

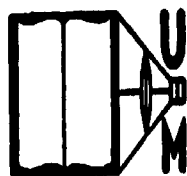
DOCTORAL DISSERTATION SERIES

TITLE METAPROJECTIVE GEOMETRY

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UNIVERSITY OF MICH. DATE 1950

DEGREE Ph.D. PUBLICATION NO. 2476



UNIVERSITY MICROFILMS

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METAPROJECTIVE GEOMETRY

by
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**A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in the
University of Michigan
1950**

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ACKNOWLEDGEMENTS

At the outset Professor G. Y. Rainich called attention to the ideas from which this dissertation grew and his interest and criticism made possible its completion. The comments and concrete suggestions made by Professor Arthur Copeland materially increased the value of this work. The continuing progress of this investigation was to a significant degree due to the efforts of Margaret C. Zimmie and Charles C. Buck.

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CHAPTER I

INTRODUCTION

This paper introduces as a formal system, metaprojective geometry--the geometry whose group consists of the correlations as well as the colineations of projective geometry. Chapter I is introductory in character, being designed to motivate the study of such a system. It should be emphasized that an approach to the study of geometry is presented in this chapter as a way of understanding and gaining a perspective of the many geometries now known; it is not the only possible or correct view. To avoid possible misconceptions, several matters should be pointed out at the outset. First, the term, metaprojective geometry applies to the object language of a geometry which is a generalization of projective geometry; it is not the meta-language of projective geometry. It is not the case that a new geometry has been discovered, for the group of transformations which defines the content of metaprojective geometry is recognized in all works on projective geometry.

One way of explaining the differences that exist between distinct geometries is by referring to the terms which are significant or meaningful in one geometry and not in the other. As an example, consider Euclidean and affine geometry. Both perpendicularity and parallelism as relations between lines belong to the

domain of Euclidean geometry. In affine geometry lines can be parallel, but it is meaningless to speak of perpendicular lines. Thus affine geometry may be thought of as an abstraction from the perpendicularity relation of Euclidean geometry.

One method of comparing two geometries is to derive one from the other by "distinguishing" or picking out a configuration from a set of equivalent configurations in the given geometry. The group of the derived geometry will be a sub-group of the group associated with the given geometry consisting of those transformations which preserve the distinguished configuration. For example, 2-dimensional affine geometry may be obtained by assigning a special role to one of the lines in the projective plane (called the line at infinity), thus restricting the group to those transformations which map the line at infinity onto itself. It was this approach to geometry that Felix Klein emphasized in his paper in the Erlangen Program.

The group of transformations associated with a geometry may be characterized in a quite different fashion. For each geometry there is a formal language in which the axioms make assertions about the undefined terms of the system; e. g., the class of points, the class of lines, the incidence, parallelism, and perpendicularity. The transformation group associated with a geometry

may be characterized as the set of 1-to-1 mappings of the undefined set of elements (primitive elements) onto itself which preserve the relations appearing as undefined terms in the axioms.

Our reference to "THE group of transformations associated with a set of axioms" represents a manner of speaking which is common in mathematics--the group, of course, is not unique. Since the set of primitive elements appears in the axioms as a variable, the group in question is associated with this set in much the same way as THE number $2x + 3$ is associated with x . Although groups associated with sets of axioms are not unique in absolute sense, they are values of a single-valued function whose argument is the undefined set in a system of axioms. However, the group associated with a categorical set of axioms is unique. In this case the isomorphism between two interpretations of the axioms induces an isomorphism between the corresponding groups.

A transformation is said to preserve a relation if the relation holds of the images of every set of elements whenever it is satisfied by these elements. Since a property or predicate is regarded as a unary relation, the principle of preservation of relations applies equally well to relations of one argument and relations of two, three or more arguments. This means that the appearance of a symbol for a set as an undefined term in the set of

axioms for a geometry limits its group to those transformations for which the image of every element of this set belongs to the set.

The study of projective geometry leads to the consideration of two groups of transformations, the colineations and the group which contains in addition, the correlations. The formulation of projective geometry in which points and hyperplanes are the primitive elements will be assumed for the present discussion. A colineation is a transformation which assigns to every point a point and to every hyperplane a hyperplane, of the space, in such a way that all the relations in the axioms are preserved. A correlation is a transformation which interchanges points and hyperplanes in such a way that incidence between points and hyperplanes is preserved. Only one of these groups characterizes projective geometry; a general theory of the interrelationships between geometries and groups of transformations must be capable of deciding which one it is. The other group poses the problem of investigating the geometry with which it is associated.

A key to the situation is given by a quotation from the English translation of Felix Klein's paper in the Erlangen Program--the translation appearing in the New York Mathematical Society Bulletin, Vol. II, 1893:

"... long before the origin of a true projective geometry the properties of figures derived by projection from a given figure were inferred from those of the given figure. But projective geometry only arose as it became customary to regard the original figure as essentially identical with all those deducible from it by projection, and to enunciate the properties transferred in the process of projection in such a way as to put in evidence their independence of the change due to the projection."

The theory presented in this chapter, the preceding paragraph in particular, associates the group of colineations with the geometry which has the class of points and the class of hyperplanes as undefined terms; since the correlations do not preserve the property of being a point. The group of transformations consisting of the correlations, as well as the colineations of projective geometry, does not characterize this geometry. The purpose of the remaining chapters of this paper is to formulate a language and a set of axioms for the geometry which has this group as its associated transformation group, and to investigate the ramifications of such a system.

CHAPTER II

TWO-DIMENSIONAL METAPROJECTIVE GEOMETRY

In this chapter the ideas discussed in the introduction are embodied in a simple formal system, two-dimensional metaprojective geometry. The complications of the general case, an n -dimensional geometry, are postponed to the next chapter so that the characteristic feature of metaprojective geometry, the impossibility of deciding whether an element belongs to the class of points or to its dual, will stand out clearly.

Two-dimensional metaprojective geometry has as a model the points and lines of the projective plane. However, the property of being a point cannot be expressed in the metaprojective language with the result that the correlations are admitted to the group of this geometry. The problem is to formulate a system which will have all the characteristics of projective geometry except that the class of "points" cannot be defined.

The first step is to make it possible to divide the elements of the metaprojective plane into two classes to which the later axioms can assign the roles of points and lines without specifying which is which. This dichotomy is accomplished by an equivalence

relation holding between elements of the geometry. Looking at the points and lines of a projective plane, we see that a point and a line can never be incident to the same point or to the same line, whereas two points are always incident to some line and dually for two lines. In metaprojective language we would say that two elements of the same kind have a common incident. The procedure, then, is to first invent axioms which will make the relation, having a common incident, an equivalence relation yielding two equivalence classes; second, by means of further axioms endow the incidence relation with the properties which will relate the equivalence classes to each other in the way points are related to lines in the projective plane. In these axioms there will be no unary relations (such as the property of being a point) and only one binary relation, incidence, which must be preserved by the transformations of the geometry. This means that the only restriction on the transformations of the group associated with these axioms is that they must preserve the incidence relation.

Since we will be concerned only with metaprojective geometry, the set of elements will not be denoted by a special symbol, the variables $X, X:i, Y, U, V, \dots$ invariably denoting elements of the geometry. A sequence of elements will be represented as follows: $(X:1, X:2, \dots, X:k)$.

The metaprojective geometry of two dimensions (non-Desargues as well as Desargues) is characterized by the six axioms:

1. Incidence is symmetric.
2. For every $(X:1, \dots, X:5)$ if $X:i$ is incident to $X:i + 1$, then $X:1$ is not incident to $X:5$.
3. For any three elements there exists a fourth incident to two of the three.
4. For any four elements there exists a fifth not incident to any of the four.
5. If $X:1, X:2, X:3, X:4$ are distinct and $X:i$ is incident to $X:i + 1$, then $X:1$ is not incident to $X:4$.
6. There exists at least one element.

Definition: A chain is a finite sequence of elements in which each element is incident to its successor. $(X:1, \dots, X:k)$; $X:i$ is incident to $X:i + 1$ is a k chain or a chain of order k . A chain is closed if its first element is incident to its last, otherwise it is open.

Discussion of the axioms is facilitated by the use of the terms just defined. The symmetry of incidence is fundamental in a geometry in which the types of elements have no individuality. For this reason Statement 1 has been taken as an axiom. Axiom 2

can be expressed--every five-chain is open. This axiom is the key to the transitivity of the relation which effects the dichotomy of the primitive elements (see Lemma 2 below). Axiom 3 guarantees that there will be no more than two equivalence classes. The three vertices and three sides of a triangle alone would satisfy all the axioms if four were not present. It is customary to exclude models as simple as this one; every line of a valid projective geometry having at least three points on it. One function then of Axiom 4, is to insure that this condition is satisfied. This matter will be discussed at greater length in Chapter III. Axiom 5 serves to restrict the system to the geometry of the plane. Another formulation of Axiom 5 is--every chain of four distinct elements is open. By means of Axiom 6 it is possible to show (Theorem 3) that the dimension of the geometry is at least two. In summary, the first three axioms guarantee that the relation, having a common incident, is an equivalence relation while Axioms 5 and 6 establish the dimension of the space as two. Further, every chain of order 3, 4, or 5 is open.

Since the elements appearing in Axiom 2 form a sequence any element may appear more than once among the $X:i$.

Lemma 1. Every chain of order three is open.

Proof: Consider any chain of order three; $X:1, X:2, X:3$.

$X:1, X:2, X:3, X:2, X:3$ is a five-chain which must be open (Axiom 2).

Therefore, $X:1$ is not incident to $X:3$.

Lemma 2. Every chain of five elements can be extended to a closed chain of six elements.

Proof: Let $(X:1, \dots, X:5)$ be a 5-chain. Considering $X:1$, $X:2$, and $X:5$, Axiom 3 guarantees the existence of an element X which is incident to two of these three elements. If X is incident to $X:1$ and $X:2$ then Lemma 1 is violated. If X is incident to $X:2$ and $X:5$ then Axiom 2 is violated. Therefore X is incident to $X:1$ and $X:5$ and $(X:1, \dots, X:5)$, X is a closed 6-chain.

Lemma 1 and Axiom 2 assert that every 3-chain and 5-chain is open. This can be generalized to any odd number of elements, giving:

Lemma 3. Every chain having an odd number of elements is open.

Proof: Consider a chain $(X:1, \dots, X:2k + 1)$ for $k = 3, 4, \dots$. $X:1$ and $X:5$ have a common incident, Z (Lemma 2). Now $(X:1, Z, X:5, \dots, X:2k + 1)$ is a chain of order $2k + 1$. This process may be continued until a five-chain connecting $X:1$ and $X:2k + 1$ is obtained. Therefore, $X:1$ is not incident to $X:2k + 1$ (Axiom 2).

Theorem 1. The binary relation, having a common incident, is an equivalence relation and there are exactly two equivalence classes.

Proof: The relation "having a common incident" is obviously reflexive and symmetric. Since Lemma 2 establishes transitivity this relation is an equivalence relation. It remains to show that there are exactly two equivalence classes. There is at least one element, X , in the geometry (Axiom 6). By Axiom 3, two of the elements X, X, X have a common incident. This gives an element Y , such that X is incident to Y . The existence of a common incident of X and Y would violate Lemma 1, hence there are at least two equivalence classes.

This result justifies the following:

Definition: X is equivalent to Y if X and Y have a common incident.

Theorem 2. There exists a five-chain whose elements are distinct.

Proof: 1. Every element has an incident. For an element, X , apply Axiom 3 to the triple X, X, X to obtain an element Z which is incident to X, X ; then, Z is the desired incident of X .

2. There exists an element, X (Axiom 6). X has an incident, U (Part 1). There exists Z , such that Z is not incident to X, U, X, U (Axiom 4). Z has an incident, W (Part 1). Consider the elements X, U and Z . Two of these elements have a common incident (Axiom 3). Suppose then, some element, Y , is incident to U and Z . By construction, the elements X, U, Y, Z, W , are distinct and form a five-chain.

Theorem 3. There exists a set of three elements which do not have a common incident, each pair of which has a common incident.

Proof: There exists a five-chain, $(X:1, \dots, X:5)$ whose elements are distinct (Theorem 2). Consider the elements, $X:1$, $X:3$ and $X:5$. Each pair has a common incident (Theorem 1). The existence of a common incident of the three would violate Axiom 5. Therefore, $X:1$, $X:3$ and $X:5$ satisfy the conditions mentioned.

The Desargues Theorem which can be deduced in n -dimensions, must be postulated in the 2-dimensional system in order to obtain a geometry whose number system is a division ring. The Desargues configuration is peculiarly adaptable to metaprojective treatment because of its inherent duality. This study may be introduced in terms of the following definition.

Definition: A linear graph of order k is a finite set in which each element is incident to exactly k -elements of the set.

A Desargues configuration is a graph of order 3 having twenty elements. The characterization of a Desargues configuration in metaprojective geometry as a unique (to within isomorphism) graph of order 3 is a problem which requires further investigation. An incomplete Desargues configuration might be defined as a set of nineteen elements which is isomorphic to a nineteen-element

subset of some Desargues configuration. The required postulate, then, would state that every incomplete Desargues configuration can be extended to a complete Desargues configuration.

The commutativity of the number system of a geometry is guaranteed by the Pappus Theorem. The Pappus configuration also lends itself to metaprojective treatment. This configuration is an eighteen-element graph of order 3. The problem of uniqueness arises here just as in the case of the Desargues configuration. The Pappus axiom for 2-dimensional metaprojective geometry could be formulated by asserting that every incomplete Pappus configuration can be extended to a complete Pappus configuration. In the presence of such an axiom the Desargues Theorem can be deduced.

Enough has been shown to indicate the nature of the structure of 2-dimensional metaprojective geometry. Our objective is to elaborate the more complex system for n -dimensional metaprojective geometry in the following chapters.

CHAPTER III

AXIOMS FOR n -DIMENSIONAL METAPROJECTIVE GEOMETRY

In the present chapter we state the axioms for metaprojective geometry of n -dimensions. Inasmuch as these axioms do not conveniently represent the 2-dimensional geometry when n is set equal to 2, 2-dimensional metaprojective geometry is treated as a special case in Chapter II. The remainder of this paper, then, deals with metaprojective geometry of dimension n greater than 2.

In addition to the symbolism previously introduced, it is convenient to write $[X:i]$ for the elements $(X:1, \dots, X:k)$; e.g., X is incident to $[X:i]$ means that X is incident to each of the elements $(X:1, \dots, X:k)$ where the range of the index has been adequately specified.

The first four axioms below are identical with Axioms 1 through 4 for the plane; the remaining axioms establish the dimension of the metaprojective geometry as $n = 3, 4, \dots$

MA1. Incidence is symmetric.

MA2. For every $(X:1, \dots, X:5)$ if $X:i$ is incident to $X:i+1$, then $X:1$ is not incident to $X:5$.

MA3. For any three elements there exists a fourth incident to two of the three.

MA4. For any four elements there exists a fifth not incident to any of the four.

MA5. If every two elements of a sequence of n elements have a common incident, then all the elements of the sequence have a common incident.

MA6. For every n elements $(X:1, \dots, X:n)$ for which there exist $(U:1, \dots, U:n)$ where $X:i$ is incident to $U:j$ if and only if $i \neq j$, if U, V are incident to $[X:i]$ then $U = V$.

MA7. There exists a finite set of elements in which every pair of elements has a common incident, the set having no common incident.

As a model for these axioms we may consider the real projective geometry of n dimensions. The "primitive elements" of metaprojective geometry may be interpreted as the points and hyperplanes and "incidence" as the incidence relation of the projective space. In these terms MA1...MA7 make true statements about the model, thus proving the consistency of the axioms.

The independence of the axioms has not been systematically investigated, however, there are three "counter-examples" designed to throw light on the nature of the formal system which incidentally show the independence of MA5, MA6 and MA7. A projective geometry, $P-k$, of k dimensions satisfies MA6 vacuously if $k < n$. In this case MA5 is not satisfied, thus demonstrating its independence of

the other axioms. On the other hand if $k > n$ MA6 alone is not satisfied, proving its independence.

A model, M , can be constructed using the elements $X:i$ and $U:j$ where both i and j take on all integral values. $X:i$ is defined to be incident to $U:j$ if $i < j$. This model clearly satisfies all the axioms except MA7, and is therefore, infinite-dimensional in an appropriate sense. M , therefore, guarantees the independence of MA7. Moreover, M serves another purpose. "If $(Y:1, \dots, Y:n)$ are sequentially independent (see Definition below), $[Y:i]$ has at most one common incident" is a false statement about M . These facts suggested the use of a set of axioms including in addition to MA1...MA5, the axiom MA6, which is logically weaker than the statement above, and some further axiom to eliminate M as a realization. It was found (see D1 below) that MA7 when adjoined to the first six axioms made it possible to prove the statement quoted above.

In Chapter II it was stated that the relation "having a common incident" decomposes the points and lines of a projective plane into the class of points and the class of lines. The same holds for the points and hyperplanes of a projective geometry of n -dimensions. Therefore, the first three axioms for the metaprojective geometry of 2-dimensions, since they are independent of dimension, become MA1, MA2, MA3 in the n -dimensional system. From these axioms it can be demonstrated, as in Chapter II, that the relation "having a

common incident" is an equivalence relation which decomposes the elements into two classes (Theorem 1). The application of this theorem will be taken for granted and it will be used in the proofs of subsequent theorems without comment.

Axiom MA4 serves essentially the same purpose as does Axiom 4 in Chapter II. MA4 makes it impossible for a Boolean Algebra to be a part of the metaprojective space. In particular a simplex (see Definition below) satisfies all the axioms except the axiom in question. MA4 is employed in the proofs of the theorems: every "line" has at least three points on it (D6) and every pair of equivalent elements have a common complement (D5). These are characteristics which distinguish a projective geometry from a Boolean Algebra.

The significance of the Axioms MA5, MA6 and MA7 is indicated by the "counter-examples" mentioned above. MA7 guarantees that the dimension of the space will be finite while MA5 and MA6 require that it be less than or equal to n and greater than or equal to n , respectively.

One variant of the formal system here discussed is of special interest. MA1 through MA7 may be thought of as axiom schemata which are transformed by the specification of n into a definite set of axioms for the metaprojective geometry of the

specified dimension. However, it is possible to formulate a system to be called finite-dimensional metaprojective geometry in which the number of dimensions does not appear explicitly. That is, finite-dimensional metaprojective geometry is a logical language in which the number of dimensions is a definable rather than an undefined term. This system has in common with the former, MA1 through MA4 and MA7 and in addition the new axioms replacing Axioms MA5 and MA6.

MA5'. For every $(X:1, \dots, X:k), (U:1, \dots, U:k)$ where $X:i$ is incident to $U:j$ if and only if $i \neq j$, there exists X incident to $[U:i]$ if and only if there exists U incident to $[X:j]$.

MA6'. For every finite sequence of elements having no common incident, each pair of which has a common incident, $(X:0, \dots, X:k)$ if U, V incident to $(X:1, \dots, X:k)$ then $U = V$.

In order to compare the two systems it is necessary to define n , the number of dimensions, in the language of finite-dimensional metaprojective geometry. We define n as one less than the number of elements in the smallest set of equivalent elements having no common incident. MA5 then is automatically satisfied. By a procedure similar to that used in the next Chapter to prove D1, it can be shown that every set has $n + 1$ elements if it has the property that every proper subset has a common incident

while the entire set does not. MA6 then is a consequence of MA6'. Thus, the results obtained in the remainder of this paper explicitly for n -dimensional metaprojective geometry also hold for finite-dimensional metaprojective geometry.

CHAPTER IV

LINEAR DEPENDENCE AND INDEPENDENCE

The relations, linear dependence and independence, and associated concepts of projective geometry are expressed in metaprojective language in this chapter and many of their elementary properties are embodied in the theorems which follow. Whenever possible, a term borrowed from another branch of mathematics is used in metaprojective geometry to denote the metaprojective analogue of its original meaning.

In projective geometry a complement of a point is a hyperplane to which it is not incident. This idea is embodied in the following:

Definition: Two elements are complementary if they are not incident to each other and if they have no common incident.

Independence is a concept which occurs widely in mathematics. In general the elements of a set may be said to be independent if each element belonging to the set fails to satisfy some relation which is shared by the other elements of the set. The independence of axioms is a conspicuous example of this. An axiom is proved to be independent of the other axioms of a system by showing that it makes a false statement about some valid interpretation of the other axioms. Further, the concept independence is hidden in the idea of a direct sum of a collection

of sub-groups of a group. In this case the sum of all but one sub-group of the collection has in common with the other sub-group only the group identity while it completely contains each of its summands. The vertices of a tetrahedron serve as the last example. The independence of the vertices is made manifest in this instance by the fact that each triple of vertices determines a plane which does not include the remaining vertex; these planes being independent by the same token.

The concept of independence of which the last example was a special case, expressed in the language of metaprojective geometry, becomes the following definition in the system of this paper:

Definition: The elements of a set are independent if each one has a complement which is a common incident of the other elements of the set. Equivalent elements which are not independent are said to be dependent. A basis consists of n plus 1 independent elements.

Definition: The dual basis B^* of a basis $B = (X:0, \dots, X:n)$, is a set $[U:i]$ such that $X:i$ is incident to $U:j$ if and only if $i \neq j$. A simplex $B B^*$, is a set of elements consisting of a basis B and its dual B^* .

Definition: An element is dependent on a set of elements equivalent to it if every common incident of the elements of the set is incident to it.

Independent points are often obtained in geometry by a step by step process; i.e., when a number of independent elements have been chosen, a point satisfying certain conditions is selected as the next. The use of a process of this type to extend a set of independent elements to a basis suggests the following:

Definition: A sequence of elements is sequentially independent if each element of the sequence has a complement which is incident to all the preceding elements; i.e., if $(X:0, \dots, X:k)$ are equivalent and there exists $(U:1, \dots, U:k)$ such that $X:i$ is incident to $U:j$ if $i < j$ and $X:i$ is not incident to $U:i$.

The definition of sequential independence leaves open the question as to whether or not this relation is symmetric; i.e., whether independence of a set of elements taken in one order implies their independence taken in any other order.

This question is answered by Theorem D1 which is the cornerstone of the work in this chapter, since many of the theorems depend on it directly or indirectly. This first theorem is preceded by three lemmas which are required in its proof.

Lemma 4.1. There exists a simplex.

Proof: By MA7 there exists a set of elements having no common incident, each pair of which has a common incident. Let B be a set with the smallest number, k , of elements for which this is true.

MA6 shows $k > n$. Since k is minimal every proper subset of B has a common incident. B^* , the set of common incidents of subsets, including all but one element of B , exhibits the independence of the elements of B . By MA6 $k < n + 1$. Thus, with $k = n + 1$, $B \ B^*$ is a simplex.

Lemma 4.2 If $[X:i]$ are independent they form a basis if and only if they have no common incident.

Proof: Assume $[X:i]$ is a basis. If $U:0$ is incident to $(X:1, \dots, X:n)$, it is unique by MA6. Any U incident to $[X:i]$ would be distinct from $U:0$ which is not incident to $X:0$. Assume there does not exist U incident to $(X:0, \dots, X:k)$. By Axioms MA5 and MA6 k can be neither less than nor greater than n . Hence $n = k$ and $[X:i]$ is a basis.

Lemma 4.3. Substitution Lemma. If X and U are complementary where U is the common incident of all but one element of a basis B , then X and the elements of B which are incident to U , together form a basis.

Proof: The Lemma follows by MA6.

Theorem D1. Sequential independence implies independence.

Proof: Given $(X:0, \dots, X:k)$ and $(U:1, \dots, U:k)$ where $X:i$ is incident to $U:j$ if $i < j$ and $X:i$ is not incident to $U:i$. Applying Lemma 3 we obtain a simplex, $B \ B^*$; the elements of B may be

assumed equivalent to $X:i$. $X:0$ is not incident to B^* by Lemma 4.2. Let U be an element of B^* not incident to $X:0$. By the Substitution Lemma, $X:0$, together with the elements of B incident to U , form a basis. Call this basis $B:0$ and its dual $B^*:0$. Thus, the first element of a set of sequentially independent elements can be inserted in a simplex. Assuming that there is a simplex $B:i$ $B^*:i$ containing $(X:0, \dots, X:i)$, we must show that $(X:0, \dots, X:i)$, $X:i+1$ can be inserted in a simplex, unless the $X:j$ have been exhausted. Again applying Lemma 4.2 we can find an element X of $B:i$ not incident to $U:i+1$. Let $S:i+1 = U:i+1$, together with the elements of $B^*:i$, incident to X . $S:i+1$ is a basis by Lemma 4.3, the Substitution Lemma. Each $X:j$, $j \leq i$, is an element of $S^*:i+1$, the dual basis of $S:i+1$, since it is incident to n elements of $S:i+1$ including $U:i+1$. By hypothesis $X:i+1$ is not incident to $U:i+1$ which is the common incident of n elements of $S^*:i+1$. We obtain $B:i+1$ by substituting $X:i+1$ into $S^*:i+1$. By the Substitution Lemma $B:i+1$ is a basis which contains $(X:0, \dots, X:i)$ since they are incident to $U:i+1$ and contains $X:i+1$ by construction. Thus, the simplex $B:i+1$, $B^*:i+1$ contains $(X:0, \dots, X:i)$. This process can be continued until $[X:j]$ is exhausted.

Theorem D2. Every element X has a complement.

Proof: We can obtain a simplex B , B^* (Lemma 4.1) in which the elements of B are equivalent to X . X is not incident to some

element of B^* (Lemma 4.2). Any such element of B^* is a complement of X .

Of fundamental importance in any theory of linear independence is the specification of conditions under which a set of elements may be extended to a basis. One such condition is stated in the following:

Theorem D3. Every set of independent elements can be extended to a basis.

Proof: Let $[X:i]$ be a set of independent elements. If $[X:i]$ is not a basis there exists U incident to $[X:i]$ (Definition of basis). U has a complement X (D1). $[X:i]$, X are independent (D1). This process will stop only when the sequence of elements is a basis.

Lemma 4.4. If X and Y are equivalent, distinctness is a necessary and sufficient condition of independence.

Proof: X and Y have a common incident. Extend Y to a basis. $Y, (Y:2, \dots, Y:n+1)$, then Y is the common incident of the related $U:j$. By MA6, Y is unique so X is not incident to all these same U 's. There exists an i such that $U:i$ is incident to Y but not incident to X .

An important generalization of D3 is the

Theorem D4. Simplex Theorem. Given $(X:1, \dots, X:r)$ independent, $(U:k, \dots, U:s)$ independent, $X:i$ incident to $U:j$ if and only if $i = j$. Then the set of X 's and U 's can be extended to a simplex.

Proof: For every $U:j$ which does not have a complement among the elements $X:i$, there exists $X:j$ satisfying the condition

that $X:j$ is not incident to $U:j$ and $X:j$ is incident to $U:k$ where $k \neq j$ (Definition of Independence applied to $[U:k]$). The total set $[X:i]$ is independent (D1). Since the hypothesis is again satisfied by the original $[U:k]$ and the enlarged set $[X:r]$, $[U:k]$ can be extended in a similar fashion. At this state we have $(X:1, \dots, X:s)$ and $(U:1, \dots, U:s)$ where $X:i$ is incident to $U:j$ if and only if $i = j$. If $s < n + 1$ there exists $U:s + 1$ which is incident to $(X:1, \dots, X:s)$ (MA5). The hypothesis is again satisfied by $(X:1, \dots, X:s)$ and $(U:1, \dots, U:s + 1)$. Thus the process can be continued; stopping only when a simplex has been obtained.

Theorem D5. Every two equivalent elements have a common complement.

Proof: Given $X:0$, $X:1$ equivalent and distinct. Extend $X:0$, $X:1$ to a simplex $[X:i]$, $[U:j]$ (Lemma 4.4. and D4). By MA4, there exists an X which is not incident to $X:0$, $X:1$, $U:0$, $U:1$. If X is equivalent to $U:i$, the theorem follows. Therefore, assume X is equivalent to $X:i$. By the Substitution Lemma, X may replace either $X:0$ or $X:1$ to form a basis. Let U be the common incident of X , $(X:2, \dots, X:n)$ then U is not incident to $X:0$, $X:1$ (Definition of Basis and MA6). Since U is not equivalent to $X:i$, it is a complement of both $X:0$ and $X:1$.

Lemma 4.5. Every set of elements with two distinct common incidents has a third common incident. If $X:1$ and $X:2$ are incident

to $n-1$ elements, there exists $X:0$, $(X:0, X:1, X:2)$, incident to the $n-1$ elements.

Proof: Suppose $X:1, X:2$ incident to $(U:3, U:n+1)$. There exists U , a common complement of $X:1, X:2$ (D5). There exists $X:0$ incident to $(U, U:3, \dots, U:n+1)$ (MA5). $X:0$ is the element required since $X:0$ is incident to U implies $X:0 \not\perp X:1, X:2$.

Theorem D6. For every two equivalent elements there exists a third such that the three are dependent.

Proof: Extend the two given elements to a simplex (Lemma 4.4 and D4). Apply Lemma 4.5. The three elements are dependent, otherwise D4 would yield $n+2$ independent elements.

The following propositions, in effect, establish for the bases of linear subspaces the properties of bases of the entire space which have already been demonstrated.

Lemma 4.6. Given $(X:1, \dots, X:r)$, $(Y:1, \dots, Y:s)$, $Y:i$ (for all i), dependent on $[X:j]$ and Z dependent on $Y:i$. Then Z dependent on $[X:j]$.

Theorem D7. $(X:1, \dots, X:r)$ independent, $(Y:1, \dots, Y:s)$ independent, $Y:i$ (for all i) dependent on $[X:j]$ implies $s < r$.

Theorem D8. $(X:1, \dots, X:r)$ independent, $(Y:1, \dots, Y:s)$ independent, $Y:i$ (for all i) dependent on $[X:j]$ then $X:i$ is dependent on $[Y:j]$ if and only if $r = s$.

CHAPTER V

FLATS

The present chapter is devoted to the metaprojective analogue of the linear subspace or flat of projective geometry. A projective flat is a set of points which is closed with respect to the relation, dependent on. A flat may be treated as a projective geometry of dimension which is lower than that of the given space. However, in metaprojective geometry the impossibility of identifying the equivalence class to which an element belongs introduces some new features into the theory of flats. In metaprojective language a projective flat could not be distinguished from the set of hyperplanes incident to it. Since we do not want to consider these two sets as distinct flats, we are led to a concept of flat (usually considered as a set of points) which will treat the sets together as a single flat. Thus we define a flat in metaprojective geometry so that it will in general contain elements of both equivalence classes. It is easier to state a definition of flat by specifying conditions under which an element does not belong to the flat as well as conditions under which an element does belong.

Definition: The set, F , of elements of the geometry is a flat if X is an element of the set-complement of F if and only if X has a geometric complement in F : i. e., X is not an element of F if and only if there exists Z , an element of F , such that Z is a complement of X .

Definition: If E is a flat, a basis of E is a sequence, $[X:i]$, of independent elements of E , such that every common incident of $[X:i]$ is an element of E : i. e., X is incident to $[X:i]$ implies X is an element of E .

Two projective flats are incident if all the points of one also belong to the other. This projective concept and a special case of it in which the flats differ in dimension by one, are carried over into metaprojective geometry in the form of the next two definitions.

Definition: F is incident to F' if for every pair of elements, X and X' , such that X is an element of F' but not an element of F and X' an element of F but not an element of F' , X is not equivalent to X' .

Definition: F and F' are consecutive if for every pair of X and X' , such that X is an element of F' , and not an element of F , and X' is an element of F and not an element of F' , X and X' are complementary.

According to this definition of flat each of the equivalence classes is a flat. These flats correspond to the null set and the set of all points in the space which are usually considered as flats in projective geometry. Also, it is obvious that the set of flats which have a basis consisting of a single element with incidence

between flats as the incidence relation, is a metaprojective geometry which is naturally isomorphic to the original. The basic property of flats is stated in the first lemma.

Lemma 5.0. If X is dependent on a sequence, $X:i$ of elements of a flat, F , then X is an element of F .

Proof: X is shown to be an element of F by proving that no element of F can be a complement of X . Consider Y any element of F which is not equivalent to X . Y is incident to $[X:i]$ (Definition of flat). Therefore, since X is dependent on $[X:i]$, X is incident to Y . We have shown that no element of F can be a complement of X . Consequently, X is an element of F (Definition of flat).

Some elementary properties of flats are introduced in the following three theorems.

Theorem F1. Every set of independent elements in a flat, F , can be extended to a basis of F .

Proof: If $(X:0, \dots, X:k)$ do not form a basis of F for some U , U incident to $[X:i]$ and U not an element of F (Transpose of Definition of basis of flat). There exists $X:k+1$, element of F , such that $X:k+1$ is a complement of U . $(X:0, \dots, X:k, X:k+1)$ are independent (D1). This process must terminate (MA7) in a basis of F .

Theorem F2. Let A and A' denote bases of a flat, F , having r and s elements respectively. If A is equivalent to A' then $r = s$. If A is not equivalent to A' then $r + s = n + 1$.

Proof: 1. Each element of one basis is dependent on the other basis by Definition of flat. Then apply D8.

2. (A, A') can be extended to a simplex, S (D4).

Consider X, U any complementary elements of S where X is equivalent to A . If X is an element of A , U is not an element of A' (Definition of flat). If X is not an element of A , then U is incident to A . Thus U is an element of A' (Definition of basis of flat). Therefore, for every such pair; X, U , one and only one of the pair is an element of F . Since there are $n + 1$ elements in each basis of S it follows that $r + s = n + 1$.

Theorem F3. If $[X:i]$ is a set of equivalent elements, the collection, F , of all elements, Z , such that Z is dependent on $[X:i]$ or Z is incident to $[X:i]$, is a flat. We will say that F is generated by $[X:i]$.

Proof: It will be shown that every element, X , belongs to F if and only if X has no complement in F .

Case 1. X is equivalent to $X:i$. To show that X is dependent on $[X:i]$ if and only if every U , which is incident to $[X:i]$, is also incident to X . The result follows immediately since this is exactly the definition of X dependent on $[X:i]$.

Case 2. X is not equivalent to $X:i$. To show that X is incident to $[X:i]$ if and only if every Y , which is dependent on $[X:i]$, is incident to X . If X is not incident to $[X:i]$, some $X:j$ is a complement

of X . If X is incident to $[X:i]$ it is incident to every Y , which is dependent on $[X:i]$ (Definition of dependent on).

The definition of incidence of flats given above was chosen because of its symmetry. In projective geometry two flats are incident if all the points of one belong to the other. This suggests the following characterization of this relation which, although unsymmetric, is useful in proving certain theorems.

Theorem F4. If F and F' are flats, F is incident to F' if and only if there exists Z such that for every X , element of F , X is equivalent to Z implies X is an element of F' .

Proof: If. Suppose $F \neq F'$. There exists Z , element of F' and not element of F (Lemma 4.1). Every X , element of F , such that X is equivalent to Z , belongs to F' (Definition of incidence of flats).

Only if. Given X element of F and X' element of F' , X equivalent to X' . Suppose X is not an element of F' . There exists U , an element of F such that U is a complement of X' (Definition of flat). Since U is not an element of F' , U is not equivalent to Z (hypothesis). Therefore, X, X' are equivalent to Z . Then X is an element of F' (hypothesis).

Lemma 5.1. If $F:1$ and $F:2$ are distinct flats, there exist $X:1, X:2$ such that $X:i$ is an element of $F:j$ if and only if $i = j$ and $X:1$ and $X:2$ are complementary.

We wish to introduce into metaprojective geometry the notion of betweenness which corresponds to the following situation in projective geometry: three flats are incident in pairs and the second is intermediate in dimension to the first and third. This is embodied in the following:

Definition: If $F:1$, $F:2$, $F:3$ are flats, $F:2$ is between $F:1$ and $F:3$ if $F:2$ is contained in the set union of $F:1$ and $F:3$ and $F:2 \neq F:1$ or $F:3$. In symbols we write $\overline{F:1, F:2, F:3}$ for " $F:2$ is between $F:1$ and $F:3$ ".

Definition: A chain of flats or F-chain is a sequence of flats, $(F:1, \dots, F:k)$, such that $\overline{F:i-1, F:i, F:i+1}$.

The two lemmas and theorem following show the equivalence of the definition of betweenness and a set of conditions which could equally well have been taken as the definition.

Lemma 5.2. If $\overline{F:1, F:2, F:3}$; X, U elements of $F:2$, X not an element of $F:1$, U not an element of $F:3$; then X is not equivalent to U .

Proof: The conditions guarantee that X, U will be distinct. Assume X equivalent to U . There exists Y such that X, Y and U are dependent and distinct. (D6). Since every element of $F:2$ which is incident to X, U , also is incident to Y , Y cannot have a complement in $F:2$. Therefore, Y is an element of $F:2$ (Definition of flat). Each of the elements X, Y, U belongs to $F:1$ or $F:3$ (hypothesis) and therefore

two of them must belong to the same flat. By the reasoning above the third element must also belong to this flat, $F:1$ or $F:3$, contradictory to the hypothesis. Consequently X is not equivalent to U .

Lemma 5.3. $\overline{F:1, F:2, F:3}$ implies $F:i$ incident to $F:j$ for $i, j = 1, 2, 3$.

Proof: There exist Z, U elements of $F:2$ where Z is not an element of $F:1$ and U is not an element of $F:3$ (Lemma 5.1). Z is not equivalent to U (Lemma 5.2).

1. To show if X is an element of $F:1$ and X is equivalent to Z , then X is an element of $F:2$; i.e., $F:1$ is incident to $F:2$. Consider X , an element of $F:1$ where X is equivalent to Z . Assume X is not an element of $F:2$. Then there exists V , an element of $F:2$, such that V is a complement of X (Definition of flat). Since V is not an element of $F:1$, V is not equivalent to U (Lemma 5.2). This means that X is not equivalent to Z contrary to our assumption. Therefore, X is an element of $F:2$.

2. To show if Y is an element of $F:2$ and Y is equivalent to Z , then Y is an element of $F:3$; i.e., $F:2$ is incident to $F:3$. If Y is not an element of $F:3$, then Z is not equivalent to Y (Lemma 5.2). This is impossible by hypothesis. Therefore Y is an element of $F:3$.

3. To show if X is an element of $F:1$ and X is equivalent to Z , then X is an element of $F:3$; i.e., $F:1$ is incident to $F:3$. Since Z was fixed throughout Parts 1 and 2, the result follows immediately.

Theorem F5. $\overline{F:1, F:2, F:3}$ if and only if the set intersection of $F:1$ and $F:3$ is contained in $F:2$, $F:1$ is incident to $F:3$ and $F:2 = F:1, F:3$.

Proof: If. To show that every element X , common to $F:1$ and $F:3$ also lies in $F:2$. Consider any element, Z , belonging to $F:2$. Z is an element of $F:1$ or $F:3$ (hypothesis) and as a consequence Z cannot be a complement of X (Definition of flat). Since it has been shown that no element of $F:2$ is a complement of X , it follows that X must be an element of $F:2$ (Definition of flat).

Only if. To show that every element of $F:2$ belongs to $F:1$ or $F:3$. Suppose the contrary. Let X be an element of $F:2$ which is not an element of $F:1$ or $F:3$. Then, in $F:1$ and $F:3$ there are element $X:1$ and $X:3$ which are complements of X . Since $X:1$ and $X:3$ are equivalent, one of them, say $X:1$ is in both $F:1$ and $F:3$ ($F:1$ and $F:3$ are incident by hypothesis). If $X:1$ is in both $F:1$ and $F:3$, it must belong to $F:2$ (hypothesis). This results in a contradiction since X and its complement $X:1$ cannot both be in $F:2$ (Definition of flat).

The next theorem is required to guarantee that "betweenness" has the customary properties of a ternary order relation.

Theorem F6. If $F:1, F:2, F:3, F:4$ is an F -chain, then $\overline{F:1, F:2, F:4}$ and $\overline{F:1, F:3, F:4}$.

Proof: 1. Show $F:1$, incident to $F:4$. Since $F:2$ and $F:3$ are distinct we can find X an element of $F:2$ and not of $F:3$, U a complement of X , an element of $F:3$ and not of $F:2$ by Lemma 5.1. But X is an element of $F:1$ and U is an element of $F:4$ since $\overline{F:1, F:2, F:3}$ and $\overline{F:2, F:3, F:4}$. Consider Y equivalent to Z , Y an element of $F:1$, Z an element of $F:4$. Y, Z must be equivalent to X or to U ; suppose Y, Z equivalent to X . Since $F:2$ is incident to $F:4$ by Lemma 5.3, and X is not an element of $F:4$, Z is an element of $F:2$. Since $\overline{F:2, F:3, F:4}$, Z is also in $F:3$. But X is in $F:1$ but not in $F:3$, Z is in $F:3$ and since $F:1$ is incident to $F:3$ then Z is in $F:1$ and therefore $F:1$ is incident to $F:4$.

2. Consider X and U as before. Consider an element in $F:2$. Since $\overline{F:1, F:2, F:3}$ it must be in $F:1$ or $F:3$. If it is in $F:1$ the theorem is proved. Suppose it is in $F:3$. Call the element Y and suppose it is equivalent to X . Since $F:1$ is incident to $F:3$ Y must be in $F:1$ for X cannot be.

Theorem F7. If $(X:0, \dots, X:k)$ satisfy the relations $X:i$ not an element of $F:i$ and $X:i$ an element of $F:i + 1$, $(X:0, \dots, X:k)$ are independent.

Proof: $X:i$ an element of $F:j$ for $j > i$ (F6). Each $X:i$ has a complement $U:i$ in $F:i$ (Definition of flat). If $i < j$, $U:j$ is incident to $X:i$ which has been shown to belong to the same flat, $F:j$ (Definition of flat). Thus the conditions for sequential independence are satisfied and the independence of $(X:0, \dots, X:k)$ is established (D1).

Theorem F8. The number of flats in an F-chain cannot exceed $n + 2$.

Proof: The existence of $(X:0, \dots, X:k)$ satisfying the conditions in F7 is guaranteed by Lemma 5.1. Since there can be at most $n + 1$ $X:i$, which are independent (F7), the number of flats cannot exceed $n + 2$.

Consecutiveness of flats is another concept for which alternative formulations are possible as indicated by the following:

Theorem F9. Two flats, $F:1$ and $F:3$, are consecutive if and only if they are incident and no flats lie between them.

Proof: Since consecutiveness of flats implies incidence, this proposition immediately reduces to the following statement: For two incident flats, $F:1$ and $F:3$, there exist X, U such that X belongs to $F:1$ and not $F:3$, U belongs to $F:3$ and not $F:1$, and X is incident to U if and only if there exists a flat $F:2$ such that $\overline{F:1, F:2, F:3}$.

If. $F:2$ is distinct from $F:1$ and $F:3$ by definition of betweenness of flats and there exist X, U such that X belongs to $F:2$ and not $F:3$ and U belongs to $F:2$ and not $F:1$ (Lemma 5.1). $\overline{F:1, F:2, F:3}$ implies that X belongs to $F:1$ and U belongs to $F:3$. Since $F:1$ and $F:3$ are incident, X is not equivalent to U and therefore, X is incident to U (Definition of flat).

Only if. Consider $U:i$ a basis of $F:1$ whose elements are equivalent to U . By $F:3$ ($U, U:i$) generate a flat, $F:2$. X is incident to U (hypothesis) and $U:i$, therefore X belongs to $F:2$. Then, X belongs to

$F:2$ and not $F:3$ and U belongs to $F:2$ and not $F:1$; demonstrating the distinctness of the three flats. Since U is not an element of $F:1$, $U:i$ must belong to $F:3$ (Definition of incidence of flats). Every element of $F:2$ is either dependent on U , $U:i$ and thus belongs to $F:3$ or incident to $U:i$ and thus belongs to $F:1$ (Construction of $F:2$). It follows, then, that $\overline{F:1, F:2, F:3}$.

Theorem F10. Every chain of flats can be extended to a maximal chain. In a maximal F -chain every two flats which have consecutive indices are consecutive.

Proof: If there is a flat which is between two flats which are adjacent in the chain, it may be adjoined to the chain. A maximal chain must be reached by this process since $F:8$ imposes a limit of $n + 2$ on the length of an F -chain. Theorem $F:9$ guarantees that two flats of a maximal F -chain will be consecutive if and only if their indices in the chain are consecutive.

This presentation of the theory of flats will be concluded by a discussion of the classification of flats. The class of projective flats of dimension k cannot be defined in metaprojective language, but the union of this class and its dual can be defined in this system.

Definition: A flat is of category k [for -1 less than or equal to k , less than or equal to $1/2 (n-1)$] if F has a basis consisting of $k + 1$ elements.

Definition: Two flats, F and F' , are complementary if their set intersection is empty.

It is possible to define an equivalence relation between flats which will divide the flats of each category into two classes which correspond to a class of projective flats of a certain dimension and the dual of this class.

Definition: Two flats are equivalent if they have a common complement.

Lemma 5.4. Let $F:1$ and $F:2$ be flats and $B:1$ a basis of $F:1$. $F:1$ and $F:2$ are complementary if and only if $F:2$ has a basis, $B:2$, such that $B:1, B:2$ form a basis of the space.

Proof: If B equal to $(B:1, B:2)$ can be extended to a simplex, B, B^* . Every element of B^* belongs to $F:1$ or $F:2$ (Definition of basis of flat). Any element which belonged to both $F:1$ and $F:2$ would be incident to B or to B^* ; contradicting the fact that they are bases of the space (Lemma 4.2). Therefore, $F:1$ and $F:2$ are complementary.

Only if. Let $B:2$ be a basis of $F:2$ which is equivalent to $B:1$, and represent the numbers of elements in $B:1$ and $B:2$ by r and s , respectively. Since $F:1$ and $F:2$ are complementary, $B:1$ and $B:2$ cannot have a common incident and $r + s$ is greater than or equal to $n + 1$. By considering bases of $F:1$ and $F:2$ which are not equivalent to $B:1$ and $B:2$, it may be shown that $r + s$ is less than or equal to $n + 1$ (F_2). Consequently, $B:1$ and $B:2$ form a basis of the space.

Theorem F11. Two flats, $F:1$ and $F:2$, are equivalent if and only if they have equivalent bases, $B:1$ and $B:2$, each having k elements.

Proof: If. Suppose that $r - 1$ elements, $X:i$, have been found such that $[B:1, X:i]$ and $[B:2, X:i]$ are independent. If $k + 1$ is less than $n + 1$, we may proceed as follows to choose the $X:r$. There exist $U:1, U:2$ such that $U:1$ is incident to $[X:i, B:1]$ and $U:2$ is incident to $[X:i, B:2]$ (MA5). $U:1$ and $U:2$ have a common complement, $X:r$ (D5). $(X:1, \dots, X:r, B:i)$ are independent by D1. This process can be continued until $([X:i], B:j)$ form a basis of the space. The $[X:i]$ so obtained generate a flat ($F3$) which is a common complement of $F:1$ and $F:2$ by Lemma 5.4.

Only if. $F:1$ and $F:2$ have a common complement, F , by hypothesis. Let B be a basis of F . $F:1$ and $F:2$ have bases, $B:1$ and $B:2$, such that $(B, B:1)$ and $(B, B:2)$ are bases of the space (Lemma 5.4). Therefore, $B:1$ and $B:2$ have the same number of elements.

The use of the term, equivalence of flats, is justified since the reflexivity and symmetry are incorporated in the definition and the transitivity follows easily from Theorem F11. Since two of every three flats in the " k " category must have equivalent bases of $k + 1$ elements, it can be inferred immediately that this equivalence relation effects a dichotomy of each of the categories of flats. These results are summed up in the following:

Theorem F12. The relation, F is equivalent to F' , is an equivalence relation which effects a dichotomy of each of the categories of flats.

CHAPTER VI

DERIVATION OF PROJECTIVE GEOMETRY

In line with the scheme laid down in the Introduction, Chapter VI is devoted to the formulation of projective geometry as a specialization of metaprojective geometry. What is needed is an axiom which will introduce a name, $S:P$, for one of the equivalence classes so that it will play the role of the class of points in the projective geometry. Every pair of elements of $S:P$ must be equivalent and every element equivalent to an element of $S:P$ must belong to $S:P$. Thus for the non-empty set, $S:P$, of elements of the metaprojective geometry we have the axiom:

MA8. If X , element of $S:P$, then Y element of $S:P$ if and only if some Z , X incident to Z and Y incident to Z .

The book, Projective Geometry by Veblen and Young is taken as a standard of reference. The objective of this chapter is to deduce their axioms of alignment and extension from MA1...MA8.

The first step is to define the class, $S:L$, of lines. The development of the theory of flats in the last chapter suggests that lines may be regarded as flats. Since the class of flats which have a basis consisting of two elements includes the lines and also their duals, the lines must be selected from this class.

Definition: A flat, L , is a line if some pair of elements of $S:P$ is a basis of L .

The propositions, MA1...MA7 of Chapter III, together with the above proposition, MA8, of this chapter, are axioms for a projective geometry, P , whose points and lines form the classes, $S:P$ and $L:P$ and in which the incidence relation is "element of".

The following is a list of the Assumptions of Alignment and Extension taken from Projective Geometry, Vol. II, p. 2, by Veblen and Young:

Assumptions of Alignment:

A1. If A and B are distinct points, there is at least one line on both A and B .

A2. If A and B are distinct points, there is not more than one line on both A and B .

A3. If A, B, C are points not all on the same line, and D and E ($D \neq E$) are points such that B, C, D are on a line and C, A, E are on a line, there is a point F such that A, B, F are on a line and also D, E, F are on a line.

Assumptions of Extension:

E0. There are at least three points on every line.

E1. There exists at least one line.

E2. All points are not on the same line.

E3. All points are not on the same plane.

E3'. If $S:3$ is a three-space, every point is on $S:3$.

The first axiom of alignment is an immediate consequence of F3. E0, the existence of three points on a line, is guaranteed by D6. Combining Lemma 4.1 with F3, we see that there must be at least one line, thus proving E1. The remaining axioms require more comprehensive discussion.

A2 may be stated as follows: If L and L' are lines having in common the distinct points, $X:1$ and $X:2$, then $L = L'$.

Proof: Each of the lines has a basis consisting of two points (Definition of line). $[X:i]$ is a basis for both flats (D8 and Definition of basis of a flat). If L and L' were distinct, we could find X, U where X is an element of L and not an element of L' and U is an element of L' and not an element of L (Lemma 5.1), and X and U are complementary. Either X or U must be equivalent to $X:i$. Suppose X is equivalent to $X:i$: since $[X:i]$ is a basis for both flats, U must belong to both L and L' (Definition of flat and basis of flat). This is a contradiction, however; because the complementary elements X, U cannot belong to the same flat. Therefore, we conclude that $L = L'$.

The third axiom of alignment may be paraphrased as follows: Given $X:0, X:1, X:2$ independent. $X:1, X:0, X:2$ dependent. $X:0, X':1, X:2$ dependent. Then, there exists $X':2$, such that $X:0, X:1, X':2$ dependent, $X':0, X':1, X':2$ dependent.

Proof: Assume all the given elements are distinct, otherwise the theorem follows immediately. Extend $X:0$, $X:1$ and $X:2$ to a simplex (D4), $(X:0, \dots, X:n)$ and $(U:0, \dots, U:n)$.

1. Independence of $X:0$, $X':0$ and $X':1$. $X:0$ dependent on $X:0$. $X:2$ dependent on $X:0$, $X':1$. $X:1$ dependent on $X':0$, $X:2$ dependent on $X':0$, $X:0$, $X':1$. This follows by Lemma 4.6. Then, $X:0$, $X:1$, $X:2$ dependent on $X:0$, $X':0$, $X':1$. By D7, $X:0$, $X':0$, $X':1$ are independent.

2. There exists U incident to $X':0$, $X':1$ and not incident to $X:0$ as just shown in Part 1. Since $X:0$ is not incident to U , $U:0$ can be replaced by U in the basis $(U:0, \dots, U:n)$, forming a new basis. Let $X':2$ be the common incident to U , $(U:2, \dots, U:n)$. $X:0$, $X:1$, $X':2$ incident to $(U:2, \dots, U:n)$. If these were independent, D4 would lead to a contradiction. $X':0$, $X':1$, $X':2$ incident to U , $(U:3, \dots, U:n)$ and likewise, D4 would lead to a contradiction if these elements were independent. Therefore, $X:0$, $X:1$, $X':2$ are dependent and $X':0$, $X':1$, $X':2$ are dependent.

In Veblen and Young, n -dimensional geometry is not elaborated in much detail, their chief concern being the geometry of 3 dimensions. A k -dimensional subspace is defined as a set of points which is obtained by joining a point to each point of a $k-1$ dimensional subspace to which it does not belong--the points on all such lines constitute the k -dimensional subspace. For each k less than n there is an axiom asserting

that not all of the points lie on a subspace of dimension k . The last axiom is a closure condition stating that all the points lie on an n -dimensional subspace. That the axioms MA1...MA8 lead to a geometry of the proper number of dimensions can be seen by referring to the theory of flats in Chapter V and Theorem F7 in particular.

Veblen and Young introduce a series of assumptions which characterize more and more restricted types of projective geometry. Now that points and lines have been defined in our system and have been shown to have the basic properties of points and lines in projective geometry, these assumptions can be translated into metaprojective language. Our objective is to find a simple and elegant formulation of the theory from which these properties can be deduced.

One of the most important problems in geometry is to characterize those geometries whose number systems are commutative. The Pappus Theorem from which the commutativity can be inferred, is not suitable as an axiom in metaprojective geometry of n -dimensions, since it involves a plane configuration. What is required is a statement involving a configuration made up of points and hyperplanes from which the Pappus Theorem can be deduced. An approach to the solution of this problem has been made for geometries of dimension 3. The Moebius configuration consisting of two mutually inscribed and circumscribed tetrahedra may be represented as a linear graph of order 4 (see Chapter II). Here again there is the problem of characterizing

uniquely a configuration as a linear graph of a specified order. Defining an incomplete Moebius configuration as a set of fifteen elements which is isomorphic to a subset of some Moebius configuration, the assertion that every incomplete Moebius configuration can be completed could be taken as an axiom, the Moebius axiom, in 3-dimensional metaprojective geometry. The Pappus Theorem and the commutativity of the number system can be deduced from the Moebius axiom.

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