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RELATIVISTIC FLUIDS AND WAVE PROPAGATION

by

Martin Sommerfield

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
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INTRODUCTION

The purpose of this paper is to derive and study a system of equations for relativistic, non-equilibrium fluids. Each chapter deals with a slightly different aspect of the theory and therefore has its own introduction in which the background, methods and main results are summarized. For the present we will give only a general outline.

We begin by developing a covariant formulation of the theory in the style used by Lichnerowicz for relativistic hydrodynamics [17]. The thermodynamic states are characterized by a relaxation variable in addition to the density and the entropy. Linear phenomenological laws are assumed and the basic results obtained by De Groot for multi-component reacting systems [1] are extended to the general theory of relativity. When diffusion effects are ignored we obtain the equations used by various authors [2],[7] to study general relaxation phenomena in Newtonian fluids.

As is well known, interesting mathematical and physical properties of a system of partial differential equations can be obtained by studying the surfaces over which its solutions can have discontinuous derivatives. Using a technique of Coburn [5,6,7] we will find a system of algebraic equations for the discontinuities corresponding to various (C^m, C^n) classes of solutions. The magnitude of the discontinuities is expressed in terms of a single unknown scalar whose behavior is studied later in the paper. From the discontinuity system we also show that the characteristic equation can be simply expressed in terms of certain limiting speeds of propagation and thus we find relativistic analogues of earlier work for the Newtonian case [1],[7]. The significance of the bicharacteristic directions, which are needed for the work of Chapter III, is

discussed.

In Chapter III we obtain a differential equation for the growth and decay of the discontinuities along the bicharacteristic curves. This equation depends on the mean curvature of the history of the wave and is valid for an arbitrary flow ahead of the disturbance. When the wave propagates into a uniform fluid, it is shown that the mean curvature and the Gaussian curvature of the wave are governed by a pair of non-linear ordinary differential equations. The exact proper time s_1 at which the singularity occurs is explicitly calculated from the three ordinary differential equations. In an appropriate series expansion of s_1 we see that the first term is the proper time that would be observed for a similar wave in flat space-time, and the second term is a correction due to the presence of the fluid. This provides the theoretical basis for a test of the general theory of relativity in which Einstein's field equations must take into account a non-vanishing stress-energy tensor.

The final chapter contains the proof of a local existence and uniqueness theorem for the Cauchy problem with sufficiently smooth data. The technique is that of Lichnerowicz and is based on Leray's results for a strictly hyperbolic system of partial differential equations.

CHAPTER I

A SYSTEM OF EQUATIONS FOR NON-EQUILIBRIUM RELATIVISTIC FLUIDS

I.I. Introduction

In this section we will obtain a system of equations for a relativistic fluid in which diffusion and a single chemical reaction are occurring. This work extends to an arbitrary space time the work of de Groot [1] for the case of special relativity. Rather than employing, as de Groot does, coordinates dependent 3-velocities and mass densities, our formulation is in terms of the tangents to world lines in 4-space and certain invariant scalar functions. One problem in extending de Groot's theory to curved space-time is providing appropriate interpretations of these functions as densities of proper mass. Once this is done, the equation of conservation can be derived. In order that this be done in a convenient way, a modified form of de Groot's definition of the barycentric 4-velocity is given. The time orientability [2] of the manifold (which is understood in special relativity) is explicitly postulated so that the barycentric 4-velocity will be time like.

A heuristic discussion of the type used by physicists is given to make the necessary identifications and justify the definitions. Then using the basic properties of the barycentric 4-velocity, the form of the stress-energy tensor is deduced. We then derive some basic results, already obtained by de Groot [1] for the case of flat space-time; namely the following: the equation of mass conservation, the existence of a scalar function called the rate of reaction, and the entropy balance equation. For the first and second of these results we assume that the proper rest mass of each atom is a constant and is independent of

the manner in which it is combined with other atoms.

The linear phenomenological laws are discussed and then the system of equations which governs a relativistic chemically reacting fluid with diffusion can be written down. In the final equations, the idea of a chemical reaction manifests itself in the conservation of mass law which is a consequence of the chemists stoichiometric equation. It is then seen that the equations studied by Stupochenko and Stakhanov [3], Yuan [4], Coburn [5,6,7] are valid only under special conditions, the simplest of which is when the microscopic velocity of all the chemical components is the same, i.e., no diffusion effects occur.

I.II. Specification of the Reacting Mixture

Consider a fluid consisting of a mixture of chemical components undergoing a single chemical reaction. In order to account for observed phenomena [8], we think of each chemical component as having a well defined velocity. Thus to specify a reacting mixture one thinks of a 4-manifold on which there are given the unit tangents $dx_a^{(k)}/ds$ for the world lines of the k^{th} chemical component and scalar functions $\rho_0^{(k)}$ which will be interpreted as giving the proper density of proper mass of the k^{th} constituent.

Definition 1: The k^{th} flow vector is denoted by $w_a^{(k)}$ and is defined as $w_a^{(k)} \equiv \rho_0^{(k)} \, dx_a^{(k)}/ds = \rho_0^{(k)} \, u_a^{(k)}$, where $\rho_0^{(k)}$ is a given scalar function and $u_a^{(k)}$ a given timelike vector field of unit length.

It is natural to try to refer the motion of the various chemical components to a common reference system which is determined by the mixture. From the mathematical point of view the simplest procedure is to

add the $w^{(k)a}$ and take the result as a reference. For our later work it will be necessary that this sum is time-like, thus we require that the $u^{(k)a}$ are such that $\Sigma_k u^{(k)a}$ is time-like. In particular, it would be sufficient to assume that the manifold is time oriented at each point and that the given $u^{(k)a}$ are future pointing. We adopt the following slightly simpler version of de Groot's procedure.

Definition 2: Let the vector $w_a \equiv \Sigma_k w_a^{(k)}$ be time-like with $w \equiv (w^a w_a)^{1/2}$. A barycentric coordinate system is a coordinate system which has the vector $u_a \equiv w_a/w$ as the tangent to one coordinate line.

From Definition 2 we can derive the following physical interpretation of w.

Proposition 1: Suppose S is a barycentric coordinate system and ρ the relative density of proper mass as measured by instruments at rest in S. Then $w = \rho$.

Proof: From Definition 2, in a barycentric coordinate system,

$$w_a = \left(\sum_k \frac{dx^0_{(k)}}{ds} \rho_{0\,(k)}, 0, 0, 0\right).$$

Let $S^0_{(k)}$ be an inertial rest frame for the k^{th} component. By definition $\rho_0^{(k)}$ has the following interpretation: If the volume of a fluid element, as measured by instruments fixed in $S^0_{(k)}$, is $d\mathring{V}^0_{(k)}$, and this fluid element contains $n_{(k)}$ molecules of the k^{th} constituent, then

$$\rho_0^{(k)}\, d\mathring{V}^0_{(k)} = n_{(k)}\, M_0^{(k)} \tag{1.1}$$

where $M_0^{(k)}$ is the rest mass. From a known result [9] for the generalized Lorentz contraction we have in a barycentric frame

(1.1a) $$\overset{\circ}{dV}{}^{0}_{(k)} = dV \frac{dt_{(k)}}{d\tau}$$

where $d\tau^2 = ds^2$ and $t = x^0$. Hence (1) is

(1.1b) $$\rho_0^{(k)} \, dV \frac{dt_{(k)}}{d\tau} = n_k \, M_0^{(k)}.$$

Now

$$w = w^a \cdot u_a = \sum \frac{dx^{(k)0}}{ds} \rho_0^{(k)}.$$

Hence

$$w \, dV = \sum_k \frac{dx^{(k)0}}{ds} \rho_0^{(k)} \, dV$$

which by (1.1b) becomes

$$w \, dV = \sum_k \frac{dx^{(k)0}}{ds} n_k \, M_0^{(k)} \left(\frac{dx^{(k)0}}{ds}\right)^{-1}$$

$$w \, dV = \sum_k \frac{dt}{ds} n_k \, M_0^{(k)} \left(\frac{dt}{ds}\right)^{-1} = \sum_k n_k \, M_0^{(k)}.$$

This proves the assertion.

I.III. Chemical Reactions and Reaction Rate

Consider the chemical reaction

(1.2) $$\alpha_1 a_1 + \cdots + \alpha_{k-1} a_{k-1} \rightarrow \alpha_k a_k + \cdots + \alpha_m a_m$$

where α_i are non-negative integers and a_k are the symbols for the k^{th} chemical constituent. For convenience we define

(1.2a) $$b_j = -\alpha_j, \qquad 1 \leq j \leq k - 1$$

(1.2b) $$b_j = \alpha_j, \qquad k \leq j \leq m.$$

As an example of (1.2)-(1.2b) we have the reaction

$$2H_2 + O_2 \rightarrow 2H_2O$$

and $b_1 = -2$, $b_2 = -1$, $b_3 = 2$.

Choose an arbitrary point P in a Newtonian reference frame, and a

volume V enclosing it. The change in the number of moles of the k^{th} constituent is governed by

$$(1.3) \qquad \frac{\partial}{\partial t}\int_V \eta^{(k)}\,dv = (-1)\int_S \eta^{(k)} v^{(k)}\,ds + \Delta N^{(k)}$$

where $\eta^{(k)}$ is the molar density of $a_{(k)}$ and $\Delta N^{(k)}$ is the rate of change of the number of moles produced in V. The chemist's notation in (1.2) means that for all k,p

$$(1.4) \qquad \frac{\Delta N^{(k)}}{\Delta N^{(p)}} = \frac{b^{(k)}}{b^{(p)}} .$$

Substituting (1.4) into (1.3) yields

$$(1.5) \quad b^{(p)}\left\{\frac{\partial}{\partial t}\int_V \eta^{(k)}\,dv + \int_S \eta^{(k)} v^{(k)}\,ds\right\} = b^{(k)}\left\{\frac{\partial}{\partial t}\int_V \eta^{(p)}\,dv + \int_S \eta^{(p)} v^{(p)}\,ds\right\}$$

or with the divergence theorem

$$(1.5a) \quad b^{(p)}\left\{\frac{\partial}{\partial t}\eta^{(k)}\,dv + \operatorname{div}(\eta^{(k)} v^{(k)}\right\} = b^{(k)}\left\{\frac{\partial}{\partial t}\eta^{(p)} + \operatorname{div}(\eta^{(p)} v^{(p)})\right\}.$$

Introducing the 4-velocity [10], (p. 130),

$$\frac{dx^{(k)\alpha}}{ds} = v_a^{(k)}\,\gamma^{(k)} \qquad \alpha = 1,\ 2,\ 3$$

$$\frac{dx_0^{(k)}}{ds} = \gamma^{(k)} \equiv \frac{1}{(1 - v^2_{(k)})^{1/2}}$$

where $dx_a^{(k)}/ds$ is the velocity of the k^{th} substance (relative to the co-ordinate system of (1.5)). Then (1.5a) takes on the 4-dimensional form

$$b^{(p)}\ \partial_a\left(\eta^{(k)}\gamma^{(k)-1}\frac{dx_a^{(k)}}{ds}\right) = b^{(k)}\ \partial_a\left(\eta^{(p)}\gamma^{(p)-1}\frac{dx_a^{(p)}}{ds}\right).$$

Now using the standard result [10], (p. 260), that $dV^0 = dV\,\gamma^{(k)}$ the last equation becomes

$$(1.6) \qquad b^{(p)}\ \partial_a\left(\eta_0^{(k)}\ \frac{dx_a^{(k)}}{ds}\right) = b^{(k)}\ \partial_a\left(\eta_0^{(p)}\ \frac{dx_a^{(p)}}{ds}\right)$$

where $\eta_0^{(k)}$ is the number density of the species k measured by an observer moving with the species k and $dx_a^{(k)}/ds$ is the velocity relative to the Newtonian coordinates of (1.5). The function $\eta_0^{(k)}$ is a well defined scalar function, $\eta_0^{(k)}\ dx_a^{(k)}/ds$ is a 4-vector and so according to the usual principle for extending classical equations to special relativity we regard (1.6) as valid in any Lorentz frame and the following as the form for general relativity:

$$(1.7) \qquad b^{(p)}\ \nabla_a\left(\eta_0^{(k)}\ \frac{dx^{(k)a}}{ds}\right) = b^{(k)}\ \nabla_a\left(\eta_0^{(p)}\ \frac{dx^{(p)a}}{ds}\right).$$

Assuming that the rest mass of all components is constant (conditions under which this is valid are discussed in [10], p. 166) we find that for all k,p

$$(1.8) \qquad M^{(p)}\ b^{(p)}\nabla_a\left(\rho_0^{(k)}\ \frac{dx^{(k)a}}{ds}\right) = M_0^{(k)}\ b^{(k)}\ \nabla_a\left(\rho_0^{(p)}\ \frac{dx^{(p)a}}{ds}\right).$$

Thus there exists a scalar function J such that for all k

$$(1.9) \qquad \nabla_a\left(\rho_0^{(k)}\ \frac{dx^{(k)a}}{ds}\right) = M_0^{(k)}\ b^{(k)}\ J,$$

where $\rho_0^{(k)}$ is the proper density of proper mass measured by an observer moving with the k^{th} chemical component.

Definition 3: Given the chemical reaction (1.2), the scalar J defined by (1.9) is called the rate of reaction.

Assuming that the rest mass of each atom is constant regardless of how it is combined with other atoms we have from the stoichiometric equation and (1.9) that

$$(1.9a) \qquad 0 = \Sigma\ b^{(k)}\ M_0^{(k)} = \sum_k \nabla_a\left(\rho_0^{(k)}\ \frac{dx^{(k)a}}{ds}\right).$$

Proposition 3: In a relativistic chemically reacting fluid we have

$$\nabla_a(\rho\, u^a) = 0. \tag{1.10}$$

Proof: From Definition 1 and Definition 2,

$$\sum_k \rho_0^{(k)} \frac{dx^{(k)a}}{ds} \equiv \sum_k w^{(k)a} = \rho\, u^a.$$

Thus (1.10) follows from (1.9a).

I.IV. Thermodynamic Considerations, Stress-Energy Tensor and Basic Equations.

Remarks: The following relation is taken over into the theory of equilibrium relativistic fluids [11],[12] directly from classical physics:

$$de = \frac{p}{\rho^2}\, d\rho + T\, ds. \tag{1.10a}$$

To generalize (1.10a) to the non-equilibrium case we take ρ, s, $C_{(k)} \equiv w_a^{(k)}\, u^a/\rho$ as thermodynamic variables and postulate the relation (Gibb's relation)

$$de = \frac{p}{\rho^2}\, d\rho + T\, ds + \sum \left.\frac{\partial e}{\partial C_{(k)}}\right)_{s,\rho} dC_{(k)}. \tag{1.10b}$$

We point out that since $C_{(k)}/M^0_{(k)}$ is the same as the mole number of the k^{th} chemical constituent, it follows that the partial of e with $C_{(k)}$ held constant is the same as the partial with the k^{th} mole number held constant. This justifies denoting the coefficient $\left.\frac{\partial e}{\partial s}\right)_{\rho, C_{(k)}}$ in (1.10b) by the symbol T which by definition is the partial of e at constant mole number and density. Similarly for the coefficient of $d\rho$ in (1.10b).

To determine the form of the stress-energy tensor T_{ab} we consider the physical situation in barycentric coordinates. By the definition of barycentric coordinates there is no net flow of matter relative to an observer

at the origin. Since there is no heat transfer or viscosity and since the chemical reaction occurs pointwise (i.e. internal energy is not exchanged between neighboring fluid elements) we have that $T_{4\alpha} = 0$, $\alpha = 1,2,3$, and T_{00} gives the density of energy as measured by an observer B whose velocity is u. The energy density has a contribution due to the mass density ρ as measured by B (see Proposition 1) and a contribution due to the internal energy of the fluid. We assume that the mechanical properties of the fluid are those of a perfect fluid so that the stress-energy tensor is [11]

$$T_{ab} = \eta u_a u_b - p g_{ab} \tag{1.10c}$$

$$\eta \equiv \rho(1 + e + \frac{p}{\rho})$$

where e is the specific internal energy.

The conservation law

$$u_b \nabla_a T^{ab} = 0 \tag{1.10d}$$

will now be used to derive the following useful relation.

Proposition 4: For a chemically reacting fluid we have

$$\rho \frac{de}{dt} + p \nabla_a u^a = 0. \tag{1.10e}$$

Proof:

$$\nabla_a(u_b T^{ab}) = u_b \nabla_a T^{ab} + T^{ab} \nabla_a u_b. \tag{1.11}$$

Hence (1.10d) is equivalent to

$$\nabla_a(u_b T^{ab}) - T^{ab} \nabla_a u_b = 0. \tag{1.12}$$

We will calculate each of these terms separately.

Recalling that $u^a u_a = 1$ gives

$$u_b T_a{}^b = u_a \rho(1 + e). \tag{1.13}$$

Writing $u^a \nabla_a \equiv d/dt$ we have

$$\nabla_a(u_b T^{ab}) = (1 + e)\nabla_a(u^a\rho) + \rho \frac{d}{dt} e. \tag{1.14}$$

The second term of (1.12) is

$$T^{ab} \nabla_a u_b = (\nabla_a u_b)(-pg^{ab}) \tag{1.15}$$

and

$$\nabla_a u_b(-p) g^{ab} = -p \nabla_a u^a. \tag{1.16}$$

Substituting (1.16) into (1.15)

$$T_b{}^a \nabla_a u^b = -p \nabla_a u^a. \tag{1.17}$$

Now putting (1.14),(1.17) into (1.12) we find

$$(1 + e)\nabla_a(u^a\rho) + \rho \frac{de}{dt} + p \nabla_a u^a = 0. \tag{1.18}$$

But for a chemically reacting fluid the first term on the left vanishes by Proposition 3; thus

$$\rho \frac{de}{dt} + p \nabla_a u^a = 0. \tag{1.19}$$

Proposition 5: For a chemically reacting fluid we have

$$\frac{de}{dt} = \rho T \frac{ds}{dt} - p \nabla_a u^a + \rho \sum_k \mu_{(k)} \frac{dC_{(k)}}{dt}$$

where $\mu_{(k)} \equiv \partial e/\partial C_{(k)}$.

Proof: The expression (1.10b) is

$$de = T\, ds + \frac{p}{\rho^2} d\rho + \sum_k \mu^{(k)} dC^{(k)}. \tag{1.20}$$

Now

$$\frac{de}{dt} \equiv u^a \partial_a e = u^a\left(\frac{\partial e}{\partial s} \partial_a s + \frac{\partial e}{\partial \rho} \partial_a \rho + \sum_k \frac{\partial e}{\partial C_{(k)}} \partial_a C_{(k)}\right) \tag{1.21}$$

$$\rho \frac{de}{dt} = \rho\, u^a\left(T \partial_a s + \frac{p}{\rho^2} \partial_a \rho + \Sigma\, \mu^{(k)} \partial_a C_{(k)}\right) \tag{1.22}$$

$$(1.23)\qquad \rho \frac{de}{dt} + p\, \nabla_a u^a = p\, \nabla_a u^a + \rho T \frac{ds}{dt} + \frac{p}{\rho} u^a \partial_a \rho + \rho \sum_k \mu_{(k)} \frac{dC_{(k)}}{dt}.$$

The first and third terms on the right side of (1.23) combine to give $p\, \nabla_a(\rho u^a)/\rho$ which vanishes by (1.10). Thus (1.23) is

$$(1.24)\qquad \rho \frac{de}{dt} + p\, \nabla_\alpha u^\alpha = \rho T \frac{ds}{dt} + \rho \sum \mu_{(k)} \frac{dC_{(k)}}{dt}$$

which is the required expression.

Proposition 6:

$$(1.24a)\qquad T \frac{ds}{dt} + \sum \mu_{(k)} \frac{dC_{(k)}}{dt} = 0.$$

Proof: If we substitute the expression for $\rho \frac{de}{dt}$ in (1.24) into (1.19), we obtain (1.24a).

The entropy balance equation is formulated in terms of

Definition 4: The flows of matter relative to the barycentric 4-velocity are defined by

$$(1.24b)\qquad H_a^{(k)} \equiv w_a^{(k)} - \left(w_s^{(k)}\, u^s\right) u_a.$$

These matter flows have the properties listed by de Groot [1], (2.23),(2.24)) for his vectors $I_a^{(k)}$, [1], ((2.21)), namely

Proposition 7: The vectors $H_a^{(k)}$ have the following properties:

(i) $H_a^{(k)}\, u^a = 0$;

(ii) $\sum_k H_a^{(k)} = 0.$

Proof: Taking the scalar product of (1.24b) with u^a and recalling $u^a u_a = 1$ proves (i). To establish (ii), sum (1.24b) over k and use Definition 2 in the result to obtain

$$\sum_k H_a^{(k)} = w_a - (w \cdot u)u_a$$

$$= w_a - \left(w_s \frac{w^s}{w}\right)\frac{w_a}{w} = 0.$$

The following can be easily verified:

Proposition 8: In a barycentric Lorentz frame $H_a^{(k)}$ reduces to de Groot vector $I_a^{(k)}$, and therefore $H_a^{(k)}$ is the form of $I_a^{(k)}$ suitable for general relativity.

Remark: For the sake of completeness it should be pointed out that the definition of barycentric coordinate system given above is also a barycentric system in de Groot's sense [1]. For in barycentric coordinates we have, according to Definition 4, that $0 = w_\alpha = \Sigma\, \rho_0^{(k)} w_\alpha^{(k)}$, $\alpha = 1,2,3$, which for the case of flat space time implies $v_\alpha = 0$, $\alpha = 1,2,3$, and thus de Groot's definition is satisfied.

Following de Groot we make

Definition 5: The thermodynamic forces are defined by

$$Y_a^{(0)} \equiv (-1)\left(\frac{\nabla_a T}{T} - \frac{du_a}{dt}\right) \qquad a = 1,\ldots,4 \tag{1.25}$$

$$Y_a^{(k)} \equiv (-1)T\nabla_a\left(\frac{\mu^{(k)}}{T}\right) \qquad a = 1,\ldots,4;\ k = 1,\ldots,m. \tag{1.26}$$

Proposition 9:

$$\frac{dC^{(k)}}{dt} = \frac{1}{\rho}\left[\nabla_a H^{(k)a} - M^{(k)} b^{(k)} J\right]. \tag{1.27}$$

Proof:

$$\nabla_a H^{(k)a} = \nabla_a\{\rho_0^{(k)} \frac{dx^{(k)a}}{ds} - (w^{(k)} \cdot u)u_a\} \tag{1.28}$$

$$= \nabla_a\left(\rho_0^{(k)} \frac{dx^{(k)a}}{ds}\right) - \nabla_a\left(\frac{\rho^{(k)}}{\rho} w^a\right). \tag{1.29}$$

Using (1.8) to evaluate the first term, we find

$$\nabla_a H^{(k)a} = M_0^{(k)}{}_b{}^{(k)} J - \nabla_a \left(\frac{\rho^{(k)}}{\rho} w^a\right) \tag{1.30}$$

$$= M_0^{(k)}{}_b{}^{(k)} J - \left(\frac{\rho^{(k)}}{\rho} \nabla_a w^a + w^a \nabla_a \frac{\rho^{(k)}}{\rho}\right) . \tag{1.31}$$

By Proposition 3, $\nabla_a w^a = 0$. By the interpretations already given to $\rho^{(k)}$, ρ, we know $\rho^{(k)}/\rho$ reduces to de Groot's $C_{(k)}$ when evaluated in a barycentric Lorentz frame. Hence we write $C_{(k)} \equiv \rho^{(k)}/\rho$ and obtain

$$\nabla_a H^{(k)a} = M_0^{(k)}{}_b{}^{(k)} J - w^a \nabla_a C_{(k)} . \tag{1.32}$$

By Proposition 2 the above relation becomes

$$\nabla_a H^{(k)a} = M_0^{(k)}{}_b{}^{(k)} J - \rho \frac{dC_{(k)}}{dt} \tag{1.33}$$

from which the conclusion follows.

Definition 6: Define the scalar function $\bar{A}$ by

$$\bar{A} \equiv \sum_k \frac{\partial e}{\partial C_{(k)}} M_0^{(k)}{}_b{}^{(k)} . \tag{1.33a}$$

Proposition 10: The entropy balance equation is

$$\rho \frac{ds}{dt} + \sum_k \nabla_a \left(\frac{1}{T} \frac{\partial e}{\partial C_{(k)}} H^{(k)a}\right) - \sum \frac{1}{T} H^{(k)a} Y_a^{(k)} - \frac{1}{T} \bar{A} J = 0. \tag{1.34}$$

Proof: We will rewrite the terms of (1.24a) using Definition 5. From Proposition 9 we have

$$\frac{\rho}{T} \sum \frac{\partial e}{\partial C_{(k)}} \frac{dC_{(k)}}{dt} = \frac{1}{T} \sum_k \frac{\partial e}{\partial C_{(k)}} \left(\nabla_a H^{(k)a} - M_0^{(k)}{}_b{}^{(k)} J\right) . \tag{1.35}$$

Rewriting (1.35) we have,

$$\frac{\rho}{T} \sum \frac{\partial e}{\partial C_{(k)}} \frac{dC_{(k)}}{dt} = \sum_k \left\{\nabla_a \left(\frac{\partial e}{\partial C_{(k)}} \frac{1}{T}\right) H^{(k)a} - H^{(k)a} \nabla_a \left(\frac{\partial e}{\partial C_{(k)}} \frac{1}{T}\right) - \frac{1}{T} \frac{\partial e}{\partial C_{(k)}} M_0^{(k)}{}_b{}^{(k)} J\right\} . \tag{1.36}$$

Using (1.26) and (1.33a) we find

$$(1.37)\qquad \frac{\rho}{T}\sum_k \frac{\partial e}{\partial C_{(k)}}\frac{dC_{(k)}}{dt} = \sum_k \left\{\nabla_a\left(\frac{\partial e}{\partial C_{(k)}}\frac{1}{T}H^{(k)a}\right) + H^{(k)a}\frac{1}{T}Y_a^{(k)}\right\} - \frac{1}{T}\bar{A}J.$$

Substituting (1.37) and (1.25) into (1.24a) yields

$$(1.38)\qquad \frac{ds}{dt} = -\frac{1}{\rho}\sum_k \left\{\nabla_a\left(\frac{1}{T}\frac{\partial e}{\partial C_{(k)}}H^{(k)a}\right) + H^{(k)a}\frac{1}{T}Y_a^{(k)}\right\} + \frac{1}{\rho T}\bar{A}J,$$

which is the desired equation.

I.V. Linear Phenomenological Principle and Basic Equations.

Certain experimentally determined physical laws, for example Fourier's Law of Heat Flow, Fick's Law of Diffusion and Ohm's Law of Electrical Conduction, express the proportionality between the flow of some quantity (e.g. heat) and the gradient of some intensive thermodynamic variable (e.g. temperature). As a generalization of these results it is postulated [8] that for systems which remain sufficiently close to equilibrium the flows are proportional to a linear combination of the affinities which appear with them in the entropy balance equation. The justification for this postulate rests on its success in predicting the results of experiments. De Groot points out [8] that a linear phenomenological law is not completely satisfactory for chemical reactions.

Applying the phenomenological principle to affinities and corresponding fluxes in the entropy balance equation (1.34) we obtain

$$(1.39)\qquad \bar{A} = \frac{\rho T}{\tilde{K}}J$$

$$(1.39a)\qquad H_a^{(k)} = \sum_{j=1}^{m}\sum_{b=1}^{4} L_{ab}^{(k)(j)}\,Y^{(j)b}$$

where $\tilde{K}$, $L_{ab}^{(j)(k)}$ are known functions of the thermodynamic variables and are referred to as phenomenological coefficients.

For convenience we define the relaxation scalar K by

$$K \equiv \frac{\tilde{K}}{T} . \tag{1.39b}$$

At this point we follow earlier workers ([3],[4],[8]) who have introduced a new thermodynamic variable which together with the phenomenological laws essentially replaces the $C_{(k)}$ (see Remark below).

Definition 7: Let ζ be a function whose values on an initial hypersurface are prescribed and which everywhere satisfies

$$\frac{d\zeta}{dt} = J. \tag{1.40}$$

The function ζ is called the relaxation variable.

Remark: By substituting (1.40) and (1.39a) into (1.27) we see that ζ and $C_{(k)}$ determine each other (up to initial values) and thus the thermodynamic variables ρ, s, ζ are equivalent to the variables ρ, s, $C_{(k)}$.

Proposition 11: For a relativistic fluid undergoing a chemical reaction in the region for which the phenomenological laws (1.39),(1.39a) are valid, the following hold:

$$\left(\frac{d\zeta}{dt}\right)^2 + K\left(\frac{\partial e}{\partial \zeta}\right)_{\rho,s}\frac{d\zeta}{dt} - \frac{K}{\rho}\sum_k \left(\frac{\partial e}{\partial C_{(k)}}\right)_{\rho,s} \nabla_a H^{(k)a} = 0 \tag{1.41}$$

$$\left(\frac{ds}{dt}\right)^2 - \frac{K}{T}\left(\frac{\partial e}{\partial \zeta}\right)^2_{\rho,s}\frac{ds}{dt} - \frac{1}{\rho T}\left(\frac{\partial e}{\partial \zeta}\right)^2_{\rho,s}\sum \left(\frac{\partial e}{\partial C_{(k)}}\right)_{\rho,s} \nabla_a H^{(k)a} = 0. \tag{1.42}$$

Proof: Regarding e as a function of s, ρ, ζ we have

$$\frac{de}{dt} = \left(\frac{\partial e}{\partial \rho}\right)_{\zeta,s}\frac{d\rho}{dt} + \left(\frac{\partial e}{\partial s}\right)_{\rho,\zeta}\frac{ds}{dt} + \left(\frac{\partial e}{\partial \zeta}\right)_{\rho,s}\frac{d\zeta}{dt} . \tag{1.43}$$

Also from (1.27)

$$(1.44)\quad \frac{de}{dt} = \frac{p}{\rho^2}\frac{d\rho}{dt} + T\frac{ds}{dt} + \sum_k \frac{\partial e}{\partial C_{(k)}}\Big)_{\rho,s} \frac{1}{\rho}\nabla_a H^{a(k)} - \frac{1}{\rho}\sum \frac{\partial e}{\partial C_{(k)}} M_0^{(k)} b^{(k)} J.$$

Using (1.33a) and (1.40), this becomes

$$(1.45)\quad \frac{de}{dt} = \frac{p}{\rho^2}\frac{d\rho}{dt} + T\frac{ds}{dt} + \sum_k \frac{\partial e}{\partial C_{(k)}}\frac{1}{\rho}\nabla_a H^{a(k)} - \frac{1}{\rho} A \frac{d\zeta}{dt}.$$

Equations (1.43),(1.45) are valid for any change of the system from one state to another. In particular, they are valid for $ds = d\rho = 0$. Hence equating the right sides of (1.43),(1.45) and putting $ds = d\rho = 0$, we find

$$(1.46)\quad \sum \frac{\partial e}{\partial C_{(k)}}\Big)_{\rho,s} \frac{1}{\rho}\nabla_a H^{a(k)} - \frac{1}{\rho} A \frac{d\zeta}{dt} = \frac{\partial e}{\partial \zeta}\Big)_{\rho,s} \frac{d\zeta}{dt}.$$

The equation (1.46) is a point function which does not depend on the choice of paths and so is valid in general. Using (1.39b) and $\frac{d\zeta}{dt} = J$ in (1.46) we obtain

$$(1.47)\quad \frac{1}{\rho}\sum \frac{\partial e}{\partial C_{(k)}}\Big)_{\rho,s} \nabla_a H^{a(k)} - \frac{1}{K}\left(\frac{d\zeta}{dt}\right)^2 = \frac{\partial e}{\partial \zeta}\Big)_{\rho,s} \frac{d\zeta}{dt}.$$

Putting (1.27) into (1.24a) yields

$$(1.48)\quad \frac{ds}{dt} = \frac{-1}{T\rho}\sum \frac{\partial e}{\partial C_{(k)}} \nabla_a H^{(k)a} + \frac{1}{T\rho}\sum \frac{\partial e}{\partial C_{(k)}} b^{(k)} M_0^{(k)} J,$$

$$(1.49)\quad \frac{ds}{dt} = \frac{-1}{T\rho}\sum \frac{\partial e}{\partial C_{(k)}} \nabla_a H^{(k)a} + \frac{1}{\rho T} A J.$$

Using (1.46)

$$(1.50)\quad \frac{ds}{dt} = \frac{(-1)A}{T\rho}\frac{d\zeta}{dt} - \frac{1}{T}\frac{\partial e}{\partial \zeta}\Big)_{\rho,s} \frac{d\zeta}{dt} + \frac{1}{\rho T} A J$$

$$(1.51)\quad \frac{ds}{dt} = \frac{-1}{T}\frac{\partial e}{\partial \zeta}\Big)_{\rho,s} \frac{d\zeta}{dt}.$$

To eliminate $\frac{d\zeta}{dt}$ we proceed as follows for $\frac{\partial e}{\partial \zeta} \neq 0$:

$$\frac{d\zeta}{dt} = -T \left(\frac{\partial e}{\partial \zeta}\right)^{-1}_{\rho,s} \frac{ds}{dt} \tag{1.52}$$

$$\left(\frac{d\zeta}{dt}\right)^2 = T^2 \left(\frac{\partial e}{\partial \zeta}\right)^2_{\rho,s} \left(\frac{ds}{dt}\right)^2 . \tag{1.53}$$

Put (1.52),(1.53) into (1.47) and get

$$\frac{1}{\rho} \sum_k \frac{\partial e}{\partial C_{(k)}} \nabla_a H^{(k)a} - \frac{1}{K} T^2 e_\zeta^{-2} \left(\frac{ds}{dt}\right)^2 = e_\zeta (-T) e_\zeta^{-1} \frac{ds}{dt} . \tag{1.54}$$

Remarks: In the special case that $\sum \frac{\partial e}{\partial C_{(k)}} \nabla_a H^{(k)a} = 0$ the equations (1.47),(1.54) reduce for Newtonian physics to the equation studied by Stakhanov and Stupochenko [3] and Coburn [6],[7]. Specifically these authors have assumed that all the chemical components have the same 3-velocity so that $H^{(k)a} = 0$, $a = 1,2,3$, which implies $\nabla_a H^{(k)a} = 0$.

Below we collect together and write out the Einstein field equations for the form of the stress-energy tensor derived in (1.10c) and the results of Proposition 3 and Proposition 11.

Proposition 12: For a relativistic fluid undergoing a chemical reaction in the region for which the linear phenomenological laws (1.39) and (1.39a) are valid the following hold:

$$R_{ik} - \frac{1}{2} R g_{ik} = \lambda T_{ik} \tag{1.55}$$

$$T_{ik} = \eta u_i u_k - p g_{ik} \tag{1.56}$$

$$\eta \equiv \rho(1 + e) + p \tag{1.57}$$

$$u^a u_a = 1. \tag{1.58}$$

Conservation of mass:

(1.59) $$\nabla_a(\rho\, u^a) = 0.$$

Rate equation:

(1.60) $$\left(\frac{d\zeta}{dt}\right)^2 = -K\,\frac{\partial e}{\partial \zeta}\,\frac{d\zeta}{dt} + \frac{K}{\rho}\sum_k \frac{\partial e}{\partial C_{(k)}}\,\nabla_a H^{(k)a}.$$

Entropy equation:

(1.61) $$\left(\frac{ds}{dt}\right)^2 = \frac{K}{T}\,e_\zeta^2\,\frac{ds}{dt} + \frac{K}{T^2\rho}\sum_k \frac{\partial e}{\partial C_{(k)}}\,\nabla_a H^{(k)a}.$$

Linear phenomenological laws:

(1.62) $$H_a^{(k)} = \sum_{j=1}^{m}\sum_{b=1}^{4} L_{ab}^{(k)(j)}\, Y^{(j)b}.$$

Various authors [11] have analyzed the equilibrium system by considering the equations of motion, i.e.

(1.63) $$\gamma_{ab}\nabla_s T^{sb} = 0$$

(1.64) $$\gamma_{ab} \equiv g_{ab} - u_a u_b$$

which follows directly from the fact that the left side of (1.55) has vanishing divergence. For a chemically reacting fluid it follows from (1.10),(1.19),(1.56)-(1.58) and (1.63) that the equations of motion are

(1.65) $$u^a\nabla_a u_b = \frac{1}{\eta}\,\gamma_b^a(A\nabla_a\rho + B\nabla_a s + C\nabla_a\zeta)$$

where

(1.66) $$A \equiv \frac{\partial p}{\partial \rho}, \quad B \equiv \frac{\partial p}{\partial s}, \quad C \equiv \frac{\partial p}{\partial \zeta}.$$

We will put the relevant equations in a form which will be convenient for much of our later work. To do this we introduce some notation.

Definition 8: Let A^a be the array

$$A^a \equiv \begin{pmatrix} u^a & \rho\delta^a_i & 0 & 0 \\ -\gamma^a_b A & \eta u^a g_{ib} & -\gamma^a_b C & -\gamma^a_b B \\ 0 & 0 & u^a & 0 \\ 0 & 0 & 0 & u^a \end{pmatrix} \tag{1.67}$$

and denote by u the 4×1 column matrix of unknown functions defined by

$$u \equiv (\rho, u^i, \zeta, s)^{tr} \tag{1.68}$$

where tr means matrix transpose.

Proposition 13: If all the chemical components have the same 4-velocity then

$$A^a \nabla_a u + D = 0 \tag{1.69}$$

where

$$D \equiv \left(0,0,Ke_\zeta, \frac{-Ke^2_\zeta}{T}\right)^{tr}. \tag{1.70}$$

Proof: Expand (1.69) by formal matrix multiplication. The resulting equations are (1.10),(1.65),(1.61),(1.62).

In the next two chapters we shall be concerned with solutions of (1.69) which have discontinuous second derivatives over some hypersurface. Thus we will study the following differentiated form of (1.69) which we write out for ease of reference:

$$A^a \nabla_b \nabla_a u + \nabla_b A^a \nabla_a u + \nabla_b D = 0. \tag{1.71}$$

CHAPTER II

DISCONTINUITY SURFACES

II.I. Introduction

From both the mathematical and physical point of view it is fruitful to study, for a given system of partial differential equations, the surfaces over which the solutions may have discontinuous derivatives. Such surfaces which are referred to as surfaces of weak discontinuities (or somewhat loosely as discontinuity surfaces) are thought of as limits of thin layers of space-time in which the derivatives of the solutions are changing rapidly. These surfaces have been investigated by a number of authors [7],[12],[18],[19] to determine their mathematical and physical properties. For example it is known, in the case of classical equilibrium hydrodynamics, that the boundary of a region of rarefying flow is a surface of weak discontinuity [20] and that discontinuity theory can be used to obtain waves which propagate relative to the medium with the velocity of sound [7].

Using a completely different approach (small perturbation theory) Stupochenko and Stakhanov [3] established the existence of certain speeds of propagation corresponding to the low and high frequency limits of plane wave solutions to the linearized equations of relaxation hydrodynamics. Later Coburn [7], using discontinuity theory, showed that for the non-linear case another limiting speed exists and that the partial differential equation for the discontinuity surface can be given a simple expression in terms of these limiting speeds. Below we will derive relativistic generalizations of these limit speeds and the equations for the associated discontinuity surfaces corresponding to

various (C^m,C^n) classes of solutions and of the relaxation scalar K. For purposes of interpretation the discontinuity conditions for the isentropic system will be derived first.

From the equation for the discontinuity surfaces we obtain certain well known curves (the bicharacteristics) which are known to play an important role in various aspects of the theory, particularly in the propagation of discontinuities. We will find a useful representation for the tangents to these curves which generalizes work due to Courant and Hilbert [18]. In addition we establish certain relationships between the bicharacteristic directions, the relativistic velocity of propagation and the Newtonian idea of wave motion.

II.II. Equation for the Surfaces of Discontinuity

We recall the following basic definitions [13].

Definition 1: Let U be an open set of R^r, V an open subset of R^s and f: U → V defined by $f(x^1,\ldots,x^r) = (f^1,\ldots,f^s)$. Suppose ϕ is a real valued function of class C^1 defined on U, and let S(c) denote the hypersurface defined by $\phi(x^1,\ldots,x^r) = c$. The function f is piecewise of class C^m $(0 \leq m \leq \infty)$ in U if and only if: (i) f is of class C^m on U - S(0), and (ii) the functions given by f and its partial derivatives, of order $\leq m$, restricted to S(c) converge uniformly as c approaches zero. The surface S(0) is called a discontinuity surface of the function f and will be denoted by Σ. If f is of class C^m on U and also piecewise C^n on U, we write $f \in (C^m,C^n)$ on U.

Definition 2: Whenever f is a function (or more generally the component, with respect to some local coordinates) defined on $U \subset R^m$ and Σ is a discontinuity hypersurface we define the jump of f across Σ, denoted

[f], by the equation

$$[f] \equiv \lim_{c\to 0^+} f - \lim_{c\to 0^-} f. \tag{2.1}$$

In the first part of this chapter we will be concerned with the algebraic decomposition of jump quantities and with applying the representations to derive a partial differential equation for the discontinuity surfaces. An elementary but basic tool which is used throughout is the following [7]:

Proposition 1: If $f, g_{ab} \in (C^1, C^3)$, then there exists a function F defined on Σ such that

$$[\nabla_a \nabla_b f] = n_a n_b F.$$

In a recent work, Coburn [15] has obtained representations for the discontinuities in the second derivatives of the metric provided the coordinate transformations on the manifold are (C^3, C^∞). Thus we will assume the foregoing smoothness requirement. We will also assume that the discontinuity hypersurfaces are nowhere null, i.e.,

$$N \equiv n^a n_a = \pm 1$$

where

$$n_a \equiv \frac{\partial_a \phi}{(|g^{bc}\partial_b\phi\partial_c\phi|)^{1/2}}.$$

Proposition 2: If $g_{ab}, u \in (C^1, C^3)$ is a solution of (1.71), then there exists a scalar $\mathfrak{C}$ and a tensor U_{mp} defined on Σ such that

$$[\nabla_{jk} u_p] = n_j n_k \{N n^m U_{mp} - n_p \mathfrak{C}\}. \tag{2.2}$$

Proof: The fact that $g_{ab} \in (C^1, C^3)$ implies that $[\Gamma^s_{kp}]$ vanishes. This and the hypothesis $u \in (C^1, C^3)$ gives

$$n^j[\nabla_{jk}\, u_p] = n^j[\partial_{jk}\, u_p] - n^j[\partial_j \Gamma^s_{kp}]u_s. \tag{2.3}$$

It is well known [15] that there exist scalars $\bar{A}$, $\bar{D}$ and a vector t tangent to Σ such that

$$n^j[\partial_j \Gamma^s_{kp}] = N\{\frac{\bar{A}}{2}\, n_k n_p n^s + \bar{D} n_k n_p t^s\} \tag{2.4}$$

which when substituted in (2.3) yields

$$n^j[\nabla_{jk}\, u_p] = n^j[\partial_{jk}\, u_p] - N n_k n_p (\frac{\bar{A}}{2}\, n^s + \bar{D} t^s) u_s. \tag{2.5}$$

Again from the class of u_p and Hadamard's Theorem [13] we know

$$v^j[\nabla_{jk}\, u_p] = 0 \tag{2.6}$$

for v^j tangent to Σ. This means there is a tensor A_{kp} such that

$$[\nabla_{jk}\, u_p] = n_j A_{kp}. \tag{2.7}$$

From (2.5),(2.7) we conclude

$$[\nabla_{jk}\, u_p] = N\{n^j[\partial_{jk}\, u_p] - N n_k n_p \mathfrak{A}\} \tag{2.8}$$

where

$$\mathfrak{A} \equiv \frac{\bar{A}}{2}\, n^s u_s + \bar{D}\, t^s u_s. \tag{2.9}$$

Also it is known [15] that $u \in (C^1, C^3)$ implies that there exists a tensor U_{kp} such that

$$[\partial_{jk}\, u_p] = n_j U_{kp}. \tag{2.10}$$

The smoothness of u_p allows us to interchange the order of differentiation to find

$$n_k U_{jp} = [\partial_{kj}\, u_p] = n_j U_{kp}$$

$$U_{kp} = N n_k n^j U_{jp}. \tag{2.11}$$

Substituting (2.11) into (2.10) and the resulting equation into (2.8)

yields the conclusion.

Suppose we take the right and left hand limits (in the sense of Definition 2) of a solution to (1.71) and subtract. Then because of Proposition 1 and Proposition 2 there exist scalars P, Q, S, k and a vector

$$U_p \equiv N\, n^a\, U_{ap} - n_p \mathcal{Q}$$

such that the resulting system can be written in the form

$$A^a\, n_a\, r + \bar{v}^{tr} = 0 \tag{2.11a}$$

where

$$r \equiv (P,\ U_j,\ Q,\ S)^{tr} \tag{2.11b}$$

$$\bar{v} \equiv k e_\zeta (0,\ 0,\ 1,\ \frac{-e_\zeta}{T}) \tag{2.11c}$$

and

$$[\nabla_b\, K] = n_b k. \tag{2.11d}$$

The system (2.11a) and the following generalize, to the inhomogeneous case, a definition originally due to Courant [18].

Definition 3: Let the matrix r in (2.11b) and $\ell \equiv (\ell_1, m_b, \ell_2, \ell_3)$, where m_b is a vector and ℓ_α, $\alpha = 1,2,3$, are scalars, denote solutions to the equations (2.11a) and

$$\ell A^a n_a + \bar{v} = 0. \tag{2.11e}$$

The equations which determine the discontinuity surfaces are obtained by studying (2.11a). The solutions to (2.11e) will play an important role in the next chapter.

In order to interpret results which will be proved subsequently for the non-equilibrium system and to motivate the choice of the continuity class of K we will first derive some results for the equilibrium systems.

Definition 4: The function L is defined on a neighborhood of Σ according to

$$L \equiv n^a u_a. \tag{2.11f}$$

Definition 5: The system for a perfect fluid consists of (1.55), (1.56),(1.58) together with the definition $\eta \equiv \rho + p$ [17].

Proposition 3: If $g_{ab} \in (C^1,C^3)$ and $u \in (C^0,C^2)$ is a solution for a perfect equilibrium fluid, then $L = 0$ or $L^2/N = A/A - 1$.

Proof: If we take the left and right hand limits of (1.55),(1.56) and subtract, we find that (2.11a) has the form

$$A^a n_a r = 0.$$

In order that this have a non-trivial solution the determinant must vanish, and this requirement proves the assertion.

Definition 6: The isentropic equilibrium system consists of (1.55)-(1.59) and

$$\frac{ds}{dt} = 0. \tag{2.12}$$

The proof of the following is essentially the same as that of Proposition 3 and is omitted.

Proposition 4: Let $g_{ik} \in (C^1,C^3)$ and $u \in (C^0,C^2)$ be a solution of the isentropic equilibrium system. Then the discontinuity surfaces are given by $L = 0$ or

$$\mathfrak{E} \equiv \frac{L^2}{N} - \frac{\rho A}{\rho A - \eta} = 0. \tag{2.12a}$$

Remarks: The velocities of propagation [17] corresponding to the results of Proposition 3 and Proposition 4 are respectively $A^{1/2}$, $(\rho A/\eta)^{1/2}$. In the non-relativistic limit it is known [6] that $\eta \to \rho$ so

that both these speeds reduce to the Newtonian speed.

The results in Proposition 3 and Proposition 4 are for equilibrium systems; however as stated in the next result, the same characteristic condition will hold for the non-equilibrium system (1.71) if K is smooth. Later we will treat the case where K is not smooth in order to include non-equilibrium effects.

Proposition 5: For $K \in (C^1,C^2)$ and $u, g_{ab} \in (C^1,C^3)$ a solution of (1.71) we have $L = 0$ or

$$\mathfrak{G} = 0. \tag{2.12b}$$

The equilibrium speeds which are introduced below are the relativistic equilibrium speed $\rho A/\eta$ (see Remark following Proposition 4) modified by non-equilibrium contributions.

Definition 7: The limiting speeds c_0, c_∞ are defined by

$$c_0^2 \equiv \frac{\rho A}{\eta} - \frac{C^2}{\rho\eta G}, \quad G \equiv e_{\zeta\zeta} \tag{2.13}$$

$$c_\infty^2 \equiv \frac{\rho A}{\eta} - \frac{BC}{\rho\eta F}, \quad F \equiv e_{\zeta\zeta}. \tag{2.14}$$

Furthermore, for ease in comparing our results to the work of others [5], we write

$$a \equiv (-1)GT \qquad b \equiv e_q F. \tag{2.15}$$

Proposition 6: If $u, g_{ab} \in (C^1,C^3)$, $K \in (C^0,C^1)$, $e_\zeta \in C^2$ is a solution to (1.71), and Σ is a non-equilibrium hypersurface, then

$$0 = a\{\frac{L^2}{N}(1 - c_0^2) + c_0^2\} + b\{\frac{L^2}{N}(1 - c_\infty^2) + c_\infty^2\}. \tag{2.16}$$

Proof: As in the proof of Proposition 3 we are led to a system of equations of the form (2.11a). The first and second equations of this system are, respectively,

$$\rho g^{st} n_s (N n^m U_{mt} - n_t \mathcal{Q}) + u^a n_a P = 0$$

$$\eta u^a n_a (N n^m U_{mi} - n_i \mathcal{Q}) + (u_i u^a - \delta_i^a) n_a (AP + BS + CQ) = 0$$

which can be rewritten

(2.17) $$\rho N (n^t n^m U_{mt} - \mathcal{Q}) + LP = 0$$

(2.18) $$\eta L (N n^m U_{mi} - n_i \mathcal{Q}) + (u_i u^a - \delta_i^a) n_a (AP + BS + CQ) = 0.$$

Forming the scalar product of (2.18) with n^i and using the resulting equation (for $L \neq 0$) to eliminate the first factor of (2.17) we find

(2.19) $$P\{\eta L^2 - (L^2 - N)\rho A\} - \rho (L^2 - N)(BS + CQ) = 0.$$

From the second and third equations of (2.11a) we have respectively

(2.20) $$LS = \frac{k e_q^2}{T}$$

(2.21) $$LQ = -k e_q .$$

Thus

(2.22) $$S = (-1) \frac{e_q}{T} Q.$$

The condition $e_\zeta \in C^2$ implies

(2.23) $$e_{\zeta\rho} P + e_{\zeta s} S + e_{\zeta\zeta} Q = 0.$$

Eliminating P, S from (2.19) by means of (2.22),(2.23), dividing the resulting equation by Q and rearranging yields (2.16).

Remarks: In the classical limit, the speed c_0 reduces to the low frequency limit speed first obtained by Stupochenko and Stakhanov [3] for the linearized Newtonian system. The second limit speed reduces to that obtained by Coburn [7] for the non-linear Newtonian case.

The same condition (2.16) results for $u \in (C^0, C^1)$, K piecewise continuous and $e_q \in C^1$.

II.III. Bicharacteristics

We begin with a standard definition [18].

Definition 8: Let $\Omega(x_a,u,\xi_a) \equiv 0$ where $\xi_a \equiv \partial_a\phi$ be the characteristic equation for a system of differential equations. The integral curves of $\partial\Omega/\partial\xi_a$ are called the bicharacteristic of the system.

Courant [18] has found another representation for the tangents to the bicharacteristics which he showed is useful in studying the propagation of discontinuities. We will now obtain a generalization of his representation. To do this we will list some obvious facts about the solutions ℓ, r to systems (2.11a), (2.11e). The first result (Proposition 7) is for propagation with equilibrium speed while Proposition 8 treats the non-equilibrium speeds.

Proposition 7: For $\mathfrak{E} = B = C = 0$ and $K \in (C^1,C^2)$ we have

$$r = n^c U_c \frac{(-\rho)}{L}\left(1, \frac{-A}{\eta L}(Lu_i - n_i), 0, 0\right) \tag{2.24}$$

$$\ell_2 = \ell_3 = 0, \quad m_b = \frac{-\rho}{\eta L}\ell_1 n_b \tag{2.25}$$

where $n^c U_c$ and ℓ_1 are undetermined.

Proof: With $k = B = C = 0$, the equations (2.11a),(2.11e) are homogeneous systems which can easily be solved to give (2.24),(2.25).

Proposition 8: If $\mathfrak{E} \neq 0$ and $K \in (C^0,C^1)$, then

$$r = ke_\zeta\left(\frac{\rho}{L}\frac{(L^2-N)}{\mathfrak{E}}\left(\frac{Be_\zeta}{T} - C\right), (n_b - Lu_b)\left(\frac{Be_\zeta}{T} - C\right), \frac{e_\zeta}{LT}, \frac{-1}{L}\right)^{tr} \tag{2.26}$$

$$\ell = \frac{ke_\zeta}{L}\left(0, 0, \frac{e_\zeta}{T}, -1\right). \tag{2.27}$$

Proof: Since $\mathfrak{E} \neq 0$, it follows from the first two equations of (2.11e) that $\ell_1 = m_i = 0$. The last two equations of (2.11e) are

$$L\ell_2 = \frac{ke_\zeta}{T} \tag{2.28}$$

$$L\ell_3 = -ke_\zeta \tag{2.29}$$

which proves (2.27). Again using $\mathfrak{E} \neq 0$ we can solve the inhomogeneous system (2.11a) in terms of k and find (2.26).

The notion which generalizes Courant's [18] representation for the tangents to the bicharacteristics is

Definition 9: Let $\bar{b}^i$ be defined by

$$\bar{b}^i \equiv (\ell A^i - n^i\bar{v})\tilde{r} \tag{2.30}$$

where for $\mathfrak{E} = 0$

$$\tilde{r} \equiv \frac{(-1)\rho}{L}\left(1, \frac{-A}{\eta L}(Lu_i - n_i), 0, 0\right)^{tr} \tag{2.31}$$

and for $\mathfrak{E} \neq 0$, we write $\tilde{r} \equiv \frac{r}{k}$.

Proposition 9: The tangent to the bicharacteristics determined by a solution u, $g_{ab} \in (C^1, C^3)$ of the equilibrium system or a solution u, $g_{ab} \in (C^1, C^3)$, $K \in (C^0, C^2)$ of the non-equilibrium system (1.71), is colinear with $\bar{b}^i$. Furthermore, for $\mathfrak{E} = 0$ we have

$$\bar{b}^i = \ell_1 \frac{(-2)\rho^2 A}{\eta L^2}\left(\frac{1}{L} u^i + n^i\right) \tag{2.31a}$$

and for $\mathfrak{E} \neq 0$

$$\bar{b}^i = \frac{k\, e_\zeta^2}{L}\left(1 + \frac{e_\zeta^2}{T^2}\right)\left(\frac{1}{L} u^i + n^i\right).$$

Proof: From Proposition 5 and Proposition 6 we see that the characteristic condition for the equilibrium speeds and the non-equilibrium speeds can be put in the form

$$\Omega \equiv \tilde{A}L^2 + \tilde{B} = 0 \tag{2.32}$$

where the functions $\tilde{A}$, $\tilde{B}$ are independent of ξ_a. Thus the tangent to the bicharacteristic is given by

$$\frac{\partial \Omega}{\partial \xi_a} = \tilde{A}\, 2L \frac{\partial L}{\partial \xi_a} = \tilde{A}\, 2L \frac{(u^a + Ln^a)}{(-\xi^s \xi_s)^{1/2}}. \tag{2.33}$$

Now substituting $k = 0$, (2.31) and (2.25) into (2.30) we find

$$\bar{b}^i \equiv \ell A^i \tilde{r} = \ell_1 \frac{(-2)\rho^2 A}{\eta L}\left(\frac{u^i}{L} + n^i\right) \tag{2.34}$$

which proves (2.31a).

Similarly, substituting (2.26),(2.27) into (2.30) we have for $\xi \neq 0$

$$\bar{b}^i = \frac{k\, e^2_\zeta}{L}\left(1 + \frac{e^2_\zeta}{T^2}\right)\left(\frac{u^i}{L} + n^i\right) \tag{2.35}$$

which proves (2.31b). The colinearity of the bicharacteristic direction and $\bar{b}^i$ follows by comparing (2.33),(2.34) and (2.35).

II.IV. Bicharacteristic Directions and the Relativistic and Classical Notions of Wave Propagation

The bicharacteristic directions obtained above can be given a simple but interesting interpretation which does not seem to have been previously noted. Let us first recall that Lichnerowicz [17] defines the relativistic speed of propagation v of the family of hypersurfaces $f(x_i)$ equals constant, by

$$v^2 \equiv \lim_{\varepsilon \to 0} \frac{\eta_1^2}{t^a t_a} = \frac{L^2}{L^2 + 1} \tag{2.36}$$

where η_1 is the distance, along the normal, from the hypersurface given by ε equals zero to a neighboring hypersurface, and t_a is the difference between the vectors representing the separation measured respectively along the normal and along u_a. We then observe that it follows easily from the relations given by Lichnerowicz that t_a lies along $\bar{b}_a$. Thus we see that definition (2.36) is equivalent to saying that the speed of propagation of a family of hypersurfaces is the speed, as measured by an

observer moving along the bicharacteristic, of a point which moves along the normal to the given family of hypersurfaces.

For applications in the next chapter we will establish the relationship between the relativistic speed v defined above and the Newtonian notion of wave propagation. Specifically, suppose that there exist local, orthonormal, flow oriented coordinates in which the hypersurface in 4-space has the representation

$$\phi \equiv \psi(x,y,z) - wt \equiv 0 \tag{2.37}$$

where $\bar{v}$ is constant. Classically (2.37) is regarded as defining a one parameter family of surfaces S(t). The speed of propagation, say "a" of S(t) is determined by the motion, parameterized by t, of a point moving along the normal, in 3-space to S(t). Then "a" must satisfy

$$\frac{d\phi}{dt} = \sum_{\alpha=1}^{3} \partial_\alpha \psi \; a \frac{\partial_\alpha \psi}{\left[\sum_{\beta=1}^{3} (\partial_\beta \psi)^2\right]^{1/2}} - w = 0. \tag{2.38}$$

Note that (2.38) expresses the orthogonality, in flat 4-space, of the normal vector $(\partial_\alpha \psi, w)$ and $\left(a \frac{\partial_\alpha \psi}{\left[\sum_{\alpha=1}^{3} (\partial_\alpha \psi)^2\right]^{1/2}}, 1\right)$.

Furthermore, a short calculation using $n^i n_i = -1$ and the fact that in flow oriented coordinates $L = n_0$, yields

$$a^2 = \frac{w^2}{\sum_{\alpha=1}^{3} (\partial_\alpha \psi)^2} = \frac{L^2}{L^2 + 1} \equiv v^2. \tag{2.39}$$

Thus if the flow oriented coordinates are regarded as an inertial frame, the classical speed agrees with the relativistic speed.

With the above as motivation we can arrive at a more abstract model of wave motion which again shows the basic role of the bicharacteristic direction and proves that v may be interpreted as the speed of the wave relative to the fluid. Thus denote by w_i the tangent to the path of a

point on the history Σ of the wave, and by N_i the vector

$$N_i \equiv \frac{\gamma_i^a n_a}{(-\gamma_i^a n_a \gamma^{si} n_s)^{1/2}} .$$

We require that the velocity of the wave relative to the flow be along $\gamma^{ai} n_a$. (This is the case in (2.39).) Thus there is an "a" such that

$$w_i - u_i = aN_i . \tag{2.40}$$

Since w_i is tangent to Σ we can solve (2.40) subject to $n^i w_i = 0$ to find

$$a = \frac{L}{(L^2 + 1)^{1/2}} = v. \tag{2.41}$$

If we substitute the value of "a" in (2.41) into (2.40) we find that w_i is along the bicharacteristic and

$$w_i = \frac{L}{L^2 + 1}\left(\frac{u_i}{L} + n_i\right) . \tag{2.42}$$

Thus the classical velocity, a, along N^i relative to u^i agrees with v.

CHAPTER III

GROWTH AND DECAY OF THE DISCONTINUITIES

III.I. Introduction

In this chapter we will consider the geometrical and analytical aspects of the growth and decay of the discontinuity. The geometrical considerations are concerned with the behavior of the mean curvature of Σ along the bicharacteristics. We begin by deriving an ordinary differential equation which governs the eigenvalues of the tensor $\nabla_a t_c$ where t^c is tangent to a geodesic. Next we show that under reasonable physical hypotheses the bicharacteristics are geodesics in the metric g_{ab}. These hypotheses also enable us to prove the integrability of the distribution $b^{\perp} \cap \Sigma$ and thereby recover a moving two dimensional wave. Using these results together with Einstein's field equations and the Bianchi identity we obtain a system of two non-linear ordinary differential equations for the mean curvature and Gaussian curvature of the wave. The solutions of this system of differential equations allow us to explicitly calculate the proper time, s_1, at which the mean curvature becomes unbounded. This in turn (see below in introduction) means that the discontinuity r becomes unbounded at s_1. The value of s_1 is compared, by an appropriate series expansion, to the corresponding Newtonian value. The role of the mean curvature in the Newtonian theory of the propagation of discontinuities was established, by quite different methods, in the work of Thomas [22].

Section III.III is devoted to obtaining and studying the differential equation which governs the discontinuity, and in which the mean curvature plays a basic role. The analytical aspect of our method originates with

the treatment of linear partial differential equations by Courant and Hilbert [18], and was applied to a special case of the non-linear Newtonian equations in Coburn and Sommerfield [16]. The solutions of the final equation become unbounded at the finite proper time s_1, if the mean curvature becomes (negatively) unbounded at s_1.

III.II. Mean Curvature and Wave Motion

When dealing with an indefinite metric it is convenient to agree on the following

Convention: Given a vector v^a, such that $v^a v_a \neq 0$, we define $\varepsilon(v)$ to be ± 1 according to whether $v^a v_a$ is greater than or less than zero. For an arbitrary vector w^a, we will refer to $w^a w_a$ as the magnitude of w^a.

Proposition 1: Let y^c, v^c be orthogonal vector fields defined on an open set U, where v^a is of constant magnitude. Furthermore suppose that the integral curve C of y^c is a geodesic. If $\nabla_a y_c$ is a symmetric tensor on C, such that on C,

$$v^a \nabla_a y_c = k v_c, \tag{3.1}$$

then along C the function k satisfies

$$\varepsilon(v)\frac{dk}{ds} + \varepsilon(v)k^2 = (-1)v^u y_s y^b v_r R^{rs}{}_{ub}. \tag{3.1a}$$

Proof: From the orthogonality of v^s, y^s on U we have

$$v^s \nabla_u y_s + y_s \nabla_u v^s = 0 \tag{3.2}$$

which by (3.1) and the symmetry of $\nabla_u y_s$ implies

$$k v_u + y_s \nabla_u v^s = 0. \tag{3.3}$$

Operating with $v^u y^b \nabla_b$ on (3.3) we find

$$(3.4)\qquad v^u[ky^b\nabla_b v_u + v_u y^b\nabla_b k + y_s y^b\nabla_{bu}v^s + (y^b\nabla_b y_s)\nabla_u v^s] = 0.$$

The first term vanishes because v^u is of constant length, and the last term vanishes along C since C is a geodesic with tangent y^s. Hence along C

$$(3.5)\qquad \varepsilon(v)y^b\nabla_b k + v^u y_s y^b\nabla_{bu}v^s = 0.$$

On the other hand, as in (3.3), we have

$$(3.6)\qquad kv_b + y_s\nabla_b v^s = 0.$$

Applying $y^b v^u\nabla_u$ to this equation and using the orthogonality of y^s and v^s we see

$$(3.7)\qquad ky^b v^u\nabla_u v_b + y^b y_s v^u\nabla_{ub}v^s + (y^b\nabla_b v^s)v^u\nabla_u y_s = 0.$$

Rewriting the last term with the aid of (3.1) yields

$$(3.8)\qquad (-1)kv_b v^u\nabla_u y^b + y^b y_s v^u\nabla_{ub}v^s + (y^b\nabla_b v^s)kv_s = 0,$$

$$(3.9)\qquad (-1)k^2\varepsilon(v) + y^b y_s v^u\nabla_{ub}v^s = 0.$$

Subtracting (3.9) from (3.5) we obtain

$$(3.10)\qquad \varepsilon(v)y^b\nabla_b k + v^u y_s y^b\nabla_{bu}v^s + \varepsilon(v)k^2 - y^b y_s v^u\nabla_{ub}v^s = 0.$$

Using the basic formula for interchanging the order of covariant differentiation, (3.10) becomes

$$(3.11)\qquad \varepsilon(v)y^b\nabla_b k + \varepsilon(v)k^2 + v^u y_s y^b v_r R^{rs}{}_{ub} = 0,$$

which is the desired equation.

Standard references [20],[22] for Newtonian fluid mechanics discuss disturbances which propagate into a region which is at rest and in which the thermodynamic variables are constant. A four dimensional formulation is the following.

Definition 1: We will say that a disturbance, with history Σ, is

propagating into a fluid which is uniform provided that at each point $m \in \Sigma$ we know $\nabla_a u_c$ and $\nabla_a \theta$ (where θ is any thermodynamic variable) vanish.

Under the conditions of Definition 1, we can prove directly from the Hamiltonian system determined by (2.32) that the bicharacteristics (i.e., the integral curves of $\bar{b}^a$) are geodesics on Σ. However, we will obtain this result and additional useful information by introducing another vector field.

Definition 2: Let M^a be defined on a 4-neighborhood U of Σ by taking M^a to be the tangent to the geodesic whose initial tangent at $m \in \Sigma$ is the vector n^a. Denote by $\hat{B}_a$ the vector field on U defined by

$$\hat{B}^a \equiv \frac{1}{B} B^a \tag{3.12}$$

where

$$B^a \equiv \frac{1}{M^s u_s} u^a + M^a \tag{3.13}$$

$$B \equiv (B^a B_a)^{1/2}. \tag{3.14}$$

Remark: On Σ, the scalars $M^s u_s$ and $B^a B_a$ have the values $L \neq 0$ and $(L^2+1)/L^2$ respectively. Thus there is a neighborhood, say U, on which $B \neq 0$, and thus $\hat{B}^a$ is well defined. It follows from Proposition 9 of Chapter II that, on Σ, the vector $\hat{B}^a$ is collinear with the tangent to the bicharacteristic.

Proposition 3: If a wave moves into a uniform fluid, then there exist, on Σ, orthogonal vector fields v^a, $w^a \in b^{\perp} \cap \Sigma$ and scalars k, ℓ such that on Σ

$$\nabla_a \hat{B}_c = (-1)(k v_a v_c + \ell w_a w_c) = \frac{1}{B} \nabla_a M_c \tag{3.15}$$

$$v^a v_a = w^a w_a = (-1). \tag{3.15a}$$

Proof: According to Definition 2, we have

$$B^c B_c = \frac{1}{(M^s u_s)^2} + \frac{2}{M^s u_s} u^c M_c - 1 \tag{3.16}$$

$$\nabla_a (B^c B_c) = (-2)(M^s u_s)^{-3} \nabla_a (M^s u_s). \tag{3.16a}$$

By hypothesis $\nabla_a u_c$ vanishes on Σ, thus on Σ

$$\nabla_a (M^c u_c) = u^c \nabla_a M_c. \tag{3.17}$$

Since M_c is tangent to the geodesic whose initial tangent is n^a, it follows from (3.16a),(3.17) that on Σ

$$n^a \nabla_a (B^c B_c) = 0. \tag{3.18}$$

Furthermore, n_a and M_a agree on Σ; thus if t^a is any vector tangent to Σ, (3.17) implies

$$t^a \nabla_a (M^c u_c) = u^c t^a \nabla_a n_c = (t^a \nabla_a L - t^a n_c \nabla_a u^c) \tag{3.19}$$

which also vanishes by the hypothesis. Thus on Σ

$$\nabla_a \hat{B}_c = \frac{1}{B} \nabla_a M_c \tag{3.20}$$

which proves the equality of the first and third members of (3.15).

To prove the first equality in (3.15) recall the well known fact [23] that $\nabla_a M_c$ is symmetric, and thus on Σ, the tensor $\nabla_a \hat{B}_c$ is also symmetric. Now $\hat{B}^c \nabla_a \hat{B}_c$ vanishes and as a consequence the linear space $b^\perp$ is an invariant subspace of $\nabla_a \hat{B}_c$. Since $\hat{B}^a$ is time like, the metric restricted to $b^\perp$ is negative definite. Thus there exists an orthogonal set $v^a, w^a, m^a \in b^\perp$ of eigenvectors of $\nabla_a \hat{B}_c$ with eigenvalues k, ℓ, μ and $v^a v_a = w^a w_a = m^a m_a = (-1)$. On the other hand from (3.20) it follows that $n^a \nabla_a \hat{B}_c$ vanishes; thus one of the eigenvalues, say μ, vanishes and we conclude that on Σ

$$\nabla_a \hat{B}_c = (-1)(k v_a v_c + \ell w_a w_c) \tag{3.21}$$

where $v^a, w^a \in b^{\perp} \cap \Sigma$. This completes the proof.

Corollary 3.1: When a wave propagates into a uniform fluid, the bicharacteristics are geodesics.

Proof: From Proposition 3 it follows that on Σ

$$\hat{B}^a \nabla_a \hat{B}_c = 0. \tag{3.22}$$

Equations (3.12),(3.13),(2.33) imply that on Σ, we have $\hat{B}^c = f(\theta)\partial\Omega/\partial\xi_c$. The conclusion follows from the hypothesis and (3.22).

We will take a moment to point out in the next proposition that a classical two dimensional wave can be recovered in the present context. Its existence will be used in the main results.

Proposition 4: The space-like distribution defined on Σ by $m \to v^a(m), w^a(m)$ is involutive at each point of Σ.

Proof: It follows from Proposition 3 that $B_c \pounds_w v^c$ and $n_c \pounds_v w^c$ vanish. Then since we know v^c, w^c span $b^{\perp} \cap \Sigma$, we conclude that $\pounds_v w^c$ is a linear combination of v^c and w^c which proves the assertion.

We will need one more intermediate result which will be used in this and in the next section.

Proposition 5: If u, $g_{ik} \in (C^1, C^3)$ is a solution of (1.69), then the Riemannian tensor is continuous across Σ, that is,

$$[R_{ijk\ell}] = 0. \tag{3.23}$$

Proof: The continuity of u, g_{ik} implies the continuity of the stress energy tensor. Then from a known result [15], we have

$$2[R_{jpqk}] = n_j \tilde{G}_{pqk} - n_p \tilde{G}_{jqk}. \tag{3.23a}$$

Substituting the known values [15] for $\tilde{G}_{pqk}$, equation (3.23a) becomes

(3.23b) $$2[R_{jpqk}] = n_j\{n_q\bar{G}_{pk} - n_k\bar{G}_{pq} + n_p\bar{G}_{qk}\} - n_p\{n_q\bar{G}_{jk} - n_k\bar{G}_{jq} + n_j\bar{G}_{qk}\}.$$

When the $\bar{G}_{jk}$ are replaced by known expressions [15], the right side of (3.23b) vanishes, proving (3.23).

Definition 3: Let D denote the standard connection induced on Σ by the connection ∇ of M^4 and write ${}^{(3)}R_{\alpha\beta\gamma\delta}$ for the Riemannian tensor corresponding to D.

Proposition 6: If a wave propagates into a uniform region, then the following holds along the integral curves of $\hat{B}^a$:

(3.24) $$\frac{d}{ds}(k\ell) + k\ell(k + \ell) = 0$$

where s is arc length.

Proof: By Proposition 5, the tensor ${}^{(4)}R_{ijk\ell}$ is well defined on Σ. Then the Gauss-Codazzi equation can be written, in Gaussian coordinates based on Σ, [23] as

(3.25) $${}^{(4)}R_{\alpha\beta\gamma\delta} = {}^{(3)}R_{\alpha\beta\gamma\delta} + N(h_{\alpha\gamma}h_{\beta\delta} - h_{\alpha\delta}h_{\beta\gamma})$$

where

(3.26) $$h_{ab} \equiv \nabla_a M_b.$$

Operating on both sides of (3.25) with the tangential operator $v^\alpha v^\gamma w^\beta w^\delta \hat{B}^\mu \nabla_\mu$ we have

(3.27) $$v^\alpha v^\gamma w^\beta w^\delta \hat{B}^\mu \nabla_\mu {}^{(4)}R_{\alpha\beta\gamma\delta} = v^\alpha v^\gamma w^\beta w^\delta \hat{B}^\mu \nabla_\mu \{{}^{(3)}R_{\alpha\beta\gamma\delta} - (h_{\alpha\gamma}h_{\beta\delta} - h_{\alpha\delta}h_{\beta\gamma})\}.$$

As each of the vectors v_a, w_a, $\hat{B}_a$ is tangent to Σ, it follows from the definition of the induced connection D that (3.27) reduces to

$$v^\alpha v^\gamma w^\beta w^\delta \hat{B}^\mu \nabla_\mu {}^{(4)}R_{\alpha\beta\gamma\delta} = v^\alpha v^\gamma w^\beta w^\delta \hat{B}^\mu D_\mu \{{}^{(3)}R_{\alpha\beta\gamma\delta} - (h_{\alpha\gamma}h_{\beta\delta} - h_{\alpha\delta}h_{\beta\gamma})\}.$$

Permute the indices γ,δ,μ cyclically, add the three equations and apply the Bianchi identity to the left side and to the first member of the right side to obtain

$$(3.28)\quad \underset{\gamma\delta\mu}{\text{Cyclic}}\; v^{\alpha}v^{\gamma}w^{\beta}w^{\delta}\hat{B}^{\mu}D_{\mu}(h_{\alpha\gamma}h_{\beta\delta} - h_{\alpha\delta}h_{\beta\gamma}) = 0.$$

The operators in (3.28) are tangential so (3.28) is an equation along Σ and therefore the hypothesis implies that $h_{\alpha\beta}$ has the form given in Proposition 3. Substituting (3.15) into (3.28), and applying the results of Proposition 3, we find that (3.28) reduces to

$$\ell\hat{B}^{\mu}D_{\mu}k + k\hat{B}^{\mu}D_{\mu}\ell + k^2\ell + \ell^2k = 0$$

$$\hat{B}^{\mu}D_{\mu}(k\ell) + k\ell(k + \ell) = 0$$

and this proves (3.24).

Proposition 7: If a wave propagates into a uniform region, then the following equation is valid along the integral curves of $\hat{B}^{a}$:

$$(3.29)\qquad \frac{d}{ds}(k + \ell) + (k + \ell)^2 - 2k\ell = f$$

where s is arc length and

$$(3.30)\qquad f \equiv \lambda\Big\{\frac{\eta(\eta + \rho A) + 2p(\eta - \rho A)}{2(\eta - \rho A)}\Big\}.$$

Proof: A choice of Gaussian coordinates based on Σ depends on a given coordinatization of Σ. For the present proof a simple choice for these coordinates is to take them to be the local Gaussian coordinates, in Σ, based on an integral 2-submanifold determined by Proposition 4. Then one set of coordinate lines, say those given by x^0, will be the integral curves of $\hat{B}^{a}$. In these coordinates the metric, on Σ, satisfies $g^{00} = g_{00} = 1$, and $g^{0\alpha} = g_{0\alpha} = 0$ for $\alpha = 1,2,3$. This form of the metric implies that on Σ we have the identities

(3.31) $$R = 2g^{\beta\gamma}R_{0\beta\gamma 0} + g^{\beta\gamma}g^{\alpha\delta}R_{\alpha\beta\gamma\delta}$$

and

(3.32) $$R_{\beta\gamma} = R_{0\beta\gamma 0} + g^{\alpha\delta}R_{\alpha\beta\gamma\delta}.$$

Substituting (3.31),(3.32) into the definition of the Einstein tensor, $G_{\sigma\tau} \equiv R_{\sigma\tau} - \frac{1}{2} g_{\sigma\tau}R$, we obtain

(3.33) $$g^{\sigma\tau}G_{\sigma\tau} = g^{\sigma\tau}R_{0\sigma\tau 0} + g^{\sigma\tau}g^{\alpha\delta}R_{\alpha\sigma\tau\delta} - \frac{3}{2} R.$$

Solving (3.31) for its last member and putting the result in (3.33) yields

(3.34) $$g^{\sigma\tau}G_{\sigma\tau} = (-1)g^{\sigma\tau}R_{0\sigma\tau 0} - \frac{1}{2} R.$$

Since v^a, w^a, n^a form a space-like orthogonal basis of $b^{\perp}$ (see Proposition 3) we have on Σ

(3.35) $$g^{\sigma\tau} = (-1)(v^{\sigma}v^{\tau} + w^{\sigma}w^{\tau} + n^{\sigma}n^{\tau}).$$

Recalling that $R_{0\sigma\tau 0} = (-1)R_{\sigma 0\tau 0}$, and then substituting (3.35) in (3.34) we find that on Σ

(3.36) $$g^{\sigma\tau}G_{\sigma\tau} = (-1)(v^{\sigma}v^{\tau} + w^{\sigma}w^{\tau} + n^{\sigma}n^{\tau})R_{\sigma 0\tau 0} - \frac{1}{2} R.$$

Using the Gaussian coordinates (on M^4) we can extend v^a to a vector field $\bar{v}^a$ defined on a 4-neighborhood U of Σ and having the property that $\bar{v}^a\hat{B}_a = 0$ on U. The x^0 coordinate lines are integral curves of $\hat{B}^a$, and by Proposition 3, the tensor $\nabla_a\hat{B}_c$ is symmetric on Σ. Then the hypotheses of Proposition 1 are satisfied so by (3.1a), with $\hat{B}^a$ playing the role of y^a, we have along the x^0 curves that

(3.37) $$\varepsilon(v)\frac{dk}{ds} + \varepsilon(v)k^2 = (-1)v^a\hat{B}_s\hat{B}^b v_r R^{rs}{}_{ab}.$$

A similar equation holds when v is replaced by w and k by ℓ. The vector field M^c extends n^c to U and satisfies $M^cB_c = 0$ on U. But $n^c\nabla_c\hat{B}_a$

vanishes on Σ, so for this field Proposition 1 yields

$$n^a \hat{B}_s \hat{B}^b n_r R^{rs}{}_{ab} = 0. \tag{3.38}$$

In the present coordinates, $v_0 = v^0 = \hat{B}^\alpha = \hat{B}_\alpha = 0$ and $B_0 = B^0 = 1$. Using this and substituting (3.37),(3.38) into (3.36) yields

$$g^{\sigma\tau} G_{\sigma\tau} + \tfrac{1}{2} R = \varepsilon(v)\left(\frac{dk}{ds} + k^2\right) + \varepsilon(w)\left(\frac{d\ell}{ds} + \ell^2\right). \tag{3.39}$$

From Einstein's field equations

$$R = (-1) g^{ab} G_{ab}. \tag{3.40}$$

Again the choice of coordinates implies

$$g^{\sigma\tau} G_{\sigma\tau} = g^{ab} G_{ab} - \hat{B}^a \hat{B}^c G_{ac}. \tag{3.41}$$

Substituting (3.40),(3.41) into (3.39) and recalling $\varepsilon(v) = \varepsilon(w) = -1$ we find

$$(-1)\left(\tfrac{1}{2} g^{ab} G_{ab} - \hat{B}^a \hat{B}^c G_{ac}\right) = \frac{d(k + \ell)}{ds} + (k + \ell)^2 - 2k\ell. \tag{3.42}$$

According to the field equations, G_{ab} is proportional to the stress energy tensor defined in (1.56). Calculating the left side of (3.42) with this definition we arrive at the desired equation (3.29).

The system (3.24),(3.29) determines $k+\ell$ and $k\ell$ for given initial values. These, as will be shown later, are essentially the mean curvature and Gaussian curvature of the two dimensional wave front. We will now solve (3.24),(3.29) and compare the results (see Remarks following Proposition 8) to the Newtonian formulas for the case of cylindrical waves, that is, waves for which $\ell(0) = 0$. We are especially interested in finding the value of s for which the solution becomes (negatively) unbounded. In the Newtonian case the formulas can be obtained without solving differential equations [21].

Before proceeding with the solution, we note that λ in (3.29) is

such that $R_{ik} - \frac{1}{2} g_{ik}R = \lambda T_{ik}$. In the proof below $\lambda < 0$. A new justification for this choice is given in the remarks after Proposition 8.

Proposition 8: A cylindrical wave moving into a uniform fluid satisfies

$$k(s) = (-1)m \tan\{ms + \tan^{-1}[\frac{(-1)k(0)}{m}]\} \tag{3.43}$$

where $m \equiv (-f)^{1/2}$. In particular, if $k(0) < 0$, then

$$\lim_{s \to s_1} k(s) = -\infty \tag{3.44}$$

where the critical time s_1 is positive and is given by

$$s_1 \equiv \frac{1}{m}\{\frac{\pi}{2} - \tan^{-1}[\frac{(-1)k(0)}{m}]\} > 0. \tag{3.45}$$

Proof: Clearly (3.24) and the initial condition $\ell(0) = 0$ are satisfied if we put $\ell(s) \equiv 0$ for all s. Then by separating the variables and integrating, keeping in mind $\lambda < 0$ (that is, $f < 0$) we find that the solution of (3.29) is (3.43). Then putting

$$s_1 = \frac{1}{m}\{\frac{\pi}{2} - \tan^{-1}[\frac{(-1)k(0)}{m}]\}$$

we see that (3.44) follows. The critical time s_1 is positive because $k(0) < 0$ and $\tan^{-1}\theta < \frac{\pi}{2}$.

Remarks: Let us compare (3.45) to the classical result. Expand $\tan^{-1}[(-1)k(0)/m]$ for the case $(-1)k(0)/m > 1$ which approximates the Newtonian limit of low energy density. We find

$$s_1 = \frac{(-1)}{k(0)} + \frac{1}{3}\frac{m^2}{k^3(0)} - \ldots + \ldots \tag{3.46}$$

The first term on the right is the proper time as would be measured in flat space-time. Remembering that $k(0) < 0$ we see that the relativistic value of s_1, which is the proper time as measured by a comoving

observer with world line $\hat{B}^a$, is less than its classical value. There is a contribution to the deficiency in both the first and second terms of (3.46).

A singularity also exists for a negative value of s_1 (which can also be calculated easily) for waves having $k(0) > 0$.

We point out that the condition $\lambda < 0$ (that is, $f < 0$) in Proposition 8 is essential. For $\lambda > 0$ there are unbounded solutions only when $k(0) < (-1)(f)^{1/2}$, which disagrees qualitatively with Newtonian theory.

The quantity which enters into the considerations of the next section is $\nabla_a n^a$. We now relate $\nabla_a n^a$ to k and ℓ.

Proposition 9: On Σ we have

$$\nabla_a n_c \equiv \nabla_a M_c + A n_a \hat{B}_c + C n_a v_c + D n_a w_c \tag{3.47}$$

where

$$A \equiv \hat{B}^a n^c \nabla_c n_a, \quad C \equiv v^a n^c \nabla_c n_a, \quad D \equiv w^a n^c \nabla_c n_a. \tag{3.48}$$

In particular, on Σ

$$\nabla_a n^a = \left(\frac{L^2 + 1}{L^2}\right)^{1/2} (k + \ell) = \frac{1}{v}(k + \ell). \tag{3.49}$$

Proof: According to Definition 2, n^a and M^a agree on Σ. Thus if t^a is any vector tangent to Σ,

$$t^a \nabla_a n^c = t^a \nabla_a M^c.$$

In addition $n_c \nabla_a n^c$, $n^a \nabla_a M^c$ and $n_c \nabla_a M^c$ vanish on Σ. Thus representing $\nabla_a n_c$ in terms of the pseudo-orthonormal set $\hat{B}_a$, n_a, v_a, w_a we have on Σ

$$\nabla_a n_c = \nabla_a M_c + A n_a \hat{B}_c + C n_a v_c + D n_a w_c \tag{3.50}$$

where A, C, D have the values given in (3.48). This proves (3.47). Equation (3.49) follows from (3.47) and (3.15).

For later reference we rephrase Proposition 8 in terms of $\nabla_a n^a$ by making use of (3.51).

Proposition 10: If a cylindrical wave moves into a uniform fluid, then

$$\nabla_a n^a(s) = \frac{(-1)m}{v} \tan\left\{ms + \tan^{-1}\left[\frac{(-1)v\nabla_a n^a(0)}{m}\right]\right\}. \tag{3.51}$$

Furthermore for $\nabla_a n^a(0) < 0$, we have

$$\lim_{s \to s_1} \nabla_a n^a(s) = -\infty \tag{3.52}$$

where

$$s_1 \equiv \frac{1}{m}\left\{\frac{\pi}{2} - \tan^{-1}\left[\frac{(-1)v\nabla_a n^a(0)}{m}\right]\right\} > 0. \tag{3.53}$$

In order to be able to make useful predictions from Proposition 10, we would like to be able to interpret the geometrical objects $\nabla_a n^a$, v_i, w_i in terms of quantities observable to an experimenter whose world line is u^i. Such an interpretation is as follows. The proof is similar to that of Proposition 3 and is omitted.

Proposition 11: The eigenvectors v_a, w_a are in $u^\perp$ and hence are observable to an experimenter with world line u_i. Also the vector N_i (see 2.39a) is the unit normal in $u^\perp$ to the 2-wave spanned by v^a, w^a and $|L|\,\nabla_a N^a = v\nabla_a n^a$ (where v is the velocity of wave propagation). In particular this means that the initial value $\nabla_a n^a(0)$ needed for predicting the behavior of $\nabla_a n^a(s)$ is observable, and has value $(k + \ell)/v$.

III.III. The Behavior of the Discontinuity Along the Bicharacteristic

One preliminary definition and calculation will be needed.

Definition 4: Let w_{bam} denote the operator defined by

$$w_{bam}u = \nabla_{bam}u - \nabla_{amb}u.$$

Proposition 12: Suppose $g_{ab}, u \in (C^1, C^3)$ is a solution of (1.69). Then on Σ we have

$$\ell A^b[w_{bam}u] = (-1)\ell_1 \frac{\rho}{L} u^b u_s n^v [\nabla_a R^s{}_{vmb}]. \tag{3.54}$$

Proof: We can write

$$w_{bam}u_v = (\nabla_{bam} - \nabla_{abm})u_v + (\nabla_{abm} - \nabla_{amb})u_v \tag{3.55}$$

and by the application of a standard formula find

$$w_{bam}u_v = (\nabla_s u_v)R^s{}_{mab} + (\nabla_m u_s)R^s{}_{vab} + \nabla_a(u_s R^s{}_{vab}). \tag{3.56}$$

Taking the jump of equation (3.56), applying Proposition 5 and remembering $u, g_{ik} \in C^1$ we find

$$[w_{bam}u_v] = u_s[\nabla_a R^s{}_{vmb}]. \tag{3.57}$$

For the scalar entries ρ, s, ζ of u we have by a similar procedure that

$$[w_{bam}\rho] = [w_{bam}s] = [w_{bam}\zeta] = 0. \tag{3.58}$$

From (3.57),(3.58),(1.67) and the form of ℓ given in Chapter II, Definition 3 it follows that

$$\ell A^b[w_{bam}u] = (\ell_1 \rho g^{vb} + \eta u^b m^v)u_s[\nabla_a R^s{}_{vmb}].$$

Since $g_{ab} \in C^1$ this becomes

$$\ell A^b[w_{bam}u] = \ell_1 \rho u_s[\nabla_a R^s{}_m] + \eta u^b m^v u_s[\nabla_a R^s{}_{vmb}]. \tag{3.59}$$

Again Einstein's field equations

$$R_{ik} - \frac{1}{2} R g_{ik} = \lambda T_{ik} \tag{3.60}$$

imply $R = (-1)\lambda T^k{}_k = \lambda\{\rho(1 + e) - 3p\}$. Thus $u \in C^1$ implies $[\nabla_a R]$ vanishes which by (3.60) gives $[\nabla_a R_{ik}] = 0$. With this last equality, (3.59) becomes

$$\ell A^b[w_{bam}] = \eta u^b m^v u_s[\nabla_a R^s{}_{vmb}]. \tag{3.61}$$

The value of m^v given in (2.25) proves (3.54).

We recall from the algebraic solution (2.24) and the definition (2.31) that the discontinuity $r \equiv n^c U_c \tilde{r}$ is known when $n^c U_c$ is known. A differential equation, along the bicharacteristic, for this latter quantity will now be derived.

Proposition 13: Let $K \in C^1$ and $g_{ik}, u \in (C^1, C^3)$ be a solution of (1.69). Then $n^c U_c$ satisfies

$$(3.62) \quad n_a n_m \frac{d}{db}(n^s U_s) + n^s U_s \{\frac{d}{db}(n_a n_m) + M_{am}\} + \ell_1 \frac{\rho}{L} u^b u_s n^v [\nabla_a R^s{}_{vmb}] = 0$$

where

$$(3.62a) \quad M_{am} \equiv \ell(\nabla_m A^b) n_b n_a \tilde{r} + \ell(\nabla_a A^b) n_b n_m \tilde{r} + n_a n_m \ell A^b \nabla_b \tilde{r} + \ell \widetilde{[\nabla_{am} A_b]} \nabla^b u$$

and

$$(3.62b) \quad \frac{d}{db} \equiv \bar{b}^s \nabla_s.$$

Proof: We know from (2.11a) and (2.24) that

$$(3.63) \quad [\nabla_{am} u] = n_a n_m n^s U_s \tilde{r}.$$

Further, according to (2.11e) the operator ℓA^b is a tangential operator and therefore it can be applied to both sides of (3.63) to obtain

$$(3.64) \quad \ell A^b \nabla_b [\nabla_{am} u] = \ell A^b \nabla_b (n_a n_m n^s U_s \tilde{r}).$$

Again, since $\ell A^b \nabla_b$ is tangential, we can, by Hadamard's Theorem, interchange this operator and the operation of taking the limits which define the jump. Hence (3.64) implies

$$(3.65) \quad [\ell A^b \nabla_b \nabla_{am} u] = \ell A^b \nabla_b (n_a n_m n^s U_s \tilde{r}).$$

Expanding the left side of (3.65) we have

$$(3.66) \quad [\ell A^b \nabla_{bam} u] = \ell [A^b \nabla_{amb} u] + \ell A^b [w_{bam} u]$$

which implies

$$(3.67)\qquad [\ell A^b \nabla_{bam} u] = \ell[\nabla_a (A^b \nabla_{mb} u) - \nabla_a A^b \nabla_{mb} u] + \ell A^b [w_{bam} u].$$

Using the fact that u satisfies (1.71) on each side of Σ, we get

$$(3.68)\quad [T^b \nabla_{bam} u] = \ell[(-1)\nabla_a (\nabla_m A^b \nabla_b u + \nabla_m D) - \nabla_a A^b \nabla_{mb} u] + \ell A^b [w_{bam} u].$$

Inspecting the form of ℓ given in (2.25) and the form of D in (1.70) we see that $\ell[\nabla_a \nabla_m D]$ vanishes. Using this fact and expressing the last term in (3.68) by use of (3.54) we find

$$(3.69)\quad [\ell A^b \nabla_{bam} u] = (-1)\ell\{[(\nabla_{am} A^b)\nabla_b u] + \nabla_m A^b [\nabla_{ab} u] + \nabla_a A^b [\nabla_{mb} u]\}$$

$$- \ell_1 \frac{\rho}{L} u^b u_s n^v [\nabla_a R^s{}_{vmb}].$$

Now expand the right hand side of (3.65) and equate the result to the right hand side of (3.69) to obtain (3.62).

The next two results will put (3.62a) in a form which is suitable for geometrical and physical interpretation.

Proposition 14: The sum of the first and second terms of (3.62a) may be written

$$(3.70)\qquad \ell(\nabla_m A^b) n_b n_a \tilde{r} + \ell(\nabla_a A^b) n_b n_m \tilde{r}$$

$$= (-2)\ell_1 \frac{\rho^2 A}{\eta L^2}\left(n_a n_c \nabla_m \frac{u^c}{L} + n_m n_c \nabla_a \frac{u^c}{L}\right).$$

Proof: Since n^a is assumed to satisfy $L^2(\eta - \rho A) - \rho A = 0$ on a 4-neighborhood of Σ it is clear that

$$(3.71)\qquad A^c n_c \tilde{r} = 0$$

$$(3.72)\qquad \ell A^c n_c = 0$$

hold on a 4-neighborhood. The matrices A^c, ℓ, $\tilde{r}$ are differentiable across Σ because $u \in C^1$ and therefore

$$(3.73)\qquad (\nabla_i A^c) n_c \tilde{r} + A^c (\nabla_i n_c)\tilde{r} + A^c n_c \nabla_i \tilde{r} = 0$$

$$\ell(\nabla_i A^c)n_c\tilde{r} + \ell A^c(\nabla_i n_c)\tilde{r} + \ell A^c n_c \nabla_i \tilde{r} = 0. \tag{3.74}$$

The third term of (3.74) vanishes due to (3.72). Then using Definition 9, Chapter II for $\bar{b}^a$, equation (3.74) becomes

$$\ell(\nabla_i A^c)n_c\tilde{r} = (-1)\bar{b}^c\nabla_i n_c. \tag{3.74a}$$

Substituting for $\bar{b}^c$ from (2.31a) and using the orthogonality of $\bar{b}^c$ and n^c, and $n^c n_c = -1$, we have

$$\ell(\nabla_i A^c)n_c\tilde{r} = (-2)\ell_1 \frac{\rho^2 A}{\eta L^2} n^c \nabla_i \left(\frac{1}{L} u_c + n_c\right).$$

Hence

$$n_a \ell(\nabla_i A^c)n_c\tilde{r} = n_a(-2)\ell_1 \frac{\rho^2 A}{\eta L^2} n^c \nabla_i \left(\frac{1}{L} u_c\right)$$

from which (3.70) follows.

In the following computation we use (2.31) to evaluate $\tilde{r}$, (2.25) for ℓ and (1.67) for A^b. The calculation is straightforward but rather long and the equations involved are cumbersome. The details are omitted.

Proposition 15: The sum of the third and fourth terms of (3.62a) may be put in the form

$$\begin{aligned} &n_a n_m \ell A^b \nabla_b \tilde{r} + \ell[\widetilde{\nabla_{am} A_b}]\nabla^b u \\ &= n_a n_m \ell_1 \{(f_1 + f_2)\tilde{b}^c \nabla_c \rho - \frac{f_2}{L} u^c \nabla_c \rho + f_3 \tilde{b}^c \nabla_c s + f_4 \tilde{b}^c \nabla_c \zeta \\ &\quad - \frac{\rho A}{\eta L^3}(\tilde{b}^c \nabla_c L - \tilde{b}^m \tilde{b}^c \nabla_c u_m) - \frac{A\rho^2}{\eta L^2} \nabla_c n^c\} \end{aligned} \tag{3.75}$$

where

$$f_1 \equiv \frac{3\rho^3 A^2(\eta + \rho A)}{L^7 \eta^2(\eta - \rho A)} \frac{\partial L}{\partial \rho}; \quad f_2 \equiv \frac{1}{\eta A}(3\rho A^2 - \eta\rho \frac{\partial A}{\partial \rho} - 2\eta A) \tag{3.76}$$

$$f_3 \equiv \frac{2\rho^4 A^3(\eta + \rho A)}{L^8 \eta^2} \frac{\partial L}{\partial s}; \quad f_4 \equiv \frac{2\rho^4 A^3(\eta + \rho A)}{L^8 \eta^2} \frac{\partial L}{\partial \zeta} \tag{3.77}$$

$$\tilde{b}^a \equiv \frac{1}{L} u^a + n^a. \tag{3.78}$$

Remarks: If we form the scalar product of (3.62) with $u^a u^m$ we obtain a linear differential equation, whose solution is determined by the integrals of the terms in (3.70) and (3.75). The function ℓ_1, which appears in each term of (3.62), is undetermined by the algebraic solution (2.25). The proper choice of ℓ_1 makes the parameter along the curve arc length. With this choice all but two of the terms in (3.75) are of the form

$$\int_{s=0}^{s} f\hat{B}^a \nabla_a \theta ds = \int_{\theta(0)}^{\theta(s)} f \, d\theta. \tag{3.79}$$

Assuming bounded values of the thermodynamic variables, this integral can become unbounded at a finite time, s_1, only when f does so. Considering the form of the functions f_i in (3.76),(3.77), this can only occur if

$$\lim_{s \to s_1} (\eta - \rho A) = 0. \tag{3.80}$$

Taking into account $L^2(\eta - \rho A) = \rho A$, there are two possibilities: Case (i) $\lim L = \pm\infty$ and Case (ii) $\lim L$ is finite. In the first case we see that the speed, L^2/L^2+1, of the wave approaches unity, the speed of radiation. In the second case, using $\eta \equiv \rho(1 + e) + p$ we see either A tends to zero or ρ and p tend to zero, all of which are ruled out on physical grounds. Since n^a is orthogonal to $\tilde{b}^a$ the terms in (3.70) and the sixth term on the right of (3.75) are observable quantities and in any case are regular by the hypothesis $u \in C^1$. Thus the singularities, if they arise, are due to the remaining term which involves the quantity $\nabla_a n^a$. Even in the Newtonian case the behavior of the analogue of this term is only known for the case of disturbances propagating into fluids at rest.

Proposition 16: Let $K \in C^1$ and $u,g,\Gamma \in (C^1,C^3)$ be a solution of

(1.69) corresponding to a disturbance propagating into a uniform fluid. Then the discontinuity is governed by

$$n^c U_c(s) = n^c U_c(0) \exp\{\frac{(-v)}{2\eta} \int_0^s \nabla_a n^a ds\} \tag{3.81}$$

$$\nabla_a n^a = \frac{1}{v}(k + \ell) \tag{3.82}$$

$$\frac{d(k + \ell)}{ds} + (k + \ell)^2 - 2k\ell = f \tag{3.83}$$

$$\frac{d(k\ell)}{ds} + k\ell(k + \ell) = 0. \tag{3.84}$$

Proof: Form the scalar product of (3.62) with $u^a u^m$. The term $u^a u^m u^b u_s n^v [\nabla_a R^s{}_{vmb}]$ vanishes because the Riemannian tensor is skew in its last two indices. By the uniformity hypotheses all the terms in (3.70),(3.75) vanish except the final term of (3.75). Making the choice

$$\ell_1 \equiv \frac{(-1)}{2} \frac{L^3}{(L^2+1)^{1/2}} \frac{1}{\rho A^2} \tag{3.85}$$

the parameter in (3.62) becomes arc length and (3.62) is

$$\frac{d}{ds}(n^s U_s) + (\tfrac{1}{2} \frac{v}{\eta} \nabla_a n^a) n^c U_c = 0 \tag{3.86}$$

which proves (3.81). Equations (3.82),(3.83),(3.84) were proved in Section III.II.

Corollary 17: A cylindrical wave solution with $u,g,\Gamma \in (C^1,C^3)$ and $K \in C^1$ and initial mean curvature negative, develops a singularity at the finite time s_1; that is,

$$\lim_{s \to s_1} n^c U_c(s) = \infty \tag{3.87}$$

where

$$s_1 \equiv \frac{1}{m}\{\frac{\pi}{2} - \tan^{-1}[\frac{(-1)v\nabla_a n^a(0)}{m}]\} > 0. \tag{3.87a}$$

Proof: The assertions follow directly from Proposition 16 and Proposition 10.

Remarks: The unboundedness of the jump in the second derivatives is interpreted to mean that first derivatives of u become discontinuous at s_1. We emphasize that the relationship of s_1 to its Newtonian value is discussed following Proposition 8.

Corollary 18: The discontinuity associated with a disturbance (not necessarily a cylindrical wave) propagating into a uniform fluid is an increasing function of proper time, provided its mean curvature is initially negative.

Proof: Rewriting (3.83) as

$$\frac{d(k + \ell)}{ds} = (-1)(k^2 + \ell^2) + f \tag{3.88}$$

and using (3.82) with v constant and $f < 0$ we have that $\nabla_a n^a$ is a decreasing function. The conclusion follows from (3.81).

CHAPTER IV

SOLUTION OF THE CAUCHY PROBLEM

IV.I. Introduction

We will obtain a local existence and uniqueness theorem for the Cauchy problem, with sufficiently smooth data, for the non-equilibrium system. The basic technique follows that developed by Lichnerowicz [12]. All the relevant notions pertaining to strictly hyperbolic Leray systems which are not presented below can be found in [12] and [25]. To facilitate the discussion we introduce two definitions.

Definition 1: A local coordinate system, x^i, will be called harmonic if

$$F^i \equiv \nabla^a \nabla_a x^i = 0$$

on the domain of the x^i.

Definition 2: The following equations, for the unknowns g_{ab}, u_a, ρ, s, ζ, together with the compatibility conditions will be referred to as system I.

$$S_{ab} \equiv R_{ab} - \tfrac{1}{2} g_{ab} R = \lambda T_{ab} \tag{4.1}$$

$$T_{ab} \equiv \eta u_a u_b - p g_{ab} \tag{4.1a}$$

$$u^a u_a = 1 \tag{4.2}$$

$$u^a \nabla_a \zeta = -K e_\zeta \tag{4.3}$$

$$u^a \nabla_a s = \frac{K e_\zeta}{T^2} \tag{4.4}$$

$$\nabla_a (\rho u^a) = 0. \tag{4.5}$$

Compatibility conditions to be satisfied on the initial surface Σ:

(4.6) $$F^r = 0$$

(4.7) $$S_{a0} = \lambda T_{a0}$$

(4.8) $$\frac{f}{\rho}\frac{\partial \rho}{\partial f} \geq 1$$

(4.8a) $$g^{00},_{u^0},g^{00} - \left(1 - \frac{f}{\rho}\frac{\partial \rho}{\partial f}\right)(u^0)^2 \neq 0.$$

To prove the existence of a solution to I, we introduce a related system which implies I.

Definition 3: System II will consist of (4.3)-(4.8) and the following:

(4.9) $$R^{(h)}_{ab} = \lambda\{\rho f u_a u_b + \left(p - \frac{\rho f}{2}\right)g_{ab}\}$$

(4.10) $$f u^c \nabla_c u_a = \gamma^c{}_a(\partial_c f - T\partial_c s - e_\zeta \partial_c \zeta)$$

where

(4.11) $$f \equiv 1 + e + \frac{p}{\rho}$$

(4.11a) $$u^a u_a = 1 \quad \text{on } \Sigma.$$

We will show that, for analytic data, the Cauchy Problem corresponding to II has a unique local solution. Then a third system will be derived from II and will be shown to be a strictly hyperbolic Leray system. Leray's Theorem gives a solution of III for sufficiently smooth data. This solution, together with the fact that II admits a solution for analytic data, allows us to infer the existence of a solution to II for sufficiently smooth data. The last solution also solves I with the same data.

IV.II. Existence Theorem

Proposition 1: A solution of system II is a solution of system I, and conversely.

Proof: We will begin by proving that II implies I. Since (4.2)-(4.8) are common to I and II we must only show (4.1) holds. It is well known [12] that a solution to (4.9) which satisfies (4.6),(4.7) together with

$$\nabla_a T^{ab} = 0 \tag{4.12}$$

is a solution of (4.1). To prove (4.12) it suffices to show

$$u_b \nabla_a T^{ab} = 0 \tag{4.12a}$$

$$\gamma_{bc} \nabla_a T^{ac} = 0. \tag{4.12b}$$

Since $u^t \gamma^c{}_t = 0$, it follows from (4.10) $u^t u^c \nabla_c u_t$ vanishes. This and (4.11a) imply $u^a u_a = 1$ everywhere. Using this last fact, we find from (4.1a),(4.2) that

$$u_b \nabla_a T^{ab} = \rho u^a \nabla_a e + (1+e) u^a \nabla_a \rho + [(1+e) + p] \nabla_a u^a. \tag{4.13}$$

From (4.5) we see that $\nabla_a u^a$ can be replaced by $(-1) u^a \nabla_a \rho / \rho$. Carrying out this substitution and simplifying the result gives

$$u_b \nabla_a T^{ab} = \rho u^a \nabla_a e - \frac{p}{\rho} u^a \nabla_a \rho. \tag{4.14}$$

Recall from (1.43) that

$$u^a \nabla_a e = \frac{p}{\rho^2} u^a \nabla_a \rho + T u^a \nabla_a s + e_\zeta u^a \nabla_a \zeta \tag{4.15}$$

and substitute from (4.3),(4.4) for the last two members of (4.15) to obtain

$$u^a \nabla_a e = \frac{p}{\rho^2} u^a \nabla_a \rho. \tag{4.16}$$

Putting (4.16) into (4.14) we find $u_b \nabla_a T^{ab}$ vanishes. Thus a solution of II implies (4.12a) holds. To prove (4.12b) we note that from (1.63) we have

$$\gamma_{bc} \nabla_a T^{ac} = \eta u^a \nabla_a u_b - \nabla_b p. \tag{4.17}$$

Carrying out operations similar to those above, (4.17) becomes

$$\gamma_{bc}\nabla_a T^{ab} = fu^a\nabla_a u_b - \gamma^a{}_b(\partial_a f - e_\zeta \partial_a \zeta - T\partial_a s).$$

Thus if (4.10) is satisfied so is (4.12b), which completes the first half of the proof.

To prove the converse, recall [12] that (4.1),(4.2),(4.5)-(4.7) imply that $F^i = 0$ everywhere. Using this to express (4.1) in harmonic coordinates we find (4.9). Finally, the geometric identity $\nabla_b S^{ab} = 0$ implies $\nabla_b T^{ab} = 0$ and hence $\gamma_{ac}\nabla_b T^{bc} = 0$, which by calculations similar to those above lead to (4.10). Thus II implies I, and the proof is complete.

In the next three propositions we will derive a system which is a consequence of II, and is a strictly hyperbolic Leray system. To avoid writing cumbersome expressions we adopt the following notation:

Convention: An expression such as E (3 in g_{ab}; 2 in f, s, u^b) means that E is a known, regular function of its arguments and depends on the partial derivatives of g_{ab} of order at most three, etc.

Definition 4: Denote by A^{ab} the tensor

$$A^{ab} \equiv \left(\frac{f}{\rho}\frac{\partial \rho}{\partial f} - 1\right)u^a u^b + g^{ab}. \tag{4.18}$$

Proposition 2: If II holds, then the scalar f satisfies

$$A^{ab}u^c\partial_{abc} f = \bar{E}(3 \text{ in } g_{ab};\ 2 \text{ in } f,\ s,\ \zeta,\ u^b). \tag{4.19}$$

Proof: Taking the divergence of (4.10) we have

$$\begin{aligned}&(\nabla_b f)u^a\nabla_a u^b + f(\nabla_b u^a)\nabla_a u^b + fu^a\nabla_{ba}u^b \\ &= (-1)[(\nabla_b u^a)u^b + u^a\nabla_b u^b](\partial_a f - e_\zeta\partial_a\zeta - T\partial_a s) \\ &\quad + \gamma^{ab}[\nabla_{ab} f - \nabla_b(e_\zeta\partial_a\zeta) - \nabla_b(T\partial_a s)].\end{aligned} \tag{4.20}$$

Applying ∇_a to both of (4.5) gives

$$\rho\nabla_{ab}u^b + (\nabla_a\rho)\nabla_b u^b + (\nabla_a u^b)\nabla_b\rho + u^b\nabla_{ab}\rho = 0. \tag{4.21}$$

From the standard formula for interchanging the order of differentiation we know

$$\nabla_{ba}u^b = \nabla_{ab}u^b - u_c R^c{}_a. \tag{4.22}$$

Put (4.21) into (4.22) to find

$$\nabla_{ba}u^b = \frac{(-1)}{\rho}[(\nabla_a\rho)\nabla_b u^b + (\nabla_a u^b)\nabla_b\rho + u^b\nabla_{ab}\rho] - u_c R^c{}_a. \tag{4.23}$$

Now considering ρ as a function of f, s, ζ, which we denote by θ_1, we can expand the last term of (4.23) and substitute the result into (4.20) to obtain

$$\begin{aligned}&(\nabla_b f)u^a\nabla_a u^b + f(\nabla_b u^a)\nabla_a u^b \\ &+ fu^a\frac{(-1)}{\rho}\Big\{(\nabla_a\rho)\nabla_b u^b + (\nabla_a u^b)\nabla_b\rho + u^b\Big[\frac{\partial\rho}{\partial\theta_k}\nabla_{ab}\theta_k + \frac{\partial^2\rho}{\partial\theta_m\partial\theta_k}\partial_a\theta_m\partial_b\theta_k\Big]\Big\} \\ &- fu_c R^c{}_a = F(1 \text{ in } u^b, f, s, \zeta) + \gamma^{ab}(\nabla_{ba}f - e_\zeta\nabla_{ba}\zeta - T\nabla_{ba}s).\end{aligned} \tag{4.24}$$

Solving (4.24) for the terms containing the second derivatives of f and using (4.18), equation (4.24) becomes

$$\begin{aligned}(-1)A^{ab}\nabla_{ab}f = &\Big(\frac{f}{\rho}\frac{\partial\rho}{\partial s}u^a u^b - \gamma^{ab}T\Big)\nabla_{ab}s + \Big(\frac{f}{\rho}\frac{\partial\rho}{\partial\zeta}u^a u^b - \gamma^{ab}e_\zeta\Big)\nabla_{ab}\zeta \\ &+ \bar{F}(1 \text{ in } u^b, f, s, \zeta;\ 2 \text{ in } g_{ab}).\end{aligned} \tag{4.25}$$

Now operate on (4.25) with $u^c\nabla_c$ to get

$$\begin{aligned}(-1)A^{ab}u^c\nabla_{cab}f = &\Big[\frac{f}{\rho}\frac{\partial\rho}{\partial s}u^a u^b - \gamma^{ab}T\Big]u^c\nabla_{cab}s \\ &+ \Big[\frac{f}{\rho}\frac{\partial\rho}{\partial s}u^a u^b - \gamma^{ab}e_\zeta\Big]u^c\nabla_{cab}\zeta \\ &+ H(2 \text{ in } u^b, f, s, \zeta; 3 \text{ in } g_{ab}).\end{aligned} \tag{4.26}$$

Differentiating (4.3), (4.4) each twice we find

$$u^c \nabla_{cab} s = D(2 \text{ in } f,s,\zeta,u^b;\ 2 \text{ in } g_{ab}) \tag{4.27}$$

$$u^c \nabla_{cab} \zeta = \bar{D}(2 \text{ in } f,s,\zeta,u^b;\ 2 \text{ in } g_{ab}). \tag{4.28}$$

Substituting (4.27),(4.28) into (4.26) yields (4.19), which completes the proof.

Proposition 3: Given a solution of II, we have

$$fA^{ab}u^c \partial_{abc} u^t = H^t(2 \text{ in } f,\zeta,s,u^b;\ 3 \text{ in } g_{ab}). \tag{4.29}$$

Proof: Apply $(-1)A^{ab}\nabla_{ab}$ to both members of (4.20) to obtain

$$\begin{aligned}(-1)fA^{ab}u^c\nabla_{abc}u^t = \gamma^{ct}(-1)A^{ab}(\nabla_{abc}f - T\nabla_{abc}s - e_\zeta \nabla_{abc}\zeta) \\ + F^t(2 \text{ in } f,s,\zeta,u^b;\ 2 \text{ in } g_{ab}).\end{aligned} \tag{4.30}$$

Operating on (4.25) with $\gamma^{ct}\nabla_c$ gives

$$\begin{aligned}(-1)A^{ab}\gamma^{ct}\nabla_{cab}f = \left(\frac{f}{\rho}\frac{\partial\rho}{\partial s}u^a u^b - \gamma^{ab}T\right)\gamma^{ct}\nabla_{cab}s \\ + \left(\frac{f}{\rho}\frac{\partial\rho}{\partial\zeta}u^a u^b - \gamma^{ab}e_\zeta\right)\gamma^{ct}\nabla_{cab}\zeta + \bar{F}^t(2 \text{ in } u^b,f,s,\zeta;\ 3 \text{ in } g_{ab}).\end{aligned} \tag{4.31}$$

Interchanging the order of differentiation in (4.22) and substituting the result for the second term on the right of (4.29) yields

$$\begin{aligned}(-1)fA^{ab}u^c\nabla_{abc}u^t = \left(\frac{f}{\rho}\frac{\partial\rho}{\partial s}u^a u^b - \gamma^{ab}T\right)\gamma^{ct}\nabla_{cab}s \\ + \left(\frac{f}{\rho}\frac{\partial\rho}{\partial\zeta}u^a u^b - \gamma^{ab}e_\zeta\right)\gamma^{ct}\nabla_{cab}\zeta + \bar{F}^t(2 \text{ in } u^b,f,\zeta,s;3 \text{ in } g_{ab}) \\ + \gamma^{ct}A^{ab}(e_\zeta\nabla_{abc} + T\nabla_{abc}s).\end{aligned} \tag{4.32}$$

Using the definition of A^{ab} in (4.18), equation (4.32) simplifies to

$$\begin{aligned}(-1)fA^{ab}u^c\nabla_{abc}u^t = \frac{f}{\rho}u^a u^b\gamma^{ct}\left(\frac{\partial\rho}{\partial s} + \frac{\partial\rho}{\partial f}T\right)\nabla_{abc}s \\ + \frac{f}{\rho}\left(\frac{\partial\rho}{\partial\zeta} + \frac{\partial\rho}{\partial f}e_\zeta\right)u^a u^b\gamma^{ct}\nabla_{abc}\zeta + \bar{F}^t(2 \text{ in } u^b,f,s,\zeta;3 \text{ in } g_{ab}).\end{aligned} \tag{4.33}$$

The terms in (4.33) involving the third derivatives of s, ζ have the form of the terms in (4.27),(4.28). Therefore, using (4.27),(4.28), equation (4.33) can be put in the form given in (4.29).

Proposition 4: When II is satisfied we have

$$u^c g^{ab} \partial_{abc} g_{\ell m} = F_{\ell m}(1 \text{ in } f,s,\zeta;\ 0 \text{ in } u^b;\ 2 \text{ in } g_{ab}). \tag{4.34}$$

Proof: Equation (4.9) implies

$$u^c \nabla_c R^{(h)}_{ab} = \chi u^c \nabla_c \{\rho f u_a u_b + g_{ab}(p - \tfrac{1}{2}\rho f)\}.$$

Using (4.5) to replace $u^c \nabla_c u_a$ by $(-1)u^a \nabla_a \rho/\rho$, and expressing $R^{(h)}_{ab}$ in terms of g_{ab} and its partial derivatives [12] we arrive at (4.34).

Definition 5: The system (4.19),(4.29),(4.34),(4.3)(4.4),(4.6)-(4.8a) will be referred to as system III.

Proposition 5: If Σ is given locally by $x^0 = 0$, then the Cauchy problem, with analytical data, for system II has a unique analytic solution provided the data satisfy the compatibility conditions and on Σ

$$g^{00},\ u^0,\ g^{00} - \left(1 - \frac{f}{\rho}\frac{\partial \rho}{\partial f}\right)(u^0)^2 \neq 0. \tag{4.35}$$

Proof: From the compatibility condition

$$R^{(h)}_{ab} = \chi[\rho f u_a u_b + (p - \tfrac{\rho f}{2}) g_{ab}], \text{ on } \Sigma \tag{4.36}$$

and the expression for $R^{(h)}_{ab}$ (see [12]) we have immediately that

$$g^{00} \partial_{00} g_{ab} = d \tag{4.37}$$

where d is known from the Cauchy data. By (4.35) we have $g^{00} \neq 0$, so $\partial_{00} g_{ab}$ is known from (4.37). Also by hypothesis $u^0 \neq 0$ on Σ and therefore from (4.3),(4.4) we can find respectively the derivatives $\partial_0 \zeta$, $\partial_0 s$ in terms of the data. Hence putting $t = 0$ in (4.10) we have

$$fu^0\partial_0 u^0 - [g^{00} - (u^0)^2]\partial_0 f = \bar{d}. \tag{4.38}$$

From (4.5)

$$0 = \rho\nabla_a u^a + u^a\left(\frac{\partial\rho}{\partial f}\,\partial_a f + \frac{\partial\rho}{\partial s}\,\partial_a s + \frac{\partial\rho}{\partial\zeta}\,\partial_a\zeta\right). \tag{4.39}$$

Again using (4.3),(4.4), equation (4.39) becomes

$$\rho\partial_0 u^0 + \frac{\partial\rho}{\partial f}\,u^0\partial_0 f = \bar{\bar{d}}. \tag{4.40}$$

Equations (4.38),(4.40) determine $\partial_0 u^0$ and $\partial_0 f$ provided the determinant of u^0 and f is non-zero. This determinant is the third member of (4.35) and thus $\partial_0 u^0$ and $\partial_0 f$ can be expressed in terms of the Cauchy data.

Putting t = 1 in (4.10) we have

$$f(u^0\partial_0 u_1 + u^\alpha\partial_\alpha u_1) = \gamma_1^c(\partial_c f - e_\zeta\partial_c\zeta - T\partial_c s). \tag{4.41}$$

Since $\partial_0 f$ is now known and u^0 is non-zero by hypothesis, we conclude that $\partial_0 u_1$ is known. Similarly for $\partial_0 u_k$ for k = 2,3. By a similar recursive argument the higher order derivatives, in the direction x^0 transverse to Σ, can be determined and therefore by the Cauchy-Kovelewski theorem a unique local analytic solution exists.

In order to fix notation and state our final results precisely, we will give some basic definitions from [12].

Definition 6: Let $V = (v_\sigma(x))$ be a matrix, defined on a 4-neighborhood W, of Σ with locally square integrable derivatives of order $\leq s(\sigma)+1$. The matrix V is called a data matrix for the system $A(x,U,\partial)U = B(x,U)$ provided the following hold:

(i) The matrix $A(x,V,\partial)$ is strictly hyperbolic in W and Σ is spacelike with respect to $A(x,V,\partial)$.

(ii) The quantities $a_\tau(x,V,\partial)v_\tau - b_\tau(x,V)$ together with their

derivatives of order $\leq t(\tau) - 1$ vanish on Σ.

If a Leray system, $AU = B$, admits such a V, it is called a strictly hyperbolic Leray system.

Remark: Suppose V is a matrix, defined on Σ, satisfying (i) and such that $a_\tau(x,V,\partial)v_\tau - b_\tau(x,V)$ vanish on Σ. Then if V is sufficiently smooth, it is known [12] that it can be extended to a 4-neighborhood so as to satisfy all the above requirements.

Definition 7: Let V be a data matrix for $AU = B$ and U a solution of this system having locally square integrable derivatives of order $\leq s(\sigma)$. If the quantities $u_\sigma - v_\sigma$ together with their derivatives of order $\leq s(\sigma) - 1$ vanish on Σ, then U is called a solution to the Cauchy problem with data V.

Remark: Leray's theorem states that the Cauchy problem for a strictly hyperbolic Leray system has a unique local solution in the class of all U having locally square integrable derivatives of order $\leq s(\sigma) + 1$.

Proposition 6: The Cauchy problem corresponding to I has a unique local solution U, in the class of all U with locally square integrable derivatives of order $\leq s(\sigma) + 1$, provided the data are sufficiently smooth, g_{ab} is a normal hyperbolic metric on Σ and Σ is space-like with respect to g_{ab}.

Proof: We will first establish uniqueness. Thus suppose U, $\bar{U}$ are two solutions of (I,d) where d denotes the data. Then by Proposition 1, U, $\bar{U}$ satisfy (II,d). From this and the method of arriving at III, it follows that U, $\bar{U}$ also satisfy III. Now on Σ, we have $F^r = 0$, so (4.7)

implies (4.9) holds on Σ. Equation (4.9) and $g^{00} \neq 0$ give the second derivatives of g_{ab} on Σ. Equations (4.3),(4.4) determine the first derivatives of ζ and s. Then using (4.8), equations (4.5),(4.10) determine the first derivatives of f and u^a. Knowing the first derivatives of f, s, ζ, the equations (4.3),(4.4) give the second derivatives of s, ζ and (4.34) yields the third derivatives of g_{ab}. Thus the derivatives of order $\leq s(\sigma) - 1$ are determined on Σ. Then by the uniqueness assertion of Leray's theorem we conclude $U = \bar{U}$.

To prove existence, we consider the assignment of Leray indices given by

$$s(g_{ab}) = 4,\ s(f) = s(s) = s(\zeta) = 3,\ s(u^a) = 3,$$
$$t(4.34) = 2,\ t(4.19) = t(4.29) = 1,\ t(4.3) = t(4.4) = 3.$$

By inspection of III it is seen that the indices satisfy Leray's requirements.

Also the set of differential operators in III, subject to the condition (4.7), coincides with a set of operators which is known from [12] to define a strictly hyperbolic system. Thus III is a strictly hyperbolic Leray system and as such its associated Cauchy problem has a unique local solution in the class specified. Now let a_k be analytic Cauchy data for II such that $a_k \to d$, and sufficiently many derivatives of a_k converge uniformly to the corresponding derivatives of d. By Proposition 5 there is a unique analytic solution S_k for (II,a_k). Let d'_k denote the Cauchy data for the corresponding problem with III. In the uniqueness proof above we showed S_k satisfies III and that d'_k is determined by d and S_k so that S_k satisfies (III,d'_k). Due to the uniform convergence of a_k and its derivatives, d'_k converges to d', which is

Cauchy data for III. Since III is a strictly hyperbolic Leray system, there exists a solution S of (III,d') such that $S_k \to S$ and $d'_k \to d'$. Hence S satisfies (II,d) which by Proposition 1 implies S satisfies (I,d). This completes the proof.

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