This dissertation has been
microfilmed exactly as received

69-12,159

KOZLOWSKI, Jr., George Aloysius, 1941-
MAPPING THEOREMS FOR HOMOTOPY.

The University of Michigan, Ph.D., 1968
Mathematics

University Microfilms, Inc., Ann Arbor, Michigan

MAPPING THEOREMS FOR HOMOTOPY

by

George Aloysius Kozlowski, Jr.

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy in
The University of Michigan
1968

Doctoral Committee:

Professor Morton Brown, Chairman
Associate Professor Ronald Douglas
Professor James Kister
Associate Professor Ronald Rosen
Associate Professor Jens Zorn

ABSTRACT

MAPPING THEOREMS FOR HOMOTOPY

by

George A. Kozlowski, Jr.

Chairman: Morton Brown

The basic problem undertaken is to give conditions on the point inverses of a proper surjective map $f: X' \to X$ between metric spaces to insure that maps induced on homotopy classes of maps will be bijections. For a space S dominated by a CW-complex both $f_{\#}: [S;X'] \to [S;X]$ (covariant case) and $f^{\#}: [X;S] \to [X';S]$ (contravariant case) are considered.

The covariant case is reduced to the study of the induced map on the homotopy groups. The map $f: X' \to X$ is called a local connection in dimension n, provided that for each $x \in X$ every neighborhood U of x contains a neighborhood V of x such that for $0 \leq k \leq n-1$ any map $S^k \to f^{-1}V$ extends to a map $B^{k+1} \to f^{-1}U$ and such that for any map $g: S^n \to f^{-1}V$ the map $fg: S^n \to V$ extends to a map $B^{n+1} \to U$.

By the use of star-refinements of open covers an approximation procedure is developed which gives

THEOREM. If $f: X' \to X$ is a local connection in dimension n, then for any pair (K,L) of finite simplicial complexes such that $\dim (\overline{K - L}) \leq n$ and any maps $g': L \to X'$ and $h: K \to X$ such that $fg' = h|L$ there is a map $g: K \to X'$ extending g' in such a way that fg is homotopic to h relative to L.

As a corollary it is then deduced that X is LC^n and that $f_{\#}: \pi_k(X',x') \to \pi_k(X,f(x'))$ is injective for $k < n$ and surjective for $k \leq n$.

If X and X' are further assumed to be ANR's, then it is shown that for any closed subset A of X $(f|\ f^{-1}A)^{\#}$: $[A;S] \to [f^{-1}A;S]$ is a bijection provided f is a local connection in dimension n, where $n \geq \text{Max}\{1 + \dim X', \dim X\}$; from this it can be deduced that a local connection in all dimensions between ANR's does not raise dimension.

If it is assumed that X' is a manifold and that $f^{-1}x$ is cellular for each $x \in X$, then it is shown that X is an n-homotopy manifold where $n = \dim X'$, provided $\dim X < \infty$ and that the following conditions are equivalent: $\dim X < \infty$, $\dim X \leq \dim X'$, X is an ANR.

The contravariant case is reduced to the case of maps into an ANR. For this case results are obtained only on the assumption that X is finite dimensional. A compact metric space A is (S,n)-<u>trivial</u> provided that for $0 \leq k \leq n$ any map of the k-fold (unreduced) suspension $\Sigma^k A$ of A into S is null homotopic.

THEOREM. If $f^{-1}x$ is (S,n)-trivial for every $x \in X$, then $f^{\#}$: $[X;S] \to [X';S]$ is surjective if $\dim X \leq n$ and bijective if $1 + \dim X \leq n$.

The theorem is then applied to deduce that if each $f^{-1}x$ is contractible or cell-like and X is finite dimensional, then $f^{\#}$ is a bijection. Also, if S is connected and n-simple for every n, X and X' are finite dimensional, and each $f^{-1}x$ has the integral Cech cohomology of a point, then $f^{\#}$: $[X;S] \to [X';S]$ is a bijection.

To My Parents

ACKNOWLEDGMENTS

I would like to thank Professor Morton Brown for his help as the director of this thesis. I would also like to thank Professor Ronald Rosen, my second reader, for his helpful comments and Professor James Kister for several useful conversations.

This research was supported by the National Science Foundation through a Graduate Fellowship and through NSF GP-6462.

TABLE OF CONTENTS

	Page
DEDICATION	ii
ACKNOWLEDGMENTS	iii
INTRODUCTION	1
Chapter	
I. THE COVARIANT CASE	
1. Local Connections	6
2. A Proof of Theorem 1	9
3. Effect on the Homotopy Groups	15
4. Local Connections between ANR's	20
5. Decompositions of Manifolds	25
II. THE CONTRAVARIANT CASE	
6. (Y,n)-trivial Spaces	27
7. A Proof of Theorem 6	34
8. Applications	39
APPENDIX	42
BIBLIOGRAPHY	48

INTRODUCTION

If $f: X' \to X$ is a proper[‡] surjective map between metric spaces and S is a space dominated by a CW-complex, then a natural problem, which has roots in classical topology, is to give conditions on the point inverses of f to insure that the induced maps on homotopy classes of 1) maps defined on S and 2) maps into S be bijections.

In the covariant case the map $f_{\#}: [S;X'] \to [S;X]$ is studied through the induced map on homotopy groups. Here Smale [21] and Price [19] have obtained apparently diverse results. The definition introduced for this case derived from an independent investigation of Price's theorem [19] which states that if X' and X are manifolds and if $f^{-1}x$ is a cellular subset of X' for each $x \in X$, then f induces isomorphisms of homotopy groups. Thus, f is said to be a <u>local connection in dimension</u> n, provided that for each $x \in X$ every neighborhood U of x contains a neighborhood V of x such that for $0 \leq k \leq n-1$ any map of $S^k \to f^{-1}V$ extends to a map $B^{k+1} \to f^{-1}U$ and such that for any map $g: S^n \to f^{-1}V$ the map fg extends to a map $B^{n+1} \to U$.

In the proof of the main theorem of Chapter I star-refinements of open covers are used to develop an approximation procedure which shows that if $f: X' \to X$ is

---

‡ Throughout this introduction let f, X', X, and S be fixed as in this paragraph. For some remarks on definitions see the last paragraph of the Introduction.

a local connection in dimension n, then for any pair (K,L) of finite simplicial complexes such that dim $(\overline{K - L}) \leq n$ and for any maps g: L → X' and h: K → X such that fg = h| L there is a map g': K → X' extending g such that fg' is homotopic to h relative to L. A similar but slightly weaker result has been announced by Price [20], who requires in the definition above that any map of S^n into $f^{-1}V$ extends to a map of B^{n+1} into $f^{-1}U$. Lacher [11,13] has also considered this condition. In Section 1 of Chapter I such a map is called a strong local connection in dimension n, the important point being that such a map onto an ANR is automatically a local connection in dimension (n+1).

It follows directly from the main theorem that X is LC^n and $f_{\#}$: $\pi_k(X',x') \to \pi_k(X,f(x'))$ is bijective for $k \leq n-1$ and surjective for $k = n$ for any $x' \in X'$ which leads to a solution of the covariant case: if f: X' → X is a local connection in dimension n, and if S is dominated by a CW-complex of dimension $\leq m$, then $f_{\#}$: $[S;X'] \to [S;X]$ is surjective if $m \leq n$ and bijective if $m+1 \leq n$. Also, it is shown that the maps considered by Smale in [21] are special cases of local connections in dimension n, which leads to an extension of his theorem.

When X and X' are further assumed to be ANR's analysis of the inverse system of neighborhoods and their inclusion maps of a fixed closed set gives information about the behavior of f restricted to inverse images of

closed sets. In this case if $f: X' \to X$ is a local connection in all dimensions, then $\dim X \leq \dim X'$, and more generally, if A is any closed subset of X the induced map $(f| f^{-1}A)^{\#}: [A;S] \to [f^{-1}A;S]$ is a bijection. Furthermore if X and X' are finite dimensional, both conclusions hold as long as f is a local connection in dimension n, where $n = \text{Max}\{1 + \dim X', \dim X\}$.

Another interesting corollary of the main theorem of Chapter I is that in the original case of a mapping $f: X' \to X$ of a manifold onto a manifold the relation between the homotopy groups is independent of the fact that X was a manifold. This allows a classification of certain decompositions of manifolds in terms of local and global homotopy properties. In fact if X' is an n-manifold, and $f^{-1}x$ is a cellular subset of X' for each $x \in X$, then provided X is finite-dimensional, X is an n-homotopy manifold and f is a homotopy equivalence. Furthermore, the following conditions are equivalent

i) X is finite dimensional
ii) $\dim X \leq n$
iii) X is an ANR

This result sharpens results of Kwun [10] and Kompaniec [9].

The analysis of the map $f^{\#}: [X;S] \to [X';S]$ (the contravariant case) is reduced to the case where S is an ANR and then specialized to the problem of giving conditions which insure that $f^{\#}$ is surjective, which has a natural restatement as a problem of extending a map $X' \to S$ over the mapping cylinder Z of f. If $h: X' \to S$

has an extension over Z, then $h|\ f^{-1}x$ is null homotopic for every point $x \in X$. Assuming this last condition one can define extensions of h over sets of a locally finite cover of Z; since any two do not necessarily agree on their common domain, one procedure for obtaining a global extension is to insure that the partial extensions are homotopic on their common domain. In case X is finite dimensional this will be so provided that iterated (unreduced) suspensions of point inverses of f do not admit essential maps into S.

A compact metric space A is said to be (S,n)-trivial, provided that for $0 \leq k \leq n$ any map of the k-fold suspension $\Sigma^k A$ of A into S is null homotopic (where $\Sigma^0 A = A$). The main theorem of Chapter II states that if X is finite dimensional and $f^{-1}x$ is (S,n)-trivial for every $x \in X$, then $f^{\#}: [X;S] \to [X';S]$ is surjective if $n \geq \dim X$ and bijective if $n \geq 1 + \dim X$. Contractible spaces and cell-like spaces, which are defined by Lacher in [11,12], are (S,n)-trivial for all n and every space S dominated by a CW-complex; also, a compact metric space A is (S,n)-trivial, if and only if the natural inclusion of S into the space of maps $A \to S$ is an n-equivalence. Also, if A is a finite dimensional compact connected Hausdorff space with integral Cech cohomology $\check{H}^q(A;Z) = 0$ for $q > 0$, and if S is connected and n-simple for every integer n, then A is (S,n)-trivial for every n. As a corollary it is deduced that if X and X' are both finite dimensional, connected, n-simple for every

n, and dominated by CW-complexes, and if $f: X' \to X$ is a monotone map such that the integral Cech cohomology $\check{H}^q(f^{-1}x;Z) = 0$ for $q > 0$ for every point $x \in X$, then f is a homotopy equivalence.

In the Appendix an analysis of $f^{\#}: [X;S] \to [X';S]$ is provided in case X and X' are compact Hausdorff spaces and S is a compact ANR.

Lastly, a word on notation and conventions is in order. Basically, the notation and definitions are those of Spanier [22]. Two exceptions are noteworthy: an ANR is a metric space which is an absolute neighborhood retract for the class of all metric spaces, and unless specified otherwise "complex" and "simplicial complex" mean "finite simplicial complex". When X is a metric space, dim X = Ind X is covering dimension, or equivalently, large inductive dimension.

CHAPTER 1

THE COVARIANT CASE

1. Local Connections

In this section further examples are presented to clarify the definition of local connection and to provide applications of the main theorem, which is stated at the close of the section.

For each integer $n \geq -1$ a map $f: X' \to X$ will be called a <u>local connection in dimension</u> n, if for every point $x \in X$ every neighborhood U of x contains a neighborhood V of x such that for $0 \leq k < n$ any map $S^k \to f^{-1}V$ extends to a map $B^{k+1} \to f^{-1}U$ and for any map $g: S^n \to f^{-1}V$ the map $fg: S^n \to V$ extends to a map $B^{n+1} \to U$. If $f: X' \to X$ is a local connection in dimension n for every n, f will be called a <u>local connection</u>. It will be useful to call a map $f: X' \to X$ a <u>strong local connection in dimension</u> n, provided that for every $x \in X$ every neighborhood U of x contains a neighborhood V of x such that for $0 \leq k \leq n$ any map $S^k \to f^{-1}V$ extends to a map $B^{k+1} \to f^{-1}U$. The case $n = -1$ in these definitions makes no requirements on the map $f: X' \to X$ and is included for the purpose of simplifying induction arguments in Section 2.

EXAMPLES.

1. If $f: X' \to X$ is a local connection in dimension n, then for any open subset U of X the same is true of the map $f \mid f^{-1}U: f^{-1}U \to U$.

2. The standard inclusion map $R^n \to S^n$ of Euclidean n-space into the n-sphere (regarded as the one-point compactification of R^n) is a local connection in dimension (n-1).

3. If X' is an ANR and f: X' → X is a closed surjective map such that $f^{-1}x$ is contractible for each point $x \in X$, then f is a local connection.

PROOF. If U is any open neighborhood of x in X, $f^{-1}U$ is an ANR; hence a homotopy of $f^{-1}x$ into itself between the identity map 1: $f^{-1}x \to f^{-1}x$ and a constant map extends to a homotopy $N \times I \to f^{-1}U$ between the inclusion map $N \to f^{-1}U$ of a neighborhood N of $f^{-1}x$ into $f^{-1}U$ and a constant map. Since f is a closed surjective map, fN is a neighborhood of x. This shows that f is a strong local connection in all dimensions and thus a local connection.

4. Let M be an n-manifold, and let G be an upper semi-continuous decomposition of M into cellular sets. Then the projection map P: M → M/G of M onto the decomposition space M/G is a local connection.

PROOF. The assertion that G is upper semi-continuous is equivalent to the assertion that P: M → M/G is a closed (surjective) map. If U is any neighborhood of a point $x \in M/G$, the cellularity of $P^{-1}x$ implies that $P^{-1}U$ contains an n-cell neighborhood Q of $P^{-1}x$. Since Q is a contractible and PQ is a neighborhood of x, it follows that P is a strong local connection in all

dimensions and thus a local connection.

5. The usual statement that a space is LC^n is equivalent to the statement that the identity map of the space is a strong local connection in dimension n. If A is an LC^{n-1} space which is a closed subset of an LC^n space X', then the inclusion map $A \to X'$ is a local connection in dimension n. If $f: X' \to X$ is a closed surjective map between paracompact Hausdorff spaces such that X' is LC^n and $f^{-1}x$ is LC^{n-1} and (n-1)-connected for every $x \in X$, then it will be shown in the proof of Theorem 3 that f is a local connection in dimension n.

6. If $f: X' \to X$ is a strong local connection in dimension (n-1), and if X is LC^n, then f is a local connection in dimension n. In particular, a strong local connection in dimension (n-1) between ANR's is a local connection in dimension n.

For a certain type of local connection $f: X' \to X$ the induced homomorphism provides a comparison between the homotopy groups of X' and X which is conveniently expressed by the main theorem of this chapter.

THEOREM 1. Let X be a metric space, and let $f: X' \to X$ be a local connection in dimension n with dense image. If L is any subcomplex of a finite simplicial complex K such that $\dim(\overline{K - L}) \leq n$, and if $g: L \to X'$ and $h: K \to X$ are maps such that $h | L = fg$, then there is a map $g': K \to X'$ such that $g' | L = g$ and fg' is homotopic to h relative to L.

2. A Proof of Theorem 1

Consider the simplicial complexes K and $L \subset K$ with the maps $h: K \to X$ and $g: L \to X'$. The proof of Theorem 1 given here is based on the possibility of finding a sequence $\{g_i \mid 1 \leq i < \infty\}$ of maps of K into X' such that $\{fg_i \mid 1 \leq i < \infty\}$ converges uniformly to h and on the possibility of finding a sequence $\{H_i \mid 1 \leq i < \infty\}$ of homotopies H_i between fg_i and fg_{i+1} such that the diameter of $H_i(p \times I)$ decreases to zero uniformly for all points $p \in K$. Lemma 1, which is also used in the proof of Theorem 3, deals with the existence of the maps g_i and partially provides for the homotopies H_i, whose existence is demonstrated by Lemma 2. These lemmas lead directly to the proof of Theorem 1, after which the section closes with some remarks concerning the extent to which fg' approximates h.

Certain conventions facilitate the statements of Lemmas 1 and 2. Let L be a subcomplex of the simplicial complex K. Provided $\dim(\overline{K - L}) \leq n$, the simplicial pair (K,L) will be called simply an n-pair. The word "simplex" will always be used to mean "closed simplex", and the subcomplex of K consisting of all simplexes of K of dimension $\leq q$ will be denoted K^q. If α is a collection of open sets in X, then a map $h: K \to X$ (a homotopy $H: K \times I \to X$) will be said to map K (resp. K x I) into α, provided that

for every simplex σ of K there is $U \in \alpha$ with $h(\sigma) \subset U$ (resp. $H(\sigma \times I) \subset U$). Associated with α is the collection $\alpha^* = \{U^* \mid U \in \alpha\}$, where $U^* = \bigcup \{U' \in \alpha \mid U \cap U' \neq \emptyset\}$. Recall that a Hausdorff space X is paracompact if and only if for every open cover α of X there is an open cover β of X such that β^* refines α.

Let α and β be open covers of a space X such that β refines α, and let $f: X' \to X$ be a map. For every integer $n \geq -1$ assertions $E(\beta,\alpha; n)$, $H(\beta,\alpha; n)$, and $H(\beta,\alpha; f; n)$ are defined as follows:

$E(\beta,\alpha; n)$. If (K,L) is any n-pair and $g: L \to X'$, $h: K \to X$ are any maps such that h extends fg and maps $\overline{K - L}$ into the collection $\{V \in \beta \mid f(X') \cap V \neq \emptyset\}$, then there is an extension $g': K \to X'$ of g such that for every simplex σ of $\overline{K - L}$ there is $U \in \alpha$ with $fg'(\sigma) \cup h(\sigma) \subset U$.

$H(\beta,\alpha; n)$. If (K,L) is any n-pair and $g: L \to X'$, $g': K \to X'$, $g'': K \to X'$ are any maps such that $g = g' \mid L = g'' \mid L$ and for every simplex σ of K there is $V \in \beta$ with $g'(\sigma) \cup g''(\sigma) \subset f^{-1}V$, then there is a homotopy $G: K \times I \to X'$ between g' and g'' relative to L which maps $K \times I$ into $f^{-1}\alpha = \{f^{-1}U \mid U \in \alpha\}$.

$H(\beta,\alpha; f; n)$. If (K,L) is any n-pair and $g: L \to X'$, $g': K \to X'$, $g'': K \to X'$ are any maps such that $g = g' \mid L = g'' \mid L$ and for every simplex σ of K there is $V \in \beta$ with $g'(\sigma) \cup g''(\sigma) \subset f^{-1}V$, then there is a homotopy $H: K \times I \to X$ between fg' and fg'' relative to L which maps

K x I into α'.

LEMMA 1. Let X be a paracompact Hausdorff space, and let f: X' → X be a strong local connection in dimension n. Then for any open cover α of X there is an open cover β of X refining α such that both $E(\beta,\alpha;\ n+1)$ and $H(\beta,\alpha;\ n)$ hold.

PROOF. For n = -1 there are no conditions on the map f, and both assertions hold for $\beta = \alpha$. Assume that the lemma is true for $n < k$, and let f: X' → X be a strong local connection in dimension k.

If α is an open cover of X, let γ be an open cover of X such that for each $W \in \gamma$ there is $U \in \alpha$ such that $W^* \subset U$ and any map $S^k \to f^{-1}(W^*)$ extends to a map $B^{k+1} \to f^{-1}U$. Let β be an open cover refining γ such that both $E(\beta,\gamma;\ k)$ and $H(\beta,\gamma;\ k-1)$ hold.

$E(\beta,\alpha;\ k+1)$ holds. If K, L, g, and h satisfy the hypotheses of the assertion, there is an extension $g'': L \cup K^k \to X'$ of g such that for every k-simplex τ of $\overline{K - L}$ there is $W(\tau) \in \gamma$ with $fg''(\tau) \cup h(\tau) \subset W(\tau)$. Let σ be any (k+1)-simplex of K - L, and let $W \in \gamma$ be such that $h(\sigma) \subset W$. If τ is a proper face of σ, then $h(\tau) \subset W \cap W(\tau)$. It follows that $fg''(\partial\sigma) \subset W^*$; hence there is $U \in \alpha$ such that $W^* \subset U$ and there is an extension $\sigma \to f^{-1}U$ of $g'' \mid \partial\sigma: \partial\sigma \to f^{-1}(W^*)$. Combining such extensions gives the map g': K → X' required by the conclusion of $E(\beta,\alpha;\ k+1)$.

$H(\beta,\alpha;\ k)$ holds. If K, L, g, g', and g'' satisfy the hypotheses of $H(\beta,\alpha;\ k)$, let G': $(L \cup K^{k-1}) \times I \to X'$

be a homotopy between $g' \mid L \cup K^{k-1}$ and $g'' \mid L \cup K^{k-1}$ satisfying the conclusion of $H(\beta,\gamma;\ k-1)$, and extend G' to G'': $K \times \{0,1\} \cup (L \cup K^{k-1}) \times I \to X'$ by $G''(p,0) = g'(p)$ and $G''(p,1) = g''(p)$ for all points $p \in K$. For any $(k-1)$-simplex τ of K there is $W(\tau) \in \gamma$ such that $G''(\tau \times I) = G'(\tau \times I) \subset f^{-1}(W(\tau))$, and for any k-simplex σ of K there is $W \in \gamma$ such that $G''(\sigma \times \{0,1\}) \subset f^{-1}W$; thus

$$G''(\partial(\sigma \times I)) \subset \cup \{f^{-1}W' \mid W' \in \gamma \text{ and } f^{-1}W' \cap f^{-1}W \neq \emptyset\} \subset f^{-1}(W^*).$$

For each k-simplex σ of $K - L$ there is $U \in \alpha$ such that $W^* \subset U$ and there is an extension $\sigma \times I \to f^{-1}U$ of $G'' \mid \partial(\sigma \times I) : \partial(\sigma \times I) \to f^{-1}(W^*)$. These extensions define an extension $G: K \times I \to X'$ of G'' which maps $K \times I$ into $f^{-1}\alpha$. This completes the proof of Lemma 1.

LEMMA 2. Let X be a paracompact Hausdorff space, and let $f: X' \to X$ be a local connection in dimension n. Then for any open cover α of X there is an open cover β refining α such that $H(\beta,\alpha;\ f;\ n)$ holds.

PROOF. If $n = -1$, there is nothing to prove. Assume that the lemma is true for $n < k$, and let $f: X' \to X$ be a local connection in dimension k. Then f is a strong local connection in dimension $k-1$. If α is an open cover of X, let γ be an open cover such that for each $W \in \gamma$ there is $U \in \alpha$ such that $W^* \subset U$ and for any map $h: S^k \to f^{-1}(W^*)$ the map fh extends to a map $B^{k+1} \to U$. By Lemma 1 there is an open cover β of X refining γ such that $H(\beta,\gamma;\ k-1)$ holds. To see that $H(\beta,\alpha;\ f;\ k)$ holds

consider K, L, g, g', and g" satisfying the hypothesis of the assertion, and let $G:(L \cup K^{k-1}) \times I \to X'$ be a homotopy between $g'|L \cup K^{k-1}$ and $g''|L \cup K^{k-1}$ as in $H(\beta,\gamma; k-1)$. Extend G to a map $G': K \times \{0,1\} \cup (L \cup K^{k-1}) \times I \to X'$ by $G'(p,0) = g'(p)$ and $G'(p,1) = g''(p)$, and observe that for every k-simplex σ of K - L there is $W \in \gamma$ with $fG'(\partial(\sigma \times I)) \subset W^*$. It follows that fG' extends to a homotopy H which maps K x I into α, which completes the proof of Lemma 2.

We now prove Theorem 1.

Let d be a metric for X. Using Lemmas 1 and 2 choose a sequence $\{\alpha_i \mid 0 \leq i < \infty\}$ of open covers of X such that for $0 \leq i < \infty$ α^*_{i+1} refines α_i, $E(\alpha_{i+1},\alpha_i; n)$ holds, $H(\alpha^*_{i+1},\alpha_i; f; n)$ holds, and $\lim_i$ mesh $\alpha_i = 0$, where mesh α_i is the supremum of the diameters with respect to d of the members of α_i. If K, L, g, and h are as in Theorem 1, choose a sequence $\{K_i \mid 1 \leq i < \infty\}$ of simplicial subdivisions of K such that for $1 \leq i < \infty$ K_{i+1} is a subdivision of K_i, and h maps K_i into the cover α_{i+1}. Using the fact $E(\alpha_{i+1},\alpha_i; n)$ holds choose a sequence $\{g_i \mid 1 \leq i < \infty\}$ of extensions $g_i: K \to X'$ of $g: L \to X'$ such that for every simplex σ of K_i there is $U \in \alpha_i$ with $fg_i(\sigma) \cup h(\sigma) \subset U$.

Set $g' = g_1$, and construct a homotopy H between h and fg' as follows. Let σ be a simplex of K_i, and let $fg_i(\sigma) \cup h(\sigma) \subset U$ for some $U \in \alpha_i$. Consider σ as a subcomplex of K_{i+1}, and observe that for every simplex τ of σ

there is $U' \in \alpha_i$ such that $fg_{i+1}(\tau) \cup h(\tau) \subset U'$; hence $fg_{i+1}(\sigma) \cup fg_i(\sigma) \subset U^*$. Since $H(\alpha_i^*, \alpha_{i-1}; f; n)$ holds, there is a homotopy H_i between fg_{i+1} and fg_i relative to L, which may be considered as a map H_i: K x [1/(i+1),1/i] → X, such that $H_i(p,1/(i+1)) = fg_{i+1}(p)$ and $H_i(p,1/i) = fg_i(p)$ for all points $p \in K$ and such that the diameter of $H_i(\sigma \times [1/(i+1),1/i])$ is $\leq$ mesh α_{i-1} for every simplex α of K_i. Then

(*) $\quad d(H_i(p,t),h(p)) \leq$ mesh α_{i-1} + mesh α_i

for all points $(p,t) \in K \times [1/(i+1),1/i]$. Define H: K x I → X by $H(p,t) = H_i(p,t)$ for $1/(i+1) \leq t \leq 1/i$ and by $H(p,0) = h(p)$ for all points $p \in K$. Since $\lim_i$ mesh $\alpha_i = 0$, there is for every positive real number r a positive integer i such that $d(H(p,t),h(p)) < r$ for every point $p \in K$ and for all t satisfying $0 < t \leq 1/i$. This implies that H is continuous at all points of K x 0, and since H is clearly continuous at all other points of K x I, the proof is completed.

ADDENDUM TO THEOREM 1. If d is any metric for X and $\epsilon > 0$, the extension g' and the homotopy H may be chosen so that for every point $p \in K$ the diameter of H(p x I) is $< \epsilon$.

In fact if $\epsilon > 2$ (mesh α_{i-1} + mesh α_i) for $1 \leq i < \infty$, then it follows from the inequality (*) that for every point (p,t) of K x I $d(H(p,t),h(p)) < \epsilon/2$ and thus that the diameter of H(p x I) is $< \epsilon$.

3. Effect on the Homotopy Groups

For any space X and any point $x \in X$ let $\pi_0(X,x)$ be the set of path components of X. For any integer $n \geq 0$ a map $f: X' \to X$ is an n-equivalence, if for every $x' \in X'$ the induced homomorphism $f_{\#}: \pi_k(X',x') \to \pi_k(X,f(x'))$ is an isomorphism for $k < n$ and an epimorphism for $k = n$.

In this section it is shown that Theorem 1 implies that if X is a metric space and $f: X' \to X$ is a local connection in dimension n with dense image, then f is an n-equivalence. From this fact and the definition of local connection in dimension n it then follows that X is LC^n. These results are stated in Theorem 2 which is used with Lemma 1 to give an extension of Smale's Mapping Theorem for Homotopy[21].

LEMMA 3. If $f: X' \to X$ is a map satisfying the conclusion of Theorem 1 for some $n \geq 0$, then f is an n-equivalence.

PROOF. If $h: I^k, \partial I^k \to X, f(x')$ is a map $(0 \leq k \leq n)$, let $K = I^k$ and $L = \partial I^k$ and define $g: \partial I^k \to X'$ by $g(\partial I^k) = x'$. By hypothesis there is a map $g': I^k, \partial I^k \to X', x'$ such that fg' is homotopic to h relative to ∂I^k, which shows that f maps the set of path components of X' onto the set of path components of X (the case when $k = 0$), and that $f_{\#}: \pi_k(X',x') \to \pi_k(X,f(x'))$ is surjective

for $k \leq n$.

If $g_0, g_1 : I^k, \partial I^k \to X', x'$ are maps such that fg_0 is homotopic to fg_1 relative to ∂I^k by a map $H : I^k \times I, \partial I^k \times I \to X, f(x')$, set $K = I^k \times I$, $L = (I^k \times \partial I) \cup (\partial I^k \times I)$, $h = H$, and define $g : L \to X'$ by $g(u,t) = f(x')$ for $(u,t) \in \partial I^k \times I$ and by $g(u,0) = g_0(u)$ and $g(u,1) = g_1(u)$ for $u \in I^k$. By hypothesis g extends to a map $g : I^k \times I, \partial I^k \times I \to X', x'$, which shows that f maps the set of path components injectively to the set of path components of X (the case when $k = 0$) and that $f_{\#} : \pi_k(X', x') \to \pi_k(X, f(x'))$ is injective for $k < n$.

THEOREM 2. If X is a metric space and $f : X' \to X$ is a local connection in dimension $n (\geq 0)$ with dense image, then X is LC^n and f is an n-equivalence.

PROOF. The second assertion follows from Theorem 1 and Lemma 3. To prove the first let $x \in X$, and let U be a neighborhood of x. Choose a neighborhood V of x in accordance with the definition of local connection in dimension n, and observe that for $0 \leq k < n$ Theorem 1 (applied to $f \mid f^{-1}V$) implies that for any map $g : S^k \to V$ there is a map $g' : S^k \to f^{-1}V$ such that fg' is homotopic to g. Since $fg' : S^k \to V$ extends to a map $B^{k+1} \to U$, g must also.

THEOREM 3. Let X' be a LC^n paracompact Hausdorff space, let X be a metric space, and let $f : X' \to X$ be a closed surjective map such that $f^{-1}x$ is LC^{n-1} and (n-1)-connected for each point $x \in X$. Then X is LC^n and

f is an n-equivalence.

PROOF. In light of Theorems 1 and 2 it suffices to show that f is a local connection in dimension n. Since f is a closed surjective map and $f^{-1}x$ is (n-1)-connected for every $x \in X$, it thus suffices to show that every open neighborhood U of $f^{-1}x$ contains an open neighborhood V of $f^{-1}x$ such that for $0 \leq k \leq n$ any map $S^k \to V$ is homotopic in U to a map $S^k \to f^{-1}x$. To show this, Lemma 1 will be applied to example 5 of Section 1.

Let A be the inverse set under f of a point of X. If U is an open neighborhood of A, let α be the cover of X' consisting of U and X' - A, let γ be an open cover of X' refining α such that $H(\gamma,\alpha; n)$ holds for the identity map 1: X' → X', and let β be an open cover of X' refining γ such that $E(\beta,\gamma; n)$ holds for the inclusion map i: A → X'.

Set $V = \bigcup \{V' \in \beta \mid V' \cap A \neq \emptyset\}$. For $0 \leq k \leq n$ triangulate S^k as a complex K, and observe that for any map h: K → V there is a subdivision K' of K such that h maps K' into $\{V' \in \beta \mid V' \cap A \neq \emptyset\}$. Since $E(\beta,\gamma; n)$ holds for i: A → X', there is a map g: K' → A such that for each simplex σ of K' there is $W \in \gamma$ with $g(\sigma) \cup h(\sigma) \subset W$. Since $H(\gamma,\alpha; n)$ holds for 1: x' → x', there is a homotopy H: K' x I → X' between h and g such that H maps K' x I into $\alpha = \{U, X' - A\}$. Since $g(K') \subset A$, $H(K' \times I) \subset U$. This completes the proof.

To complete the analysis of the covariant case the following fact is used. If f: X' → X is an

n-equivalence and W is a CW-complex of dimension $\leq m$, then the induced map $f_{\#}$: $[W;X'] \to [W;X]$ is surjective if $m \leq n$ and bijective if $m+1 \leq n$; if f is a weak homotopy equivalence (that is an n-equivalence for every integer n), then $f_{\#}$: $[W;X'] \to [W;X]$ is a bijection for any CW-complex W. It follows from this that one can deduce that the preceding assertions are true if W is replaced by a space S which is merely dominated by a CW-complex (m now being the dimension of the dominating CW-complex).

LEMMA 4. If S is dominated by a CW-complex W of dimension $\leq m$, and f: $X' \to X$ is an n-equivalence, then $f_{\#}$: $[S;X'] \to [S;X]$ is surjective if $m \leq n$ and bijective if $m+1 \leq n$. If f is a weak homotopy equivalence, then $f_{\#}$: $[S;X'] \to [S;X]$ is a bijection for any CW-complex W.

PROOF. Suppose g: $S \to W$ and h: $W \to S$ are maps such that hg is homotopic to the identity on S. Consider the diagram

$$\begin{array}{ccccc} [S;X'] & \to & [W;X'] & \to & [S;X'] \\ \downarrow & & \downarrow & & \downarrow \\ [S;X] & \to & [W;X] & \to & [S;X] \end{array}$$

where the vertical arrows are induced by f and where $[S;X'] \to [W;X']$ and $[S;X] \to [W;X]$ are induced by h while $[W;X'] \to [S;X']$ and $[W;X] \to [S;X]$ are induced by g. Since the compositions $[S;X'] \to [W;X'] \to [S;X']$ and $[S;X] \to [W;X] \to [S;X]$ are the respective identities, $[W;X] \to [S;X]$ is surjective and $[S;X'] \to [W;X']$ is injective. It follows immediately from the commutativity of the diagram that $f_{\#}$: $[S;X'] \to [S;X]$ is surjective (or injective) whenever $f_{\#}$: $[W;X'] \to [W;X]$ is.

COROLLARY TO THEOREM 2. Let X be a metric space, let $f: X' \to X$ be a local connection in dimension $n \leq k$ with dense image (where k is a positive integer or ∞), and let S be a space dominated by a CW-complex of dimension $\leq m$. Then $f_{\#}: [S;X'] \to [S;X]$ is surjective if $m \leq k$ and bijective if $m+1 \leq k$.

4. Local Connections between ANR's.

If $f: X' \to X$ is a local connection and A is a closed subset of X, then $f | f^{-1}A: f^{-1}A \to A$ is not necessarily a local connection. A trivial example is provided by a map f of R^2 onto itself whose only non-degenerate point inverse is some cellular set C which is not path connected: f is a local connection but $f | f^{-1}fC$ is not. For a restricted but still useful class of local connections $f: X' \to X$ there is an effective comparison of the homotopy classes of maps of a closed subset A of X into any ANR with those of maps defined on $f^{-1}A$, the technique for showing this being an extension of the proof that certain local connections do not raise dimension.

Two standard facts are used in this section. A map between ANR's which is an n-equivalence for every integer n is a homotopy equivalence, and a metric space X has dimension $\leq n$ if and only if every map of a closed subset of X into S^n extends to a map of X into S^n.

If X and Y are spaces, $[X;Y]$ denotes the set of homotopy classes of all maps of X into Y, and if $f: X' \to X$ is a map, $f^{\#}$ denotes the induced function $[X;Y] \to [X';Y]$. For Y fixed, the assignments $X \to [X;Y]$, $f \to f^{\#}$ define a contravariant functor from the category of spaces and maps to the category of sets and

functions; hence, for a subset A of X the inverse system ν consisting of all neighborhoods N of A in X and their inclusion maps gives rise to a direct system whose objects are the sets [N;Y] and whose projection maps are the functions induced by the inclusion maps. The inclusion maps $A \to N$ ($N \in \nu$) induce a function $\underset{N \in \nu}{\text{dir. lim.}}[N;Y] \to [A;Y]$ and for any map $f: X' \to X$ there is a commutative diagram

$$\begin{array}{ccccc} [X;Y] & \to & \underset{N \in \nu}{\text{dir. lim.}}[N;Y] & \to & [A;Y] \\ \downarrow & & \downarrow & & \downarrow \\ [X;Y] & \to & \underset{N \in \nu}{\text{dir. lim.}}[f^{-1}N;Y] & \to & [f^{-1}A;Y] \end{array}$$

where the horizontal arrows are induced by inclusion maps and the vertical arrows by appropriate restrictions of f.

THEOREM 4. Let X' and X be ANR's, and let $f: X' \to X$ be a local connection with dense image.

A. Then $\dim X \leq \dim X'$.

B. If f is a closed map, then for any closed subset A of X and any ANR Y $(f|\ f^{-1}A)^{\#}: [A;Y] \to [f^{-1}A;Y]$ is a bijection.

PROOF OF ASSERTION A. If $\dim X' = \infty$, there is nothing to prove. Suppose $\dim X' \leq n$. Then for any closed subset A' of X' the inclusion map $A' \to X'$ induces a surjection $[X';S^n] \to [A';S^n]$.

Let A be a closed subset of X, and refer to the diagram above. For any open subset U of X $f|\ f^{-1}U: f^{-1}U \to U$ is a weak homotopy equivalence between

ANR's and thus a homotopy equivalence. It follows from this and the fact that the open neighborhoods of A comprise a cofinal set in ν that the vertical arrows of the left square are bijections. If Y is an ANR, then the upper horizontal arrow of the right square is a bijection.

Set $Y = S^n$. Since the set of closed neighborhoods of A is cofinal in ν, the lower horizontal arrow of the left square is a surjection. Then the upper horizontal arrow of the left square is also a surjection. This implies that amy map $A \to S^n$ extends to a map $X \to S^n$; hence $\dim X \leq n$.

PROOF OF ASSERTION B. Refer to the right square of the diagram. It follows from the assertions of the second paragraph of the Proof of Assertion A that the left vertical arrow of this square is a bijection. Since f is a closed surjective map, the sets $f^{-1}N$ $(N \in \nu)$ comprise a cofinal family in the system of all neighborhoods of $f^{-1}A$.

This implies that the lower horizontal arrow is a bijection. The proof of assertion B now follows from the commutativity of the square.

ADDENDUM TO THEOREM 4. If X and X' are finite dimensional, then Assertions A and B are valid provided f is a local connection in dimension n, where $n \geq \text{Max}[1 + \dim X', \dim X]$.

In fact for any ANR X there is an open cover α of X such that the dimension of the nerve $K(\alpha)$ of α does

not exceed the dimension of X and $K(\alpha)$ dominates X. It follows from the Corollary to Theorem 2 that $f_{\#}$: $[X;X'] \to [X;X]$ is surjective and $f_{\#}$: $[X';X'] \to [X';X]$ is injective. Thus there is $g: X \to X'$ such that gf is homotopic to the identity on X, and since fgf is homotopic to f, gf is homotopic to the identity on X'. Since this argument applies to $f|\ f^{-1}U: f^{-1}U \to U$ for any open subset U of X, the proof of Theorem 4 is valid under the hypotheses of the Addendum.

By means of the Addendum and a result of Milnor [17] Theorem 4B can be sharpened.

COROLLARY TO THEOREM 4. Let $f: X' \to X$ be a closed surjective map between ANR's. If f is a local connection in dimension n and $n \geq \text{Max}(1 + \dim X', \dim X)$, then for any space Y which is dominated by a CW-complex $(f|\ f^{-1}A)^{\#}$: $[A;Y] \to [f^{-1}A;Y]$ is a bijection.

PROOF. Let Y be dominated by a CW-complex. By Milnor's result Y has the homotopy type of a (possibly infinite) simplicial complex Y' with the metric topology. Let $g: Y \to Y'$ be a homotopy equivalence. Since Y' is known to be an ANR, and since for any closed subset A of X there is a commutative square

$$\begin{array}{ccc} [A;Y] & \to & [A;Y'] \\ \downarrow & & \downarrow \\ [f^{-1}A;Y] & \to & [f^{-1}A;Y'] \end{array}$$

in which the horizontal arrows are bijections induced by g and the vertical arrows are maps induced by f, the proof

follows from Theorem 4B applied to Y'.

5. Decompositions of Manifolds

The purpose of this section is to give a classification in terms of local and global homotopy invariants of cellular decompositions of manifolds whose decomposition spaces are finite dimensional.

Following a suggestion of K.W. Kwun a finite dimensional separable metric ANR X is called a k-homotopy manifold if every point $x \in X$ has arbitrarily small pairs V,U of connected open neighborhoods ($V \subset U$) such that for any connected open neighborhood $W \subset V$ of x the image in $\pi_i(U - x)$ of $\pi_i(W - x)$ under the homomorphism induced by the inclusion map $W - x \to U - x$ is isomorphic to $\pi_i(S^k)$ for all i. As has been pointed out by Curtis in [4] the original definition (see e.g. [3]) makes no distinction between manifolds and manifolds with boundary.

THEOREM 5. Let M be an n-manifold, and let G be an upper semi-continuous decomposition of M into cellular sets. If the decomposition space M/G is finite dimensional, then M/G is an n-homotopy manifold, and the projection map $P: M \to M/G$ is a homotopy equivalence. Furthermore, the following three conditions are equivalent:

1. M/G is an ANR.
2. $\dim M/G \leq n$.
3. M/G is finite dimensional.

PROOF. Since M/G is a separable metric space,

Theorem 2 may be applied to the projection map $P: M \to M/G$, which was shown in Example 5 to be a local connection. Since any finite dimensional LC^∞ metric space is an ANR (for a proof see [7]), the equivalence of conditions 1, 2, and 3 follows from the fact that M/G is LC^∞ and from Theorem 4A.

Since $P: M \to M/G$ is a weak homotopy equivalence between ANR's, P is a homotopy equivalence.

That M/G possesses the local characteristics of an n-homotopy manifold follows without essential change from the arguments Kwum has given in [10] except that applications of Smale's theorem are replaced with those of Theorem 2. The details will be omitted.

CHAPTER II

THE CONTRAVARIANT CASE

6. (Y,n)-trivial Spaces

In this chapter homotopy classes of maps into ANR's are considered. Because of the usual extension properties associated with ANR's there is no need (for the theorems anticipated here) to consider open sets nesting at point inverses of maps. It is sufficient to consider the point inverses without regard to any containing space.

In this section the (variable) range space Y will always be assumed to be an ANR.

A space A will be said to be (Y,n)-trivial, if $[\Sigma^q A; Y] = 0$ for $0 \leq q \leq n$. A space A which is (Y,n)-trivial for all n will be said to be Y-trivial.

EXAMPLES.

1. If A is contractible, then A is Y-trivial for any Y.

PROOF. Since $[C;Y] = 0$ for any contractible space C, the assertion follows from the fact that the suspension of a contractible space is contractible.

2. A space has been called cell-like by Lacher [11,12], if it can be embedded as a cellular subset of some manifold, or equivalently, as a cellular subset of some sphere S^n. Lacher has shown, for example, that every

compact contractible finite dimensional metric space is cell-like.

If A is cell-like, then A is Y-trivial for any ANR Y.

PROOF. Since $[C;Y] = 0$ whenever Y is an ANR and C is cell-like, the assertion follows once it is shown that the suspension of a cell-like space is cell-like, and this follows from the fact that a subset A of S^n is cellular if and only if $S^n - A$ is homeomorphic to R^n (which follows from the results of [2]). In fact by identifying S^{n+1} with $\Sigma(S^n)$ it follows from the fact that $S^n - A$ is homeomorphic to R^n that $S^{n+1} - \Sigma A = \Sigma(S^{n+1}) - \Sigma A$ which is homeomorphic to $S^n \times R^1 - A \times R^1 = (S^n - A) \times R^1$ which is homeomorphic to $R^n \times R^1 = R^{n+1}$.

I am grateful to Professor M. Brown for providing this alternate proof that the suspension of a cellular set is a cellular set.

3. If A is any connected space such that $[A;S^1] = 0$, then A is S^1-trivial.

PROOF. It suffices to show that the conditions imply $[\Sigma A;S^1] = 0$.

Let $p: R^1 \to S^1$ be the exponential map assigning $t \to \exp(2\pi i t)$. If $f: \Sigma A \to S^1$ is any map, regard f as a homotopy $F: A \times I \to S^1$ between two constant maps. Since $p: R^1 \to S^1$ has the covering homotopy property, and since A is connected, there is a homotopy $F': A \times I \to R^1$ between

two constant maps such that $pF' = F$; F' defines a map $f': \Sigma A \to R^1$ such that $pf' = f$, which shows that f is homotopic to a constant map.

4. If Y is a connected ANR which is r-simple for every r (see e.g. [22]), and A is a finite dimensional compact connected Hausdorff space such that $\overset{\vee}{H}{}^q(A;Z) = 0$ for $q > 0$, then A is Y-trivial.

PROOF. By a Mayer-Vietoris argument it follows that $\overset{\vee}{H}{}^q(\Sigma^m A;Z) = 0$ for all $m,q > 0$, and by the Universal Coefficient Theorem for Cech cohomology on compact Hausdorff spaces (see [22]) it follows that for each $m \geq 0$ $\overset{\vee}{H}{}^q(\Sigma^m A;G) = 0$ for all $q > 0$ and for any Abelian group G. Since Hu has shown (see [6,7] and in particular Theorem 8.1 of [6]) that for a connected finite dimensional compact Hausdorff space X and an ANR Y as above the condition $\overset{\vee}{H}{}^q(X;\pi_q(Y)) = 0$ for all $q > 0$ suffices to show that $[X;Y] = 0$, it follows that A is Y-trivial.

Two examples indicate the consequences of weakening the hypotheses above. On the one hand Kahn [8] has shown that there exist infinite dimensional compact connected Hausdorff spaces A such that $\overset{\vee}{H}{}^q(A;Z) = 0$, and $[A;S^3] \neq 0$. He has also shown that this pathology persists for certain other spheres of arbitrarily high dimension. On the other hand if $A = Y$ is an (homologically) acyclic 2-dimensional polyhedron with non-trivial fundamental group, then $\overset{\vee}{H}{}^q(A;Z) = 0$ for $q > 0$, but $[A;Y] \neq 0$. An example of such a polyhedron has been given by Newman [18].

5. The notion of (Y,n)-triviality is somewhat clarified by considering mapping spaces (always appearing here with the compact-open topology). Let A be a compact Hausdorff space and Y an ANR which is connected and topologically complete. Suppose [A;Y] = 0; this implies that restriction defines a surjective map $\rho\colon Y^{CA} \to Y^{A}$ (where CA is the cone on A) whose domain has the homotopy type of the space of all maps of a one point space into Y.

The space A is (Y,n)-trivial if and only if ρ is an n-equivalence.

PROOF. The proof consists of showing that (Y^{CA},ρ,Y^{A}) is an Hurewicz fibration and then using the homotopy sequence of a fibration to complete the analysis. It will be convenient for a later application to have a more general statement available concerning fibrations defined by restriction in a mapping space; this is treated separately in the lemma below.

Assuming that (Y^{CA},ρ,Y^{A}) is an Hurewicz fibration one uses the exact sequence of the fibration to conclude that ρ is an n-equivalence if and only if all the fibers of ρ are (n-1)-connected as follows. Let $y_0 \in Y$ and $b_0\colon A \to Y$, $e_0\colon CA \to Y$ be constant maps whose images are y_0. Since Y^{A} is path connected, all the fibers of ρ have the same homotopy type; consider in particular $F = \rho^{-1}(b_0)$. Let q be a non-negative integer, and let $p_0 \in S^q$ be fixed. Define spaces $F_i (1 \leq i \leq 5)$ as follows:

F_1 consists of all maps $S^q, p_0 \to F, e_0$;

F_2 consists of all maps

$S^q \times CA, (p_0 \times CA) \cup (S^q \times A) \to Y, y_0$;

F_3 consists of all maps

$u: S^q \times I \times A, (p_0 \times I \times A) \cup (S^q \times 0 \times A) \to Y, y_0$ such that for all $p \in S^q$ $u(p \times 1 \times A)$ is a point of Y;

F_4 consists of all maps

$u: B^{q+1} \times A, p_0 \times A \to Y, y_0$ such that $u(p \times A)$ is a point of Y for each $p \in S^q$.

F_5 consists of all maps $\Sigma^{q+1}A, p_0 \to Y, y_0$. Here the (q+1)-fold suspension of A is considered to be defined by a map σ defined on $B^{q+1} \times A$ which identifies the set $p \times A$ with p for each $p \in S^q$.

If $E: Y^{CA} \times CA \to Y$ is the evaluation map, then the homeomorphism $\psi: (Y^{CA})^{S^q} \to Y^{(S^q \times CA)}$ defined by $\psi(g) = E(g \times 1_{CA})$ maps F_1 onto F_2. If $\gamma: I \times A, 1 \times A \to CA$, (cone point) is the standard map defining the cone on A, then $1_{S^q} \times \gamma$ induces a homeomorphism of F_2 onto F_3. If $k: S^q \times I, (S^q \times 0) \cup (p_0 \times I) \to B^{q+1}, p_0$ is defined by $k(p,t) = (1-t)p_0 + tp$ for $p \in S^q$, $t \in I$, then $(k \times 1_A)$ induces a homeomorphism of F_4 onto F_3. Lastly, the identification map $\sigma: B^{q+1} \times A \to \Sigma^{q+1}A$ induces a homeomorphism of F_5 onto F_4. Thus F_1 is homeomorphic to F_5. For $q = 0$ it follows that F is 0-connected if and only if $[\Sigma A; Y] = 0$, and since $\pi_q(F, e_0)$ consists of the path components of F_1 for $q > 0$, it follows that $\pi_q(F, e_0) = 0$ if and only if $[\Sigma^{q+1}A; Y] = 0$, which completes the proof of the assertion.

LEMMA 5. Let X be a compact Hausdorff space,

A a closed subspace of X, and Y an ANR which is connected and topologically complete. Let B be the space of all maps of A into Y which extend over X. Then (Y^X,ρ,B) is an Hurewicz fibration, where ρ: $Y^X \to B$ is defined by restriction.

PROOF. Since Y^X and B are metrizable, it suffices to show that (Y^X,ρ,B) is a fibration for the class of all metric spaces. To this end let P be a metric space and assume maps f': $0 \times P \to Y^X$, F: $I \times P \to B$ such that $\rho f' = F|\ 0 \times P$. F induces a map Φ: $I \times P \times A \to Y$, and f' induces a map φ': $0 \times P \times X \to Y$; by hypothesis $\varphi' | 0 \times P \times A = \Phi | 0 \times P \times A$. Since Y is an absolute neighborhood retract for the class of all paracompact Hausdorff spaces [14], the combined map $(0 \times P \times X) \cup (I \times P \times A) \to Y$ defined by Φ and φ' has an extension Φ'' over a neighborhood of $(0 \times P \times X) \cup (I \times P \times A)$ in $I \times P \times X$. There is then an open neighborhood N of $P \times A$ in $P \times X$ such that Φ'' is defined on $(0 \times P \times X) \cup (I \times N)$. Let u: $P \times X \to I$ be a map such that $u((P \times X) - N) = 0$ and $u(P \times A) = 1$, and define Φ': $I \times P \times X \to Y$ by $\Phi'(t,p,x) = \Phi''(tu(p,x),p,x)$. Then Φ' induces a homotopy F': $I \times P \to Y^X$ extending f' and covering F. This completes the proof.

The notion of (Y,n)-triviality is useful for the analysis of $f^{\#}$: $[X;Y] \to [X';Y]$ whenever X is finite dimensional. If Z is the mapping cylinder of f, the assertion that $f^{\#}$ is surjective is equivalent to the assertion that any map h: $X' \to Y$ extends to a map of Z into Y. Let r be

the natural retraction of Z onto X. For a closed set A of X and a map g: $A \to Y$ such that $h|\ f^{-1}A$ is homotopic to $g(f|\ f^{-1}A)$ the problem of extending g and the homotopy to a map g': $X \to Y$ and a homotopy between h and $g'f$ is the same as that of extending a certain map $X' \cup r^{-1}A \to Y$ to a map $Z \to Y$. The theorem that follows thus shows that certain induced maps exhibit a strong kind of surjectivity.

THEOREM 6. Let f: $X' \to X$ be a proper map of a metric space onto a finite dimensional metric space X. Let Y be an ANR, and suppose that $f^{-1}x$ is (Y,dim X)-trivial for each $x \in X$. Then for any map h: $X' \to Y$ and any closed subset A of X any partial extension $X' \cup r^{-1}A \to Y$ of h has an extension to a map $Z \to Y$.

7. A Proof of Theorem 6.

The elements of the proof of Theorem 6 are emphasized by a similar result. The notation is that of Theorem 6.

THEOREM 7. Let $f: X' \to X$ be a closed map of a paracompact Hausdorff space onto a finite dimensional metric space. Let Y be a neighborhood extension space for the class of paracompact Hausdorff spaces, and suppose that $[\Sigma^q(f^{-1}x);Y] = 0$ for $0 \leq q \leq \dim X$ for every point $x \in X$. Then for any map $h: X' \to Y$ and any closed subset A of X any partial extension $X' \cup r^{-1}A \to Y$ of h has an extension to a map $Z \to Y$.

Theorem 6 and Theorem 7 are not logically comparable. For metric spaces (or even for k-spaces) a map $f: X' \to X$ is proper if and only if it is a closed map and every point inverse is compact, and an ANR is a neighborhood extension space for the class of all paracompact Hausdorff spaces if and only if it is topologically complete [14](see also [1]).

Let $f: X' \to X$ be a closed surjective map. Recall that the mapping cylinder Z(f) of f may be obtained from $X' \times I$ by identifying the point $(x',0)$ with the point x' for each $x' \in X'$ and by identifying all points $(x',1)$ such that $f(x') = x$ with the point x for each $x \in X$. The quotient map $X' \times I \to Z(f)$ will be denoted ζ, and the

standard retraction of $Z(f)$ onto X (which maps $\zeta(x',t)$ to $f(x')$) will be denoted r. The <u>suspension space</u> $\Sigma(f)$ of f is the space obtained from $X' \times B^1$ by identifying $(x',0)$ with x' for all $x' \in X'$ and by identifying all points (x',u) such that $f(x') = x$ with the point (x,u) for all $x \in X$ and $u \in S^0$. The quotient map $X' \times B^1 \to \Sigma(f)$ will be denoted σ, and the unique map $\Sigma(f) \to X$ which maps the point $\sigma(x',u)$ to $f(x')$ $(x' \in X', u \in B^1)$ will be denoted π. Note that $\pi^{-1}x$ may be naturally identified with $\Sigma(f^{-1}x)$. All the maps ζ, r, σ, π are closed maps, and if $f: X' \to X$ is proper, then ζ, r, σ, and π are also. From this it follows as a consequence of a theorem of Michael [16] that if X and X' are paracompact, $Z(f)$ and $\Sigma(f)$ are also; similarly a result of Hanai (see [5]) implies that if f is proper and X and X' are metrizable, then $Z(f)$ and $\Sigma(f)$ are metrizable.

The proof of Theorem 6 and Theorem 7 is accomplished by combining partial extensions of h. The main step of the argument is to compare two partial extensions, which is accomplished by induction on the dimension of X and the following lemma.

LEMMA 6. Let $f: X' \to X$ be a closed surjective map and let Y be any space. Let H_+ and H_- be maps $Z(f) \to Y$ such that $H_+ | X' = H_- | X'$, and define a map $H: \Sigma(f) \to Y$ by $H\sigma(x',\pm t) = H_{\pm}\zeta(x',\pm t)$ for all points $x' \in X'$ and $t \in I$.

If $H: \Sigma(f) \to Y$ extends over $Z(\pi)$, then H_+ is

homotopic to H_- relative to X'.

PROOF. Consider the diagram

$$\begin{array}{ccc} & \zeta \times 1_I & \\ X' \times I \times I & \rightarrow & Z(f) \times I \\ {\scriptstyle 1_{X'} \times \psi} \downarrow & & \downarrow {\scriptstyle \eta} \\ & \theta & \\ X' \times B^1 \times I & \rightarrow & Z(\pi) \\ {\scriptstyle 1_{X'} \times \varphi} \uparrow & & \| \\ & \zeta'(\sigma \times 1_I) & \\ X' \times B^1 \times I & \rightarrow & Z(\pi) \end{array}$$

in which φ, ψ, θ, and η have not yet been defined, and ζ' is the quotient map $\Sigma(f) \times I \to Z(\pi)$ for the mapping cylinder of the map $\pi\colon \Sigma(f) \to X$. The map $\varphi\colon B^1 \times I, B^1 \times 1 \to B^1 \times I, (0,1)$ is chosen to be any relative homeomorphism of $B^1 \times I$ onto itself such that $\varphi^{-1}(0,1) = B^1 \times 1$, and $\theta\colon X' \times B^1 \times I \to Z(\pi)$ is defined by commutativity of the lower square. The map $\psi\colon I \times I, I \times 0 \to B^1 \times I, (0,0)$ is chosen to be any relative homeomorphism of $I \times I$ onto $B^1 \times I$ such that $\psi^{-1}(0,0) = I \times 0$. The map $\eta\colon Z(f) \times I \to Z(\pi)$ is then defined so as to make the square commutative.

A straightforward verification shows that if $K\colon Z(\pi) \to Y$ is an extension of $H\colon \Sigma(\pi) \to Y$ over $Z(\pi)$, then $K\eta\colon Z(f) \times I \to Y$ is a homotopy between H_+ and H_- relative to X', which completes the proof of the lemma.

The proof of Theorem 7 is the same as the proof of Theorem 6 which now follows except that the phrase "Y is an ANR" is replaced by "Y is a neighborhood extension

space for the class of paracompact Hausdorff spaces".

PROOF OF THEOREM 6. Let T(n) denote the statement of Theorem 6 with the additional hypothesis that dim $X \leq n$. T(-1) is trivial; suppose T(n-1) is true (where $n \geq 0$), and let $f: X' \to X,Y$, $h: X' \to Y$, and $A \subset X$ be as in the statement of the theorem.

Let $X' \cup r^{-1}A \to Y$ be an extension of $h: X' \to Y$; since Y is an ANR, there is an extension H_N of this map over a neighborhood N of $X' \cup r^{-1}A$ in Z, and since $r: Z \to X$ is a closed map, there is an open neighborhood U of A in X such that $r^{-1}(\overline{U}) \subset N$, where $\overline{U}$ is the closure of U in X. Let $H_U = H_N | X' \cup r^{-1}(\overline{U})$. For each point $x \in X - A$ let $H_x: X' \cup r^{-1}x \to Y$ be an extension of h (which exists because $h | f^{-1}x$ is homotopic to a constant map). For precisely the same reasons as above it follows that for every point $x \in X - U$ there is a closed neighborhood N(x) of x such that $N(x) \subset X - U$ and H_x has an extension over $X' \cup r^{-1}(N(x))$. Let γ be a closed locally finite cover of $X - U$ refining $\{N(x) | x \in X - U\}$, and assume $\gamma = \{C_j | j \in J\}$, where J is well-ordered by a relation $\leq$. For each $j \in J$ let $X_j = \overline{U} \cup \{C_k | k \leq j, k \neq j\}$.

The following assertion comprises the inductive step of the argument. For any $j \in J$ any extension $X' \cup r^{-1}(X_j) \to Y$ of $H_U: X' \cup r^{-1}(\overline{U}) \to Y$ extends over $X' \cup r^{-1}(X_j \cup C_j)$. To establish the assertion (which clearly completes the proof) let $j \in J$ and an extension

$H\colon X' \cup r^{-1}(X_j) \to Y$ of H_U be fixed. There is an extension $H_V\colon X' \cup r^{-1}V \to Y$ of H_U for some neighborhood V of X_j in X, and since $\operatorname{Ind} X = \dim X \leq n$, V contains an open neighborhood V_j of X_j such that $\operatorname{Ind}(\overline{V}_j - V_j) \leq n-1$.

Let $A = C_j \cap (\overline{V}_j - V_j)$, let $H_j\colon X' \cup r^{-1}C_j \to Y$ be an extension of h, and let $H_+ = H_V | \ r^{-1}A$ and $H_- = H_j | \ r^{-1}A$ define a map $H\colon \Sigma(f | \ f^{-1}A) \to Y$ as in Lemma 6. Since $\dim A \leq n-1$, and since $[\Sigma^q(\pi^{-1}a); Y] = 0$ for $0 \leq q \leq \dim A$ for each point $a \in A$, it follows from $T(n-1)$ and the lemma that H_+ is homotopic to H_- relative to $f^{-1}A$. This implies that $H_V | \ X' \cup r^{-1}A$ is homotopic to $H_j | \ X' \cup r^{-1}A$ relative to X'; since the latter map is the restriction of a map $X' \cup r^{-1}(C_j - V_j) \to Y$, it follows that $H_V | \ X' \cup r^{-1}A$ also extends to a map $X' \cup r^{-1}(C_j - V_j) \to Y$. This map and $H_V | \ X' \cup r^{-1}(X_j) \cup r^{-1}(C_j \cap \overline{V}_j) \to Y$ then combine to give an extension $X' \cup r^{-1}(X_j \cup C_j) \to Y$ of H_U, which completes the proof of the assertion.

8. Applications

In the lemma and theorem let $f: X' \to X$ be a proper map of a metric space onto a finite dimensional metric space.

LEMMA 7. Let Y be an ANR, and suppose $[\Sigma^q(f^{-1}x);Y] = 0$ for $1 \leq q \leq 1 + \dim X$. Then $f^{\#}: [X;Y] \to [X';Y]$ is injective.

THEOREM 8. Suppose $\dim X \leq n$. Let Y be an ANR, and suppose for each $x \in X$ that $f^{-1}x$ is $(Y,n+1)$-trivial. Then $f^{\#}: [X;Y] \to [X';Y]$ is bijective.

PROOFS. The theorem follows from Theorem 6 and the lemma.

To establish the lemma let $g_0, g_1: X \to Y$ be maps and suppose H' is a homotopy between $g_0 f$ and $g_1 f$, regarded as a map $X' \times B^1 \to Y$. H' defines a map $H: \Sigma(f) \to Y$, which by Theorem 6 extends to a map $K: Z(\pi) \to Y$, where $\pi: \Sigma(f) \to X$ is the map defined in the last section. The mapping cylinder $Z(\pi)$ of π contains the mapping cylinder of $\pi | X \times S^0: X \times S^0 \to X$, and this cylinder is naturally identified with $X \times B^1$; thus with these identifications $K | Z(\pi | X \times S^0): X \times B^1 \to Y$ defines a homotopy between g_0 and g_1.

THEOREM 9. Let $f: X' \to X$ be a proper map between finite-dimensional metric spaces such that $f^{-1}x$ is cell-like for each $x \in X$. If Y is any space dominated by a

CW-complex, then $f^{\#}$: $[X;Y] \to [X';Y]$ is a bijection. If it is also assumed that X and X' are dominated by CW-complexes, then f is a homotopy equivalence.

PROOF. Compare the Proof of the Corollary to Theorem 4. By Milnor's result [17] there is an ANR Y', and there is a homotopy equivalence $g: Y \to Y'$. There is then a commutative square

$$\begin{array}{ccc} [X;Y] & \to & [X;Y'] \\ \downarrow & & \downarrow \\ [X';Y] & \to & [X';Y'] \end{array}$$

in which the horizontal arrows are bijections induced by f, and the vertical arrows are maps induced by f. By Theorem 8 and Example 2 of Section 6 $f^{\#}$: $[X;Y'] \to [X';Y']$ is a bijection; hence the first part of the theorem follows from the commutativity of the diagram.

If X and X' are dominated by CW-complexes, $f^{\#}$: $[X;X] \to [X';X]$ and $f^{\#}$: $[X;X'] \to [X';X']$ are bijections, from which it follows easily that f is a homotopy equivalence. This completes the proof.

The proof of Theorem 10 is the same as the preceding proof except that the result of Example 4 of Section 6 is used instead of Example 2.

THEOREM 10. Let $f: X' \to X$ be a proper monotone surjective map between finite dimensional metric spaces such that $\check{H}^q(f^{-1}x;Z) = 0$ for all $q > 0$ for each $x \in X$. If Y is any connected space which is n-simple for all n and which is dominated by a CW-complex, then

$f^{\#}$: $[X;Y] \to [X';Y]$ is a bijection. If it is also assumed that X and X' are connected, n-simple for every n, and dominated by CW-complexes, then f is a homotopy equivalence.

APPENDIX

THEOREM. Let $f: X' \to X$ be a surjective map of a compact Hausdorff space onto a finite-dimensional compact Hausdorff space. Let Y be a compact ANR, and suppose $[\Sigma^q(f^{-1}x);Y] = 0$ for $0 \leq q \leq \dim X$ for every $x \in X$. Then for any map $h: X' \to Y$ and any closed subset A of X any extension $X' \cup r^{-1}A \to Y$ of h has an extension $Z(f) \to Y$.

The proof is accomplished by changing the extension problem of the theorem to a selection problem which can be solved by means of a theorem of Michael [15]. The portions of Michael's work which are relevant to this procedure can be summarized as follows. If X and E are topological spaces and σ is some family of non-empty closed subsets of E, a <u>carrier</u> is a function $c: X \to \sigma$ (the image of $x \in X$ under this carrier is denoted E_x), and a <u>selection</u> for this carrier is a map $\varphi: X \to E$ such that $\varphi(x) \in E_x$ for each $x \in X$. The carrier $c: X \to \sigma$ is <u>lower semi-continuous</u> (l.s.c.), provided $\{x \in X | E_x \cap V \neq \phi\}$ is open in X for every open subset V of E. The family σ is <u>equi</u>-LC^n, if for every $u \in E$ every neighborhood U of u contains a neighborhood V of u such that for all $k \leq u$ and for every $E' \in \sigma$ every map $S^k \to E' \cap V$ extends to a map $B^{k+1} \to E' \cap U$. If X is a paracompact Hausdorff space and A is a closed subspace of X, then $\dim_X(X - A) \leq n$, if $\dim F \leq n$ for every subset F of $X - A$ which is closed in X.

In [15] Michael established Theorem M. Let X be a paracompact Hausdorff space, A a closed subset of X with $\dim_X(X - A) \leq n$, E a complete metric space, σ an equi-LC^n family of non-empty closed subsets of E such that every member of σ is (n-1)-connected, and $c: X \to \sigma$ an l.s.c. carrier. Then every selection for $c|A$ can be extended to a selection for c.

Let $f: X' \to X$ and $h: X' \to Y$ be as in the theorem. Let E be the space of all maps $H: X' \times I \to Y$ such that $H(x',0) = h(x')$ for all $x' \in X'$. It is well known that the topology of E is the same as the uniform topology induced in E by any metric for Y and that E is complete with respect to any induced metric.

Let $A' = f^{-1}A$ and consider a fixed extension $H_A: X' \cup r^{-1}A \to Y$. This extension composed with the restriction to $(X' \times 0) \cup (A' \times I)$ of the standard map $\zeta: X' \times I \to Z(f)$ defines the map $H'_A: (X' \times 0) \cup (A' \times I) \to Y$. The map H'_A has an extension $H''_A: X' \times I \to Y$ because Y has the homotopy extension property.

For each $x \in A$ let $E_x = \{H''_A\}$ and for each $x \in X - A$ let E_x be the subspace of E consisting of all maps $H: X' \times I \to Y$ such that $H|(X' \times 0) \cup (A' \times I) = H'_A$ and $H(x',1) = H(x'',1)$ whenever $f(x') = f(x'')$. A standard argument establishes that each E_x is closed in E. Let c be the carrier defined by $c(x) = E_x$.

LEMMA. With any selection $\varphi: X \to E$ for c is associated an extension $H: Z(f) \to Y$ of $H_A: X' \cup r^{-1}A \to Y$.

PROOF. If φ: $X \to E$ is a map, φ defines a map Φ: $X \times X' \times I \to Y$ by $\Phi(x,x',t) = \varphi(x)(x',t)$. Let H': $X' \times I \to Y$ be the map defined by $H'(x',t) = \Phi(fx',x',t)$. Suppose φ is a selection for c. Then for every $x' \in X'$ it follows from the definition of E that $H'(x',0) = \varphi(fx')(x',0) = h(x')$. If $(x',t) \in A' \times I$, then $H'(x',t) = \varphi(fx')(x',t) = H''_A(x',t) = H'_A(x',t)$ by the definition of c and H''_A. If $x \in X$ and $x',x'' \in f^{-1}x$, then $\varphi(f(x')) = \varphi(f(x''))$; hence $H'(x',1) = H'(x'',1)$. Then H': $X' \times I \to Y$ defines a map H: $Z(f) \to Y$ which extends H_A. This completes the proof of the lemma.

To verify that the conditions of Michael's Theorem are met it will be convenient to fix a metric d for Y, and let $N(H;\varepsilon)$ be the ε-neighborhood of $H \in E$ in the induced metric. Also let $n = \dim X$.

LEMMA. c is l.s.c.

PROOF. Let W be open in E, $x \in X$, and $H \in W \cap c(x_0)$. Assume W is an ε-neighborhood of H: $X' \times I \to Y$ and let $0 < \delta < \varepsilon/2$. Set $y_0 = H(f^{-1}(x_0) \times 1)$, and let V be an open neighborhood of y_0 which is inessential in the δ-neighborhood N of y_0 in Y. There is an open neighborhood U of x_0, and there is a non-degenerate interval $[t_0,1]$ such that $H(\overline{f^{-1}U \times [t_0,1]}) \subset V$. Then $c(x) \cap W \neq \emptyset$ for every $x \in U - A$. In fact for a fixed $x \in U - A$ there is t with $t_0 < t < 1$ and there is a homotopy $f^{-1}(x) \times [t,1] \times I \to N$ of $H|\ f^{-1}(x) \times [t,1]$ and a constant map. Combine this with the constant homotopy of H

restricted to the frontier of $f^{-1}(U - A) \times [t_0,1]$ and apply Borsuk's extension theorem to conclude that there is a map H': $f^{-1}(U - A) \times [t_0,1] \to N$ such that $H'(f^{-1}(x) \times 1)$ is a point and $H' = H$ on the frontier of $f^{-1}(U - A) \times [t_0,1]$. Then H' extends via H to a map $X' \times I \to Y$ which is contained in $c(x) \cap W$ (since it agrees with H on $(X' \times 0) \cup (A' \times I)$ and differs from H only in a set of diameter $< \epsilon/2$). It follows immediately that $\{x \in X | E_x \cap W \neq \phi\}$ is open in X.

LEMMA. The collection $\{E_x | x \in X\}$ is equi-locally contractible.

PROOF. This means that for every $H \in E$ every neighborhood U of H contains a neighborhood V of H such that the inclusion map $V \cap E_x \to U \cap E_x$ for every $x \in X$ is inessential.

For every positive δ let $W(\delta) = \{(y_1,y_2) \in Y \times Y | d(y_1,y_2) < \delta\}$. It follows from the compactness of Y that for every positive ϵ there is a positive δ such that the maps p_1: $W(\delta) \to Y$ and p_2: $W(\delta) \to Y$ defined by $p_i(y_1,y_2) = y_i$ $(i = 1,2)$ are ϵ-homotopic keeping the diagonal of Y fixed. Let L: $W(\delta) \times I \to Y$ be such a homotopy. If H is any member of E and N is the δ-neighborhood of H in E, then an ϵ-homotopy Φ: $N \times I \to E$ such that $\Phi(N \times 1) = H$ is defined for $(H',t') \in N \times I$ by $\Phi(H',t')(x',t) = L(H'(x',t),H(x',t),t')$. If $H \in E_x$ for some x, then $\Phi((N \cap E_x) \times I \subset E_x$.

To establish the lemma let $H \in E$, and let

$N(H;2\varepsilon)$ be the 2ε-neighborhood of H in E. Choose δ to work for ε as above, and observe that if $N(H;\delta/2) \cap E_x \neq \phi$, then there is $H' \in E_k$ such that $N(H;\delta/2) \cap E_x \subset N(H';\delta) \cap E_x$; since $N(H';\delta) \cap E_x$ is inessential in $N(H';\varepsilon) \cap E_x$, it follows that $N(H;\delta/2) \cap E_x$ is inessential in $N(H;2\varepsilon) \cap E_x$ for every $x \in X$. This completes the proof of the lemma.

LEMMA. E_x is $(n-1)$-connected for every $x \in X$.

PROOF. Fix $x \in X$ and let B_x be the space of all maps $H: (X' \times 0) \cup (A' \times I) \cup (f^{-1}(x) \times I) \to Y$ such that $H = H'_A$ on $(X' \times 0) \cup (A' \times I)$ and $H(f^{-1}(x) \times 1)$ is some point of Y. Restriction defines a map $p: E_x \to B_x$ which by the Lemma appearing in Example 5 of Section 6 is an Hurewicz fibration. Since B_x is homeomorphic to the space of all maps $H: c(f^{-1}x) \to Y$ such that $H|\, f^{-1}x = h|\, f^{-1}x$, it follows by Example 5 of Section 6 that B_x is $(n-1)$-connected. To show that E_x is $(n-1)$-connected it suffices to show that a typical fiber of $p:E_x \to B_x$ is an AR.

Fix $H_x \in B_x$ for $x \in X - A$. The fiber $F = p^{-1}(H_x)$ consists of the maps $H: X' \times I \to Y$ which extend H_x. Let P_0 be a closed subset of the metric space P, and let $\varphi_0: P_0 \to F$ be a map. Then the map $\Phi_0: P_0 \times X' \times I \to Y$ defined by $\Phi_0(z,x',t) = \varphi_0(z)(x',t)$ can be extended to a map

$$\Phi_1: (P \times [(X' \times 0) \cup (A' \times I) \cup (f^{-1}(x) \times I)]) \cup (P_0 \times X' \times I) \to Y$$

by defining $\Phi_1(z,x',t) = H_x(x',t)$ for $z \in P - P_0$. Then $\Phi_1|[(P_0 \times X') \cup (P \times A) \cup (P \times f^{-1}(x))] \times I$ may be regarded

as a partial homotopy of $\Phi_1 | P \times X' \times 0$; by the homotopy extension property Φ_1 extends to a map $\Phi: P \times X' \times I \to Y$. Then the map $\varphi: P \to F$ defined by $\varphi(z)(x',t) = \Phi(z,x',t)$ for $(z,x',t) \in P \times X' \times I$ is an extension of $\varphi_0: P_0 \to F$. This shows that F is an AR. Thus E_x is (n-1)-connected for every $x \in X$.

All the conditions of Theorem M are met; hence there is a selection for $c: X \to \sigma$. By the first lemma of the sequence the proof is complete.

BIBLIOGRAPHY

1. R. Arens, Extensions of functions on fully normal spaces, Pacific J. Math. 2 (1951), 11-22.

2. M. Brown, A proof of the generalized Schoenflies theorem, Bull. Amer. Math. Soc. 66 (1960), 74-76.

3. M. L. Curtis, Homotopy manifolds, topology of 3-manifolds and related topics (Proc. the Univ. of Georgia Institute, 1961) 102-104. Prentice Hall, Englewood Cliffs, N. J., 1962.

4. _____, Shrinking continua in 3-space, Proc. Cambridge Philos. Soc. 57 (1961), 432-433.

5. J. Dugundji, Topology, Allyn and Bacon, Boston 1966.

6. S.-T. Hu, Mappings of a normal space into an absolute neighborhood retract, Trans. Amer. Math. Soc. 64 (1948), 336-358.

7. _____, Theory of Retracts, Wayne State University Press, Detroit, 1965.

8. D. S. Kahn, An example in Cech cohomology, Proc. Amer. Math. Soc. 16 (1965), 584.

9. V. P. Kompaniec, A homotopy criterion for a pointlike mapping, Ukrain. Math. Z. 18 (1966), 3-10 (Russian).

10. K. W. Kwun, A fundamental theorem on decompositions of the sphere into points and tame arcs, Proc. Amer. Math. Soc. 12 (1961), 47-50.

11. R. C. Lacher, Cell-like spaces and mappings, (to appear).

12. _____, Cell-like spaces, (to appear).

13. _____, Cell-like mappings I, (to appear).

14. E. A. Michael, Some extension theorems for continuous functions, Pacific J. Math. 3 (1953), 789-806.

15. _____, Continuous selections II, Ann. of Math. 64 (1956), 562-580.

16. _____, Another note on paracompact spaces, Proc. Amer. Math. Soc. 8 (1957), 822-828.

17. J. Milnor, On Spaces having the homotopy type of a CW-complex, Trans. Amer. Math. Soc. 90 (1959), 272-280.

18. M. H. A. Newman, Boundaries of ULC sets in Euclidean n-space, Proc. Nat. Acad. Sci. U.S.A. 34 (1948), 193-196.

19. T. M. Price, A necessary condition that a cellular upper semi-continuous decomposition of E^n yield E^n, Trans. Amer. Math. Soc. 122 (1966), 427-435.

20. _____, On decompositions and homotopy groups, Notices Amer. Math. Soc. 14 (1967) 274.

21. S. Smale, A Vietoris mapping theorem for homotopy, Proc. Amer. Math. Soc. 8 (1957) 604-610.

22. E. H. Spanier, Algebraic Topology, Mc Graw - Hill, New York, 1966.