:P\longrightarrow T$ between g and h such that h_v is stationary on A and for each $v\in I$, h_v covers F.

Proof. Let E' be the induced bundle $F^{-1}(T)$ [13, p. 47].

$$\begin{array}{ccc} E' & \xrightarrow{q_2} & T \\ \downarrow{\scriptstyle q_1} & & \downarrow{\scriptstyle \pi_1} \\ P & \xrightarrow{F} & M \end{array}$$

By definition:

$$E'=\left\{(p,t)\in P\times T \;\middle|\; F(p)=\pi_1(t)\right\}.$$

$$q_2(p,t)=t, \qquad q_1(p,t)=p.$$

Since P is contractible, E' is a product $P\times S^{n-1}$ [13, p. 53] with q_1 being the projection of E' onto P. Let $\pi':E'\longrightarrow S^{n-1}$ be the other projection.

Define $\bar{g}:P\longrightarrow E'\subset P\times T$ by $\bar{g}(p)=(p,g(p))$ and let $\bar{\bar{g}}:P\longrightarrow S^{n-1}$ be the composition $\pi'\bar{g}$. Similarly, define $\bar{h}$ and $\bar{\bar{h}}$ from h.

Apply the previous lemma to obtain a homotopy

$\bar{\bar{h}}_v: P \longrightarrow S^{n-1}$ between $\bar{\bar{g}}$ and $\bar{\bar{h}}$ which is stationary on A. Define $\bar{h}_v: P \longrightarrow E' = P \times S^{n-1}$ by $\bar{h}_v(p) = (p, \bar{\bar{h}}_v(p))$ and $h_v: P \longrightarrow T$ by $h_v = q_2 \bar{h}_v$. It can be quickly checked that h_v satisfies the lemma. q.e.d.

To prove Theorem B it is sufficient by Proposition 1.2 to show that (E, π, T) has covering homotopy property for cubes. Suppose, then, we are given a homotopy $f_v: P \longrightarrow T$ and a map $f: P \longrightarrow E$ covering f_o where P is a cube. We will construct a covering homotopy $\bar{f}_v: P \longrightarrow E$.

Application of Proposition 3.1 yields a covering map $\bar{h}: P \times I \longrightarrow E$ of $\pi_1 f_v$ such that $\bar{h}(p,0) = \bar{f}(p)$. Lemma 3.4 then yields a homotopy $H_u: P \times I \longrightarrow T$ such that (1) $H_o(p,v) = \pi \bar{h}(p,v)$, (2) $H_1(p,v) = f_v(p)$, (3) $H_u(p,v)$ covers $\pi_1 f_v(p)$ for each $u \in I$, and (4) H_u is stationary on $P \times 0$. By (3) H_u is a vertical homotopy so Proposition 3.2 applies to yield a homotopy $\bar{H}_u: P \times I \longrightarrow E$ of $\bar{h}$ which covers H_u.

We assert that $\bar{H}_1(p,v)$ can be taken as the desired covering homotopy $\bar{f}_v(p)$. From $H_1(p,v) = f_v(p)$ it follows that $\bar{H}_1(v,p)$ covers $f_v(p)$. Since H_u is stationary on $P \times 0$, $\bar{H}_u$ is also. Then $\bar{H}_1(p,0) = \bar{H}_o(p,0) = \bar{h}(p,0) = \bar{f}(p)$ or $\bar{H}_1(p,v)$ is a homotopy of $\bar{f}(p)$. This shows that Theorem B follows from Propositions 3.1 and 3.2.

SECTION 4

PROOF OF PROPOSITION 3.1

We will need two lemmas. By $v \perp w$ it is meant that the vectors v and w are perpendicular.

Lemma 4.1. Let $n > 1$ and S^{n-1} be the unit vectors of Euclidean n-space E^n considered as a vector space. Suppose P is a cube and a map $w: P \longrightarrow S^{n-1}$ is given. Then there exists a map $u: P \longrightarrow S^{n-1}$ such that for all $p \in P$, $u(p) \perp w(p)$.

This lemma is not true for a general polyhedron. In particular, if $P = S^{n-1}$ and w is the identity, the existence of such a map u implies the existence of a unit vector field on S^{n-1}. It is well known that this is impossible for odd n.

Proof of 4.1. Let $V_{n,2}$ be the Stiefel manifold [13, p.33] of ordered orthogonal unit 2-frames in E^n. With a projection p_1 sending a 2-frame into its first vector, $V_{n,2}$ becomes an (n-2)-sphere bundle over S^{n-1}. Let E' be the induced bundle $w^{-1}(V_{n,2})$.

$$\begin{array}{ccc} E' & \xrightarrow{f} & V_{n,2} \\ \downarrow & & \downarrow p_1 \\ P & \xrightarrow{w} & S^{n-1} \end{array}$$

Since P is contractible E' is a product. Let $s: P \longrightarrow E'$ be any cross-section, and let $p_2: V_{n,2} \longrightarrow S^{n-1}$ send a 2-frame into its second vector. Then the composition $u = p_2 f s$ has the desired property. q.e.d.

Lemma 4.2. Given y_0, $0 < y_0 \leq 1/2$, there exists a real continuous differentiable function $\beta(y)$ defined on I such that (1) $\beta(0)= \beta(1)=0$, (2) $\beta'(0)=0$, and (3) for $y_0 \leq y \leq 1$, $\beta'(y) \geq 4$.

Proof. Consider the function:

$$r(y)=0 \qquad 0 \leq y \leq y_0/2$$

$$r(y)=-\frac{8}{y_0}(1-y_0)y+r(1-y_0) \qquad y_0/2 \leq y \leq y_0$$

$$r(y)=4y-4 \qquad y_0 \leq y \leq 1$$

Note that r(y) could be taken as $\beta(y)$ except for the fact that it has corners at $y=y_0/2$ and $y=y_0$. By "rounding off the corners" of r(y) the desired function can be obtained.

In order to prove 3.1 it is sufficient by Proposition 1.4 to show that $(\bar{p}^{-1}(U_0),\bar{p},U_0)$ has the CHP where U_0 is a coordinate neighborhood of M. Since U_0 is homeomorphic to E^n we can identify the two spaces under this homeomorphism. Thus $U_0=E^n \subset M$. $T_0(U_0)$ is a product space $E^n \times E^n$ where the first factor comes from the base point and the second from the direction and magnitude of a vector. We identify each of the two factors of $T_0(U_0)$ with a single E^n whose elements we consider as vectors. If S^{n-1} is the space of unit vectors of E^n, $T(U_0)=E^n \times S^{n-1}$ The magnitude of a vector v of E^n is written $|v|$.

For convenience the following new convention is used in this section and the next. The derivative of a regular curve at a point of U_0 will not carry the base point. That is, it is now the projection of the old

derivative onto the second factor of $T_o(U_o)$. This is possible since most of the analysis in these sections is concerned with $T_o(U_o)$ and U_o.

By Proposition 1.2 it is enough to show that $(\bar{p}^{-1}(U_o),\bar{p},U_o)$ has the covering homotopy property for cubes. Let $h_v:P\longrightarrow U_o$ be a given homotopy with $\bar{h}:P\longrightarrow\bar{p}^{-1}(U_o)$ covering h_o where P is a cube. We will construct a covering homotopy $\bar{h}_v:P\longrightarrow\bar{p}^{-1}(U_o)$.

Choose J with $0\leq J<1$ such that for all $p\in P$ and $t\in[J,1]$, $\bar{h}(p)(t)\in U_o$. Then choose J_o with $J\leq J_o<1$ so that for all $p\in P$ and $t\in[J_o,1]$,

$$|\bar{h}'(p)(t)-\bar{h}'(p)(1)|<\frac{|\bar{h}'(p)(1)|}{10}$$

The following choices are motivated by the need to insure the regularity of the covering homotopy curves we are constructing. Let

$$K=\max\left\{|h_v(p)-h_o(p)| \;\middle|\; v\in I,\ p\in P\right\}.$$

If $K=0$, let $y_o=1/2$. Otherwise let

$$y_o=\min\left\{\frac{1}{2},\ \frac{|\bar{h}'(p)(1)|(1-J_o)}{6K} \;\middle|\; p\in P\right\}.$$

The compactness of P yields that $y_o>0$.

Taking y_o as above, let $\beta(y)$ be the function given by Lemma 4.2.

By taking $\omega(p)=\dfrac{\bar{h}'(p)(1)}{|\bar{h}'(p)(1)|}$, Lemma 4.1 yields a map $u:P\longrightarrow S^{n-1}$ such that $u(p)\perp\bar{h}'(p)(1)$.

We define the desired covering homotopy $\bar{h}_v:P\longrightarrow\bar{p}^{-1}(U_o)$ as follows. For $0\leq t\leq J_o$ set $\bar{h}_v(p)(t)=\bar{h}(p)(t)$. For $J_o\leq t\leq 1$ let $s=s(t)=(t-J_o)/(1-J_o)$; then set

$$\bar{h}_v(p)(t)=\bar{h}(p)(t)+s^2[h_v(p)-h_o(p)]+$$
$$\beta(s)|h_v(p)-h_o(p)|u(p).$$

Here $\bar{h}(p)$ is to be taken distinguished (see Section 2), but $\bar{h}_v(p)$, in general won't be. Note that for $t \geq J_o$ all the terms used to define $\bar{h}_v(p)$ lie in U_o and hence the additions make sense.

The following properties of $\bar{h}_v$ can be readily checked:

1) $\bar{h}_v(p)(t)$ is continuous in v, p, and t.

2) $\bar{h}_v(p)(1)=h_v(p)$.

3) $\bar{h}_o(p)=\bar{h}(p)$.

The derivative of $\bar{h}_v(p)(t)$ for $t \geq J_o$ can be computed to be:

$$\bar{h}_v'(p)(t)=\bar{h}'(p)(t)+s'2s[h_v(p)-h_o(p)]+$$
$$s'\beta'(s)|h_v(p)-h_o(p)|u(p).$$

Then it can be seen:

4) $\bar{h}_v(p)$ is differentiable.

5) $\bar{h}_v'(p)(0)/|\bar{h}_v'(p)(0)|=x_o$. The derivative meant here is in the sense of Section 2.

The following requires proof:

6) $\bar{h}_v(p)$ is a regular curve.

For this it is sufficient to show that $\bar{h}_v'(p)(t)\neq 0$ for $t \geq J_o$. For such t we can write $\bar{h}_v'(p)(t)=A_1+A_2$ where

$$A_1=\bar{h}'(p)(t)+s'\beta'(s)|h_v(p)-h_o(p)|u(p)$$

and

$$A_2=2s's[h_v(p)-h_o(p)].$$

We will divide the proof into two parts.

Case I, $s \leq y_o$: By the choice of y_o, $|A_2| \leq (1/3)|\bar{h}'(p)(1)|$. On the other hand, from the facts that $\bar{h}'(p)(t)$ is close (by the choice of J_o) to $\bar{h}'(p)(1)$ and $u(p) \perp \bar{h}'(p)(1)$ it follows that $|A_1| \geq (1/2)|\bar{h}'(p)(1)|$. From the triangle inequality we have that $\bar{h}_v'(p)(t) \neq 0$.

Case II, $s \geq y_o$: It can be shown without much trouble that $|A_1| \geq 4|h_v(p) - h_o(p)|s'$ using $\beta'(s) \geq 4$. On the other hand, $|A_2| \leq 2|h_v(p) - h_o(p)|s'$. Then, by the triangle inequality, $\bar{h}_v'(p)(t) \neq 0$. This proves (6).

Properties (1), (4), (5) and (6) imply that $\bar{h}_v(p)$ is really an element of E, (2) says that $\bar{h}_v$ covers h_v and (3) that $\bar{h}_v$ is a homotopy of $\bar{h}$. Therefore we have proved 3.1.

SECTION 5

PROOF OF PROPOSITION 3.2

Let U_o be a coordinate neighborhood on M, and let $V=T(U_o)$. By the argument used to prove Proposition 1.4, it is sufficient to prove 3.2 for the case where $\{f_v(p) \mid v \in I, p \in P\} \subset V$. The notation and conventions of the last section will be continued.

Let $\pi_2 : T_o(U_o) = E^n \times E^n \longrightarrow E^n$ be the projection onto the second factor. Then $\pi_2(T(U_o)) \subset S^{n-1}$. The angle between two vectors of S^{n-1} is a continuous function of the vectors. This fact, together with the compactness of P, justifies the following choice. Pick $\varepsilon > 0$ such that for all $p \in P$ and $|v-v'| \leq \varepsilon$, the angle (measured in radians) between $\pi_2 f_v(p)$ and $\pi_2 f_{v'}(p)$ is less than 1/10.

For $v \leq \varepsilon$ let $\alpha_v(p)$ be the oriented angle from $\pi_2 f_o(p)$ to $\pi_2 f_v(p)$. By our choice of ε, $\alpha_v(p) < 1/10$.

For $v \leq \varepsilon$, and $t \in I$, we will define $Q_v(p,t)$ to be the following rotation of E^n; that is, we are defining a map $Q_v : P \times I \longrightarrow R_n$ where R_n is the rotation group of E^n. If $\pi_2 f_o(p) = \pi_2 f_v(p)$ let $Q_v(p,t) = e$, the identity rotation. Otherwise let $Q_v(p,t)$ rotate V, the unique plane determined by $\pi_2 f_o(p)$ and $\pi_2 f_v(p)$, through the angle $t\alpha_v(p)$ and leave $V^{\perp}$, the orthogonal complement of V, fixed.

That $Q_v(p,t)$ is continuous in v, p, and t and has a continuous first derivative in t, $Q_v'(p,t)$, is easily seen. Later, in fact, we will have occasion to compute this derivative.

Choose J, $0 \leq J < 1$, such that

$$\left\{\overline{f}(p)(t) \;\middle|\; p \in P,\ t \in [J,1]\right\} \subset U_o.$$

Lemma 5.1. There exists a J_o with $J \leq J_o < 1$ such that for all $p \in P$ and $t \in [J_o,1]$,

$$\left|\frac{\overline{f}(p)(t)-\overline{f}(p)(1)}{J-1}\right| \leq \frac{4}{3}|\overline{f}'(p)(1)|.$$

Proof. It follows from the definition that

$$\overline{f}'(p)(1)=\lim_{t\to 1}\frac{\overline{f}(p)(t)-\overline{f}(p)(1)}{t-1}$$

so by the compactness of P there exists a J_o with $J \leq J_o < 1$ such that for $t \in [J_o,1]$,

$$\left|\frac{\overline{f}(p)(t)-\overline{f}(p)(1)}{t-1} - \overline{f}'(p)(1)\right| \leq \frac{1}{3}|\overline{f}'(p)(1)|.$$

Then by the triangle inequality

$$\left|\frac{\overline{f}(p)(t)-\overline{f}(p)(1)}{t-1}\right| \leq \frac{4}{3}|\overline{f}'(p)(1)|.$$

Also clearly for $J_o \leq t \leq 1$,

$$\left|\frac{\overline{f}(p)(t)-\overline{f}(p)(1)}{J_o-1}\right| \leq \left|\frac{\overline{f}(p)(t)-\overline{f}(p)(1)}{t-1}\right|.$$

These last two inequalities yield the lemma.

Choosing J_o by 5.1 the desired covering homotopy is defined as follows for $v \leq \mathcal{E}$.

For $0 \leq t \leq J_o$ set $\overline{f}_v(p)(t)=\overline{f}(p)(t)$.

For $J_o \leq t \leq 1$ let $s=s(t)=(t-J_o)/(1-J_o)$; then set $\overline{f}_v(p)(t)=[\overline{f}(p)(t)-\overline{f}(p)(1)]Q_v(p,s)[e+(s^2-s)Q_v'(p,0)]+\overline{f}(p)(1)$. Here $Q_v(p,s)$, $Q_v'(p,0)$ and e are to be considered as transformations acting on the right. The curve $\overline{f}(p)$ is taken distinguished, but in general, $\overline{f}_v(p)$ won't be.

The following properties of the covering homotopy

can be quickly checked.

1) $\overline{f}_v(p)(t)$ is continuous in v, p, and t.

2) $\overline{f}_0(p)=\overline{f}(p)$

3) $\overline{f}_v(p)$ is stationary on a subpolyhedron A of P if $f_v(p)$ is.

The first derivative of $\overline{f}_v(p)$ can be quickly computed as follows for $t \geq J_0$:

$$\overline{f}_v'(p)(t)=\overline{f}'(p)(t)Q_v(p,s)[e+(s^2-s)Q_v'(p,0)]+$$
$$s'[\overline{f}(p)(t)-\overline{f}(p)(1)]\{Q_v'(p,s)[e+(s^2-s)Q_v'(p,0)]+$$
$$Q_v(p,s)(2s-1)Q_v'(p,0)\}.$$

Using this it can be further checked that:

4) $\overline{f}_v(p)(t)$ is differentiable.

5) $\overline{f}_v'(p)(0)/|\overline{f}_v'(p)(0)|=x_0$ in the sense of Section 2.

6) $\overline{f}_v'(p)(1)/|\overline{f}_v'(p)(1)|=f_v(p)$ again in the sense of Section 2.

The following requires proof:

7) The curve $\overline{f}_v(p)(t)$ is regular.

For the proof of (7) we will use:

Lemma 5.2. Let $0 \leq t \leq 1$. Then (a) the transformation $Q_v'(p,t)$ reduces the magnitude of a vector to less than 1/10 of its original magnitude and (b) the transformation $e+(t^2-t)Q_v'(p,0)$ does not change the magnitude of a vector by a factor of more than 1/10.

Proof. For given p and v let coordinates $x_1, \cdots, x^n$ of E^n be chosen so that V (from the definition $Q_v(p,t)$) is the x_1-x_2 plane and the direction of $\pi_2 f_0(p)$

coincides with the x_2 axis. Then with this system suitably oriented $Q_v(p,t)$ can be represented in the matrix form,

$$\begin{pmatrix} \cos[t\alpha_v(p)] & -\sin[t\alpha_v(p)] & 0 & \cdots & 0 \\ \sin[t\alpha_v(p)] & \cos[t\alpha_v(p)] & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Then $Q_v'(p,t)$ will be of the form,

$$-\alpha v(p)\begin{pmatrix} \sin[t\alpha_v(p)] & \cos[t\alpha_v(p)] & 0 & \cdots & 0 \\ -\cos[t\alpha_v(p)] & \sin[t\alpha_v(p)] & 0 & & 0 \\ 0 & 0 & 0 & & 0 \\ & & & & \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

If $\beta_1, \beta_2, \cdots, \beta_n$ are the components of a vector β in the above system then

$$|\beta Q_v'(p,t)| = |\alpha_v(p)|[(\beta_1)^2+(\beta_2)^2]^{1/2} \leq |\alpha_v(p)||\beta|$$

This yields (a) since $|\alpha_v(p)| < 1/10$. (b) follows from from (a) and 5.2 is proved.

To prove (7) it is clearly sufficient to show that $\overline{f}_v'(p)(t) \neq 0$ for $t \geq J_o$.

Let $\overline{f}_v'(p)(t) = A_1 + A_2$ where

$$A_1 = \overline{f}'(p)(t) Q_v(p,s)[e+(s^2-s)Q_v'(p,0)]$$

and

$$A_2 = \left[\frac{\overline{f}(p)(t)-\overline{f}(p)(1)}{J_o-1}\right]\left\{Q_v'(p,s)[e+(s^2-s)Q_v'(p,0)] + Q_v(p,s)(2s-1)Q_v'(p,0)\right\}.$$

From Lemma 5.2 (b) and the fact that $Q_v(p,s)$ does not change the magnitude of a vector it follows that $|A_1| \geq (9/10)|\bar{f}'(p)(1)|$. On the other hand, from the choice of J_o and Lemma 5.2 it can be shown without much trouble that $|A_2| \leq (1/2)|\bar{f}'(p)(1)|$. From the triangle inequality this implies that $\bar{f}_v'(p)(t) \neq 0$ and hence (7).

Properties (1), (4), (5) and (7) imply that $\bar{f}_v(p)$ really belongs to E, (2) says that $\bar{f}_v$ is a homotopy of $\bar{f}$ and (6) that $\bar{f}_v$ covers f_v. Lastly, (3) is the stationary property demanded by 3.2. Thus $\bar{f}_v$ is a satisfactory covering homotopy for $v \leq \varepsilon$.

The above construction may be repeated if $\varepsilon < 1$ using $\bar{f}_\varepsilon(p)$ instead of $\bar{f}(p)$ and using a new value for J_o, if necessary. This yields a covering homotopy for $v \leq 2\varepsilon$. Iteration yields a full $\bar{f}_v$ and the proposition is proved.

SECTION 6

ON THE TOPOLOGY OF THE FIBER Γ

We recall some definitions and theorems of [11]. Let X be a space and $x_0 \in X$. The path space of X written $E_{x_0}(X)$ or sometimes E_{x_0} is the space of all curves (or paths) on X which start at x_0, with the compact open topology. Define $p: E_{x_0} \longrightarrow X$ by sending a path onto its endpoint, i.e., let $p(f)=f(1)$. The loop space of X at x_0, $p^{-1}(x_0)$ is denoted by $\Omega(X)$ or Ω. It is shown in [11, p. 479–481] that (E_{x_0}, p, X) has the CHP and that E_{x_0} is contractible.

Using the notation of the previous sections let M be a manifold, $T=T(M)$, $E=E(M)$ and $\Gamma = \Gamma(M) = \pi^{-1}(x_0)$. Define a map $\varphi: E \longrightarrow E_{x_0}(T)$ by $\varphi(g)(t)=g'(t)/|g'(t)|$ for $g \in E$. It can be seen that φ is continuous as follows. Let E' be the set $E_{x_0}(T)$ endowed with the metric topology $d^*(f,g)=\max \{ d[f(t),g(t)] \mid t \in I \}$. Then φ can be factored through E' by maps $\varphi_1: E \longrightarrow E'$ and $\varphi_2: E' \longrightarrow E_{x_0}$, where φ_2 is the identity. It is well known that the metric topology and the compact open topology are equivalent on E_{x_0}. See for example [4, p. 55]. Hence φ_2 is continuous. From the topology of E it follows that φ_1 is continuous. Thus φ is continuous. Let $\overline{\varphi}$ be φ restricted to Γ. Then $\overline{\varphi}(\Gamma) \subset \Omega$. This map is the same as $\overline{\varphi}$ of the Introduction.

A map between two spaces is a weak homotopy equivalence [after 10, p. 279] if it induces an isomorphism of the homotopy groups of the spaces. The following theorem

is well-known. A proof can be found in [14, p. 113].

Theorem 6.1. A weak homotopy equivalence induces isomorphisms of the singular homology groups of the spaces involved.

Theorem C and 6.1 yield that $\overline{\varphi}$ induces isomorphisms of the singular homology groups of Γ and Ω. This is of interest because a certain amount of attention has been given to the problem of determining the singular homology of loop spaces. For example see [11] and [15].

The proof of Theorem C requires the following lemma.

Lemma 6.2. If M is a manifold the space E(M) is homotopically trivial.

Proof. Consider first the case where M is Euclidean n-space E^n. Assume $\pi_1 x_o$ to be the origin of a coordinate system of E^n and let $\pi_2 x_o = \overline{x}_o$ where π_2 is the projection of $T=E^n \times S^{n-1}$ onto S^{n-1}.

For some $k \geq o$ let $f: S^k \longrightarrow E$ be given. To prove the lemma for E^n it is sufficient to show that f is is homotopic to a constant. Since S^k is compact we can choose $J>0$ close enough to 0 so that for all $p \in S^k$ and $t \in [0,J]$,

$$|f'(p)(t)-f'(p)(0)| < |f'(p)(0)|.$$

Then for $v \in [0,1/2]$ let $f_v(p)(t)=f(p)[t-2(1-J)vt]$. The curve f(p) is to be distinguished, but $f_v(p)$ won't be, in general. This homotopy merely contracts f(p) into a curve whose tangent is fairly close to a constant.

Define $e(t)$ as the fixed path of E given by $\bar{x}_o t$. Then for $v \in [1/2,1]$ define

$$f_v(p)(t)=(2-2v)f_{1/2}(p)(t)+(2v-1)e(t).$$

where $f_{1/2}(p)(t)$ is the non-distinguished curve given by the previous homotopy.

It can be checked that $f_v(p)$ is really contained in E, that $f_0(p)=f(p)$ and that $f_1(p)=e$. It is the selection of J that yields the necessary regularity of $f_v(p)$ for $v \geqslant 1/2$.

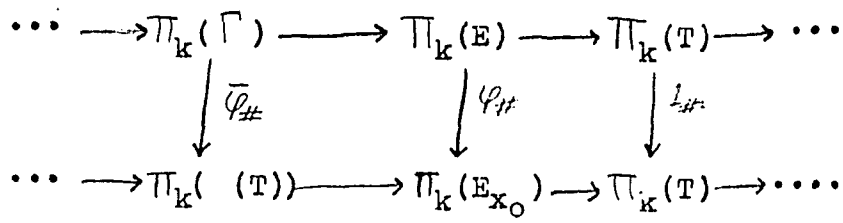
We have proved the lemma for $M=E^n$. The proof for a general M goes as follows. As before, let $f:S^k \longrightarrow E$. Now for $v \leq 1/2$ let $f_v(p)$ be a "shortening" of $f(p)$ so that for all $p \in S^k$, $f_{1/2}(p)$ lies in a certain coordinate neighborhood about x_o. For $v \in [1/2,1]$ the homotopy is the same as the total homotopy for E^n. q.e.d.

Theorem C is proved as follows.

From the definition of φ it follows easily that φ commutes with the identity of T, i.e., $p\varphi = \pi$ or

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & E_{x_o} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} \\ T & \xrightarrow{1} & T \end{array}$$

commutes. Then, φ induces a homomorphism of the homotopy sequence of E into that of E_{x_o}. We have the following commutative diagram with the horizontal sequences exact.



From $\pi_k(E)=\pi_k(E_{x_0})=0$ for all k (using 6.2), it follows that $\bar{\varphi}_{\#}$ is an isomorphism for all k. This proves Theorem C.

SECTION 7

CLASSES OF REGULAR CURVES ON A MANIFOLD

Two regular curves on a manifold M are said to be regularly homotopic if they are homotopic and the homotopy $g_v: I \longrightarrow M$ can be chosen such that for each $v \in I$, g_v is a regular curve, $g_v'(0)=g_o'(0)$, and $g_v'(1)=g_o'(1)$. A regular curve g on M will be called closed if $g'(0)=g'(1)$. It will be said to be at a point y_o of T if $g'(0)/|g'(0)| = y_o$. Two closed regular curves on M are freely regularly homotopic if they are homotopic and the homotopy $g_v: I \longrightarrow M$ can be chosen so that for each $v \in I$, g_v is a regular closed curve. Regular homotopy (free regular homotopy) is an equivalence relation and a class (free class) of regular curves on M will mean an equivalence class with respect to this relation.

M. Morse has investigated the behavior of locally simple sensed closed curves (or L-S-curves) under L-S-deformations. For definitions and discussion see [7, 8, and 9]. In these articles he classified L-S-curves on closed 2-manifolds and E^2 into equivalence classes under L-S deformations. He has noted the similarity between this study and the classification of closed regular curves with free regular homotopies playing the role of L-S-deformations. The results of this section are parallel to Morse's.

From the definition of regular homotopy it follows that if M is a manifold two curves of $\Gamma(M)$ are regularly homotopic if and only if they lie in the same arcwise

connected component of Γ, i.e., in the same element of $\pi_0(\Gamma)$. Using this fact Theorem A is the case n=0 of the following:

Theorem 7.1. If M is a manifold, there exists an isomorphism q from $\pi_n(\Gamma(M))$ to $\pi_{n+1}(T(M))$.

Proof. Let $\bar{\varphi}_{\#}: \pi_n(\Gamma(M)) \longrightarrow \pi_n(\Omega(T))$ be the isomorphism of Theorem C, and $s: \pi_n(\Omega(T)) \longrightarrow \pi_{n+1}(T)$ be the Hurewics isomorphism [16, p. 210]. The composition of these two maps $q = s\bar{\varphi}_{\#}$ gives the isomorphism demanded by the theorem. The rest of this section will be devoted to special cases of Theorem A.

Theorem 7.2. Let the dimension of a manifold M be greater than 2 and $x_0 \in T(M)$. Then two regular closed curves on M at x_0 are regularly homotopic if and only if they are homotopic with fixed end points.

As we shall see, this theorem is far from true for 2-manifolds.

Proof of 7.2. The "only if" part is immediate from the definition of regular homotopy.

Consider the exact homotopy sequence of T. Then

$$\pi_1(S^{n-1}) \longrightarrow \pi_1(T) \xrightarrow{\pi_{1_\#}} \pi_1(M) \longrightarrow \pi_0(S^{n-1})$$

is exact. Since $\pi_1(S^{n-1}) = \pi_0(S^{n-1}) = 0$ for $n > 2$, $\pi_{1_\#}: \pi_1(T) \approx \pi_1(M)$. With this, Theorem A yields that $\pi_{1_\#} q: \pi_0(\Gamma) \approx \pi_1(M)$. Moreover, from the definitions of π_1 and q we can consider $\pi_{1_\#} q$ to be just the map induced by sending a curve into itself. Then 7.2 follows

immediately from the definitions of $\pi_0(\Gamma)$ and $\pi_1(M)$.

Regular curve classes on some 2-manifolds will now be investigated.

a) The plane. The following definition is due to Whitney [17]: If f is a regular closed curve in E^2, its rotation number $\gamma(f)$ is the total angle which $\pi_2 f'(t)$ turns as t traverses I. The function

$$f^*(t) = \pi_2 f'(t) / |\pi_2 f'(t)|$$

is a map of I into the unit circle. $\gamma(f)$ is 2π times the degree of this map.

Theorem 7.3 (Whitney–Graustein). Two regular closed curves on the plane are freely regularly homotopic if and only if they have the same rotation number.

Proof. Because a translation of a regular closed curve in the plane is a regular homotopy and preserves its rotation number it is sufficient to consider curves of $\Gamma(E^2)$.

Consider the isomorphisms

$$\pi_0(\Gamma) \xrightarrow{q} \pi_1(T) \xrightarrow{\pi_{2\#}} \pi_1(S^1)$$

where π_2 is the projection of $T=E^2 \times S^1$ onto S^1. Let f be the element of Γ represented by $e^{2\pi i t}$ in complex coordinates such that the base point of Γ is $x_0=(1,2\pi i)$. If $h \in \Gamma$, $\bar{h}$ will be the element of $\pi_0(\Gamma)$ containing h. Then $\pi_{2\#} q(\bar{f})$ will be a generator of $\pi_1(S^1)$ say e. If g is any element of Γ, $\pi_{2\#} q(\bar{g})$ will be of the form me. From the definition of rotation number it follows that m is the rotation number of g. Since $\pi_{2\#} q$ is an isomorphism

onto, this proves 7.3.

For the case of L-S-curves in the plane see [7].

b) The 2-sphere S^2. It is known that $\pi_1(T(S^2))$ is cyclic of order 2; for example see [13]. Hence, by Theorem A we can put regular closed curves of S^2 at a point x_o into two classes under regular homotopy equivalence. For the case of L-S-curves on S^2, see [8].

c) The torus T^2. From the exact homotopy sequence of the tangent bundle $T(T^2)$, it can be deduced that $\pi_1(T(T^2))$ is Z+Z+Z (Z is the infinite cyclic group). Then similar remarks to those of (b) apply.

d) The projective plane P^2. It can be proved that $\pi_1(T(P^2))$ is cyclic of order four. Hence, there are four classes of regular closed curves at a point x_o on P^2. For the case of L-S-curves see [9].

Remark: By taking Γ as a fiber over a different point of T one can obtain results similar to those of Sections 6 and 7 for non-closed curves.

SECTION 8

REGULAR CURVES PERPENDICULAR TO A SUBMANIFOLD

Let N be a regularly imbedded submanifold of a manifold M and let T be the unit tangent manifold of M. Let V be the normal bundle of N with respect to M; that is, V is the subspace of T which consists of all vectors which have their base points in N and are normal to N. Let x_o be a point of V and Ω be the loop space of T at x_o. Denote by Ω_V the subspace of $E_{x_o}(T)$ which consists of the paths ending in V.

Let Γ_N be the subspace of E(M) of curves whose final tangent is in V; i.e., $\Gamma_N = \pi^{-1}(V)$ where π is the map of Theorem B. Let π restricted to Γ_N be still denoted by π. Then we have:

Theorem 8.1. The triple (Γ_N, π, V) has the CHP.

Proof. This theorem is an easy consequence of Theorem B. In fact, let the homotopy $h_v: P \longrightarrow V$ be given with $\bar{h}: P \longrightarrow \Gamma_N$ covering h_o. Theorem B yields a covering homotopy $\bar{h}_v: P \longrightarrow E$. But since $\bar{h}_v$ covers h_v we have that $\bar{h}_v(p) \in \Gamma_N$ for all $v \in I$ and $p \in P$ and so $\bar{h}_v: P \longrightarrow \Gamma_N$. This proves 8.1.

Similarly, (Ω_V, p, V) has the CHP. Let $\tilde{\varphi}$ be the map φ of Section 6 restricted to Γ_N. Then

$$\begin{array}{ccc} \Gamma_N & \xrightarrow{\tilde{\varphi}} & \Omega_V \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} \\ V & \xrightarrow{1} & V \end{array}$$

commutes so $\tilde{\varphi}$ induces a homomorphism of the exact

sequence of Γ_N into that of Ω_V. We have

$$\begin{array}{ccccccc} \cdots \longrightarrow & \pi_k(\Gamma) & \longrightarrow & \pi_k(\Gamma_N) & \longrightarrow & \pi_k(V) & \longrightarrow \cdots \\ & \downarrow \bar{\varphi}_{\#} & & \downarrow \tilde{\varphi}_{\#} & & \downarrow 1_{\#} & \\ \cdots \longrightarrow & \pi_k(\Omega) & \longrightarrow & \pi_k(\Omega_V) & \longrightarrow & \pi_k(V) & \longrightarrow \cdots \end{array}$$

By Theorem C, $\bar{\varphi}_{\#}$ is an isomorphism onto and, of course, $1_{\#}$ is the identity isomorphism. From this and the "Five" Lemma [2, p. 16], it follows that $\tilde{\varphi}_{\#}$ is also an isomorphism onto. We have proved:

Theorem 8.2. The map $\tilde{\varphi}: \Gamma_N \to \Omega_V$ defined above is a weak homotopy equivalence. Hence, by Theorem 6.1 it induces an isomorphism between the singular homology groups of Γ_N and Ω_V.

SECTION 9

INTEGRAL CURVES OF ω_0

Suppose (x,y) are Cartesian coordinates in E^2. Let $T=E^2\times S^1$ be the unit tangent bundle of E^2 with coordinates (x,y,Θ). If $f=(x(t),y(t))$ is a regular curve in E^2, then φf (See Introduction and Section 6) is a curve on T which appears in coordinates as

$$f(t)=(x(t),y(t),\Theta(t)) \qquad \Theta(t)=\arccos\frac{x'(t)}{[x'(t)^2+y'(t)^2]^{1/2}}$$

It can be easily checked that $\varphi f(t)$ is an integral curve of the canonical 1-form [1] on T, $\omega_0=\sin\Theta\,dx-\cos\theta\,dy$. One might hope to get a characterization of regular plane curves this way. Unfortunately, however, ω_0 admits a type of integral curve which is not the image under φ of a plane regular curve. The following is a class of such curves.

$$g(t)=(x(t),y(t),\theta(t)) \qquad \begin{array}{l} x(t)=\text{arbitrary constant} \\ y(t)=\text{arbitrary constant} \\ \theta(t)=\text{arbitrary} \end{array}$$

By using local coordinates the preceding statements may be modified to hold for any 2-manifold instead of E^2. Thus it is not sufficient for the study of regular curves on a 2-manifold M to study integral curves of ω_0 on (TM). However, there is still the question as to what can be said about integral curves of ω_0. This section was written as an attempt to answer this question.

Throughout the rest of Section 9 we will assume

that M is a given manifold of dimension three. It seems very likely that the theory here generalizes to manifolds of higher dimension. However, because the treatment of 3-manifolds is so much simpler, we confine ourselves to this case.

A kind of curve essentially the same as the "stuckweise glatt" curves of [12] is considered here. A curve f on M is called a <u>parametrized piecewise regular curve</u> if there exist real numbers t_i for $i=0,1,\cdots,k$ with $t_0=0$, $t_k=1$, and $t_i < t_{i+1}$ such that for each $i<k$, f restricted to $[t_i,t_{i+1}]$ is either constant or regular in the sense that $|f'(t)|\neq 0$ for $t\in[t_i,t_{i+1}]$. We say that such a curve is <u>distinguished</u> if its parameter is proportional to arc-length [12]. By changing the parameter one can associate to each parametrized piecewise regular curve a unique distinguished parametrized piecewise regular curve [see 12]. Two parametrized piecewise regular curves will be called <u>equivalent</u> if their associated distinguished curves are the same. A <u>piecewise regular curve</u> is an equivalence class of parametrized piecewise regular curves. Each such curve will have a unique distinguished representative. Oftentimes we will identify a piecewise regular curve with its distinguished representative.

As will be shown by an example at the end of the section, the theorems here hold only if some restriction is placed on the 1-forms.

Suppose ω is a 1-form on a 3-manifold M and

(x,y,z) are coordinates of a coordinate neighborhood U of M. In U ω can be written in the form $Pdx+Qdy+Rdz$ where P, Q and R are real differentiable functions of x, y and z. The 1-form ω will belong to Class A if U can be chosen so that

$$PP'+QQ'+RR'\neq 0$$

in U where

$$P'=R_y-Q_z \qquad Q'=P_z-R_x \qquad R'=Q_x-P_y$$

Let q_0 be a fixed point of M, and let ω be a 1-form on M of Class A. Denote by E_ω the space of all piecewise regular curves on M which start at q_0 and are integral curves of ω. Let E_ω be a metric space under the metric

$$d(f,g)=\max\left\{\bar{d}(f(t),g(t)) \;\middle|\; t\in I\right\}$$

where $\bar{d}$ is any fixed metric on M. Define a map $p:E_\omega\longrightarrow M$ by $p(g)=g(1)$. Let $p^{-1}(q_0)$ be denoted by Ω_ω.

Theorem 9.1. The triple (E_ω,p,M) has the CHP.

Roughly speaking, Theorem 9.1 is proved as follows. First, by a classical theorem, there are local coordinates (x,y,z) about a point of M such that in them ω assumes an especially simple form. Here the fact that ω is in Class A is used. By Proposition 1.4 we reduce the proof, in a sense, to this local situation. Then the local coordinates thus obtained are used to write down explicitly the desired covering homotopy equations.

We break the definition of the covering homotopy curves into four parts according to values of the parameter t. In general, this curve will turn out to have a corner

at $t=1/4$, $t=1/2$, and $t=3/4$. The first part of the constructed curve is merely a reparametrization of the given covering curve. Then the construction is such that at $t=1/2$, the z coordinate of the covering homotopy has moved to a position over the z-coordinate of the given homotopy. At $t=3/4$ the x-coordinate has undergone a similar motion, and finally, at $t=1$, the y-coordinate of the covering homotopy projects into the y-coordinate of the given homotopy in M.

Proof of 9.1. Let $q \in M$. Take a coordinate neighborhood U of q with coordinates (x,y,z) which is given from the definition of Class A. Then in U $\omega=Pdx+Qdy+Rdz$ and $PP'+QQ'+RR' \neq 0$.

We appeal now to a classical result of the theory of differential equations which for example may be found in [3] . This is, if as above, $PP'+QQ'+RR' \neq 0$ at q then there exist differentiable functions u, v, and w of x, y, and z defined in a neighborhood of q such that $\omega=du+vdw$. Furthermore, it can be checked using [3] that the Jacobian determinant

$$\frac{\partial(u,v,w)}{\partial(x,y,z)}=PP'+QQ'+RR'.$$

Since the latter is not zero at q, there is a neighborhood V of p in U with coordinates (u,v,w) such that in V $\omega=du+vdw$. By 1.4 it is sufficient to prove the CHP for the triple $(p^{-1}(V),p,V)$. For convenience we will change the coordinates u, v, and w into x, y, and z respectively. So now x, y, and z are coordinates of V such that $\omega=dx+ydz$.

Let $h_v:P\longrightarrow V$ be a given homotopy with $\bar{h}:P\longrightarrow p^{-1}(V)$ covering h_o. We will construct a covering homotopy $\bar{h}_v:P\longrightarrow p^{-1}(V)$. To describe these maps in the coordinate system (x,y,z) we use the following notation:

$$h_v(p)=(x_v(p),y_v(p),z_v(p))$$

$$\bar{h}_v(p)(t)=(\bar{x}_v(p)(t),\bar{y}_v(p)(t),\bar{z}_v(p)(t))$$

Then: for $0\leq t\leq 1/4$ let

$$\bar{h}_v(p)(t)=\bar{h}(p)(4t)$$

for $1/4\leq t\leq 1/2$ let $s=s(t)=4t-1$ and let

$$\bar{x}_v(p)(t)=x_o(p)-[z_v(p)-z_o(p)]y_o(p)s$$

$$\bar{y}_v(p)(t)=y_o(p)$$

$$\bar{z}_v(p)(t)=z_o(p)+[z_v(p)-z_o(p)]s$$

for $1/2\leq t\leq 3/4$ let $s=s(t)=4t-2$. Then let

$$\tilde{x}_v(p)=x_o(p)-y_o(p)[z_v(p)-z_o(p)]$$

$$\bar{x}_v(p)(t)=(36s^5-45s^4-20s^3+30s^2)[x_v(p)-\tilde{x}_v(p)]+$$
$$60|x_v(p)-\tilde{x}_v(p)|^{1/2}y_o(p)(s^3-s)+\tilde{x}_v(p)$$

$$\bar{y}_v(p)(t)=Sg|x_v(p)-\tilde{x}_v(p)|^{1/2}(s^2-s)+y_o(p)$$

Sg=sign of $x_v(p)-\tilde{x}_v(p)$

$$\bar{z}_v(p)(t)=-60|x_v(p)-\tilde{x}_v(p)|^{1/2}(s^3-s)+z_v(p)$$

for $3/4\leq t\leq 1$ let $s=s(t)=4t-3$ and

$$\bar{x}_v(p)(t)=x_v(p)$$

$$\bar{y}_v(p)(t)=[y_v(p)-y_o(p)]s+y_o(p)$$

$$\bar{z}_v(p)(t)=z_v(p)$$

In order to be sure that these equations define a satisfactory covering homotopy it must be checked that (1) $\bar{h}_v(p)$ is a curve in E_ω (for each p and v), (2) $\bar{h}_v(p)$ is a homotopy of $\bar{h}(p)$ or $\bar{h}_0(p)=\bar{h}(p)$, and (3) $\bar{h}_V(p)$ covers $h_V(p)$

or $\bar{h}_v(p)(1)=h_v(p)$.

We will check (1) first. It is easy to note that $\bar{h}_v(p)(0)=q_o$. Also, $\bar{h}_v(p)(t)$ is clearly continuous and piecewise regular between the values t=0, 1/4, 1/3, 3/4, and 1. It is necessary to check that $\bar{h}_v(p)(t)$ is well-defined at t=1/4, 1/2, and 3/4 since at each of these values $\bar{h}_v(p)(t)$ is defined in two different ways. By substituting these values of t into the appropriate equations it can be seen that where the definitions overlap they agree. To complete the proof of (1) it needs to be shown that $\bar{h}_v(p)(t)$ satisfies $\omega=0$ or, in other words,

$$\frac{d\bar{x}_v(p)(t)}{dt}+\bar{y}_v(p)(t)\frac{d\bar{z}_v(p)(t)=0}{dt}=0$$

identically for all v, p, and t. This is trivial for $t\leq 1/4$. For $1/4\leq t\leq 1$ to prove that this differential equation is satisfied it is sufficient to make three computations, one for $1/4\leq t\leq 1/2$, one for $1/2\leq t\leq 3/4$ and one for $3/4\leq t\leq 1$. These are not difficult and will be left for the reader.

To show (2) we set v=0, getting

$$\bar{h}_o(p)(t)=\bar{h}(p)(4t) \qquad 0\leq t\leq 1/4$$
$$\bar{x}_o(p)(t)=x_o(p)$$
$$\bar{y}_o(p)(t)=y_o(p) \qquad 1/4\leq t\leq 1$$
$$\bar{z}_o(p)(t)=z_o(p)$$

This is a parametrized piecewise regular curve whose associated distinguished curve is exactly the given curve $\bar{h}(p)(t)$.

It is trivially checked that (3) holds. q.e.d.

The space E_ω is contractible to a point. The deformation accomplishing this is $D: E_\omega \times I \longrightarrow E_\omega$ defined by $D(f,v)=f(vt)$. Define a map $i: E_\omega \longrightarrow E_{q_0}(M)$ (See Section 6) by letting $i(f)$ be the distinguished representative of f. Then i is continuous by the argument used to show that φ was continuous in Section 6. Let $\bar{i}$ be i restricted to Ω_ω where $\Omega_\omega = p^{-1}(q_0) \subset E_\omega$. The argument used in Section 6 to show that $\bar{\varphi}: \Gamma \longrightarrow \Omega(T)$ was a weak homotopy equivalence may now be used to show that $\bar{i}: \Omega_\omega \longrightarrow \Omega(M)$ is also a weak homotopy equivalence. This proves Theorem D.

The following example shows how Theorems 9.1 and D fail for a form ω_0 which is completely integrable. Let $M=E^3$ and $\omega_0 = xdx+ydy+zdz$ in a given Cartesian coordinate system (x,y,z) of E^3. Take for q_0 any point at distance $d>0$ from the origin of E^3. Then any integral curve of ω_0 starting at q_0 stays on the surface of the 2-sphere $x^2+y^2+z^2=d^2$. Clearly, the conclusions of Theorems 9.1 and D fail in this case. Actually, Theorem 9.1 is false for any 1-form which is completely integrable at a certain point $q_0 \in E^3$. For then short curves at q_0 must lie on a surface of E^3 and the covering homotopy property can't possibly hold.

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