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APPLICATIONS OF OPTIMAL CONTROL THEORY TO MACROECONOMIC STABILIZATION POLICY

by

Christopher Frederick Baum

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Economics)
in The University of Michigan
1977

Doctoral Committee :

Professor E. Philip Howrey, Chairman
Professor Robert S. Holbrook
Associate Professor Daniel L. Rubinfeld
Professor W. Allan Spivey

DEDICATION

This dissertation is dedicated to my wife, MaryAnn Albertines-Baum, whose encouragement, support, and understanding was a vital addition to my own energies in times of frustration and difficulty.

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This study would have been impossible without the assistance and impetus provided by a number of individuals. This acknowledgement can only begin to recognize these contributions. Phil Howrey's interest in my work and endurance in reviewing its progress was invaluable, and managed to convert my research efforts from aimless wanderings to a directed goal. Bob Holbrook's thoughts and critiques of control theoretic issues were extremely helpful in designing the research. Dan Rubinfeld and Allan Spivey both provided useful perspectives from their points of view.

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CHAPTER I

INTRODUCTION

1. Rationale and Objectives

The techniques of macroeconomic stabilization policy have been widely studied within the framework of large-scale economic forecasting models. Through deterministic and stochastic simulation techniques, the properties of these models and the interactions between multiple targets and instruments have been rigorously examined. The very complexity of these forecasting models tends to obscure the underlying linkages between the instruments of stabilization policy and target values for the crucial variables. Furthermore, the lag structures and other dynamic relationships require that such linkages may only be analyzed via multi-period simulations.

The study of stabilization policy via such simulations has inherent limitations, not the least of which is a lack of comparability between various models' performance. To go beyond the simple equation-by-equation comparisons of tracking error, a normative statement about relative preferences must be made: e.g. a measure of "welfare", or

conversely "loss", associated with a given simulation's outcomes. This approach enables an entire model's performance to be summarized in a scalar measure, comparable over models for identical welfare functions and horizons.

The application of a "welfare" measure to a simulation model has gained support in some current research. In the simplest case, it involves the examination of policy multipliers and, based on these multipliers, construction of a number of "reasonable" policies which are then simulated. The resultant paths of target variables are then "costed" via the welfare function and the best policy selected on the basis of minimum cost. A more sophisticated approach "inverts" the model, endogenizing instruments and exogenizing target variables, to compute the required instrument settings directly. The targets and instruments are then "costed" as before. In either procedure, the additive uncertainty associated with equation disturbances is typically ignored unless stochastic simulations are performed. It may be recognized that this approach involves two distinct steps, separating the solution of the model from the evaluation of the loss associated with that solution. This separability implies that the solutions produced are not "optimal", in the sense that they do not involve an explicit optimization procedure, but rather rely

on *ad hoc* policy determination.

Two faults may be found with this approach when the object of research is the determination of optimal stabilization policy. First, the procedure is inelegant and somewhat clumsy. Rather than an integrated approach which clearly has as its objective the determination of optimal policy, this process involves the inversion of behavioral equations, which may be a complex matter in the presence of multiple instruments entering each of several equations. Furthermore, only the magnitudes of the appropriate policy settings are given; little (if any) information is available regarding the sensitivity of policy to differing exogenous conditions. Secondly, and perhaps more serious, is the loss of information in the operation of this two-part procedure. When the original model is estimated, not only the additive uncertainty but also the parametric uncertainty is quantified. The numerous econometric techniques which have been developed for improved model estimation are, to a large extent, concerned with refining the estimates of variances of the estimated coefficients. If an estimated model is used for stabilization purposes and all estimates of parametric uncertainty are disregarded, a large part of the information gained in the estimation procedure is discarded. A spurious sense of precision is lent to the results by this

exclusive use of point estimates. Furthermore, the relative stability of the various parameter estimates is ignored; all multipliers take on their estimated means, and there is no consideration of the "loss" involved with the use of a policy whose coefficient estimates possess a very wide confidence interval (a concern analogous to risk aversion by policymakers).

These two critiques are clearly fabricated to argue for a more sophisticated and integrated approach to the problem. Instead of testing (via simulation) alternative stimuli, then running that "optimal" instrument set, and computing its cost, we may combine all these problems into one exercise. I refer to the optimal control of economic systems, in which the economist (like any rational being) exhibits maximizing behavior. In this approach, a scalar-valued welfare function is specified and maximized subject to the dynamic constraints of the economic model at hand. Through various analytic solution techniques (both dynamic programming and Lagrangean multiplier methods have been used) the discrete-time control problem may be solved iteratively for each period in the planning horizon. The result will be an optimal trajectory for each of the model's endogenous variables, based upon the values of the exogenous variables and parameters. Perhaps more importantly, the

optimal control "rules" which produce these trajectories (expressing instrument settings in terms of lagged target and instrument values) can be analyzed. To summarize the model's performance, the scalar welfare loss associated with the solution is computed. In this way, the optimal performance of a model with a given set of instruments may be reduced to a single number. Experimentation will readily indicate how sensitive this "loss" may be to alternative instrument sets, coefficients, and policy horizons.

In this study, we will discuss some of the applications of optimal control theory to the design and implementation of macroeconomic stabilization policy. Attention will be directed to the limitations of the various methods currently utilized, and to possible extensions of that repertoire. Finally, we will consider the techniques by which suboptimal situations may be evaluated vis-a-vis optima, and examine the difficulties involved in adapting largely linear techniques to the control of nonlinear stochastic systems.

2. Outline of the Study

The plan of this study proceeds as follows. The fundamental concerns which confront the practitioner of optimal control experiments are discussed in Chapter II. These include the nature of the model, nature of the objective function, and the type of uncertainty to be included. This chapter also discusses the basic algorithm used, and contrasts its characteristics with others found in the literature. Chapter III presents the model to be subjected to control, giving its background and detailed specification. Estimates of the linear and nonlinear forms of the model are presented here. Some of the relevant characteristics of the model are analyzed in Chapter IV, including the results of dynamic simulations, multiplier analysis, and the computation of stability properties. This chapter also details the procedures required to calculate restricted reduced form estimates, and contrasts these with their structural counterparts.

The results of optimal control policy experiments performed on the linear model are presented in Chapter V. A set of "certainty equivalence" control experiments is discussed, followed by their fully stochastic counterparts. A useful application of feedback control techniques, the

computation of analytic derivatives of the feedback rule, is considered. These derivatives are then put to use in one of three suboptimal control evaluations. These experiments compare "steady-state" feedback rules, the modelling of historical policies, and arbitrary feedback rules with their fully optimal counterparts. Finally, Chapter VI presents the results of optimal stabilization policy experiments conducted with the nonlinear version of the model, and contrasts these results with those obtained from a one-time linearization of the same structure. A discussion of the difficulties of incorporating parametric uncertainty into the control of nonlinear systems and some suggestions for further research are given.

CHAPTER II

OPTIMAL CONTROL OF MACROECONOMIC MODELS

0. Introduction

In this chapter, the fundamental considerations which arise in applying optimal control to a macroeconomic model are discussed. The first section examines the nature of the model to be subjected to control. The procedures used for controlling linear and nonlinear models are contrasted, and their applicability to large models is considered. A parallel concern is discussed in Section Two : the nature of the objective function. As we shall see, much of the research done has utilized a quadratic objective function. The third section considers the nature of the uncertainty to be included : basically additive only, or additive and multiplicative. In Section Four, all of these concerns are combined in an exposition of the Linear-Quadratic-Gaussian dynamic programming algorithm for closed-loop, or feedback, control. This algorithm will be used exclusively in the remainder of this study. Section Five discusses the essential similarities and differences between this approach and the Theil "open-loop" control. The results of some recent research indicating when the two approaches are

equivalent are discussed in this concluding section.

1. Nature of the model to be controlled

As the theory and practice of econometrics has advanced, macroeconomic models have become more complex, using more complicated specifications as well as dealing with economic relations at a disaggregated level. Even a relatively modest-sized national forecasting model, such as Hymans and Shapiro's "Michigan Model" (1973), is characterized by sophisticated distributed lag structures and nonlinearities in the variables. Many models also include complex equations which are nonlinear in the parameters as well. A major concern in the application of optimal control techniques to an operational macroeconomic model must certainly be the nature of that model.

Two types of problems arise : one of degree, the other modifying the fundamental approach to control. The first set of problems are those related to the sheer size of the model. As Pindyck has aptly stated (1973, p. 7), we are used to thinking of a model's size as being the number of endogenous variables. However, the order of each equation (that is, the maximum lag on any endogenous or control variable) is just as important. When a model is set up for

closed-loop optimal control, the usual practice involves transforming the behavioral equations and identities into "state vector" form. In this form, each lagged endogenous and control variable is redefined as a new variable by introducing a set of identities. This permits the entire system to be reduced to first order in the endogenous and control variables. The state vector then contains all current endogenous and control variables of importance to the system at a point in time, while the specification of the lagged state vector's values suffices to determine initial conditions. The transformation of an economic model into state vector, or state variable form is adequately discussed in Pindyck (1973, Appendix) and Chow (1975, pp. 153-154).

The size of a model in state vector form will therefore depend not only on the number of equations, but will also depend on their order. It may be readily seen that a model with a large number of high-order distributed lags will introduce a large number of elements into the state vector, and thus cause the dimension of the problem to be greatly increased. Further problems will arise if a popular algorithm such as Chow's (loc. cit.) is utilized without modification. This procedure requires the specification of the maximum lags in the model on both endogenous and control

variables and creates state variables for all endogenous and control variables up to that order. Thus, if a four-period-lagged level of income appears in one equation, state variables will be created for consumption lagged one, two, three, and four times, even though none of these lagged values appeared in the original model. This feature of the algorithm may obviously be circumvented, but it indicates the sensitivity of control solution techniques to the order of the model.

Another element of the solution procedure which becomes sensitive to size is the variance-covariance matrix of estimated coefficients required in stochastic control. Even though the reduced form coefficient matrix for all state variables and exogenous variables may be of modest size, their covariance matrix will not be very tractable. A six-equation model with fifteen predetermined variables will have 90 reduced form coefficients, not all nonzero; but those coefficients will have a variance-covariance matrix containing 4085 distinct elements. Thus, if stochastic control is to be performed, considerations of size become quite important.

The second set of problems goes beyond the dimension of the model to be solved. To this point, we have been

discussing the size of a model which can be expressed as a linear reduced form. A more critical concern than size (which can always be circumvented to some degree by programming) are the inherent nonlinearities present in many operational macroeconomic models. Since the models' tracking ability may often be a result of carefully specified nonlinearities, linearization may drastically change a given model's dynamic behavior. This is clearly a concern specific to the individual model since models differ widely in their use of nonlinearities. Two routes may be taken : the first, perhaps repugnant to the modeler, involves either estimating a linear model or linearizing (once and for all) an inherently nonlinear model. Either approach may impose serious constraints, before or after the fact, on the econometrician or forecaster who wants to use optimal control techniques.

The second route involves dealing with the model as estimated by utilizing a nonlinear control algorithm. As in model estimation, the introduction of nonlinear solution techniques imposes significant multiples of cost and effort on the straightforward linear program. Well-defined analytical solutions must give way to iterative approximations, and the inherent concerns over speed and existence of convergence, appropriateness of initial

estimates, and the like. The most telling argument for linear control may perhaps be the existence of a cost constraint : the iterative techniques required to solve a large nonlinear model may prove excessively costly, and thus preclude experimentation. However, this may represent one point on a continuum of falling costs of computation, which makes possible today what would have been unthinkably expensive (if indeed tractable) a short time ago. The real concern is that of the tradeoff involved : is nonlinear control worth the added computational expense? That, unfortunately, is a question that cannot be answered within the context of control theory alone, but rather must involve the nature of the economic model's nonlinearities and the payoff expected from experimenting with a more "correct" model. Many researchers have suggested that trials must be made with a given model to examine just how sensitive the optimal policy is to linearization. This involves the specification of the linearized form of the nonlinear model, and computation of optimal solutions for both linear and nonlinear models. To some degree, an exposition of this practice is presented in Chapters V and VI of this study.

2. Specification of the Objective Function

The closed-loop optimal control with which we are

concerned can be viewed as a classical optimization problem, in which we minimize an objective function subject to the constraints given by the structure of the macroeconomic model. This objective function usually contains as arguments the deviations of certain endogenous variables, denoted target variables, from their "desired" values. As in estimation theory, the simplicity of a "least squares" approach suggests the use of a quadratic function in these deviations. Most optimal control experiments have, therefore, made use of a quadratic objective (or welfare cost) function, composed at least in part of the weighted sum of squared deviations of controlled variables from desired values.

The reasonableness of a quadratic cost function applied to a linear model is partially due to the resulting linearity of feedback control rules derived in the solution. However, a restriction that must be considered is the peculiar symmetry of the quadratic function, wherein a 4% unemployment rate target will create equal "loss" if either 5% or 3% were realized. This can be solved in part by specifying less attainable targets, so that achieved values always fall on one side of the target. It should be noted that the quadratic cost function need not limit the user too strenuously, since any endogenous variable or control

variable may be "costed". Thus, we may indicate both a desired rate of inflation and a desired rate of growth of the money supply, and by varying the weights attached to these targets alter the attainment of each. Since the generalized cost function is a quadratic form, off-diagonal elements of the matrix of the form may be used to penalize the cross product of the deviations of target variables from specified target values.

We see, then, that almost any set of preferences may be modelled within the constraint of a quadratic objective function. As Holbrook (1974, p. 158) has pointed out, artificial variables may be added to the function for a nonlinear control problem which yield results similar to a "piece-wise" preference function (see Pindyck, 1973, p. 40). Thus, small deviations may be given zero cost, while larger deviations may be costed at increasing - and asymmetric - rates. The limit to specifying the cost function seems to lie in being able to substantiate one's choice for a more complex alternative, keeping in mind that the the choice of welfare function must be made with some concern over what lies within the feasible frontier for the macroeconomic model at hand.

3. Nature of the Model's Uncertainty

The third consideration in performing optimal control is perhaps the most crucial in terms of defining the nature (and complexity) of the problem to be solved. The type of uncertainty to be taken into account in controlling the model must be specified. Every econometrically estimated model contains, as part of its inherent structure, estimates of the stochastic properties of the error terms and estimated parameters. When a model is used for forecasting or simulation, these stochastic properties are often ignored, and the experiment performed as if the true model were known without error. It should be readily apparent that optimal control of such a system is a trivial task, involving only the "inversion" of the model to obtain the control variables as functions of desired targets. If there exists a large enough set of independent instruments with independent effects on targets, a set of consistent target variables may be attained exactly. Welfare cost - the value of the objective function - will be minimized at a value of zero.

This type of experiment is clearly an academic exercise, and totally ignores any concept of random behavior on the part of economic agents. We might reject it, then,

and decide to include the knowledge we have regarding the behavioral equations' error terms in our calculations. If the only random processes are assumed to be additive error terms possessing the usual Gaussian properties, a linear model subject to optimal control will possess the "first period certainty equivalence" of Theil (1958). The first period setting of the control variables based upon a deterministic solution will be appropriate, no matter what random shocks may occur in that period. In Theil's approach, this "open loop" control consists of calculating T control settings for a T-period horizon, and making use of the period one setting. After observing period one's realized values, a (T-1) period horizon (from periods two through T) is specified, and the first period setting of that horizon used to determine period two's optimal policy. The process would be repeated for horizons of length (T-2), (T-3), ... 1. In each optimization, the first period setting which is calculated by ignoring the random disturbances will be optimal. The expected welfare cost will not, of course, be zero; one obvious cause is the existence of positive probabilities of nonzero random disturbances occuring during the horizon.

This "certainty equivalence" control has a parallel in the "closed-loop" or "feedback" algorithm discussed by

Pindyck (1973, Ch. 3) and Chow (1975, Ch. 7-8). In this approach, the T-period horizon is treated as T separate problems. Using Bellman's (1957) principle of optimality in dynamic programming, the period T problem is solved as a functional form. That is, rather than calculating the optimal settings for policy instruments, optimal feedback rules are calculated, which allow the policymaker contemplating period T to make use of all observations through period (T-1) in designing the optimal period T policy. Having derived such a feedback rule for period T, one may then step back and calculate a rule for period (T-1), knowing (via the aforementioned principle) that future decisions will be made optimally. Thus, the process is continued backwards in time until period one's optimal rule is calculated as a function of initial conditions; which, being known, permit the calculation of the optimal policy settings for period one.

As with the open-loop approach, only the first-period policy settings calculated via the feedback rule will be appropriate. Although T periods of control settings may be calculated via the feedback rules, they are only correct in an expectational sense. However, no matter what random shock may be realized in period one, the period two feedback rule will be optimal, and will suffice at that time to

generate period two's setting as a function of observed period one conditions. Using the same logic, period three's, ..., T's feedback rules may be considered optimal. Thus, in the closed-loop approach, we only need perform one set of calculations over the T-period horizon, under the simplifying assumption that our knowledge of the stochastic processes and the values of exogenous variables will not be enhanced during the horizon.

This type of optimal control, in open-loop or closed-loop form, we will designate as "certainty equivalence" control. It has been the technique used in a great deal of the research done with macroeconomic models; for instance, Pindyck's (1973) study deals with it exclusively. This preference has certainly been dictated by the relative ease with which it may be performed. The major alternative, with which this study is primarily concerned, we designate as "stochastic" control. It takes into account not only all the stochastic properties of error terms, but also considers the uncertainty associated with each of the model's estimated parameters. A quantum leap in complexity must be made at this point, as the values of all previously "known" coefficient matrices must be replaced by their expectations. This is naturally much more readily achieved in theory than in practice. The "expected" nature of control trajectories

is even more apparent in stochastic control. As before, the first period settings are optimal, but further settings are conditional upon disturbances assuming zero values and all parameters equalling their point estimates. This implies that the future settings will never be used in the presence of uncertainty. However, the feedback rules used to generate those settings may be used during the horizon to calculate fully optimal instrument values.

In stochastic control, the welfare cost may be expected to exceed zero for two reasons. As in certainty equivalence control, there will be a nonzero probability of random disturbances occuring in each period. Added to this expected cost we must add the likelihood of parameters assuming values other than their point estimates. It should be noted, however, that the certainty equivalence result - that open-loop or closed-loop control will yield the same first-period control settings - does not in general carry over to the stochastic control case. This topic will be more fully discussed in Section Five.

The advantages of a stochastic control solution for policy analysis should be evident. If the policymaker accepts a particular econometric model, he or she also accepts the precision implied about the coefficients'

estimated values. The considerable amount of information gathered in the estimation procedure will certainly make the control solution more realistic. Two caveats must be offered to this argument, however. First, the performance of stochastic control adds a significant burden of cost and complexity to all phases of experimenting with a given model. Pindyck was concerned that the solution of a general stochastic control problem "could be an insurmountable task" (1973, p. 38). This conclusion may have been altered in the last four years by vast increases in computational capability, decreases in costs, and the emergence of new algorithms; but it still remains that a stochastic problem is more costly and thus less attractive to the prospective researcher than its deterministic counterpart. We may hope that further advances in the computational sphere will render more of these problems tractable.

The second and more serious concern is the one type of random error not currently integrated into optimal control calculations. We refer to the problem of model specification error; as is well understood, significance tests are posited on the maintained hypothesis that a specified nonrandom relationship exists. This maintained hypothesis is rarely tested, and indeed classical statistics does not give us a straightforward test of such a

hypothesis. Some researchers have suggested that classically estimated econometric models cannot meaningfully be used to analyze stabilization policy (cf. Lucas, 1976). Their arguments rest on the hypothesis that behavioral relations are fundamentally altered in response to changes in policy, so that current techniques are rendered worthless. The work that has been done in analyzing the effects of specification error on macroeconomic optimal control has stopped short of these radical critiques, and suggested instead that a set of models be tested, assuming that the "correct" model is contained within the set. Chow suggests the use of a minimax criterion to evaluate the strategy of testing competing models (1977). The most reasonable conclusion might be to present all optimal control results conditional upon the model, as is currently the practice with simulation results.

4. The Linear-Quadratic-Gaussian Dynamic Programming Algorithm

In the previous three sections, several considerations encountered in the application of optimal control to a macroeconomic model were discussed. The discussions of model and objective function specifications and of the nature of the model's uncertainty have necessarily alluded

to the technique used to solve the optimal control problem. This section formally presents that technique as the Linear-Quadratic-Gaussian (L-Q-G) dynamic programming algorithm, and indicates how it may be adapted to the control of a nonlinear system. These expositions are made acknowledging a debt to Chow's (1975) writings, in which an extremely clear development of these topics is made.

To digress slightly, we should introduce the optimal control problem as a classical optimization problem. Using the familiar mathematical economics techniques, we may view it as a constrained optimization problem : to minimize the loss function subject to the constraints posed by the economic model. The usual application of Lagrangian multipliers to this class of problem may be performed. The only feature differentiating this type of problem from those commonly found in economic theory - for instance, the standard consumer choice or profit maximization problems - is its dynamic nature, with a finite horizon in discrete time. Examples of this approach are to be found in Pindyck (1973, Ch. 2), in which a "discrete-time minimum principle" is developed and in Chow (1975, Ch. 7).

The technique used in this study is analogous to the Lagrangian multiplier approach, but differs in its

structure. We speak of the method of dynamic programming, illustrated by Bellman (1957) and favored by control engineers and economists such as Aoki (1967). The objective function, as we have previously stated, is a quadratic form in the deviations of the state vector from its desired value; expected losses are summed over the T-period horizon to yield the scalar expected loss, or "welfare" measure W:

$$(2.1)\quad W = SUM\{t=1,T\}\ E\ \{\ \{y(t) - z(t)\}'\ K(t)\ \{y(t) - z(t)\}\ \}$$

The operator SUM{t=1,T} indicates that the following expression is to be summed over all the time periods from one through T. The vector y(t) denotes the state vector : that is, the vector of (at least) current endogenous and control variables. (Depending on the order of the econometric model, the state vector may also contain lagged endogenous and lagged control variables; however, these would never be targeted by the objective function). The vector z(t) contains the target values for elements of y(t); only those variables whose values we are concerned with need be specified. K(t), the matrix of the quadratic form, is usually a diagonal matrix, although it need not be so. It may be specified to vary over time; commonly, however, it will be chosen as constant over the horizon, specifying

constant relative "weights", or tradeoffs, between the target variables at every point in time (and thus no discounting of future values).

The optimization problem posed by optimal control is the minimization of the objective function above subject to the constraints of a macroeconomic model. We will assume that we have produced the reduced form of a linear model:

$$(2.2) \quad y(t) = A(t)\, y(t-1) + C(t)\, x(t) + b(t) + u(t)$$

In this case, we have already written the model in state vector form, subsuming all lags higher than one in the state vector. The matrices A(t) and C(t) contain the reduced form coefficients on the lagged state variables and current control variables, respectively. We might consider these coefficients as functions of time, but will henceforth consider them as time-invariant. The vector b(t) contains the product of all exogenous (non-control) variables entering the reduced form equations with their respective coefficients. Thus, all truly autonomous variables and equations' intercepts may be subsumed in a single vector. The vector u(t) contains the additive disturbance terms for the reduced form equations.

Formally, we may consider the L-Q-G control problem as involving the choice of control vectors $\{x(1), x(2), \ldots, x(T)\}$ to minimize the mathematical expectation of W. This problem has initial conditions given by a lagged state vector $y(0)$. We must at this point specify the system's stochastic properties. At its simplest, we will examine the "certainty equivalent" case of coefficients known with certainty; the elements of $A(t)$, $C(t)$ and the coefficients in $b(t)$ are fixed and denoted as A, C, and p. We then make the assumptions that $E\{u(t)\} = 0$ and $E\{u(t)u(t)'\} = V$; that is, the system's random disturbances have a zero mean vector and a variance-covariance matrix V. One further assumption is characteristically made : that we minimize the expected value of W, with the expectation conditional on the information available at time zero. This precludes any "learning" about the stochastic properties of the model during the horizon. (For a discussion of "learning" in a control framework, see Chapter 11 of Chow (1975)).

The dynamic programming "closed-loop" solution proceeds as follows. The problem for period T may be solved as a static problem in the initial conditions $y(T-1)$, yielding the optimal settings $x^*(T)$ as a function of $y(T-1)$. The last two periods' optimal values may then be found as functions of the initial conditions $y(T-2)$. In truth, only

one set of values {x(T-1)} need be solved for at this time, since the optimal x*(T) has already been determined. This reduces the order of the problem to solving for only one period's optimal x(t) at each point in time. The principle of optimality in dynamic programming ensures that this technique is appropriate, since at each stage all future policies will be optimally determined. By the use of this recursive procedure every vector x*(t) may then be determined, back to x*(1) as a function of initial conditions y(0). The most important features to be noted are the backwards recursion used to solve the problem, and the closed form nature of the process. The solution is expressed as a function of conditions at each point in time, not as a set of optimal values per se. This guarantees, as we have mentioned previously, that policy may be made according to optimal feedback rules, which take into account the state of the economy.

Mathematically, the solution proceeds as follows. In the last period, we want to minimize the Tth term of W, which we shall denote as w(T), the loss incurred in and following period T. Assuming that all expectations are conditional on time zero, we minimize:

$$(2.3)\quad w(T) = E(0)\ \{\ \{y(T) - z(T)\}'\ K\ \{y(T) - z(T)\}\ \}$$

Or, setting $H(T) = K$, $h(T) = \{K\ z(T)\}$,
and $c(T) = \{z(T)'\ K\ z(T)\}$,

$$(2.4)\quad w(T) = E(0)\ \{\ y(T)'\ H(T)\ y(T) - 2\ y(T)'\ h(T) + c(T)\ \}$$

Substituting the model (2.2) for $y(T)$ and taking expectations, we may write $w(T)$ as a function of $y(T-1)$, $x(T)$, and $u(T)$ alone. This function may be partially differentiated with respect to the vector $x(T)$, yielding:

$$(2.5)\quad dw(T)/dx(T) = 2\ C'H(T)\ \{Ay(T-1) + Cx(T) + b\} - 2C'h(T)$$

This expression, when set equal to zero and solved for $x^*(T)$, the optimal policy, yields:

$$(2.6)\quad x^*(T) = -\{C'H(T)C\}^{-1}\ \{C'H(T)A\ y(T-1) + C'\{H(T)b-h(T)\}\ \}$$

or, written in more compact form,

$$(2.7)\quad x^*(T) = G(T)\ y(T-1) + g(T)$$

which displays the optimal setting for period T as a linear function of the observed conditions y(T-1). This setting is thus a feedback rule for optimal policy, consisting of the matrix G and vector g. These are in turn functions of all the coefficients, target values, and exogenous variables in the model.

The solution to the two-, three-, or T-period problem is characterized by the same formulae being recursively applied. The two-period problem seeks to minimize w(T-1), the total loss incurred in and after period (T-1). This can be written as the sum of w(T) and the additional loss incurred in period (T-1), knowing that x(T) has been optimally chosen. Likewise, matrix H and vectors h and c must be expressed in terms of the two-period problem. The solution will yield an optimal x*(T-1) as a function of y(T-2). The explicit forms of the backward recursion equations are given in Chow (1975, pp. 177-179), and in Chapter V of this study. A complete solution to the T-period problem will consist of the T feedback rules {G(t), g(t)}, the expected total welfare cost W, and the expected trajectories of the control variables {x*(1), ..., x*(T)} and of the state vector.

The procedure thus far has included only the additive

uncertainty associated with u(t). Let us now consider the general stochastic problem posed by unknown parameters : that is, assume that we have estimated the elements of A, C and b(t) and their joint distributions. The solution involves minimizing welfare cost subject to the model, with the stochastic nature of the model's parameters correctly included. Again, a crucial assumption must be stated. In eqn. (2.3), the expectation is stated conditional on time (T-1), but we took the expectation conditional on time zero, with no learning assumed. That assumption becomes even more important in this context; although the parameters' expectations in the Tth period should be conditional on time (T-1), we are approximating their joint distributions at time (T-1) by their distributions as known at time zero. As Chow points out (1975, p. 230), this approximation cannot overstate the efficacy of optimal control, but may only yield a conservative estimate of the gain from its application. This assumption will be made henceforth in this study. The stochastic control problem can thus be stated analogous to the "certainty equivalent" problem. The same basic formulas are utilized, with the differences lying solely in how they are evaluated; that is, involving the substitution of parameters' expectations for their point estimates.

Finally, let us consider the possible applications of the L-Q-G algorithm to the control of a nonlinear model. This same set of techniques may be used to calculate optimal linear feedback rules for a nonlinear system. First, of course, the system must be linearized around a tentative solution to produce an approximate reduced form. Many computational techniques have been used to linearize a nonlinear system. Chow's procedure numerically differentiates the model in the neighborhood of the tentative solution to provide estimates of reduced form parameters. The optimal control then provides a first-stage estimate of vectors $x^*(t)$; these may then be substituted through the original nonlinear system to give new values for the state vector. A second-stage linearized reduced form may then be calculated, and another optimization performed. Convergence (within an arbitrarily small limit) of the optimal control variables' settings $x^*(t)$, from one iteration to the next, denotes that an approximately optimal solution has been found.

Several concerns must be noted. First, this procedure assumes only additive uncertainty, and will require significant modification to incorporate parametric uncertainty. Second, this "certainty equivalent" analogue is of necessity only an approximation to the optimal policy.

As Howrey and Kelejian (1971) have noted, the characteristics of nonlinear structural equations do not in general carry over to a linearized reduced form. The nonlinearities of the model destroy the certainty equivalence of a linear model's solution. The details are discussed in Chow (1975, pp. 289-292). However, as he points out, the equations will probably be close to optimal if the linear approximation is reasonable, that is, if the model is not drastically nonlinear. Unfortunately, these caveats reveal the current state of the art in control of nonlinear models. No non-approximative method exists, and qualitative judgments must be made to indicate how appropriate a control solution based on a linearized model may be.

5. Comparison of Closed-loop and Open-loop Control

A great deal of discussion has taken place in the literature regarding the relative merits of "closed-loop" versus "open-loop" control solution techniques (see, for instance, Chow (1975, p. 160, p. 173), Fair (1974), Holbrook (1974), etc.). In favor of the closed-loop, or feedback approach, Chow argues for the smaller dimensionality of that technique, and suggests that the feedback rules themselves may be studied using various techniques of dynamic analysis.

Arrayed in favor of the open-loop approach are economists and control engineers such as Norman, Norman, and Palash (1975), Fair (1974), and Craine and Havenner (1973). Much of their discussion has centered on the choice of a computer algorithm to be used in solving the open-loop problem. Characteristically, the open-loop advocates have been concerned with the large nonlinear models currently used by forecasters. Many have suggested that only a numerical solution can feasibly be obtained for a model of operational size; the analytical solutions given by the feedback methods have been assumed too cumbersome.

We shall not dwell further on this portion of the controversy, except to note that advances in computer technology and software have made analytical solutions increasingly accessible. The parametric uncertainty solutions presented in Chapter V are expensive relative to single-equation regressions, or simulations of the same model, but not exorbitantly so. Chow has applied certainty equivalence techniques to control the nonlinear "Michigan Model" (1977), which although not comparable in size to some, is a working forecasting model of disaggregated nature. There should be no doubt that continuing declines in the price of computing services will render larger and larger experiments feasible.

One important concern must not be overlooked, at a much more fundamental level than that of controlling operational forecasting models of the macroeconomy. That is, are the open-loop and closed-loop approaches merely alternative technical implementations of the same optimization problem? Much of the discussion has failed to address this topic, relying on Norman's (1974) theorem which proves the logical equivalence of the approaches in a known coefficients framework. Theil's original work (1964) dealt with non-autoregressive models and known coefficients. Those who have extended his methods have often implicitly assumed that violation of these conditions does not alter the similarities between closed- and open-loop approaches. The Norman theorem has served as a foundation for such inferences.

In a forthcoming article, Holbrook and Howrey (1977), denoted as H-H, demonstrate that the policy rules derived in open- and closed-loop stochastic control will indeed differ, under quite general assumptions. The remainder of this section draws heavily from the results of that research.

In their exposition, H-H consider a single-equation policy model containing a lagged dependent variable, current and lagged control variable, an exogenous variable, and an

additive disturbance. Using only a two-period horizon, they analytically demonstrate that the two first-period instrument settings will be equivalent:

1) if there are no lagged dependent variables in the model;

2) if there is no uncertainty about the contemporaneous effects of policy; or

3) if various combinations of variances and covariances are fortuitously zero.

The latter possibility aside, these quite striking results suggest that, for a "reasonable" model, whenever parameter uncertainty is considered, the Theil procedure cannot be assumed a substitute for feedback control. The "reasonableness" of the model relies only upon the presence of lagged endogenous variables and/or unknown policy multipliers. Since the former condition involves even known values of lagged endogenous coefficients, H-H argue that "... the explicit consideration of parameter uncertainty destroys the equivalence of the Theil and Chow procedures, for all practical purposes." (1977, p. 9)

The question might be raised, of course, of why we claim that the Theil procedure may no longer be considered a substitute for Chow's technique, and not vice versa. H-H go on to indicate precisely why it is that the Theil procedure is found wanting. The Chow feedback procedure explicitly

assumes that each period's policy setting will be made after observing last period's outcomes. The Theil procedure, on the other hand, is optimization over a T-period horizon with the inherent assumption that all T policy settings will be followed. Although all practitioners advocate reoptimizing for a (T-1), (T-2), etc. period horizon so that only the first-period (certainty equivalent) settings are used, their derivation is based upon assumptions inconsistent with practice. This (not surprisingly) results in a different implied behavior than that of the closed-loop, continuous adjustment framework. It may also be demonstrated that the feedback approach involves a smaller expected loss than the Theil procedure. This rather noteworthy conclusion actually demonstrates the non-optimality of open-loop control, as compared to closed-loop control in which the policymaking assumptions are satisfied.

H-H also indicate how, with assumptions regarding intercorrelation of coefficients, the intuitive result may be obtained: that the non-feedback policy action will be more conservative than its feedback (and thus correctible) counterpart. With uncertainty only about the coefficient of the control variable, it is possible to prove that the Theil policy setting will differ more widely from the Chow policy setting, the longer is the policy horizon. Again, the

implicit (if not honored) assumption that Theil settings are chosen once and for all hinders the policymaker.

H-H conclude that the expected outcome of feedback control "will be superior to that of the Theil policy over both the short- and the long-run." (1977, p. 16) This research should make clear the result that open-loop and closed-loop methods are not the substitutes that many have incautiously believed them to be. Whatever the cost advantages and feasibility of performing open-loop control, considerable thought must be given as to whether the experiment is really what it is stated to be. The analytical solution to the constrained optimization control problem in macroeconomics may be quite different in nature from the numerical solution of its open-loop counterpart.

CHAPTER III

A MODEL OF THE U.S. ECONOMY

0. Introduction

After examining the fundamental issues and concerns involved with the use of optimal control theory in the analysis of economic policy, we turn now to an exposition of the model to be controlled. This chapter gives a detailed description of that model and its variants. The first section explains the rationale for development of this particular model, and compares it to several similar models used in previous macroeconomic research. Section Two contains the details of the model's specification, explaining some of the economic foundations underlying the current form of the model. Finally, Section Three presents the estimates of both the linear and nonlinear forms of the model.

1. Rationale for Development of the Model

In examining the applications of optimal control theory to macroeconomic stabilization, it must be noted that any study is to a large degree model-specific. Accordingly, we

now present the basic model to be utilized: a quarterly econometric model of the U.S. macroeconomy. The model is relatively small in comparison with the large-scale forecasting models currently in use, but in its simplicity it retains many of their characteristics as basic features. Examination of the model's structure reveals a fairly standard form, with highly aggregative behavioral equations utilized to represent the major neo-Keynesian components of macroeconomic theory. Several sectoral aggregate demand equations structure the fundamental multiplier-accelerator framework. Asset demand functions for highly stylized asset categories express the price and income effects of interest rates and wealth accumulation of financial assets. The behavior of the aggregate price, wage and unemployment levels completes the basic structure of the model.

The model's design and specification draws heavily from the presentations of Pindyck (1973), Pindyck & Rubinfeld (1976) and Howrey (1975). Pindyck's original work laid out a small, quarterly, and wholly linear model of the U.S. economy. His analysis involved optimal control of this structure, taking into account additive uncertainty alone. Pindyck and Rubinfeld upgraded this model by adding nonlinear specifications of several variables' growth rates. Both these forms bear a resemblance to that employed by

Howrey, who built a very similar framework for the study of spectral properties. Although the design of my study's model is very closely related to all three of these models - and especially Pindyck-Rubinfeld's (P-R) - it was decided not to merely take their model as the point of departure. The matters to be examined in this study required slightly greater detail in some sectors, and the recent revisions in national income accounts data made comparability with series published in P-R (pp. 412-416) very difficult. Nevertheless, my debt to Pindyck and Pindyck-Rubinfeld's studies must be clearly acknowledged. The exposition they present, especially the fundamental logic underlying the structure of their models, proved invaluable while designing this model.

As was indicated earlier, this study attempts to examine a number of state-of-the-art control techniques. Accordingly, as the development of general stochastic control is much more advanced for linear systems, a feasible model must possess a linearized counterpart. The model exposited here is nonlinear in fundamental structure, like P-R's, but unlike their model has been specified and estimated in completely linear form so that the more advanced linear control techniques may be directly applied. Two points should be made. First, there exist alternative

linearizations of any given model; for instance, a purely statistical linearization involving Taylor-series expansion may be performed. It must then be noted that the resulting linearized model, although statistically equivalent to the nonlinear version from which it was derived in the neighborhood of the expansion, will not in general be equivalent to a model specified and estimated with the constraint of linearity. That is, the linearized model presented here has been refined beyond the statistical linearization, since its specification has been tempered by consideration of best fit and *a priori* parameter constraints. The end result of this process necessarily bears a close resemblance to its nonlinear parent but of necessity possesses certain crucial differences. For instance, the wage-price-employment sector specification differs widely between linear and nonlinear versions. An additional variable was added to the linear price equation to help keep it on track in a system context. Various *ad hoc* respecifications were used, and will be pointed out.

2. Specification of the Model

The model's structure consists of eleven behavioral equations plus a number of identities. The foreign sector is assumed to have zero net demand, so that gross national product and total income (Y) may be expressed simply as the sum of consumption, investment, and government aggregate demand. Consumption (C) is treated as an aggregate, while three separate components of investment are identified: non-residential investment (INR), residential investment (IR) and investment in inventories (IIN). Governmental expenditures (G) are considered wholly exogenous. All components of aggregate demand are expressed in real terms. The national income identity can thus be written as:

(3.1) $C + INR + IR + IIN + G = Y.$

In addition to government spending, fiscal policy in the model involves the manipulation of two tax/transfer parameters. Disposable personal income (YD) is defined by the identity

(3.2) $YD = Y - T$

where T represents total net tax receipts. The tax

mechanism used is of a linear form, defined by

(3.3) T = TXRATE * Y + TO

where TXRATE is a decimal fraction, representing an average tax rate, and TO is a surtax (in constant dollars). The latter parameter is utilized to include a wholly linear tax instrument for the purposes of linear control. It should be noted that this surtax may be translated into a surcharge rate, where {TO/(T-TO)} expresses that rate as a decimal fraction. This simplistic tax mechanism, adopted from Pindyck & Rubinfeld (p. 373) permits the definition of disposable personal income to be rewritten as :

(3.4) YD = (1 - TXRATE) * (C + INR + IR + IIN + G) - TO

Let us now consider the specification of the sectoral aggregate demand equations. A consumption function of fairly standard form expresses consumption demand as a Koyck lag on the level of disposable income and the change in the level of "wealth". The latter term provides for the inclusion of non-income effects on consumption demand through the Pigou effect, adding another channel of effectiveness for monetary policy to the traditional Keynesian interest rate-investment channel. Pindyck-

Rubinfeld's specification (1976, p. 374) uses the level of "wealth"; high collinearity between income and "wealth" prompted a somewhat ad hoc differencing of the "wealth" series. The functional form used is a Koyck lag, where we express

(3.5) C = F{ DWEALTH, YD, C(-1) }

wherein the operator 'D' denotes first difference.

The categories of gross private investment demand are treated in a parallel fashion, all containing one or more lagged dependent variables. The non-residential investment (INR) component, measuring plant and equipment expenditures, is taken to depend on the level of income with a lag, representing the lag between investment decisions and their implementation. An acceleration effect is included, measuring the first difference of income; and the effects of changes in the price of capital are captured by inclusion of a lagged long-term interest rate. Two lags on the level of INR are included as well. Thus the functional form may be written as:

(3.6) INR = F{ DY, Y(-2), RL(-2), INR(-1), INR(-2) }

Pindyck and Rubinfeld go through the derivation of a similar behavioral equation at some length (cf. Pindyck, pp. 47-48, Pindyck-Rubinfeld, pp. 374-375). It should be noted that their first-differencing of non-residential investment is a special case of this function, as the above equation may be written as a "quasi-" or "semi-difference" in the level of INR, by transferring both lagged INR terms, with their estimated coefficients, to the left side of the equation. Like Pindyck & Rubinfeld, we omit any detailed investigation of depreciation, corporate profits and retained earnings, dividends, etc. Business fixed investment is always a weak point in a highly aggregative model, and a greatly improved performance may only be bought at the cost of extreme complexity, which is contrary to the aims of this study.

It must also be accepted that highly accurate forecasts of residential investment (IR) may only be constructed with the consideration of a large number of specialized market conditions measures, such as mortgage rates, governmental policies regarding insured and veterans' loans, zoning regulations, construction costs and availability of materials, etc. These conditions are proxied by formulating this demand as a function of disposable income (as it more clearly represents consumers' effective buying power of housing services) rather than total income. In addition, a

lagged measure of the spread between long-term (RL) and short-term private rates (RCP) is used, as well as the current short-term rate and two lags in IR itself. This specification is designed to capture both the current effect of the short rate on credit availability to facilitate ongoing construction, and the delayed effect of past mortgage market conditions on current construction. The function then is represented by:

(3.7) IR = F{ YD, RCP, (RL-RCP)(-2), IR(-1), IR(-2) }

Inventory investment (IIN), or the change in the stock of business inventories, has often been related (for the want of more specific a priori statements) to the levels and differences of income, disposable income, consumption, etc. Our equation measures inventory investment as a semi-difference in the semi-differences of disposable income and consumption. The equation may be fitted in the first-differences of YD and C as well, the choice being made on econometric rather than theoretical grounds. The functional form using the former treatment becomes:

(3.8) IIN = F{ YD, YD(-1), C, C(-1), IIN(-1) }

Leaving the sectoral demand equation block of the

model, we now consider the asset demand equations. Three broad categories of assets are explicitly recognized in this model: money, short-term financial assets (both public and private), and long-term financial assets. Implicitly, at least one major category is excluded from consideration by Walras' Law: the demand for real capital. In the actual implementation of such a framework, however, several pragmatic concerns intrude. First, it is somewhat unreasonable to apply any of the well-known Walras' Law constraints (cf. Brainard-Tobin, 1968), since literally hundreds of major asset categories are being excluded from direct consideration. For example, only the narrowly defined money supply (M-1) appears in the model, while the numerous other near-monies comprising M-2, M-3, et cetera are largely ignored. No direct consideration is made of corporate equities, although any complete model must surely account for accumulation and decumulation of this important category to satisfy flow-of-funds constraints. But in the (possibly misguided) spirit of most neo-Keynesian econometric models, the development of this model only entails a sketchy formulation of assets and interest rates.

Secondly, it is certainly apparent that asset demand functions should not differ in design from those functions measuring the demand for any good. That is, the quantity of

assets demanded should be expressed as a function of own and other prices, income, etc. An inconsistency appears here if one interest rate pertains to the omitted asset class; in our model, we omit the capital stock, but include a proxy for its price, the long-term interest rate. So completely parallel estimation, with asset categories regressed on prices and income, is not feasible. Furthermore, the form in which the model is usually written (e.g. the equilibrium demand for money regressed on an interest rate and income) serves well in the theoretical IS/LM framework, but must be inverted and formalized to predict the "own" interest rate. It was found that estimation followed by inversion - in essence, approximating the regression of X on Y - was vastly inferior to the direct regression of interest rates on asset stocks, income, etc. The predictions made by the inverted form of the equation were characteristically not stable in a deterministic simulation framework. Therefore, the "asset demand" functions for money and short-term financial assets were estimated in inverted form, while the long-term financial asset function was inverted for the long interest rate and specified as a pure term structure relationship.

The demand function for government securities of all maturities is considered to include the short-term (three month) Treasury bill rate (RTB). A naive term structure

assumes that movements in this short rate will be closely paralleled by movements in long-term Government rates. The inverted demand function thus will yield RTB as a function of levels and/or differences in income, outstanding Government securities, and the aggregate price index. Any model of this simplicity necessarily omits a complete price expectations mechanism; suffice it to say that several countervailing effects of price movements on interest rates are being represented here. A proxy for the real, rather than nominal, rate of interest was tried, but without much success. Inclusion of the stock of money or the change in that stock rendered the equation unusable. The linearized; form of this equation was most successful specified as:

(3.9) RTB = F{ DP, DY, B }

where as previously the "D" operator denotes first differencing.

The specification of this model with the restrictions of linearity relaxed involved only minor modifications. First of all, proportional movements in the aggregate price index were found to be better predictors than levels or differences; thus the first difference of the natural logarithm of P was utilized. Secondly, scale effects in the

movements of real income were removed by substituting a discrete percentage change for the difference used in the linear form. No alteration was made in the specification of the stock of securities variable. Therefore, the nonlinear form of the Treasury bill rate equation becomes:

(3.10) RTB = F{ DLNP, DY/Y, B }

where the "LN" operator denotes natural logarithm.

The demand for money equation should theoretically be specified in a parallel fashion; however, it was not possible to obtain satisfactory single-equation parameter estimates in the linear form of the equation when prices were introduced, in level or difference form, nor when the level of the equilibrium money supply was utilized. The interest rate used as an "own price" for money, by measuring the return on a near-money with a measurable risk component, was the rate on short-term commercial paper (RCP). The linear functional form found to be superior can thus be expressed as:

(3.11) RCP = F{ DY, Y(-2), DM }

wherein explicit price terms are suppressed.

The nonlinear form of the RCP equation is an even simpler specification. Here, the imperfect linear Taylor-series approximations for both Y and M percentage changes are replaced by the more acceptable discrete percentage changes themselves. Therefore, the nonlinear form of this function becomes:

(3.12) RCP = F{ DY/Y, DM/M }

Again, price terms, in both level and percentage change form, were tried and found wanting.

The long-term interest rate, completing the battery of asset prices explained by the model, is assumed to adjust to the short-term rates, being a weighted average of those rates. It should be noted that, following the standard Keynesian framework, it is this rate that appears most prominently in the investment demand equations. The rate used is the quoted rate on long-term bond issues (grade Aaa), labeled RL. This term structure equation explains the quasi-difference of RL by first differences in RTB and RCP, along with a second-order lag in RCP. The functional form is written as:

(3.13) RL = F{ DRTB, DRCP, RCP(-2), RL(-1) }

One significant addition to Pindyck & Rubinfeld's model embodied in this structure is the presence of an explicit supply of money equation. In their model, the money supply (narrowly defined) itself is a controllable policy instrument. Here, the supply of money is endogenous, affected by two instruments of the Federal Reserve, and the short-term Government securities rate (RTB). The instruments of monetary control include the level of the monetary base, which is assumed to be perfectly controllable, and the bank discount rate (RDISCT). This function, estimated with the change in the money supply as the dependent variable, takes the form:

(3.14) DM = F{ RTB, DBASE, RDISCT }

The implicit assumption made is that changes in levels of interest rates alter the rate of growth of the money supply, rather than the level of the equilibrium money supply.

Leaving the asset demand and money supply functions, we turn now to the last block of behavioral equations: those concerned with wages, prices, and unemployment. Quite logically, these equations are the least tractable in linear form, as they inherently involve ratio variables. A much more sophisticated specification is feasible with equations

nonlinear in the variables. To consider the linear versions of these equations, let us turn first to the money wage level (W$) function. This variable is an index of average dollar compensation per manhour in manufacturing. In the linear form, the money wage adjusts with a distributed lag to changes in the current level of income, and past changes in the price index and the unemployment rate. In addition, a time trend is included to capture the effect of productivity gains on wages. We would expect past price changes to be reflected in the money wage level, via cost of living adjustments. The level of income (Y) appears as a measure of aggregate demand, yielding a derived demand for labor. The unemployment rate (expressed in per cent) will give some indication of labor supply; we would expect high unemployment rates in recent times to depress (or at least slow the increase) of the money wage level. Thus the wage equation is of the form:

(3.15) W$ = F{ Y, P(-1), UR(-1), TIME, W$(-1) }

The nonlinear form of the money wage equation is significantly different from its linear counterpart. First of all, the most satisfactory dependent variable is not the level of W$, but rather the first difference of its logarithm: its discrete percentage change. Naturally, the

explanatory variables are defined in somewhat similar terms. The income variable is expressed in discrete percentage change form. Price effects are introduced by a current percentage change and a weighted average of lagged percentage changes, both in logarithmic form. This mechanism aids in capturing the "pass-through" mechanism hypothesized by many researchers, denoting the reflection of price movements in wage levels. Finally, the unemployment rate term in the nonlinear form is represented by a weighted average of current and lagged unemployment rates, in logarithmic form. The final nonlinear form, measuring the percentage change in the money wage level, can thus be written:

(3.16) DLNW$ = F{ DY/Y, DLNP, DLNPAV4(-1), LNURAV5 }

wherein the "AVn" operator indicates the simple moving average of "n" terms (current plus (n-1) lagged terms).

The linear price equation explains the level of the implicit GNP deflator by a relatively simple mechanism. Experiments with the wage-price sector indicated that explanation of the first difference of P was not feasible; likewise, equations in the level of P proved quite unsatisfactory when very few exogenous influences impinged

upon the sector. Thus, it was decided to include an exogenous measure of price movements, PCRUD, which is an index of wholesale raw materials prices, excluding foods and feeds. Especially in recent years, this index contributed a great deal to the explanation of inflation. This, of course, is rather ad hoc, defining P in terms of one of its large components; but satisfactory linear estimates could not be produced in the absence of this modification. Other explanatory variables include the money wage index, proxying cost pressures, as well as the lagged unemployement rate, which serves in a sense to capture the degree of demand pressure. The price index adjusts to these variables with a distributed lag, so that a lagged dependent variable is included. This yields the functional form:

(3.17) P = F{ PCRUD, W$, UR(-1), P(-1) }

The nonlinear form of this equation is quite similar, explaining the discrete percentage change in the price level by current and a weighted average of lagged percentage changes in the money wage. Again, the simple first difference in the level of income was found to be most satisfactory. The raw materials price index was not included in the nonlinear form of the price equation. Thus the nonlinear function form becomes:

(3.18) DLNP = F{ DLNW$, DLNW$AV4, DY }

Completing the trinity of equations in this bloc of the model is the unemployment rate equation. As is clearly the case, a fully linear specification is a very naive model of this complex process. A crude form of Okun's Law is embodied in this function, in which the global unemployment rate adjusts to the gap between actual and potential gross national product. Various more sophisticated specifications were found to be inferior in their dynamic response to controlling influences; for instance, inclusion of the level of the money wage yielded signs contrary to a priori beliefs. The "GNP gap" (in reversed form, as a shortfall) is measured by comparing the level of income with the Council of Economic Advisors' potential GNP series, YPOT. The unemployment rate function in linear form can thus be written as:

(3.19) UR = F{ (Y-YPOT), UR(-1) }

The changes in specification made when linearizing restrictions are relaxed are quite significant. Rather than the level of the wage, the logarithm of the money wage is introduced. To capture the supply and demand effects of past wage levels, a moving average of percentage changes in

the wage level is also used. Finally, rather than the "GNP gap" between actual and potential output utilized in the linear form, the logarithm of their ratio is used - measuring a concept akin to a capacity utilization ratio. This measure, like the percentage wage changes, has the advantage of being free of scale effects. The nonlinear form of the unemployment rate equation thus becomes:

(3.20) UR = F{ LNW$, DLNW$AV4, LN(Y/YPOT) }

Left to be explained at this point are the several identities which close the system. Two have already been presented: the definition of gross national product or income: Y (eqn. 3.1) and the definition of disposable income YD (eqn. 3.2). Added to these we have dynamic identities relating levels and differences of the price level:

(3.21) P = DP + P(-1)

The money supply:

(3.22) M = DM + M(-1)

And the stock of government bonds:

(3.23) B = DB + B(-1)

The last identity of the model is a very important addition, differentiating this model from Pindyck-Rubinfeld's and, indeed, from most small neo-Keynesian macroeconometric models. This is the government budget constraint, of the sort developed by Christ (1968) and Blinder-Solow (1974). Most treatments of the budget constraint deal with purely theoretical models which contain a stock of consols, each with value (1/R), where R is the appropriate rate of interest. The modification used here is to regard the nominal surplus or deficit as being financed by changes in the stock of outside or "high-powered" money (the reserve base) and/or the stock of government securities:

(3.24) DBASE$ + DB$ = P{ G - T }

wherein BASE$ and B$ are the nominal stocks of reserve base and bonds, respectively. Two simplifying assumptions are then made to transform this equation into a more tractable form. First, both sides are divided by the price level P, placing the entire identity into real terms. This causes certain difficulties of valuation; but it is assumed that at any point in time, the policy authorities can identify the

change in real stocks needed to finance a deficit or surplus, even though all variables in their control will be in nominal terms.

The second assumption to be made is a more crucial one. Whereas the former only was introduced to preserve linearity, this assumption is necessary to prevent the introduction of policymakers' behavior into the model. Clearly, any given deficit will be financed by both monetary and bond issuances, in some proportion depending on a complex set of political and economic considerations. Since that proportion is not stable over time, it was judged unrealistic to model it by a habit persistence mechanism. Therefore, the assumption was made to make bond finance the residual instrument. A given surplus or deficit, arising from given spending and tax levels, is combined with the change in the reserve base selected by the monetary authorities. The net effect of these policy settings is to leave some surplus or deficit to be financed via bond issuance or retirement. Thus, the government budget constraint inherent in this model can be rewritten as:

(3.25) $DB = G - T - DBASE$

where DB and DBASE are the first differences of the real

stocks of bonds and reserve base, respectively. Substituting in the tax structure (from eqn. 3.3), this may be usefully written as:

(3.26) DB = G - TXRATE * Y - TO - DBASE

in this form, it is clearly the case that the budget constraint may be written as a function of exogenous policy variables and the level of income. It must be noted that the varibles DB and B are constructed series, taken from an initial B(0) and repeated applications of eqns. (3.25) and (3.23). The budget constraint implies a set of mechanics linking Federal Reserve bond holdings and reserve base issuances, which will not generally hold for any but a stylized view of the central bank.

3. Estimation of the Model

Estimation of the model, in both linear and nonlinear forms, was performed with the Michigan Terminal System computing facilities. Ordinary-least-squares (OLS) and two-stage least squares (TSLS) estimates were obtained from the MIDAS statistical package, while estimates involving first- and second-order autocorrelation corrections were produced by the TSP computer program. The latter corrections,

involving application of the Cochrane-Orcutt iterative estimation technique combined with TSLS (TSCORC and TSCORC2 for first and second order correction, respectively) were applied to TSLS estimates when those estimates revealed significant presence of autocorrelation. This procedure was not used in the presence of lagged dependent variables, and is a simplified version of the technique described in Pindyck-Rubinfeld (1976, pp. 410-411).

All data were obtained from the University of Michigan's Research Seminar in Quantitative Economics (RSQE) data bank. A list of mnemonics and their detailed descriptions and sources is given as an Appendix to this study.

As previously stated, data series were constructed for gross national product Y (omitting the foreign sector), consistent with eqn. 3.1, and for disposable income YD, consistent with eqn. 3.2. The latter estimate, derived by removing net taxes (T) from Y, required an estimate of the average tax rate (TXRATE). This parameter was estimated by regressing net taxes on income (Y), without an intercept term, using ordinary least squares. This regression yielded:

(3.27) Estimated linear equation for : T

Variable	Coefficient	Std.Err.	T-Stat.
Y	.22182	.00259	85.53
R-Squared		.8562	
Std. Err. of Regression		23.035	
Fit Period (N)		1954.1-1976.2 (90)	
Durbin-Watson		.2911	

This estimate, in which the tax surcharge variable was suppressed, yielded the constructed series for YD as:

(3.28) YD = .77818 * Y

We now present the estimates of the eleven behavioral equations in the model. As with any major estimation project, the reported estimates comprise a minute fraction of the number of alternative specifications which were tried and discarded. Some mention is made of the more noteworthy failures. We now turn to the estimates of the behavioral equations.

First, the consumption function: [1]

(3.29) Estimated linear equation for : C			
Variable	Coefficient	Std.Err.	T-Stat.
{DM+DB}	.13649	.03926	3.477
YD	.09778	.02855	3.425
C(-1)	.89781	.03379	26.57
Constant	-8.382	3.1558	2.656
R-Squared		.9989	
Std. Err. of Regression		4.5403	
Fit Period (N)		1954.4-1976.2 (87)	
Durbin-Watson		1.6745	

It should be noted that the coefficient on the change in "wealth" (as defined, DM + DB) has the expected positive sign, supporting the wealth effect on consumption advanced

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[1] In these estimates, R-Squared is the unadjusted coefficient of determination. The full sample for all regressions is 1954.1-1976.2, with leading observations subtracted for lags and autoregressive transformations. "N" is the number of sample observations. "TSCORC" or "TSCORC2" indicates that first- or second-order Cochrane-Orcutt transformations have been applied. The Durbin-Watson statistic is presented for all estimates, although it is biased in those estimates which contain lagged values of the dependent variable.

from a priori grounds. As in the Pindyck & Rubinfeld (P-R) model, disposable income has a very small impact effect on consumption. With the equation in steady-state form, however, it may be seen that this yields an impact marginal propensity to consume of approximately 0.90.

The estimated equation for non-residential investment was:

(3.30) Estimated linear equation for : INR

Variable	Coefficient	Std.Err.	T-Stat.
DY	.07269	.01828	3.976
Y(-2)	.02782	.00703	3.957
RL(-2)	-1.221	.33025	3.697
INR(-1)	1.2840	.08594	14.94
INR(-2)	-.4374	.07550	5.793
Constant	-4.978	1.3972	3.563
R-Squared		.9972	
Std. Err. of Regression		1.3397	
Fit Period (N)		1954.3-1976.2 (88)	
Durbin-Watson		2.2158	

Although this equation's specification is much less complex than the P-P version, in which an Almon lag is fitted to the

income term, it expresses qualitatively similar results. Our equation includes both the lagged level and difference in income, introducing both multiplier and accelerator effects. A shorter lag (2 periods rather than P-R's 4 period lag) was used on the long-term interest rate; our results show a much stronger and more significant effect than theirs. It must be noted, however, that their equation was fitted in differenced form.

The estimated equation for residential investment was:

(3.31) - Estimated linear equation for : IR

Variable	Coefficient	Std.Err.	T-Stat.
YD	.01112	.00225	4.942
RCP	-.5751	.17293	3.326
{RL-RCP}(-2)	.46089	.22038	2.091
IR(-1)	1.2291	.10297	11.94
IR(-2)	-.3668	.09729	3.770
Constant	.29577	.87601	0.338
R-Squared		.9752	
Std. Err. of Regression		1.3790	
Fit Period (N)		1954.4-1976.2 (87)	
Durbin-Watson		1.8569	

The equation is again quite similar to Pindyck and Rubinfeld's with the addition on our part of two lags on the dependent variable. However theoretically disagreeable this specification may be, it does yield much better tracking of the actual series. All variables are significant; inclusion of only the spread between long and short rates proved superior to an equation containing both the spread and the level of one or both rates. However, it was deemed advantageous to include the short rate, as its inclusion

provides for an immediate effect of monetary policy on investment demand. Without this channel monetary actions were seen to be implausibly weak.

The last category of investment expenditures, inventory investment, has the following estimated equation:

(3.32) Estimated linear equation for : IIN

Variable	Coefficient	Std.Err.	T-Stat.
YD	.75892	.06419	11.82
YD(-1)	-.7129	.07308	9.755
C	-.9404	.11750	8.003
C(-1)	.89274	.12006	7.326
IIN(-1)	.70600	.06822	10.35
Constant	-3.310	1.5778	2.098
R-Squared		.8692	
Std. Err. of Regression		2.6268	
Fit Period (N)		1954.4-1976.2 (87)	
Durbin-Watson		2.0096	

The coefficients on YD current and lagged and C current and lagged, respectively, were very nearly equal and opposite to each other within each pair; however, the restrictions of fitting first differences were found to be excessively

tight. The two-quarter differences preferred by P-R were not found superior in our model, as the dynamic instability they referred to was not excessively bothersome.

We now present the estimated asset demand functions, in inverted form so as to normalize on the three interest rates determined within the model. First, the linear and nonlinear estimates for the Treasury bill rate:

(3.33) Estimated linear equation for : RTB

Variable	Coefficient	Std.Err.	T-Stat.
DP	.55588	.21398	2.598
DY	.00116	.00623	0.187
B	.00358	.00162	2.215
Constant	.15365	1.6774	0.092
R-Squared		.9017	
Std. Err. of Regression		.55622	
Fit Period (N)		1955.1-1976.2 (86)	
Durbin-Watson		1.9297	TSCORC2

In this formulation, we obtain the expected positive sign on the change in prices - stating that rising prices lead to an increase in the nominal interest rate - and a positive sign on the change in real income. The level of bonds is seen to

be positively related to the level of the interest rate. This result is in accordance with the usually stated inverse relationship between interest rates and bond prices; in essence the interest rate RTB is not the "own price" of government bonds but some function of that price's reciprocal. The equation was estimated with a second-order Cochrane-Orcutt autoregressive transformation. First- and second-order estimated autocorrelation coefficients were: RHO1 = 1.0976, RHO2 = -0.2897.

The nonlinear version of this equation takes advantage of the relaxation of linear constraints by introducing both price and income terms in percentage change form. The estimated equation is:

(3.34) Estimated nonlinear equation for : RTB			
Variable	Coefficient	Std.Err.	T-Stat.
DLNP	.43252	.04349	9.945
{DY%Y}	-34.88	10.509	3.319
B	.00373	.00044	8.477
Constant	-.8921	.41182	2.166
R-Squared		.8041	
Std. Err. of Regression		.80834	
Fit Period (N)		1954.4-1976.2 (87)	
Durbin-Watson		1.4197	

The signs of coefficients in this nonlinear specification are the same as those obtained in the linear model with the exception of the income term. The discrete percentage change in the price level increases the Treasury bill rate, causing the nominal rate to diverge from the "real" rate, while the level of Government bonds is positively related to the interest rate. The fit of this specification is somewhat worse than that of the linear form (where simple differences were used) when the estimated standard errors of regression are compared.

Turning to the inverted money demand function, we have

the linear and nonlinear estimates for the commercial paper rate:

(3.35) Estimated linear equation for : RCP

Variable	Coefficient	Std.Err.	T-Stat.
DY	.00467	.00791	0.590
Y(-2)	.00821	.00103	7.996
DM	-.2045	.07041	2.904
Constant	-2.541	.97107	2.617
R-Squared		.9013	
Std. Err. of Regression		.65288	
Fit Period (N)		1955.2-1976.2 (85)	
Durbin-Watson		1.7763	TSCORC2

Although it would have been theoretically preferable to estimate all asset demand functions in parallel form, it was found not feasible to include price terms in this equation without undesirable signs on the income and money variables. Also, although the stock of money (rather than its first difference) would have been preferable, this too resulted in nonadmissable estimates from a priori grounds. The income terms (forming a rough approximation to a linearized percentage change) have the expected positive sign on their ratio, and the first difference in the money stock is

negative and significant. It should be noted that this latter specification is analogous to Pindyck's (1973, p.49), and applied for the same rationale, that of avoiding collinearity with income terms. The equation was estimated with a second-order Cochrane-Orcutt transformation, in which the estimated autocorrelation coefficients were: RHO1 = 0.9607, RHO2 = -0.3091.

In the nonlinear specification of this equation, an even simpler specification is utilized:

(3.36) Estimated nonlinear equation for : RCP

Variable	Coefficient	Std.Err.	T-Stat.
{DY%Y}	6.5214	7.3081	0.892
{DM%M}	-38.13	16.046	2.376
Constant	5.3243	.79658	6.684
R-Squared		.8866	
Std. Err. of Regression		.69595	
Fit Period (N)		1955.2-1976.2 (85)	
Durbin-Watson		1.7797	TSCORC2

In this formulation, similar results are obtained for the income variable; the positive sign corresponds to a positively-sloped LM curve for a given money stock.

However, the coefficient is not significant in the reported version of this equation. This version was obtained by applying a second-order autoregressive correction of the Cochrane-Orcutt form. The estimated values for the autocorrelation coefficients were: RHO1 = 1.1578, RHO2 = -0.2527.

Completing the discussion of estimated asset demand functions, we have the implicit demand function for long-term private securities, inverted to yield estimates of the long-term interest rate RL. This function may be described as implicit, as it does not contain the stock of such securities per se but rather expresses their inverse price as a weighted average of short interest rates. This, then, is the model's term structure relationship, the long-term interest rate equation:

(3.37) Estimated linear equation for : RL

Variable	Coefficient	Std.Err.	T-Stat.
DRTB	.13049	.07787	1.676
DRCP	.14012	.05895	2.377
RCP(-2)	.09732	.01438	6.768
RL(-1)	.91552	.01578	58.02
Constant	.04634	.04451	1.041
R-Squared		.9955	
Std. Err. of Regression		.12528	
Fit Period (N)		1954.4-1976.2 (87)	
Durbin-Watson		1.8200	

In this function, as in previous specifications, a lagged dependent variable is included. It should be noted that, unlike P-P's specification, no income terms are present; a pure term structure relationship is maintained. The signs are as expected, with all interest rates moving in unison.

The last stock determination equation in the model is that expressing the supply of money relation:

(3.38) Estimated linear equation for : DM			
Variable	Coefficient	Std.Err.	T-Stat.
RTB	.63628	.25974	2.450
DBASE	2.6490	.18165	14.58
RDISCT	-.9437	.28901	3.265
Constant	1.0885	.30720	3.543
R-Squared		.7524	
Std. Err. of Regression		.89733	
Fit Period (N)		1954.4-1976.2 (87)	
Durbin-Watson		1.4670	

As expected, higher interest rates on government bonds will ceteris paribus cause an increase in the rate of growth of the money supply. The reserve base coefficient (or "money multiplier") has a magnitude of 2.65 in level form, in accordance with the majority of similar estimates. Finally, the discount rate is negatively related; the monetary authorities may shrink the money supply by raising the level of this tool of monetary control.

Entering the wage-price-employment sector of the model, we arrive at several of the most complex specifications. Since all three of these equations are reestimated in

nonlinear form, we present the linear set as a block followed by the nonlinear alternatives. First, the money wage equation:

(3.39) Estimated linear equation for : W$

Variable	Coefficient	Std.Err.	T-Stat.
Y	.00643	.00338	1.902
P(-1)	.43213	.12883	3.354
UR(-1)	-.2068	.06824	3.030
W$(-1)	.75061	.09042	8.301
TIME	.02142	.02030	1.055
Constant	-14.64	4.6702	3.134
R-Squared		.9999	
Std. Err. of Regression		.46123	
Fit Period (N)		1954.4-1976.2 (87)	
Durbin-Watson		1.6675	

This equation explains the level of the money wage index via demand pressure through Y, cost-of-living adjustments to changes in the price level - both positive influences - and in response to the lagged unemployment rate, which lowers the money wage. This latter effect may be considered to represent labor's weakened bargaining position in periods of high unemployment. The trend effect of productivity gains

are captured through the TIME variable. Although insignificant, the variable is retained because of its significant contribution to the system's dynamic tracking.

The price equation in linear form is:

(3.40) Estimated linear equation for : P

Variable	Coefficient	Std.Err.	T-Stat.
PCRUD	.03054	.00352	8.664
W$	.10151	.01985	5.115
UR(-1)	-.0659	.04113	1.603
P(-1)	.77990	.04477	17.42
Constant	5.1589	1.1305	4.563
R-Squared		.9998	
Std. Err. of Regression		.28590	
Fit Period (N)		1954.4-1976.2 (87)	
Durbin-Watson		1.2151	

This equation explains changes in the overall price level with reference to the raw materials price index, PCRUD, and the level of the money wage index - both of which are highly significant. The lagged unemployment rate is marginally significant, but we include it on a priori grounds as a measure of demand pressure (or the lack thereof).

The last linear form in this bloc of equations yields the estimated unemployment rate:

(3.41) Estimated linear equation for : UR

Variable	Coefficient	Std.Err.	T-Stat.
(Y-YPOT)	-.0079	.00109	7.254
UR(-1)	.71741	.04331	16.565
Constant	1.1365	.19376	5.866
R-Squared		.9375	
Std. Err. of Regression		.32008	
Fit Period (N)		1954.4-1976.2 (87)	
Durbin-Watson		0.7263	

This equation, as indicated previously, is fundamentally a statement of Okun's Law, relating actual and potential output to the unemployemnt rate (which adjusts with a distributed lag). The coefficient on (Y-YPOT), the "GNP gap", has the expected negative sign. The gap, usually a negative magnitude, is increased by a higher level of real income, which is associated with high employment and a diminution of the unemployment rate. Interesting to note is the "full-employment" level of UR; if the gap is zero, denoting operation at full capacity, the implied rate of unemployment is 4.02%. This figure might not be taken

seriously as an estimate of this decade's "natural rate", but it agrees with earlier estimates of a "full-employment" level.

This completes the discussion of the linear forms of the wage-price-employment equations. Turning now to the nonlinear forms, we have first the nonlinear money wage equation, in percentage change form:

(3.42) Estimated nonlinear equation for : DLNW$

Variable	Coefficient	Std.Err.	T-Stat.
{DY/Y}	9.2828	22.057	.4209
DLNP	.40929	.12983	3.153
DLNPAV4(-1)	.30769	.13932	2.209
LNUPAV5	-1.002	.91859	1.091
Constant	4.5810	1.4402	3.181
R-Squared		.5703	
Std. Err. of Regression		1.5474	
Fit Period (N)		1955.2-1976.2 (85)	
Durbin-Watson		2.2091	

The nonlinear money wage equation has the same sign pattern as its linear counterpart. It should be noted that much longer lags were used, capturing the effects of current

price changes and the average price change over the past four quarters. Similarly, an average unemployment rate (in logarithmic form) over current and the past four quarters impacts upon the wage level. The latter coefficient is marginally significant, but in line with prior expectations developed in the linear model.

In the nonlinear price equation, the major explanation is derived from current wage levels:

(3.43) Estimated nonlinear equation for : DLNP

Variable	Coefficient	Std.Err.	T-Stat.
DLNW$	1.2246	.20747	5.903
DLNW$AV4	.11893	.19452	0.611
DY	-.0175	.02365	0.740
Constant	-3.763	.86896	4.330
R-Squared		.5980	
Std. Err. of Regression		1.9445	
Fit Period (N)		1955.2-1976.2 (85)	
Durbin-Watson		2.1988	

Lagged percentage changes in the money wage (up to four quarters previous) enter this equation, although the coefficient's significance is nil. The first difference in

the level of income enters with the same sign as that term in the linear equation. The overall fit may appear weak, but it must be considered that percentage changes fit more poorly than differences or levels.

The last equation in this nonlinear bloc serves to explain the behavior of the unemployment rate:

(3.44) Estimated nonlinear equation for : UR

Variable	Coefficient	Std.Err.	T-Stat.
LNW$	.58120	.35150	1.654
DLNW$AV4	-.1503	.06290	2.389
LN(Y/YPOT)	-25.89	2.1919	11.81
Constant	2.2229	1.4115	1.575
R-Squared		.9726	
Std. Err. of Regression		.22234	
Fit Period (N)		1955.4-1976.2 (83)	
Durbin-Watson		2.2099	TSCORC2

This specification is considerably more complex than the linear form of eqn. (3.41), including a moving average of wage effects. The equation yields the expected positive sign between the wage level and unemployment - noting also that the average of past wage changes reduces the jobless

rate. The log of Y/YPOT plays the same role as the "GNP gap" in the linear specification. A higher "capacity utilization" ratio yields lower unemployment, although with diminishing marginal effect, through the logarithmic transform. The reported equation was obtained with a second-order Cochrane-Orcutt transformation. The estimated autocorrelation coefficients were RHO1 = 1.2843, RHO2 = -.5730.

CHAPTER IV

CHARACTERISTICS OF THE MODEL

0. Introduction

In this chapter, we analyze some of the relevant characteristics of the model detailed in Chapter III. The standard criterion of "tracking" accuracy are presented in Section One, which summarizes the results of dynamic simulations performed with both linear and nonlinear versions of the model. Several multiplier experiments are also described, in which the model's sensitivity to fiscal and monetary shocks is measured. Section Two analyzes this sensitivity in a formal sense via an examination of the dynamic stability properties of the linear model. Section Three details the procedures required to calculate the reduced form coefficient estimates, residuals, and estimated variance-covariance matrices from the structural form of the model. Finally, Section Four of the chapter examines the structural versus reduced form estimates of the coefficients, and their standard errors.

1. Dynamic Simulation and Multiplier Analysis

Although it must be stated that expanded presentations of the model's properties are not a major goal of this study, a certain familiarization with the tracking accuracy of the model is deemed necessary if we are to comprehend its behavior when subject to control. Estimated equations' specifications were tempered largely by the consideration of system performance, in which the interactions among the dynamic equations of the system were carefully analyzed. To summarize such performance, we present several sets of statistics for both linear and nonlinear models. To test the models' system performance, both static and dynamic simulations were performed. In static simulation, historical initial conditions are used to construct each period's forecast. In dynamic simulation, this reinitialization is not performed, so that disturbances accumulate from period to period. This implies that dynamic simulation is a much more severe test of a model's behavior. Accordingly, our first experiments involved dynamically simulating the models from 1955.1 through 1975.2, with all government policy variables taking on their historical values, and the tax surcharge set at zero. The following tables report several relevant statistics computed from the linear and nonlinear models' tracking performance.

Table 4.1

Dynamic Simulation Characteristics of the Linear Model [1]

1955.1 - 1976.2

Variable	Mean	RMS Error	RMSE % Mean
C	573.68	21.582	3.7620
INF	91.234	6.5298	7.1572
IP	41.205	4.8827	11.850
IIN	6.8906	7.7863	113.00
RTB	4.2115	0.9952	23.630
RCP	4.9611	0.9765	19.684
RL	5.5952	0.3594	6.4227
DM	0.0858	1.0031	1169.5
WS	102.10	3.6649	3.5895
P	82.675	1.7343	2.0978
UR	5.2147	0.9508	18.233
Y	927.37	29.217	3.1506
YD	721.67	22.733	3.1501
DB	6.9200	12.914	186.57
M	225.17	10.655	4.7322
R	1012.9	57.228	5.6497

Table 4.2

Dynamic Simulation Characteristics of the Nonlinear Model [1]

1955.4 - 1976.2

Variable	Mean	RMS Error	RMSE % Mean
C	580.24	20.954	3.6113
INR	92.366	11.922	12.908
IR	41.408	6.0069	14.507
IIN	6.8783	7.9366	115.39
RTB	4.3087	1.2764	29.623
RCP	5.0693	1.5305	30.193
RL	5.6879	1.1070	19.463
DM	.08729	1.0314	1181.6
WS	103.71	4.3378	4.1826
DLNWS	5.6942	2.0481	35.968
P	83.465	4.4424	5.3224
DLNP	3.7398	2.0077	53.684
UR	5.2437	1.1144	21.252
Y	937.55	37.729	4.0242
YD	729.58	29.358	4.0239
DB	6.6839	15.266	228.40

It is somewhat interesting to note that with few exceptions the linear model out-performs the nonlinear model in terms of these summary statistics. This may partially be caused by the more ambitious goals of the nonlinear model (to explain, for instance, percentage rates of growth rather than levels or first differences). Another factor influencing this comparison may well be the increased volatility of that model with respect to monetary and wage-price feedback. These feedback effects are markedly weak in the linear model, and may be demonstrated more fully in the nonlinear counterpart.

Going beyond these general statistics on model performance, we wish to analyze the model's behavior when "shocked" by government policy actions. That will, after all, be the environment in which the model must perform when subject to control. Two basic experiments are discussed: first, the elevation of government expenditures $1.0 billion above the historical path in 1959.1 and all quarters

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[1] In Tables 4.1 and 4.2, "Variable" refers to the endogenous variable calculated via dynamic simulations; "Mean" is the variable's average value over the simulation period. "RMS Error" is the root-mean-squared forecast error. "RMSE % Mean" expresses the RMS error as a per cent of the variable's mean value. This statistic has little meaning for a variable with both positive and negative values, such as DM.

following. Secondly, an increase in the change in the reserve base (by $1.0 billion) in 1959.1 alone is simulated. The fundamental distinction between the government expenditure variable - which enters in level form - and the reserve base variable, which enters the model only in differenced form - should be noted. Since changes in the base cause changes in the equilibrium quantity of money, and those changes enter the FCP equation, only changes in the base affect the level of activity. A one-shot increase in the base (and quantity of money) will not have a lasting effect on economic actions. For that reason, there is a difference between the interpretation of dynamic multipliers for fiscal versus monetary policies. The following tables present dynamic multipliers for the linear and nonlinear models, respectively, for both the fiscal and monetary policy experiments outlined.

Table 4.3

Dynamic Multipliers for the Linear Model [1]

Government Expenditures $1 billion above historical path

1959.1 - 1967.1

Period	DY	DUR	DPRICE
0	3.0869	-0.0243	0.0020
1	2.3040	-0.0356	0.0068
2	2.2092	-0.0430	0.0128
3	2.3359	-0.0492	0.0198
4	2.4763	-0.0548	0.0274
5	2.6411	-0.0601	0.0356
6	2.8037	-0.0652	0.0443
7	2.9661	-0.0702	0.0536
8	3.1248	-0.0750	0.0633
12	3.6863	-0.0924	0.1063
16	4.1145	-0.1070	0.1541
20	4.5217	-0.1190	0.2045
24	4.8213	-0.1288	0.2559
28	5.0601	-0.1365	0.3068
32	5.2466	-0.1427	0.3564

Table 4.4

Dynamic Multipliers for the Linear Model [1]

Reserve Base $1 billion above historical path

1959.1 - 1967.1

Period	DY	DUR	DPRICE
0	0.9927	-0.0078	0.0006
1	1.0708	-0.0140	0.0024
2	1.8269	-0.0245	0.0054
3	1.6118	-0.0303	0.0096
4	1.3652	-0.0325	0.0143
5	1.1853	-0.0326	0.0189
6	1.0344	-0.0316	0.0234
7	0.9273	-0.0299	0.0274
8	0.8333	-0.0281	0.0311
12	0.5828	-0.0206	0.0418
16	0.4321	-0.0151	0.0476
20	0.3384	-0.0114	0.0501
24	0.2505	-0.0086	0.0505
28	0.1807	-0.0063	0.0494
32	0.1218	-0.0045	0.0473

The government expenditure multipliers for the linear model indicate a very strongly Keynesian response to fiscal actions. The impact multiplier for Y is sizable, with feedback from the monetary sector reducing its impact over subsequent quarters. The long-run level is effectively obtained by quarter 32 where a level of approximately 5.50 persists. That same policy has the effect of depressing the long-run unemployment rate by about 4%, or about one-tenth of a point. The price level is seen to rise extremely slowly, showing little change over the first 20 quarters. Thereafter, the effect builds up to approximately 0.5%.

The monetary policy effects in this Keynesian structure are, as expected, less powerful. A $1 billion change in the reserve base brings about a roughly equal increase in Y in the current and following quarters, and then the maximum response in the second quarter (as postulated, given the lags in investment sector demand response to interest rate adjustments). This response dies off sharply after two to three years, and eventually goes to zero. The monetary

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[1] In Tables 4.3 and 4.4, "Period" refers to the delay length of the dynamic multiplier, zero corresponding to the impact multiplier. "DY", "DUR", and "DPRICE" are the simulated deviations of those endogenous variables from their control simulation levels, in which policy variables took on their historical values.

policy effect on the unemployment rate is virtually nil, never reducing that rate by more than 0.03%. The initial effect on the price level is very small, followed by a slowly rising multiplier, which peaks at roughly one-twentieth of an index point. Thus, the presupposed weakness of the monetary instrument implied by the strongly Keynesian specification of the model is borne out.

A careful examination of the model's reaction to both fiscal and monetary shocks points out one of its basic weaknesses - the lack of interaction between the wage-price-unemployment sector and the rest of the model. The model behaves much like a textbook IS/LM model with no price terms. Since the sector in question is largely recursive, with prices only impinging upon the Treasury bill rate, very little interaction takes place. Implausibly small changes in the price index and unemployment rate result from policy stimuli; on the contrary, real expenditure and interest rate multipliers are of reasonable size. It should be noted that the money wage index multipliers (not reported above) react more strongly than price level multipliers, although still short of reasonable expectations. Clearly, the structure of this sector does not enhance its capabilities to react to external influences in a plausible manner; and control of these targets will be hampered by these weak linkages.

Finally, we present the dynamic characteristics of the nonlinear model, as revealed by the dynamic multipliers for that model. The following tables report the results of the same fiscal and monetary policy experiments carried out with the linear model and reported in an earlier portion of this section. First, we present the multipliers for an increase in government expenditures, followed by those produced by a change in the reserve base.

Table 4.5

Dynamic Multipliers for the Nonlinear Model [1]

Government Expenditures $1 billion above historical path

1959.1 - 1967.1

Period	DY	DUR	DPRICE
0	3.1462	-0.1171	0.0007
1	2.3215	-0.0879	0.0048
2	2.0959	-0.0796	0.0107
3	2.3267	-0.0890	0.0191
4	2.5432	-0.0982	0.0310
5	2.7231	-0.1034	0.0445
6	2.9150	-0.1119	0.0611
7	3.1060	-0.1201	0.0807
8	3.2920	-0.1263	0.1033
12	3.9954	-0.1499	0.2278
16	4.6379	-0.1736	0.4166
20	5.2004	-0.1926	0.6771
24	5.6995	-0.2116	1.1116
28	6.1577	-0.2203	1.4319
32	6.6057	-0.2271	1.9472

[1] See note to Table 4.3 for definitions of table headings.

Table 4.6

Dynamic Multipliers for the Nonlinear Model [1]

Reserve Base $1 billion above historical path

1959.1 - 1967.1

Period	DY	DUR	DPRICE
0	0.0415	-0.0016	0.0
1	0.1821	-0.0068	0.0000
2	1.6489	-0.0609	0.0004
3	2.2095	-0.0759	0.0026
4	1.4644	-0.0555	0.0067
5	1.1675	-0.0441	0.0122
6	1.0256	-0.0395	0.0193
7	0.8958	-0.0350	0.0274
8	0.7747	-0.0307	0.0361
12	0.5418	-0.0236	0.0783
16	0.4500	-0.0208	0.1290
20	0.3740	-0.0184	0.1863
24	0.3171	-0.0166	0.2482
28	0.2878	-0.0151	0.2963
32	0.2690	-0.0139	0.3809

[1] See note to Table 4.3 for definitions of table headings.

Table 4.6

Dynamic Multipliers for the Nonlinear Model [1]

Reserve Base $1 billion above historical path

1959.1 - 1967.1

Period	DY	DUR	DPRICE
0	0.0415	-0.0016	0.0
1	0.1821	-0.0068	0.0000
2	1.6489	-0.0609	0.0004
3	2.2095	-0.0759	0.0026
4	1.4644	-0.0555	0.0067
5	1.1675	-0.0441	0.0122
6	1.0256	-0.0395	0.0193
7	0.8958	-0.0350	0.0274
8	0.7747	-0.0307	0.0361
12	0.5418	-0.0236	0.0783
16	0.4500	-0.0208	0.1290
20	0.3740	-0.0184	0.1863
24	0.3171	-0.0166	0.2482
28	0.2878	-0.0151	0.2963
32	0.2690	-0.0139	0.3809

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[1] See note to Table 4.3 for definitions of table headings.

The nonlinear model's government expenditure multipliers are quite similar to those of the linear model. The impact multiplier for Y is even larger than its linear counterpart, again falling in quarters one and two, recovering its impact effect after eight quarters. The 32-period delay effect is over $1.3 billion higher for the nonlinear model. The unemployment rate multipliers are sizable in the initial quarters, rising to about one-quarter of a per cent in absolute value. The long lags in the nonlinear model's wage-price sector cause the price level multiplier to rise very slowly, and then accelerate to a level of almost two index points after 32 quarters. This behavior, while still short of expectations, is somewhat more plausible than that observed in the linear model.

Considering the nonlinear model's monetary policy multipliers, we may note a much more delayed response of income to a change in the base as compared to the linear model. Policy effects are sizable in quarter two and peak in quarter three, at a level of over two billion dollars. The response as compared with the linear model is thus slightly stronger and distributed over more lagged periods. The unemployment rate multipliers, on the other hand, peak earlier - in quarter three - and slowly fall. Price effects are negligible in quarters one and two and steadily rising

thereafter, accelerating after eight quarters. Here, too, the nonlinear model is seen to have a more respectable response in terms of price movements, although still quite small. As mentioned earlier, the wage-price sector suffers from weak linkages to the real sector, even when its specification (in this nonlinear version of the model) is made much more complex.

2. Analysis of Stability Properties

At this point, it is instructive to note that several multipliers, such as the linear model's government spending multiplier for the level of income, do not appear to converge to a stable long run value, as we might expect them to do in a stable model. To formally consider the model's properties, a detailed analysis of the stability of this linear system was performed. This involved the calculation of the characteristic roots, or eigenvalues, corresponding to the matrix of reduced form coefficients on lagged endogenous variables (augmented by identity and null matrices). For a system of linear difference equations to be globally stable, it is necessary that all characteristic roots of this matrix be less than unity in absolute value. The modulus of each conjugate pair of complex roots must satisfy this condition. A system is neither stable nor

unstable if roots of precisely unity exist; in that borderline case, the levels of variables will not be stationary processes. The fundamental structure of the system, in which some variables appear in levels and others in differences, implies the existence of one or more eigenvalues of exactly unity (see Chow, 1975, pp. 104-105). The following table lists the nineteen non-zero eigenvalues of the system giving the moduli of complex roots and the cycle length implied by those roots.

Table 4.7

Non-zero Eigenvalues of the Linear Model [1]

Real	Imag [2]	Modulus	Period [3]
1.00000			
-0.1058	0.20414	0.22993	5.7507
0.44373	0.14377	0.46643	20.053
0.69067	0.17044	0.71139	25.970
0.99416			
0.97038			
0.95418			
0.91772			
0.71254			
0.72343			
0.10449			
0.60308			
0.2632 -5	0.2913 -3	0.2913 -3	4.0231
-0.510 -5			
0.598 -10			

As we have noted, there exists a root of exactly unity, corresponding to the stochastic equation for DM, estimated in differenced form. A number of other real roots have a magnitude in excess of 0.90. The largest complex roots have a modulus of roughly 0.71, with large real components causing a long-term (5 to 6 year) cycle. The shorter cycles in the model are caused by very weak roots. Thus, we might expect the model to exhibit a significant amount of "business cycle" behavior, with less significant short-run fluctuations. The unitary eigenvalue will cause the levels of variables to be non-stationary processes, even though their respective differences may be stable.

3. Construction of the Reduced Form

In this section, we consider the formal derivation of the linear model's restricted reduced form from the structural estimates. The system of structural equations (which in our case includes eleven behavioral equations, and

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1 In the entries of Table 4.7, the figure "-n" following a root indicates that its value has an exponent of ten raised to the power "-n".
2 "Imag" reports the imaginary part of the complex roots, in absolute value.
3 "Period" gives the length of the cycle implied by the complex roots, in calendar quarters per cycle.

five identities) may be written in matrix form as:

$$(4.1) \quad B\, y(t) = G\, z(t) + u(t)$$

where: y is the NY by 1 vector of current endogenous variables ; z is the NZ by 1 vector of "predetermined" variables; and u is the NY by 1 vector of equation disturbances. The predetermined variables include the lagged endogenous variables, the current and lagged control variables, and the exogenous variables (including a "variable" for the constant term). B is a NY by NY matrix of coefficients on current endogenous variables; some of these coefficients are estimated while others (corresponding to the main diagonal and identities) are known with certainty. The G matrix is of dimension NY by NZ, and contains the remainder of the coefficients estimated in the model as well as unitary coefficients corresponding to identities.

The reduced form of this system is defined as:

$$(4.2) \quad y(t) = PI\, z(t) + v(t)$$

where PI is the NY by NZ matrix of estimated reduced form coefficients, and v is the NY by 1 vector of reduced form

disturbances. These reduced form coefficients are related to the structural coefficients by:

$$(4.3) \quad PI = B^{-1} G$$

and the reduced form disturbances are related to the structural disturbances by:

$$(4.4) \quad v(t) = B^{-1} u(t)$$

for any non-singular B matrix. The computation of the PI matrix and vector v is thus a straightforward task, involving only matrix multiplication and inversion. The elements of PI of special interest are those columns corresponding to current control variables; these portions of the matrix are the "impact", or zero-period delay, dynamic multipliers for the policy variables. We proceed now to the more challenging task of computing the estimated variance-covariance matrices for PI and v.

To compute the estimated variance-covariance matrix for the vector of reduced form residuals v, two approaches may be taken. The more formal approach derives the matrix from equation (4.4). If S(u) is defined as the variance-covariance matrix of vector u, then

$$(4.5) \quad S(u) = E \{ u - \underline{u} \} \{ u - \underline{u} \}'$$

where $\underline{u}$ is the vector of means and the operator E denotes mathematical expectation. Since the residuals have a mean of zero by assumption, we express this as

$$(4.6) \quad S(u) = E \{ u\ u' \}$$

Likewise, we may define S(v):

$$(4.7) \quad S(v) = E \{ v\ v' \}$$

From equation (4.4), this may be rewritten as:

$$(4.8) \quad S(v) = E \{ B^{-1} u \} \{ u' (B^{-1})' \}$$

or, rearranging terms,

$$(4.9) \quad S(v) = \{ B^{-1} S(u) (B^{-1})' \}$$

This details the first approach; mere application of this formula will yield the S(v) matrix. However, the S(u) matrix must be computed separately unless a full system method of estimation is used. This latter point provides the rationale for the second approach, which directly

calculates the S(v) matrix as the moments matrix of reduced form residuals. These residuals are those calculated in a "static" simulation of the entire model over the full sample; that is, a sequence of one-period-ahead forecasts. It may be readily noted that simulation of the full system takes into account precisely the information inherent in eqn. (4.9), that is, the inter-relationships between current endogenous variables contained in the B matrix. This latter approach is more tractable computationally, as any simulation program will calculate these sequences of residuals v, from which the sample moments may be readily obtained.

Computation of the second reduced form variance-covariance matrix, that corresponding to the PI matrix, requires much more effort. The method used is the asymptotic approximation of Goldberger, Nagar, and Odeh (1961, pp. 556-561), henceforth GNO. As the derivation of their formula is quite lengthy, it will not be reproduced here. Chow discusses their technique (1975, pp. 244-247), but applies it in a slightly different manner, as we shall see. Let S(B | G) be the matrix of sample covariances of the estimates {B | G}, taken row by row. This defines the partitions of S(B | G) as the estimated covariances between two rows of {B | G}, that is, between two equations'

estimates. In this study, we utilized only single-equation estimation techniques, so that the S(B | G) matrix will be block-diagonal in structure. Each block corresponds to the estimated variance-covariance matrix calculated in the estimation of that structural equation (which will be null for the identities). Chow's technique and algorithm considers the matrix of sample covariances of {B | G} column by column; thus, the analogous S matrix will not be block-diagonal, and a great deal of computational complexity will result compared with the former approach.

Using a technique analogous to that described in the definition of the S(v) matrix, the GNO approximation computes the matrix of sample covariances of reduced form coefficients. We may write:

$$(4.10) \quad S(PI) = \{ L \quad S(B \mid G) \quad L' \}$$

in which the uncertainty associated with the rows of PI (that is, with each reduced form equation's coefficients) is computed as a quadratic form of the matrix of sample covariances of the rows of {B | G}. The matrix L is then defined as:

$$(4.11) \quad L = \{ (B^{-1})' \otimes (PI' \mid I) \}$$

where I is the identity matrix of order NZ, and the operator "⊓" signifies the Kronecker matrix product.

It should be clear that actual computation of S(PI) need not utilize this formula directly; in our case, where S(B | G) is not only block-diagonal but contains several null blocks (corresponding to the identities), a simpler approach suggests itself. The calculation of the quadratic form may be broken up into several smaller portions; for instance, each partition of S(PI), which corresponds to the covariance between estimates of equation i and equation j, consists of a sum of terms. There will be one term for each non-null block in S(B | G), that is, for each structural equation; these terms will have the form:

$$(4.12) \quad T(i) = (PI' \mid I)\, S(i)\, (PI' \mid I)'$$

where T(i) is the ith term, including the ith diagonal block S(i) of S(B | G). Each partition of S(PI) will be a weighted sum of these terms T(i), where the weights are the scalar elements of $\{(B^{-1})'\}$. Although this procedure may appear considerably less elegant than the direct application of eqn. (4.10), one need only calculate the dimensions of matrices L and S(B | G) to note that for a model of any size, the problem will be computationally very cumbersome,

if indeed feasible. Since only approximately half of the symmetric S(PI) is of interest, computation of more than its lower triangular form should be avoided.

Thus, we have indicated how all the information needed to control an economic system may be generated from the structural estimates and static simulations performed in the course of designing a model. We turn in the next section to the implementation of these procedures for the model of Chapter III.

4. Structural versus Reduced Form Estimates

The computation of reduced form estimates for the model of Chapter III yielded several interesting insights into matters of concern when working with an econometric model. The previous section detailed the calculation of the restricted reduced form estimates from the structural equation estimates. A question remains to be answered at this point: why use a restricted, rather than unrestricted, reduced form? We stated that the reduced form coefficient matrix PI of eqn. (4.3) is restricted since a priori restrictions have been placed on the predetermined variables: not every predetermined variable enters every equation. Furthermore, current endogenous variables enter

the equations as well, necessitating the use of a simultaneous equation estimation method. The alternative, considerably simpler from the estimation point of view, would involve regressing each endogenous variable on all predetermined variables in the system. This approach, obviating the need for estimation procedures more complex than ordinary least squares techniques, yields the unrestricted reduced form of the model, calculating a coefficient matrix PI* directly. Furthermore, if a system method of estimation is used, the variance-covariance matrices of reduced form residuals and coefficient estimates may be obtained directly.

There are pitfalls to be encountered in the unrestricted reduced form approach. For instance, the estimates in PI* are constructed subject to the criterion of minimum forecast error; the restrictions utilized in the construction of PI should lead to a poorer in-sample forecast, just as inclusion of variables increases the unadjusted coefficient of determination in regression. Thus, we might prefer to use PI* estimates to construct "best" forecasts. However, this goal is at odds with the goal of generating accurate parameter estimates. In any sizable timeseries model, the high correlations among predetermined variables will reduce our confidence in

individual parameters' point estimates. Thus, any attempt to trace economic linkages through the model's structure will be hampered by large estimated standard errors - which may explain the unpopularity of the unrestricted reduced form approach in operational forecasting and policy analysis. Furthermore, the imprecision of individual estimates may seriouly hamper our ability to control the model and comprehend the linkages being used to effect policy impacts on target variables. For these reasons, we make no further mention of unrestricted reduced form estimates.

In confining our study to a restricted reduced form, we encounter a fundamental failing of most macroeconometric models. The precision of such models has customarily been judged with reference to the significance of the structural estimates, and by calculation of measures of in-sample forecast accuracy via deterministic simulations. Neither set of statistics measures precision as we are concerned with it in the context of control. The statistical significance of the reduced form estimates is that with which with which we should be concerned, as it considers our degree of certainty about the point estimates in PI. Heretofore, concern about that significance has often been lacking, as calculation of the distribution of the reduced

form estimates has required a very large number of stochastic simulations. These simulations, in which one parameter in one structural equation is perturbed by a random shock, and the effect averaged over numerous repetitions of this process, are costly and tedious, as the process must be repeated for each parameter to be examined.

This discussion clearly argues for an analytical solution to the problem. For instance, if one equation in a model is reestimated, all estimated variances and covariances based on stochastic simulation results may be affected. This process may be circumvented for linear models if the GNO approximation is applied directly to the structural estimates and their estimated variance-covariance matrices. Although the calculations required for a sizable model are indeed numerous, the computational effort is less than that required if stochastic simulation is used, and the process may be readily repeated if a new set of estimates is prepared.

The procedures described in Section 3 were applied to the structural estimates of (B | G) presented in Chapter III, their estimated variance-covariance matrices, and their estimated reduced form residuals calculated via static simulation. The GNO approximation was used to calculate the

estimated variances and covariances for the 15 rows and 27 non-null columns of PI. Thus, the covariance between estimates of reduced form equations i and j appear in a 27 by 27 matrix, with {(NY)(NY+1)/2} such partitions making up S(PI). For this system, there are 136 such partitions. For any partition where i = j, the diagonal elements are estimated variances of the PI coefficients. Of special interest from the control point of view are the ratios of PI elements for current control variables to their respective standard errors. Since the confidence with which we accept these point estimates will help to determine our degree of reliance on the instrument, an examination of these ratios (denoted as "R") may be useful.

For government purchases, four of the ratios are in excess of four : those for Y, YD, UR, and IIN. Three more are greater than three : those for DB, B, and C. On the other hand, the R's for all interest rates, DM, and M are less than unity in absolute value. This pattern is repeated for the tax surcharge variable. Thus, we may conclude that fiscal policy is very effective with respect to the level of activity, but not very certain as a tool to control interest rates. This conclusion extends to the wage and price variables (W$ and P); both have R's between one and two for either fiscal tool.

Turning to the monetary variables, we see a quite different pattern. The reserve base has an R exceeding four for DB, B, DM and M. However, the R's for Y, YD, and RCP lie between two and three, with R(RL) just below two. Surprisingly, the R(RTB) is less than unity. For the discount rate, an even weaker pattern is revealed. The R's for DM and M are between three and four, while the ratios for Y, YD, DB, and B are all less than two. The R(RTB) is near zero. Thus, we may note that we have significantly less confidence in monetary policy's ability to systematically affect the level of activity; the target best suited for monetary instruments is the stock of money itself, which is reassuring but not very useful. Monetary tools may be used with some confidence to control their "own" rate of interest, RCP; however, the long and complex linkages between asset demand equations dictates that we may effectively control the quantity of government securities but not their price (RTB). We have somewhat weaker control over the term structure of interest rates via RL.

We see, then, a very Keynesian set of results, suggesting via the reduced form that fiscal policy will dominate monetary policy, not only in terms of deterministic multipliers but also in terms of the model's stochastic properties. Although the structure and specification of a

model that yields these results may be taken to task, it must be kept in mind that this study is concerned with analyzing the properties of a reasonable macroeconometric model. In that vein, a rabidly Keynesian reliance on fiscal policy is no more unreasonable than the monetarists' alternative in the Anderson-Jordan species of "reduced form" model. But one critical concern of methodology must be noted. The monetarist-fiscalist debate has largely been concerned with the structural estimates of large Keynesian models held up against the reduced form St. Louis models. The monetarists have claimed, and rightly so, that this does not involve meeting them on their own ground - that of the reduced form of the models. The techniques discussed and results presented in this section have given us the ability to make a meaningful comparison. The Anderson-Jordan equation may be viewed as a severely restricted reduced form of a structural macro model; we could compare this equation's estimates and their confidence intervals directly with those of the restricted reduced form of the macro model of Chapter III. Although the techniques are only currently applicable to linear (or linearized) models of modest size, they may point the way to a useful consideration of a reduced form model's stochastic properties.

CHAPTER V

OPTIMAL STABILIZATION POLICY WITH THE LINEAR MODEL

0. Introduction

This chapter presents the results of optimal stabilization policy experiments conducted with the linear version of the model. The first section presents a set of "certainty equivalence" optimal control experiments - that is, solutions in which only additive uncertainty is taken into account. In these solutions, the number of usable policy tools cannot exceed the number of distinct targets. Section Two generalizes the process to include the full parametric uncertainty of the reduced form coefficients. This fully stochastic optimal control is used in an exposition of the relative strengths of the several policies, singly and jointly. A number of experiments are presented to explore the most satisfactory instrument-target combinations.

The third section considers an application of closed-loop control in the form of analytic derivatives. These derivatives, which may be obtained as a by-product of the optimal solution, measure the sensitivity of the feedback

rules to two additional sources of uncertainty. These sources, not heretofore considered, include the policymaker's uncertainties about target specification, and the effects of exogenous variable forecast errors. In Section Four, these derivatives are put to use in one of the applications of a much wider principle : the evaluation of sub-optimal solutions, and the proper methods to be used in comparing the loss associated with those solutions with that of the optimal policy set. Three types of suboptimal policy are studied : 1) the effects of using "steady-state" feedback rules in place of the optima; 2) the ex post comparison of historical policies with the optimal policies for that period; and 3) the effects of arbitrary feedback rules vis-a-vis their truly optimal counterparts.

1. Certainty Equivalence Optimal Control Experiments

In this section, the results of a number of "certainty equivalence" optimal control solutions are presented to give a first view of the linear model's characteristics when subject to control. These solutions are "certainty equivalent" since they involve only the additive uncertainty derived from each behavioral equation's disturbance term. With only additive uncertainty, the Theil-Norman "first period certainty equivalence" theorem (1974, pp. 209-215)

holds: that is, the feedback rules derived in closed-loop control will yield an expected trajectory for the state variables identical to that derived from a deterministic solution. In practice, the Theil open-loop solution will produce a set of T state vectors which are optimal only for the deterministic case. An optimal stochastic open-loop solution would be recalculated each period; thus, only the first period settings of the control variables would be used, and these settings are equivalent to the certainty - i.e. deterministic - values.

In the Chow closed-loop optimal solution, a set of T feedback rules are produced, and an expected trajectory for the state vector is calculated. This trajectory will, with no parametric uncertainty, be identical to the Theil solution, and thus equivalent to the deterministic solution. The difference lies in the fact that the feedback rules calculated via the closed-loop approach are truly optimal if the stochastic assumptions are accepted. Although the expected trajectory would not actually be realized in the presence of random disturbances, the feedback rules will in theory govern the policymaker's behavior throughout the T-period horizon to achieve a true optimum for the specified problem.

The following "certainty equivalent" solutions are presented to give an indication of the dynamic model's characteristics undisturbed by our relative uncertainty about policy coefficients' values. Although that uncertainty makes the problem much more interesting and realistic, we will abstract from it for the time being. One consequence of that abstraction is the well-known concern with enumeration of targets and instruments. Unlike Pindyck's algorithm (1973, p. 28), in which all control variables had to be "costed", or entered in the welfare function, our approach does not require the specification of desired policy variable trajectories. In general, a count must be made of the number of targets versus the number of independent policy variables. Using Chow's procedure, the number of instruments may never exceed the number of targets: i.e. fiscal and monetary tools may not be jointly used to stabilize Y in the absence of parametric uncertainty. This becomes evident when the form of the solution is examined; the matrix (C'HC), which must be inverted, can have rank no greater than the rank of H. In period T, H(T) = K(T); if only one target is specified, K will be null except for one diagonal element, and (C'HC) will be of dimension NX while of rank one. Thus, the first limitation of certainty equivalence control lies in the number of targets and instruments.

This example also suggests that a mere tally of targets and instruments will not suffice. In addition, independent linkages must exist. Two fiscal instruments which have parallel effects on every target variable may not be used to control two independent targets. Therefore, the addition of another policy implement may not enlarge the instrument set, if its effects cannot be separated from those of other policy tools. This problem, as we'll see, is very likely to occur when dealing with a highly aggregated model. Its counterpart is equally perplexing : the nonuniqueness of target variables. In a simple model, abstracting from parametric uncertainty may reveal the existence of very rigid linkages between endogenous variables, wherein controlling one variable essentially suffices to control several. The most obvious example is the simple Keynesian model with autonomous investment; consumption and income comprise only one target as their static multipliers differ only by the (fixed) marginal propensity to consume. Thus, one (and only one) instrument may be used to drive either variable along a desired path; independent goals cannot be realized. In the parlance of the earlier problem, the rank of (C'HC) is also constrained by the rank of C. The lack of independent linkages will be seen as the rank of C falling short of the number of instruments. Thus, a second limitation presents itself as the requirement to use only

independent target and instrument combinations. This, unfortunately, is not a clear cut empirical statement : lack of independence, like multicollinearity of regressors, is only respected in its absence. A "high degree" of interdependence may or may not permit the computation of meaningful solutions, depending on the precision and criterion of smallness inherent in the algorithm.

A third statement should be made regarding certainty equivalence solutions in which: 1) only endogenous variables are targeted, and 2) the numbers of independent targets and instruments are equal. In this case, the welfare loss incurred will be purely a function of the additive disturbances, and the expected deviation between controlled and desired trajectories will be zero for every period. This, of course, is the consequence of "certainty equivalence" assumptions; with known multipliers, one need only calculate how wide a gap is to be closed, and close it completely (and immediately) with policy actions irregardless of its magnitude. It follows from this behavior that the optimal certainty-equivalent feedback rule will have a constant feedback matrix G; i.e. consider an optimal rule :

$$(5.1) \quad x^*(t) = G(t)\, y(t-1) + g(t)$$

where x* is the optimal period t policy on observing the state vector in period (t-1). Then matrix G(t) will be constant for all t; the "steady-state" feedback rule is always present. This is a direct consequence of the "close the full gap every period" strategy detailed above. There is no need to adjust the optimal reaction to an observed shock via a measurement of the gap, as would be the case if parameters were not assumed known with certainty.

With these characteristics of certainty equivalence control in mind, let us now consider the several control solutions performed with the linear model. All of the optimal policy experiments described in this section were run for twenty quarters of historical time, starting in 1959.1, 1964.1, or 1969.1. The government spending, tax surcharge, and reserve base policy instruments were examined singly for each of these starting points. The single target specified in each case is the attainment of a "full-employment" level of income. The target path was generated by specifying a constant 1% quarterly growth (4.06% compound annual rate) on a base of the YPOT measure in the initial conditions. Due to changes in the trend rate of growth of YPOT, our target series does not correspond exactly to YPOT, and usually lies above the C.E.A. series.

Since the aims of this study do not emphasize certainty equivalence experiments, this rather naive constant Y growth goal is used in most of the experiments reported. The first three involve controlling Y with the government spending instrument (G). The total welfare cost (derived solely from the additive disturbances) and mean instrument value in each case are given below. Runs four through six use the tax surcharge variable (T) to drive Y along the constant-growth path. Runs seven through nine make use only of the change in the reserve base (DBASE) to accomplish the same objective. Since the central bank discount rate enters the model parallel to the change in the reserve base, experiments using the former variable are not reported here.

Table 5.1 [1]

Certainty Equivalence Control Solutions of the Linear Model

One target, one instrument, 20 quarters			
Inst	Start per.	Total W	Inst mean
G	1959.1	1548.600	198.756 (183.33)
G	1964.1	1548.600	228.318 (229.77)
G	1969.1	1548.600	265.775 (252.40)
T	1959.1	1548.600	-15.277 (0.0)
T	1964.1	1548.600	6.27001 (0.0)
T	1969.1	1548.600	-17.372 (0.0)
DBASE	1959.1	1548.600	4.58235 (.17582)
DBASE	1964.1	1548.600	0.85506 (.45313)
DBASE	1969.1	1548.600	4.22523 (.22799)

The summary statistics presented above tell only a small portion of the story of these experiments. One obvious conclusion is that the one target may be equally well attained by any of the instruments used, as the total welfare cost is 1548.6 in all runs, regardless of instrument or starting point. This is the intuitive result of assuming no parametric uncertainty and a constant variance for each equation's residuals over time. The loss incurred is purely the result of our uncertainty about the additive disturbances, and that loss will be a fixed value for any alternative twenty-quarter optimization with one target and one instrument. This does not imply, however, that the path of the instrument and the time paths of non-targeted endogenous variables will be at all similar. We turn now to the examination of some of that detail.

When government spending is the instrument, a distinct pattern emerges. In run one, G is perturbed from its initial condition of $174.82 billion to a first-quarter

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1 In Table 5.1, the single target is 1% quarterly growth of Y from the 1958.4 level of YPOT. "Inst" indicates which instrument is being used to control; "Start per." indicates the starting quarter of the experiment. "Total W" measures the value of the objective function over the twenty-quarter horizon used. "Inst mean" reports the mean value of the control variable; its historical mean is given below in parentheses.

level of $186.56 billion (annual rate). Thereafter, the optimal increases in G taper off reaching an almost stable level in quarters eight through eleven. Small amounts of stimulus are added thereafter to place G on a steady growth path. Turning to other endogenous variables, we see that the unemployment rate is quickly brought down to below 4% by quarter nine, and near the steady-state value of 3.5% by the end of the solution. Since the target path for Y is slightly above that of YPOT, we achieve a lower rate of unemployment than the 4.02% consistent with Y equal to YPOT. The money wage index is rising steadily, in line with rising income and unemployment, while the price level is following a slower (but steady) trend. The solution is achieved with steady growth in the levels of consumption and business fixed investment; higher levels of interest rates deter any sizable expansion of residential investment. Since increases in the base are held at their historical (and non-expansive) levels, the burden of financing increased government spending is only partially assumed by higher tax levels; most of the deficit generated appears as sharp increases in bond stocks. The dynamics of this process show a very quick increase in bond issuances in the first eight quarters, tapering off later as induced taxes carry a greater share of the burden. The money stock remains at near-historical levels, influenced by exogenous reserve base

and discount rate policies.

This series of observations completes our cursory inspection of a single control solution. Since we cannot consider each experiment at even this level of detail, let us mention only the highlights of control via tax or monetary policies. When the tax surcharge variable is used, a pattern similar to that of government spending emerges. Large rebates characterize the first eight quarters of the optimal policy, tapering off to zero action after twelve quarters and actually imposed as a surtax henceforth. The major impact of policy is seen as the initial shock to drive Y from a very suboptimal position to the full-employment path; subsequent policy actions are designed to stabilize income growth around that path. Tax surcharges may thus be considered gross substitutes for adjustments of government spending levels, an expected result given the nature of the model's fiscal sector.

On the other hand, when a monetary instrument is utilized to control Y, substantially different results are produced. The target is achieved with the same precision, but the detailed effects are quite dissimilar (not to mention implausible). When the change in the reserve base is used in run seven (starting in 1959.1) a huge increase in

the base is required to drive Y to YPOT in the first control quarter. Since the impact multiplier on income is near unity, a $46.39 billion increase in the base is used to close a $45.15 billion gap. This raises the money supply by $123.42 billion, drives the interest rate on commercial paper to -21.99%, and increases residential investment by almost 50%. A very strong cyclical pattern of base changes may be observed; the quarter three optimal change is $-29.13 billion, followed in quarter five by a $35.54 billion increase. Periodic behavior also is revealed in the level of RCP, which becomes negative every four quarters, reflecting the basic cycle in the dynamic system.

Analysis of these clearly unreasonable results indicates a very useful set of guidelines concerning use of the model. In the normal sense, the model is stable; multiplier experiments will show that the response to a monetary stimulus is bounded. However, when the problem is inverted to perform optimal control, the system's behavior may be quite different, especially if no constraints have been placed on instrument settings. This later case arises here because of the ultra- Keynesian nature of the model; the money multiplier's magnitude requires that a great deal of money be injected to achieve a goods-market target level immediately. This enters the area of instrument

instability, which has been exhaustively examined by Holbrook (1972, pp. 57-65). We shall say no more at this point except to anticipate that our willingness to use monetary instruments will in general depend not only on the magnitude of their estimated coefficients, but also on those coefficients' level of significance.

We turn now to several alternative certainty equivalence control experiments. The first four relate to specification of alternate targets (that is, other than Y) with fiscal or monetary instruments utilized. The last three runs are dual-target, dual-instrument solutions in which two targets and two instruments are used. All experiments are performed with twenty-quarter horizons, starting in 1964.1. A discussion of these results follows their presentation.

Table 5.2 [1]

Certainty Equivalence Control Solutions of the Linear Model

Inst	Start per.	Total W	Inst mean
With UR vs. one instrument for 20 quarters			
G	1964.1	2.18957	227.108 (229.77)
DBASE	1964.1	2.18957	3.90085 (.45313)
With P vs. one instrument for 20 quarters			
G	1964.1	1.60081	233.931 (229.77)
DBASE	1964.1	1.60081	-1.0922 (.45313)
With {P,Y} vs. two instruments for 20 quarters			
G	1964.1	7.127E9 [2]	-266297. (229.77)
DBASE			769109.0 (.45313)

(continued)

With {RCP,Y} vs. two instruments for 20 quarters			
G	1964.1	1552.022	221.887 (229.77)
DBASE			2.63766 (.45313)
T	1964.1	1552.022	14.8978 (0.0)
DBASE			2.65007 (.45313)

Again, the summary statistics present only a fraction of the information to be found in these experiments. Let us discuss the most important results: in runs one and two, the straightforward Y - UR relation in the model was exploited to hold the unemployment rate at its "full employment" level via fiscal or monetary instruments. Either instrument will, as before, incur the same total welfare cost. In the first case, the leap from 5.6% unemployment to 4% is achieved by adding $60 billion to government spending - causing large

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[1] In Table 5.2, the targets include: for UR, a constant 4.0%; for P and Y, 1% quarterly growth; and for RCP, a constant (1963.4) level of 3.907%. For the definitions of table headings, see note to Table 5.1. In the {P,Y} run, a relative weight of 0.9 was given to P. In the {RCP,Y} runs, RCP's weight relative to Y was 0.5.

[2] The notation "F9" indicates that the figure given must be multiplied by ten to the ninth power.

disruptions in subsequent quarters. The instrument is systematically varied around the $220 billion level ($20 billion above initial conditions) to achieve the desired result. On the other hand, when the reserve base is the instrument, implausible changes are required to make the same leap, increasing the base by $184 billion in the current quarter, decreasing it by $164 billion in the next quarter, and so on. Changes of this magnitude (causing $400 billion changes in the money stock) create very unrealistic movements of interest rates, and in general only slightly damped cycles in the instrument and in non-targeted endogenous variables. However, since the sole target is UR, none of these fluctuations are "costed"; the low welfare cost is derived from the small residual variance of the UR equation.

In runs three and four, another portion of the wage-price- employment sector is targeted by specifying the desired rate of change in the price level at 1.0% compound growth per quarter. The fiscal experiment results in very wide swings in the level of G, Y, and inventory investment; here, too, we may consider that a degree of instrument instability intrudes. The linkage between P and the level of aggregate demand is not sufficiently tight to yield coherent results, given the model's structure. In the

monetary experiment, an even more severe problem of instability may be noted. This problem can perhaps be traced to the near-recursive structure of the wage-price-unemployment sector. Somewhat counter to what we might expect *a priori*, the effects of monetary expansion on prices work only via income and the unemployment rate, in that order. Thus, neither G nor DBASE alone may be used to control successfully movements in the price level, unless the trajectories of all other state variables (including the policy instrument) are truly of no consequence.

The remaining three experiments summarized above are two-target, two-instrument runs. A series of these were attempted, but several proved unworkable for the reasons described earlier. That is, an absence of independent (or nearly independent) linkages between instruments and targets was discovered. In the first experiment attempted, the price level and the level of income were targeted to be controlled by the government spending and tax surcharge variables. The computations revealed a nearly singular (C'HC) matrix in the very first period of the horizon. Thus, we may conclude that the two fiscal instruments appear in such a parallel fashion as to preclude their joint usage - at least in terms of controlling P and Y. In general, we should not be surprised that deterministic control reveals

the existence of only one effective fiscal tool in our highly aggregated model.

An attempt to circumvent this problem by using both fiscal and monetary instruments brings out some of the borderline cases. In run five above, P and Y are controlled with G and DBASE. Upon closer inspection, however, we note that the total welfare cost is in the billions of units. A glance at the target and controlled paths shows wildly varying cycles of both instruments and targets with neither target being closely approached thus, we find that controllability must not be considered as an absolute; in this case, the linkages between and among instruments and targets are sufficiently well defined to permit computation of a control solution, but not clear enough to achieve a usable solution. We find, then, that the {P,Y} combination is not controllable in a deterministic setting with the available instruments.

Returning to more feasible experiments, we now consider runs six and seven, in which we target both the growth of Y and the maintainence of a stable interest rate (RCP). We might think of this experiment as fiscal stimulus coordinated with a interest rate target for monetary policy, holding interest rates fixed to encourage the expansion of

investment demand. In run six, government expenditures are the fiscal tool, coupled with a change in the reserve base. The Theil-Norman theorem holds, and both targets are achieved precisely. The total welfare cost is seen to be only marginally higher than that of the one-instrument, one-target solutions of Table 5.1. This suggests that the addition of an extra target and instrument add very little to the cost of control (given judicious selection of the set). Indeed, the task of reducing fluctuations in the level of income may be made easier by the maintainence of stable interest rates. Both instruments are used in very modest amounts, with slight upward trends in their levels. In run seven, the tax surcharge replaced government spending; a very similar set of results was obtained. The exception is the optimal setting of the fiscal instrument. Because of the indirect effects of tax policy, a rebate is used very heavily initially, then gradually a cycle in income brings the optimal tax level to a surcharge. This cyclical behavior persists throughout the horizon suggesting the complexities of designing tax policies when spending levels are held at their non-stimulative levels - i.e. uncoordinated with the tax stimulus.

2. Stochastic Optimal Control Experiments

This section generalizes the results of the previous section by considering a crucial source of uncertainty - that relating to the model's estimated parameters. As discussed in Chapter IV, the full variance-covariance matrices of reduced form parameters (both within and across equations) are used in the determination of optimal policy. The degree of uncertainty associated with each instrument and its linkages to targets will therefore be taken into account. Intuitively, we would expect a more cautious approach to policymaking, with the optimal policy tempered by considerations of possible "slippage" when a given instrument is used.

Using the Chow closed-loop approach, a set of T feedback rules are calculated, as in the certainty equivalent case. Now, however, the expected trajectory for the state vector embodies two assumptions : that all additive disturbances take on their mean value of zero, and that all parameters in the model are equal to their estimated values. Since neither of these assumptions will in general be satisfied for any given replication of the model, the expected trajectory will never be realised. However, the feedback rules will still be optimal, and if

followed faithfully will produce the minimum expected loss if the stochastic assumptions are satisfied.

In the parametric uncertainty solution, a more intuitively reasonable result is obtained - the addition of a stochastically independent instrument adds to the policymaker's tool kit, even if only one target has been specified. (Stochastic independence is not meant to imply independence in the statistical sense, but rather the existence of a unique linkage). A stochastically independent instrument's linkage may be indistinguishable from that of another instrument when parametric uncertainty is ignored. For instance, we have seen that, in general, government expenditures and a surtax are perfect substitutes under certainty equivalence; that is, their effects cannot be distinguished from one another. In the stochastic case, though, the two may be complementary. The policymaker will in general make use of both of them to reduce the expected loss associated with utilizing one instrument vigorously. Thus, the previous section's conclusions regarding independent linkages must be modified to include stochastic independence. We are not concerned with the rank of (C'HC) but rather with the rank of E(C'HC) - that is, its expectation.

One other difference between certainty equivalent and parametric uncertainty solutions should be noted. Under quite general conditions, the matrix G of the optimal feedback rule will not be constant, implying that the steady-state feedback rule will only be reached asymptotically. The optimal reaction to an observed shock will depend in general on the magnitude of the disturbance when parametric uncertainty is taken into account.

With these characteristics of optimal control in the presence of parametric uncertainty duly noted, let us examine several stochastic control solutions performed with the linear model. The first experiments were run for twenty quarters of historical time, starting in 1959.1. The single target specified in each case was the attainment of a "full-employment" level of income, specified as in the certainty equivalence experiments. One run was performed using all four available instruments; it may be considered the benchmark in which all available policy tools were utilized. Subsequent runs present the results when each instrument is used singly; the last two involve using two of the instruments simultaneously. The total expected welfare cost, mean deviation from target path (and its standard deviation) and mean instrument values in each case are given below.

Table 5.3

Parametric Uncertainty Control Solutions for the Linear Model [1]

One target over a horizon of 20 quarters			
Inst	Total W	Eps. mean	Inst mean
All [2]	2512.190	-.0118 (.0406)	313.157 135.773 -2.1639 2.41397
G	2892.550	-.1270 (.2061)	198.729
T	3246.670	-.3287 (.5358)	-15.272
DBASE	1773316.	-16.75 (7.777)	2.28027
RDISCT	1161255.	17.50 (24.66)	-15.504
{G,T}	2549.910	-.0104 (.0391)	308.714 140.165
{G,DBASE}	2836.040	-.1273 (.2045)	207.927 -2.5942

The summary statistics reported in the table give only an outline of the results of stochastic optimal control. Their most striking feature is the variation in total welfare cost - that is, differing values of the objective function - for the different instrument sets. The cost is seen to depend not only on the number of instruments used, but on the choice of instruments. Interestingly, the "all instrument" case has a cost not substantially lower than the best one-instrument case (in which G alone is used). Furthermore, there are wide discrepancies between the costs incurred in these several one-instrument runs. One caution must be noted : because of the differing stochastic assumptions in these runs and the certainty equivalent runs, the total cost measures are not directly comparable. Controlling a model subject to improper assumptions is a type of suboptimal control, and will be dealt with in Section Four of this chapter.

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1 In Table 5.3, the single target is 1% quarterly growth of Y from 1959.1 through 1963.4. "Inst" indicates which instrument is being used to control; "All" implies the order {G,T,DBASE, RDISCT}. Their historical mean values are {183.33, 0.0, 0.17582, 3.2228}, respectively. "Total W" measures the value of the objective function. "Eps. mean" reports the expected mean deviation of Y from its target; the standard deviation of that quantity is given below in parentheses. "Inst mean" reports the mean expected value of the control variable or variables.

2 For "All" instruments, the instrument means are reported in the order {G,T, DBASE, RDISCT}.

We may readily note that the strongly Keynesian flavor of this model, noted in its specification, estimation and certainty equivalence control, carries over to stochastic optimal control. Examining the first five experiments, we see that all instruments' use results in a loss of the same order of magnitude as that resulting from either fiscal instrument. Either monetary instrument results in a tremendously higher loss (as in the certainty equivalent cases), caused by both deterministic and stochastic features of the model. Let us first examine the "all instrument" run. The target is achieved with only a $10 million mean error, with a good deal of that attributable to a $151 million first-period shortfall. Government spending is used vigorously, increasing from an initial condition of $174.82 billion to a first-quarter level of $222.43 billion. Thereafter, steady increases are used. The tax surcharge is increased to $45.51 billion in the first quarter (almost equalling the increase in government spending) and raised steadily thereafter, in line with government spending increases.[1] Monetary instruments are used in a contractionary sense; the base is steadily reduced by two to three billion dollars per quarter, resulting in a steady

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1 This seemingly marginally balanced-budget change in fiscal tools may be used to infer something about the political persuasions of the Amdahl 470/V6.

shrinkage of the money supply. The discount rate, however, is steadily lowered to counteract this effect.

To examine the impact on non-targeted endogenous variables let us first consider the components of aggregate demand. The major impetus of policy is evidenced by a shift toward public sector spending; consumption and residential investment both decline in response to higher taxes and rising interest rates. Non-residential investment picks up slightly in response to higher levels of spending. All three interest rates rise, with the Treasury bill rate rising only slightly in response to initially increased bond issuances. Wages, prices, and the unemployment rate behave as expected, the first two rising steadily, and the unemployment rate falling to its steady-state value of 3.5% (as in the certainty equivalent case). Thus, we see a greater emphasis on public spending, with initial deficits being turned into surpluses in the fifth quarter and henceforth. By a careful choice of fiscal instruments, the desired level of income is reached, with private spending held down subsequently by monetary restraint and government surpluses.

Using this experiment as a benchmark, let us consider the highlights of the single-instrument runs. We see first

of all that government spending alone may be used to achieve "full employment" with only slightly higher cost; the mean deviation from the target is still only a quite respectable $127 million. Without control of the surtax, however, a quite different pattern emerges. All private aggregate demand components increase strongly; the three interest rates rise, reflecting the exogenous nature of monetary instruments constrained to their historical path. Government bonds are issued at a very rapid rate throughout the horizon, tapering off as induced taxes increase to narrow the deficit. On balance, therefore, private "wealth" increases. We may also note that fairly small increases in government spending suffice to close the GNP gap, as we found in the certainty equivalent case.

When the only available instrument is the tax surcharge, a much higher loss is incurred, and the mean deviation from target widens to almost one third of a billion dollars. Otherwise, the experiment is very similar to the government spending run. The more volatile nature of tax policy, due to its indirect effects, may be clearly seen; a rebate of $18.53 billion in the first quarter is increased to $42.50 billion in quarter five in an attempt to bring income up to the desired level. A threatened overshooting of the target path is then countered by

reducing the rebate, and eventually imposing a $7.60 billion surtax in quarter fifteen. These wide swings are reflected in the government's budget position and in the level of government securities, contributing to the generally greater instability of tax policy compared to spending policy.

Turning to the monetary instruments, we see a striking inability of policy to achieve the target level of income. Using the reserve base, the control value falls almost $17 billion short of the target path on the average, never coming closer than a $5.7 billion shortfall. This is achieved by huge initial increases in the base, which drive the money supply up $40.2 billion in the initial quarter, requiring an unrealistic commercial paper rate of -5.1% to clear the market. Although the unreasonably wide swings in levels of spending and interest rates experienced in certainty equivalence control are avoided in this run, use of the reserve base alone does not suffice to stabilize income at a full-employment level. In run five, when only the discount rate is used, similar results emerge. Even more unrealistic RCP levels are caused by yet larger increases in the stock of money, reflecting the relative impotence of this instrument in both deterministic and stochastic terms. Interestingly, the control values of income start below the target, and then increase markedly

above it, reflecting the inability of the discount rate alone to stabilize income at the target rate of growth. As we might expect, the optimal discount rate is sizably negative throughout the horizon.

Let us now consider the last two runs summarized above, in which two instruments are used. In experiment six, the fiscal instruments alone are used, and the expected loss is only marginally higher than that of the "all instrument" case. This indicates that addition of the two monetary tools adds little to our ability to control the model; and, indeed, a larger mean and standard error of the deviation from target path are experienced when they are added (in run one). The essentially fiscal character of the model, in both deterministic and stochastic terms, manifests itself here. Compared to run one, we see an even greater reliance on public sector activity, with over-balanced government budgets steadily retiring government debt. As in the "all-instrument" run, disposable income actually declines while total income grows along its "optimal" path.

In the last experiment, the reserve base and government spending are used, with results quite similar to the results for government spending alone. Only a small improvement in expected loss is noted, and the mean expected deviation from

target is fractionally higher. As in run one, modest increases in government spending are coupled with decreases in the reserve base to reduce private spending. Sizable increases in the stock of bonds are required to counteract these activities. It seems evident that, given the model, addition of a monetary instrument does not greatly enhance the policymaker's capabilities.

In the rest of this section, we present the results of several multi-target, multi-instrument experiments. As the number of possible experiments of this type is virtually limitless, we include here only a sampling of possibilities. Ultimately, the choice of experiments must be linked to the direction of research to be performed with a particular model. These experiments were again performed over twenty quarters, starting in 1959.1. Details of the target paths specified are given in the table, as are the components of the instrument set for each experiment. The other summary statistics reported correspond to those given in the previous table.

Table 5.4

Parametric Uncertainty Control Solutions for the Linear Model [1]

Inst	Total W	Eps. means	Inst means
With {P,Y} vs. two instruments for 20 quarters			
{G,DBASE}	5095.707	-2.683	207.932
(.9)		-.1003	-2.5936
{G,I}	4809.087	-2.678	308.751
(.9)		.0160	140.202
With {RCP,Y} vs. two instruments for 20 quarters			
{G,DBASE}	2890.690	1.414	204.110
(.5)		-.1329	-1.3100
{T,DBASE}	334436.0	-22.18	152.547
(.5)		.7440	44.1213
With {W$,Y} vs. two instruments for 20 quarters			
{G,T}	7455.278	4.131	308.371
(.75)		-.3754	139.882
With {UR,Y} vs. two instruments for 20 quarters			
{G,DBASE}	2849.475	.0675	207.926
(.9)		-.1286	-2.5941
{G,T}	2563.321	.0643	308.713
(.9)		-.0117	140.165

In the first pair of experiments, the price level and income were both targeted to grow at 1%: income roughly on its potential path, and the price level from its 1958.4 level. Attainment of this rate of inflation was given a weight of 0.9 relative to the weight on income. The first experiment used government spending and monetary policy to achieve these goals. The income goal was quite closely met, but the rate of growth of the price level falls far short of the target. Once again, modest increases in government spending were coupled with constant decreases in the reserve base. Interest rates rise, choking off residential investment. The bond stock was rapidly increased to meet the government deficit.

When purely fiscal tools were used to achieve these goals, the underachieving of prices carried over. However, income is much more closely controlled. A very different detail of results emerges, though, as large balanced-budget increases in spending and taxes augment public spending at

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1 In Table 5.4, one of the targets is 1% quarterly growth of Y along its potential path, with the horizon extending from 1959.1 through 1963.4. The other variables' targets include: for P and WS, 1% quarterly growth; for RCP, a constant (1958.4) level of 3.213%; and for UR, a constant 4.0%. Y was always given a weight of unity; the weight on the other target is reported in the "Inst" column. For the definitions of table headings and the instruments' mean values, see note to Table 5.3.

the expense of consumption and residential investment. Bonds were retired in response to a growing surplus, causing the Treasury bill rate to fall while the commercial paper rate rose. Despite a rising long rate, business investment turned upward strongly. We again may note that widely varying states of the economy may be generated by quite similar control experiments.

In the second pair of experiments, the second target was RCP; its stability was given a relative weight of 0.5. Although the {P,Y} pair could be controlled with similar precision by either instrument set, the {RCP,Y} pair proved much more sensitive to choice of instruments. Run three, using spending and reserve base policies, achieved the targets quite closely, although the interest rate deviated markedly above its target by an average 1.4 percentage points. This was achieved by the same combination of moderate spending increases coupled with declines in the base. No untoward fluctuations were observed.

When the tax surcharge replaced spending as the fiscal instrument, a completely different story was told. The huge expected loss was primarily caused by a negative interest rate throughout most of the horizon. Consumption decreased sharply, while RL also took on negative values. Tremendous

($40 billion) additions to the reserve base were coupled with $150 billion surtax levels. This caused the entire stock of government debt to be retired by quarter five, with negative levels observed thereafter. The differences between the efficacy of fiscal instruments in a stochastic framework are evident. Far from being the perfect substitutes they appear to be in certainty equivalence terms, their differing stochastic multipliers cause them to be used in very different ways.

In run five, joint control of W$ and Y was tested, using only fiscal tools. A relative weight of 0.75 was given to steady growth of the money wage level. It may be recalled that neither of the pairs {W$,Y} or {UR,Y} could be controlled in certainty equivalence terms due to the lack of independent linkages. This run indicates that the former pair is indeed controllable, but not very precisely. Income's target was missed by an average $375 million, while the money wage level drifted far below its target trend. This illustrates that a seemingly reasonable set of goals may not be internally consistent. We may target money wage and income growth, but the particular targets we set need not lie on the attainable frontier; and will not therefore be met.

The last two runs extend the process of comparison with certainty equivalent results by targeting both UR and Y, which are rigidly linked by the unemployment rate equation. In this case, we find ourselves paradoxically trying to increase unemployment, since the desired income growth alone will tend to drive unemployment below the specified target of 4%. The necessity of specifying meaningful targets must be reiterated. In these experiments, even the high relative weight of 0.9 placed on UR fails to have much effect. Income was driven close to its target path, and the unemployment rate inexorably fell below its respective target in the second half of the horizon. Both fiscal-monetary and purely fiscal instrument sets were similarly successful, with the fiscal set having a slightly lower expected cost. Both sets achieved similar results for UR, but the fiscal set achieved the target path for income with much greater precision. The effects on other endogenous variables are roughly similar to those described earlier in fiscal-monetary or purely fiscal policy experiments.

This concludes our presentation of ordinary stochastic control experiments. The few runs presented have given a cursory indication of the model's performance under stochastic control. The focus of this study now turns to applications of stochastic control beyond the

straightforward computation of ordinary control solutions. It should be clear that the behavior of a macroeconomic model may be much more rigorously analyzed in the context of fully stochastic control. Artificial restrictions (such as equivalence of the numbers of targets and instruments) may be lifted, and the confidence which the policy analyst may place in the policy instruments may be rigorously quantified. Although it may be argued that such confidence is ill-placed, given the state of the art in econometric modelling, stochastic control techniques allow us to examine the workability of a model, taking the estimates and degree of uncertainty as a given.

3. Sensitivity of Control Solutions to Target Specification and Exogenous Variable Forecasts

In the discussion of various types of uncertainty in Chapter II, one source of uncertainty relevant to the policymaker was omitted. When an optimal control solution is sought, the researcher must possess an accurate macroeconomic model with well specified policy linkages. But if the control solution is actually performed over future periods, analogous to an *ex ante* model simulation, various unknown quantities must be utilized to implement the solution. The most obvious unknowns are the future values

of all exogenous variables in the model. While some (such as the ubiquitous time trend) may be known with relative certainty, the necessity to project exogenous variables' values over the horizon introduces considerable (if largely immeasurable) uncertainty into the calculations. This, of course, is not at all restricted to the application of control, as any forecaster deals with this problem in operating his or her model.

The forecaster performing deterministic simulations may readily calculate the sensitivity of the forecast to a badly chosen exogenous variable value by multiplier calculations. This, however, requires a control solution and an alternate solution to be performed for each sensitivity calculation. That is, to determine the change in $y(t)$ given a different $x(t+j)$, a separate perturbation of the x path, and solution of the entire model, must be performed for each y-x pair and future time j to be examined. Although these results can clearly be obtained by any reliable model simulation program, the number of required perturbations of a sizable model becomes very large.

Since the solution of an econometric model involves numerical methods, it should not be surprising that such perturbations are required to measure dynamic sensitivities.

Neither should it surprise the reader that such sensitivities - or derivatives - may be analytically calculated within a closed-loop analytical solution to the optimal control problem. In Baum and Howrey's (1977) modification of Chow's (1976) optimal control algoritha, these derivatives may be obtained as part of a standard optimal control solution. We will deal here with two types of derivatives: with respect to targets and exogenous variables, respectively.

We have indicated why a researcher may be concerned with the sensitivity of a model simulation to forecasted exogenous values. It should be even more evident that a practitioner of stochastic control would be interested in measuring the robustness of that stochastic solution with respect to forecast errors. Thus, one set of measures of sensitivity calculates the response of the feedback rule intercept, g(t) in eqn. (5.1), to the exogenous variables' values in periods (t+j), j=0,1,...T. This set of dynamic derivatives will help the researcher understand what effect a poorly specified exogenous variable 4,8,... quarters hence will have on this period's policy prescription.

The rationale for computing the second set of derivatives lies closer to the realm of control theory. We

may calculate the sensitivity of g(t) with respect to the vector z(t+j) - the target vector - for any j in T. The effect of altering the target path might readily be calculated by solving another control problem. But without the need of solving a multitude of alternate paths, the policymaker may inspect this set of derivatives to find what effect a different set of goals will have on this period's optimal policy. This might be quite relevant in examining now, in 1977.3, what effect the coming of a Republican administration in 1980.1 might have on a Carter administration control solution run through 1984.4, assuming that the two groups possess differing targets for key variables.

In the remainder of this section, we will first discuss the theoretical background underlying the computation of these derivatives, and then present a sample of sensitivity measures calculated in one of the stochastic control experiments of Section Two. Let us first derive the measures of the feedback rules' sensitivity to specifications of the target vector z(t), as these expressions are more easily manipulated. We will then present the derivatives of the feedback rules with respect to the exogenous variables' values.

The derivatives' form may be found in the "updating equations" which direct the backward recursion of the L-Q-G algorithm. These equations were mentioned, but not explicitly presented, in Section Four of Chapter II. They may be found in Chow (1975, pp. 177-179). As we stated in Chapter II, the vectors h(T) and k(T) are defined in period T as {K z(T)}, and the feedback rule's intercept vector defined (in eqn. 2.6-2.7) as:

$$(5.2) \quad g(T) = - E \{ \{C'H(T)C\}^{-1} \{C'H(T)b(T) - C'h(T)\} \}$$

Thus, the derivative of vector g(T) with respect to vector z(T) is:

$$(5.3) \quad dg(T)/dz(T) = E\{ \{C'H(T)C\}^{-1} C'K \}$$

where we may also note that H(T) = K; so that this set of derivatives depends only on control variables' coefficients and the target weight matrix.

The updating equation used to compute h(T-1) can be written as:

$$(5.4) \quad h(T-1) = k(T-1) + \{A + CG(T)\}'\{h(T) - H(T)b\}$$

so that we may compute:

$$(5.5) \quad dh(T-1)/dz(T) = \{A + CG(T)\}' K$$

by differentiating h(T) with respect to z(T) in turn. Let us at this point define $R(t) = \{A + CG(t)\}'$ for ease of notation. Then, by applying equation (5.2) stepped back to period (T-1), we may express g(T-1) as a function of h(T-1); and, substituting eqn. (5.4), calculate:

$$(5.6) \quad dg(T-1)/dz(T) = E\{ \{C'H(T-1)C\}^{-1} C' R(T) K \}$$

By continuing this recursive process one more step, we may show that:

$$(5.7) \quad dh(T-2)/dz(T) = R(T-1) R(T) K$$

and, analogous to eqn. (5.6),

$$(5.8) \quad dg(T-2)/dz(T) = E\{ \{C'H(T-2)C\}^{-1} C' R(T-1) R(T) K \}$$

This should provide enough detail to indicate the underlying pattern of these derivatives' form. We have illustrated the sensitivity of the Tth, (T-1)th, and (T-2)th feedback rules to the terminal target vector z(T). It should be clear that

the same formulas apply to any of the feedback rules and target vectors. Thus, if we wish to examine the first feedback rule's sensitivities to target specification, we calculate the following:

$$(5.9)\quad dg(1)/dz(1) = E\{ \{C'H(1)C\}^{-1} C' K \}$$

$$(5.10)\quad dg(1)/dz(2) = E\{ \{C'H(1)C\}^{-1} C' R(2) K \}$$

$$(5.11)\quad dg(1)/dz(3) = E\{ \{C'H(1)C\}^{-1} C' R(2) R(3) K \}$$

Or, in the general form,

$$(5.12)\quad dg(1)/dz(tau) = E\{\{C'H(1)C\}^{-1} C'\{R(2)R(3)\ldots R(tau)\}K\}$$

for any tau greater than one. Comparing the form of these derivatives to the solution of the L-Q-G algorithm, it is evident that little extra need be calculated. Nearly all of the expressions appearing in the derivatives are computed in the process of calculating g(t), so that the formation of these sensitivity measures require nothing but organization and storage.

We now consider the more complex task of calculating

derivatives of the feedback rules with respect to the exogenous variables. In Chapter II, we subsumed the exogenous variables and their estimated coefficients in a time-varying vector b(t) which contained their product. For this exposition, we must express them separately; thus, we define b(t) = p w(t), where p is the NY by NW matrix of coefficients, and w(t) is the NW by one vector of exogenous variables at time t. Beginning once again with period T, the derivative of vector g(T) with respect to vector w(T) is:

$$(5.13)\quad dg(T)/dw(T) = - E\{ \{C'H(T)C\}^{-1} C'H(T)\ p \}$$

This illustrates that these derivatives depend only on control and exogenous variables' coefficients and the target weight matrix.

If we examine eqn. (5.4), the updating equation for h(T-1), we see that we may compute:

$$(5.14)\quad dh(T-1)/dw(T) = - E\{ R(T)\ H(T)\ p \}$$

making use of the definition of R(t). This may then be substituted into the g(t) updating equation to produce:

$$(5.15)\quad dg(T-1)/dw(T) = - E\{ \{C'H(T-1)C\}^{-1} C' R(T) H(T) p \}$$

By continuing for one more iteration, we see that:

$$(5.16)\quad dh(T-2)/dw(T) = - E\{ R(T-1) R(T) H(T) p \}$$

and, analogous to eqn. (5.15),

$$(5.17)\quad dg(T-2)/dw(T) = - E\{\{C'H(T-2)C\}^{-1} C'R(T-1)R(T)H(T)p\}$$

As we indicated for the derivatives with respect to the target vectors, these formulas will apply equally well to the first feedback rule. Its sensitivities to exogenous variable specification we may calculate as:

$$(5.18)\quad dg(1)/dw(1) = - E\{ \{C'H(1)C\}^{-1} C' H(1) p \}$$

$$(5.19)\quad dg(1)/dw(2) = - E\{ \{C'H(1)C\}^{-1} C' R(2) H(2) p \}$$

$$(5.20)\quad dg(1)/dw(3) = - E\{ \{C'H(1)C\}^{-1} C' R(2) R(3) H(3) p \}$$

Or, in the general form,

$$(5.21)\quad dg(1)/dw(tau) = - E\{ \{C'H(1)C\}^{-1} C'\{R(2)R(3)\ldots R(tau)\} H(tau)\ p \}$$

for any tau greater than one. As in the case of the derivatives with respect to target specification, little extra need be calculated to produce these derivatives, as virtually all their components are available in the process of computing the control solution.

At this point, we might consider the uses to which these sensitivity measures could be put. As suggested earlier, they provide the researcher with a great deal of information about the robustness of his or her control solution. That is, they indicate how much the first period (or conceptually, any period) of the solution will be altered by changing the specified targets or forecasted exogenous variables at some future time. We also will use these derivatives in one of the forms of evaluation of suboptimal control solutions; in particular, in computing the "steady-state" suboptimum. This use will be discussed in Section Four.

The derivatives of g(1) with respect to z(t) and w(t) were calculated for a number of the stochastic control experiments in Section Two. (They were not calculated for

the certainty equivalence runs in Section One, as they are uniformly zero under conditions of certainty equivalence). Like most parts of a solution, they may not be fully illustrated here. However, a sample of the derivatives computed in the first run presented in Table 5.3 are shown. In that experiment, all four instruments were used to drive Y to a "full-employment" path over a twenty quarter horizon. Since the matrix K is null except for the diagonal element corresponding to Y, only one column of the dg/dz matrix (which is of dimension four by 36, the order of the state vector) is non-null. We present the magnitude of g(1), of x*(1), and of the first four periods of {dg/dz}' in the following table.

Table 5.5

Derivatives of the Feedback Rule

with respect to Target Vectors [1]

Intercept Vector g of feedback rule one			
G	T	DBASE	RDISCT
694.925	573.614	-5.0522	-4.9567
Optimal Policy Vector for period one			
222.434	45.5120	-1.0159	4.78328
{dg(1)/dz(tau)}' for tau = 1, ..., 4			
.927465	.779983	.004622	-.00219
-.00102	-.00229	.001196	-.00056
-.00127	-.00227	-.00136	-.00045
-.00109	-.00186	-.00127	-.00040

As may readily be seen, the model's dynamic structure implies a very low sensitivity to future target specification in all but the current period. An increase of one billion dollars in period two's target would only

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[1] The statistics presented in this table come from the "all instrument" run of Table 5.3. All values in the table are in billions of dollars.

perturb the optimal instrument settings by one to two million dollars for the fiscal instruments - paltry amounts - and would increase the base by slightly over one million dollars, or less than 0.02% of its optimal setting. The effect on the discount rate is negligible. Although further periods' derivatives do not decay as rapidly due to the model's cyclical structure, they are never as great as the fourth-period values.

We now present a subset of the derivatives of g(1) with respect to w(t) obtained in the same experiment. There were four exogenous "variables" in the experiment. The first two, the time trend and the constant terms' unity vector, are not really "variables". However, the derivatives associated with the constant term may be of interest, as they may be interpreted as measuring the model's response to an "adjustment constant" alteration of the estimated intercepts. The latter two variables were PCRUD and YPOT. The dg/dw matrix is thus of dimension four by four. We present the first three periods of {dg/dw}' for each of the exogenous variables in turn in the following table.

Table 5.6

Derivatives of the Feedback Rule

with respect to Exogenous Variables [1]

{dg(1)/dw(tau)}' for tau = 1,2,3			
With respect to TIME			
.000091	.000060	-.00005	.00105
-.00003	-.00002	.000054	0.0
-.00002	0.0	.000035	0.0
With respect to Const			
16.3365	10.8517	-3.809	.045048
-1.4474	-1.8368	.01583	-.05697
-.27843	-.28519	.17051	-.05844
With respect to PCRUD			
.001604	.001030	-.00125	.014705
-.00027	-.00019	.000392	.000018
-.00003	-.00001	.000068	.000012
With respect to YPOT			
.000012	0.0	-.00002	0.0
0.0	0.0	-.00002	0.0
0.0	0.0	-.00002	0.0

As may be noted, very small values appear for these derivatives as well. The large values with respect to the constant terms indicate the result of doubling all intercepts - a sizable change. The price series shows very little sensitivity, while the model's most crucial exogenous variable, potential GNP, has almost no sensitivity whatsoever associated with its future values. Given the specification of the model, we may conclude that imperfect forecasts of exogenous variables will have little effect on the optimal policy settings. This need not imply that these derivatives are of no value or consequence; in general, they need not be damped so quickly. That phenomenon is linked to the short lags utilized in specification of the linear model, and the resulting short cycles (cf. Section Two of Chapter Four). As we shall see in the next section, it is very useful to derive a "rule" for g(t): that is, to express it as a linear combination in future values of the targets and exogenous variables.

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1 The statistics' source and units are given in the note to Table 5.5. Elements less than 0.00001 are given the value 0.0.

4. Evaluation and Comparison of Sub-optimal and Optimal Control Solutions

In this section, we discuss the evaluation of various sub-optimal control solutions, and consider how the loss associated with those solutions may be properly quantified. The purpose of this quantification is, of course, the desire to compare optimal and less than optimal solutions in numerical terms. If we return to the stochastic control experiments presented in Section Two, we see that some suboptimal solutions have already been presented. The "All instrument" run of Table 5.3 is optimal in the sense of utilizing all available policy tools in the instrument set. The following four single-instrument experiments are "optimal" for the given instrument set, but may be considered examples of suboptimal runs. The run using only government expenditures implicitly constrains the three other instruments to take on arbitrary values : in this case, their historical values. Thus, this run must be suboptimal compared to an experiment in which all instruments are utilized. As we may note, the "All instrument" run has the lowest welfare cost of any of these runs. Any experiment which involves the imposition of constraints upon the instruments will necessarily incur a higher welfare cost.

It is therefore possible to consider various "suboptima" merely by performing alternative control runs in which differing subsets of the full instrument set are used. This is a straightforward application of optimal control techniques, and may be readily used to compare alternatives such as the well-worn monetary versus fiscal policy debate. We will not pursue this type of suboptimal policy evaluation any further, as its performance and comparison with optima raise no new matters of substance.

We turn to the evaluation of three alternative types of suboptimal policy, all of which do require additions to the fundamental control algorithm. These are: 1) a suboptimum consisting of a repeated "steady-state" feedback rule; 2) the ex post comparison of historical policy with optimal policy for that period; and 3) the effect of utilizing an arbitrary feedback rule. In all three cases, calculation of the expected loss must be altered to include the assumption that the feedback rule being evaluated may not be optimally chosen. We now discuss the nature of those alterations, and the three alternative types of suboptima.

The computational formulas underlying the L-Q-G algorithm are the backwards recursion equations or "updating equations" discussed in Chapter II. These equations as

presented by Chow (1975, pp. 228-232) are the solution to the optimal policy problem exposited in Chapter II. As a by-product of the solution, the expected value of the objective function (or "expected loss") may be numerically calculated. In Chapter II, we discussed minimizing w(t), the expected loss incurred in and following period t. That expression for period one, w(1), is equivalent to W, the total value of the objective function. Thus, "updating equations" are defined for H(t), h(t), and w(t). If suboptimal policy is to be evaluated, these updating equations must be modified to account for the suboptimal nature of the feedback rules. The computational formulas presented earlier take advantage of the feedback rules' optimality by removing common terms. We must investigate four areas in which changes must be made to accomodate suboptimal feedback rules.

The first modification must be made in the updating of H(t). As noted in Section Three, its updating equation, which defines H(T-1), arises in the two-period problem of minimizing w(T-1). The full equation may be expressed as:

$$(5.22)\quad H(T-1) = E\{\{A+CG(T)\}'H(T)\{A+CG(T)\}\} + K(T-1)$$

which may be expanded to read:

$$(5.23)\quad H(T-1) = E\{ A'H(T)A + G'(T)C'H(T)A + A'H(T)CG(T) + G(T)'C'H(T)CG(T) \} + K(T-1)$$

The expression may be considerably simplified if G(T) was optimally chosen, as then we know that:

$$(5.24)\quad G(T) = - E\{ \{C'H(T)C\}^{-1} C'H(T)A \}$$

This quantity may be substituted into the last two terms in braces of eqn. (5.23) to cancel them against each other, yielding the optimal updating equation:

$$(5.25)\quad H(T-1) = E\{ A'H(T)A + G(T)'C'H(T)A \} + K(T-1)$$

This exposition should make it clear that eqn. (5.25) is only valid if indeed G(t) is chosen optimally. In general, the last two terms in braces of eqn. (5.23) must be included to compute the H(t) pertaining to suboptimal G(t)s.

The second modification involves a parallel change in the updating equation for h(T-1). The full equation, before substitution, reads:

$$(5.26)\quad h(T-1) = E\{ K(T-1)z(T-1) - \{A+CG(T)\}' \{H(T)Cg(T) + H(T)b(T) - h(T)\} \}$$

which may be expanded and rearranged into the form:

$$(5.27)\quad h(T-1) = E\{K(T-1)z(T-1) - \{A+CG(T)\}'\{H(T)b(T) - h(T)\} - \{A'H(T)C + G(T)'C'H(T)C\}g(T)\}$$

In this expression, the last bracketed term will disappear if G(T) is optimally chosen, and its value from eqn. (5.24) substituted in. Thus, for an optimal G(t), we have the updating equation:

$$(5.28)\quad h(T-1) = E\{K(T-1)z(T-1) + \{A+CG(T)\}'\{h(T) - H(T)b(T)\}\}$$

Here, too, it should be clear that eqn. (5.28) only applies to the optimally chosen G(t). For evaluation of an arbitrary G(t), the last term in eqn. (5.27) must be included to correctly update h(t).

The third change which must be made in the updating equations involves the expression c(t), which is something of a catchall term for leftover portions of w(t). This expression does not appear prominently in the normal exposition of the algorithm, as it does not enter the formulas for G(t) or g(t). Nevertheless, its correct evaluation is crucial if the objective function is to be

evaluated for suboptimal rules. In the two-period problem, c(t) appears as:

$$(5.29)\quad c(T-1) = E\{\{b(T)+Cg(T)\}'H(T)\{b(T)+Cg(T)\} - 2\{b(T)+Cg(T)\}'h(T) + z(T-1)'Kz(T-1) + u(T)'H(T)u(T) + c(T)\}$$

This formidable expression may be made less menacing if g(T) is optimally chosen, for in that case we know that g(T) is:

$$(5.30)\quad g(T) = -E\{\{C'H(T)C\}^{-1}\{C'H(T)b(T) - C'h(T)\}\}$$

and several terms containing g(T) which appear in the quadratic form in H(T) cancel with each other. Therefore, the optimal updating equation for c(t) becomes:

$$(5.31)\quad c(T) = E\{g(T)'C'\{H(T)b(T)-h(T)\} - 2b(T)'h(T) + b(T)'H(T)b(T) + z(T-1)'Kz(T-1) + u(T)'H(T)u(T) + c(T)\}$$

which may not seem like a great improvement. Indeed, Chow presents the previous form in his exposition (loc. cit.), but implements the shortened form of eqn. (5.31) in his algorithm. Although the differences may not be readily apparent, it is the previous, complete form which must be

used to update c(t) in the presence of suboptimal feedback rules; i.e. when q(t) is not optimally chosen.

The fourth and last modification involves a much more complex and obscure alteration of the algorithm, involving the proper "costing" of the initial conditions y(0). Intuitively, this is the loss incurred due to the state of the system immediately prior to the initial control period. We will not present the explicit expressions here, but rather indicate what is involved. When the recursion formulas are iterated back to w(1), the total expected loss over the horizon {w(1)} may be written as a quadratic form in y(0), the lagged state vector. The manipulations performed on that form involve the cancellation of several terms given the optimal values of G(2) and q(2) given in eqns. (5.24) and (5.30). When the feedback rule is not optimally determined, these cancellations may not be made. The entire expression for w(1) must be evaluated to yield a correct measure of the loss incurred by y(0).

We have presented the theoretical basis for four modifications to the basic L-Q-G algorithm as implemented by Chow. These modifications enable the correct "costing" of an arbitrary sequence of feedback rules; that is, a measure of the expected loss associated with a suboptimal policy.

The computational algorithm used in this study (Baum-Howrey, 1977) contains these modifications, as well as a set of options which allow various non-optimal sequences of feedback rules to be introduced. The basic framework for suboptimal control of a linear macroeconomic model involves two "passes" over the horizon. In the first pass, the standard optimal control solution is performed; optionally, the analytic derivatives presented in Section Three may be calculated. To initiate the second pass, a sequence of non-optimal feedback rules is defined for some set of tau periods (tau less than or equal to T) of the T-period horizon. The non-optimal feedback rules specified are utilized in evaluating the first tau periods of the horizon, while the rest of the horizon is calculated under optimal conditions. The total expected welfare cost for the suboptimum may then be measured, and directly compared with the optimal counterpart calculated in pass one.

In the remainder of this section, we discuss three types of non-optimal feedback rules which have been used with the linear model. Examples of each type's expected loss relative to the optimal value are given. The first type of suboptimum which we consider is the "steady-state" feedback rule. Chow considers the theory of a steady state solution (1975, pp. 170-172) for a a system known with

certainty. He finds that an equivalence between the numbers of target variables and independent instruments is sufficient for the existence of a steady state, that is, a value $\underline{G}$ upon which G(t) will converge. Given that the recursion process moves backwards in time, the steady state will exist for small t, as he points out. The results he gives will not be altered by our considering a model with unknown parameters, since this will only require that the expectations of parameters be included in place of the parameters themselves. We will not be concerned with the existence of a steady-state value for g, as we do not generally consider time-invariant targets.

The type of suboptimum we may model with steady-state rules involves the repeated application of the first-period rule for the first tau periods of the T-period horizon. Even with our relatively short twenty-period horizons, G(t) for t=3,2,1 approaches a stable position $\underline{G}$. Therefore, the assumption that G(1) is a steady-state value is not too bad from a superficial view. The application of the first-period rule must be carefully defined in terms of the way in which we construct the intercept vector g(t). Clearly, we cannot merely replicate g(1), which was calculated conditional on w(t) and z(t) for all future t. The definition we use implies that a constant formula for g(t)

be used. In eqn. (5.30), we see g(T) as a linear form in w(T) and z(T), appearing indirectly in b(T) and h(T). For any t, g(t) may be calculated as a linear combination in the exogenous variables w(mu) and targets z(mu), for mu = (t, t+1, t+2, ... T). The weights in this linear combination are the derivatives calculated in Section Three.

This suboptimum may thus be calculated by making two "passes" - the first, optimal, in which the welfare cost, $G(^1)$ and derivatives are calculated; the second suboptimal. To the extent that derivatives of g(t) with respect to w(mu), z(mu), mu greater than T are nonzero, the linear approximation will be poor. Since we may only compute derivatives for mu less than or equal to T, this concern may seem academic. However, it merely reflects the more general concern over choice of the appropriate horizon. We always make the assumption that T is chosen as "large enough" so that terminal effects do not dominate. We see that the derivatives' values seem to decline very rapidly for our model, so that we might expect derivatives near the horizon's end to be inconsequential in this case. Application of the linear combination does require that targets and exogenous variables be specified for periods beyond T. In general, to calculate g(tau), (tau-1) additional periods of w(t) and z(t) must be available.

Three experiments with this steady-state suboptimum were performed. The first two relate to the "All instrument" run of Table 5.3. In the first experiment, the first ten periods (tau equal to 10) were performed suboptimally; in the second run, all twenty periods were suboptimal. The third experiment replicates run three of Table 5.4, in which {RCP,Y} were controlled by {G,DBASE}. The three are given below.

Table 5.7

Suboptimal Control Solutions for the Linear Model [1]

Y as target, tau = 10 quarters			
Inst	Total W	Eps. means	Inst means
All	2514.167	-.0134 (.0401)	316.453 141.459 -1.9649 2.30535
Y as target, tau = 20 quarters			
All	2526.828	-.0177 (.0321)	319.580 148.087 -1.7257 2.21941
{RCP,Y} as targets, tau = 20 quarters			
{G,DBASE} (.5)	2893.149	1.514 -.1169	204.275 -1.4942

It may be readily noted that none of the runs indicate a marked difference between optimal and suboptimal values of W. The first two runs should be compared against the optimal welfare cost of 2512.189. Clearly, setting tau equal to one would incur only that cost. Increasing tau corresponds to prolonging the deviation from the optimal trajectory. For tau equal to 10, only 0.08% higher cost is incurred. For tau equal to 20, implying that the whole horizon is performed suboptimally, a slightly higher 0.58% deviation is incurred. In the latter case, the system is given no chance to regain the optimal path, but the cost of this constraint seems minimal. In the third experiment, the cost should be compared with the optimal value of 2890.690. Again, a slightly higher cost (0.09% higher) is incurred, but the difference for a completely suboptimal horizon is very slight.

The conclusion that we may draw is quite relevant to policy analysis. If a model is as "well-behaved" as the linear model in the sense of possessing a near-steady state

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1 The runs in Table 5.7 are steady-state suboptima performed on the optimal experiments of Tables 5.3 (run 1) and 5.4 (run 3). For definitions of targets and table headings, see notes to those tables.

G(t), repeated optimizations may be avoided to some degree. Once an optimal solution is obtained, the steady-state rule given by G(1) and the matrices of derivatives defining g(1) may be used to approximate optimal feedback rules. For some situations, this would apparently be a quite reasonable approximation.

We now consider an alternative type of suboptima - that defined by the ex post "costing" of historical policy using the optimal policy as a benchmark. It must be noted that, in general, there is no single accepted procedure for evaluating historical vis-a-vis optimal policy. There are several reasons for this lack of consensus, not the least of which is the necessity of specifying one's assumptions about policymakers' own behavior. In the context of feedback control, we may discuss alternative assumptions in terms of their implications for the feedback rule matrices G(t) and g(t). I am indebted to E.P. Howrey for aid in developing the following taxonomy.

First, we may consider alternative A : setting G(t) = 0 for all t in tau, and setting g(t) = x*(t), that is, equal to the optimal instrument settings derived from fully optimal control. This will replicate the optimal instrument settings, of course, and the expected target trajectories as

a result. If we use Chow's (1975, pp. 156-157) decomposition of the control problem into deterministic and stochastic parts, we see that the same results will be realised for the deterministic portion of both the optimal and suboptimal experiments. However, even in the certainty equivalence case, the stochastic portion of expected loss will not be replicated for tau greater than one. Setting G(t) null alters the fundamental dynamic properties of the controlled system : in particular, the matrix of autoregressive coefficients on y(t-1), y(t-2)... will be changed. With certainty equivalence, it may be shown that a higher expected welfare cost will result from this strategy versus its fully optimal counterpart. In the presence of parameter uncertainty, this relation cannot be proven, but the expected cost derived from equations' disturbances will surely differ from the optimal.

This artificial case is, we contend, an appropriate benchmark against which to measure optimal policy. Its non-optimal expected welfare cost may be attributed to its non-feedback nature. In essence, alternative A corresponds to evaluating the cost of calculating optimal feedback rules, and the expected (conditional on time zero) instrument settings - and then applying those settings. In the optimal case, only the first period setting would ever be used;

additive errors will cause a different setting to be chosen for period two. If one ignored the need to make those in-course corrections, we would have alternative A.

Alternative B may be considered as mechanistically a minor variant on A. It too, uses a null G(t) matrix for all t in tau, but sets g(t) to the historical values of x(t), designated as x(t)h. Two things must be considered about this solution. Most importantly, the implication is made that policy was formulated with no feedback from variables in the model. Policy shapers did not consider present and past values of GNP, UR, DP or other endogenous variables in choosing policy settings. Although this may seem to border on the ridiculous, in one sense it may be reasonable. If policymakers are most strongly motivated by variables not included in the model (for example, the political climate), their feedback behavior may be weak, or in any case extremely variable. If no stable relationship can be observed by economic researchers linking current conditions of economic variables and future policy settings, the assumption of exogeneity may be pragmatic if not wholly pleasing.

Secondly, from our consideration of alternative A, it may be noted that our replacement of optimal G(t) with a

null matrix will alter the dynamic properties of the system, thus modifying the stochastic component of welfare cost. Alternative B also might be expected to incur a deterministic cost in terms of the expected target trajectories. Unless we have very modest goals, the specified targets will be different than the historical levels of those variables; and a model estimated from those historical data will not yield historical instrument settings associated with near-optimal target levels. Thus, we might expect a higher deterministic cost, related to the expected nonattainment of targets vis-a-vis the optimal case. In the certainty equivalence case, this suggests that alternative B must cost more than alternative A, which in turn exceeds the optimal expected cost. The excess of B's expected cost over that of A presumably quantifies the cost incurred by following the exogenous historical levels of policy, over and above the cost incurred by optimally determining policy setting rules and blindly following them.

In a fully stochastic case, this intuitive ranking need not be reproduced. In going from optimal G(t) to null G(t), a cost is incurred, depending on the dynamic stucture of the system. Choosing g(t) equal to the optimal x(t) will minimize the deterministic part of the solution, while x(t)h (historical settings) cannot. However, if x(t)* exceeds

x(t) h, a higher stochastic cost will be incurred, derived from our uncertainty about coefficients' values. Since that inequality characteristically is satisfied, the stochastic portion of B may be less than that of A. The net effect can go either way. Thus, no intuitive result follows. Exogenous policy actions observed in past history may well have a lower expected cost than optimal settings followed without feedback actions. In this case, we may want to consider alternatives A and B relative to the optimal expected cost.

To summarize the discussion thus far, we may note that both alternatives A and B use a null G(t) matrix, and various values of the intercept vector g(t). A different dynamic behavior (and thus different expected loss) is implied by modification of the G(t) matrix from its optimal value. As we have noted, a null G(t) implies that policymakers do not react to past economic conditions. As both of these implications have undesirable properties, our second pair of alternatives is designed to avoid those implications, but of necessity involve further assumptions.

The first, alternative C, makes use of the optimally contructed G(t) matrix, and computes a modified intercept vector, denoted as g(t). This vector is computed as:

(5.32) $\underline{g}(t) = x(t)\{hist\} - G(t)\{opt\}\ y(t-1)\{hist\}$

where $x(t)\{hist\}$ is the historical value of policy in period t; $y(t-1)\{hist\}$ is the historical value of the state vector in period $(t-1)$; and $G(t)\{opt\}$ is the feedback rule matrix computed in optimal control. The use of $G(t)\{opt\}$ will imply that the autoregressive behavior of the system will not be altered in this experiment. By rearranging the expression for $\underline{g}(t)$, we may note that the intercept vector is defined at that level which would have brought forth the historical policy given the historical initial conditions.

Two points must be noted. First, the implicit assumption is made that historical policymakers actually behaved according to the optimal feedback coefficients from an ex post control solution. This admits that policymakers are not totally unaware of past conditions, but assumes that their behavior is linear and mechanistic, determined by a set of coefficients which could not possibly have been estimated at the time. We might argue that a policymaker possessing a great deal of foresight might behave this way, but we can hardly fault those responsible for policy for not possessing the information we do after the fact. Second, we must note that the expected trajectory of the state vector for the tau suboptimal periods need not replicate historical

values of its variables for any period but the first. The trajectory might be expected to drift away from the historical values as the effects of residuals accumulate (analogous to a dynamic simulation). All in all, though, we might consider this alternative to be the most realistic in terms of representing historical policy actions.

The last alternative is a variant of the third. Alternative D substitutes for G(t){opt} a matrix G(t){est}, where the {est} denotes that this feedback matrix has been estimated from historical data. This substitution is meant to accomodate the objection that policymakers' behavior could not have been based upon ex post optimal feedback coefficients except by fortuitous coincidence. This introduces some ambiguity, however, as there are any number of alternate G(t) estimates that we might use. Ideally, policy variables should be regressed upon lagged values of the state vector. This regression, for a model such as the linear model, cannot be computed directly for two reasons. First, there are numerous identities, in each period and between periods, linking elements of the state vector. Second, even when such perfect combinations have been omitted, a condition of near-perfect multicollinearity or zero residual variance could (and did) result. This obstacle can be circumvented by utilizing stepwise

regression techniques to select the most powerful variables. The results of those regressions for the four policy instruments of the linear model are presented as a separate Appendix.

It should be clear that no analytical statement may be made about the results of evaluating historical policy with alternative D. The modification of the G(t) matrix (characteristically, to a much simpler form) will alter the autoregressive behavior of the system; and the lack of any optimal values in the constructed feedback rule will guarantee that something other than the optimal value of expected loss will result. Little else may be said analytically, since for any but the simplest case, evaluation of the expressions involved becomes prohibitively complex. For that reason, we turn now to the numerical results obtained by computing alternatives A through D for one of our previously presented experiments. In each case, the first ten periods of the twenty-quarter horizon are performed suboptimally.

Table 5.8

Suboptimal Control Solutions for the Linear Model [1]

Evaluating Historical Policy versus Optimal Policy

Y as target, tau = 10 quarters			
Alternative	Total W	Eps. mean	Inst means
Base Run	2512.190	-.0118 (.0406)	313.157 135.773 -2.1639 2.41397
A	6631832.	-.0118 (.0406)	313.157 135.773 -2.1639 2.41397
B	4856618.	-29.15 (30.27)	229.111 45.1420 -1.3333 3.28590
C	31896.57	-26.44 (28.38)	236.681 54.3826 -1.3499 3.08109
D	34311.74	-26.87 (28.23)	231.587 48.8070 -1.3042 3.24106

The first conclusion that we may draw is that a very high penalty applies to modification of the autoregressive structure of the model. Even though the Base Run and alternative A have identical deterministic results, the latter cost is 2500 times larger, due entirely to the stochastic component of expected loss. Non-feedback policy is shown to incur a much greater cost than appropriate use of the feedback rules. Interestingly, alternative B exhibits a large shortfall from the deterministic target path, and yet possesses a welfare cost 27 per cent lower. The mean deviation from the target path is -29.15 billion, with much less active use of fiscal tools during the suboptimal period. However, as suggested earlier, these smaller x(t) settings may well reduce expected loss by more than the deterministic component increases - as is happening in this case.

Turning to alternatives C and D, we note that the use of either non-null G(t) matrix reduces expected loss a hundredfold relative to the first two alternatives. The

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[1] The runs in Table 5.8 (alternatives A-D) are suboptimal control solutions performed using run 1 of Table 5.3 (reproduced above) as the base run. For definitions of the target path, instrument set, and table headings, see the notes to Table 5.3.

deterministic portions of both C and D are similar to that of B, with mean shortfalls of $26 billion, and much less vigorous instrument use compared to alternative A. The expected loss in both cases is only thirteen times the optimal (Base Run) value. As might be expected, the optimal G(t) matrix used in alternative C incurs a slightly lower cost than the estimated feedback matrix used in alternative D.

We have presented four alternative methods of evaluating optimal policy vis-a-vis historical policy, counting the non-feedback benchmark of alternative A. Each alternate technique makes use of quite different assumptions regarding policymakers' behavior, ranging from total exogeneity to approximated endogeneity. Although we may not claim that any of these techniques provides "the" way to evaluate historical policy, their differences and underlying rationales should help the policy analyst decide how such an evaluation might be reasonably performed.

We now may consider a third type of suboptimal control: that in which an arbitrary feedback rule is used in place of the optimal rule, and the cost of that substitution evaluated. This has two facets: first, a completely arbitrary rule (such as "close half the GNP gap with

additional government spending this quarter") may be costed. The implications of doing that should be clear from the previous discussions of historical policy evaluation. Since an infinite number of alternative rules could be proposed and tested, we will not consider any actual evaluation of this sort. The crucial point is that the modifications to the algorithm made for historical policy evaluation will serve for this purpose as well.

Second, and perhaps more important from a context of policy evaluation, the "arbitrary" rule to be evaluated may only be arbitrary in the sense of being not fully optimal. If we evaluate the cost of using a certainty equivalence feedback rule in the context of stochastic control, we ultimately may measure the "cost" of certainty equivalence control. Alternatively, this is the cost of misspecification: how bad an approximation to the stochastic control results will be made if we ignore parameter uncertainty, and specify the model's parameters as known? It should be stated emphatically that this cost cannot be simply derived from two control solutions, such as those presented in Sections One and Two, although, some authors have suggested that such a comparison could be meaningfully made (see Chow, 1975, p.271-276). In this example, Chow presents three control solutions for a two-equation model :

certainty equivalence, parametric uncertainty and parametric uncertainty with "learning". He discusses the magnitudes of the feedback rule coefficients, and the first-period policy settings implied by the rules, and then considers the magnitudes of the expected loss from each solution. Chow suggests that a "computer simulation experiment" could be used (op.cit., p. 277) to compare these expected welfare loss figures. It is our contention that these comparisons should be made as a suboptimal evaluation as outlined above.

It should be clear that two control solutions performed on the same model, horizon, and objective function, but differing with respect to parameter uncertainty, are not directly comparable. They are two separate problems, subject to fundamentally different constraints. For this reason, a meaningful comparison can be only be made by considering one experiment as a suboptimal case of the other. The techniques used to evaluate the suboptimal solution (which, in the case of certainty equivalence, will be a steady-state rule, not equal to the fully optimal rule) are identical to those discussed in earlier portions of this section. As an example, we reproduce a one-target, one-instrument run presented in both Sections One and Two. Following the original runs, we report the "suboptimal" case, in which the certainty equivalence feedback rule is

evaluated as a suboptimal modification of the parametric uncertainty case.

Table 5.9

Suboptimal Control Solutions for the Linear Model [1]

Evaluating Certainty Equivalence versus Parametric Uncertainty Solutions

Y as target, tau = 10 quarters			
Alternative	Total W	Eps. mean	Inst Mean
Cert.Equiv.	1548.600	0.0 (0.0)	198.756
Par.Uncert.	2892.550	-.1270 (.2061)	198.729
Suboptimal	2894.299	-.0287 (.0477)	198.746

In this experiment, we see that the use of certainty equivalence rules for the first ten quarters of a twenty-quarter horizon imposes an infinitesimally higher expected loss than the fully optimal stochastic solution. Interestingly enough, the mean deviation from target path

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[1] The first two runs are run 1 of Table 5.1 and run 2 of Table 5.3, respectively. The third run is the suboptimum defined by using the certainty equivalence feedback rule in the parametric uncertainty case. For definition of the target path, instrument set and table headings, see notes to the tables mentioned.

and its standard error are smaller than those relating to parametric uncertainty. This is plausible, in the sense that the latter solution trades off deterministic achievement of the target for a lesser expected cost derived from imperfect instruments. As may be expected, the suboptimal case involves a fractionally larger mean instrument setting than the parametric uncertainty case, reflecting that imprecision.

In conclusion, the alteration of the feedback control algorithm for the evaluation of suboptimal policy may be fruitfully applied to several topics of interest. In the last case, we see that the steady-state feedback rule derived from certainty equivalence control may be costed. In the case of our linear model, the cost of applying a feedback rule quite similar to the parametric uncertainty rule may be considered negligible. This may not be the case, of course, for a different target set, horizon, or indeed a different model. We reach an important conclusion: that this cost may, and indeed should, be evaluated for each case in which the application of optimal control is considered. It is not enough to merely suggest that it appears (from expected paths, for example) that parametric uncertainty may reasonably be ignored. This latter conclusion should only be made after the effects of

certainty equivalence vis-a-vis parametric uncertainty control are rigorously measured.

CHAPTER VI

OPTIMAL STABILIZATION POLICY WITH THE NONLINEAR MODEL

0. Introduction

This chapter presents the results of optimal stabilization policy experiments conducted with the nonlinear version of the model. Section One presents a "certainty equivalence" optimal control experiment - that is, a solution in which only additive uncertainty is taken into account. Section Two contrasts these results with the certainty equivalence results obtained from a linearized version of the model. The difficulties encountered in considering parametric uncertainty in the control of a nonlinear model are discussed.

1. Certainty Equivalence Optimal Control Experiments

In this section, the results of a "certainty equivalence" control solution with the nonlinear model are presented. This solution is "certainty equivalent" since it involves only the additive uncertainty derived from each behavioral equation's disturbance term. The optimality of this solution is discussed, depending heavily on the

discussion in Chow (1975, Ch. 12).

As we suggested in Chapter Two, optimal control of a wholly deterministic linear system is an almost trivial exercise, depending only on the existence of enough independent linkages between targets and instruments. Chow points out (op. cit., p. 280) that a deterministic nonlinear model may be linearized with a Taylor-series approximation, and the resulting linear form subjected to the standard linear control algorithm. The only modification (apart from the linearization itself) is the necessity to iterate on the optimal policy vectors $x^*(t)$. This involves the initial specification of a tentative path for $x(t)$, which is substituted into the model to provide $y(t)$. The linear approximation is performed around these tentative values and linear feedback rules derived as the solution to the linear control problem. The implied $x^*(t)$ are computed and compared with the tentative $x(t)$. This process may be repeated, comparing first-round $x^*(t)$'s with second-round values, until $dx^*(t)$ (between iterations) converges to an arbitrarily small tolerance. This entire technique involves little more than the addition of a model solution algorithm (such as the Gauss-Seidel routine) to a linear control program.

In considering a similar problem with uncertainty, however, several cautions must be noted. One might expect that the "certainty equivalence" theorem quoted in Chapter V would carry over to control of a nonlinear model. Unfortunately, as Chow points out (op. cit., p. 290) this argument is "almost correct but not quite" - even with a strictly additive random disturbance and known parameters. First of all, the values of future y's and x's are unknown, so that the linearization is not performed around the exact optimal solution. This implies that the linear model will be equivalent to the nonlinear model in the neighborhood of a point - which will not, in general, be the optimal point. This error will not occur in the deterministic case, as there the two forms of the model will be equivalent at the optimal point by construction. Secondly, the entire dynamic programming algorithm rests upon the assumption that future decisions will be made optimally. If the exact optima cannot be located, that future optimality of decisions cannot exist, and we must contend with a second-best situation. However, even in the absence of true optimality for such an approach, we may consider that it would be a good approximation. The degree to which it fails is, as might be imagined, closely linked to the degree of nonlinearity of the model in the neighborhood of the solution. Use of a largely linear model will cause the

degree of error to be relatively small, and the resulting feedback control equations approximately optimal. The same conclusion will be reached for a model possessing very small error variances; in the limit, if these variances are shrunk to zero, the problem becomes deterministic.

We will now consider the implementation of the method described for an actual model : the nonlinear version of the model in Chapter III. The algorithm used is Baum-Howrey's (1977) modification of Butters-Chow's original program (1975). The fundamental structure of the model is specified by coding each of the 19 equations in FORTRAN, using a standardized notation wherein all current endogenous variables are designated D(J), all current control variables X(J), and so on. The equations are themselves, with other appropriate statements, a subroutine of the nonlinear control program named MODEL. This subroutine must be compiled in the usual manner and linked to the remainder of the program to perform a control experiment. A listing of the MODEL subroutine used in this study is given as an Appendix.

Following specification of the model, the other required inputs must be assembled. To perform certainty equivalence control on a nonlinear model, the estimated

variance-covariance matrix of the structural disturbances must be provided. Unless a system method of estimation was utilized, only the diagonal elements of this matrix will be produced in the estimation process. The specification of the objective function is identical to that of the linear model, as we still assume that a quadratic objective function is used to evaluate the control solution. The only additional information required, beyond that needed for the linear model, is a vector of trial y values for period one, and a matrix of trial x values for all periods of the horizon. These trial values are needed only for the initial solution of the model via the Gauss-Seidel algorithm. Historical values may be used; their "reasonableness" only serves to shorten the range over which the algorithm must search for the optimal policy.

The resulting Gauss-Seidel solution should be identical to the results of a dynamic simulation performed with the model, such as those reported in Chapter IV. This provides a useful check on the coding of the model. Following this solution, each equation of the model is numerically differentiated with respect to each right-hand-side variable. This is done by perturbing the latter variable one epsilon above and below its previous value, evaluating the vector of left-hand-side values which results from each

perturbation, and calculating the derivative as the difference in the function values divided by (2 epsilon). Epsilon is characteristically set as a small fraction of the variable being perturbed (such as one-tenth of one per cent). It should be noted that these first partial derivatives are calculated without re-solving the model for each perturbation, whch would be prohibitively costly. Instead, the model's derivatives with respect to both current endogenous and predetermined values are computed by perturbing the model in the neighborhood of its simultaneous solution. The assumption is made that this neighborhood is small enough to ensure that the function evaluations performed are approximately equivalent to those derived from a full simultaneous solution. The derivatives obtained by this method are considered as structural parameters of the model. The linearized reduced form used in control is derived from these structural parameters by the usual techniques (see Section Three of Chapter IV). This linearized reduced form is then used in a version of the standard linear control algorithm to derive feedback rules for each point in the horizon.

It should be noted that this process entails one linearization of the model for each period of the horizon. The magnitude of this task becomes enormous when it is

realized that the differentiating variables include all current endogenous variables as well as all variables in the lagged state vector. Thus, for the 19 equations of the nonlinear model, at least {19(19+19*6)} derivatives must be calculated for the zero-order through sixth-order lagged values of the endogenous variables. In addition, the model must be differentiated with respect to the current and lagged control variables, and the exogenous variables (either directly or indirectly). Thus, the problem is seen to be one requiring considerably more computational resources than control of the linear model.

This conclusion becomes all the more evident when it is noted that all these computations, including the calculation of the expected state vector trajectory and welfare cost, comprise only one iteration over the system. To determine the approximately optimal policy, at least one additional iteration must be performed (even on a truly linear model). This is required since the approximate optimality may only be defined in terms of the convergence of targeted variables' values from one iteration to the next. Depending on the tolerance criterion used to measure this convergence, a model may need more than two iterations for a stable policy set to be defined. In tests on the nonlinear "Michigan Model", Chow (1976) found that three iterations

over the system were required to achieve convergence. The horizon was seventeen quarters, and the numbers of simultaneous equations and control variables were 61 and six, respectively.

We turn now to the presentation of an optimal control experiment performed with the nonlinear model. Because of budgetary constraints and the considerable cost of these experiments, we decided to choose one experiment to represent nonlinear control. The natural choice was the one-target, one-instrument run similar to those reported in Chapter V. There, the objective function costed deviations of income from its potential level. Given the specification of the nonlinear model, an analogous objective was specified: that the ratio (Y/YPOT) equal unity. Government spending was chosen as the most representative instrument. It was decided to control the model over the eight-quarter period 1974.1 - 1975.4. The starting quarter represents the first downturn in income in that recessionary period; initial conditions thus represent a slowing rate of growth of most aggregates. We ask the question therefore: could we have prevented the recession, asssuming that the economy's behavior is faithfully captured by our model? The following table presents the results.

Table 6.1

Certainty Equivalence Control Solution for the Nonlinear Model [1]

One target over the horizon 1974.1 - 1975.4			
Inst	Total W	Eps. mean	Inst mean
G	.0004634	0.0 (.00003)	307.636 13.0005

From the table above, we may note that welfare cost was very small, compared to the magnitudes reported for linear control solutions. This is partially due to the type of uncertainty represented; since single-equation estimation was used to construct the model, the full variance-covariance matrix of structural disturbances was not available. Accordingly, only the diagonal elements of this matrix (the squared standard errors of regression) were considered in computing the optimal control. It should also be noted that, as cost may be divided into deterministic and

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[1] In this table, the single target is the maintainence of {Y/YPOT} at unity. "Inst" indicates the instrument being used. "Total W" is the value of the objective function over the eight-quarter horizon used. "Eps. mean" reports the mean deviation from target, with standard deviation below. "Inst mean" reports the mean value of the control variable, with its standard deviation below. The historical mean value of G was $258.66 billion.

stochastic components, that the cost reported was almost entirely caused by stochastic elements. We see that the mean deviation from target was effectively zero, with a very small standard error. Not surprisingly, our one target was achievable with the single instrument used. The optimal instrument value jumped from $251.86 billion in the last pre-control period to 281.19 billion in the first control period, advancing by smaller increments through the next several quarters and peaking in quarter seven (1975.3) at $317.57 billion. This may be compared with the historical value of $265.14 billion for that quarter.

This optimal solution was achieved in two iterations over the eight-quarter horizon, with a fairly loose (one per cent) convergence criterion between iterations. The starting values for endogenous variables, the tentative path for the control variable, and the values of exogenous variables were their historical levels. This implied that the Gauss-Seidel solution around which the model was linearized was equal to a historical dynamic simulation path. The complete solution required approximatedly three minutes of central processing time on the Michigan Terminal System's Amdahl 470/V6 computer. Since no recognition was made of the numerous lagged endogenous variables whch did not appear in any of the structural equations, the state

vector had 115 elements, and matrices A(t) and H(t) were of order 115.

To further characterize this single glimpse of the nonlinear model under control, let us consider several of the endogenous variables' paths. Consumption and nonresidential investment rose along the trend of income, while residential investment fell 20% (in response to higher interest rates), then recovered by the end of the horizon. Inventory investment was strictly positive, but declining. With monetary policy tools exogenous, the Treasury bill rate rose sharply in quarter two and a small amount thereafter. The commercial paper rate initially rose about one per cent, then slowly declined. Turning to the wage-price-unemployment sector, the unemployment rate fell from 4.8% to 3.6% initially, then stabilized at 4.0% (dutifully obeying the Okun's Law built into the model). Performance of wages and prices was less satisfactory. The inflation rate fell from 9.2% in 1973.4 to 6.1% in the first controlled quarter, then rose steadily to almost 8.0%. Peculiarly, the money wage index accelerated from 8.2% to 8.9%, exceeding the inflation rate over the control horizon. Clearly, something less than 100% "pass-through" of labor costs is being modeled.

We may see, then, that even in the simplest of certainty equivalence experiments with the nonlinear model, there is a great deal of interesting behavior to study. The presence in the model of percentage rates of change of prices and wages makes it much easier to comprehend - and judge - the model's performance. In the next section, we present an alternative experiment based upon the same horizon, with which this experiment may be contrasted.

2. Comparison with the Linearized Model

We now consider a way in which the results of Section One may be compared with the familiar linear control results presented in Chapter V. It is not apparent that a direct comparison of the results in Tables 5.1 and 6.1 may be made. Apart from the differing assumptions regarding uncertainty, there also exist fundamental differences in structure between the two forms (linear and nonlinear) of Chapter III's model. These differences even extend to the targeted variable: the "GNP gap" in the linear model, the ratio of GNP to potential GNP in the nonlinear version. These differences and others imply that direct comparisons ought not to be made.

Alternatively, we could linearize the nonlinear model

at the starting point of a projected horizon, and use the derived reduced form coefficient matrices as input into the linear model. This is feasible in theory, but in practice runs afoul of a different treatment of the exogenous variables in the two cases. The linear model utilizes constant coefficients p on the exogenous variables w(t), where b(t) = p w(t). In the nonlinear algorithm, this distinction is not preserved, and only the b(t) vectors are available. This suggests that the easiest "linear" control solution may be obtained from a modified nonlinear algorithm: one which computes A(t) and C(t) for period one only, and b(t) for each period of the horizon. These modifications were made, enabling us to perform a fixed coefficient solution.

Such a solution was computed for the nonlinear model, with the same horizon, instrument set, and objective function used in the previous section. The results of that experiment are presented in the following table.

Table 6.2

Certainty Equivalence "Linearized" Control Solution for the Nonlinear Model [1]

One target over the horizon 1974.1 - 1975.4			
Inst	Total W	Eps. mean	Inst mean
G	.0004958	.003526 (.00254)	305.758 11.7949

From this experiment, we may first note that a slightly higher welfare cost (7.0% higher) was realized in this alternative. The mean deviation from target was still very small - about 0.35 per cent of the target ratio of unity. A slightly lower mean instrument setting was recorded. The initial value of G was equivalent to that experienced in the nonlinear optimal run, as we would expect. Repeated use of the first-period linearization of the model caused the quarter eight setting to equal $312.77 billion, as compared with $316.67 billion in the optimal case.

To consider individual variables' paths, we see the major components of aggregate demand behaving very similarly

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[1] For definitions of table headings, see notes to Table 6.1.

in this experiment vis-a-vis the fully optimal run. Interest rates and the rates of change of prices and money wages followed much the same pattern. The unemployment rate diverged slightly, rising to 4.16%, compared with 3.97% in the optimal case. This suggests that the UR equation's coefficients are time-varying in a meaningful sense, even in the relatively short interval that we have modeled here. The target ratio of (Y/YPOT) falls monotonically to a quarter eight level of 0.9929, causing income to fall short of potential by about $10 billion in that quarter. A slightly smaller government bond stock results from the less expansive fiscal policy in this experiment.

In this approximately optimal linearized optimal control experiment, we have avoided a great deal of the computation required in the fully nonlinear counterpart. The inevitable question is whether this approximation is acceptable. From the results presented above, we would conclude that it is not an erroneous or misleading alternative. We noted significant divergence between the two runs in only one key variable, the unemployment rate. That divergence, as well as the degree of shortfall noted from the target ratio, was monotonic and quite predictable. We must qualify all this by noting that this comparison was made over a short horizon, with the single target following

a smooth path, and being controlled by the most powerful instrument in the model. Certainly no generalization should be made to more complex models, or to those with more significant nonlinearities. We might conclude, however, that the single linearization may suffice quite well for the length of horizon commonly used for short term policy forecasts - usually six to eight quarters. Over that range, optimal control of even a complex and sizable macroeconomic model may reasonably be performed. Depending upon the researcher's resources and the model's dimensions, it may be preferable to use a single linearization, or perhaps (as has been suggested by E.P. Howrey) several linearizations spaced over the horizon, as an alternative to the costlier quarter-by-quarter linearization suggested by theory. Certainly experiments ought to be performed to determine exactly how robust the feedback rules may be with respect to the frequency of linearizations performed upon the model. The results from our nonlinear model suggest that the rules are indeed robust, given the caveats stated above.

To complete the discussion of nonlinear optimal control techniques, we must consider the feasibility of extending parametric uncertainty techniques to the control of nonlinear systems. Chow (1975, pp. 291-295) discusses some of the problems inherent in this extension. His discussion

alludes to the same inexact conclusions stated earlier : that if the linear approximation is "reasonable," the feedback rules derived from it should be "approximately optimal." In a later paper, Chow (1977) suggests that the uncertainty associated with random parameters may be taken into account by dividing the problem into deterministic and stochastic components. The former part of the problem may be solved by the methods of Section One. The stochastic part of the problem requires the use of "numerical integration or Monte Carlo methods" (p. 13) to calculate the required expectations. Again, an iterative procedure would be used to derive the approximately optimal solution. Chow also suggests an approximation to this technique which would ease the computational burden of calculating these expectations, stating that "These calculations are by no means difficult using the computers available today" (p. 20).

If we accept Chow's conclusions, we may foresee the time when even a large macroeconomic model may routinely be subjected to control as a standard portion of research. The advent of this millenium will certainly fall short of providing a panacea for those researchers who steadfastly ignore the dictates of economic theory and econometric theory in the specification and estimation of their models.

It should, however, allow a much more systematic and illuminative approach to policy modelling than the techniques in use today. The use of optimal control techniques requires the economist to make explicit his or her assumptions about the behavior of the economy, and guarantees that an experiment is truly reproducible, without the knowledge of "fudge factors" which intrude between the computed and published versions of model simulations. We may, and should, regard the conclusions reached from macroeconomic models with a healthy skepticism. But this study should help illustrate a means whereby a model may be subjected to tests more rigorous and revealing than mere simulations; and in doing so, hopefully adds to our repertoire of modelling techniques.

APPENDICES

Appendix I : Data Mnemonics and Sources

This appendix presents a concordance between the mnemonics used in this study and the formal definitions of the variables. As indicated earlier, all data were obtained from the R.S.Q.E. data bank, originating from the National Bureau of Economic Research data base (updated through 1976.2).

Mnemonic	Description	Source
C	Personal Consumption Expenditures	SCB #1.2
INP	Nonresidential Fixed Investment	SCB #1.2
IR	Fixed Investment in Residential Structures	SCB #1.2
IIN	Change in the Stock of Business Inventories	SCB #1.2
PTB	90 day Treasury Bill Rate	FRB M.M.
RCP	Interest Rate on 4-6 Month Commercial Paper	FRB M.M.
RL	Corporate Aaa Bond Interest Rate	FRB B.Y.
M$	Demand Deposits and Currency, nominal	FRB M.S.
M	{ M$ / P } : Real Money Stock	
W$	Compensation per Manhour, Business Nonfarm Sector	E&E IND.

Mnemonic	Description	Source
P	Implicit Price Deflator for Gross National Product	SCB #7.1
UR	Civilian Unemployment Rate, Males 20 and over, per cent	E&E B.S.
TP$	Personal Tax, Non-tax Payments to Fed. Govt. : nominal	SCB #3.2
TC$	Corporate Profits Tax Accruals, Fed. Govt. : nominal	SCB #3.2
TR$	Fed. Govt. Transfer Payments to Persons, nominal	SCB #3.2
TI$	Contributions for Social Insurance : nominal	SCB #3.2
TB$	Indirect Business Taxes to Fed. Govt. : nominal	SCB #3.2
T	Total Net Real Taxes: { TP$ + TC$ + TI$ + TB$ - TR$ } / P	
GD$	Federal Government Expenditures : Defense	SCB #3.2
GO$	Federal Government Expenditures : Nondefense	SCB #3.2
GS$	State & Local Government Expenditures	SCB #3.4
PG	Implicit Price Deflator for Government Purchases	SCB #7.1
G	Total Government Expenditures: { GD$ + GO$ + GS$ } / PG	
Y	Real Gross Domestic Product : { C + INR + IR + IIN + G }	

Mnemonic	Description	Source
TXRATE	Estimated proportion of Y taken by Net Taxes; slope of the linear tax function	
TO	"Surtax" levied on disposable income; zero for all historical periods. Intercept of the linear tax function.	
YD	Real Disposable Income : { 1.0 - TXRATE } Y	
DB	Change in the Stock of Federal Government Securities : { G - T - DBASE }	
B	Stock of Federal Government Securities : B(T) = B(T-1) + DB(T)	
PCRMD	Wholesale Price Index for Raw Materials less Raw Foodstocks and Feedstuffs	LABOR
BASE$	Monetary Base (Member Bank Reserves plus Currency)	FRB ST.L.
BASE	Real Monetary Base : {BASE$ / P }	
RDISCT	Fed. Res. Bank of N. Y. Discount Rate	FRB I.R.
WEALTH	Total liquid "Wealth" of Public : { M + B }	
YPOT	Council of Economic Advisors' "Potential GNP" series	F.R.P.
TIME	Linear time trend; 1954.1 = 1.	

Sources :

SCB	Survey of Current Business, U.S. Department of Commerce. Numbers refer to "white pages" tables.
FRB	Federal Reserve Bulletin M.M. : Money Market Conditions B.Y. : Bond Yields M.S. : Money Stock Measures I.R. : Interest Rates
FRB ST.L.	Federal Reserve Bank of St. Louis
E&E	Employment and Earnings, U.S. Department of Labor. IND. : Compensation Indices E.S. : Employment Status
LABOR	U. S. Department of Labor
E.R.P.	Economic Report of the President, Council of Economic Advisors.

Appendix II : Estimated Feedback Regression Equations

This appendix presents the estimated equations used in constructing G(t){est}, the estimated feedback matrix used in Alternative D of Section Four, Chapter Five. There is one equation for each of the four control variables.

(A2.1) Estimated linear equation for : G

Variable	Coefficient	Std.Err.	T-Stat.
DM(-1)	-.4047	.15283	2.648
BL(-1)	-1.437	.31426	4.573
B(-1)	.01209	.00289	4.176
INR(-2)	.18167	.04207	4.318
T(-1)	-.1444	.02446	5.904
Y(-1)	.02016	.00742	2.717
G(-1)	.96213	.02602	36.98
R-Squared		.9978	
Std. Err. of Regression		1.9979	
Fit Period (N)		1954.4-1976.2 (87)	
Durbin-Watson		2.0725	

(A2.2) Estimated linear equation for : T

Variable	Coefficient	Std.Err.	T-Stat.
DM(-1)	.90968	.49567	1.835
P(-1)	-.3050	.13850	2.202
Y(-2)	-.2091	.06902	3.030
Y(-1)	.30961	.06879	4.501
T(-1)	.67160	.09096	7.383
R-Squared		.9878	
Std. Err. of Regression		5.5534	
Fit Period (N)		1954.4-1976.2 (87)	
Durbin-Watson		2.1028	

(A2.3) Estimated linear equation for : DBASE

Variable	Coefficient	Std.Err.	T-Stat.
M(-1)	-.0067	.00166	4.071
B(-1)	.00173	.00038	4.573
Y(-1)	.00101	.00091	1.110
DBASE(-1)	-.0565	.11568	0.488
RCP(-1)	-.2681	.04614	5.811
RL(-1)	.05580	.07448	0.749
R-Squared		.5552	
Std. Err. of Regression		.37925	
Fit Period (N)		1954.4-1976.2 (87)	
Durbin-Watson		1.8427	

(A2.4) Estimated linear equation for : RDISCT

Variable	Coefficient	Std.Err.	T-Stat.
RS(-1)	.33865	.08603	3.937
RCP(-2)	-.0988	.05225	1.891
PL(-2)	-.5618	.18093	3.105
G(-1)	-.0081	.00239	3.408
Y(-1)	.00303	.00070	4.315
RDISCT(-1)	.68375	.10405	6.571
RCP(-1)	-.0746	.08241	0.905
RL(-1)	.51320	.18246	2.813
R-Squared		.98147	
Std. Err. of Regression		.22254	
Fit Period (N)		1954.4-1976.2 (87)	
Durbin-Watson		1.7248	

Appendix III : The Coded Nonlinear Model

This appendix contains a listing of the FORTRAN source code for the nonlinear model, as described in Section One, Chapter VI.

```
C
C*CFB    MODEL
C
      SUBROUTINE MODEL(NY,NX,NW,D,Y,YL,XL,X,W,*)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION D(NY),Y(NY),X(NX),YL(NY,1),XL(NX,1),W(NW)
      DIMENSION P(50)
      DATA P/-8.3822,.13469,.09778,.89781,-5.1945,
     *.082500,.028557,-1.2095,1.2857,-.44663,
     *1.7369,.0055095,.80076,1.2856,-.43430,
     *-3.3102,.75992,-.94037,.70600,-.71292,
     *.89274,-.89212,.43252,-34.884,.0037268,
     *5.47884,3.44908,-56.6209,1.0885,.53628,
     *2.6490,-.94369,.046338,.13049,.14012,
     *.097320,.91552,4.5810,9.2828,.40929,
     *-1.0021,.30769,2.22287,.581194,-25.8877,
     *-.150268,-3.7634,1.2246,-.017513,.11893/
C
C***EQUATIONS OF TSLS NONLINEAR MODEL - VERS 15 NOV 76
C
C*****SET UP FOR CHOW NONLINEAR PGM
C
      IEQN = 1
C*  CONSUMPTION
      D(1) = P(1) + P(2)*(Y(7) + Y(17)) + P(3)*Y(15) +
     * P(4)*YL(1,1)
C
      IEQN = 2
C*  NONRESIDENTIAL INVESTMENT
      D(2) = P(5) + P(6)*(Y(14)-YL(14,1)) + P(7)*YL(14,2)
     * + P(8)*YL(8,3) + P(9)*YL(2,1) + P(10)*YL(2,2)
C
      IEQN = 3
C*  RESIDENTIAL INVESTMENT
      D(3) = P(11) + P(12)*Y(15) + P(13)*(YL(8,2)-YL(6,2)) +
     *P(14)*YL(3,1) + P(15)*YL(3,2)
C
```

```
      IEQN = 4
C*   INVENTORY INVESTMENT
      D(4) = P(16) + P(17)*Y(15) + P(18)*Y(1) +
     * P(19)*YL(4,1) + P(20)*YL(15,1) + P(21)*YL(1,1)
C
      IEQN = 5
C*   TREASURY BILL RATE
      D(5) = P(22) + P(23)*Y(12) +
     * P(24)*((Y(14)-YL(14,1))/Y(14)) +
     * P(25)*Y(19)
C
      IEQN = 6
C*   COMMERCIAL PAPER RATE
      D(6) = .086344*P(26) + P(27)*(((Y(14)-YL(14,1))/Y(14))
     * -.913656*((YL(14,1)-YL(14,2))/YL(14,1))) + P(28)*
     *(((Y(18)-YL(18,1))/Y(18)) - .913656*
     *((YL(18,1)-YL(18,2))/YL(18,1))) + .913656*YL(6,1)
C
      IEQN = 7
C*   CHANGE IN THE MONEY SUPPLY
      D(7) = P(29) + P(30)*Y(5) + P(31)*X(4) + P(32)*X(5)
C
      IEQN = 8
C*   AAA BOND INTEREST RATE
      D(8) = P(33) + P(34)*(Y(5)-YL(5,1)) +
     * P(35)*(Y(6)-YL(6,1)) +
     * P(36)*YL(6,2) + P(37)*YL(8,1)
C
      IEQN = 9
C*   D LN WAGE$ INDEX
      D(9) = P(38) + P(39)*((Y(14)-YL(14,1))/Y(14))
     *   + P(40)*Y(12) +
     *P(41)*(.2*(DLOG(Y(11))+DLOG(YL(11,1))+DLOG(YL(11,2))+
     *DLOG(YL(11,3))+DLOG(YL(11,4)))) + P(42)*(.25*(YL(12,1)+
     *YL(12,2)+YL(12,3)+YL(12,4)))
C
      IEQN = 10
C*   WAGE$ INDEX
      D(10) = DEXP(Y(9)/400. + DLOG(YL(10,1)))
C
```

```
      IEQN = 11
C*   UNEMPLOYMENT RATE
      D(11) = .2887542*P(43) + P(44)*(DLOG(Y(10))
     * - 1.2842722*DLOG(YL(10,1)) + .5730264*DLOG(YL(10,2)))
     * + P(45)*(DLOG(Y(16)) - 1.2842722*DLOG(YL(16,1))
     *  + .5730264*DLOG(YL(16,2))) + .25*P(46)
     * *(YL(9,1) - .2842722*YL(9,2) +
     * .2887542*(YL(9,3)+YL(9,4)) - .7112458*YL(9,5) +
     * .5730264*YL(9,6)) + 1.2842722*YL(11,1) -
     * .5730264*YL(11,2)
C
      IEQN = 12
C*   D LN PRICE INDEX
      D(12) = P(47) + P(48)*Y(9) + P(49)*(Y(14)-YL(14,1)) +
     * P(50)*(.25*(YL(9,1)+YL(9,2)+YL(9,3)+YL(9,4)))
C
      IEQN=13
C*   PRICE INDEX
      D(13) = DEXP(Y(12)/400. + DLOG(YL(13,1)))
C
      IEQN = 14
C*   NATIONAL INCOME
      D(14) = Y(1) + Y(2) + Y(3) + Y(4) + X(1)
C
      IEQN = 15
C*   DISPOSABLE INCOME
      D(15) = (1. - Y(2))*Y(14) - X(3)
C
      IEQN = 16
C*   RATIO OF INCOME TO POTENTIAL INCOME
      D(16) = Y(14) / W(1)
C
      IEQN = 17
C* CHANGE IN THE STOCK OF GOVERNMENT BONDS
C*   (GOVERNMENT BUDGET CONSTRAINT)
      D(17) = X(1) - X(2)*Y(14) - X(3) - X(4)
C
      IEQN = 18
C*   STOCK OF MONEY
      D(18) = Y(7) + YL(18,1)
C
      IEQN = 19
C*   STOCK OF GOVERNMENT BONDS
      D(19) = Y(17) + YL(19,1)
C
      RETURN
      END
```

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