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PERKEL, Manley, 1948-
ON FINITE GROUPS ACTING ON POLYGONAL GRAPHS.

The University of Michigan, Ph.D., 1977
Mathematics

Xerox University Microfilms, Ann Arbor, Michigan 48106

ON FINITE GROUPS ACTING ON POLYGONAL GRAPHS

by
Manley Perkel

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
1977

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To the memory of my dear friend and colleague

Vicente Landázuri Narváez

ACKNOWLEDGMENTS

I would like to take this opportunity to thank my advisor, Professor Donald G. Higman, for all his guidance and help over the past years. I would also like to express my gratitude to Professor William M. Kantor for his assistance during the summer of 1975 and since.

My thanks also to Bob Scott for permitting me to use his coset enumeration program which proved invaluable for some sections of this work.

Lastly, I thank my wife, Barbara, for her invaluable inspiration and for typing this manuscript.

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NOTATION

Most of the notation is introduced in the chapter when first used. The following is a supplementary list.

Z_n	The cyclic group of order n
D_n	The dihedral group of order n
Σ_n	The symmetric group of degree n
A_n	The alternating group of degree n
Q_8	The quaternion group of order 8
$GF(p^n)$	The finite field of order p^n, p a prime
$PSL(2,p^n)$	The projective special linear group over the field $GF(p^n)$
$\|G\|$	The order of a group G
$\|g\|$	The order of the element g
$\|G\|_p$	The highest power of the prime p dividing $\|G\|$
$\pi(G)$	The set of primes dividing $\|G\|$
$H \leq G$	H is a subgroup of G
$\|G:H\|$	The index of H in G
$\langle X \rangle$	The subgroup generated by a subset X
x^y	$y^{-1}xy$
$N_G(H)$	$\{g \in G: H^g = H\}$, the normalizer of H
$C_G(H)$	$\{g \in G: h^g = h$ for all $h \in H\}$, the centralizer of H
$Z(G)$	The center of the group G
$Syl_p(G)$	The set of Sylow p-subgroups of G
Involution	An element of order 2 of a group
$\mathbb{C}$	The set of complex numbers
$\mathbb{Q}$	The set of rational numbers
$x \tilde{G} y$	$x = y^g$ for some $g \in G$

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CHAPTER I

INTRODUCTION

In this thesis we investigate the action of (finite) groups G on (finite) polygonal graphs. A polygonal graph is a pair $(\mathcal{H},\mathcal{E})$ consisting of an undirected graph $\mathcal{H}$ (no loops or multiple edges) which is regular, connected and has girth m for some $m \geq 3$, and a set $\mathcal{E}$ of m-gons of $\mathcal{H}$ such that every 2-claw of $\mathcal{H}$ is contained in an unique element of $\mathcal{E}$. If $\mathcal{E}$ is the set of all m-gons of $\mathcal{H}$, so that there is an unique m-gon in $\mathcal{H}$ on every 2-claw, then we write $\mathcal{H}$ for $(\mathcal{H},\mathcal{E})$ and call $\mathcal{H}$ a strict polygonal graph. If we wish to emphasize the integer m, then we call $(\mathcal{H},\mathcal{E})$ an m-gon-graph (respectively, a strict m-gon-graph).

Examples of polygonal graphs not arising from regular solids are known mainly for $m \leq 6$ and with valency $k \leq 5$. Fewer examples with $m > 6$ or $k > 5$ are known, the most notable arising from J_1, Janko's first simple group ($m = 5$ and $k = 11$).

Our goal is two-fold. On the one hand we would like to know if any information about a group which acts on a polygonal graph can be obtained from this action. On the other hand, can any information about the polygonal graph itself be obtained because of the group acting on it? By making additional assumptions on the group or the

graph, or both, we shall see that we can indeed obtain further information about them.

In Chapter I we give the definitions and prove some of the results we shall be assuming for the rest of the thesis. In Chapter II we investigate and give examples of polygonal graphs of valency 3, and note a relationship between the graphs and regular maps, and between their groups and polyhedral groups. In Chapter III we give examples of valency greater than 3 arising both from regular solids as well as the groups PSL(2,q).

In Chapter IV we assume that m is odd, and make certain assumptions about involutions which fix 2-claws and m-gons, and we prove two main theorems. In the one we assume that $k = q+1$ for some prime power q, and that for x a vertex of $\mathcal{H}$, $PSL(2,q) \leq G_x^{\Delta(x)} \leq P\Gamma L(2,q)$, where $\Delta(x)$ is the set of points adjacent to x. We prove that $k = 3, 4, 5, 6$ or 10. In the other we assume that k is odd, G_x is 3-transitive on $\Delta(x)$ and $\mathcal{H}$ contains no strict polygonal graphs of valency 3 as subgraphs. We prove that $k = 5$ and $G_x \simeq A_5$. We then get a characterization of $PSL(2,31)$ in its action on a strict 5-gon-graph.

In Chapter V, we give a characterization of J_1 in terms of its action on a 5-gon-graph of valency 11 with point stabilizer isomorphic with PSL(2,11). Finally in Chapter VI we investigate certain special cases of rank 4 groups with a 3-transitive suborbit, applying some of the results of Chapter IV to them, whence we will have come full circle as it was precisely the study of one of these cases which led to the concept of polygonal graph and the subject matter of this thesis!

In Section 1 of this chapter we give the basic definitions and in Section 2 we prove some elementary results on polygonal graphs which will be used repeatedly in the thesis.

Section 1. Definitions and Notation

All groups and graphs to be considered will be finite. By a graph $\mathcal{H}$ (with no loops or multiple edges) we mean a finite, non-empty set Ω (called the set of vertices or points) together with a set $E \subseteq \Omega \times \Omega$ (whose elements are called edges), where $E = \{(x, y): x, y \in \Omega, x \neq y\}$. $\mathcal{H}$ is undirected if $(x, y) \in E \Rightarrow (y, x) \in E$. If $(x, y) \in E$ we say that y is adjacent to x and write $x \sim y$. A path of length n in $\mathcal{H}$ is a sequence $(x_0, x_1, \ldots, x_n)$ of $n+1$ vertices $x_i \in \Omega$ such that $(x_i, x_{i+1}) \in E$ for all $i = 0, \ldots, n-1$, and $x_i \neq x_{i+2}$ for $i = 0, \ldots, n-2$. The above path is called a circuit of length n if $x_0 = x_n$, and $x_{n-1} \neq x_1$, in which case we write $(x_1, x_2, \ldots, x_n)$. It is called a simple path (respectively a simple circuit) if $x_i \neq x_j$ for any $i \neq j$, $0 \leq i, j \leq n$ (except of course $x_0 = x_n$ in the case of a circuit).

Remark. If $\mathcal{H}$ is undirected and $(x_1, \ldots, x_n)$ is a circuit of length n, then we shall not distinguish this circuit from the circuits $(x_i, \ldots, x_n, x_1, \ldots, x_{i-1})$ and $(x_i, \ldots, x_1, x_n, \ldots, x_{i+1})$, for $1 \leq i \leq n$; for our purposes these circuits are considered to be the same.

$\mathcal{H}$ is said to be connected if for any two points $x, y \in \Omega$ there is a path from x to y, i.e. a path $(x_0, x_1, \ldots, x_n)$, for some n, with $x = x_0$ and $y = x_n$. The minimum such n is called the distance from x to y (0 if $x = y$) denoted by $d(x, y)$, and the diameter of $\mathcal{H}$ is the maximum of the distances from x to y taken over all pairs $(x, y) \in \Omega \times \Omega$. The girth of $\mathcal{H}$ is the minimum of the lengths of all circuits of $\mathcal{H}$. For $x \in \Omega$, define $\Delta_i(x) = \{y: d(x, y) = i\}$ for all $i \geq 0$. Of course, if the diameter of $\mathcal{H}$ is n, then $\Delta_i(x) = \phi$ for $i > n$. Often we will just write $\Delta(x)$ for $\Delta_1(x)$, the points adjacent to x.

A subgraph of $\mathcal{H}$ is a graph $\overline{\mathcal{H}}$ whose vertices are $\overline{\Omega} \subseteq \Omega$ and whose edges are $\overline{E} \subseteq E \cap \overline{\Omega} \times \overline{\Omega}$. If $\overline{E} = E \cap \overline{\Omega} \times \overline{\Omega}$ we say that $\overline{\mathcal{H}}$ is an induced subgraph, or $\overline{\mathcal{H}}$ is the subgraph of $\mathcal{H}$ induced by $\overline{\Omega}$. The link of x in $\mathcal{H}$, $x \in \Omega$, denoted by link $(x,\mathcal{H})$, is the subgraph induced by $\Delta(x)$. An m-claw, denoted by $(x_0: x_1, \ldots, x_m)$, is a subgraph $\overline{\mathcal{H}}$ of $\mathcal{H}$ on the $m+1$ distinct points $\{x_0, x_1, \ldots, x_m\}$ where $\{x_1, \ldots, x_m\}$ are adjacent to x_0 and there are no further adjacencies between the x_i's in $\overline{\mathcal{H}}$. Note that the definition of m-claw is independent of the order of the vertices $x_1, \ldots, x_m$. If we wish to take into account the order of these vertices we call the m-claw an ordered m-claw, so that for example, the ordered 2-claw $(x_0: x_1, x_2)$ is not equal to the ordered 2-claw $(x_0: x_2, x_1)$.

A connected component of a graph $\mathcal{H}$ is a maximal (with respect to set inclusion of vertices and edges) connected subgraph of $\mathcal{H}$. Where it is unlikely to cause confusion we shall

often say the "subgraph of $\mathcal{H}$ on $\overline{\Omega} \subseteq \Omega$" instead of the "subgraph of $\mathcal{H}$ induced by $\overline{\Omega} \subseteq \Omega$" and shall write $\overline{\Omega}$ for this subgraph as well.

The <u>valency</u> of a vertex x of $\mathcal{H}$ is $|\Delta(x)|$. $\mathcal{H}$ is said to be <u>regular</u> (of valency k) if $|\Delta(x)| = k$ for all $x \in \Omega$. If $k = 3$, $\mathcal{H}$ is called a <u>cubic</u> graph, while if $k = 2$, $\mathcal{H}$ is connected and $|\Omega| = m \geq 3$, $\mathcal{H}$ is called an <u>m-gon</u>. An m-gon, with $m = 3, 4, 5, \ldots$ will be called, respectively, a <u>triangle</u>, <u>rectangle</u>, <u>pentagon</u>, ... The automorphism group of $\mathcal{H}$ will be denoted by $\mathrm{Aut}(\mathcal{H})$.

A <u>polygonal graph</u> is a pair $(\mathcal{H},\&)$ consisting of a graph $\mathcal{H}$ (undirected, no loops or multiple edges) which is regular, connected and has girth m for some $m \geq 3$, and a set $\&$ of m-gons of $\mathcal{H}$ such that every 2-claw of $\mathcal{H}$ is contained in an unique element of $\&$. If $\&$ is the set of all m-gons of $\mathcal{H}$, so that there is in $\mathcal{H}$ an unique m-gon on every one of its 2-claws, then we write $\mathcal{H}$ for $(\mathcal{H},\&)$ and call $\mathcal{H}$ a <u>strict polygonal graph</u>. If we wish to emphasize the integer m, then we call $(\mathcal{H},\&)$ an <u>m-gon-graph</u> (respectively a <u>strict m-gon-graph</u>). For example, if the valency is k, then with $k = 2$, an m-gon-graph is just an m-gon, while if k is arbitrary and $m = 3$ then $\mathcal{H}$ is just the complete graph on $k+1$ vertices (or the 1-skeleton of the k-dimensional tetrahedron). For this reason we shall assume that $k > 2$ and $m > 3$. Strict m-gon-graphs with $m = 4, 5, 6, \ldots$ will also be called <u>rectagraphs</u>, <u>pentagraphs</u>, <u>hexagraphs</u>, ...

If G is a group acting on a set Ω, then we shall denote by x^g the image of $x \in \Omega$ by an element $g \in G$. $\Omega(g)$ will

denote the set of points of Ω fixed by $g \in G$, and for $H \subseteq G$, $\Omega(H) = \bigcap_{g \in H} \Omega(g)$. If $\Delta \subseteq \Omega$, then G_Δ will denote the setwise stabilizer of Δ, i.e. the subgroup of G consisting of those elements which fix the set Δ. If $\Delta = \{x, y, z, \ldots\}$, we will also write $G_{\{x, y, z, \ldots\}}$ for G_Δ. $G_{[\Delta]} = G_{xyz\ldots}$ will denote the pointwise stabilizer of Δ, i.e. the largest subgroup of G which fixes every point of Δ. G_Δ^Δ will denote the group of permutations induced by G_Δ on Δ, so that $G_\Delta^\Delta \simeq G_\Delta/G_{[\Delta]}$.

If G is transitive on Ω, then the number of orbits of G_x on Ω, for any $x \in \Omega$, is called the <u>rank</u> of G on Ω, and the orbits of G_x on Ω are called the <u>suborbits</u>. Clearly $\{x\}$ is a suborbit for G_x, called a trivial suborbit. If $\Delta \subseteq \Omega - \{x\}$ is a non-trivial suborbit for G_x, then we can construct a graph $\mathcal{H} = \mathcal{H}(\Delta)$ as follows. The vertices of $\mathcal{H}$ are the points of Ω, and the edge set of $\mathcal{H}$ is $E = \{(x^g, y^g): g \in G\}$ where y is any point of Δ. Clearly since Δ is a suborbit, $\Delta = \Delta_1(x)$. Also $G^\Omega \leq \mathrm{Aut}(\mathcal{H})$ and G_x is transitive on $\Delta(x)$. We call $\mathcal{H}$ the graph constructed with respect to the suborbit Δ.

If $G \leq \mathrm{Aut}(\mathcal{H})$ is a group of automorphisms of a graph $\mathcal{H}$ with vertex set Ω, if $x \in \Omega$ and $g \in G_x$, then we denote by $\Omega_i(g)$ the set $\Omega(g) \cap \Delta_i(x)$, $i \geq 1$, i.e. $\Omega_i(g)$ is the set of vertices at distance i from x fixed by g.

<u>Remark</u>. By our convention on m-claws, if $G \leq \mathrm{Aut}(\mathcal{H})$, then G is transitive on ordered m-claws of $\mathcal{H} \Leftrightarrow G$ is transitive on Ω and G_x is m-transitive on $\Delta(x)$.

We close this section by mentioning that by a (finite) regular nearfield we shall mean a nearfield constructed from a (finite) field as in Theorem 20.7.2 of M. Hall, Jr. [17].

Section 2. Preliminary Results

Lemma 1.1. Let $\mathcal{H}$ be a connected, undirected graph (no loops or multiple edges) and suppose that every 2-claw of $\mathcal{H}$ is contained in an unique m-gon of $\mathcal{H}$. If the girth of $\mathcal{H}$ is m then $\mathcal{H}$ is a strict m-gon-graph.

Proof. All we need to show is that $\mathcal{H}$ is regular. So let x be a vertex of $\mathcal{H}$ of valency k, where k is the maximal valency of all vertices of $\mathcal{H}$. Let $\Delta(x) = \{y_1, \ldots, y_k\}$. Suppose that the valency of y_1 is $\ell < k$ and let $\Delta(y_1) = \{x = x_1, x_2, \ldots, x_\ell\}$.

On each of the 2-claws $(x: y_1, y_i)$, $2 \leq i \leq k$, there is in $\mathcal{H}$ an unique m-gon Π_i, say. Since there are $\ell < k$ vertices of $\mathcal{H}$ adjacent to y_1, some vertex x_j, $2 \leq j \leq \ell$ must occur in at least two of the m-gons Π_i. But then there are two m-gons in $\mathcal{H}$ on the 2-claw $(y_1: x, x_j)$, a contradiction.

Thus y_1 has valency k, and since $\mathcal{H}$ is connected, so does every vertex of $\mathcal{H}$. Thus $\mathcal{H}$ is regular. □

Lemma 1.2. Suppose that $\mathcal{H}$ is a strict m-gon-graph on Ω and that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two induced subgraphs of $\mathcal{H}$ which are also strict m-gon-graphs, on Ω_1 and Ω_2 respectively, where Ω_1, Ω_2

are subsets of Ω with $\Omega_1 \cap \Omega_2 \neq \phi$. Then connected components of the subgraph of $\mathcal{H}$ induced by $\Omega_1 \cap \Omega_2$ are points, edges, or strict m-gon-graphs.

Proof. Let C be a connected component of $\Omega_1 \cap \Omega_2$. If C is not a point or an edge, then C contains 2-claws. On each 2-claw of C there is an unique m-gon Π in $\mathcal{H}$. However the 2-claw is in both $\mathcal{H}_1$ and $\mathcal{H}_2$ by definition of C, so that Π is in $\mathcal{H}_1$ and in $\mathcal{H}_2$, whence Π is in C. Thus each 2-claw of C is on a (necessarily unique) m-gon of C.

Now apply Lemma 1.1. □

Lemma 1.3. Let $\mathcal{H}$ be a graph of girth $m \geq 3$, and $G \leq \mathrm{Aut}(\mathcal{H})$. Suppose that $(x: y, z)$ is a 2-claw in $\mathcal{H}$ and $g \in G_{xyz}$. If g fixes the vertices of an m-gon of $\mathcal{H}$ on $(x: y, z)$ setwise, then g fixes these vertices pointwise.

Proof. Clear, since the girth of $\mathcal{H}$ is m. □

Lemma 1.4. Let $\mathcal{H}$ be a strict m-gon-graph and $G \leq \mathrm{Aut}(\mathcal{H})$ with $\Omega(G) \neq \phi$. Then connected components of the subgraph of $\mathcal{H}$ induced by $\Omega(G)$ are points, edges, or strict m-gon-graphs.

Proof. Let C be a connected component of $\Omega(G)$ which is not a point or an edge. Then for $g \in G$, C is contained in a connected component $C(g)$ of $\Omega(g)$, and clearly $C = \bigcap_{g \in G} C(g)$.

Now by Lemma 1.3, the unique m-gon in $\mathcal{H}$ on any 2-claw of

$C(g)$ is in $C(g)$. Thus by Lemma 1.1, $C(g)$ is a strict m-gon-graph whence by Lemma 1.2, $C = \bigcap_{g \in G} C(g)$ is also a strict m-gon-graph. □

Lemma 1.5. Let $\mathcal{H}$ be a strict m-gon-graph and $G \leq \mathrm{Aut}(\mathcal{H})$. For x a vertex of $\mathcal{H}$, G_x is faithful on $\Delta(x)$.

Proof. Let $g \in G_x$ and suppose g fixes $\Delta(x)$ pointwise. We show by induction that g fixes $\Delta_n(x)$ pointwise for all $n \geq 2$, and thus, since $\mathcal{H}$ is connected, g fixes $\mathcal{H}$.

So suppose g fixes $\Delta_i(x)$ for $i < n$. Let $y \in \Delta_n(x)$ and choose $u \in \Delta_{n-1}(x)$, $v \in \Delta_{n-2}(x)$ with $v \sim u \sim y$. Let $\Pi = (y, u, v, w, \ldots)$ be the unique m-gon on the 2-claw $(u: v, y)$. Then Π is also the unique m-gon on the 2-claw $(v: u, w)$ and since $w \in \Delta_i(x)$ for some $i < n$, g fixes this latter 2-claw, and hence Π, pointwise by Lemma 1.3. Thus g fixes y. Since y was arbitrary, g fixes $\Delta_n(x)$.

So g fixes every vertex of $\mathcal{H}$, and thus $g = 1$. □

Remark. Lemma 1.5 is false for general (non-strict) polygonal graphs: a counterexample is given by the Petersen graph and its full automorphism group (see Chapter II, Section 1).

The following t-transitive version of a theorem of Jordan (see [32], Theorem 3.7) is given without proof.

Lemma 1.6. Let the group G act t-transitively on the set Ω. Let S be a Sylow subgroup of the stabilizer of some t points of Ω. Then $N_G(S)$ is t-transitive on $\Omega(S)$. □

Although the definition of m-gon-graphs assumes the girth is m, there are interesting examples of graphs in which sets of m-gons, a unique one for each 2-claw, can be singled out, but having girth less than m. For example consider the graphs $\mathcal{H}_q$ to be defined in Chapter III. For $q > 41$ their girths are unknown although they satisfy all other requirements of the definition of polygonal graphs. Another example is given by J_1 acting on the cosets of PSL(2,11) and defining a graph with respect to the suborbit of length 12. Certain hexagons, a unique one for each 2-claw, can be singled out yet the girth of the graph is 5. (Do not confuse this graph with the one defined with respect to the suborbit of length 11 which <u>is</u> a 5-gon-graph -- see Chapter V.)

As shall be seen in Chapter IV, particular interest pertains to the problem of when a polygonal graph of valency > 3 (and not arising from a regular solid) can contain a cubic strict polygonal graph as a subgraph. Only a finite number of the latter graphs are known (see Chapter II) so that the aforementioned problem and the question of finiteness of the number of cubic strict polygonal graphs are two of the outstanding problems arising from this thesis.

We conclude this chapter by mentioning that m-gon-graphs give rise to incidence structures belonging to the diagram $\overset{\bullet}{\underline{\quad}}\overset{(m)}{\underline{\quad}}\overset{\bullet}{\underline{\quad}}\overset{c}{\underline{\quad}}\overset{\bullet}{\underline{\quad}}$ of F. Buekenhout [8].

CHAPTER II

CUBIC POLYGONAL GRAPHS

In this chapter we investigate, and give examples of, polygonal graphs of valency three. We shall be concerned mainly with strict polygonal graphs, although we shall give an example of a polygonal graph which is not strict (the Petersen graph). We shall also be restricting our attention almost exclusively to the cases in which the automorphism group of the graph is at least transitive on 2-claws, although some examples which fail to meet this requirement will also be mentioned. In Section 1 we list all of the possible examples of cubic m-gon-graphs with $m \leq 5$.

In Section 2 we investigate a class of graphs which are closely related to the cubic polygonal graphs. These are graphs having an m-gon as constant link. (See Chapter I, Section 1, for the definition of the link of a graph.) Such graphs are known to exist for all m (M. Brown and R. Connelly, [7]), and their duals will possibly be examples of strict m-gon-graphs of valency 3 (see Section 2 for the definition of dual).

In Section 3 we consider the case where the automorphism group of the strict polygonal graph contains a subgroup which is transitive on 2-claws but not on ordered 2-claws. This amounts to having a group of automorphisms of the graph which is transitive on the vertices and such that the stabilizer of any vertex is simply

transitive (i.e. regular) on the three vertices adjacent to the given vertex. In this case the group can be shown to be a homomorphic image of a certain polyhedral group. Also, information about the kernel of this homomorphism can be derived and using this information, we can exclude many of the groups PSL(2,p) as such subgroups of the automorphism group of a p-gon-graph, p a prime, as well as indicate for which p, PSL(2,p) might occur as such a subgroup.

In Section 4 we note a connection between cubic polygonal graphs and regular maps (see H.S.M. Coxeter and W.O.J. Moser [11], Section 3.2 and Chapter 8), from which relationship some examples of cubic polygonal graphs can be obtained. For convenience we have listed in Appendix 1 the groups (with their generators and relations) which have been obtained in this chapter as automorphism groups of polygonal graphs, as well as (where practical) the intersection matrices which correspond to the suborbit with respect to which the polygonal graph is defined.

Section 1. Examples with m = 4 and 5

With m = 4, the most obvious example is the graph consisting of the vertices and edges of a 3-dimensional cube with automorphism group isomorphic with $\Sigma_4 \times Z_2$ (H.S.M. Coxeter and W.O.J. Moser [11], Section 4.4). In fact this rectagraph can be defined as being the unique connected cubic graph (undirected, no loops or multiple edges) of girth 4 and having exactly one rectangle on each 2-claw.

This can easily be shown by direct construction and so the proof will be omitted.

A 4-gon-graph not equal to the above rectagraph can be seen to have to contain exactly two rectangles on each 2-claw. Again direct construction shows that the only connected cubic graph of girth 4 and having exactly two rectangles on each 2-claw is the complete bipartite graph on 6 vertices. This graph can be realised as a regular hexagon with an edge from any vertex to its antipode. However, it is not a 4-gon-graph for, if it was, let $\mathfrak{E}$ be a corresponding set of N rectangles, one on each 2-claw, as in the definition of 4-gon-graph. Count the number of pairs (C,Π), where C is a 2-claw of the graph and Π is a rectangle in $\mathfrak{E}$ with Π on C, in two ways. There are exactly 18 2-claws, each in a unique member of $\mathfrak{E}$. There are N rectangles in $\mathfrak{E}$ each containing 4 2-claws. Thus $4N = 18$, contradicting N being an integer.

With $m = 5$ there are exactly two 5-gon-graphs of valency 3 with automorphism groups transitive on 2-claws. These are (a) the Petersen graph, and (b) the dodecahedral graph (which is also a cubic pentagraph). We consider these separately.

(a) Consider $G = \Sigma_5$ acting on the set Ω of two-element subsets of $\{1, 2, 3, 4, 5\}$, so $|\Omega| = 10$. Let $x = \{1, 2\}$, $u = \{3, 4\}$. Define a graph $\mathcal{H}$ on Ω whose edges are $\{(x, u)^g: g \in G\}$. Alternatively, and equivalently, join pair $\{i, j\}$ to pair $\{i', j'\}$ if and only if $\{i, j\} \cap \{i', j'\} = \phi$. Then $\mathcal{H}$ is an undirected, connected, regular graph of valency 3

isomorphic with the Petersen graph. (See Figure 2.1. For the more standard diagram of the Petersen graph, see Figure 2.2.) Note that the girth of $\mathcal{H}$ is 5 and that there are exactly two pentagons on each 2-claw.

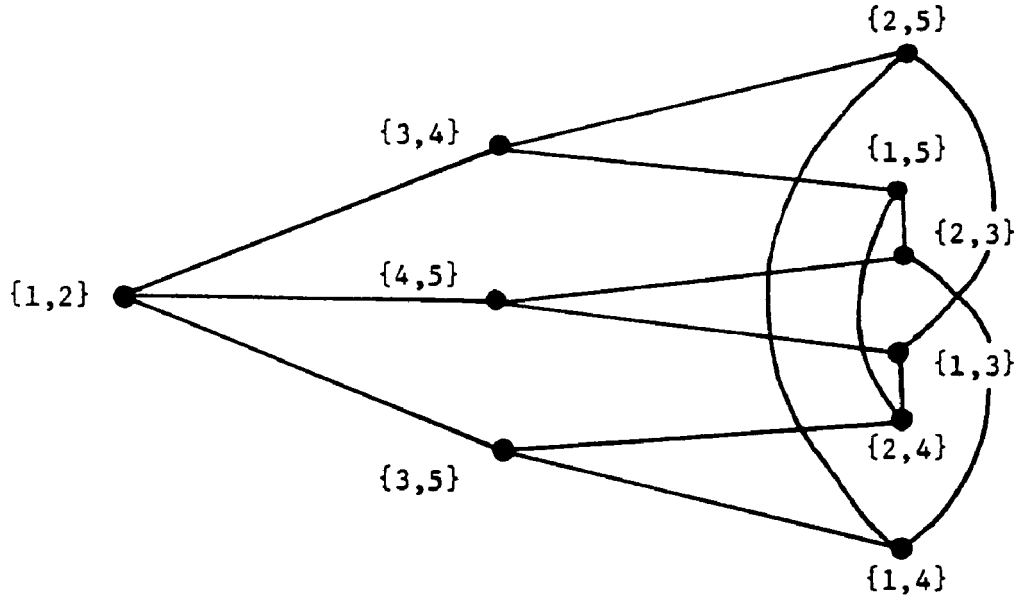


Figure 2.1. A 5-gon-graph of Valency 3

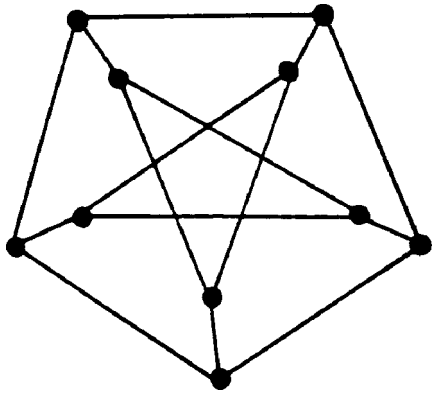


Figure 2.2. The Petersen Graph

Direct construction can be used to show that $\mathcal{H}$ is the unique (up to isomorphism) undirected, connected, cubic graph of girth 5 having exactly two pentagons on each 2-claw. Further, $\mathcal{H}$ is a 5-gon-graph; for the set $\&$ pick any pentagon and its images under the group A_5. Then $\&$ has 6 pentagons, a unique one for each 2-claw, so that $(\mathcal{H},\&)$ is a 5-gon-graph. Altogether $\mathcal{H}$ has 12 pentagons and with $\&'$ the set of 6 pentagons of $\mathcal{H}$ not in $\&$, $(\mathcal{H},\&')$ is also a 5-gon-graph.

$G \leq \mathrm{Aut}(\mathcal{H})$ and clearly G is transitive on Ω. $G_x \simeq D_{12}$ is 3-transitive on $\Delta(x)$; in fact $G_x^{\Delta(x)} \simeq \Sigma_3$, so G_x is not faithful on $\Delta(x)$, its kernel being $\langle (1\ 2) \rangle \leq G$. Thus $\mathrm{Aut}(\mathcal{H})_x$ is not faithful on $\Delta(x)$, so any non-identity element in the kernel of $\mathrm{Aut}(\mathcal{H})_x^{\Delta(x)}$ must interchange the 2 pentagons on any 2-claw $(x: y, z)$, $y, z \in \Delta(x)$, since the girth is 5, and it is easy to see that an element fixing all of the vertices of one pentagon, must fix all the vertices of $\mathcal{H}$, and hence be the identity. Thus this kernel has order 2, and so $\mathrm{Aut}(\mathcal{H}) = G$.

G has a subgroup $H \simeq A_5$, and H is also transitive on Ω. Furthermore, $H_x \simeq \Sigma_3$ is faithful on $\Delta(x)$. Note also that G and H are the only subgroups of $\mathrm{Aut}(\mathcal{H})$ whose one point stabilizers are 3-transitive on the points adjacent to the given point. This fact will be used in Chapter V, as well as the following two facts. (i) There is an element of odd order in H, e.g. $(1\ 3\ 2\ 4\ 5)$, which takes x to u. (ii) Every involution in $\mathrm{Aut}(\mathcal{H})$ fixes some point of Ω.

(b) Consider the group $H = A_5$ acting on the set of ordered pairs $\Omega = \{(i, j): i \neq j, i, j \in \{1, 2, 3, 4, 5\}\}$. (For convenience we shall write ij for the pair (i, j).) Then $|\Omega| = 20$. Let $x = 12$ and $u = 34$, and define a graph $\mathcal{H}$ on Ω whose edges are $\{x^g u^g: g \in H\}$. Then $\mathcal{H}$ is an undirected, connected, regular graph of valency 3 isomorphic with the graph of vertices and edges of a regular dodecahedron and we shall thus call it the dodecahedral graph. $\mathcal{H}$ has girth 5 and on each 2-claw there is an unique pentagon. (See Figure 2.3. For the more usual diagram of the dodecahedral graph, see Figure 2.4.)

Once again, direct construction can be used to verify that $\mathcal{H}$ is (up to isomorphism) the unique, undirected, connected, cubic graph of girth 5 having an unique pentagon on each 2-claw. Thus $\mathcal{H}$ is the unique pentagraph of valency 3.

$H \leq \mathrm{Aut}(\mathcal{H})$, is transitive on Ω, and $H_x = \langle (3\ 4\ 5) \rangle$ is simple transitive on $\Delta(x)$. Let s be the following permutation on Ω: $(ij)^s = ji$. Clearly s is an involution and s preserves adjacency in $\mathcal{H}$, so $s \in \mathrm{Aut}(\mathcal{H})$. Also, for $g \in H$, $(ij)^{gs} = j^g i^g = (ij)^{sg}$, so s centralizes H.

Thus $G = H \times \langle s \rangle \leq \mathrm{Aut}(\mathcal{H})$. Furthermore, $G_x = \langle (3\ 4\ 5), (1\ 2)(4\ 5).s \rangle \simeq \Sigma_3$ is faithful and acts 3-transitively on $\Delta(x)$. But by Lemma 1.5 , $\mathrm{Aut}(\mathcal{H})_x$ is also faithful on $\Delta(x)$ and so $G = \mathrm{Aut}(\mathcal{H}) = H \times \langle s \rangle$.

We note the following facts about $\mathrm{Aut}(\mathcal{H})$, some of which will be used in Chapter V. (i) There is an involution in H. viz. $(1\ 3)(2\ 4)$, which interchanges x and u. (ii) $\mathcal{H}$ has 30 edges,

G is transitive on these edges, and so $|G_{\{x,u\}}| = 4$.

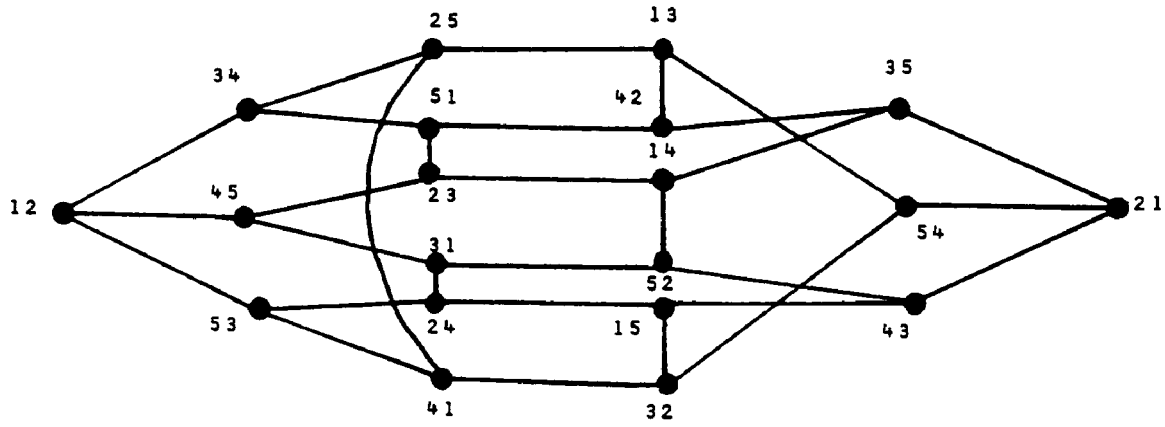


Figure 2.3. The Pentagraph of Valency 3

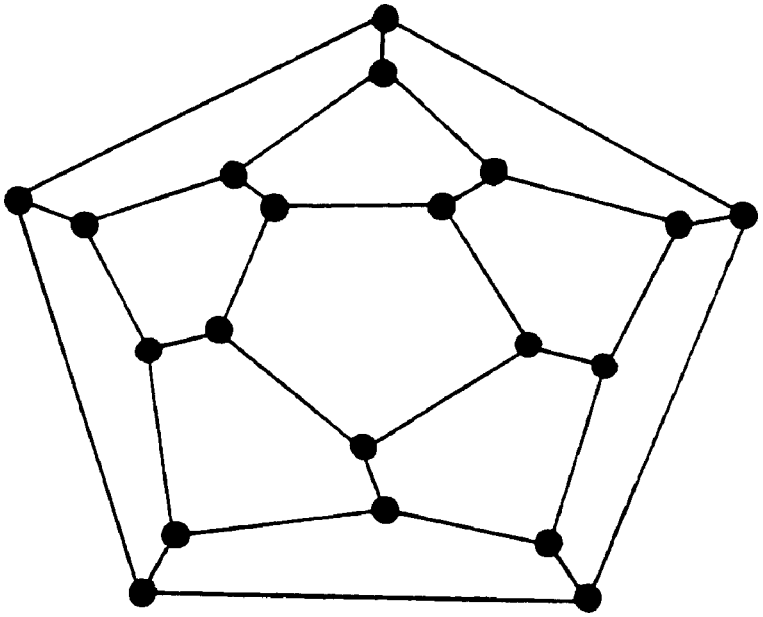


Figure 2.4. The Dodecahedral Graph

Section 2. A Related Class of Graphs

For $m \geq 3$, let Π_m (the πolygonal graphs of valency 3) be the set of isomorphism classes of finite, strict m-gon-graphs of valency 3, and let T_m (the τriangular graphs of valency m) be the

set of isomorphism classes of finite graphs which are undirected, have no loops or multiple edges, are connected, regular of valency m, and have constant link C_m, an m-gon.

Example. $\Pi_3 = \{\text{tetrahedron}\}$, $T_3 = \{\text{tetrahedron}\}$.

$\Pi_4 = \{\text{cube}\}$, $T_4 = \{\text{octahedron}\}$.

$\Pi_5 = \{\text{dodecahedron}\}$, $T_5 = \{\text{icosahedron}\}$.

If A is a graph, then $A \in \Pi_m$ or $A \in T_m$ will mean that A is in one of the isomorphism classes of Π_m, respectively T_m.

For $A \in \Pi_m$, $D(A)$, the dual of A, is the graph whose vertices are the m-gons of A, two vertices being adjacent if and only if the corresponding m-gons have an edge in common. Note that because the girth of A is m, and A has no multiple edges, two m-gons of A have at most one edge in common, so that $D(A)$ is undirected with no loops or multiple edges.

For $B \in T_m$, $D(B)$, the dual of B, is the graph whose vertices are the triangles of B, two vertices being adjacent if and only if the corresponding triangles have an edge in common. Note that B has no multiple edges, so two triangles share at most one edge and hence $D(B)$ is undirected and has no loops or multiple edges.

Remark. For $A \in \Pi_m$ or T_m, $A \simeq D(A)$ if and only if $m = 3$ and A is a tetrahedron. In this case $\mathrm{Aut}(A) \simeq \Sigma_4$. Also $D(\Pi_4) = T_4$, and $D(\Pi_5) = T_5$, and vice versa.

Henceforth assume that $A \in \Pi_m$ and $B \in T_m$. Let n_A, e_A and p_A be the number of vertices, edges and m-gons of A, respectively, and n_B, e_B and t_B the number of vertices, edges and triangles of B, respectively. Then $e_A = 3n_A/2$, $p_A = 3n_A/m$, $e_B = mn_B/2$, and $t_B = mn_B/3$.

Lemma 2.1. If two m-gons in A have a point x in common, then they have an edge, with x as one endpoint, in common.

Proof. Valency of A is 3. □

Lemma 2.2. If two m-gons in A have two points x and y in common, then $x \sim y$, and they have no other common edge.

Proof. Girth of A is m. □

Lemma 2.3. Each edge of A lies in exactly two distinct m-gons of A.

Proof. Follows from the valency of A being 3, and the uniqueness of m-gons on 2-claws of A. □

Lemma 2.4. Three m-gons in A have at most one point in common.

Proof. Lemmas 2.2 and 2.3. □

Corresponding to the above four lemmas, we have the following.

Lemma 2.5. If two triangles of B have two points x and y in common, then clearly $x \sim y$. □

Lemma 2.6. Each edge of B lies in exactly two triangles of B.

Proof. Let (x, y) be an edge of B. Since link (x, B) is an m-gon, there are at least two triangles on (x, y). Suppose (x, y, z_1), (x, y, z_2) and (x, y, z_3) are three triangles on (x, y). Then link (x, B) contains the 3-claw $(y: z_1, z_2, z_3)$, a contradiction. □

Lemma 2.7. Three triangles have at most one point mutually in common.

Proof. Lemmas 2.5 and 2.6. □

Lemma 2.8. (i) If $D(A) \in T_m$, then $D(D(A)) \simeq A$.

(ii) Similarly, if $D(B) \in \Pi_m$, then $D(D(B)) \simeq B$.

Proof. (i) We need to show there is a one-to-one correspondence ϕ between points of A and triangles of $D(A)$ such that two points of A are adjacent if and only if their corresponding triangles under ϕ share an edge.

Since the valency of A is 3, if x is a point of A then by uniqueness of m-gons on 2-claws, exactly three m-gons of A contain x, and these three m-gons form a triangle in $D(A)$. Thus we have defined a map ϕ from Ω, the set of vertices of A, to Γ,

the set of triangles of $D(A)$.

If $x, y \in \Omega$, with $\phi(x) = \phi(y)$ then the three m-gons on x and y coincide, which contradicts Lemma 2.4 unless $x = y$. So ϕ is one-to-one.

Now, by definition of $D(A)$, $p_A = n_{D(A)}$, so $3n_A/m = n_{D(A)}$. Thus $mn_{D(A)}/3 = n_A$, which gives $|\Omega| = |\Gamma|$, and so ϕ is a one-to-one correspondence.

Now suppose $x, y \in \Omega$ with $x \sim y$. By Lemma 2.3, $\phi(x)$ and $\phi(y)$ have an edge in common.

Suppose $x \neq y$ and $\phi(x)$ and $\phi(y)$ have an edge in common. Then two of the m-gons (in A) on x are also on y, and they share an edge. So by Lemma 2.2, $x \sim y$.

Hence $D(D(A)) \simeq A$.

(ii) Here we need to show that there is a one-to-one correspondence ψ between vertices of B and m-gons of $D(B)$ such that two vertices x and y of B are adjacent if and only if $\psi(x)$ and $\psi(y)$ share an edge.

Let Ω be the vertices of B, Γ the m-gons of $D(B)$. If $x \in \Omega$, then since link (x, B) is an m-gon, there are exactly m triangles which have x as a point in common. Further, these m triangles constitute an m-gon in $D(B)$. Thus we have defined a map $\psi: \Omega \to \Gamma$.

If $x, y \in \Omega$ with $\psi(x) = \psi(y)$, then by Lemma 2.7, $x = y$, and so ψ is one-to-one.

Now by definition of $D(B)$, $t_B = n_{D(B)}$, so $mn_B/3 = n_{D(B)}$,

giving $|\Omega| = |\Gamma|$ and ψ is a one-to-one correspondence.

Suppose $x, y \in \Omega$ with $x \sim y$. Then, by Lemma 2.6, there are exactly two triangles on the edge (x, y); these two triangles thus define an edge of the m-gons $\psi(x)$ and $\psi(y)$ in $D(B)$ and so $\psi(x)$ and $\psi(y)$ have an edge in common. Conversely, suppose $\psi(x)$ and $\psi(y)$ have an edge in common. Then two of the triangles on x are also on y, and share an edge, which implies $x \sim y$ (Lemma 2.5)

Thus $D(D(B)) \simeq B$. □

Corollary 2.1. (i) If $D(A) \in T_m$, then $\mathrm{Aut}(A) \simeq \mathrm{Aut}(D(A))$. (ii) If $D(B) \in \Pi_m$, then $\mathrm{Aut}(B) \simeq \mathrm{Aut}(D(B))$.

Proof. Clearly $\mathrm{Aut}(C)$ is isomorphic to a subgroup of $\mathrm{Aut}(D(C))$, for any $C \in \Pi_m$ or T_m. Thus for $A \in \Pi_m$, $\mathrm{Aut}(A) \leq \mathrm{Aut}(D(A)) \leq \mathrm{Aut}(D(D(A))) \simeq \mathrm{Aut}(A)$ by Lemma 2.8. The result follows similarly for $B \in T_m$. □

The question that naturally arises is, under what conditions is $D(A) \in T_m$ for $A \in \Pi_m$, or $D(B) \in \Pi_m$ for $B \in T_m$? An answer to this will be given for a special case in Section 4 (Theorem 2.2).

In [7], Theorem 3, Brown and Connelly give a method for constructing graphs in T_m for all $m \geq 6$. In summary, their method consists of taking two disjoint, equal copies of a graph with a simple path of length $m-3$ as constant link (such graphs are proved to exist in earlier sections of the paper), and, for any isomorphism of their boundary subgraphs, joining together corresponding

components of their boundaries by means of annuli of the correct length. (For the relevant definitions of boundary and annulus, see [7].) Graphs with simple paths of length greater than three as constant link and having sharply transitive automorphism groups can be constructed using the methods of A. Blass, F. Harary and Z. Miller [6]. Unfortunately even when these latter graphs are used in Brown and Connelly's construction, the resulting graphs in T_m often do not have automorphism groups which are transitive on triangles, and as a result their duals, even if in Π_m, often do not have automorphism groups transitive on points.

Example. Consider a tesselation of the plane by equilateral triangles of side 1. (Call these the 1-triangles.) Pick any triangle, and any vertex on it. With the vertex as origin we can choose X and Y coordinate axes at 60° to each other, intersecting with the origin at their zeros, and such that a side of the given triangle is parallel to one of the axes. Then every vertex of every 1-triangle can be given a unique coordinate (m, n), m, n integers, in the obvious way.

Choose two positive integers p and q and define a relation $\equiv$ on the vertices of the triangles by $(m, n) \equiv (m_1, n_1)$ if and only if $m \equiv m_1 \pmod{p}$ and $n \equiv n_1 \pmod{q}$. Clearly $\equiv$ is an equivalence relation. Define a graph whose vertices are the equivalence classes of $\equiv$, two classes being adjacent if some point in the one and some point in the other are two distinct vertices of some 1-triangle. Then it is not difficult to see that if p and

q are ≥ 4, this graph is in T_6. Further, it has pq vertices, $3pq$ edges and $2pq$ triangles, and its automorphism group is transitive on its points and on its triangles but will contain the rotations on any one triangle if and only if $p = q$, in which case it also contains the full symmetry group ($\simeq \Sigma_3$) of that triangle. It is readily seen that its dual (which we shall call $\mathcal{H}(p,q)$) is in Π_6 because $p, q \geq 4$. So $\mathrm{Aut}(\mathcal{H}(p,q))$ is transitive on the vertices of $\mathcal{H}(p,q)$, and is transitive on ordered 2-claws if and only if $p = q$ (by Corollary 2.1). $\mathcal{H}(p,q)$ has $2pq$ vertices, $3pq$ edges and pq hexagons.

It may be mentioned in conclusion, that the graphs in T_6 constructed above with $p = 4$ and $q \geq 4$, can also be obtained by Brown and Connelly's construction using the graphs C^2_{2q} as graphs with constant link a simple path of length 3. ($C_r^{\,2}$ is the graph constructed from an r-gon by drawing an edge between every second point on the r-gon. Of course, using $C_r^{\,2}$ in the construction, $r \geq 7$, r odd, we can also get graphs in T_6.)

Section 3. Simply Transitive Point Stabilizers

Throughout this section we shall assume that $\mathcal{H}$ is a strict m-gon-graph, $m \geq 3$, $G \leq \mathrm{Aut}(\mathcal{H})$ is transitive on the vertices Ω of $\mathcal{H}$, and for any $x \in \Omega$, G_x is simply transitive on $\Delta(x)$. Let G_1 and G_2 be the abstract groups given in terms of generators and relations as follows:

$$G_1 = G_1(m) = \langle g, h : g^2 = h^3 = (hg)^m = 1 \rangle, \text{ and}$$

$$G_2 = G_2(m) = \langle g, h : g^2 = h^3 = (hgh^{-1}g)^{m/2} = 1 \rangle \text{ (for } m \text{ even)}.$$

Remark. These are the polyhedral groups of H.S.M. Coxeter and W.O.J. Moser ([11], Sections 4.4 and 6.4). For $m \geq 6$ it is known that these groups are infinite, while $G_1(3) \simeq A_4$, $G_1(4) \simeq \Sigma_4$, $G_1(5) \simeq A_5$, and $G_2(4) \simeq A_4 \times Z_2$.

We shall prove the following.

Theorem 2.1. There is a normal subgroup N of $G = G_1$ or $G = G_2$ such that $G \simeq G/N$, and no non-trivial element of N has length less than $2m+1$ in the generators g and h of G. □

As a consequence of Theorem 2.1 we have the following.

Corollary 2.2. If $(m, 6) = 1$, then G is perfect, i.e. $G = G'$. In particular, G is non-solvable.

Proof. Suppose there is a proper normal subgroup K of G such that $1 \neq G/K = \overline{G}$ is abelian. Let $\overline{\ }$ denote images in $\overline{G}$. Then $\overline{gh} = \overline{hg}$ implies that $1 = (\overline{gh})^m = \overline{g^m h^m} = \overline{gh^\varepsilon}$, with $\varepsilon \neq 0$ since $(m, 6) = 1$. Thus $\overline{g} = \overline{h}^{-\varepsilon}$, but $\overline{g}$ and $\overline{h}$ have relatively prime orders, whence $\overline{g} = \overline{h} = 1$. But $\overline{G} = \langle \overline{g}, \overline{h} \rangle$ by Theorem 2.1, so $\overline{G} = 1$, a contradiction. □

We shall prove Theorem 2.1 by a series of lemmas. Pick a point $0 \in \Omega$ and let $\Delta(0) = \{1, 2, 3\}$. Let $H = G_0$. By Lemma 1.5 , H is faithful on $\Delta(0)$, and since H is simply transitive on $\Delta(0)$, $H = \langle h \rangle$ where $h^3 = 1$.

Lemma 2.9. There is an involution $g \in G$ such that $G = \langle H, g \rangle$.

Proof. There is an element $g \in G$ which interchanges 0 and 1, so that $g^2 \in H$ and fixes 1. Thus $g^2 = 1$. Since $\mathcal{H}$ is connected, $G = \langle H, g \rangle$. □

Since H is the stabilizer of the point $0 \in \Omega$, we may identify the set Ω with the set of right cosets of H in G such that 0 corresponds to H, and 0^k corresponds to Hk for all $k \in G$. We shall adopt this point of view from now on when convenient, so for example we have $\Delta(H) = \{Hg, Hgh, Hgh^{-1}\}$.

Lemma 2.10. For any $k \in G$, $\Delta(Hk) = \{Hgk, Hghk, Hgh^{-1}k\}$. □

Let G be the abstract group defined by $\langle g, h: g^2 = h^3 = 1 \rangle$. Then by Lemma 2.9, there is a function ψ from G onto G such that $\psi(g) = g$, and $\psi(h) = h$, and G acts on $\mathcal{H}$ via ψ. Call a word of the form $h^{\varepsilon}g \in G$, where $\varepsilon = 1$ or -1, a term of G. Then by suitable conjugation by g and h, every non-trivial word of G is conjugate to a word of the form $h^{\varepsilon_n} g\, h^{\varepsilon_{n-1}} g \ldots h^{\varepsilon_1} g$ (i.e. a product of n terms of G) where the positive integer n

is unique and $\varepsilon_i = \pm 1$ $(1 \leq i \leq n)$. A word of this form will be called a term word (or an n-term word if we wish to emphasize the integer n). Two term words $w_1 = h^{\varepsilon_n} g \ldots h^{\varepsilon_1} g$ and $w_2 = h^{\delta_p} g \ldots h^{\delta_1} g$ $(\varepsilon_i = \pm 1,\ \delta_i = \pm 1)$ will be called equivalent, if $n = p$, and the n-tuple $(\varepsilon_n, \ldots, \varepsilon_1)$ equals the n-tuple $(-1)^s\ (\delta_{\sigma(n)}, \ldots, \delta_{\sigma(1)})$, for some permutation σ in the dihedral group of order $2n$ in Σ_n containing the n-cycle $\tau = (1\ 2 \ldots n)$, and $s = 0$ or 1 according as σ is or is not a power of τ. (i.e. w_1 is equivalent to w_2 if w_2 is a product of the terms of w_1 or $(w_1^g)^{-1}$ permuted cyclically.) This defines an equivalence relation on the set of all term words in G and it can also be seen that two n-term words w_1 and w_2 are equivalent if and only if w_1 is conjugate to w_2 or w_2^{-1} in G, so that if $w \in K = \ker \psi$ is an n-term word in K, then the equivalence class of w is also in K.

Lemma 2.11. There is a mapping ϕ from the set of equivalence classes of term words in K onto the set of equivalence classes of circuits in $\mathcal{H}$. Under ϕ, an equivalence class of n-term words in K gets mapped to an equivalence class of circuits of length n in $\mathcal{H}$.

Note. Two paths P_1 and P_2 in $\mathcal{H}$ are said to be equivalent if $P_1^f = P_2$ for some $f \in G$. This defines an equivalence relation on paths of $\mathcal{H}$, and since G is transitive on edges of $\mathcal{H}$, every path is equivalent to one containing the edge (H, Hg).

Proof. Let $w = h^{\varepsilon_n} g\, h^{\varepsilon_{n-1}} g \ldots h^{\varepsilon_1} g$ be an n-term word of K. Since $K = \ker \psi$, $\psi(w) = 1 \in G$. Define the elements $k_i = h^{\varepsilon_i} g \ldots h^{\varepsilon_1} g \in G$, for $i = 1, \ldots, n$, and let $k_i = \psi(k_i) = h^{\varepsilon_i} g \ldots h^{\varepsilon_1} g \in G$. Then we have the sequence $\{Hk_i : i = 1,\ldots,n\}$ of vertices of $\mathcal{H}$ with $Hk_1 = Hg$ and $Hk_n = H$. It is clear from Lemma 2.10 that $Hk_{i+1} \sim Hk_i$ for $1 \leq i \leq n-1$, so that $P = (Hk_1,\ldots,Hk_n)$ will be a circuit of length n containing the edge (H, Hg) provided we show that $Hk_i \neq Hk_{i+2}$ for $1 \leq i \leq n-2$, $Hk_1 \neq Hk_{n-1}$ and $Hk_2 \neq Hk_n$.

So suppose for some $i \leq n-2$ that $Hk_i = Hk_{i+2}$. Then clearly $H = Hgh^{\varepsilon_{i+1}} gh^{\varepsilon_i}$, so that $Hg = Hgh^{\varepsilon_{i+1}}$, which contradicts the transitivity of H on $\Delta(H)$, since $\varepsilon_{i+1} \neq 0$. Since $Hk_{n-1} = Hh^{\varepsilon_{n-1}} g \ldots h^{\varepsilon_1} g = Hgh^{-\varepsilon_n}$ we see similarly that $Hk_{n-1} \neq Hk_1$, and likewise that $Hk_2 \neq Hk_n$. So P is a circuit.

Now suppose that $w_1 \in K$ is an n-term word equivalent to w. Then $w_1 = h^{\varepsilon_i} g\, h^{\varepsilon_{i-1}} g \ldots h^{\varepsilon_1} g\, h^{\varepsilon_n} g \ldots h^{\varepsilon_{i+1}} g$, or $w_1 = h^{-\varepsilon_i} g\, h^{-\varepsilon_{i+1}} g \ldots h^{-\varepsilon_n} g\, h^{-\varepsilon_1} g \ldots h^{-\varepsilon_{i-1}} g$, for some i, $1 \leq i \leq n$. In the first case we get, as above, a circuit $P_1 = (Hg, Hgh^{\varepsilon_{i+1}} g, \ldots, Hgh^{\varepsilon_1} gh^{\varepsilon_n} g \ldots h^{\varepsilon_{i+1}} g, \ldots, H)$, so that $P = P_1^f$ with $f = gh^{-\varepsilon_{i+1}} gh^{-\varepsilon_{i+2}} \ldots gh^{-\varepsilon_n}$

$= h^{\varepsilon_i} g \ldots h^{\varepsilon_1} g \in G$, whence P_1 is equivalent to P.

In the second case, we get a circuit

$$P_2 = (Hg, Hgh^{-\varepsilon_{i-1}} g, \ldots, Hgh^{-\varepsilon_1} g \ldots h^{-\varepsilon_{i-1}} g, Hgh^{-\varepsilon_n} gh^{-\varepsilon_1} \ldots h^{-\varepsilon_{i-1}} g, \ldots, H),$$

so with

$$f = gh^{\varepsilon_{i-1}} \ldots gh^{\varepsilon_1} = h^{-\varepsilon_i} g\, h^{-\varepsilon_{i+1}} g \ldots h^{-\varepsilon_n} g \in G ,$$

$$P_2{}^f = (Hh^{\varepsilon_{i-1}} g \ldots gh^{\varepsilon_1}, Hh^{\varepsilon_{i-2}} g \ldots gh^{\varepsilon_1}, \ldots ,$$

$$Hh^{\varepsilon_n} g\, h^{\varepsilon_{n-1}} g \ldots h^{\varepsilon_1}, Hh^{\varepsilon_{n-1}} g\, h^{\varepsilon_{n-2}} g \ldots h^{\varepsilon_1}, \ldots ,$$

$$Hgh^{\varepsilon_{i-1}} \ldots gh^{\varepsilon_1}).$$

Hence $P_2{}^{fg} = P$ so P_2 is equivalent to P.

So let $\phi[\omega] = [P]$, where $[\omega]$ and $[P]$ denote the equivalence classes of ω in K, and P in $\mathcal{H}$. Then ϕ is a well-defined map from the set of equivalence classes of term words in K to the set of equivalence classes of circuits in $\mathcal{H}$ (in which n-term words go to circuits of length n).

To show that ϕ is onto, suppose that $P = (Hk_1, \ldots, Hk_n)$ is a circuit in $\mathcal{H}$ containing the edge (H, Hg), where we may suppose without loss of generality that $Hk_1 = H$ and $Hk_2 = Hg$. Then, since $Hk_3 \neq Hk_1$, we see by Lemma 2.10 that $Hk_3 = Hgh^{\varepsilon_2}g$, where $\varepsilon_2 = \pm 1$. Similarly, we can prove by induction that for $3 \leq i \leq n$, $Hk_i = Hgh^{\varepsilon_{i-1}}g \ldots h^{\varepsilon_2}g$, where $\varepsilon_j = \pm 1$, $2 \leq j \leq i-1$. But Hk_n is adjacent to Hk_1, so again by Lemma 2.10, $H = Hgh^{\varepsilon_n}g \ldots h^{\varepsilon_2}g$, where again $\varepsilon_n = \pm 1$ (since $\varepsilon_n = 0$ implies $H = Hk_1 = Hk_{n-1}$, a contradiction). Hence $gh^{\varepsilon_n}g \ldots h^{\varepsilon_2}g = h^{-\varepsilon_1}$, for some $\varepsilon_1 \in \{0, 1, -1\}$.

$\varepsilon_1 = 0$ implies that $Hk_n = Hk_2$, a contradiction. Thus $\varepsilon_1 = \pm 1$ and $h^{\varepsilon_n}g \ldots h^{\varepsilon_1}g = 1$. Thus $\omega = h^{\varepsilon_n}g \ldots h^{\varepsilon_1}g$ is an n-term word in K, and clearly $\phi[\omega] = [P]$. □

Remark. It is possible to modify the notion of equivalence of term words so that ϕ will be a one to one correspondence, but this is not needed here. However the following corollary is true.

Corollary 2.3. Let C be the set of equivalence classes of simple circuits of $\mathcal{H}$. Then $\phi|_{\phi^{-1}(C)}$ is a one to one correspondence from $\phi^{-1}(C)$ onto C. □

Corollary 2.4. If w is an n-term word in K, then $n \geq m$.

Proof. If $n < m$, then $\phi[w]$ is a class of circuits in $\mathcal{H}$ of length less than m, which contradicts the girth of $\mathcal{H}$ being m. □

We now prove Theorem 2.1.

Proof of Theorem 2.1. Since there are circuits in $\mathcal{H}$ of length m, there are m-term words in K, but no n-term word for $n < m$, by Lemma 2.11 and Corollary 2.4. Also any word in K of length $\leq 2m$ which is not a term word is conjugate to an n-term word in K for $n < m$, a contradiction. Thus the only words in K of length $\leq 2m$ must be m-term words. So let $w = h^{\varepsilon_m} g h^{\varepsilon_{m-1}} g \ldots h^{\varepsilon_2} g h^{\varepsilon_1} g$ be an m-term word in K, $\varepsilon_i = \pm 1$.

Clearly we are done if $\varepsilon_i = 1$ for all i, or if $\varepsilon_i = -1$ for all i. So we may assume, up to equivalence, that $w = h^{-1} g h^{\varepsilon_{m-1}} g \ldots h^{\varepsilon_{t+2}} g h^{-1} g (hg)^t$, where $t < m$ is the maximal power of the term hg or $h^{-1}g$ which occurs in any m-term

word equivalent to w.

Let ϕ be as in Lemma 2.11. Then $\phi[w] = [P]$, where $P = (Hg, Hghg, \ldots, Hg(hg)^t, \ldots, Hgh, H)$, since $h^{\varepsilon_{m-1}}g \ldots h^{-1}g(hg)^t = gh$, and since the girth of $\mathcal{H}$ is m, P is an m-gon. If $t > 1$, let $w_1 = (hg)^{t-1} h^{-1}gh^{\varepsilon_{m-1}}g \ldots h^{\varepsilon_{t+2}} gh^{-1}ghg$, so that $w_1 \in K$ and $w_1 \in [w]$. Thus $\phi[w_1] = \phi[w]$. But by definition of ϕ, $\phi[w_1] = [P_1]$, where $P_1 = (Hg, Hghg, \ldots, Hgh^{-1}, H)$, since $(hg)^{t-2} h^{-1}gh^{\varepsilon_{m-1}}g \ldots h^{\varepsilon_{t+2}}gh^{-1}ghg = gh^{-1}$, and P_1 is also an m-gon.

Now both P and P_1 contain the 2-claw $(Hg: H, Hghg)$, so by uniqueness of m-gons on 2-claws, we must have in fact that $P = P_1$. But this is impossible since $\Delta(H) \cap P = \{Hg, Hgh\}$, whereas $\Delta(H) \cap P_1 = \{Hg, Hgh^{-1}\}$.

Hence $t = 1$, and so m is even and $w = (h^{-1}ghg)^{m/2}$.

This completes the proof of the theorem. □

Remark. There is a converse to Theorem 2.1, viz. if $N \triangleleft G_1$ (or $N \triangleleft G_2$) is such that no non-trivial word in N has length $\leq 2m$, then there is a strict m-gon-graph $\mathcal{H}$ of valency 3 with $\mathrm{Aut}(\mathcal{H})$ transitive on 2-claws of $\mathcal{H}$. It is worth remarking that both Theorem 2.1 and this converse hold for infinite strict m-gon-graphs and infinite quotients G_1/N or G_2/N, as well.

Example. It is known (H.S.M. Coxeter and W.O.J. Moser [11], Section 7.5) that for any prime $p > 2$ the group $G = PSL(2,p)$ is a quotient of the group $G_1(p)$ via the map

$g \mapsto g \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $h \mapsto h \equiv \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$,

(all matrices modulo $\left\langle\!\left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle\!\right\rangle$). Let N be the kernel of this map. Now,

$$hg \equiv \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad h^{-1}g \equiv \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} , \text{ so that}$$

$$k = h^{-1}ghg \equiv \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} .$$

Let $\{u_n\}$ be the Fibonacci sequence defined by $u_{n+2} = u_{n+1} + u_n$, $u_1 = 1$, $u_2 = 1$. It can be shown that

$u_n = \frac{1}{\sqrt{5}} [(\frac{1 + \sqrt{5}}{2})^n - (\frac{1 - \sqrt{5}}{2})^n]$, so that (for $p \neq 5$) if the Legendre symbol $(\frac{5}{p}) = 1$ then $p | u_{p-1}$, whereas if $(\frac{5}{p}) = -1$ then $p | u_{p+1}$. Of course, it is possible that $p | u_n$ for some $n < p-1$ (or $n < p+1$); the minimal such n satisfies $n | p-1$ (respectively $n | p+1$). It is an easy induction argument to show that for any integer $n > 0$,

$$k^n \equiv \begin{pmatrix} u_{2n+1} & u_{2n} \\ u_{2n} & u_{2n-1} \end{pmatrix} .$$

Now suppose that $p > 5$ and write $p = 2n+1$. Then if $(\frac{5}{p}) = 1$, $p | u_{2n}$. Thus $u_{2n+1} \equiv u_{2n-1} \pmod{p}$, and since the determinant of k^n is 1, $u_{2n+1}u_{2n-1} \equiv 1 \pmod{p}$, whence

$u_{2n+1} \equiv u_{2n-1} \equiv \pm 1 \pmod{p}$. Thus the order of k divides n. If $(\frac{5}{p}) = -1$, $p|u_{2n+2}$, and so $u_{2n+3} \equiv u_{2n+1} \equiv \pm 1 \pmod{p}$, and thus the order of k divides $n+1$.

Therefore, if $(\frac{5}{p}) = 1$, $(h^{-1}ghg)^n$ is a $2n$-term word in N and since $2n < p$, and all elements of order 3 in G are conjugate, G does not act on a strict p-gon-graph $\mathcal{H}$ with $G_x = \langle h \rangle$ for some vertex x of $\mathcal{H}$ with $x^g \in \Delta(x)$.

If $(\frac{5}{p}) = -1$, then let r be minimal such that $p|u_r$, so $r|p+1$. Suppose $r < p+1$. If r is even, then $k^{r/2} = 1$, as before, so that $(h^{-1}ghg)^{r/2}$ is an r-term word in N, and so, since $r < p$, we get once again the conclusion of the previous paragraph. If r is odd, write $r = 2t+1$. Then

$$k^t \equiv \begin{pmatrix} u_{2t+1} & u_{2t} \\ u_{2t} & u_{2t-1} \end{pmatrix} \quad \text{so that} \quad k^t h^{-1} g \equiv \begin{pmatrix} u_{2t+1} & u_{2t+2} \\ u_{2t} & u_{2t+1} \end{pmatrix}$$

$$\equiv \begin{pmatrix} 0 & u_{2t+2} \\ u_{2t} & 0 \end{pmatrix} \quad \text{in } G.$$

Since then $u_{2t+2}u_{2t} \equiv -1 \pmod{p}$, we have that $(k^t h^{-1} g)^2 \equiv 1$. Also, $k^r \equiv 1$, since $p|u_{2r}$, so that $(h^{-1}ghg)^t h^{-1} g \equiv (gh^{-1}gh)^t gh^{-1}$. Thus

$$1 \equiv [(h^{-1}ghg)^t h^{-1} g]^2 \equiv (h^{-1}ghg)^t h^{-1} g (gh^{-1}gh)^t gh^{-1}$$

$$\equiv (h^{-1}ghg)^t hg(hgh^{-1}g)^{t-1} hgh^{-1} .$$

Conjugating by hgh^{-1}, we get $1 \equiv (hg)^3 (h^{-1}ghg)^{t-1} hg(hgh^{-1}g)^{t-1}$. Thus $(hg)^3 (h^{-1}ghg)^{t-1} hg(hgh^{-1}g)^{t-1}$ is a $4t$-term word in N. Since $4t = 2r-2$, and $r|p+1$, $4t \leq p-1 < p$, so we again get that G does not act on a strict p-gon-graph $\mathcal{H}$ with $G_x = \langle h \rangle$ for some vertex x of $\mathcal{H}$ with $x^g \in \Delta(x)$.

Suppose $r = p+1$. The first value of $p > 5$ for which this is true is $p = 7$. It can indeed be shown that $PSL(2,7)$ acts on a heptagraph $\mathcal{H}$ of valency 3. The vertices of $\mathcal{H}$ are the cosets of $H = \langle h \rangle$ in G and the graph is defined with respect to the suborbit (double coset) HgH. $\mathcal{H}$ has 56 vertices and $Aut(\mathcal{H}) \simeq PGL(2,7)$.

It is not known whether or not the remaining groups $G = PSL(2,p)$ with $r = p+1$ (the next two values are $p = 23$ and $p = 43$) act on p-gon-graphs with $G_x = \langle h \rangle$ and $x^g \in \Delta(x)$.

Of course, it is possible that $G = PSL(2,p)$ acts on some strict m-gon-graph of valency 3 with $m \neq p$, and with $G_x \simeq Z_3$ for some vertex x of $\mathcal{H}$. An example of this is given by $PSL(2,13)$ acting on a heptagraph of valency 3 whose full automorphism group is $PGL(2,13)$. This example will be mentioned again in Section 4.

Example. It is known ([11] Section 7.6) that the group $PSL(2,8)$ is also a quotient of the group $G_1(7)$. It can be shown that this leads to a heptagraph $\mathcal{H}$ of valency 3 as above on which $PSL(2,8)$ acts. $\mathcal{H}$ has 168 vertices and $Aut(\mathcal{H}) \simeq Z_2 \times PSL(2,8)$. What is perhaps surprising is that, as we shall see in the next section, $PSL(2,8)$ is the full automorphism group of another 7-gon-graph of valency 3 on 84 vertices.

Section 4. Triply Transitive Point Stabilizers

In [11], Section 3.2, Coxeter and Moser define a map to be the "decomposition of an unbounded surface into N_2 non-overlapping

simply-connected regions (called faces) by N_1 arcs (called edges) joining pairs of N_0 points (called vertices). It is understood that no two edges have a common interior point, and that every edge belongs to just two faces." In their Chapter 8, a map is said to be regular of type $\{p,q\}$ if p edges belong to a face (i.e. the faces are p-gons) q edges to a vertex (i.e. the graph of vertices and edges of the map is regular of valency q), and if the automorphism group of the map contains an element which cyclically permutes the successive edges of one face and an element which cyclically permutes the successive edges meeting at one vertex of this face, so is transitive on vertices, edges and faces.

Since every edge of a map belongs to just two faces, whereas in an m-gon-graph of valency k every edge belongs to $k-1$ m-gons, we see that for $q > 3$ the vertices and edges of the (regular) maps $\{p,q\}$ are not strict polygonal graphs. However for $q = 3$, many of the known maps $\{p,3\}$ do turn out to be cubic polygonal graphs. For example, the maps $\{6,3\}_{2q}$ of [11], Table 8, are the graphs $\mathcal{H}(q,q)$ of Section 2 of this chapter.

In particular, the following maps of Table 8 of [11] were shown to be strict m-gon-graphs of valency 3 using the coset enumeration program of R. Scott [31]. In each case, the group given is the full automorphism group of the graph and is transitive on ordered 2-claws. (We only give those examples with $m \geq 7$.)

TABLE 2.1

EXAMPLES FROM MAPS

Map*	Group	Value of m	Number of Vertices
$\{7,3\}_8$	PGL(2,7)	7	56 (obtained in Section 3)
$\{7,3\}_9$	PSL(2,8)	7	84 (compare with Section 3)
$\{7,3\}_{14}$ or $\{7,3\}_{12}$	PGL(2,13)	7	364
$\{7,3\}_{13}$	PSL(2,13)	7	182
$\{7,3\}_{15}$	PSL(2,29)	7	2030
$\{8,3\}_8$	$Z_2 \times$ PGL(2,7)	8	112
$\{8,3\}_{10}$	$G^{3,8,10}$	8	720 (see below for $G^{3,8,10}$)
$\{8,3\}_{11}$	PGL(2,23)	8	2024
$\{9,3\}_{10}$	$\Sigma_3 \times$ PSL(2,19)	9	3420

*See [11], Chapter 8, Section 6, for the definition of a map of type $\{p,q\}_r$.

Other examples of strict m-gon-graphs of valency 3 for $m \geq 7$ are given in Table 2.2.

TABLE 2.2

OTHER EXAMPLES OF CUBIC POLYGONAL GRAPHS

Group	Value of m	Number of Vertices
$Z_2 \times PSL(2,8)$	7	168 (see Section 3)
$Z_2 \times PGL(2,23)$	8	4048 (see $\{8,3\}_{11}$ above)
$Z_2 \times PSL(2,19)$*	9	1140

*The map $\{9,3\}_9$, with group PSL(2,19), leads to a cubic graph with two nonagons on each 2-claw. Compare with the Petersen graph and the dodecahedral graph.

Now let $\mathcal{H}$ be a strict m-gon-graph of valency 3 and suppose that $G = Aut(\mathcal{H})$ is transitive on ordered 2-claws of $\mathcal{H}$. Let Ω be the vertices of $\mathcal{H}$. Let $0 \in \Omega$ and $\Delta(0) = \{1, 2, 3\}$, and let $H = G_0 \simeq \Sigma_3$ (by Lemma 1.5). As we had in Section 3 of this chapter (Lemma 2.9), we have the following.

Lemma 2.12. There is an involution $g \in G$ such that $G = \langle H, g \rangle$.

Proof. Let $\Delta(1) = \{0, 2', 3'\}$, with $P = (2, 0, 1, 2', \ldots)$ and $(3, 0, 1, 3', \ldots)$ the unique m-gons on $(0: 1, 2)$ and $(0: 1, 3)$, respectively. Then there is a $g \in G$ such that g carries the ordered 3-claw $(0: 1, 2, 3)$ onto the ordered 3-claw $(1: 0, 2', 3')$. Thus $P^g = (2', 1, 0, (2')^g, \ldots)$, so that by

uniqueness of P on $(1: 2', 0)$ we have $P = P^g$, whence $(2')^g = 2$. Similarly $(3')^g = 3$, and so g^2 fixes 0, 1, 2 and 3, and so $g^2 = 1$.

Since $\mathcal{H}$ is connected, $G = \langle H, g \rangle$. □

Let $H = \langle h, t \rangle$ where $h^3 = 1$, $t^2 = 1$ and $h^t = h^{-1}$, and we may suppose that $t = (0)(1)(2\ 3)(2'\ 3')\ \ldots$. Then $gt = (0\ 1)(2\ 3')(3\ 2')\ \ldots$ so that $(gt)^2 = 1$ and so g commutes with t.

Now if $t \notin \langle g,h \rangle$, then it is not difficult to see that $\langle g,h \rangle$ is a subgroup of G of index 2, transitive on Ω and such that $\langle g,h \rangle_0 = \langle h \rangle$, and thus the results of Section 3 and Theorem 2.1 apply to $\langle g,h \rangle$. An example of this type is given by the heptagraph coming from the map $\{7,13\}_{12}$ with group $PGL(2,13) = \langle A, B, C: A^3 = B^7 = C^{12} = (AB)^2 = (BC)^2 = (CA)^2 = (ABC)^2 = 1 \rangle$ ([11], Section 7.5). Here, $g = AB$, $h = A$, and $t = BC$, and it can be shown that $\langle g,h \rangle = \langle A,B \rangle$ has index 2 in $PGL(2,13)$, so that $t \notin \langle g,h \rangle \simeq PSL(2,13)$.

Consider the element ght. It is seen that

$$(ght)^n = \begin{cases} (ghgh^{-1})^{n/2}, & \text{if } n \text{ is even} \\ (ghgh^{-1})^{(n-1)/2}ght, & \text{if } n \text{ is odd.} \end{cases}$$

So if $|ght| = \ell$ is odd, then $t = (ghgh^{-1})^{(\ell-1)/2}gh \in \langle g,h \rangle$. If $\mathcal{H}$ is the graph of a map of type $\{m,3\}_\ell$ of [11], Chapter 8, then from Equation (8.61) of [11], we see that ℓ being odd

corresponds to non-orientability of the surface of the map. An example of this is the map $\{7,3\}_{13}$ with group PSL(2,13) which does not have a subgroup of index 2.

If $t \in \langle g,h \rangle = G$ and G does have a subgroup $\tilde{G}$ of index 2 which is transitive on Ω and such that $\tilde{G}_0$ is simply transitive on $\Delta(0)$, then it can be seen that $\tilde{G} = \langle h,gt \rangle$ and the results of Section 3 apply to $\tilde{G}$. An example of this is given by the octagraph coming from the map $\{8,3\}_{11}$, with group $PGL(2,23) = \langle A, B, C: A^3 = B^8 = C^{11} = (AB)^2 = (BC)^2 = (CA)^2 = (ABC)^2 = 1 \rangle$ ([11], Section 7.5). Here $g = AB$, $h = A$, and $t = BC$. It can be shown that $\langle g,h \rangle = \langle A,B \rangle = PGL(2,23)$, so $t \in \langle g,h \rangle$, while $\langle h,gt \rangle = \langle A,C \rangle = PSL(2,23)$ has index 2 in PGL(2,23).

Remark. The map $\{p,3\}_r$ has group $G = G^{3,p,r} = \langle A, B, C: A^3 = B^p = C^r = (AB)^2 = (BC)^2 = (CA)^2 = (ABC)^2 = 1 \rangle$, (see [11], Section 6 of Chapter 8). With $g = AB$, $h = A$ and $t = BC$, $H = \langle h,t \rangle \simeq \Sigma_3$, $g^t = g$, $(gh)^p = 1$, and $(ght)^r = 1$. With p even and r odd, it can be shown that $\langle g,h \rangle = \langle A,B \rangle = G$, and $\langle h,gt \rangle = \langle A,C \rangle$ has index 2 in G, so that $t \in \langle g,h \rangle$ and G has a subgroup of index 2 in G, transitive on the vertices of the map; with p odd and r even, $\langle g,h \rangle$ has index 2 in G, $\langle h,gt \rangle = G$, so that $t \notin \langle g,h \rangle$ and again G has a subgroup of index 2 transitive on the vertices of the map; if both p and r are odd then $\langle g,h \rangle = \langle h,gt \rangle = G$ and G does not have a subgroup of index 2 transitive on the vertices of the map; finally, if p and q are both even then $\langle g,h \rangle$ and $\langle h,gt \rangle$ both are sub-

groups of index 2 in G, transitive on the vertices of the map.

We conclude this chapter with the following result.

Theorem 2.2. Let A be the graph of vertices and edges of a regular map on an orientable surface. Suppose that $A \in \Pi_m$ for some $m \geq 6$, and $G = \mathrm{Aut}(A)$ is transitive on ordered 2-claws of A (i.e., G is transitive and $G_x^{\Delta(x)} \simeq \Sigma_3$ for any vertex x of A). If $m = 6$, 7 or 9, then the dual of A, $D(A) \in T_m$, and for each vertex α of $D(A)$, G_α is isomorphic with D_{2m}, the full group of automorphisms of link $(\alpha, D(A))$.

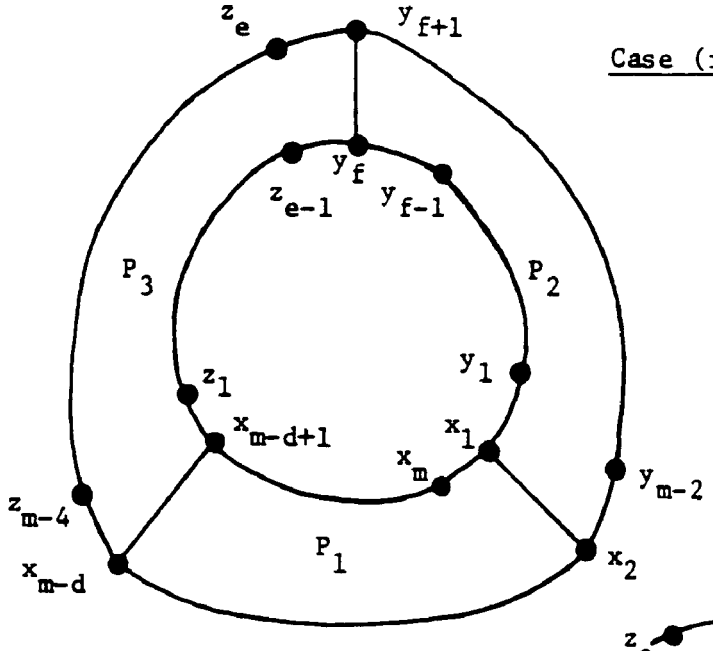
Proof. It is clear (see Lemma 2.3) that the valency of $D(A)$ is m. Also if (x: y,z) is any 2-claw of some m-gon P of A, then there are two distinct, unique m-gons R_1 and R_2, neither equal to P, on the edges (x, y) and (x,z) of P, respectively. So since the valency of x is 3, R_1 and R_2 have an edge in common. In this way we see that for α any vertex of $D(A)$, link$(\alpha, D(A))$ contains a subgraph $C(\alpha)$ which is an m-gon. Thus if there are no further adjacencies between any two vertices of link$(\alpha, D(A))$, we are done. So we may assume that there exist at least two points β and γ in link$(\alpha, D(A))$ with β adjacent to γ in link$(\alpha, D(A))$ but not adjacent in $C(\alpha)$. This implies that in A we have three distinct m-gons P_1, P_2, and P_3, such that the pairs P_1 and P_2, P_2 and P_3, P_3 and P_1, each have an edge in common, but P_1, P_2 and P_3 do not have a vertex in common.

Let $P_1 = (x_1, x_2, \ldots, x_m)$ and $P_2 = (x_2, x_1, y_1, \ldots, y_{m-2})$ be the m-gons P_1 and P_2. Then we can let either

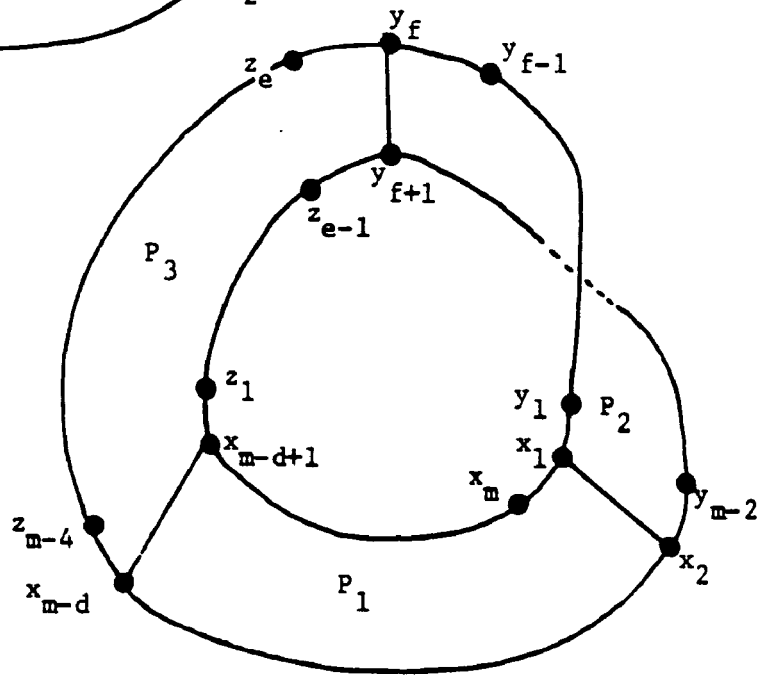
(i) $P_3 = (x_{m-d}, x_{m-d+1}, z_1, \ldots, z_{e-1}, y_f, y_{f+1}, z_e, \ldots, z_{m-4})$, or

(ii) $P_3 = (x_{m-d}, x_{m-d+1}, z_1, \ldots, z_{e-1}, y_{f+1}, y_f, z_e, \ldots, z_{m-4})$,

for some positive integers d, e and f. The following diagrams represent these cases.



Case (i)



Case (ii)

It is clear, since the surface of the map is orientable, that case (ii) cannot occur. For all such triples P_1, P_2 and P_3 of m-gons in A as in case (i), choose one triple for which d is minimal. Then $d \leq [\frac{m-2}{2}]$, where [n] denotes the greatest integer less than or equal to n.

If one of d or m-d-2 is even, then since G is transitive on ordered 2-claws, there is an element $g \in G$ which either fixes $x_{(2m-d+2)/2}$ and interchanges $x_{(2m-d)/2}$ with $x_{(2m-d+4)/2}$ (in the case d is even), or fixes $x_{(m-d+2)/2}$ and interchanges $x_{(m-d)/2}$ with $x_{(m-d+4)/2}$ (in the case m-d-2 is even). If both d and m-d-2 are odd, then there is a $g \in G$ which interchanges $x_{(2m-d+1)/2}$ with $x_{(2m-d+3)/2}$, and takes $x_{(2m-d-1)/2}$ to $x_{(2m-d+5)/2}$ (where, if d = 1 or 3, $x_{m+1} = x_1$, $x_{m+2} = x_2$). In either case, by uniqueness of m-gons on 2-claws, g fixes P_1, and interchanges the ordered 2-claw $(x_{m-d+1}: x_{m-d+2}, x_{m-d})$ with the ordered 2-claw $(x_1: x_m, x_2)$. Since the valency of A is three, we then see that g interchanges z_1 and y_1, and hence the ordered 2-claws $(x_{m-d+1}: z_1, x_{m-d})$ and $(x_1: y_1, x_2)$ as well.

Thus g interchanges P_2 and P_3, and so the image under g of the edge (y_f, y_{f+1}) is common to P_2 and P_3, whence by Lemma 2.2, $(y_f, y_{f+1})^g = (y_f, y_{f+1})$. Hence e = f.

Similarly, there is an element of G fixing P_2 and interchanging P_1 and P_3, whence d = e = f.

Now by uniqueness of m-gons on 2-claws, and since the girth of A is m, the circuit $(x_1, x_m, \ldots, x_{m-d+1}, z_1, \ldots z_{e-1}, y_f, y_{f-1}, \ldots, y_1)$ must have length greater than m, so

$3d \geq m+1$. Thus if m is odd, $(m+1)/3 \leq d \leq (m-3)/2$, giving $m \geq 11$, a contradiction. If m is even, $(m+1)/3 \leq d \leq (m-2)/2$, giving $m \geq 8$, a contradiction.

The second part of the theorem now follows easily. □

CHAPTER III

POLYGONAL GRAPHS OF VALENCY GREATER THAN THREE

In this chapter we construct some polygonal graphs of valency greater than 3, and investigate the possibility of further examples arising from the groups $PSL(2,q)$ for suitable prime powers q. All examples considered here, except for one, have automorphism groups whose one point stabilizers are 3-transitive on the points adjacent to that point. (Another example, failing to meet this latter condition, will be studied in Chapter V.)

The most obvious examples which come to mind are the recta-graphs of valency k, for all $k \geq 4$, consisting of the vertices and edges (i.e., the 1-skeleton) of the k-dimensional cube. These are investigated briefly in Section 1. Also in Section 1 we shall mention one more example of a polygonal graph of valency 4 (other than the 4-dimensional cube) which also arises from a regular solid, while in Section 2 we construct polygonal graphs of valency 5 from the action of the groups $PSL(2,q)$, for suitable prime powers q, on subgroups isomorphic with A_5. In Section 3 two more examples of polygonal graphs are given.

For convenience we have listed in Appendix 2 some of the groups which have been obtained in this chapter as automorphism groups of polygonal graphs, as well as (where known) the intersection matrices

which correspond to the suborbit with respect to which the polygonal graph is defined.

Section 1. Examples Arising From Regular Solids

For any $k \geq 1$, consider the following set Ω of points of k-dimensional real Euclidean space: $\Omega = \{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k): \varepsilon_i \in \{0,1\}$ for all i, $1 \leq i \leq k\}$. Then $|\Omega| = 2^k$ and Ω can be regarded as the set of vertices of a k-dimensional unit cube. On Ω define a graph $\mathcal{H}(k)$ by letting $x \in \Omega$ be adjacent to $y \in \Omega$ if and only if x and y differ in exactly one coordinate. Then it is clear that for $k \geq 2$, $\mathcal{H}(k)$ is a rectagraph of valency k, and that $\mathcal{H}(k)$ contains, as a subgraph, $\mathcal{H}(k')$ for any $1 \leq k' < k$. ($\mathcal{H}(3)$ is the rectagraph coming from the 3-dimensional cube mentioned in Section 1 of Chapter II.)

Let $e_i \in \Omega$ $(1 \leq i \leq k)$ be the k-tuple with 1 for the ith coordinate and 0 for every other coordinate. Denote by 0 the element of Ω with a zero for every coordinate. Then $\Delta(0) = \{e_1, \ldots, e_k\}$. Further, it is clear that any permutation of the k coordinate positions preserves adjacency in $\mathcal{H}$, and so if $G(k) = \mathrm{Aut}(\mathcal{H}(k))$ then (Lemma 1.5) $G(k)_0 \simeq \Sigma_k$.

Let s_i $(1 \leq i \leq k)$ be the permutation on Ω induced by pointwise addition, modulo 2, of e_i, i.e., $s_i: x \to x + e_i \pmod 2$. Then it is clear that $s_i^2 = 1$ and $s_i s_j = s_j s_i$ for all $1 \leq i \leq j \leq k$, and that each s_i preserves adjacency

in $\mathcal{H}$, and so $S = \langle s_i : i = 1, \ldots, k \rangle \leq G(k)$. Also, the s_i are independent functions on Ω and so S is elementary abelian of order 2^k and is clearly regular on Ω. Further $G(k)_0$ acts on the set $\{s_i\}$ and hence normalizes S, so that $G(k) = S.G(k)_0$.

Hence $\mathrm{Aut}(\mathcal{H}(k))$ is isomorphic with the wreath product $Z_2 \wr \Sigma_k$ of Z_2 by Σ_k of order $2^k.k!$ afforded by the obvious action of Σ_k on $Z_2^{\,k}$.

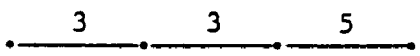
Another example of a rectagraph of valency $k \geq 5$ arising from $\mathcal{H}(k)$ is obtained as follows.

For any point $(\varepsilon_1, \ldots, \varepsilon_k)$ of $\mathcal{H}(k)$, the antipodal point is $(1-\varepsilon_1, \ldots, 1-\varepsilon_k)$ (or $(1, \ldots, 1) + (\varepsilon_1, \ldots, \varepsilon_k)$, modulo 2). Let $\mathcal{H}'(k)$, for $k \geq 5$, be the graph obtained from $\mathcal{H}(k)$ by identifying antipodal points. Then it can be checked that $\mathcal{H}'(k)$ is a rectagraph of valency k on 2^{k-1} vertices whose automorphism group is isomorphic with $\mathrm{Aut}(\mathcal{H}(k))/Z(\mathrm{Aut}(\mathcal{H}(k)))$ $[Z(\mathrm{Aut}(\mathcal{H}(k))) = s_1 s_2 \ldots s_k]$. Alternatively, $\mathrm{Aut}(\mathcal{H}'(k))$ is isomorphic with the semidirect product of $Z_2^{\,k-1}$ by Σ_k. (The action of Σ_k on $Z_2^{\,k-1}$ can be described as follows: Let $\{a_1, \ldots, a_{k-1}\}$ be a generating set for $Z_2^{\,k-1}$. Put $a_k = a_1 a_2 \ldots a_{k-1}$. Then it can be checked that $a_i \mapsto a_{i\sigma}$ is an automorphism of $Z_2^{\,k-1}$ for any $\sigma \in \Sigma_k$.)

We now consider another polygonal graph arising from a regular solid.

There is a pentagraph $\mathcal{H}$ of valency 4 with 600 vertices

and 720 pentagons obtained by considering the points and edges of the 4-dimensional polytope called the 120-cell (see H.S.M. Coxeter [12] and H.P. Manning [29]). This is a regular solid in 4-dimensional real space which has 120 dodecahedra as its 3-dimensional "faces" or "cells". The dual of this regular solid is the 600-cell which has 600 tetrahedra as its 3-dimensional cells. The vertices of the latter solid can be viewed as the roots of the root system whose Coxeter graph is given below.



(See Chapter 5 of C. T. Benson and L. C. Grove [5]; also compare the coordinates of the vertices of the 600-cell given in Section 8.7 of H. S. M. Coxeter [12] with the vectors of this root system.)

The 120 roots of this system can be regarded as a subset of the ring of quaternions (see C. T. Benson and L. C. Grove [5], Exercises 5.14, 5.15, and 5.16, and E. Witt [33]) and as such form a subgroup of the group of units of the quaternions isomorphic with $SL(2,5)$. The corresponding Coxeter group, which is also the group of the 120-cell, is a non-crystallographic reflection group.

Let $H = SL(2,5) \wr Z_2$, the wreath product of $SL(2,5)$ by Z_2. The elements of H can be written in the form (h_1, h_2) or $(h_1, h_2)t$, where $h_1, h_2 \in SL(2,5)$ and t is the involutory

automorphism of $SL(2,5) \times SL(2,5)$ which takes (h_1, h_2) to (h_2, h_1). The center of H, $Z(H)$, is $(-1,-1)$.

Consider the following four elements of $SL(2,5)$.

$$s_1 = \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}, \qquad s_2 = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix},$$

$$s_3 = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}, \quad \text{and} \quad s_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The following statements can be verified by direct calculation.

$$s_1^5 = 1, \quad \text{and} \quad s_2^2 = s_3^2 = s_4^2 = -1. \tag{3.1.1}$$

$$(-s_1^{-1}s_2)^3 = (s_1s_2)^3 = (s_2s_3)^3 = (s_3s_4)^5 = 1,$$

$$s_1s_3 = s_3s_1^{-1}, \qquad s_1s_4 = s_4s_1^{-1},$$

$$\text{and} \quad s_2s_4 = -s_4s_2. \tag{3.1.2}$$

$$(s_2s_3s_4)^5 = 1. \tag{3.1.3}$$

$$(s_1s_2s_3s_4)^3 = (s_1^{-1}s_2s_3s_4)^5 = -1. \tag{3.1.4}$$

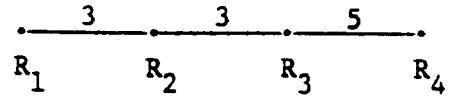
Let $r_1 = (s_1, -s_1^{-1})t$, $r_2 = (s_2, s_2)t$, $r_3 = (s_3, s_3)t$, and $r_4 = (s_4, s_4)t$, $r_1, r_2, r_3, r_4 \in H$. Then from (3.1.1) we see that $r_i^2 \in Z(H)$ for all $i = 1, 2, 3, 4$.

Let R be the subgroup of H generated by $\{r_i : i = 1,2,3,4\}$. From (3.1.3), $(r_2r_3r_4)^5 = t \in R$, and from (3.1.4), $(r_1r_2r_3r_4)^{15} = (-1, 1) \in R$.

Also from (3.1.4), $(r_1r_2r_3r_4)^{10} = (s_1s_2s_3s_4, 1) \in R$ and has order 3, and $(r_1r_2r_3r_4)^3 = (1, (-s_1^{-1}s_2s_3s_4)^3) \in R$ and has order 5. Conjugating this last element by $t \in R$ implies that $((-s_1^{-1}s_2s_3s_4)^3, 1) \in R$. But $((-s_1^{-1}s_2s_3s_4)^3, 1).(s_1s_2s_3s_4, 1)$ is an element whose square is $(-1, 1)$, and so these two elements generate the subgroup $\{(h, 1): h \in SL(2,5)\}$ of H. So conjugating by t shows that $R = H$.

Finally, from (3.1.2) we see that $(r_1r_2)^3 = (r_2r_3)^3 = (r_3r_4)^5 = 1$, and that modulo $Z(H)$, r_1 commutes with r_3 and r_4, and r_2 commutes with r_4.

So let $G = H/Z(H)$ and $R_i \in G$ the images of $r_i \in H$ for $i = 1, 2, 3, 4$. Then we have shown that G is a reflection group whose Coxeter graph (or Dynkin diagram) is shown below:



G has order $(120)^2$ and is the automorphism group of the previously mentioned pentagraph $\mathcal{H}$ of valency 4. Further, the stabilizer of a point of $\mathcal{H}$ is isomorphic with Σ_4. Note too, that $\mathcal{H}$ contains, as subgraphs, dodecahedral graphs of valency 3.

Section 2. Valency 5: Examples Arising From PSL(2,q)

Consider $G_q = PSL(2, q)$, where $q = p^n$ for some prime p and integer $n \geq 1$, and suppose that $q \equiv \pm 1 \pmod 5$ and $q \equiv \pm 1 \pmod 8$ (or equivalently $q^2 \equiv 1 \pmod{80}$) and that $3 \nmid q$. Then there is

a subgroup $H \leq G_q$ with $H \simeq A_5$ and $N_{G_q}(H) = H$ (see L. E. Dickson [14]). Consider the action of G_q on the set Ω of right cosets of H in G, so that $|\Omega| = q(q-1)(q+1)/120$.

Present H as a group of permutations on $\{1, 2, 3, 4, 5\}$ in the usual way and let $K = H_{\{5\}} \simeq A_4$.

Since $q \equiv \pm 1 \pmod 8$, $N_G(K) \simeq \Sigma_4$ (L. E. Dickson [14]) and we may present $N_G(K)$ as a permutation group on $\{1, 2, 3, 4\}$ in the usual way, consistent with the previous action of K on $\{1, 2, 3, 4, 5\}$.

Let $g = (1\ 2) \in N_{G_q}(K) - H$, so $|g| = 2$, $g \in N_{G_q}(L)$ and $H \cap H^g = K$.

Let $\alpha = Hg \in \Omega$ and define a graph $\mathcal{H}_q$ with respect to the suborbit of α under H, i.e., whose vertices are the elements of Ω and whose edge set is $\{(Hx, Hgx): x \in G\}$. Then $G_q \leq \mathrm{Aut}(\mathcal{H}_q)$, and we shall see that, at least for certain values of q, and possibly for many other q, the graphs $\mathcal{H}_q$ are polygonal graphs.

Lemma 3.1. The valency of $\mathcal{H}_q$ is 5.

Proof. As coset representatives for K in H we may take $\{h_0 = 1, h_1 = h, h_2, h_3, h_4\}$ where each h_i $(1 \leq i \leq 4)$ is an involution in the normalizer (in H) of one of the four distinct Sylow 3-subgroups L_i $(i = 1, 2, 3, 4)$ of K; viz. let $L = L_1 = \langle (1\ 2\ 3) \rangle$, $L_2 = \langle (1\ 3\ 4) \rangle$, $L_3 = \langle (1\ 2\ 4) \rangle$, and $L_4 = \langle (2\ 3\ 4) \rangle$, and let $h_1 = h = (1\ 2)(4\ 5)$, $h_2 = (1\ 3)(2\ 5)$, $h_3 = (1\ 2)(3\ 5)$, and $h_4 = (2\ 3)(1\ 5)$. Consider the orbit of α under H. Since

each element of H can be written as kh_i, $0 \leq i \leq 4$, for some $k \in K$, we have $\alpha^{kh_i} = Hgkh_i = Hgh_i$. Further, $Hgh_i = Hgh_j \Rightarrow h_i h_j^{-1} \in H \cap H^g = K \Rightarrow Kh_i = Kh_j \Rightarrow i = j$. Thus $HgH = \bigcup_{i=0}^{4} Hgh_i$.

So the orbit of α under H consists of the five cosets Hg, Hgh, Hgh_2, Hgh_3, Hgh_4, and H acts on this orbit as a permutation group on 5 letters in the usual way. Thus these latter 5 cosets are in fact $\Delta(\alpha)$ and the valency of $\mathcal{H}_q$ is 5. □

Note that $\mathcal{H}_q$ is undirected, and is connected if and only if $G_q = \langle H, g \rangle$. Also, since $|N_{G_q}(K):K| = 2$, there is only one suborbit of length 5 and so $\mathcal{H}_q$ is independent of the choice of $g \in N_{G_q}(K) - H$.

For any $x \in G_q$, let $N(x)$ be the number of points of Ω fixed by x. x fixes a coset H_y $(y \in G_q)$ if and only if $x \in H^y$, so $N(x)$ is the number of conjugates of H which contain x. Now let $z = (1\ 2\ 3) \in H$. Then z fixes exactly two points adjacent to H, viz. α and $\beta = Hgh$, since $\langle g, h \rangle \leq N(\langle z \rangle)$. So $z \in H^g$ and H^{gh}, and thus z fixes exactly two points adjacent to α (one of which is H) and two points adjacent to β (again one is H). Thus, continuing in this way we see that the fixed points of z lie in (possibly many, disjoint) simple circuits of $\mathcal{H}_q$. Clearly these circuits (if there are more than one) all have the same length so in particular if $N = N(z)$ is prime, then there is just one circuit of fixed points of z.

To calculate N, we count the number of pairs (z', H'), where $z' \in H'$, $z' \underset{G}{\sim} z$, and $H' \underset{G}{\sim} H$, in two ways.

If $z' \underset{G}{\sim} z$, then $N(z') = N$. There are $q(q+(-1)^{\varepsilon})$ conjugates of z in $PSL(2,q)$, where $\varepsilon = 0$ or 1 is defined by $q \equiv (-1)^{\varepsilon} \pmod 3$, and each lies in N conjugates of H. Also there are $q(q-1)(q+1)/120$ conjugates of H, each of which contains 20 conjugates of z.

Hence, $Nq(q+(-1)^{\varepsilon}) = 20[q(q-1)(q+1)/120]$, and therefore

$$N = \frac{q - (-1)^{\varepsilon}}{6} . \tag{3.2.1}$$

Remark. It is simple to show that those values of q, $q^2 \equiv 1 \pmod{80}$, for which N is prime satisfy the following congruences $q = 31$ and $N = 5$; $q \equiv 79 \pmod{120}$ and $N \equiv 13 \pmod{20}$; $q \equiv 41 \pmod{120}$ and $N \equiv 7 \pmod{20}$.

Now consider the following elements t, z, h and g of $PSL(2,\mathbb{C}) = SL(2,\mathbb{C})/C$, where $C = \left\langle\!\left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle\!\right\rangle$ is the center of $SL(2,\mathbb{C})$.

$$t = \frac{1}{\sqrt{-3}} \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & -1 \end{pmatrix} C ,$$

$$z = \frac{1}{2} \begin{pmatrix} -1 + \sqrt{-3} & 0 \\ 0 & -1 - \sqrt{-3} \end{pmatrix} C ,$$

$$h = \frac{\sqrt{2}}{4} \begin{pmatrix} 0 & -\sqrt{5} + \sqrt{-3} \\ \sqrt{5} + \sqrt{-3} & 0 \end{pmatrix} C , \text{ and} \tag{3.2.2}$$

$$g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} C .$$

Then we have the following:

$|t| = |h| = |g| = 2$, $|z| = 3$, and $\langle z \rangle^h = \langle z \rangle$. Also $|tt^z| = 2$, so that $K = \langle t,z \rangle \simeq A_4$.

Let $w = tz$, so that $|w| = 3$, and let $v = htz = hw$. Then elementary but tedious calculations show that

(i) $|v| = 5$ and so $H = \langle h,w \rangle \simeq A_5$, and

(ii) $z = v^{-2}whw^{-1} \in H$, so that $H = \langle K,h \rangle$.

Further, $z^g = (z)^{-1}$ and $t^g = t$, so g normalizes, but does not centralize K, and so $g \notin H$ and $\langle K,g \rangle \simeq \Sigma_4$.

We thus have faithful projective representations of A_5 and Σ_4 in $PSL(2, \mathbb{Q}(\sqrt{10}, \sqrt{-6}))$. [In the usual presentations of A_5 and Σ_4 as permutation groups on the sets of integers $\{1,2,3,4,5\}$ and $\{1,2,3,4\}$, respectively, we can identify the above elements as follows: $t = (1\ 2)(3\ 4)$, $z = (1\ 2\ 3)$, $h = (1\ 2)(4\ 5)$, and in its action on K (but not on the rest of H), $g = (1\ 2)$. When convenient we shall view the elements t, z, h and g in this way.]

We again consider the groups $G_q = PSL(2,q)$, $q^2 \equiv 1 \pmod{80}$, and $3 \nmid q$.

Lemma 3.2. 5 and 2 are non-zero squares in $GF(q)$, the field of q elements, and if $q \equiv 1 \pmod 3$, so is -3.

Proof. By the congruences on q, $p \neq 2$, 3, or 5. Let ω be a generator for the multiplicative group $GF(q)^* = GF(q) - \{0\}$ of the field. Then $\omega^{\frac{q-1}{p-1}}$ generates $GF(p)^*$, the multiplicative group

of the prime field GF(p) in GF(q). Now

$$\frac{q-1}{p-1} = \frac{p^n-1}{p-1} = p^{n-1} + \ldots + p+1 \equiv n \pmod 2 \quad \text{since } p \text{ is odd.}$$

So if n is even, $\frac{q-1}{p-1} \equiv 0 \pmod 2$ and all elements of GF(p)* are squares in GF(q), and we are done in this case. So suppose n is odd.

If $q \equiv 1 \pmod 3$ then $p \equiv 1 \pmod 3$ and then the Legendre symbol $(\frac{-3}{p})$ equals 1 (E. G. H. Landau [27]), so -3 is a square in GF(p) and thus in GF(q).

Similarly, $p \equiv \pm 1 \pmod 5$, and $(\frac{5}{p}) = 1$, whence 5 is a square in GF(p). Also, $p \equiv \pm 1 \pmod 8$ and so $(\frac{2}{p}) = 1$ and 2 is also a square in GF(p). □

Now suppose that $q \equiv 1 \pmod 3$, so that by Lemma 3.2 we can choose $\sqrt{2}$, $\sqrt{5}$ and $\sqrt{-3}$ in GF(q). Then we may regard the elements t, h, g and z as elements of G_q, and H and K as subgroups of G_q. So let $H = H$, $K = K$, $h = h$, $z = z$, and $g = g$ be the previously defined subgroups and elements of G_q (and write t for t), and consider the graph $\mathcal{H}_q$ whose vertices Ω are the right cosets of H in G_q, as before.

It is worth noting at this stage that although there are two conjugacy classes of subgroups isomorphic with A_5 in G_q, they are all conjugate in PGL(2,q), so that up to graph isomorphism, the graph $\mathcal{H}_q$ is independent of the choice of $H \simeq A_5$ in G_q.

Now the element $z \in H$ has order 3 and as noted before its fixed points lie in disjoint simple circuits, so consider the

circuit of fixed points of z which contains the 2-claw $(H: Hg, Hgh)$. The vertex $H(gh)^2$ is adjacent to Hgh, since $(H, Hgh)^{gh} = (Hgh, H(gh))^2$, is fixed by z, and therefore lies in the circuit. Similarly so is the vertex $H(hg)^2$ adjacent to $Hg = Hhg$, and in the curcuit, and in fact we can see in a similar fashion that all the vertices in this curcuit can be written as cosets $H(hg)^i$ or $H(gh)^i$ for some i. So if the length of this circuit is ℓ, then $H(gh)^\ell = H$ and thus $(gh)^\ell \in H$. But gh centralizes z, so $(gh)^\ell \in C_H(\langle z \rangle) = \langle z \rangle$. Thus $|gh| = \ell$ or 3ℓ.

Now $$gh = \frac{\sqrt{2}}{4}\begin{pmatrix} \sqrt{5} + \sqrt{-3} & 0 \\ 0 & \sqrt{5} - \sqrt{-3} \end{pmatrix} C \quad .$$

Let $\eta = \dfrac{\sqrt{5} + \sqrt{-3}}{\sqrt{8}} \in \mathbb{Q}(\sqrt{10}, \sqrt{-6})$.

Using the fact that $\eta^\ell = \pm 1$ or $\eta^{3\ell} = \pm 1$ in $GF(q)$, one can proceed by direct calculation to find the possible values of the characteristic p of $GF(q)$ for a given value of ℓ. However, the following lemma, for which the author wishes to thank J.N. Ridley [30], allows the calculation of p to be carried out much more readily, especially if a calculator or computer is available.

Lemma 3.3. Define sequences $\{A_m\}$, $\{a_m\}$, $\{B_m\}$ and $\{b_m\}$ $(m \geq 1)$ by

$$\eta^{2m-1} = \frac{A_m\sqrt{5} + a_m\sqrt{-3}}{2^m\sqrt{2}} \quad , \quad \text{and}$$

$$\eta^{2m} = \frac{B_m + b_m\sqrt{-15}}{2^{m+1}} \quad .$$

Then (i) A_m, a_m, B_m, and b_m are integers for all $m \geq 1$, and a_m and b_m satisfy the following recurrence relations: $a_{m+2} = a_{m+1} - 4a_m$, $a_1 = 1$, $a_2 = 3$, and $b_{m+2} = b_{m+1} - 4b_m$, $b_1 = 1$, $b_2 = 1$.

(ii) Furthermore, if K is a finite field of characteristic relatively prime to 30, and containing $\sqrt{5}$, $\sqrt{-3}$ and $\sqrt{2}$, then

$\eta^{2m-1} = \pm 1$ in K if and only if $\text{char } K \mid a_m$, and

$\eta^{2m} = \pm 1$ in K if and only if $\text{char } K \mid b_m$.

Proof. Direct calculation shows that $A_1 = 1$, $A_2 = -1$, $a_1 = 1$, $a_2 = 3$, and, since

$$\frac{A_{m+1}\sqrt{5} + a_{m+1}\sqrt{-3}}{2^{m+1}\sqrt{2}} = \frac{A_m\sqrt{5} + a_m\sqrt{-3}}{2^m\sqrt{2}}\ \eta^2$$

$$= \left(\frac{A_m\sqrt{5} + a_m\sqrt{-3}}{2^m\sqrt{2}}\right)\left(\frac{1 + \sqrt{-15}}{4}\right)$$

$$= [\tfrac{1}{2}(A_m - 3a_m)\sqrt{5} + \tfrac{1}{2}(5A_m + a_m)\sqrt{-3}]/2^{m+1}\sqrt{2} \ ,$$

it follows that $\begin{pmatrix} A_{m+1} \\ a_{m+1} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -3 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} A_m \\ a_m \end{pmatrix}$.

It now follows that the sequences $\{A_m\}$ and $\{a_m\}$ both satisfy the recurrence relation with polynomial $X^2 - X + 4$, so, since A_1, A_2, a_1, a_2 are all integers, so must A_m and a_m be for all $m \geq 1$.

The result for B_m and b_m is proved similarly.

(ii) From elementary complex number theory we see that

$$\left(\frac{A_m\sqrt{5} + a_m\sqrt{-3}}{2^m\sqrt{2}}\right)^{-1} = \frac{A_m\sqrt{5} - a_m\sqrt{-3}}{2^m\sqrt{2}} ,$$

so that $\eta^{2m-1} = \pm 1$ in K if and only if $\eta^{2m-1} = (\eta^{2m-1})^{-1}$ if and only if $a_m = 0$.

The result for b_m is proved similarly. □

The first few values of a_m and b_m are tabulated in Table 3.1 and Table 3.2, respectively.

TABLE 3.1

VALUES OF a_m

m	1	2	3	4	5	6	7	8	9	10	11
a_m	1	3	-1	-13	-3^2	43	79	-3.31	-409	-37	3.13.41
$2m-1$	1	3	5	7	9	11	13	15	17	19	21

TABLE 3.2

VALUES OF b_m

m	1	2	3	4	5	6	7	8	9
b_m	1	1	-3	-7	5	3.11	13	-119	$-3^2.19$
$2m$	2	4	6	8	10	12	14	16	18

Theorem 3.1. Let $q^2 \equiv 1 \pmod{80}$, $3 \nmid q$, $q = p^n$ and G_q, $\mathcal{H}_q$, H, K, g, h, z be as defined for Lemma 3.1. Let $\varepsilon = 0$ or 1 with $q \equiv (-1)^{\varepsilon} \pmod 3$. If a simple circuit of fixed points of z has length ℓ, then if ℓ is odd, $p \mid a_{\frac{1}{2}(\ell+1)}$ or $p \mid a_{\frac{1}{2}(3\ell+1)}$, while if ℓ is even, $p \mid b_{\ell/2}$ or $p \mid b_{3\ell/2}$. In either case $\ell \left| \frac{q-(-1)^{\varepsilon}}{6} \right.$.

Proof. If $q \equiv 1 \pmod 3$, then the theorem follows from equation (3.2.1), Lemma 3.3, and the material just preceding Lemma 3.3. So suppose $q \equiv -1 \pmod 3$, then $q^2 \equiv 1 \pmod 3$ and we may embed $PSL(2,q)$ in $PSL(2,q^2)$. Then all the previous results hold for $PSL(2,q^2)$, and since all subgroups of $PSL(2,q^2)$ isomorphic with A_5 are conjugate in $PGL(2,q^2)$, $\mathcal{H}_q$ is an induced subgraph of $\mathcal{H}_{q^2}$ and we get the conclusion of the proposition for G_q in this case as well. □

Let $G = G_q$. Recall that $\alpha = Hg$ and $\beta = Hgh$, so that $G_\alpha = H^g$, $G_\beta = H^{gh}$, and $G_{\alpha\beta} = H^g \cap H^{gh}$.

Lemma 3.4. Assume the hypotheses of Theorem 3.1 (or Lemma 3.1). Except for $p = 11$, $|G_{\alpha\beta}| = 3$, so that $|\Delta_2(H)| = 20$, and the girth of $\mathcal{H}_q$ is ≥ 5.

Proof. $|G_{\alpha\beta}| = 3 \Leftrightarrow H \cap H^{ghg} = L = \langle z \rangle$, for, $|G_{\alpha\beta}| = |G_{\alpha\beta}^g| = |H \cap H^{ghg}|$, and clearly $L \leq H \cap H^{ghg}$ since $ghg \in N_G(L)$.

Now $H \neq H^{ghg}$, for otherwise $h^g \in N(H) = H$ which would imply that g normalizes $\langle K,h \rangle = H$. Suppose for the purposes

of obtaining a contradiction that $L < H \cap H^{ghg} < H$. Now the only subgroups of H which contain L properly are:

(i) $N_H(L) = \langle h \rangle L$ of order 6;

(ii) K of order 12; and

(iii) K^h of order 12.

We consider each of these separately.

(i) $H \cap H^{ghg} = \langle h \rangle L \simeq \Sigma_3$. Then $(\langle h \rangle L)^{ghg} \leq H$, and $L < (\langle h \rangle L)^{ghg}$ which has order 6. Thus $(\langle h \rangle L)^{ghg} = \langle h \rangle L$ and so $h^g \in N_G(\langle h \rangle L)$. Hence $h^{h^g} = h, zh$ or $z^{-1}h$, and so $(gh)^4 \in \langle z \rangle$. But then z fixes a simple circuit of length 4, so by Theorem 3.1, $p|b_2$ or $p|b_6$. A glance at Table 3.2 shows that the only possibility is $p = 11$.

(ii) $H \cap H^{ghg} = K$. Then K fixes the vertices H, Hg and $Hghg$ of $\mathcal{H}_q$. But since $|N_G(K):K| = 2$ it is a simple matter to see that K fixes exactly 2 points of Ω. Thus $H = Hghg$ or $Hg = Hghg$. The latter is impossible since $H \neq Hg$, while the former is also impossible since $h^g \in H$ would imply that g normalizes $\langle K,h \rangle = H$, a contradiction.

(iii) $H \cap H^{ghg} = K^h$. Then K^h fixes the vertices H, Hgh, and $Hghg$, so as in the preceding case we have that $H = Hghg$ or $Hgh = Hghg$. The former possibility leads to a contradiction as before. In the latter case we have that $(gh)^3 \in C_H(\langle z \rangle) = \langle z \rangle$, and so z fixes a simple curcuit of length 3. Thus, by Theorem 3.1, $p|a_2$ or $p|a_5$. From Table 3.1 we see that this implies

that $p = 3$, a contradiction.

What we have in fact shown is that $|G_{H,Hghg}| = 3$, and since $Hghg \in \Delta_2(H)$, $|\Delta_2(H)| = |H: G_{H,Hghg}| = 20$. Now by counting the number of pairs (γ,δ), with $\gamma, \delta \in \Omega$, $\gamma \in \Delta(H)$, $\delta \in \Delta_2(H)$ and $\gamma \sim \delta$ in two ways, we see that each $\delta \in \Delta_2(H)$ is adjacent to exactly one $\gamma \in \Delta(H)$ and so the girth of $\mathcal{H}_q$ is ≥ 5. □

We now determine some polygonal graphs from the graphs $\mathcal{H}_q$.

Theorem 3.2. $\mathcal{H}_{31}$ is a pentagraph of valency 5 with $G = G_{31} = PSL(2,31) \leq Aut(\mathcal{H}_{31})$ transitive on Ω and such that the stabilizer G_α of a point α of $\mathcal{H}_{31}$ is 3-transitive on $\Delta(\alpha)$. Further, if a graph $\mathcal{H}_q$ is a pentagraph, then $q = 31$.

Proof. $\mathcal{H}_{31}$ is connected since H is maximal in G_{31} and so $G_{31} = \langle H,g \rangle$. By equation (3.2.1) $N = 5$ and so $\mathcal{H}_{31}$ contains a pentagon fixed pointwise by the element z of order 3 in H. Thus by Lemma 3.4, the girth of $\mathcal{H}_{31}$ is 5. We now show the uniqueness of pentagons on 2-claws.

Lemma 3.5. Let $\gamma = H \in \Omega$, and δ_1, δ_2 any two points of Ω adjacent to γ in the graph $\mathcal{H}_{31}$. Then there is a unique pentagon in $\mathcal{H}_{31}$ on the 2-claw $(\gamma: \delta_1,\delta_2)$.

Proof. In $H = G_\gamma$ there is an element w of order 3 which fixes γ, δ_1 and δ_2, and hence a pentagon $(\delta_1,\gamma,\delta_2,\varepsilon_2,\varepsilon_1)$ say, containing the 2-claw $(\gamma: \delta_1, \delta_2)$, where $\varepsilon_1, \varepsilon_2 \in \Delta_2(\gamma)$.

We claim that this is the unique pentagon in $\mathcal{H}_{31}$ containing this 2-claw.

If not, there is another pentagon on γ, δ_1, and δ_2. Since w fixes just 5 points of Ω, there are in fact 3 more pentagons on $(\gamma: \delta_1, \delta_2)$, permuted transitively by w. Furthermore if $(\delta_1, \gamma, \delta_2, \varepsilon_3, \varepsilon_4)$ is one of these, it is clear that $\varepsilon_3 \neq \varepsilon_2$, and $\varepsilon_4 \neq \varepsilon_1$, since $\mathcal{H}_{31}$ has no rectangles and $\varepsilon_3 \neq \varepsilon_1$, $\varepsilon_4 \neq \varepsilon_2$ since there are no triangles.

Now since H is 3-transitive on $\Delta(\gamma) = \{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5\}$, it is clear that for any two points $\delta_i \neq \delta_j$ there are 4 pentagons containing them (and γ too since there are no rectangles). So by considering the 4 pentagons on each of the pairs of points δ_1 and δ_i, $i = 2, 3, 4, 5$, it can be seen that G has rank 3 and degree 26 on Ω. No such rank 3 group exists (D. G. Higman [19]), and anyway the degree of G is $|G_{31}: 60| = 248$. This contradiction proves the lemma. □

So now the first statement of the theorem follows.

Now suppose that $\mathcal{H}_q$ is a pentagraph. Then there is a unique pentagon on the 2-claw $(H: Hg, Hgh)$ which must therefore be fixed by z, and so the circuit of fixed points of z is a pentagon. Hence by Theorem 3.1 and Table 3.1 we see that $p = 31$ and $q = 31^n$. Now since $\mathcal{H}_q$ is connected, $G_q = \langle H, g \rangle \simeq PSL(2,31)$. Thus $q = 31$ and the theorem has been proved. □

Remark. It is readily seen that $\mathcal{H}_q$ is an induced subgraph of $\mathcal{H}_q k$, for any integer $k \geq 1$. Further if $\mathcal{H}_q$ is connected then it is a connected component of $\mathcal{H}_q k$; in any event the connected components of $\mathcal{H}_q$ are connected components of $\mathcal{H}_q k$ and the vertices of $\mathcal{H}_q$ form a block of imprimitivity for the action of $G_q k$ on the vertices of $\mathcal{H}_q k$.

Now the question that naturally arises is what is $\mathrm{Aut}(\mathcal{H}_q)$? The following lemma will be used to show that at least for the pentagraph $\mathcal{H}_{31}$ we have $\mathrm{Aut}(\mathcal{H}_{31}) \simeq G_{31}$.

Lemma 3.6. $\mathcal{H}_{31}$ does not contain a cubic pentagraph as a subgraph.

Proof. Recall from Chapter II that a cubic pentagraph is a dodecahedral graph with automorphism group isomorphic with $A_5 \times Z_2$. Suppose that Γ is a dodecahedral graph which is a subgraph of $\mathcal{H}_{31}$. By Theorem 3.2, $G = G_{31}$ is transitive on ordered 3-claws of $\mathcal{H}_{31}$ so if C_1 and C_2 are any two ordered 3-claws of Γ there is an element $f \in G$ such that $C_1{}^f = C_2$. But then $\Gamma^f \cap \Gamma$ contains C_2 and so by Lemma 1.2, $\Gamma^f \cap \Gamma$ is also a dodecahedral graph.

Thus $\Gamma^f \cap \Gamma = \Gamma$ and so $\Gamma^f = \Gamma$. (This could also have been proved by observing that Γ is uniquely determined by any of its 3-claws.) So $G_\Gamma{}^\Gamma$ is transitive on Γ and transitive on ordered 3-claws of Γ, and hence $G_\Gamma{}^\Gamma \simeq A_5 \times Z_2$. By a Sylow 2-subgroup of PSL(2,31) does not contain a section isomorphic with

$Z_2 \times Z_2 \times Z_2$ (which is isomorphic with a Sylow 2-subgroup of $G_\Gamma{}^\Gamma$), and so we have a contradiction.

Remark. The conclusions of Theorem 3.2, and Lemma 3.6 are sufficient to characterize the group PSL(2,31) and the graph $\mathcal{H}_{31}$. This will be done in Chapter IV, Section 3 (Theorem 4.3 and Corollary 4.6).

The intersection matrix for $\mathcal{H}_{31}$ is given in Appendix 2.

Theorem 3.3. $\mathrm{Aut}(\mathcal{H}_{31}) = G_{31}$.

Proof. Let $G = \mathrm{Aut}(\mathcal{H}_{31})$, and $\alpha \in \Omega$. Since $G_{31} \leq G$, we have that $G_\alpha^{\Delta(\alpha)}$ is isomorphic to a subgroup of Σ_5 containing A_5. So suppose $G_\alpha^{\Delta(\alpha)} \simeq \Sigma_5$.

Let $s \in G_\alpha^{\Delta(\alpha)}$ be an involution fixing 3 points of $\Delta(\alpha)$. Then by Lemma 1.4, the connected component of $\Omega(s)$ which contains α is a cubic pentagraph. This contradicts Lemma 3.6.

Hence $G_\alpha^{\Delta(\alpha)} \simeq A_5$, and by Lemma 1.5, $G_\alpha \simeq A_5$, whence $|G| = |G_\alpha||\Omega| = |G_{31}|$, and $G = G_{31}$. □

Analogous to Theorem 3.2, Lemma 3.6, and Theorem 3.3, we have the following.

Theorem 3.4. Let $\mathcal{H}_{11}'$ be that connected component of $\mathcal{H}_{11}{}^2$ which contains the vertex $H \in \Omega$. Then $\mathcal{H}_{11}'$ is a rectagraph of valency 5 on 22 vertices with $G = \mathrm{PGL}(2,11) \leq \mathrm{Aut}(\mathcal{H}_{11}')$ tran-

sitive on $\Omega' \subseteq \Omega$, where Ω' is the set of vertices of $\mathcal{H}_{11}$, and such that the stabilizer G_α of a vertex α of $\mathcal{H}_{11}$ is 3-transitive on $\Delta(\alpha)$. Further, if a graph $\mathcal{H}_q$ has a connected component $\mathcal{H}'_q$ which is a rectagraph, then $q = 11^{2k}$ for some integer $k \geq 1$ and $\mathcal{H}'_q$ is isomorphic with $\mathcal{H}_{11}$.

Proof. The proof follows by first observing that although PSL(2,11) does not contain a subgroup isomorphic to Σ_4, PGL(2,11) does [14] and so $\langle H, g \rangle \simeq PGL(2,11) = G$; secondly by noting that z lies in exactly four G conjugates of H so that z fixes precisely four points of $\mathcal{H}_{11}$; and finally an easy argument then shows that this rectangle of fixed points of z is unique on any one of its 2-claws. The second statement of the proposition was shown in Lemma 3.4. □

Lemma 3.7. $\mathcal{H}_{11}$ does not contain a cubic rectagraph (i.e. the vertices and edges of the 3-dimensional cube) as a subgraph.

Proof. The proof follows as in Lemma 3.6 since the automorphism group of the cube is isomorphic with $\Sigma_4 \times Z_2$ which contains subgroups $Z_2 \times Z_2 \times Z_2$, whereas PGL(2,11) has dihedral Sylow 2-subgroups. □

We now have, as before,

Theorem 3.5. $\mathrm{Aut}(\mathcal{H}'_{11}) = PGL(2,11)$. □

Remark. $\mathcal{H}_{11}'$ is also the incidence graph of the unique 2 - (11, 5, 2) design (see for example, P. J. Cameron [10], Theorem 4.5).

The intersection matrix for $\mathcal{H}_{11}'$ is given in Appendix 2.

The following result was verified by direct calculation working with 2×2 matrices over the field of 41 elements.

Theorem 3.6. $\mathcal{H}_{41}$ is a heptagraph with $|\mathrm{Aut}(\mathcal{H}_{41}) : \mathrm{PSL}(2,41)| = 1$ or 2, and $\mathrm{Aut}(\mathcal{H}_{41})$ transitive on 3-claws of $\mathcal{H}_{41}$. □

The question as to whether any other of the graphs $\mathcal{H}_q$ (or connected components thereof) are polygonal graphs will be answered in the affirmative if we can show that the girth of $\mathcal{H}_q$ is m, where m is the length of circuits of fixed points of elements of order 3 (m is uniquely defined because all elements of order 3 are conjugate in G_q). For then we can take for the distinguished set $\mathcal{E}$ of m-gons these circuits of fixed points of elements of order 3. (Of course it is just possible that for certain of the $\mathcal{H}_q$ the girth will be $\bar{m}$ less than m, but it may still be possible to choose a set $\mathcal{E}$ of $\bar{m}$-gons so as to make $\mathcal{H}_q$ an $\bar{m}$-gon-graph.)

In order to ensure that the graphs are connected we observe the following:

If $p \equiv 1 \pmod{80}$, $\mathcal{H}_{p^n}$ is connected if and only if $n = 1$.

If $p \not\equiv \pm 1 \pmod{5}$, $\mathcal{H}_{p^n}$ is connected if and only if $n = 2$.

(Of course in this case n must be even to define $\mathcal{H}_{p^n}$.)

If $p \equiv \pm 1 \pmod 5$, but $p \not\equiv \pm 1 \pmod 8$, then $\mathcal{H}_{p^k}$ (for k even) is never connected, but if we construct graphs $\mathcal{H}'_{p^n}$ of valency 5 from the groups $PGL(2, p^n)$ as we constructed $\mathcal{H}_{p^n}$ before (which we can do as $PGL(2,p)$ contains subgroups isomorphic to Σ_4 [14]), then $\mathcal{H}'_{p^n}$ is connected if and only if $n = 1$, and in fact $\mathcal{H}'_p$ is a connected component of $\mathcal{H}_{p^k}$.

In the event that a graph $\mathcal{H}_q$ or $\mathcal{H}'_q$ is an m-gon-graph, the question arises as to whether or not it contains as a subgraph a strict m-gon-graph of valency 3. The importance of this will become apparent in Chapter IV (Theorem 4.2; see also Lemmas 3.6 and 3.7). I do not know the answer to this question even in the case of the heptagraph of Theorem 3.6.

We conclude this section with some remarks concerning the case $p = 3$. As $9^{2n} \equiv 1 \pmod{80}$, we can construct graphs $\mathcal{H}_{3^{2n}}$ of valency 5 as we did before for $p \neq 3$. However, $PSL(2,9) \simeq A_6$, so each connected component of $\mathcal{H}_{3^{2n}}$ is isomorphic to the complete graph on 6 vertices.

Section 3. Two More Examples

A rectagraph of valency 4 on 14 vertices is obtained from the incidence graph of the unique $2 - (7, 4, 2)$ design (see for example, P.J. Cameron [10], Theorem 4.5). This graph has automorphism group isomorphic with $PGL(2,7)$ and the stabilizer of a vertex is isomorphic with Σ_4.

A 5-gon-graph, which is not a pentagraph, of valency 6 can be obtained from the group $G = PSL(2,19)$ by letting G act on the right cosets of a subgroup $H \simeq A_5$. There is a suborbit of length 6 (on which H acts as does $PSL(2,5)$ on 6 points), and the graph $\mathcal{H}$ defined with respect to this suborbit has 3 pentagons on each 2-claw. The stabilizer of a 2-claw is an involution which fixes an unique pentagon on that 2-claw (and in fact the vertices of this pentagon are precisely the 5 points fixed by this involution), so that with $\mathcal{E}$ the set of pentagons fixed pointwise by involutions of G, $(\mathcal{H},\mathcal{E})$ is a 5-gon-graph of valency 6. Note that the stabilizer of a vertex in this example is 5/2-transitive on the vertices adjacent to the given vertex.

For $q \equiv \pm 1 \pmod{20}$, $PSL(2,q)$ has a subgroup isomorphic with A_5 and in its action on the right cosets of such a subgroup there is exactly one suborbit of length 6. It is no doubt possible that an analysis similar to that given in the previous section might give examples of polygonal graphs of valency 6 defined with respect to this suborbit, with the circuits of fixed points of involutions as the set of distinguished polygons. However we shall not do this here.

It is worth noting that in both examples of this section, the groups are rank 4 permutation groups on the vertices of the graphs (see Appendix 2).

CHAPTER IV

BOUNDING THE VALENCY OF POLYGONAL GRAPHS WITH m ODD

In this chapter we shall show that under certain conditions on the stabilizer of a point of the automorphism group of a polygonal graph with m odd, and on its involutions which fix a 2-claw, the valency of the polygonal graph is bounded.

So we shall assume that $(\mathcal{H}, \mathcal{E})$ is an m-gon-graph of valency $k \geq 3$ on a set Ω, with m odd, $m \geq 5$, and that $G \leq \mathrm{Aut}(\mathcal{H})$ is a group of automorphisms of $\mathcal{H}$ transitive on Ω. We also suppose that for any 2-claw $(x:y,z)$, $x, y, z \in \Omega$, every involution in G_{xyz} fixes (pointwise) the m-gon in $\mathcal{E}$ on $(x:y,z)$, but no other m-gon on $(x:y,z)$. Note that this latter hypothesis is automatically satisfied if $\mathcal{H}$ is a strict m-gon-graph (even if m is even), and in the case that G_{xyz} has no involutions we interpret this hypothesis to mean that G_{xyz} fixes the m-gon in $\mathcal{E}$ on $(x:y,z)$, and no other m-gon on $(x:y,z)$.

We then shall prove the following two theorems.

Theorem 4.1. Let $x \in \Omega$. Suppose that for some prime p and integer $n > 0$, $\mathrm{PSL}(2,p^n) \leq G_x^{\Delta(x)} \leq \mathrm{P\Gamma L}(2,p^n)$ on p^n+1 points. Then either $k = 3$ and $G_x \simeq \Sigma_3$, $k = 4$ and $G_x \simeq A_4$ or Σ_4, $k = 5$ and $G_x \simeq A_5$ or Σ_5, $k = 6$ and $G_x \simeq \mathrm{PSL}(2,5)$, or $k = 10$ and $G_x \simeq \mathrm{PSL}(2,9)$ or $\mathrm{PSL}(2,9)\langle\alpha\rangle$, where α is the non-trivial automorphism of the field of 9 elements.

Theorem 4.2. If k is odd, G_x is 3-transitive on $\Delta(x)$ for $x \in \Omega$, and $\mathcal{H}$ contains no strict m-gon-graph of valency 3 as a subgraph, then $k = 5$ and $G_x \simeq A_5$.

All the examples of polygonal graphs obtained in Chapters II and III (with m odd) satisfy the hypotheses of Theorem 4.1 (except the Petersen graph). As for Theorem 4.2, the only example arising from Chapters II and III which satisfy its hypotheses is the graph $\mathcal{H}_{31}$ with $G = PSL(2,31)$ (see Proposition 3.2 and Lemma 3.6); however if any other of the graphs $\mathcal{H}_q$ of Chapter III (with m odd) are polygonal graphs (with $\mathcal{E}$ the set of m-gons fixed by elements of order 3), then with $G = PSL(2,q)$ these groups and graphs will also be examples for Theorem 4.2.

If we remove the restriction that m be odd, then all of the examples of Chapters II and III, except the Petersen graph and the hexagraphs $\mathcal{H}_{p,q}$ with $p \neq q$, would satisfy the remaining hypotheses of Theorem 4.1, and in addition to $\mathcal{H}_{31}$, the graph $\mathcal{H}'_{11}$ with $G = PGL(2,11)$ would satisfy the remaining hypotheses of Theorem 4.2 (see Proposition 3.4 and Lemma 3.7).

Remark. I know of no example of a polygonal graph with k and m odd $(k > 3)$ which does contain a strict m-gon-graph of valency 3 as a subgraph (whether or not $G_x^{\Delta(x)}$ is 3-transitive), so it is quite possible that the hypothesis on strict m-gon-graphs in Theorem 4.2 may be unnecessary. With k even and m odd, the

120-cell (Chapter III, Section 1) contains dodecahedral subgraphs, and with m even, the k-dimensional cube (k even or odd) contains the ℓ-dimensional cube as a subgraph for all $\ell < k$.

In Section 1 we prove some of the basic lemmas needed for the proofs of the theorems which are then proved in Sections 2 and 3, respectively. Finally, in Section 4, we investigate further groups acting on pentagraphs of valency 5 which satisfy the hypotheses of Theorem 4.2 and characterize $PSL(2,31)$ and $\mathcal{H}_{31}$ as the only possible group and graph. So for Sections 1, 2 and 3 of this chapter we shall be making the assumptions given in the second paragraph of the chapter.

Section 1. Preliminary Lemmas

Lemma 4.1. If $\Pi \in \mathfrak{G}$ is the m-gon containing $(x:y,z)$ then G_{xyz} fixes Π pointwise (and no other m-gon on $(x:y,z)$).

Proof. Let $H = \langle t: t \text{ an involution in } G_{xyz} \rangle$. H char G_{xyz}, so G_{xyz} acts on the fixed points of H. But H fixes Π and no other m-gon on $(x:y,z)$, so G_{xyz} also fixes Π. □

As was shown for strict m-gon-graphs (Lemma 1.5) we can now also show:

Lemma 4.2. G_x is faithful on $\Delta(x)$.

Proof. Suppose $g \in G_x$ fixes $\Delta(x)$ pointwise. Assume g fixes $\Delta_i(x)$ for all $i < n$. We show g fixes $\Delta_n(x)$, whence by induction g fixes $\mathcal{H}$ (since $\mathcal{H}$ is connected), which implies that $g = 1$.

Take $y \in \Delta_n(x)$, $u \in \Delta_{n-1}(x)$, $v \in \Delta_{n-2}(x)$ with $v \sim u \sim y$. Let $\Pi = (y,u,v,w,\ldots)$ be the element of $\mathcal{E}$ on $(u:v,y)$. Then Π is also the m-gon in $\mathcal{E}$ containing $(v:u,w)$, so by Lemma 4.1, Π is fixed by G_{uvw}. But by the inductive hypothesis $g \in G_{uvw}$ because $w \in \Delta_i(x)$ for some $i \leq n-1$. So g fixes Π, whence g fixes y. Since y was arbitrary in $\Delta_n(x)$, this completes the proof. □

Lemma 4.3. Let X be a 2-transitive Frobenius group, V the Frobenius kernel, σ an involutory automorphism of X such that $M^\sigma = M$ for some complement $M \leq X$.

(i) If σ is inner, then σ centralizes M.

(ii) If σ is outer, then there is a nearfield $(N, +, \circ)$ with $V \simeq (N, +)$, $M \simeq (N - \{0\}, \circ)$ and $\sigma \in \mathrm{Aut}(N)$.

Proof. (For properties of Frobenius groups used here see D.H. Gorenstein [15], Theorems 2.7.6 and 10.3.1).

(i) Suppose σ is inner. Then there is $x \in X$ so that conjugation by x induces σ on X. Since $X = MV$, $x = mv$, say, with $m \in M$ and $v \in V$. Since $M^\sigma = M$, we have for all $n \in M$, $n^x \in M$, so $n^x = n^{mv} = n^m(v^{n^m})^{-1}v \in M$. Thus $v^{n^m} = v$ for all $n \in M$. Thus $n^m \in C_M(v)$ for some $n \neq 1$ and so $v = 1$.

Now $\sigma^2 = 1$, so $x^2 \in Z(X) = 1$ and thus $m^2 = 1$. Hence

$x = m$ is the unique involution in M, so $x \in Z(M)$ and σ centralizes M.

(ii) Suppose σ is outer. We claim that σ fixes an element $u \neq 1$ of V. If $|V|$ is even, since $V^{\sigma} = V$ and σ fixes $1 \in V$, σ must fix some $u \neq 1$ in V.

So suppose $|V|$ is odd and σ fixes no $u \neq 1$ in V. V is abelian (see, for example, M. Hall, Jr. [17], Section 20.7). Then σ fixes vv^{σ} for all $v \in V$, so $vv^{\sigma} = 1$, and hence $v^{\sigma} = v^{-1}$, for all $v \in V$. Thus for any $m \in M$, $v^{\sigma m} = (v^{m})^{-1}$. Also $v^{m} \in V$, so $(v^{m})^{\sigma} = (v^{m})^{-1}$.

Thus $(v^{m})^{\sigma} = v^{\sigma m^{\sigma}} = v^{\sigma m}$. Hence $m^{\sigma}m^{-1} \in C_{M}(v^{\sigma})$, since $m^{\sigma} \in M$, so that for $v \neq 1$, $C_{M}(v^{\sigma}) = 1$ gives $m^{\sigma} = m$. This is true for all $m \in M$, so σ fixes M elementwise. Now clearly if t is the unique involution in M (which exists since $|M| = |V| - 1$ is even) then $m^{t} = m$ for all $m \in M$ and $v^{t} = v^{-1}$ for all $v \in V$, so that conjugation by t induces σ on X. This contradicts σ being outer.

So there is a $u \in V$ with $u^{\sigma} = u \neq 1$.

Now define multiplication $\circ$ on V as follows: $1 \circ v = v \circ 1 = 1$ for all $v \in V$. If $v_1, v_2 \in V - \{1\}$, then there are unique elements $m_1, m_2 \in M$ with $v_1 = u^{m_1}$ and $v_2 = u^{m_2}$. Define $v_1 \circ v_2 = u^{m_1 m_2}$. Clearly this makes V into a nearfield N with operations $+$ and $\circ$, where $+$ is the multiplication of V in X, 1 is the $+$ identity (so denote it by 0_N), and u is the $\circ$ identity (so denote it by 1_N). Since $(u^{m})^{\sigma} = u^{\sigma m^{\sigma}} = u^{m^{\sigma}}$, it is now an easy matter to show that σ is an automorphism of N. $\square$

Lemma 4.4. Let Y be a rank 3 Frobenius group contained in a 2-transitive Frobenius group X with kernel V. Let σ be an involutary automorphism of Y such that $\hat{M}^\sigma = \hat{M}$ for some complement $\hat{M} \leq Y$. Let M be a complement in X with $\hat{M} < M$.

(i) If σ is induced by an inner automorphism of X, then σ centralizes $\hat{M}$.

(ii) If σ is not induced by an inner automorphism of X, then we again get the conclusion of Lemma 4.3(ii).

Proof. (i) The proof of Lemma 4.3(i) goes through with minor changes.

(ii) Necessarily, $|V|$ is odd. The proof of Lemma 4.3(ii) again goes through with some minor changes to show that there is an element $u \in V$ with $u^\sigma = u \neq 1$, so we get the nearfield N as before. It remains to show that $\sigma \in \mathrm{Aut}(N)$.

Let $S = \{u^m : m \in \hat{M}\}$, so that $|S| = \frac{1}{2}(|V| - 1)$. Suppose $1 \neq v \in V - S$. Now $1 \notin S \cup S^{-1}v$ and also $v \notin S \cup S^{-1}v$, where $S^{-1} = \{s^{-1} : s \in S\}$. Thus $|S \cup S^{-1}v| \leq |V| - 2$ and hence $S \cap S^{-1}v \neq \phi$. Thus there is $s, t \in S$ such that $t = s^{-1}v$, i.e., $v = st$.

Hence for all $v \in V$, either $v = 1$, $v \in S$, or $v = st$ for $s, t \in S$. Now since $\hat{M}^\sigma = \hat{M}$, $(u^m)^\sigma = u^{m^\sigma}$ implies that S is fixed by σ, i.e. $S^\sigma = S$. Further, for $m_1, m_2 \in \hat{M}$, $(u^{m_1} \circ u^{m_2})^\sigma = (u^{m_1 m_2})^\sigma = u^{m_1^\sigma m_2^\sigma} = u^{m_1^\sigma} \circ u^{m_2^\sigma} = (u^{m_1})^\sigma \circ (u^{m_2})^\sigma$ implies that $(s \circ t)^\sigma = s^\sigma \circ t^\sigma$ for all $s, t \in S$.

It is now an easy calculation to see that $(v \circ w)^\sigma = v^\sigma \circ w^\sigma$ for all $v, w \in V$ and thus $\sigma \in \mathrm{Aut}(N)$. □

Lemma 4.5. Let H be the multiplicative group of the field $N = GF(p^n)$ of order p^n, p a prime and $n \geq 1$, and let $A \leq Aut(N)$ be a subgroup of the automorphism group of N. Then H is characteristic in $G = AH$.

Proof. We show that H is the unique cyclic subgroup of G of order p^n-1. So suppose $K \neq H$ is cyclic of order p^n-1, and let $K = \langle k \rangle$, with $k = \sigma h$ for some $\sigma \in A$, $h \in H$. Suppose $|\sigma| = a$, with $1 < a|n$. (Result is clear for $n = 1$.) $k^2 = \sigma h \sigma h = \sigma^2 h^{\sigma} h$. Similarly, $k^a = \sigma^a h^{\sigma^{a-1}} h^{\sigma^{a-2}} \ldots h^{\sigma} h = h^{\sigma^{a-1}} \ldots h^{\sigma} h \in H$. Thus $(k^a)^{\sigma} = k^a$ since H is abelian, so $k^a \in C_H(\sigma)$. But $|C_H(\sigma)| = p^{n/a} - 1$, so $k^{a(p^{n/a} - 1)} = 1$, and so $a(p^{n/a} - 1) \equiv 0(p^n - 1)$.

Now $p^n - 1 = (p^{n/a} - 1)[(p^{n/a})^{a-1} + (p^{n/a})^{a-2} + \ldots + p^{n/a} + 1]$ $> a(p^{n/a} - 1)$. This contradiction proves the lemma. □

Lemma 4.6. Let n be an even integer and p an odd prime. Let N be the regular nearfield of order p^n with center isomorphic to the field of $p^{n/2}$ elements. Let H be the multiplicative group of N and $A \leq Aut(N)$ with $|A|$ odd. Then H is characteristic in $G = AH$.

Proof. First suppose $p^n \neq 9$. Suppose that N is constructed from the field $GF(p^n)$ of order p^n. Then it can be deduced from H. Lüneburg [28] that $Aut(N) \simeq Aut(GF(p^n))$. Let U be the set of squares in $GF(p^n) - \{0\}$, so that $U \leq H$ and $|H:U| = 2$. We claim that it suffices to prove that $U^{\alpha} \leq H$ for any $\alpha \in Aut(G)$,

for if $\alpha \in \mathrm{Aut}(G)$ and $U^\alpha \leq H$, suppose $H^\alpha \neq H$. Then $H, H^\alpha \lhd G$ and $U^\alpha = H \cap H^\alpha$. Further $|H^\alpha H : H| = |H^\alpha : H \cap H^\alpha| = 2$, which contradicts the fact that $|G:H| = |A|$ is odd.

Now U is a cyclic subgroup of order $\frac{p^n-1}{2}$, so that we will be done if we show that every cyclic subgroup of G of order $|U|$ lies in H. So suppose V is a cyclic subgroup of G of order $\frac{p^n-1}{2}$ and $V \not\leq H$. Then $V = \langle k \rangle$, say, where $k = \sigma h$, $\sigma \in A$ and $h \in H$. Suppose $|\sigma| = a$, odd with $a \geq 3$. $k^a = h^{\sigma^{a-1}} h^{\sigma^{a-2}} \ldots h^\sigma h$. Now $h \in U$ if and only if $h^{\sigma^i} \in U$ for any i. Thus $h^{\sigma^i} h^{\sigma^{i-1}} \in U$ for all i. Thus $k^{2a} = h^{\sigma^{a-1}} h^{\sigma^{a-2}} \ldots h^\sigma h h^{\sigma^{a-1}} \ldots h^\sigma h \in U$. Further, $(k^{2a})^{\sigma^2} = h^\sigma h \ldots h^{\sigma^3} h^{\sigma^2} h^\sigma \ldots h^{\sigma^3} h^{\sigma^2} = k^{2a}$ since U is abelian, so $k^{2a} \in C_U(\sigma^2)$. Thus $k^{2a} \in C_U(\sigma)$, since $\langle \sigma \rangle = \langle \sigma^2 \rangle$ as a is odd. But $|C_H(\sigma)| = p^{n/a} - 1$ and $|C_U(\sigma)| = \frac{1}{2}(p^{n/a} - 1)$, since $\frac{p^n-1}{p^{n/a}-1} = [(p^{n/a})^{a-1} + \ldots + p^{n/a} + 1] \equiv 1 \pmod 2$, so that $C_H(\sigma) \not\leq U$.

Hence $k^{a(p^{n/a}-1)} = 1$ and so $a(p^{n/a} - 1) \equiv 0(\frac{1}{2}(p^n - 1))$. But $p^n - 1 = (p^{n/a} - 1)[(p^{n/a})^{a-1} + \ldots + p^{n/a} + 1]$, so that since $p \geq 3$ and $n/a \geq 2$ we have

$$p^n - 1 \geq (p^{n/a} - 1)[9(a-1) + 1] = (p^{n/a} - 1)(9a - 8)$$
$$> 2a(p^{n/a} - 1), \quad \text{since } a > 2, \text{ a contradiction.}$$

Thus $a = 1$ and $\sigma = 1$. But then $V \leq H$ and we are done in this case.

Now suppose $p^n = 9$. Then the regular nearfield of order 9 has an automorphism of order 3 (P. Dembowski [13], 5.2.2), but

in this case $|H| = 8$ and so H is the characteristic Sylow 2-subgroup of $G = AH$.

Hence the lemma is proved. □

Remark. It can be shown that if N is the regular nearfield of order q^2, q a prime power, with center the field of q elements, then except for the case $q = 3$, the cyclic subgroup U of order $\frac{q^2-1}{2}$ of the multiplicative group N^* of N is in fact the unique cyclic subgroup of N^* of order $\frac{q^2-1}{2}$. This is not true for $q = 3$.

Corollary 4.1. With the hypotheses of Lemma 4.5, H is the unique subgroup of G isomorphic with H.

Proof. This is what was proven in the proof of Lemma 4.5. □

Corollary 4.2. With the hypotheses of Lemma 4.6, H is the unique subgroup of G isomorphic with H.

Proof. Clear, since if $H_1 \simeq H$, then either $H_1 = H$ or from what was proven in Lemma 4.6, $|HH_1:H| = 2$, a contradiction. □

Corollary 4.3. Let N, H and A be as in the hypotheses of Lemma 4.5 or Lemma 4.6. Let V be the additive group of N. Suppose $A'H'$ is a subgroup of AHV such that $A' \simeq A$, $H' \simeq H$ and $A'H' \simeq AH$. Then H' is conjugate in AHV to H.

Proof. Write $A = B \times C$ where $\pi(|B|) = \pi(g.c.d.(|A|, |H|))$, and $A' = B' \times C'$ where $|B'| = |B|$. Now AHV is solvable and both BH and $B'H'$ are Hall $\pi(|BH|)$ subgroups of AHV. All such are conjugate (D. Gorenstein [15], Theorem 6.4.1), and so $(B'H')^g = BH$, for some $g \in AHV$. But then $(H')^g \leq BH$, so that by Corollaries 4.1 and 4.2, applied with B in place of A, $(H')^g = H$. □

Remark. We mention without proof that the direct analogues of Lemma 4.5 and Corollaries 4.1 and 4.3, with H the squares of the multiplicative group of a field with p^n elements, hold true.

Section 2. Proof of Theorem 4.1

Let $x \in \Omega$ and $u, v \in \Delta(x)$. Let $\Pi \in \mathfrak{G}$ be the unique m-gon in $\mathfrak{G}$ on $(x:u,v)$. Let u', v' be the two points in Π at a distance $\frac{m-1}{2}$ from x. We now have three cases to consider.

Case 1. $PGL(2,p^n) \leq G_x^{\Delta(x)} = G_x$, by Lemma 4.2, and G_x is 3-transitive on $\Delta(x)$. G_{xuv} fixes Π and hence u' and v', so $G_{xuv} \leq G_{u'v'}$. Now $G_{u'v'}$ has a characteristic subgroup V of order p^n, and since no non-identity element of V fixes more than the one point v' of $\Delta(u')$, $G_{xuv} \cap V = \{1\}$ and G_{xuv} is a complement to V in $G_{u'v'}$. Also $G_{xuv} = AH$ where $A \simeq G_{xuvw}$ for some $w \in \Delta(x)$, and by Corollary 4.3, HV is a Frobenius group of order $p^n(p^n-1)$ with H isomorphic to the multiplicative group of

a field of order p^n, and A is isomorphic to a subgroup of the automorphism group of this field. Hence by Lemma 4.5, H is characteristic in G_{xuv}. There is an involution $\sigma \in G_x$ which interchanges u and v, hence normalizes G_{xuv}, and since Π is the unique m-gon on $(x:u,v)$ fixed by G_{xuv}, σ acts on Π and hence also interchanges u' and v'.

So σ normalizes H and V, and thus HV. Furthermore, unless $p^n = 2$ or 3, we can choose σ so that it inverts (but does not centralize) H (for example, we can choose $\sigma \in PSL(2,p^n)$). So if $p^n \neq 2$ or 3, then by Lemma 4.3(i) σ is not an inner automorphism of HV, whence by Lemma 4.3(ii) σ is an involutory field automorphism on a field N of p^n elements. Thus $p^n = r^2$ and therefore $\tau^\sigma = \tau^r = \tau^{-1}$ for all $\tau \in N - \{0\}$. Therefore $\tau^{r+1} = 1$ for all $\tau \in N - \{0\}$, so $r^2-1 \mid r+1$, from which we get $r = 2$. Thus $p^n = 4$. Hence in this case we get either $p = 2$, $n = 1$, or $p = 3$, $n = 1$, or $p = 2$, $n = 2$.

Thus either $k = 3$ and $G_x = PGL(2,2) \simeq \Sigma_3$, or

$k = 4$ and $G_x = PGL(2,3) \simeq \Sigma_4$, or

$k = 5$ and $G_x = PGL(2,4) \simeq A_5$,

or $G_x = P\Gamma L(2,4) \simeq \Sigma_5$.

Case 2. G_x is 3-transitive on $\Delta(x)$, but $G_x \not\geq PGL(2,p^n)$, so that n is even and p is odd. First suppose that $p^n \neq 9$. As in Case 1, we have that $G_{xuv} \leq G_{u'v'}$, and again $G_{u'v'}$ has a characteristic subgroup V of order p^n which is complemented by

G_{xuv}. Now $G_{xuv} = AH$ where $A \simeq G_{xuvw}$, and by Corollary 4.3, HV is a Frobenius group of order $p^n(p^n-1)$ with H isomorphic to the multiplicative group of a regular nearfield of order p^n (whose center is isomorphic to the field of $p^{n/2}$ elements), and A is isomorphic to a subgroup of the automorphism group of this nearfield. (Note that $|A|$ is odd, or else G_x would contain $PGL(2,p^n)$.) Hence by Lemma 4.6, H is characteristic in G_{xuv}.

Again, there is an involution $\sigma \in G_x$ which interchanges u and v, and also u' and v'. So σ normalizes H and V, and thus HV. Furthermore σ does not centralize H. Thus by Lemma 4.3(ii), σ is an involutory nearfield automorphism on a nearfield N of p^n elements, with center $Z(N)$ isomorphic to the field F of $p^{n/2}$ elements. However we can again choose σ so that it inverts the center of H, and thus inverts the center of $N - \{0\}$ which is $F - \{0\}$, and also σ centralizes F (as F is the fixed field of σ). Thus $\tau^\sigma = \tau = \tau^{-1}$ for all $\tau \in F - \{0\}$, and so $\tau^2 = 1$ for all $\tau \in F - \{0\}$. Hence $p^{n/2} = 3$, and so $p^n = 9$, a contradiction.

Now suppose $p^n = 9$. Then we may regard G_x as the following subgroup of $P\Gamma L(2,9)$ acting on 10 points: $G_x = \langle\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \alpha\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$: $ad - bc$ a square in $GF(9)$, $a'd' - b'c'$ a non-square in $GF(9)$, and $1 \neq \alpha \in \mathrm{Aut}(GF(9))\rangle \leq P\Gamma L(2,9)$. Let $GF(9) = \{a + ib: a, b \in GF(3), i^2 = -1\}$. The squares in $GF(9)$ are $\{\pm 1, \pm i\} = S$, say, and $i^\alpha = i^3 = -i$. Define a binary operation on $GF(9)$ by

$$w \circ u = \begin{cases} wu, & \text{if } u \in S, \\ w^3u, & \text{if } u \notin S. \end{cases}$$

Then $N = (GF(9), +, \circ)$ is the regular nearfield of 9 elements, and $Aut(N) \simeq \Sigma_3$ (P. Dembowski [13], 5.2.2).

Without loss of generality,

$$G_{xuv} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}, \alpha \begin{pmatrix} 1 & 0 \\ 0 & a' \end{pmatrix} : a \in S, \quad a' \notin S \right\}.$$

Let τ be the following map on N:

$$\tau : \begin{cases} 0 \mapsto 0 \\ \pm 1 \mapsto \pm 1 \\ \pm i \mapsto \pm(i-1) \mapsto \pm(i+1) \end{cases}.$$

Then $\tau \in Aut(N)$, $|\tau| = 3$, and $Aut(N) = \langle \alpha, \tau \rangle$. For the element $\sigma \in G_x$ which interchanges u and v, and also u' and v', choose $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Then $\left[\alpha \begin{pmatrix} 1 & 0 \\ 0 & i+1 \end{pmatrix} \right]^{\sigma} = \alpha \begin{pmatrix} i+1 & 0 \\ 0 & 1 \end{pmatrix} \equiv \alpha \begin{pmatrix} 1 & 0 \\ 0 & i-1 \end{pmatrix}$.

So σ does not centralize G_{xuv}.

Further, $(i+1)^{\alpha} = i^3 + 1 = -i + 1$; $(i+1)^{\alpha\tau} = -i - 1$; and $(i+1)^{\alpha\tau^2} = -i$. Thus no involution of $Aut(N)$ agrees with σ on $\alpha \begin{pmatrix} 1 & 0 \\ 0 & i+1 \end{pmatrix}$, contradicting Lemma 4.3. Thus Case 2 does not occur.

Case 3. $G_x = PSL(2, p^n) \langle \alpha \rangle$, $\langle \alpha \rangle \leq Aut\ GF(p^n)$, and p odd. We proceed exactly as in Case 1, observing that H is isomorphic to the squares in the multiplicative group of a field of p^n elements

and $A \simeq \langle \alpha \rangle$, using the remark at the end of Section 1, and using Lemma 4.4 in place of Lemma 4.3.

Then if $p^n \neq 3$ or 5, σ inverts but does not centralize H and so $(\tau^2)^\sigma = (\tau^2)^{p^{n/2}} = \tau^{-2}$ for all $\tau \in N - \{0\}$, N a field of p^n elements.

This gives $p^n - 1 | 2p^{n/2} + 2$, whence $p^n = 9$. Thus either

$k = 4$ and $G_x = PSL(2,3) \simeq A_4$, or

$k = 6$ and $G_x = PSL(2,5)$, or

$k = 10$ and $G_x = PSL(2,9)$,

or $G_x = PSL(2,9)\langle \alpha \rangle$, $1 \neq \alpha \in Aut(GF(9))$. □

Section 3. Proof of Theorem 4.2

Suppose Theorem 4.2 is false and from the set of pairs $(\mathcal{H}, G)$ of polygonal graphs $\mathcal{H}$ and groups $G \leq Aut(\mathcal{H})$ satisfying the hypotheses, choose a counterexample with $|\Omega|$ a minimum, and $|G|$ a minimum.

By Lemma 4.2, $k \neq 3$ and if $k = 5$ then $G_x \simeq \Sigma_5$. But then by Lemma 4.7, which follows below, if $u, v, w \in \Delta(x)$, G_{xuvw} has order 2 and on $\Omega(G_{xuvw})$ there will be a subgraph of valency 3 which is a strict m-gon-graph, contradicting the hypotheses of the theorem. Thus $k > 5$.

So choose $x \in \Omega$ and $u, v, w \in \Delta(x)$. Let $K = G_{xuvw}$. If $K = 1$, then G_x is sharply 3-transitive on $\Delta(x)$, so by D. Gorenstein and D.R. Hughes [16], Theorem 4.1 applies and we get a contradiction. So we may assume that $K \neq 1$.

Lemma 4.7. Let L be K or a 2-subgroup of K. Then connected components of the induced subgraph of $\mathcal{H}$ whose points are $\Omega(L)$ are regular, and if such a connected component has valency ≥ 2 it is a strict m-gon-graph.

Proof. Take a connected component Γ of $\Omega(L)$. From the points of Γ pick one, y say, whose valency n in Γ is maximal, and let $\{y_1,\ldots,y_n\} = \Omega_y(L)$. We claim that the valency of each y_i is n (in Γ).

Suppose on the contrary that y_1, say, has valency $\ell < n$ in Γ, and let $\Omega_{y_1}(L) = \{y,z_2,\ldots,z_\ell\}$. Since there are no triangles in $\mathcal{H}$, $(y{:}y_1, y_i)$ is a 2-claw for $2 \leq i \leq n$, so let Π_i be the unique element of $\mathfrak{E}$ on $(y{:}y_1, y_i)$. Then L fixes each Π_i pointwise, so each Π_i is in fact in Γ. Thus the points not equal to y in Π_i $(2 \leq i \leq n)$ which are adjacent to y_1 must lie in $\Omega_{y_1}(L)$. Since $\ell < n$, some z_j $(2 \leq j \leq \ell)$ occurs in at least two of the Π_i, both of which would then contain the 2-claw $(y_1{:}\, y, z_j)$, which contradicts the hypotheses on the set $\mathfrak{E}$. □

Let $S \in Syl_2(K)$ (possibly $S = 1$), and let $N = N_G(S)$. Let Γ be that connected component of the induced subgraph on $\Omega(S)$ containing x, u, v and w, so that by Lemma 4.7, Γ is a strict m-gon-graph of valency $\ell \geq 4$. Since k is odd, so is ℓ.

By Lemma 1.6, $N_x = N_{G_x}(S)$ is 3-transitive on $\Omega_x(S)$.

Now let Π_1, Π_2 be the elements of $\mathcal{E}$ on $(x: u, v)$ and $(x: u, w)$ respectively. Let v_1 and w_1 be the points in Π_1 and Π_2 other than x which are adjacent to u. Since G is transitive on ordered 3-claws, there is a $g \in G$ with $(x:u, v, w)^g = (u: x, v_1, w_1)$ and $u^g = x$, $v^g = v_1$, $w^g = w_1$. Then $K^g = (G_{xuvw})^g = G_{uxv_1w_1} = K$, by Lemma 4.1, so $g \in N(K)$. Thus by Sylow's theorem, there is $h \in K$ with $S^{gh} = S$, and thus $gh \in N_G(S)$. Hence $\Omega(S)^{gh} = \Omega(S)$ and it is clear that $(x: u, v, w)^{gh} = (u: x, v_1, w_1)$ so we may assume without loss of generality that $g \in N$ and hence $\Gamma = \Gamma^g$. Hence N is transitive on Γ as m is odd.

So by minimality of $\mathcal{H}$ and G, either (a) $\ell = 5$, or (b) $S = 1$.

Case (a). $\ell = 5$. Note that by the hypotheses of the theorem, $|\Omega_x(K)| \geq 4$, so that $|\Omega_x(K)| = 4$ or 5. Let $\Omega_x(S) = \{u, v, w, y, z\}$. If $|\Omega_x(K)| = 4$, say $\Omega_x(K) = \{u, v, w, y\}$, then Γ has a subgraph $\mathfrak{h}$ which is a strict m-gon-graph of valency 4 on $\{x, u, v, w, y\}$ by Lemma 4.7. Now $S \in \mathrm{Syl}_2(G_{xwyz})$ and $\{w, y, z\} \subset \Omega_x(G_{xwyz}) \subset \{u, v, w, y, z\}$. So some conjugate $\mathfrak{h}^k \neq \mathfrak{h}$ of $\mathfrak{h}$, $k \in G_x$, is a subgraph of Γ of valency 4, which is a strict m-gon-graph of valency 4 on $\{x, u, w, y, z\}$ or $\{x, v, w, y, z\}$. Then by Lemma 1.2, $\mathfrak{h} \cap \mathfrak{h}^k$ contains a strict m-gon-graph of valency 3, a contradiction.

So $|\Omega_x(K)| = 5$. Take $T < S$ of maximal order with respect to fixing > 5 points of $\Delta(x)$. By W.M. Kantor ([26], corollary

to, and proof of, Theorem 1), $N_{G_x}(T)^{\Omega_x(T)} \geq PSL(2,16)$, which is 3-transitive. Clearly $|\Omega_x(T)|$ is odd. So if we show that $N_G(T)$ is transitive on that connected component Γ' of the induced subgraph on $\Omega(T)$ which contains x, then the minimality of $\mathcal{H}$ and G would imply that $T = 1$.

Now $T < S \leq G_{uxv_1w_1} = K^g$ is also of maximal order with respect to fixing > 5 points of $\Delta(u)$, for if not, there is $T_1 < S$ with $|T| < |T_1|$, and T_1 fixes > 5 points of $\Delta(u)$; then, however, $S > T_1^{g^{-1}}$, and $T_1^{g^{-1}}$ fixes > 5 points of $\Delta(x)$, contradicting the maximality of T, since $|T_1^{g^{-1}}| > |T|$. So again, by W.M. Kantor [26], $N_{G_u}(T)$ is 3-transitive on $\Omega_u(T)$. Thus there is an element of $N_G(T)$ taking x to v_1. In a similar way, we see that since m is odd, $N_G(T)$ is transitive on the m-gon Π_1, so that by connectivity of Γ', $N_G(T)$ is transitive on Γ'.

Hence $T = 1$ and thus all involutions of G_x fix 1 or 5 points of $\Delta(x)$.

By F. Buekenhout ([9], Theorem 3), either

(i) $|\Delta(x)| = k = 17$ and $|P\Gamma L(2,16):G_x| = 1$ or 2, or

(ii) $k = 9$ and $G_x \simeq A_9$, or

(iii) $k = 7$ and $G_x \simeq \Sigma_7$.

Theorem 4.1 excludes (i). In (ii) and (iii), the stabilizer of 3 points fixes exactly those 3 points, whereas $|\Omega_x(K)| = 5$. Thus Case (a) does not occur.

Case (b). $S = 1$. Then involutions of G_x fix 1 point of $\Delta(x)$ and so by H. Bender [4], $P\Gamma L(2,2^j) \geq G_x \rhd PSL(2,2^j)$, for some j. But again, by Theorem 4.1, this possibility leads to a contradiction.

This proves Theorem 4.2. □

Corollary 4.4. Let $\mathcal{H}$ be a strict m-gon-graph, m odd, of valency k, odd, and let $G \leq Aut(\mathcal{H})$ be transitive on vertices of $\mathcal{H}$ and G_x be 3-transitive on $\Delta(x)$. If $\mathcal{H}$ contains no m-gon-graph as a subgraph of valency 3, then $k = 5$ and $G_x \simeq A_5$.

Proof. Theorem 4.2. □

Section 4. Pentagraphs of Valency 5 and a Characterization of PSL(2,31)

As was pointed out in the introduction to this chapter, the group $PSL(2,31)$ acting on the graph $\mathcal{H}_{31}$ of Chapter III provides an example of a pentagraph of valency 5 satisfying the hypotheses of Theorem 4.2. We now proceed to show that in fact if $\mathcal{H}$ is a pentagraph of valency 5, and $G \leq Aut(\mathcal{H})$, satisfying the hypotheses of Theorem 4.2, then $G \simeq PSL(2,31)$ and $\mathcal{H} \simeq \mathcal{H}_{31}$. The final identification of $PSL(2,31)$ is made by using the coset enumeration program of R. Scott [31].

We first prove the following general result about m-gon-graphs with m odd.

Lemma 4.8. Let $(\mathcal{H},\mathcal{E})$ be an m-gon-graph with m odd, and $G \leq \mathrm{Aut}(\mathcal{H})$ such that for any vertex x of $\mathcal{H}$, G_x is transitive on $\Delta(x)$. Then G is transitive on the vertices of $\mathcal{H}$.

Proof. Given any two points x, y on an m-gon Π with $y \in \Delta_2(x)$ there is a unique point z on Π with (z: x, y) a 2-claw of Π. Since G_z is transitive on $\Delta(z)$, we can move x to y, fixing z. In the same way we can move around Π two "steps" at a time, and so since m is odd, G is transitive on Π. Hence, by connectivity of $\mathcal{H}$, G is transitive on the vertices of $\mathcal{H}$. □

Lemma 4.9. Let $(\mathcal{H},\mathcal{E})$ be a 5-gon-graph of valency 5, and $G \leq \mathrm{Aut}(\mathcal{H})$, and suppose that for any 2-claw (x: y, z), G_{xyz} fixes the pentagon in $\mathcal{E}$ on (x: y, z). Then

(i) $G_x^{\Delta(x)}$ is 3-transitive on $\Delta(x)$ if and only if $G_x \simeq A_5$.

(ii) If $G_x^{\Delta(x)} \simeq A_5$, then $\mathcal{H}$ is a pentagraph.

Proof. (i) Clearly $G_x \simeq A_5$ implies G_x is 3-transitive on $\Delta(x)$.

Conversely, suppose $G_x^{\Delta(x)}$ is 3-transitive on $\Delta(x)$. Then as was shown in Lemma 4.2, G_x is faithful on $\Delta(x)$, so that $G_x \simeq A_5$ or Σ_5. So suppose $G_x \simeq \Sigma_5$. Then involutions in G_x fix either one or three points of $\Delta(x)$, and there is an involution $t \in G_x$ such that t fixes the three points u, v, w $\in \Delta(x)$ and interchanges the remaining two points y, z $\in \Delta(x)$. We now have two possible cases, viz., that t fixes more than one pentagon on

$(x: u, v)$, or t fixes exactly one pentagon on $(x: u, v)$. To avoid repetition, we quote a result which will be proved and used in Chapter V. In the former case, t fixes exactly two pentagons on $(x: u, v)$ (see Lemma 5.1), so that if we let Γ be that connected component of the induced subgraph of $\Omega(t)$ which contains x, Γ is isomorphic to the Petersen graph (Lemma 5.2). In the latter case, Γ is a cubic graph of girth 5 with a unique pentagon on each 2-claw, so that by Chapter II, Γ is isomorphic to a dodecahedral graph.

Since $C_{G_x}(t)$ is isomorphic to a dihedral group of order 12, and $\langle t \rangle = G_{xuvw}$, $(C_G(t)^\Gamma)_x$ is isomorphic with Σ_3 and is 3-transitive on $\Omega_x(t)$. Similarly, $(C_G(t)^\Gamma)_{u'}$ is 3-transitive on $\Omega_{u'}(t)$ for any $u' \in \Gamma$, so by Lemma 4.8, $C_G(t)^\Gamma$ is transitive on the vertices of Γ.

Now in the case of a Petersen graph, the stabilizer of a point in the full automorphism group is isomorphic with D_{12}, so that if Γ is a Petersen graph, $C_G(t)^\Gamma$ is isomorphic with A_5, while if Γ is a dodecahedral graph, $C_G(t)^\Gamma \simeq A_5 \times Z_2$.

Let $\Pi = (x, y, y', z', z)$ be the pentagon in $\mathcal{E}$ on $(x: y,z)$, so that because G_{xyz} fixes Π, and t normalizes G_{xyz}, t interchanges y' and z'. Hence t normalizes $G_{y'z'} \simeq \Sigma_4$. Note that $C_{G_x}(t)$ contains a subgroup H isomorphic with Σ_3 which fixes y and z, and also y' and z', and so $H \leq G_{y'z'}$. Thus t does not act as an inner automorphism of $G_{y'z'}$. Suppose $G_{y'z'} \leq C_G(t)$. Then $t \notin G_{y'z'}$ implies that $\Sigma_4 \simeq G^\Gamma_{y'z'} \leq C_G(t)^\Gamma$.

But neither A_5 nor $A_5 \times Z_2$ contains a subgroup isomorphic with Σ_4. Thus $G_{y',z'} \not\leq C_G(t)$.

Thus t must be an outer automorphism of Σ_4. No such automorphism, however, exists (B. Huppert [22], Satz II.5.5). This contradiction proves (i).

(ii) By Lemma 4.8, G is transitive on the vertices Ω of $\mathcal{H}$. We claim that G_x is transitive on $\Delta_2(x)$ for any $x \in \Omega$.

Clearly it suffices to show that if $y \in \Delta(x)$ and $z_1, z_2 \in \Delta(y)$, then there is a $g \in G_x$ with $z_1^g = z_2$. Since $G_y^{\Delta(y)} \simeq A_5$ is (at least) 2-transitive on $\Delta(y)$, there is a $g \in G_y$ such that g fixes x and $z_1^g = z_2$, which is what we wanted to show.

Now in the same way as was done in the proof of Lemma 3.5 of Chapter III, we can show the uniqueness of pentagons on 2-claws of $\mathcal{H}$. Thus $\mathcal{H}$ is a pentagraph. □

Corollary 4.5. Let $(\mathcal{H}, \mathcal{E})$ be a 5-gon-graph of valency 5 and $G \leq \mathrm{Aut}(\mathcal{H})$ with $G_x^{\Delta(x)}$ 3-transitive for all vertices x. Then for any 2-claw $(x: y, z)$, G_{xyz} fixes the pentagon in $\mathcal{E}$ on $(x: y, z)$ if and only if $\mathcal{H}$ is a pentagraph.

Proof. Lemma 4.9. □

Remark. Lemma 4.9 provides another proof of Theorem 3.3, namely, that $\mathrm{Aut}(\mathcal{H}_{31}) = \mathrm{PSL}(2,31)$.

From now on assume that $\mathcal{H}$ is a pentagraph of valency 5, and $G \leq \mathrm{Aut}(\mathcal{H})$ with $G_x^{\Delta(x)} \simeq A_5$ for all x in the set Ω of vertices of $\mathcal{H}$, so by Lemma 4.8, G is transitive on Ω. Pick a point $0 \in \Omega$ and let $\Delta(0) = \{1,2,3,4,5\}$. In its action on $\Delta(0)$, we represent $H = G_0$ by its usual representation as a permutation group of degree 5. So let $t = (1\ 2)(3\ 4)$, $z = (1\ 2\ 3)$ and $h = (1\ 2)(4\ 5)$, where $t, z, h \in H$. Then $G_{0,5} = K = \langle t,z \rangle \simeq A_4$ and $H = \langle K,h \rangle \simeq A_5$.

We first note the following.

Lemma 4.10. H is transitive on $\Delta_2(0)$; in fact there is a one to one correspondence between the points of $\Delta_2(0)$ and the set of ordered pairs of unequal points of $\Delta(0)$ preserving the action of H.

Proof. Since there are no rectangles in $\mathcal{H}$, given any point $u \in \Delta_2(0)$ there is a unique point $i \in \Delta(0)$ with $(i: 0, u)$ a 2-claw. By uniqueness of pentagons on 2-claws, there is a $v \in \Delta_2(0)$ and a $j \neq i$, $j \in \Delta(0)$, with $(0, i, u, v, j)$ the unique pentagon on any of its 2-claws. Let u correspond to the ordered pair (i, j) and v to the pair (j, i). This defines a map from $\Delta_2(0)$ to the set of pairs (i, j), $i, j \in \Delta(0)$, $i \neq j$. Again by uniqueness of pentagons on 2-claws of the form $(0: i, j)$, we see that this map is in fact one to one, hence onto, and is thus a one to one correspondence.

Now if $g \in H$, then the pentagon on the 2-claw $(0: i, j)$ clearly gets mapped onto the pentagon on $(0: i^g, j^g)$ by g, so that the correspondence preserves the action of H. This proves the lemma. □

From now on we will denote the point in $\Delta_2(0)$ which corresponds to the pair (i, j), $i, j \in \Delta(0)$, by the symbol ij, so that $(ij)^k = i^k j^k$ for $k \in H$.

Lemma 4.11. For $ij \in \Delta_2(0)$, $|\Delta(ij) \cap \Delta_2(0)| = 1$.

Proof. $ji \in \Delta(ij) \cap \Delta_2(0)$. If $i'j' \in \Delta(ij) \cap \Delta_2(0)$ with $i'j' \neq ij$, then there will be two pentagons on the 2-claw $(i: 0, ij)$, viz. $(0, i, ij, ji, j)$ and $(0, i, ij, i'j', i')$, a contradiction. □

Lemma 4.12. There is an involution $g \in G$ which interchanges the points 0 and 5 (and hence normalizes K).

Proof. Since G is transitive on ordered 3-claws of $\mathcal{H}$, there is a $g \in G$ with $(0: 5, 4, 3,)^g = (5: 0, 54, 53)$ such that $0^g = 5$, $5^g = 0$, $4^g = 54$ and $3^g = 53$. But then it can readily be seen, by uniqueness of pentagons on 2-claws, that $54^g = 4$, $53^g = 3$ and g fixes 45 and 35.

Thus $g^2 \in H_{5,4,3} = \{1\}$. □

Since $\mathcal{H}$ is connected $G = \langle H, g \rangle$. $L = \langle z \rangle$ fixes the pentagon (0, 5, 54, 45, 4) pointwise, so we have $L = \langle z \rangle = G_{4,0,5} = G_{0,5,54} = (G_{4,0,5})^g = L^g$, so that $z^g = z$ or $z^g = z^{-1}$. Also $0^{gh} = 4$, $0^{(gh)^2} = 45$, $0^{(gh)^3} = 54$, $0^{(gh)^4} = 5$ and $0^{(gh)^5} = 0$, so that $(gh)^5 \in H$ but $(gh)^i \notin H$ for $1 \leq i \leq 4$. Hence $(gh)^5 \in N_H(L) = L\langle h \rangle$. In fact, $(gh)^5 \in C_H(L) = \langle z \rangle$, for if $(gh)^5 = h$, zh or $z^{-1}h$, then $g^{(hg)^2} = 1$, z or z^{-1}; but $g^{(hg)^2}$ is an involution.

Suppose $g \in C_G(L)$. Then since h inverts z we have that $g^{(hg)^2}$ is an involution in $C_H(L) = L$, a contradiction. Thus $g \notin C_G(L)$. We have by Lemma 4.12 that $g \in N_G(K)$, and so g does not centralize K. If g acts as an inner automorphism of K, then since $Z(K) = 1$, there is an involution $k \in K$ such that $gk \in C_G(K)$. If $k = (1\ 2)(3\ 4)$, then $gk = (0\ 5)(4\ 53)\ \ldots$ which does not centralize z. If $k = (1\ 3)(2\ 4)$, then $gk = (0\ 5)(4\ 52\ \ldots)\ \ldots$ which also does not centralize z. Similarly, if $k = (1\ 4)(2\ 3)$, gk does not centralize z. Thus g is not inner on K.

Hence $G_{\{0,5\}} = \langle K, g \rangle \simeq \Sigma_4$. Now as a product of disjoint cycles,

$$g = (0\ 5)(4\ 54)(3\ 53)(45)(35)\ \ldots\ ,$$

$$z = (0)(5)(4)(54)(1\ 2\ 3)(51\ 52\ 53)\ \ldots\ ,$$

and $$t = (0)(5)(1\ 2)(3\ 4)(53\ 54)(51\ 52)\ \ldots\ .$$

So $z^g = (0)(5)(4)(54)(1^g\ 2^g\ 53)(51^g\ 52^g\ 3)\ \ldots$, so since $1^g, 2^g \in \Delta(0^g) = \Delta(5) = \{0, 51, 52, 53, 54\}$, and $g \notin C_G(z)$, we

have that g contains the cycles (1, 52) and (2, 51). Hence $z^g = z^{-1}$ and $t^g = t$. [We can think of g as the transposition (1 2) in its action on K (but not on the rest of H).]

Now for $\varepsilon = 0$, 1 or -1, let $G(\varepsilon)$ be the abstract group defined in terms of generators and relations as follows:

$$G(\varepsilon) = \langle g, h, t, z : g^2 = h^2 = t^2 = z^3 = (gt)^2 = (gz)^2 = (hz)^2 = (tz)^3 = (th)^3 = (gh)^5 z^{-\varepsilon} = 1 \rangle.$$

Then, $K(\varepsilon) = \langle t, z \rangle \simeq A_4$; since $|gtz| = 4$ we have $\langle K(\varepsilon), g \rangle \simeq \Sigma_4$; and since $|htz| = 5$ we have $H(\varepsilon) = \langle K(\varepsilon), h \rangle = \langle h, t, z \rangle \simeq A_5$. Also $z^3 = (gz)^2 = (hz)^2 = (tz)^3 = 1 \Leftrightarrow (z^{-1})^3 = (gz^{-1})^2 = (hz^{-1})^2 = (tz^{-1})^3 = 1$, so that $G(1) \simeq G(-1)$.

What we have shown in the previous paragraphs is that under the identification ϕ which maps $g \to g$, $h \to h$, $t \to t$, and $z \to z$, G is a quotient group of $G(\varepsilon)$, for $\varepsilon = 0$ or 1. Further, $G(\varepsilon)$ (for the correct ε) acts on $\mathcal{H}$ via ϕ. Let N be the kernel of $\phi : G(\varepsilon) \to G$.

Consider first $\varepsilon = 0$. $G(0)$ contains the subgroup $R = \langle g, h, t \rangle$ which can be seen to be a homomorphic image of the finite reflection group $S = \langle s_1, s_2, s_3 : s_1^2 = s_2^2 = s_3^2 = (s_1 s_2)^5 = (s_1 s_3)^2 = (s_2 s_3)^3 = 1 \rangle$ which is the group of the dodecahedron (and hence of the dodecahedral graph) isomorphic with $A_5 \times Z_2$ (see Chapter II). This reflection group has two non-trivial normal subgroups, viz. the center $\langle (s_3 s_2 s_1)^5 \rangle$ of order two whose image is $N_1 = \langle (thg)^5 \rangle \leq R$, and $\langle s_1 s_2, s_2 s_3 \rangle \simeq A_5$ whose image is $N_2 = \langle gh, ht \rangle \leq R$. Under the induced action of $G(0)$ ht fixes 0 while gh does not, so that $\phi(N_2) \neq \{1\}$. Now it is easy to check that $0^{(thg)^3} = 43^g$, while $0^{(ght)^2} = 34$, so that if $(thg)^5 = 1$ in G then

then $43^g = 34$. But then by considering the image of the pentagon (0, 3, 34, 43, 4) under g we will have that 53 is adjacent to 43, which contradicts Lemma 4.11 as 53 is adjacent to 35. Thus $\phi(N_1) \neq \{1\}$. Hence $R \simeq S$ and $R \cap N = \{1\}$.

Let Λ be the orbit of R in Ω containing the point 0. It can be seen that, since $g \in R$, $\Lambda \cap \Delta(0) = \{3, 4, 5\}$ or $\Delta(0)$. But, since $\mathcal{H}$ contains no rectangles, any element of R taking 2 to 3 (or 1 to 3) must fix 0 and therefore be in $H = H(0)$. Now $\langle h,t \rangle \leq H \cap R \leq H$ and since $\langle h,t \rangle$ is maximal in H and does not contain any element taking 2 to 3, or 1 to 3, if $\Lambda \cap \Delta(0) = \Delta(0)$ then $H \cap R = H \simeq A_5$. But then $H = N_2$, but $gh \in N_2$ does not fix 0, so this is not possible.

Thus $\Sigma_3 \simeq \langle h,t \rangle = H \cap R$, $\Lambda \cap \Delta(0) = \{3, 4, 5\}$ and $|\Lambda| = |R|/|R \cap H| = 20$. Hence we also see that for any $x \in \Lambda$, Λ contains exactly three points adjacent to x. Further as a product of disjoint cycles, $gh = (0\ 4\ 45\ 54\ 5) \ldots$ so that Λ contains the vertices of the (unique) pentagon on the 2-claw (0: 4, 5), and hence the vertices of the pentagon on any 2-claw (x: y, z) for which $x, y, z \in \Lambda$. Thus any connected component of the subgraph of $\mathcal{H}$ induced by the vertices of Λ has valency 3 and has a unique pentagon on any of its 2-claws. Thus Λ induces a dodecahedral graph in $\mathcal{H}$.

We have thus proved that if $\varepsilon = 0$, then $\mathcal{H}$ contains a dodecahedral subgraph.

Now consider $\varepsilon = 1$. Using a coset enumeration program of R. Scott [31], the index of $H(1) \simeq A_5$ in $G(1)$ is 248 (equal

to the number of vertices of the graph $\mathcal{H}_{31}$ of Chapter III). Thus $|G(1)| = |\mathrm{PSL}(2,31)|$. The matrices of Chapter III, equations (3.2.2), over the field of 31 elements, with $\sqrt{-3} = 11$, $\sqrt{2} = 8$ and $\sqrt{5} = 6$, become

$$g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad h = \begin{pmatrix} 0 & 10 \\ 3 & 0 \end{pmatrix},$$

$$t = \begin{pmatrix} -14 & 12 \\ 12 & 14 \end{pmatrix}, \qquad z = \begin{pmatrix} 5 & 0 \\ 0 & -6 \end{pmatrix},$$

and these satisfy the relations for $G(1)$. Further, it can be checked that

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \left[{}_z tz^{-1}(hg)^2 tz \right]_g , \quad \text{and so by}$$

H.S.M. Coxeter and W.O.J. Moser ([11], Section 7.5), these matrices generate $\mathrm{PSL}(2,31)$ over the field of 31 elements. Hence $G(1) \simeq \mathrm{PSL}(2,31)$. Then clearly too $\mathcal{H}$ is isomorphic with the graph $\mathcal{H}_{31}$ of Chapter III, Section 2.

We have thus proved the following.

Theorem 4.3. Let $\mathcal{H}$ be a pentagraph of valency 5, $G \leq \mathrm{Aut}(\mathcal{H})$ with $G_x^{\Delta(x)} \simeq A_5$ for all vertices x of $\mathcal{H}$.

Then either (i) $\mathcal{H}$ contains dodecahedral subgraphs, or (ii) $G \simeq \mathrm{PSL}(2,31)$ and $\mathcal{H} \simeq \mathcal{H}_{31}$. □

This together with Theorem 4.2 gives the following characterization of PSL(2,31).

Corollary 4.6. (Characterization of PSL(2,31).) Let $(\mathcal{H}, \mathcal{E})$ be a 5-gon-graph with the valency of $\mathcal{H}$ odd. Let $G \leq \mathrm{Aut}(\mathcal{H})$ be such that G_x is 3-transitive on $\Delta(x)$ for all vertices x of $\mathcal{H}$. Suppose for all 2-claws $(x: y, z)$, every involution in G_{xyz} fixes the pentagon in $\mathcal{E}$ on $(x: y, z)$ and no other pentagon on $(x: y, z)$ (where, if there are no involutions in G_{xyz} we interpret this to mean that G_{xyz} fixes the pentagon in $\mathcal{E}$ on $(x: y,z)$ and no other pentagon on $(x: y, z)$). Then the following are equivalent:

(i) $G \simeq \mathrm{PSL}(2,31)$ and $\mathcal{H} \simeq \mathcal{H}_{31}$.

(ii) $\mathcal{H}$ contains no dodecahedral subgraph.

Proof. For (i) implies (ii), See Lemma 3.6.

For (ii) implies (i), see Lemma 4.8, Theorem 4.2, Lemma 4.9 and Theorem 4.3. □

Remark. We have that as an abstract group PSL(2,31) has generators and relations given by $G(1)$. Another set of generators and relations for PSL(2,31) which can be derived from this is $\mathrm{PSL}(2,31) = \langle A, B, C: A^2 = B^3 = (AB)^5 = C^2 = (CB)^4 = (AC)^5 BCBCB^{-1} = 1 \rangle$, where $\langle A,B \rangle \simeq A_5$, $\langle B,C \rangle \simeq \Sigma_4$, by letting $C = g$, $A = h$, and $B = z^{tzt}$. Then $z = (B^{-1})^{CB^{-1}}$ and $t = A^{BAB}$. (As an element of $\langle A,B \rangle$ and

$\langle B,C \rangle$, B acts like (1 3 4).)

Finally, a remark about $G(0)$. I do not know whether or not $G(0)$ is finite. By letting $R_1 = g$, $R_2 = gz$, $R_3 = h$ and $R_4 = t$, we see that $G(0)$ is a quotient of the infinite Coxeter group $\langle r_1, r_2, r_3, r_4 : r_i^2 = (r_1r_2)^3 = (r_1r_3)^5 = (r_1r_4)^2 = (r_2r_3)^{15} = (r_2r_4)^4 = (r_3r_4)^3 = 1, \quad i = 1, 2, 3, 4 \rangle$ under the obvious identification; however in $G(0)$ we also have, for example, $(R_2R_3)^3 = (R_1R_3)^3$. If $G(0)$ were finite, then it would lead to an example of a pentagraph of valency 5 with dodecahedral subgraphs. However we have shown the following.

Theorem 4.4. With the hypotheses of Theorem 4.3, if $\mathcal{K}$ contains a dodecahedral subgraph, then $G \simeq G(0)/N$, where N is a normal subgroup of $G(0)$ intersecting $R = \langle g, h, t \rangle$ trivially. □

CHAPTER V

A CHARACTERIZATION OF J_1

It is well known (see for example D.G. Higman [20], p. 37) that $G = J_1$, Janko's first simple group, has a primitive permutation representation on a set Ω of degree 266, rank 5 and subdegrees 1, 11, 110, 132, and 12. The stabilizer of a point in this representation is isomorphic with PSL(2,11), and the intersection matrix M with respect to the suborbit of length 11 is

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 11 & 0 & 1 & 0 & 0 \\ 0 & 10 & 4 & 5 & 0 \\ 0 & 0 & 6 & 5 & 11 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

so that if we take $\mathcal{H}$ to be the graph defined with respect to this suborbit, it is readily seen that the girth of $\mathcal{H}$ is 5 and that there are 4 pentagons on each 2-claw. So if $x \in \Omega$, then $G_x \simeq$ PSL(2,11) is 2-transitive on $\Delta(x)$ of degree 11.

In this 2-transitive representation of G_x we have that if $y \in \Delta(x)$, then $G_{xy} \simeq A_5$ acts on the remaining 10 points of $\Delta(x) - \{y\}$ as does A_5 on the set of 2-element subsets of $\{1, 2, 3, 4, 5\}$, so that for $y \neq z \in \Delta(x)$, $G_{xyz} \simeq \Sigma_3$ and

$O_3(G_{xyz}) \simeq Z_3$ fix no points of $\Delta(x) - \{y,z\}$. Now since subgroups of $G_y \simeq PSL(2,11)$ isomorphic with Z_3 fix exactly 2 points of $\Delta(y)$, it is clear that G_{xyz} does not fix all 4 pentagons on the 2-claw $(x:y,z)$ pointwise; nor does $O_3(G_{xyz})$. Thus $O_3(G_{xyz})$ (and hence also G_{xyz}) fixes exactly one pentagon on $(x:y,z)$. Since G is transitive on ordered 2-claws, we have for any 2-claw of $\mathcal{H}$ that its pointwise stabilizer fixes a unique pentagon on that 2-claw, and further, if Π is one such pentagon and C is any of its 2-claws, then Π is the unique pentagon on C fixed by the pointwise stabilizer of C.

So let $\&$ be the set of pentagons of $\mathcal{H}$ fixed by pointwise stabilizers of 2-claws. Then we have the following.

Proposition 5.1. $(\mathcal{H},\&)$ is a 5-gon-graph whose automorphism group $G = J_1$ is transitive on the vertices Ω of $\mathcal{H}$, and $\mathcal{H}$ and G satisfy the following properties:

(a) The valency of $\mathcal{H}$ is 11.

(b) For any $x \in \Omega$, $G_x^{\Delta(x)} \simeq PSL(2,11)$.

(c) For some (and hence every, by (b)) 2-claw $(x:y,z)$, $x, y, z \in \Omega$, G_{xyz} fixes the element of $\&$ on $(x:y,z)$. □

This is the example of a polygonal graph alluded to in the introduction to Chapter III.

We shall show that the properties (a), (b) and (c) of the proposition characterize the 5-gon-graph $(\mathcal{H},\&)$ and the group J_1.

The characterization is in terms of Janko's original characterization of his simple group.

Theorem 5.1. (Z. Janko [24]) Let G be a finite group with the following properties:

(a) Sylow 2-subgroups of G are abelian;

(b) G has no subgroup of index 2; and

(c) G contains an involution t such that $C_G(t) = \langle t \rangle \times F$, where $F \simeq A_5$.

Then G is a simple group isomorphic with J_1. □

More precisely we shall prove the following.

Theorem 5.2. Let $(\mathcal{H},\&)$ be a 5-gon-graph and $G \leq \mathrm{Aut}(\mathcal{H})$ such that G and $\mathcal{H}$ satisfy properties (a), (b) and (c) of Proposition 5.1. Then $G \simeq J_1$, Janko's first simple group.

Remark. Since in its 2-transitive representation of degree 11 subgroups of PSL(2,11) isomorphic with Σ_3 fix either 0 or 2 points, it is clear that if we just assume (a) and (b) of Proposition 5.1, then the fixed points of G_{xyz} are disjoint circuits, one of which contains the 2-claw $(x:y,z)$. Statement (c) asserts that this circuit is in fact the pentagon in $\&$ on $(x:y,z)$. What is more is that this pentagon in $\&$ is then characterized as the unique pentagon on $(x:y,z)$ fixed by G_{xyz}. So as an easy consequence of the theorem we have:

Corollary 5.1. Let $\mathcal{H}$ be a connected, regular graph (undirected, no loops or multiple edges) of girth 5 and $G \leq \mathrm{Aut}(\mathcal{H})$ such that G and $\mathcal{H}$ satisfy statements (a) and (b) of Proposition 5.1, and (c') for some 2-claw $(x:y,z)$, $x, y, z \in \Omega$, G_{xyz} fixes a pentagon on $(x:y,z)$.

Then with $\mathcal{E}$ the set of pentagons fixed by pointwise stabilizers of 2-claws, $(\mathcal{H},\mathcal{E})$ is a 5-gon-graph and $G \simeq J_1$. □

Proof of Theorem 5.2. (For many of the properties of the Petersen and dodecahedral graphs used in this proof, refer to Chapter II.)

In exactly the same way as was done in Lemma 4.2, we can show that if $x \in \Omega$, then G_x is faithful on $\Delta(x)$ so that $G_x \simeq \mathrm{PSL}(2,11)$. Also it is clear that G is transitive on vertices and on edges of $\mathcal{H}$, as in Lemma 4.8.

Let $t \in G_x$ be an involution. Then t fixes exactly 3 points of $\Delta(x)$. Note that all involutions of G_x are conjugate in G_x and that G_x is 2-transitive on $\Delta(x)$. Let $\Omega_x(t) = \{u, v, w\}$ so that t fixes the pentagon in $\mathcal{E}$ on $(x:u,v)$. Note also that $G_{xuvw} = \langle t \rangle$. We now have two possible cases, viz.

Case (i): t fixes more than one pentagon on $(x:u,v)$; and

Case (ii): t fixes exactly one pentagon on $(x:u,v)$.

In either case, t fixes exactly the same number of pentagons on $(x:u,v)$ as it does on $(x:v,w)$ and $(x:u,w)$, since

$C_{G_x}(t)^{\Omega_x(t)} \simeq \Sigma_3$ is transitive on these three 2-claws, and also every involution in G_y, for any $y \in \Omega$, fixes the same number of pentagons on its fixed 2-claws as t does on $(x:u,v)$.

We now consider each of the above two cases separately and shall show that the former leads to the required result, while the latter case cannot occur.

Case (i).

Lemma 5.1. t fixes exactly two pentagons on $(x,:u,v)$.

Proof. Since t fixes at least two pentagons on $(x:u,v)$, we have that t fixes (x, u, u_1', v_1', v) and (x, u, u_2', v_2', v). Now $u_1' \not\sim v_2'$, $v_1' \not\sim u'_2$, $u_2' \neq v_1'$ and $v_2' \neq u_1'$ since the girth of $\mathcal{H}$ is 5.

So t is in G_u and fixes the three distinct points x, u_1', and u_2' (and hence no other points) adjacent to u and so there can be no further pentagons on $(x:u,v)$ fixed by t. □

Let Γ be that connected component of the induced subgraph of $\Omega(t)$ which contains x.

Lemma 5.2. Γ is isomorphic to the Petersen graph.

Proof. Clearly Γ is a cubic graph of girth 5 and having 2 pentagons on each 2-claw. Now refer to Chapter II, Section 1. □

Since $C_{G_x}(t)$ is isomorphic to a dihedral group of order 12, and $\langle t \rangle = G_{xuvw}$, $(C_G(t)^\Gamma)_x$ is isomorphic with Σ_3 and is 3-transitive on $\Omega_x(t)$. So also $C_G(t)^\Gamma$ is transitive on Γ.

Now the only subgroups of $\mathrm{Aut}(\Gamma)$ which are transitive on Γ, and are such that the stabilizer of a point in Γ is 3-transitive on the points (in Γ) adjacent to that point, are isomorphic to A_5 or $\Sigma_5 \simeq \mathrm{Aut}(\Gamma)$. But $\mathrm{Aut}(\Gamma)_x \simeq D_{12}$.

Thus $C_G(t)^\Gamma$ is isomorphic with A_5, i.e., $C_G(t)/\langle t \rangle \simeq A_5$, whence $C_G(t) \simeq SL(2,5)$ or $A_5 \times Z_2$. However $C_{G_x}(t)$ contains a subgroup isomorphic with $Z_2 \times Z_2$ and so $C_G(t) = \langle t \rangle \times K$, where $K \simeq A_5$.

Let $S \in \mathrm{Syl}_2(G)$ be a Sylow 2-subgroup of G such that $T \leq S$, where T is a Sylow 2-subgroup of $C_G(t)$, so $T \simeq Z_2 \times Z_2 \times Z_2$. If $S > T$, then pick $t_1 \in N_S(T) - T$. Now t_1 centralizes some involution $s \neq t$ in T, and so $|C_G(s)|_2 \geq 16$. But every involution in $C_G(t)^\Gamma$ fixes some point of the Petersen graph Γ (see Chapter II), and so s fixes a point of $\mathcal{H}$. Hence $s \underset{G}{\sim} t$. Thus $C_G(s) \simeq A_5 \times Z_2$, so $|C_G(s)|_2 = 8$, a contradiction.

Thus $S = T$, and Sylow 2-subgroups of G are abelian of order 8.

Now suppose for the purposes of obtaining a contradiction that G contains a subgroup H of index 2. Then $G_x \leq H$ since G_x is simple.

Lemma 5.3. There is an element $h \in G$ of odd order such that $x^h = u$.

Proof. From properties of $\mathrm{Aut}(\Gamma)$ (Chapter II) there is even an element $h \in C_G(t)$ such that $x^h = u$ and h has odd order. □

Clearly the element h of Lemma 5.3 is in the subgroup H. Thus $\langle G_x, h \rangle \leq H$. But since $\mathcal{H}$ is connected we have $\langle G_x, h \rangle = G$, a contradiction.

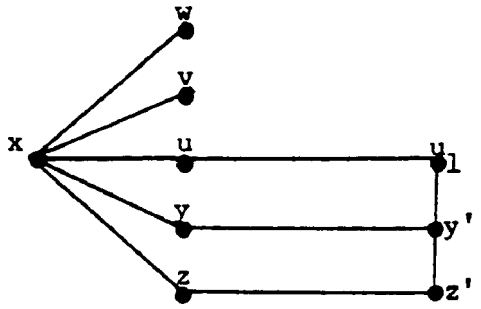
Thus by Theorem 5.1, G is isomorphic with J_1.

This completes Case (i) and Theorem 5.2 will be proven if we show that Case (ii) cannot occur, as we now proceed to do.

Case (ii). Again let Γ be that connected component of $\Omega(t)$ containing x. Then Γ is a cubic graph of girth 5 with a unique pentagon on each 2-claw, so by Chapter II, Γ is a dodecahedral graph. Further, since $(C_G(t)^\Gamma)_x$ is 3-transitive on $\Omega_x(t)$, by Chapter II $C_G(t)^\Gamma = \mathrm{Aut}(\Gamma) = \bar{K} \times \langle \bar{s} \rangle$ with $\bar{K} \simeq A_5$ and $\bar{s}^2 = 1$.

Now $C_{G_x}(t)$ contains a subgroup $H \simeq \Sigma_3$ such that H fixes two points $y, z \in \Delta(x)$ with $\{y,z\} \cap \{u,v,w\} = \phi$, and $y^t = z$. Let (x, y, y', z', z) be the unique pentagon in $\mathcal{G}$ fixed by H, so that $(y')^t = z'$, and hence t normalizes $G_{y',z'} \simeq A_5$.

Suppose $G_{y',z'} \leq C(t)$. Then $G_{y',z'}$ acts on $\Omega(t)$ and clearly $G^\Gamma_{y',z'} = \bar{K}$. Thus there is an involution in $G_{y',z'}$ which interchanges x and u (Chapter II), and fixes y', but not y as $y \not\sim u$. So there is a pentagon (x, u, u_1, y', y) as shown below.



Further, there is an involution $t' \in H$ which fixes x, u, y, z, y' and z' (and interchanges v and w). Since the girth of $\mathcal{H}$ is 5, t' fixes u_1 as well. Thus t' fixes two pentagons on the 2-claw (y:x,y'), a contradiction.

Thus $G_{y'z'} \not\leq C(t)$. Further, t is not acting on $G_{y'z'}$ as an inner automorphism since it centralizes a subgroup of $G_{y'z'}$ (viz. H) isomorphic to Σ_3. Thus $G_{\{y',z'\}} = \langle t, G_{y'z'} \rangle \simeq \Sigma_5$. So by transitivity of G on edges, $G_{\{x,u\}} \simeq \Sigma_5$ too. But $(C_G(t)^{\Gamma})_{\{x,u\}}$ has order 4, so that $(C_G(t))_{\{x,u\}} \leq G_{\{x,u\}}$ has order 8 and is thus a Sylow 2-subgroup of $G_{\{x,u\}}$. Hence $C_G(t)$ contains a subgroup isomorphic with D_8.

We now further investigate the structure of $C_G(t)$.

Lemma 5.4. $C_G(t)$ is isomorphic with one of $A_5 \times Z_2 \times Z_2$, $A_5 \times Z_4$, $SL(2,5) \times Z_2$, or $SL(2,5) \circ Z_4$ (central product).

Proof. We have $C_G(t)/\langle t \rangle = \overline{K} \times \langle \overline{s} \rangle$, where $\overline{K} \simeq A_5$ and $\overline{s}^2 = 1$. Let $C = C(t)$ and $K \leq C$ be the inverse image of $\overline{K}$, so $K/\langle t \rangle = \overline{K} \simeq A_5$ whence $K \simeq SL(2,5)$ or $A_5 \times Z_2$, and $|C:K| = 2$. Now $Aut(SL(2,5)) \simeq PGL(2,5)$ and $Aut(A_5 \times Z_2) \leq Aut(A_5) = \Sigma_5$, and $N_C(K)/C_C(K) = C/C_C(K)$ is isomorphic to a subgroup of $Aut(K)$. Also, $t \in C_C(K)$, so $|C_C(K)| \geq 2$, and since $|Z(K)| = 2$ we see that $|C_C(K)| = 2$ or 4.

If $C_C(K) = \langle t \rangle$, then $C/C_C(K) \simeq A_5 \times Z_2 \leq Aut(K)$. But neither $PGL(2,5)$ nor Σ_5 has such a subgroup, so $|C_C(K)| = 4$. So let $C_C(K) = \langle s,t \rangle$ where either $s^2 = 1$ or $s^2 = t$, and $[s,t] = 1$. Then $s \notin K$, so $C = K\langle s \rangle$.

If $K \simeq A_5 \times Z_2$ then since $\langle t \rangle = Z(K)$, we have $C \simeq A_5 \times C_C(K)$ leading to one of the first two of the groups listed in the statement of the lemma.

So suppose $K \simeq SL(2,5)$. If $\langle s \rangle \cap K = 1$ (i.e., $s^2 = 1$), then $C = K \times \langle s \rangle$, the third group in the list. If $\langle s \rangle \cap K \neq 1$, then $s^2 = t$ and $C_C(K) = \langle s \rangle \simeq Z_4$. Hence $C = K \circ \langle s \rangle$ (central product) $\simeq SL(2,5) \circ Z_4$. □

Now of the groups listed in Lemma 5.4, none except $SL(2,5) \circ Z_4$ has dihedral subgroups of order 8. So $C(t) \simeq SL(2,5) \circ Z_4$.

Let S be a Sylow 2-subgroup of $C(t)$, so $S = Q \circ Z$, where $Q \simeq Q_8$ and $Z = \langle \gamma : \gamma^4 = 1, \gamma^2 = t \rangle \simeq Z_4$, with $\langle t \rangle = Z(Q)$. $Z(S) = Z$, so t is the unique involution in $Z(S)$. Let P be a Sylow 2-subgroup of G with $P \geq S$. If $t \notin Z(P)$, there is an

involution $t_1 \in Z(P)$, so $t_1 \in C(t)$ and t_1 centralizes S. Thus $t_1 = t$, a contradiction. Thus $t \in Z(P)$ and hence $P \leq C(t)$, whence $P = S$. So S is a Sylow 2-subgroup of G.

Clearly $N_G(S) \leq N_G(Z(S)) \leq C(t) \leq C_G(Z)$, so in fact $N_G(Z) = C_G(Z)$, and thus γ is not conjugate in G to γ^{-1}.

Lemma 5.5. If $t^g \in S$ for some $g \in G$, then $t^g = t$, i.e., $g \in C(t)$.

Proof. If $t^g \in Z(S)$, we are done. So suppose $t \notin Z(S^{g^{-1}})$. Let $\gamma_1 \in S^{g^{-1}}$ such that $\langle \gamma_1 \rangle = Z(S^{g^{-1}})$, so $\gamma_1 \in C(t)$. Thus there is a $c \in C(t)$ with $\gamma_1^c \in S - Z(S)$. But γ_1^c is an element of order 4 in S, and it is easily checked that all elements of order 4 in $S - Z(S)$ are conjugate to their inverses. Thus $\gamma_1^c \sim (\gamma_1^{-1})^c$. But either $\gamma_1^c \sim \gamma_1 \sim \gamma$, or $\gamma_1^c \sim \gamma_1 \sim \gamma^{-1}$, contradicting the fact that $\gamma \not\sim \gamma^{-1}$. □

Now $S' = \langle t \rangle$, since $S/\langle t \rangle \cong Z_2 \times Z_2 \times Z_2$. So if S_1 is any other Sylow 2-subgroup of G, $S \cap S_1' = 1$ or $\langle t \rangle$, by Lemma 5.5. Also by properties of SL(2,5) and the fact that $N_G(S) \leq C(t)$, we have that $S \cap N_G(S)' = Q > \langle t \rangle$. Hence by Grün's 1st Theorem (see D. Gorenstein [15], p. 252), $S \cap G' = Q$.

Thus G has a normal subgroup L of index 2 whose Sylow 2-subgroups are isomorphic to Q, a quaternion group. By simplicity of PSL(2,11), $L \geq G_x$, a contradiction of the structure of Sylow 2-subgroups of PSL(2,11). This contradiction proves the result. □

CHAPTER VI

AN APPLICATION TO RANK 4 GROUPS

Let G be a transitive rank 4 permutation group on a set Ω with subdegrees $1, k > 2, \ell$ and m, i.e. G is transitive on Ω, and for $x \in \Omega$, G_x has orbits of length $k > 2$, ℓ and m on $\Omega - \{x\}$. (For the terminology used in this chapter, viz. rank, subdegrees, suborbits, pairing of suborbits, intersection matrices, etc., see D.G. Higman [18] and [20].) So for $x \in \Omega$, we can write $\Omega = \{x\} \cup \Delta_1(x) \cup \Delta_2(x) \cup \Delta_3(x)$, a disjoint union, where each $\Delta_i(x)$ is a G_x orbit with $\Delta_i(x)^g = \Delta_i(x^g)$ for all $g \in G$, and $|\Delta_1(x)| = k$, $|\Delta_2(x)| = \ell$ and $|\Delta_3(x)| = m$. We shall assume that $\ell = k(k-1)$ (more about this assumption later).

In Section 1, we gather together some of the notation and results from D.G. Higman [18] and [20] which we shall be using in this chapter. In Section 2 we assume that the stabilizer of a point x is 3-transitive on $\Delta_1(x)$ and make certain assumptions on the stabilizer of an edge. Three distinct cases arise. In the first of these we get a pentagraph of valency k to which some of the results of Chapter IV apply. In the second case we obtain explicit values for k while in the third case we get numerical constraints on k and one of the other parameters associated with the group.

I might mention that it was precisely the first case which initiated the study of polygonal graphs and the topics studied in this thesis.

Section 1. Preliminaries

Associated with each suborbit $\Delta_i(x)$, $i = 1, 2, 3$ (for some $x \in G$), we have graphs $\mathcal{H}_i$, $i = 1, 2, 3$, independent of $x \in G$, defined as follows. The vertices of $\mathcal{H}_i$ are the elements of Ω, and the edge set of $\mathcal{H}_i$ is $\{(x, y)^g: g \in G\}$ for $y \in \Delta_i(x)$ (and independent of choice of $y \in \Delta_i(x)$). The valency of $\mathcal{H}_i = |\Delta_i(x)|$ and $\mathcal{H}_i$ is undirected or directed according as $\Delta_i(x)$ is or is not self-paired.

Since $k > 2$, and we are assuming $\ell = k(k-1)$, at least one of $\Delta_1(x)$ and $\Delta_2(x)$ is self-paired. By S. Iwasaki ([23], Proposition 1.3), $\Delta_1(x)$ is self-paired and so $\mathcal{H}_1$ is undirected. ($\Delta_2(x)$ might be self-paired: by Proposition 1.2 of [23] this is the case if G is primitive, or even if $\mathcal{H}_1$ is connected.)

We now suppose that $\mathcal{H}_1$ is connected and that for some $y \in \Delta_2(x)$, $|\Delta_1(x) \cap \Delta_1(y)| > 0$. This is equivalent to saying that the points of $\Delta_2(x)$ are at distance 2 from x in $\mathcal{H}_1$. So now all sub-orbits are self-paired. Corresponding to each (non-trivial) suborbit we have the intersection matrices M_i which are given below (see D. G. Higman [18]).

$$M_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ k & k_{11} & k_{12} & k_{13} \\ 0 & k_{21} & k_{22} & k_{23} \\ 0 & k_{31} & k_{32} & k_{33} \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & k_{21} & k_{22} & k_{23} \\ \ell & \ell_{21} & \ell_{22} & \ell_{23} \\ 0 & \ell_{31} & \ell_{32} & \ell_{33} \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & k_{31} & k_{32} & k_{33} \\ 0 & \ell_{31} & \ell_{32} & \ell_{33} \\ m & m_{31} & m_{32} & m_{33} \end{pmatrix},$$

where for $1 \leq i, j \leq 3$, the intersection number $k_{ij} = |\Delta_i(x) \cap \Delta_1(y)|$ for some $y \in \Delta_j(x)$ = the number of edges from a point $y \in \Delta_j(x)$ to $\Delta_i(x)$ in the graph $\mathcal{H}_1$. For $1 \leq i \leq 2$, $1 \leq j \leq 3$, $\ell_{ij} = |\Delta_i(x) \cap \Delta_2(y)|$, and for $1 \leq j \leq 3$, $m_{3j} = |\Delta_3(x) \cap \Delta_3(y)|$ for $y \in \Delta_j(x)$. The latter intersection numbers also have interpretations as the number of edges from $y \in \Delta_j(x)$ to $\Delta_i(x)$ in the graphs $\mathcal{H}_2$ and $\mathcal{H}_3$ respectively.

The parameter conditions are as follows:

$$k_{21}k = k_{12}\ell, \quad k_{32}\ell = k_{23}m, \quad \text{and} \quad k_{31}k = k_{13}m.$$

$$\ell_{21}k = k_{22}\ell, \quad \ell_{31}k = k_{23}m, \quad \text{and} \quad \ell_{32}\ell = \ell_{23}m.$$

$$\ell_{31}k = k_{32}\ell, \quad m_{31}k = k_{33}m, \quad \text{and} \quad m_{32}\ell = \ell_{33}m.$$

$$(k_{12} - k_{13})\ell_{22} = k_{21}k_{12} + k_{22}^2 + k_{23}k_{32} - (k_{11} - k_{13})k_{22} - (k - k_{13}) - k_{13}\ell.$$

Further, $\mathcal{H}_1$ is connected if and only if none of the pairs (k_{12}, k_{13}),

(k_{13}, k_{23}) and (k_{12}, k_{32}) is equal to $(0, 0)$; $\mathcal{H}_2$ is connected if and only if none of (ℓ_{21}, ℓ_{23}), (ℓ_{23}, k_{23}) and (ℓ_{21}, ℓ_{31}) is equal to $(0, 0)$; and $\mathcal{H}_3$ is connected if and only if none of (m_{32}, m_{31}), (m_{31}, ℓ_{31}), (m_{32}, k_{32}) is equal to $(0, 0)$. Note that G is primitive if and only if $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ are all connected.

To summarize, we are assuming that $\ell = k(k-1)$, $\mathcal{H}_1$ is connected and that $k_{12} > 0$.

Thus $k_{21}k = k_{12}k(k-1)$ which implies that $k_{21} = k_{12}(k-1)$. Now $k_{12} \geq 2 \Rightarrow k_{21} \geq 2(k-1) > k$, which is not possible since the column sums of M_1 are k. Hence $k_{12} = 1$ and $k_{21} = k-1$. Therefore $k_{11} = k_{31} = 0$.

Let $a = k_{23}$ and $b = k_{22}$. Then $k_{32} = am/k(k-1)$, and $b = k - k_{12} - k_{32} = k-1 - am/k(k-1)$. Finally, since $k_{31} = 0$, we have $k_{13} = 0$, $k_{33} = k-a$, and

$$M_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ k & 0 & 1 & 0 \\ 0 & k-1 & b & a \\ 0 & 0 & \frac{am}{k(k-1)} & k-a \end{pmatrix} .$$

Since $\mathcal{H}_1$ is connected, $a \neq 0$. Note that $k_{11} = 0$ and $k_{12} = 1$ respectively imply that $\mathcal{H}_1$ has no triangles or rectangles. The graph polynomial for $\mathcal{H}_1$ (which is the minimal polynomial of M_1 divided by $X-k$) is $F(X) = X^3 + AX^2 + BX + C$, where

$A = (k_{13} + k_{23}) - (k_{11} + k_{22}) = a-b$,

$B = k_{13}(k_{21} - k_{22} + 1) + k_{11}(k_{22} - k_{23}) - k_{12}(k_{21} - k_{23}) - k$

$= a + 1 - 2k,$

$C = k_{22}(k - k_{13}) - k_{23}(k - k_{12}) = a - k(a-b),$ and

$D = k_{12}k_{23} + k_{13}(k - k_{22}) = a,$ where D is such that $F(M) = DJ$,

$$J = \begin{pmatrix} 1 & 1 & 1 & 1 \\ k & k & k & k \\ \ell & \ell & \ell & \ell \\ m & m & m & m \end{pmatrix} .$$

Thus $F(X) = X^3 + (a-b)X^2 + (a+1-2k)X + a - k(a-b)$. Also, one of the roots of $F(X)$ is an integer, by D.G. Higman [18] or S. Iwasaki [23].

Let s be an integer root of $F(X)$. Then

$$8k(2s + a - b) = 8[s^3 + (a-b)s^2 + (a+1)s + a]$$

$$= (2s + a - b)^3 - (a-b)(2s + a - b)^2 + (4(a+1) - (a-b)^2).$$

$$(2s + a - b) + \delta, \qquad (6.1.1)$$

where $\delta = 8a + (a-b)^3 - 4(a+1)(a-b)$, and $2s + a - b \mid \delta$. (6.1.2)

We conclude this section with the following result.

Lemma 6.1. For any $y \in \Omega$, G_y is 2-transitive (of degree k) on $\Delta_1(y)$.

Proof. We know that G_y is transitive on $\Delta_1(y)$. Pick $x \in \Delta_1(y)$. We must show that G_{yx} is transitive on $\Delta_1(y) - \{x\}$. So take

$u, v \in \Delta_1(y) - \{x\}$, $u \neq v$. Then since $u, v \in \Delta_2(x)$ (because $k_{11} = 0$) there is a $g \in G_x$ with $u^g = v$. But $k_{21} = 1$ implies that $\{y\} = \Delta_1(x) \cap \Delta_1(u) = \Delta_1(x) \cap \Delta_1(v)$, so that $y^g \in [\Delta_1(x) \cap \Delta_1(u)]^g = \Delta_1(x) \cap \Delta_1(v) = \{y\}$. Therefore $y^g = y$ and so $g \in G_{xy} = G_{yx}$. □

Section 2. A Triply Transitive Suborbit

P.J. Cameron has shown (see for example [10], Theorem 4.4) that if $\mathcal{H}$ is an undirected, regular, connected graph of valency v and if $G \leq \mathrm{Aut}(\mathcal{H})$ is such that G_x is 3-transitive on $\Delta(x)$ and transitive on $\Delta_2(x)$ for x a vertex of $\mathcal{H}$, then either (i) $\mathcal{H}$ is explicitly known, or (ii) precise information is known about v, and $\mathcal{H}$ is strongly regular or obtained from a strongly regular graph, or (iii) for any $y \in \Delta_2(x)$, $|\Delta(x) \cap \Delta(y)| \leq 2$. It would be fair to say that most of this thesis has been devoted to studying possibility (iii) (especially $|\Delta(x) \cap \Delta(y)| = 1$) in the case that $\mathcal{H}$ is of a special type, viz. a polygonal graph.

Moreover ([10], Thereom 5.2), in the situation of the previous paragraph, if G_x is 3-transitive on $\Delta(x)$, or if G_x has rank at most 3 on $\Delta_2(x)$, and if G is primitive, then either (i) G has rank 3 or is doubly transitive, or (ii) $|\Delta(x) \cap \Delta(y)| \leq 2$. The case of a rank 3 group with $|\Delta(x) \cap \Delta(y)| = 1$ has been settled by D.G. Higman [19,21], M. Aschbacher [1], and W.M. Kantor [25]. Note that when G_x is transitive on $\Delta(x)$ and $\Delta_2(x)$, $|\Delta(x) \cap \Delta(y)| = 1$ for $y \in \Delta_2(x)$ is equivalent to

$|\Delta_2(x)| = |\Delta(x)|(|\Delta(x)| - 1)$ (or $\ell = k(k-1)$ in the notation of Section 1 of this chapter). So this chapter can be regarded as an investigation into the case of a rank 4 group with $|\Delta(x) \cap \Delta(y)| = 1$.

Henceforth we shall assume the hypotheses of the first paragraph of this chapter, that $\mathcal{H}_1$ is connected and $k_{12} > 0$, that for all $y \in \Omega$, G_y is 3-transitive on $\Delta_1(y)$, as well as the following hypothesis:

(H) For $z \in \Delta_1(y)$, $G_{yz}^{\Delta_1(z) - \{y\}}$ is isomorphic as a (2-transitive) permutation group to $G_{yz}^{\Delta_1(y) - \{z\}}$, of degree $k - 1$.

Remark. The case of a rank 4 group with a 5-transitive suborbit has been handled by E. Bannai [2] and [3].

Let $x \in \Delta_1(y) - \{z\}$. Then by ($H$), G_{xyz} has exactly two orbits on $\Delta_1(z) - \{y\}$, of lengths 1 and $k-2$. We thus have three distinct cases.

Case 1. $b = 1$ and $m = k(k-1)(k-2)/a$.

Case 2. $b = k-1$ and $m = k(k-1)/a$.

Case 3. $b = 0$ and $m = k(k-1)^2/a$.

We consider these three cases separately.

Case 1. $b = 1$, $m = k(k-1)(k-2)/a$, and $F(X) = X^3 + (a-1)X^2 + (a + 1 - 2k)X + a - k(a-1)$.

In this case every 2-claw of $\mathcal{H}_1$ lies in an unique pentagon, so that $\mathcal{H}_1$ is a pentagraph.

Lemma 6.2. Suppose that $\mathcal{H}_1$ contains a dodecahedral subgraph Γ and let $x \in \Omega$. Then either (i) G_{xy} has a section isomorphic with Σ_3 for any $y \in \Delta(x)$, or (ii) $6|a$.

Proof. Let $(x: u, v, w)$ be a 3-claw of Γ. Since G is transitive on ordered 3-claws of $\mathcal{H}_1$ we can see by applying Lemma 1.2, and the material on dodecahedral graphs of Chapter II, that if $K = G_\Gamma$, then $K^\Gamma = \mathrm{Aut}(\Gamma) \simeq A_5 \times Z_2$. Let $1 \neq \tau \in Z(K^\Gamma)$ so that $\tau^2 = 1$. In K, there is a subgroup L such that $L^\Gamma \simeq \Sigma_3$ and L fixes x.

Since L^Γ fixes x^τ, if $x^\tau \in \Delta(x)$ then (i) follows. Similarly if $x^\tau \in \Delta_2(x)$, then since $\mathcal{H}_1$ has no rectangles, L fixes pointwise a pentagon on x and hence two points of $\Delta(x)$. So (i) follows in this case too.

So suppose $x^\tau \in \Delta_3(x)$. Then $|\Delta_3(x)| = k(k-1)(k-2)/a = |G_x: G_{x^\tau}|$. Also, since G_x is 3-transitive on $\Delta(x)$ we have $k(k-1)(k-2) = |G_x: G_{xuvw}|$. Now by Lemmas 1.4, and 1.2, G_{xuvw} fixes Γ and so $G_{xuvw} \leq G_{xx^\tau}$ and $|G_{xx^\tau}: G_{xuvw}| = a$. Also $|L: G_{xuvw}| = 6$, and $L \leq G_{xx^\tau}$. Thus $|G_{xx^\tau}: L| = a/6$, whence $6|a$.

Thus (ii) follows and the lemma is proved. □

Now in the general situation of Case 1, suppose that $G_{xuvw} = 1$ for some 3-claw $(x: u, v, w)$ of $\mathcal{H}_1$. Then we can apply Theorem 4.1 of Chapter IV to get that $k = 3$, 4, or 5. Now if $k = 3$, then $\mathcal{H}_1$ is a dodecahedral graph which has diameter 5, contradicting $\mathcal{H}_1$ has diameter 3. So $k \neq 3$.

Suppose that $k = 4$. Then $F(X) = X^3 + (a-1)X^2 + (a-7)X + 4-3a$. Let s be an integer root of $F(X)$. Then $a(s^2 + s - 3) = -s^3 + s^2 + 7s - 4 = -s(s^2 + s - 3) + 2(s^2 + s - 3) + 2s + 2$. Thus $s(s+1)-3 \mid 2(s+1)$, so that $s(s+1)-3 \mid 6$.

This, together with the fact that a is an integer with $1 \leq a \leq 4$, gives the two possibilities $s = 2$, $a = 2$, and $s = -1$, $a = 3$. It is now a simple matter to show that in both of these cases, $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ are all connected so that G is primitive. However the former case contradicts Proposition 3.3(i), and the latter case, Proposition 3.2(iv) of S. Iwasaki [23].

Suppose that $k = 5$. Then by Lemma 4.9 and Corollary 4.6 of Chapter IV, $G_x \simeq A_5$ and $\mathcal{H}_1$ contains dodecahedral subgraphs (as the group PSL(2,31) has rank 9 on the graph $\mathcal{H}_{31}$ of Chapter III). Since $1 \leq a \leq 5$, $6 \nmid a$. Also for $y \in \Delta(x)$, $G_{xy} \simeq A_4$ has no section isomorphic with Σ_3. This contradicts Lemma 6.2. Thus $G_{xuvw} \neq 1$.

So we have the following.

Proposition 6.1. In the situation of Case 1,

(i) $k \geq 6$, and if k is odd then $\mathcal{H}_1$ contains dodecahedral subgraphs;

(ii) $\mathcal{H}_1$ contains an induced subgraph which is a pentagraph of valency ℓ, for some ℓ with $3 \leq \ell < k$;

(iii) $a \geq 6$.

Proof. (i) By the material preceding the proposition and Theorem 4.2, $k \geq 6$, and if k is odd we must have that $\mathcal{H}_1$ contains dodecahedral subgraphs.

(ii) We have shown that for any 3-claw (x: u, v, w) of $\mathcal{H}_1$, $G_{xuvw} \neq 1$. Thus by Lemma 1.4, (ii) follows.

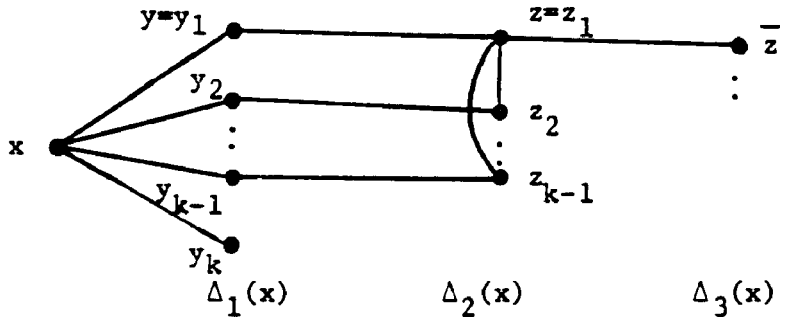
(iii) From (6.1.1) and (6.1.2) with $b = 1$, we have $\delta = (a-3)(a^2 - 4a - 1)$, $2s+a-1 \mid \delta$, and $8k = (2s+a-1)^2 - (a-1)(2s+a-1) + 4(a+1) - (a-1)^2 + \delta/(2s+a-1)$, where s is an integer root of $F(X)$. It can be seen that $s = -(a-1)/2$ if and only if $a = 3$.

Using this we can show that $a \neq 1$, 2, 4 or 5. For example, with $a = 5$, $\delta = 8$ and $2s+4 \mid 8$. Thus $2s+4 = -8, -4, -2, -1, 1, 2, 4$ or 8. Thus gives $s = -6, -4, -3, -1, 0$ or 2, from which $k = 1$ or 2, a contradiction.

Suppose $a = 3$. Then it can be verified that $\ell_{22} = 3(k-2) = \ell_{23}$, $\ell_{21} = k-1$, $\ell_{31} = (k-1)(k-2)$, $m_{32} = (k-1)(k-2)(k-3)/3 = m_{31}$, so that $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$ are connected, whence G is primitive. Then by Proposition 3.2(iv) of S. Iwasaki [23], we get a contradiction, so that $a \geq 6$. □

Case 2. $m = \frac{k(k-1)}{a}$ and $b = k-2$. Let $\bar{z}$ be the unique point in $\Delta_3(x) \cap \Delta_1(z)$ so that G_{xyz} fixes $\bar{z}$.

Let $\Delta_1(x) = \{y = y_1, y_2, \ldots, y_k\}$. Since $b = k-2$, and in $\mathcal{H}_1$ there are no quadrangles or triangles we may assume without loss of generality that $\Delta_1(z) \cap \Delta_1(y_i) \neq \phi$ for $i = 2, 3, \ldots, b+1 = k-1$. Let $\{z_i\} = \Delta_1(z) \cap \Delta_1(y_i)$, $2 \leq i \leq k-1$, and write $z = z_1$.



Lemma 6.3. Either $a \leq 2$, or $k = 6$ or 56 and G is imprimitive.

Proof. Suppose $a > 2$. Now $\{z_1\} = \Delta_1(\bar{z}) \cap \Delta_1(y_1)$ and if $\Delta_1(\bar{z}) \cap \Delta_1(y_k) \neq \phi$, then $|\Delta_1(\bar{z}) \cap \Delta_1(y_k)| = 1$. So $a > 2$ implies that for some j, $2 \leq j \leq k-1$, $\Delta_1(\bar{z}) \cap \Delta_1(y_j) \neq \phi$. We may assume $j = 2$.

Now G_{xyz} is transitive on the $k-2$ points of $\Delta_2(x) \cap \Delta_1(z) = \{z_2, \ldots, z_{k-1}\}$, and fixes $\bar{z}$. So there is $g_i \in G_{xyz}$ with $z_2^{g_i} = z_i$, $3 \leq i \leq k-1$. Then since $\{y_2\} = \Delta_1(x) \cap \Delta_1(z_2)$, $\{y_2^{g_i}\} = \Delta_1(x) \cap \Delta_1(z_i) = \{y_i\}$. But then, $\Delta_1(\bar{z}) \cap \Delta_1(y_i) = \Delta_1(\bar{z}) \cap \Delta_1(y_2^{g_i}) = [\Delta_1(\bar{z}) \cap \Delta_1(y_2)]^{g_i} \neq \phi$. So $\Delta_1(\bar{z}) \cap \Delta_1(y_j) \neq \phi$ for all j, $2 \leq j \leq k-1$, and hence $a \geq k-1$.

If $a = k-1$, then $m = k$. This is impossible by S. Iwasaki ([23], Proposition 3.5), since $k \geq 3$.

Thus $a = k$ and $m = k-1$. The following proposition shows

that this is impossible except for $k = 6$, or $k = 56$, in which case G is imprimitive.

Proposition 6.2. If there exists a transitive rank 4 permutation group with subdegrees 1, $k > 1$, $\ell = k(k-1)$, and $m = k-1$, for which the graph $\mathcal{H}$ constructed with respect to the suborbit of length k is connected, then $k = 2$, 6, or 56 and the group is imprimitive.

Proof. Let

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ k & k_{11} & k_{12} & k_{13} \\ 0 & k_{21} & k_{22} & k_{23} \\ 0 & k_{31} & k_{32} & k_{33} \end{pmatrix}$$

be the intersection matrix for $\mathcal{H}$. The parameter conditions give:

$k_{21}k = k_{12}k(k-1) \Rightarrow$ either $k_{12} = k_{21} = 0$, or $k_{21} = k-1$ and $k_{12} = 1$;

$k_{31}k = k_{13}(k-1) \Rightarrow$ either $k_{31} = k_{13} = 0$, or $k_{31} = k-1$ and $k_{13} = k$;

$k_{32}k(k-1) = k_{23}(k-1) \Rightarrow$ either $k_{32} = k_{23} = 0$, or $k_{32} = 1$ and $k_{23} = k$.

Now $\mathcal{H}$ is connected if and only if the pairs (k_{12}, k_{13}), (k_{13}, k_{23}) and (k_{12}, k_{32}) are all not $(0, 0)$. Together with the above possible parameters we see that this implies that $k_{12} = 1$, $k_{13} = 0$ and $k_{23} = k$, and

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ k & 0 & 1 & 0 \\ 0 & k-1 & k-2 & k \\ 0 & 0 & 1 & 0 \end{pmatrix} .$$

The graph polynomial is:

$F(X) = X^3 + AX^2 + BX + C$, where A, B, C are given in terms of k and the k_{ij}'s as in Section 1. Let s, t and u be the roots of $F(X)$. Then we know (see, for example, D. G. Higman [18] or S. Iwasaki [23], Lemma 2.3) that at least one of s, t and u is an integer, and there are positive integers f_1, f_2 and f_3 such that the following three equations hold:

$$f_1 + f_2 + f_3 = k + \ell + m = k^2 + k - 1. \tag{6.2.1}$$

$$sf_1 + tf_2 + uf_3 = -k. \tag{6.2.2}$$

$$s^2f_1 + t^2f_2 + u^2f_3 = k(\ell + m + 1) = k^3. \tag{6.2.3}$$

Now $F(X) = X^3 + 2X^2 + (1-k)X - k = X^3 + 2X^2 + X - k(X+1)$

$= (X+1)(X^2 + X - k)$.

Let $s = -1$, $t = \frac{-1 + \sqrt{1 + 4k}}{2}$ and $u = \frac{-1 - \sqrt{1 + 4k}}{2}$, so $tu = -k$ and $t + u = -1$. Multiplying (6.2.2) by u gives $-uf_1 - kf_2 + u^2f_3 = -ku$, and multiplying (6.2.2) by t gives $-tf_1 + t^2f_2 - kf_3 = -kt$.

So $-f_1(u+t) + u^2f_3 + t^2f_2 - kf_2 - kf_3 = -k(u+t)$, and thus from (6.2.3) we get $k^3 - kf_2 - kf_3 = k$.

Hence $f_2 + f_3 = k^2 - 1$, and (6.2.1) gives $f_1 = k$. So (6.2.2) becomes $tf_2 + uf_3 = 0$, so $tf_2 = -uf_3 = -u[k^2-1-f_2]$, whence $f_2 = \frac{k^2 - 1}{2\sqrt{1+4k}} + \frac{k^2 - 1}{2}$. Thus $1+4k$ is a square and $32f_2 = (1+4k)^{3/2} - 2(1+4k)^{1/2} - \frac{15}{(1+4k)^{1/2}} + 16(k^2-1)$.

Therefore, $1+4k \mid 225$, so $1+4k = 1, 9, 25$ or 225. This gives $k = 2, 6$ or 56.

For $k = 2$, the group is clearly imprimitive. For $k = 6$, the intersection matrix of the graph constructed with respect to the suborbit of length $k-1$ is (similar to)

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 5 & 4 & 0 & 0 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

so this graph is not connected.

For $k = 56$, the intersection matrix of the graph constructed with respect to the suborbit of length $k-1$ is (similar to)

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 55 & 54 & 0 & 0 \\ 0 & 0 & 54 & 55 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

so this graph is also not connected.

Thus the group is imprimitive. □

This completes Lemma 6.3. □

From Proposition 6.2 we can conclude immediately:

Corollary 6.1. There exists no primitive rank 4 permutation group with subdegrees 1, k, k(k-1), k-1. □

Proposition 6.3. In Case 2, $k = 6$ or 56 and G is imprimitive.

Proof.

If $k \neq 6$ or 56 then by Lemma 6.3, $a \leq 2$.

If $a = 2$ then $\Delta_1(\bar{z}) \cap \Delta_1(y_k) \neq \phi$, say $\{z_k\} = \Delta_1(\bar{z}) \cap \Delta_1(y_k)$.

Now we have the following. If $a = 1$, $|\Delta_2(y) \cap \Delta_1(\bar{z})| = k-2$, and $\Delta_2(y) \cap \Delta_1(\bar{z}) \subseteq \Delta_3(x)$, while if $a = 2$, $\Delta_2(y) \cap \Delta_1(\bar{z}) \subseteq \Delta_3(x) \cup \{z_k\}$. Thus $|\Delta_1(\bar{z}) \cap (\Delta_2(y) \cap \Delta_3(x))| = k-3$ or $k-2$. Hence the points of $\Delta_2(y) \cap \Delta_3(x)$, of which there are $k-1$, together with adjacency induced from $\mathcal{H}_1$, form a subgraph of $\mathcal{H}_1$ with the following three properties:

(a) every vertex has constant valency equal to $k-3$ or to $k-2$,

(b) there are no triangles, and

(c) there are no quadrangles.

Clearly (a), (b), and (c), and the fact that there are $k-1$ vertices implies that $k-1 = 1$ or 2, so $k = 3$.

Thus $F(X) = X^3 - 4X + 1$ $\quad (a=1)$, or

$F(X) = X^3 + X^2 - 3X - 1$ $\quad (a=2)$.

In both cases, however, $F(s) \not\equiv 0 \pmod 3$ for any integer s, so $F(X)$ has no integer root.

This is a contradiction. □

Corollary 6.2. There exists no primitive group G in Case 2. □

Case 3. $b = 0$ and $m = k(k-1)^2/a$. So the girth of $\mathcal{H}_1$ is > 5.

Let $x \in \Omega$ and $\{y_1, \ldots, y_k\} = \Delta_1(x)$. Write $Y_i = \Delta_1(y_i) \cap \Delta_2(x)$, $1 \leq i \leq k$. Then $|Y_i| = k-1$. Let $z \in Y_1$ and put $Z = \Delta_1(z) \cap \Delta_3(x)$ so that $|Z| = k-1$. Write $Z = \{z_1, \ldots, z_{k-1}\}$.

Note the following in $\mathcal{H}_1$.

(i) Suppose for some $z_i \in Z$, $1 \leq i \leq k-1$, and some j, $2 \leq j \leq k$, that $\Delta_1(z_i) \cap Y_j \neq \phi$. Then $|\Delta_1(z_i) \cap Y_j| = 1$ since there are no rectangles. (In fact this holds for any $u \in \Delta_3(x)$ with u in place of z_i.)

(ii) $\Delta_1(z_i) \cap Y_1 = \{z\}$ for $1 \leq i \leq k-1$.

(iii) For any j, $1 \leq j \leq k$, such that $\Delta_1(z_i) \cap Y_j \neq \phi$ (for some i), we have $\Delta_2(z_i) \cap Y_j = \phi$ since there are no pentagons or triangles. (Again this holds for any $u \in \Delta_3(x)$ with u in place of z_i.)

(iv) If $z_i, z_j \in Z$, $1 \leq i \neq j \leq k-1$, then $\Delta_1(z_i) \cap \Delta_2(z_j) = \phi$.

For a subset $A \subseteq \Omega$, put $\Delta_i(A) = \bigcup_{a \in A} \Delta_i(a)$, $i = 1, 2, 3$. Since we are supposing $G_{yz}^{\Delta_1(z)-\{y\}} \simeq G_{yz}^{\Delta_1(y)-\{z\}}$ as permutation groups, G_{xyz} fixes a point, say $z_1 \in Z$, and is transitive on $Z - \{z_1\}$.

By Iwasaki ([23], Lemma 3.1), $a \geq 2$. Note that each of the $k-1$ points in Z is adjacent to $a-1$ points $\neq z$ in $\Delta_2(x)$, and

since there are no rectangles, there are $(a-1)(k-1)$ distinct points in $\Delta_1(Z) \cap \Delta_2(x)$ other than z.

Since $|\Delta_2(z) \cap Y_j| = |\Delta_2(z) \cap \Delta_1(y_j) \cap \Delta_2(x)| = |\Delta_2(z) \cap \Delta_1(y_j)| - 1 = a-1$ for $2 \leq j \leq k$, we have $(\Delta_1(Z) \cap \Delta_2(x)) - \{z\} = \bigcup_{j=2}^{k} (\Delta_2(z) \cap Y_j)$, a disjoint union of $k-1$ sets of size $a-1$.

Suppose without loss of generality that $\Delta_1(z_1) \cap Y_j \neq \phi$ for $j = 2, \ldots, a$. Then it is clear that since G_{xyz} fixes z_1, G_{xyz} induces a group of permutations on $\{Y_2, \ldots, Y_a\}$. Let $d = |\Delta_1(z_2) \cap (\bigcup_{j=2}^{a} Y_j)|$. Take $g_i \in G_{xyz}$ such that $z_2^{g_i} = z_i$, $3 \leq i \leq k-1$. Then we see that $d = |\Delta_1(z_i) \cap (\bigcup_{j=2}^{a} Y_j)|$, i.e. every element of $Z - \{z_1\}$ is "adjacent" to the same number, d, of distinct Y_j's, $2 \leq j \leq a$, by (i).

We count in 2 ways the number N of pairs (u, v), $v \in \Delta_1(u)$, with $u \in \bigcup_{j=2}^{a} Y_j$, $v \in Z - \{z_1\}$. For each j there are $(a-1)$ u's in Y_j which are adjacent to exactly $(a-1)$ distinct $v \in Z$, since $|\Delta_2(z) \cap Y_j| = a-1$, and there are no rectangles. One of these v's is z_1. So $N = (a-1)(a-2)$. Also for each $v \in Z - \{z_1\}$ there are exactly d u's in $\bigcup_{j=2}^{a} Y_j$, from before, with $v \in \Delta_1(u)$. So, since $|Z - \{z_1\}| = k-2$, we have $N = d(k-2)$.

Thus $(a-1)(a-2) = d(k-2)$.

Proposition 6.4. (i) $k-2 \mid (a-1)(a-2)$ and $a \mid k(k-1)^2$.

(ii) Either $2 \leq a \leq k/2$ and G is primitive, or $a = k$ and G is imprimitive.

Proof. (i) Clear from above, and $m = k(k-1)^2/a$.

(ii) Suppose $a = k$. Then the graph $\mathcal{H}_2$ constructed with respect to the suborbit of length $k(k-1)$ has intersection matrix (from M_2) similar to

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ k(k-1) & k^2-k-1 & 0 & 0 \\ 0 & 0 & k-1 & k \\ 0 & 0 & (k-1)^2 & k(k-2) \end{pmatrix},$$

so $\mathcal{H}_2$ is not connected and hence G is imprimitive.

So suppose $a < k$. (We have already observed that $a \geq 2$.) Then there is a $v \in \Delta_3(x) \cap \Delta_1(z_1)$. Since $|\Delta_1(v) \cap \Delta_2(x)| = a = |\Delta_1(z_1) \cap \Delta_2(x)|$, by (iii) $2a \leq k$, i.e. $a \leq k/2$.

Now consider the intersection matrices M_2 and M_3 (see Section 1). Then $\ell_{22} = a(k-1) - 1$. Hence from the parameter conditions we get $\ell_{21} = 0$, $\ell_{23} = a(k-a)$, $\ell_{31} = (k-1)^2$, $\ell_{32} = (k-a)(k-1)$, $\ell_{33} = k^2-k-a-ak+a^2$, $m_{31} = (k-1)^2(k-a)/a$, $m_{32} = (k-1)\ell_{33}/a$, and $m_{33} = m-1-(k-a)-\ell_{33}$. So $\mathcal{H}_3$ is connected and $\mathcal{H}_2$ is connected if and only if $k \neq a$. □

Proposition 6.4 provides no arithmetic information in the case that $a = 2$ (equivalently, $d = 0$). This case, however, is partially handled by the following proposition.

Proposition 6.5. Under the conditions of Case 3, if $a = 2$, then $k \in \{4, 7, 16, 56, 154, 379, 742, 1379, 3404, 6671, 166754\}$ and G is primitive.

Proof. Suppose $a = 2$. Then G is primitive (Proposition 6.4). Also $F(X) = X^3 + 2X^2 + (3-2k)X + 2 - 2k = (X+1)(X^2 + X + 2 - 2k)$. Let $s = -1$, $t = \frac{-1+\sqrt{8k-7}}{2}$, $u = \frac{-1-\sqrt{8k-7}}{2}$ be the roots of $F(X)$, so $u+t = -1$, $ut = 2(1-k)$. As in Proposition 6.2, there are positive integers f_1, f_2 and f_3 such that

$$f_1 + f_2 + f_3 = \frac{k(k^2+1)}{2} , \tag{6.2.4}$$

$$-f_1 + tf_2 + uf_3 = -k \quad , \quad \text{and} \tag{6.2.5}$$

$$f_1 + t^2f_2 + u^2f_3 = \frac{k}{2}[k^3 - k + 2] . \tag{6.2.6}$$

Multiplying (6.2.5) first by t and then by u, and then adding, gives $f_1 + t^2f_2 + u^2f_3 + 2(1-k)(f_2 + f_3) = k$, which together with (6.2.6) gives $\frac{k}{2}[k^3 - k + 2] + 2(1-k)(f_2 + f_3) = k$, which reduces to $f_2 + f_3 = \frac{k^2}{4}(k+1)$. Thus either $k \equiv 0 \pmod 2$ or $k \equiv -1 \pmod 4$. So by equation (6.2.4), $f_1 = \frac{k(k^2+1)}{2} - \frac{k^2(k+1)}{4} = \frac{k(k^2-k+2)}{4}$. Thus $tf_2 + uf_3 = -k + \frac{k(k^2-k+2)}{4} = \frac{k(k-2)(k+1)}{4}$, whence $tf_2 = \frac{k}{4}(k-2)(k+1) - u[\frac{k^2}{4}(k+1) - f_2]$. So $(t-u)f_2 = \frac{k(k+1)}{4}[k-2-ku]$, giving $\sqrt{8k-7}\,f_2 = \frac{k(k+1)}{8}[3k-4+k\sqrt{8k-7}]$, so $8k-7$ is a square. Now $2^9k(k+1)(3k-4) = 3(8k-7)^3 + 55(8k-7)^2 + 73(8k-7) - 1155$, so $2^{12}f_2 = 3(8k-7)^{5/2} + 55(8k-7)^{3/2} + 73(8k-7)^{1/2} - \frac{1155}{(8k-7)^{1/2}} + 2^9k^2(k+1)$. Thus $8k-7 \mid 3^2.5^2.7^2.11^2$, giving $8k-7 = 1, 9, 25, 49, 121, 225, 441, 1089, 1225, 3025, 5929, 11025, 27225, 53361, 148225$ or 1334025. Thus $k = 1, 2, 4, 7, 16, 29, 56, 137, 154, 379, 742, 1379, 3404, 6671, 18529$ or 166754.

Since $k \equiv -1 \pmod 4$ when k is odd, this leaves k as in the statement of the lemma. □

There are now two more possibilities which need to be considered. These are obtained by replacing hypothesis (H) with either of the following:

(H') The two 2-transitive representations of G_{yz} on $\Delta_1(y) - \{z\}$ and on $\Delta_1(z) - \{y\}$ are inequivalent but have the same permutation character.

(H'') The two 2-transitive representations of G_{yz} on $\Delta_1(y) - \{z\}$ and on $\Delta_1(z) - \{y\}$ are inequivalent and have different permutation characters.

Under (H''), for $x \in \Delta_1(y) - \{z\}$, G_{xyz} is transitive on $\Delta_1(y) - \{x, z\}$ and $\Delta_1(z) - \{y\}$ and thus, since $\mathcal{H}_1$ is assumed to be connected, $b = 0$ and G is transitive on paths of length 3. An example of this is given by the action of $G = \Sigma_7$ on the set Ω of 3-element subsets of $\{1, 2, \ldots, 7\}$. G has rank 4, degree 35 and subdegrees $1, k = 4, \ell = 12$ and $m = 18$. For $x \in \Omega$, $G_x \simeq \Sigma_3 \times \Sigma_4$ and $G_x^{\Delta_1(x)} \simeq \Sigma_4$. It can easily be verified that G_{xyz} is transitive on $\Delta_1(z) - \{y\}$ for $y \in \Delta_1(x)$, $z \in \Delta_2(x)$. In this example $a = 2$.

Under (H'), for $x \in \Delta_1(y) - \{z\}$, G_{xyz} is transitive on $\Delta_1(y) - \{x, z\}$ and has two orbits on $\Delta_1(z) - \{y\}$, neither of which is a single point. I know of no examples in this situation.

APPENDIX 1

EXAMPLES OF CUBIC POLYGONAL GRAPHS

Only examples of m-gon-graphs with $m \geq 7$ are listed here. For examples with $m \leq 6$ see Chapter II, Sections 1 and 2.

For each example the following information is provided: The group G; the stabilizer of a vertex $H \simeq \Sigma_3$; the degree and rank of G; the "type" of the graph $\mathcal{H}$ with the corresponding map where applicable (see Chapter II, Section 4); and finally the intersection matrix M (or portion thereof) corresponding to the suborbit with respect to which $\mathcal{H}$ is defined.

The group $G^{p,q,r} = \langle A, B, C: A^p = B^q = C^r = (AB)^2 = (BC)^2 = (CA)^2 = (ABC)^2 = 1 \rangle$. In all cases, $\mathcal{H}$ is defined with respect to the suborbit (i.e. double coset) HDH, where $D = AB$, except for (vii) where $D = AC$.

Heptagraphs

The intersection matrices are all of the form

$$M = \left(\begin{array}{c|c} M_1 & \cdots \\ \hline \cdots & \cdots \end{array}\right), \quad \text{where}$$

$$M_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

(i) $G = G^{3,7,8} \simeq PGL(2,7)$.

$H = \langle A, BC \rangle$, degree = 56, rank = 13, map = $\{7,3\}_8$.

(ii) $G = G^{3,7,9} \simeq PSL(2,8)$.

$H = \langle A, BC \rangle$, degree = 84, rank = 17, map = $\{7,3\}_9$.

(iii) $G = G^{3,7,9} \times Z_2 \simeq PSL(2,8) \times Z_2$, $Z_2 = \langle E: E^2 = 1 \rangle$.

$H = \langle A, BCE \rangle$, degree = 168, rank = 34.

(iv) $G = G^{3,7,13} \simeq PSL(2,13)$.

$H = \langle A, BC \rangle$, degree = 182, rank = 34, map = $\{7,3\}_{13}$.

(v) $G = G^{3,7,12} \simeq G^{3,7,14} \simeq PGL(2,13)$.

$H = \langle A, BC \rangle$, degree = 364, rank = 69, map = $\{7,3\}_{12}$ or $\{7,3\}_{14}$.

(vi) $G = G^{3,7,15} \simeq PSL(2,29)$.

$H = \langle A, BC \rangle$, degree = 2030, rank = 347, map = $\{7,3\}_{15}$.

Octagraphs

The intersection matrices are all of the form

$M = \left(\begin{array}{c|c} M_2 & \cdots \\ \hline \cdots & \cdots \end{array} \right)$, where

$$M_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * & * & * \end{pmatrix} .$$

(vii) $G = G^{3,7,8} \times Z_2 \simeq PGL(2,7) \times Z_2$, $Z_2 = \langle E: E^2 = 1 \rangle$.

$H = \langle A, BCE \rangle$, degree = 112, rank = 26, map = $\{8,3\}_8$.

(viii) $G = G^{3,8,10}$.

$H = \langle A, BC \rangle$, degree = 720, rank = 146, map = $\{8,3\}_{10}$.

(ix) $G = G^{3,8,11} \simeq PGL(2,23)$.

$H = \langle A, BC \rangle$, degree = 2024, rank = 351, map = $\{8,3\}_{11}$.

(x) $G = G^{3,8,11} \times Z_2 \simeq PGL(2,23) \times Z_2$, $Z_2 = \langle E: E^2 = 1 \rangle$.

$H = \langle A, BCE \rangle$, degree = 4048, rank = 702.

Nonagraphs

The intersection matrices are all of the form

$$M = \left(\begin{array}{c|c} M_3 & \cdots \\ \hline \cdots & \cdots \end{array} \right) , \text{ where}$$

$$M_3 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} .$$

(xi) $G = G^{3,9,9} \times Z_2 \simeq PSL(2,19) \times Z_2$, $Z_2 = \langle E: E^2 = 1 \rangle$.

$H = \langle A, BCE \rangle$, degree = 1140, rank = 202,

("doubling" of the map $\{9,3\}_9$).

(xii) $G = G^{3,9,9} \times \Sigma_3 \simeq PSL(2,19) \times \Sigma_3$, $\Sigma_3 =$

$\langle E, F: E^2 = F^3 = (EF)^2 = 1 \rangle$.

$H = \langle AF, BCEF \rangle$, degree = 3420, rank = 583, map = $\{9,3\}_{10}$.

APPENDIX 2

EXAMPLES OF POLYGONAL GRAPHS OF VALENCY ≥ 4

We list here the examples of polygonal graphs of Chapter III, Sections 2 and 3. For the examples arising from regular solids, see Chapter III, Section 1. The format is the same as for the examples given in Appendix 1.

(i) $G = PGL(2,7)$.

$H \simeq \Sigma_4$, degree = 14, rank = 4.

$\mathcal{H}$ is a rectagraph of valency 4.

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 0 & 3 & 0 & 4 \\ 0 & 0 & 2 & 0 \end{pmatrix} .$$

(ii) $G = PGL(2,11) = \langle g, h, t, z: g^2 = h^2 = t^2 = z^3 = (tz)^3 = (ht)^3 = (hz)^2 = (gt)^2 = (gz)^2 = (gh)^4 z^{-1} = 1 \rangle$.

$H = \langle h, t, z \rangle \simeq A_5$, degree = 22, rank = 4.

$\mathcal{H}$, defined with respect to the suborbit HgH, is a rectagraph of valency 5.

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 5 & 0 & 2 & 0 \\ 0 & 4 & 0 & 5 \\ 0 & 0 & 3 & 0 \end{pmatrix} .$$

(iii) $G = PSL(2,19)$.

$H \simeq A_5$, degree = 57, rank = 4.

$(\mathcal{H},\&)$ is a 5-gon-graph of valency 6 with $\&$ the set of pentagons fixed pointwise by involutions in G.

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 6 & 0 & 1 & 0 \\ 0 & 5 & 3 & 3 \\ 0 & 0 & 2 & 3 \end{pmatrix} .$$

(iv) $G = PSL(2,31) = \langle g, h, t, z: g^2 = h^2 = t^2 = z^3 = (tz)^3 = (ht)^3 = (hz)^2 = (gt)^2 = (gz)^2 = (gh)^5 z^{-1} = 1 \rangle$.

$H = \langle h, t, z \rangle \simeq A_5$, degree = 248, rank = 9.

$\mathcal{H}$ defined with respect to the suborbit HgH, is a pentagraph of valency 5.

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 2 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 2 & 5 & 2 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 2 \end{pmatrix} .$$

(v) $G = PSL(2,41)$ is generated by the matrices

$$g \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \qquad h \equiv \begin{pmatrix} 15 & 10 \\ 2 & -15 \end{pmatrix} ,$$

$$t \equiv \begin{pmatrix} -6 & -2 \\ -2 & 6 \end{pmatrix} , \qquad z \equiv \begin{pmatrix} 0 & 9 \\ 9 & -1 \end{pmatrix} .$$

$H = \langle h, t, z \rangle \simeq A_5$, degree = 574, rank = 16.

$\mathcal{K}$, defined with respect to the suborbit HgH, is an heptagraph of valency 5.

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