DOCTORAL DISSERTATION SERIES

TITLE BRANCH POINTS OF SOLUTIONS OF EQUATIONS IN BANACH SPACE.

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UNIVERSITY of MICHIGAN DATE JULY 1948

DEGREE Ph. D. PUBLICATION NO. 1096



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1949

BRANCH POINTS OF SOLUTIONS OF EQUATIONS IN BANACH SPACE

A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in the University of Michigan.

July, 1948

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Acknowledgment

The writer wishes to express appreciation for the help given in the preparation of this thesis by the following persons: Professor E. H. Rothe, who acted as adviser for the work, Professor T. H. Hildebrandt, and Professor H. Samelson.

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Chapter 1: Introduction

F. Riesz [13][1], S. Banach [2], and others have generalized the Fredholm theory of linear integral equations to Banach space. However, except for a short note by R. Caccioppoli [3], nothing has been done regarding the corresponding generalization for the theory of ramifications of solutions arising in the non-linear integral equation theory as developed by E. Schmidt [17], and also by A. Liapounoff [12]. The object of the thesis is to fill this gap.

Schmidt considers the problem of determining the local solutions $u(s)$ of the integral equation

$$(1)\quad u(s) - \int_a^b C(s,t)u(t)dt + F(s)v(s) + \int_a^b K(s,t)v(t)dt + \sum_{m+n\geq 2} W_{mn}[u,v] = 0$$

when the function $v(s)$ is given. The series $\sum_{m+n\geq 2} W_{mn}[u,v]$ is what Schmidt calls an integral power series and consists of terms of the following kind:

$$[u(s)]^{\alpha_0}[v(s)]^{\beta_0}\int_a^b\cdots\int_a^b G(s,t_1,\ldots,t_\kappa)[u(t_1)]^{\alpha_1}\cdots[u(t_\kappa)]^{\alpha_\kappa}[v(t_1)]^{\beta_1}\cdots[v(t_\kappa)]^{\beta_\kappa}dt_1\cdots dt_\kappa$$

(See Chapter 4 of this thesis for a detailed definition of integral power series.)

Schmidt proves that if the operator

$$(2)\qquad u(s) - \int_a^b C(s,t)u(t)dt$$

acting on $u(s)$ is non-singular, i.e., if the operator is such that the condition

[1] Numbers in brackets refer to the bibliography.

$$u(s) - \int_a^b v(s,t)u(t)dt = 0$$

implies that $u(s) \equiv 0$, then equation (1) has a unique solution $u(s)$ when $v(s)$ is given. This corresponds in the finite dimensional case to a system of equations with a non-vanishing Jacobean.

It is also proved that if the operator (2) is singular, then for a given function $v(s)$ which has a sufficiently small maximum in the closed interval $\overline{ab}$, the number of local solutions $u(s)$ is equal to the number of local solutions of a set of n "Verzweigungsgleichungen" (ramification equations) in n variables derived from equation (1). The number n is the dimension of the null space of the operator (2), i.e., the dimension of the linear space of functions which are mapped into zero by the operator.

In the investigation of the number of local solutions of the Verzweigungsgleichungen, one special case is considered. It is shown that if the null space of the operator (2) has dimension one, then the number of solutions $u(s)$ is equal to the number of local solutions of the single Verzweigungsgleichung

$$(3) \qquad \sum_{n=2}^{\infty} L_n Z^n + \sum_{m=0}^{\infty} Z^m \sum_{n=1}^{\infty} V_n^m = 0$$

where L_n is a complex constant, Z is a complex variable, and the complex numbers V_n^m are continuous functions of $v(s)$ which vanish when $v(s) \equiv 0$. That is, the number of solutions $u(s)$ is equal to the number of zeros of the function

$$(4) \qquad \sum_{n=2}^{\infty} L_n Z^n + \sum_{m=0}^{\infty} Z^m \sum_{n=1}^{\infty} V_n^m$$

in a certain circle c in the complex plane with center zero. The number

of zeros means the sum of the multiplicities of the zero-points in C of the function (4).

If k is the smallest integer such that $A_k \neq 0$, then by applying Rouché's Theorem for complex variable theory, (See Knopp [9], p. 111), Schmidt proves that if the maximum of $v(s)$ is sufficiently small and if the radius of circle C is sufficiently small, then the number of zeros in C of the function (4) is equal to k.

The generalization to Banach space of the non-singular case considered by Schmidt, i.e., the case in which the operator is non-singular is contained in the work of Hildebrandt and Groves [6]. We consider here the problem of generalizing the singular case to Banach space, i.e., we obtain a theory of ramifications points or branch points of solutions of non-linear equations in Banach space which is a generalization of Schmidt's work.

In Chapter 3, we consider the equation in Banach space X

$$(I - C)x + Sy + T(x,y) = 0 \quad , \tag{5}$$

where I is the identity mapping, C is linear and completely continuous, S is a continuous transformation satisfying a Lipschitz condition, and T is a continuous mapping from $X \times X$ into X with certain kth-order properties where $k \geq 2$. The hypotheses are such that if T is independent of y, then the transformation $(I - C)$ is the differential at 0 of the transformation $(I - C + T)$. (See the beginning of Chapter 3 of this thesis for a precise description of transformations S and T.) We assume that the null space of $(I - C)$ is different from θ, the additive identity of X, i.e., we assume that the transformation $(I - C)$ is singular. The problem is to determine the local solutions x when y is given.

Analogous to Schmidt's set of Verzweigungsgleichungen, a mapping M of Euclidean n-space into itself is obtained. Essentially, this mapping is obtained by a consideration of the projection of equation (5) in the null space of $(I - U)$. It is shown that the number of local solutions of (5) is equal to the number of points in the set $M^{-1}(0)$ where 0 is the zero of the n-space. Thus the problem is reduced to the investigation of a mapping of n-space into itself.

Instead of investigating the number of points in the set $M^{-1}(0)$, we study the topological degree of M at 0, which is a reasonable generalization of the number of solutions of the equation

$$M(x) = 0 \quad ,$$

each solution counted with its "multiplicity". In this work, the topological degree of M at 0 is determined in two cases: first it is proved that if the dimension of the null space of the operator $(I - U)$ is one, and if a non-zero condition which is a generalization of Schmidt's condition: $\lambda_k \neq 0$, is satisfied, then the topological degree is equal to k. Secondly, if the dimension of the null space is two, if the same type of non-zero condition is satisfied for λ_2, and if a certain condition on the constants in the mapping is satisfied, then the topolgical degree of mapping M is four.

At the end of Chapter 3 is appended a brief discussion of the relationship between the number of points in the set $M^{-1}(0)$ and the topological degree at 0 of M.

In chapter 4, it is proved that equation (1), the equation studied by Schmidt, is a special case of our equation (5). Also some extensions of

Schmidt's work are obtained by applying the results of chapter 3. Schmidt's results can be applied to an integral equation of the form

$$u(s) + \int_a^b c(s,t)u(t)dt + F(s)v(s) + \int_a^b K(s,t)G[u(t), v(t)]dt = 0 \quad ,$$

where G is an analytic function of two variables. In chapter 4, we show that the results of chapter 3 make it possible to deal with this equation if G has third derivativew with respect to its two variables.

Using the results of chapter 3, we define in chapter 5 a multiplicity for the solutions of certain equations of the type

$$(1 - C + T)x = y \quad , \tag{6}$$

i.e., equations which are special cases of equation (5). This multiplicity is defined to be the topolgical degree of the mapping M. It is proved that if the Leray-Schander topological degree (See Leray and Schander [11]) is defined for the transformation $I - C + T$, then the Leray-Schander topological degree is either equal to the multiplicity defined or is equal to this multiplicity times the factor (-1).

Finally, in the appendix, we prove two lemmas which are stated but without detailed proofs by Leray and Schander. To the writer's knowledge, there are no complete proofs of these lemmas in the literature.

Chapter 2: Some Preliminary Theorems

For the discussion which follows, we shall need certain known theorems from the theory of Banach spaces. For convenience, we state these theorems in that form in which they will be used here.

Let X be a complex Banach space[2]. In this work, $x, y, z, u, v, w, \cdots$ will be used to denote elements of X; $a, b, c, \cdots$ will denote scalars; and $f, g, h, \cdots$ will denote elements of X^*, the Banach space of linear functionals on X. Capital letters $U, I, R, S, T, \cdots$ will be used to denote continuous transformations (not necessarily linear) of X on $X + X$ into X. The symbols $\|x\|$, $\|f\|$, $\|\theta\|$ will denote the norms of x, f, and θ respectively.

First, certain parts of the theory of operators of the form

$$I - U$$

where I is the identity and U is linear and completely continuous will be used[3]. We introduce the notation:

$X^n = (I - U)^n X$ where n is a positive integer

X_n = null space of $(I - U)^n$ where n is a positive integer,

i.e., X_n is the linear space of elements x in X such that

$$(I - U)^n x = \theta \quad .$$

[2] See S. Banach [2].

[3] See F. Riesz [13], T. H. Hildebrandt [5], and S. Banach [2]. The material from [5] and [2] is stated in the form in which it was given by Professor Hildebrandt in the course, Linear Spaces (Mathematics 285), University of Michigan, Fall Term, 1945-1946.

Theorem 2.1. $X_p \subseteq X_{p+s}$; $X^p \supseteq X^{p+s}$ where p and s are arbitrary positive integers.

Theorem 2.2. For each n , X_n is a finite-dimensional linear space, and X^n is a closed, linear space.

Theorem 2.3. There exists a positive integer m , called the index of the transformation (I - C), which is minimal with respect to the following two properties

a) $X_m = X_{m+r}$ for r , any positive integer.

b) $X^m = X^{m+r}$ for r , any positive integer.

It can be proved that C^* , the transformation conjugate to C and acting on X^* , is also linear and completely continuous (See Banach [2], pp. 99-100 for a definition of C^*) . Hence the analogs of Theorems 2.1, 2.2, 2.3 hold for $I^* - C^*$ where I^* is the identity transformation on X^* . Moreover, if X_1^* is the null space of $I^* - C^*$, we have:

Theorem 2.4. Dimension of X_1^* = Dimension of X_1 .

Theorem 2.5. If m is the index of the transformation I - C , then

$$X = X_m + X^m \quad \text{(direct sum)}$$

i.e., if x is an element of X , then $x = x_m + x^m$ where x_m , x^m are elements of X_m , X^m respectively, and the decomposition

$$x = x_m + x^m$$

is unique.

Now by Theorem 2.1, the null space X_1 of $(I - C)$ is contained in X_m and if $m > 1$, then there exists a linear space Y_1 such that

$$X_m = X_1 + Y_1 \qquad \text{(direct sum)}^4 \quad .$$

Denoting $Y_1 + X_m$ by X_1^1, we obtain:

Theorem 2.6. $X = X_1 + X_1^1$ (direct sum)

Now we consider an arbitrary element x of X. If by the decomposition of Theorem 2.6,

$$x = x_1 + x_1^1$$

then the following transformations, E_1 and E_1^1 may be defined:

$$E_1 : x \longrightarrow x_1$$

$$E_1^1 : x \longrightarrow x_1^1 \quad .$$

Theorem 2.7. E_1, E_1^1 are projections, i.e., E_1, E_1^1 are linear, continuous transformations, and $E_1 E_1^1$ and $E_1^1 E_1$ are both the zero transformation. Also

$$E_1^2 = E_1 \qquad (E_1^1)^2 = E_1^1$$

Since X_1 is finite dimensional (Theorem 2.2), there exists a basis $[x_1, \cdots, x_n]$ of X_1 where $\|x_i\| = 1$ for $i = 1, 2, \cdots, n$ and n is the dimension of X_1. Then we have

Theorem 2.8. There exists a set of linear functionals $f_1, f_2, \cdots, f_n$, elements of X^*, such that

[4] See P. Halmos [4].

$$E_1(x) = \sum_{i=1}^{n} f_i(x)x_i$$

for every x element of X.

Now let $g_1, g_2, \cdots g_n$ be a basis for X_1^* (Cf. Theorem 2.4). Then it can be proved that there exist elements $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n$ of X with the following property:

$$g_i(\bar{x}_j) = \delta_{ij} \quad \text{(Kronecker delta)} .$$

We define the transformation on X:

$$C_0x = Cx + \sum_{i=1}^{n} (f_i, x)\,\bar{x}_i$$

where (f_i, x) is another notation for $f_i(x)$.

Theorem 2.9. There exists $(I - C_0)^{-1}$ which we denote by R, and R is linear, continuous. X_m, X^m are invariant subspaces under the transformation R, and R is of the form $I - S$ where I is the identity transformation and S is linear and completely continuous.

Theorem 2.10. If x is an arbitrary element of X, then

$$R(I - C)x = x - E_1x$$

We shall also need an implicit function theorem for Banach spaces.

Theorem 2.11. Let X, Y, Z be Banach spaces. Let U, V, W, be open sets in X, Y, Z, respectively. Consider $W_0 = UxVxW$ and a function $\mathcal{L}$ defined on W_0 such that

$$\mathcal{L} : W_0 \longrightarrow X .$$

We make the following hypotheses:

H_1) There exists $(x_0, y_0, z_0) \in U \times V \times W$ such that

$$x_0 = \mathcal{L}(x_0, y_0, z_0)$$

H_2) There exists $k < 1$ such that

$$\| \mathcal{L}(x_1, y, z) - \mathcal{L}(x_2, y, z) \| \leq k \| x_1 - x_2 \|$$

for every (x_1, y, z), (x_2, y, z) in W_0.

H_3) $\mathcal{L}$ is uniformly continuous on the product space W_0.

Then the following conclusions hold:

C_1) For each (y, z) in $V \times W$, there is at most one point (x, y, z) in W_0 which is a solution of the equation

$$(A) \qquad x = \mathcal{L}(x, y, z)$$

C_2) There exist open sets $V_1(y_0)$ and $W_1(z_0)$, neighborhoods of x_0 and z_0 respectively, and a function

$$F : V_1 \times W_1 \longrightarrow X$$

such that the point $(F(y, z), y, z)$ is an element of W_0 and is a solution of equation (A) for every (y, z) in $V_1 \times W_1$.

C_3) The solution $F(y, z)$ of equation (A) is uniformly continuous on its domain of definition.

This theorem is a restatement of Theorem 2, pp. 134-135 of T. H. Hildebrandt and L. M. Graves [6]. The theorem will not be proved here. We only point out that in the proof, the function $F(y, z)$ is defined as the limit of a sequence of functions $X_m(y, z)$ defined as follows:

$$X_1(y, z) = \mathcal{L}(x_0, y, z)$$

$$X_{m+1}(y, z) = \mathcal{L}(X_m(y, z), y, z) \quad .$$

That is, the function $F(y, z)$, whose existence is to be proved, is actually constructed in the proof.

In the work which follows, we shall need the notion of the topological degree at a point of a continuous mapping, relative to the closure of a bounded, open set in Euclidean n-space. In Alexandroff and Hopf [1] (hereafter to be referred to as A-H), p. 474, the degree at a point of a continuous mapping is defined relative to a finite, Euclidean, algebraic complex. We extend the definition given there so as to obtain a definition of topological degree relative to the closure of a bounded, open set in the following way:

We prove first:

Lemma 1. Let P be a polyhedron in an oriented n-dimensional Euclidean space E^n which is the union of a finite number of Euclidean n-simplices $[\sigma^n_j]$ such that if $j \neq i$,

$$\sigma^n_j \cap \sigma^n_i = \sigma^q \tag{1}$$

where $q \leq n-1$ and σ^q may be the null set. Let K be the algebraic complex which is obtained by associating with each σ^n_j the coefficient $+1$ and associating arbitrary integers with the simplices of dimension $< n$ which are in P. Then the point set boundary of P is the same as the point set of the algebraic boundary of K.

Proof: We prove first that each $(n-1)$-simplex of K is on the point set boundary of one or two n-simplices of K. From the definition of P, it follows that each $(n-1)$-simplex is on the point set boundary of at least one n-simplex.

Now, let σ^{n-1} be an $(n-1)$-simplex in K. This simplex σ^{n-1} is contained in a $(n-1)$-dimensional hyperplane H_n which is uniquely determined by σ^{n-1}. The hyperplane H_n determines uniquely two disjoint open sets O_1 and O_2 which are both disjoint from H_n and which are such that

$$E^n = O_1 \cup H_n \cup O_2 \quad .$$

We prove that in O_i $(i = 1, 2)$, there exists at most one n-simplex of K whose boundary contains σ^{n-1}. This will complete the proof of the statement above.

Suppose there exist two simplices σ^n_1, σ^n_2 of K such that $\sigma^n_i \subset O_1$ and such that the point set boundary of σ^n_i contains σ^{n-1} for $i = 1, 2$. Suppose

$$\sigma^{n-1} = \overline{v_1 \cdots\cdots v_n}$$

Then

$$\sigma^n_1 = \overline{v_0 v_1 v_2 \cdots v_n}$$

and

$$\sigma^n_2 = \overline{w_0 v_1 \cdots\cdots v_n}$$

where $v_0 \in O_1$, $w_0 \in O_1$. Let p be a point in the interior of σ^{n-1}. (If $n = 1$, p is the simplex σ^{n-1} itself.) Then there exists $d > 0$ such that

$$\rho\left(p, U_{j=1}^{n} \overline{v_0 v_1 \cdots \hat{v}_j \cdots v_n}\right) > d$$

and

$$\rho\left(p, U_{j=1}^{n} \overline{w_0 v_1 \cdots \hat{v}_j \cdots v_n}\right) > d$$

where ρ is the metric function of Euclidean n-space and the notation $\overline{w_0 v_1 \cdots \hat{v}_j \cdots v_n}$ means the simplex $\overline{w_0 v_1 \cdots v_{j-2} v_{j-1} v_{j+1} v_{j+2} \cdots v_n}$.

Now let N be a spherical neighborhood of p of radius $\frac{d}{2}$. We shall show that

$$N \cap O_1 \subset \sigma_1^n$$

for $i = 1, 2$. Since σ_1^n, σ_2^n are Euclidean n-simplices, they are convex cells by A-H, Satz V, p. 613. Hence by A-H, p. 611, Satz II, σ_1^n is the intersection of the sets

$$A_1 = [(t_1, \cdots, t_n) / \sum_{i=1}^{n} a_{i1} t_i \geqslant 0]$$

$$A_2 = [(t_1, \cdots, t_n) / \sum_{i=1}^{n} a_{i2} t_i \geqslant 0]$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$A_n = [(t_1, \cdots, t_n) / \sum_{i=1}^{n} a_{in} t_i \geqslant 0]$$

The simplex σ_2^n is the intersection of the sets:

$$A_1 = [(t_1, \cdots, t_n) / \sum_{i=1}^{n} a_{i1} t_i \geqslant 0]$$

$$B_2 = [(t_1, \cdots, t_n) / \sum_{i=1}^{n} b_{i2} t_i \geqslant 0]$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$B_n = [(t_1, \cdots, t_n) / \sum_{i=1}^{n} b_{in} t_i \geqslant 0]$$

The equation

$$\sum_{i=1}^{n} a_{i1} t_i = 0$$

is the equation of the $(n-1)$-dimensional hyperplane H_n passing through the $(n-1)$-simplex σ^{n-1}. Now if the point $x_o = (t_{o1}, t_{o2}, \cdots, t_{on})$ is in $N \cap O_1$, this means that

$$(2) \qquad \sum_{i=1}^{n} a_{i1}\, t_{oi} > 0 \qquad \text{(condition that } x_o \text{ is in } O_1\text{)}$$

and that

$$(3) \qquad \sum_{i=1}^{n} (p, x_o) < \frac{d}{2} \qquad \text{(condition that } x_o \text{ is in } N\text{)} .$$

Inequality (2) shows that x_o is in A_1. From the definition of d and from inequality (3), it follows that x_o is in the sets $A_2, \cdots A_n$, $B_2, \cdots, B_n$.

Thus the open set $N \cap O_1$ is contained in $\sigma^n_1 \cap \sigma^n_2$. This is a contradiction to the equation (1) in the hypothesis of Lemma 1.

Thus we have shown that each $(n-1)$-simplex of K is on the point set boundary of one or two n-simplices of K. From the definition of the algebraic boundary operator ∂, it follows now that, since all the n-simplices of K have coefficient $+1$ in K, then an $(n-1)$-simplex σ^{n-1} of K has a non-zero coefficient in ∂K if and only if σ^{n-1} is in the point set boundary of exactly one n-simplex of K.

Now we prove that an $(n-1)$-simplex σ^{n-1} is in the point set boundary of just one n-simplex σ^n of K if and only if σ^{n-1} is in the point set boundary of P. The proof of the Lemma follows immediately from this statement and the preceding parts of the proof.

Suppose σ^{n-1}_o is in the point set boundary of just one n-simplex σ^n_o. The $(n-1)$-dimensional hyperplane H_{n_o}, that is determined by σ^{n-1}_o determines disjoint open sets O_{10}, O_{20} which are disjoint from H_{no} and have the property that

$$E^n = O_{10} \cup H_{no} \cup O_{20}$$

The simplex σ^n_o is contained in either $O_{10} \cup H_{no}$ or $O_{20} \cup H_{no}$. Suppose that

$$\sigma^n_o \subset O_{10} \cup H_{no}$$

Let p be an arbitrary point of σ^{n-1}_o . If N is any neighborhood of p then $N \cap O_{10} \neq \emptyset$ and $N \cap O_{20} \neq \emptyset$. Since $\sigma^n_o \cap O_{20} = \emptyset$ and since there is no other n-simplex of K which has σ^{n-1}_o in its point set boundary, then $N \cap O_{20}$ contains points of $E^n - P$. Hence p is a boundary point of P . Hence σ^{n-1} is in the point set boundary of K .

Now suppose σ^{n-1}_o is in the point set boundary of K . Then σ^{n-1}_o can be in the point set boundary of just one n-simplex of K . For if σ^{n-1}_o were on the boundary of two n-simplices of K , then an interior point of σ^{n-1}_o would be interior to K .

This completes the proof of Lemma 1.

Next we prove:

Lemma 2. Given a bounded open set S in Euclidean n-space E^n , a closed set of points $Q \subset S$, and $\varepsilon > 0$, then there exists a polyhedron P which has the following properties:

a) The polyhedron P is the union of a finite number of Euclidean n-simplices σ^n_i such that if $i \neq j$

$$\sigma^n_i \cap \sigma^n_j = \sigma^q$$

where $q > n$ and σ^q may be the null set.

b) The polyhedron P is contained in S .

c) Each point p of the point set boundary of P is distance $< \varepsilon$ from the boundary S^1 of the set S.

d) The set Q is contained in the interior of P.

Proof: Let K be a Euclidean n-simplex such that $\bar{S}$, the closure of S, is contained in the interior of K. By A-H, Satz V, p. 141, there exists an arbitrarily fine simplicial subdivision of K. Hence there exists a simplicial subdivision of K which has the following properties:

1) Each simplex in the subdivision has diameter $< \frac{\varepsilon}{2}$.

2) If d is the minimum distance from Q to S^1, then each simplex in the subdivision has diameter $< \frac{d}{4}$. Let K_1 be the complex consisting of those n-simplices of this subdivision of K which together with all their faces are contained in the set S. We shall prove that the point set $\bar{K}_1$ of K_1 is the desired polyhedron.

Conditions a) and b) are obviously satisfied. We prove that condition c) is satisfied as follows: suppose p is a point of the boundary of the polyhedron. Since it is a point on the boundary, then it must be contained in some $(n-1)$-simplex of K_1. If p is a vertex of this $(n-1)$-simplex, then it is clear that p is within ε of the boundary S^1 of S because since p is on the boundary of K_1, the open star of p is not in the interior of S, i.e., there exists a simplex in the open star of p which has a non-null intersection with S^1. Since each simplex of K_1 has diameter less than $\varepsilon/2$, the point p is certainly a distance less than ε from S^1.

Now suppose p is a point of the boundary of the polyhedron and p is not a vertex of any simplex of K_1. Then p is contained in the interior of an r-simplex $\sigma^r (r \geq 1)$ of K_1. Let q be a vertex of this r-simplex. Since each simplex of K_1 has diameter less than $\varepsilon/2$, then the distance

from p to q is less than $\varepsilon/2$. On the other hand, the open star of q is not contained in K_1 because then p would be in this open set and hence could not be on the boundary of the point set of K_1. Thus there exists a simplex in the open star of q which has a non-null intersection with S^1. Since each simplex of K_1 has diameter less than $\varepsilon/2$, then the distance from the point q to S^1 is less than $\varepsilon/2$. Since p is distance less than $\varepsilon/2$ from q, then the distance from p to S^1 is less than .

Now we prove that condition d) is satisfied. Let q be an arbitrary point of Q. Since q is contained in the complex K, it is contained in some n-simplex σ_1^n of K. Since $q \in S$, this means:

$$\sigma_1^n \cap S \neq \emptyset$$

The diameter of σ_1^n is less than $\frac{d}{4}$ where d is the minimum distance from Q to S^1. Hence

$$\sigma_1^n \cap S^1 = \emptyset$$

Hence, since σ_1^n is a connected set, σ_1^n is contained in S. Thus σ_1^n is contained in $\bar{K}_1$, and the open star of σ_1^n is contained in $\bar{K}_1$.

This completes the proof that condition d) is satisfied; hence the proof of Lemma 2 is complete.

Now we consider a bounded open set S in E^n, an oriented n-dimensional Euclidean space, a continuous mapping f taking $\bar{S} = S \cup S^1$, the closure of S, into E^n, and a point o in E^n such that o is not an element of $f(S^1)$. We wish to define a topological degree of f relative to $\bar{S}$ and to the point 0. First since S^1 is a closed, compact set, $f(S^1)$ is also closed and compact. Hence there exists a minimum distance $d > 0$ between o and $f(S^1)$.

Since f is a continuous mapping, and $\overline{S}$ is compact, f is uniformly continuous on $\overline{S}$. Hence there exists $\delta > 0$ such that if x_1, x_2 are elements of $\overline{S}$ and

$$\rho\,[x_1, x_2] < \delta \tag{1}$$

then

$$\rho\,[f(x_1), f(x_2)] < \frac{d}{2} \tag{2}$$

Now we apply Lemma 2 to the open set S; we take ε such that $\varepsilon \leq \delta/2$ and the set Q to be $f^{-1}(o)$. If P^1 is the boundary of the resulting polyhedron P, then the point o is not in $f(P^1)$ because the distance from o to $f(S^1)$ is d and since each point of P^1 is distance less than $\frac{\delta}{2}$ from $f(S^1)$, then for each point p in P^1, the distance from $f(p)$ to $f(S^1)$ must be less than $\frac{d}{2}$. Hence the point $f(p)$ is a distance greater than $\frac{d}{2}$ from the point o. Now if K is the algebraic complex which is obtained from P in the manner described in the statement of Lemma 1, then by the conclusion of Lemma 1, the point set $\overline{K}$ of the algebraic boundary K of K is P^1. Hence the point o is not in $f(K)$. This fact makes it possible to set up the following definition:

Definition 2.1. The topological degree of f relative to $\overline{S}$ and the point o is the topological degree of f relative to the complex K and the point o.

In order to justify this definition, we must prove that the topological degree given by the definition is independent of the choice of the polyhedron P. Let P_1 and P_2 be two polyhedra satisfying the required condi-

tions and hence having the property that the topological degree at o of the mapping f relative to the set $\overline{S}$ may be defined by using either P_1 or P_2. We shall prove that the topological degree at o of f relative to P_1 is equal to the topological degree at o of f relative to P_2. This will prove that Definition 2.1 is independent of the choice of the polyhedron P.

We consider first the point set $P_1 \cap P_2$. By A-H, Augabe 1, this set is a finite polyhedron. Moreover, since the o-points of f are interior points of P_1 and P_2, then all the o-points are interior points of $P_1 \cap P_2$. Hence the topological degree at o of f relative to $P_1 \cap P_2$ (i.e., the topological degree defined in A-H, p. 474) is defined. Since all the o-points of f are in $P_1 \cap P_2$, then from the definition of topological degree (A-H, p. 474) and Satz 11, p. 472, A-H, it follows that the topological degree at o of f relative to P_1 is equal to the topological degree at o of f relative to $P_1 \cap P_2$. By the same argument, the topological degree at o of f relative to P_2 is equal to the topological degree at o of f relative to $P_1 \cap P_2$. Hence the topological degree at o of f relative to P_1 is equal to the topological degree at of of f relative to P_2.

It is clear from Definition 2.1 that an analogue of the general Kronecker Existence Theorem (See A-H, pp. 467-468 for a statement of this theorem in terms of the topological degree of a mapping relative to a finite Euclidean algebraic complex and a point.) holds for the topological degree given in Definition 2.1.

We prove also the following analogue of Rouché's Theorem, (See A-H, p. 459)

<u>Theorem 2.12</u> Let f_1 and f_2 be two continuous mappings of $\overline{S}$ into E^n; if for each point $p \in S'$, the following condition holds:

$$\rho\,[f_1(p),\, f_2(p)] < \rho\,[f_1(p),\, o]$$

then the topological degree of f_2 relative to $\bar{S}$ and the point o is defined and is equal to the topological degree of f_1 relative to S and the point o.

Proof: The function

$$\rho\,[f_1(p),\, o] - \rho\,[f_1(p),\, f_2(p)]$$

is a continuous function on S^1, and since S^1 is compact, the function is uniformly continuous. From the hypothesis, it follows that

$$\min_{p \in S^1}\left\{\rho\,[f_1(p),\, o] - \rho\,[f_1(p),\, f_2(p)]\right\} = M > 0 \quad .$$

Hence by the uniform continuity, there exists $\varepsilon_o > 0$ such that if

$$\rho\,(p,\, S^1) < \varepsilon_o$$

then

$$\rho\,[f_1(p),\, o] - \rho\,[f_1(p),\, f_2(p)] \geq \frac{M}{2} > 0 \tag{3}$$

Now the quantities d and δ defined for the mapping f in the discussion preceding Definition 2.1 (See inequalities (1) and (2)) are defined for f_1 and f_2. Let δ_1, δ_2 be the corresponding δ's for f_1 and f_2 and let δ_o be the smaller of δ_1 and δ_2.

We apply Lemma 2 to S and let the ε of Lemma 2 be a number $< \varepsilon_o$ and also $< \frac{\delta_o}{2}$. Let the set Q be $[f_1^{-1}(o)] \cup [f_2^{-1}(o)]$. Then the topological degrees of f_1 and f_2 relative to S and o may be defined relative to the polyhedron P given by Lemma 2. Since every point on the boundary P^1 of polyhedron P has distance $< \varepsilon_o$ from S^1, inequality (3) holds for all

$p \in P^1$. Hence by Rouché's Theorem (See A-H, p. 459), the topological degree of f_1 relative to K and o is equal to the topological degree of f_2 relative to K and o where K is the algebraic complex associated with polyhedron P . This assertion together with Definition 2.1 completes the proof of the theorem.

Chapter 3: The Problem in Banach Space

We consider the equation in Banach space X

$$(3.1) \qquad (I - C)x + Sy + T(x, y) = \theta$$

where θ is the additive identity of X, I is the identity mapping of X onto itself, and the following assumptions are made:

Assumption 3.1. C is a linear, completely continuous mapping of X into itself.

Assumption 3.2. S is a continuous mapping of a subset of X into X with the properties:

a) $S(\theta) = \theta$

b) There exists a neighborhood $N_1(\theta)$ of θ, of radius r_1, in which S satisfies a Lipschitz condition, i.e., there exists a constant $B > 0$ such that if y_1, y_2 are elements of $N_1(\theta)$, then

$$\| Sy_1 - Sy_2 \| < B \| y_1 - y_2 \|$$

Assumption 3.3. T is a mapping of a subset of $X \times X$ into X with the following properties:

a) $T(\theta, \theta) = \theta$

b) There exist neighborhoods N_2 of $x = \theta$, of radius r_2 and N_3 of $y = \theta$ of radius r_3 and there exists a constant $M > 0$ such that if x_1, x_2 are elements of N_2 and y_1, y_2 are elements of N_3, then

$$\| T(x_1, y_1) - T(x_2, y_2) \| < M\left\{\|x_1\| + \|x_2\| + \|y_1\| + \|y_2\|\right\}\left\{\|x_1 - x_2\| + \|y_1 - y_2\|\right\}$$

c) The following kth order condition is imposed upon T : there exists a maximal integer $k \geq 2$ such that if a is a scalar, then

$$T[ax, \theta] = a^k T^1[a, x]$$

where T^1 is a continuous mapping of a subset of the product space $Z \times X$ into X . (Z will denote always the space of complex numbers.) This product space contains points of the form $(0, x)$ where 0 is the zero of Z .

The object of the discussion is to obtain information concerning the local solutions x of equation (3.1) when the transformation $I - C$ is singular and y is given. This is the same as determining the local solutions x_1 , x_1' (using the notation of Chapter 2) of the equation

$$(3.1') \qquad (I - C)(x_1 + x_1') + Sy + T(x_1 + x_1', y) = \theta$$

when y is given. Since x_1 is an element of the null space of $(I - C)$, equation (3.1') may be written

$$(3.1'') \qquad (I - C)x_1' + Sy + T(x_1 + x_1', y) = \theta$$

Applying the transformation R , defined in Theorem 2.9, to equation (3.1''), we obtain:

$$(3.2) \qquad R(I - C)x_1' + RSy + RT(x_1 + x_1', y) = \theta$$

Now by Theorems 2.10 and 2.7

$$R(I - C)x_1' = x_1' - E_1 x_1' = x_1'$$

Hence equation (3.2) becomes:

$$x_1' = -RSy - RT(x_1 + x_1', y) \tag{3.3}$$

Applying E_1', the transformation defined in Theorem 2.7, to equation (3.3) we obtain, since $E_1' x_1' = x_1'$:

$$x_1' = -E_1'RSy - E_1' RT(x_1 + x_1', y) \tag{3.4}$$

Applying the transformation E_1, defined in Theorem 2.7, to equation (3.3) and using Theorem 2.10, we obtain:

$$\theta_1 = E_1 RSy + E_1 RT[x_1 + x_1', y] \tag{3.5}$$

where θ_1 is the additive identity of X_1.

Definition 3.1. Two systems of equations Σ_1 and Σ_2 are equivalent if each solution of system Σ_1 is a solution of system Σ_2 and each solution of Σ_2 is solution of Σ_1.

It is clear from their derivation that the system of equations (3.4) and (3.5) are equivalent to the equation (3.1") in terms of solutions x_1, x_1'.

Now we wish to solve equation (3.4) for x_1' in terms of y and x.

Lemma 3.1. There exist neighborhoods U, V, W of $x_1' = \theta$, $x_1 = \theta$ and $y = \theta$ in X_1', X_1, and X respectively such that the function

$$-E_1' RSy - E_1' RT[x_1 + x_1', y]$$

may be regarded as a mapping from $U \times V \times W$ into X_1' and as such satisfies the hypotheses of Theorem 2.11.

Proof: First, from Theorem 2.2, it follows that X_1 and X_1' are Banach spaces. Hypothesis H_1) of Theorem 2.11 is satisfied because, from the definition of the transformations C, S, T, it follows that $x_1' = x_1 = y = \theta$ is the required initial solution.

We show that hypotheses H_2) and H_3) are satisfied by deriving an inequality. First since E_1, E_1' are linear and continuous by Theorem 2.7, then there exist neighborhoods U_1, V_1, W_1 of $x_1' = \theta$, $x_1 = \theta$, and $y = \theta$ such that: if $x_1' \in V_1$, $x_1 \in V_1$, $y \in W_1$, then $x_1 + x_1' \in N_2$, $y \in N_3$ where N_2, N_3 are the neighborhoods described in b) of Assumption 3.3. We also assume that W_1 is contained in N_1, the neighborhood described in Assumption 3.2. Now if x_1', $x_{11}' \in U_1$, x_1, $x_{11} \in V_1$, and y, $y_1 \in W_1$, then the following inequalities hold:

$$\begin{aligned}
&\| -E_1'RSy - E_1'RT[x_1+x_1', y] + E_1'RSy_1 + E_1'RT[x_{11}+x_{11}', y] \| \\
&\leq \| E_1'RSy_1 - E_1'RSy \| + \| E_1'RT[x_{11}+x_{11}', y_1] - E_1'RT[x_1+x_1', y] \| \\
&\leq \| E_1' \| \| R \| \| Sy_1 - Sy \| + \| E_1' \| \| R \| \| T[x_{11}+x_{11}', y_1] - T[x_1+x_1', y] \| \\
&\leq \| E_1' \| \| R \| B \| y_1 - y \| + \| E_1' \| \| R \| M \{ \| x_{11}+x_{11}' \| + \| y_1 \| + \| x_1+x_1' \| + \| y \| \} \\
&\qquad \{ \| x_{11} - x_1 \| + \| x_{11}' - x_1' \| + \| y_1 - y \| \}
\end{aligned}$$

The last of these inequalities shows that the mapping is uniformly continuous in $U_1 \times V_1 \times W_1$. Hence hypothesis H_3) is satisfied. Now if in this last inequality we take $x_{11} = x_1$ and $y = y_1$, then the inequality becomes:

$$\begin{aligned}
&\| -E_1'RSy - E_1'RT[x_1+x_1', y] + E_1'RSy + E_1'RT[x_1+x_{11}', y] \| \\
&\leq \| E_1' \| \| R \| M \{ \| x_1 + x_1' \| + \| x_1 + x_1'' \| + 2\| y \| \} \{ \| x_{11}' - x_1' \| \}
\end{aligned}$$

Since $\| E_1' \| \| R \| M = K_0$, a constant, then if neighborhoods U, V, W of $x_1' = \theta$, $x_1 = \theta$ and $y = \theta$ respectively are chosen such that $U \subset U_1$, $V \subset V_1$, $W \subset W_1$ and such that if x_{11}', $x_1' \in U$, $x_1 \in V$, $y \in W$, then

$$\left\{ \|x_1 + x_1'\| + \|x_1 + x_{11}'\| + 2\|y\| \right\} < \frac{1}{K_0}$$

then the inequality above shows that hypothesis H_2) is satisfied if x_1', $x_{11}' \in U$, $x_1 \in V$, $y \in W$.

qed.

It should be remarked at this point that the remainder of the discussion will concern only the domain $U \times V \times W$ or subsets thereof which will be specified later. Thus we deal only with local properties of equation 3.1.

Lemma 3.1 permits us to apply Theorem 2.11 to equation (3.4). Doing so, we obtain the unique solution

$$x_1' = F(y, x_1) \tag{3.6}$$

Substituting from equation (3.6) into equation (3.5), we obtain:

$$\theta_1 = E_1 RSy + E_1 RT[x_1 + F(y, x_1), y] \tag{3.7}$$

Now the system of equations (3.3) and (3.5) is equivalent to the system of equations (3.5) and (3.7). Hence we obtain finally:

Lemma 3.2. For fixed y, equation (3.1") is equivalent to the pair of equations (3.6) and (3.7) in terms of solutions x_1, x_1'.

Hence the study of the solutions x of equation (3.1) is reduced to a study of the solutions x_1, x_1' of the pair of equations (3.6) and (3.7). Since equation (3.6) gives x_1' explicitly in terms of x_1 and y, we are actually concerned only with the solutions x_1 of equation (3.7). Equation (3.7) is an equation in complex Euclidean n-space where n is the dimension of the null space of $(I - U)$. Hence, in order to obtain information concerning the solutions x_1 of equation (3.7), we investigate, for fixed y,

the topological degree at θ_1 of the mapping

$$E_1 RSy + E_1 RT[x_1 + F(y, x_1), y]$$

which is the right side of equation (3.7). This is a mapping of complex Euclidean n-space X_1 into itself.

To determine the topological order, we obtain first some information concerning the form of $F(y, x_1)$.

Lemma 3.3. There exists a neighborhood N''_a of 0 in the complex plane and a neighborhood N''_{v_1} of θ_1 in X_1 such that if x_1 is of the form av_1 where a, v_1 are in N''_a, N''_{v_1} respectively, then

$$F(\theta, av_1) = a^k G(a, v_1)$$

where $G(a, v_1)$ is a continuous mapping from $N''_a * N''_{v_1}$ into X_1' and k is the integer described in c) of Assumption 3.3.

Proof: $F(\theta, x_1)$ is the solution of equation (3.4) when $y = \theta$, i.e., the equation

$$x_1' = -E_1' RT(x_1 + x_1', \theta) \tag{3.4'}$$

Equation (3.4) is solved for x_1' in terms of x_1 and y by applying Theorem 2.11 [See Lemma 3.1]. The solution is obtained by applying the approximating method of the proof of Theorem 2.11, and is hence the limit of a sequence of elements of X which are functions of x_1 and y. We construct now the sequence for the case when $y = \theta$. When $y = \theta$, equation (3.4) becomes equation (3.4') above. For convenience of notation, we write equation (3.4') as

$$x_1' = -E_1' RT[x_1 + x_1']$$

and we write the solution as $F(x_1)$.

The solution $F(x_1)$ is the limit of the sequence $x_1'^{(n)}$ defined as follows:

$$x_1'^{(1)} = -E_1' \, RT[x_1]$$

$$x_1'^{(2)} = -E_1' \, RT[x_1 + x_1'^{(1)}] = -E_1' \, RT[x_1 - E_1' \, RT(x_1)]$$

$$\cdots\cdots\cdots$$

$$x_1'^{(n)} = -E_1' \, RT[x_1 + x_1'^{(n)}] \qquad (1)$$

Now we show by induction that if $a \neq 0$, then there exists a function $G_n(a, v_1)$ such that

$$x_1'^{(n)} = a^k \, G_n(a, v_1)$$

and that there exists $\lim_{a \to 0} G_n(a, v_1)$ for each n .

$$x_1'^{(1)} = -E_1' RT[av_1] = a^k \, [-E_1' \, RT'(a, v_1)]$$

by Assumption 3.3 c). We define $G_1(a, v_1) = -E_1' \, RT' \, (a, v_1)$. By Assumption 3.3 c) , T' is continuous in its domain; hence from the definition, it is clear that there exists $\lim_{a \to 0} G_1(a, v_1)$.

Nowsuppose that the statement holds for $x_1'^{(n)}$. Then from equation (1), it follows that:

$$x_1'^{(n+1)} = -E_1' \, RT[av_1 + a^k \, G_n(a, v_1)]$$

$$= a^k \left\{ -E_1' \, RT' \, [a, v_1 + a^{k-1} \, G_n(a, v_1)] \right\}$$

We define $G_{n+1}(a, v_1) = -E_1' \, RT' \, [a, v_1 + a^{k-1} \, G_n(a, v_1)]$. Since by the induction hypothesis, there exists $\lim_{a \to 0} G_n(a, v_1)$, then from the definition

of $G_{n+1}(a, v_1)$, it is clear that there exists $\lim_{a \to 0} G_{n+1}(a, v_1)$. We note further that $\lim_{a \to 0} G_n(a, v_1) = -E_1' \, RT' \, [0, v_1]$ for all n . Also from the definition of $G_n(a, v_1)$, it is clear that each $G_n(a, v_1)$ is continuous in the domain in which it is defined.

Thus we have, for a bounded neighborhood N_a of $a = 0$ and a bounded neighborhood N_{v_1} of $v_1 = \theta$ in X_1 , a function $G_n(a, v_1)$ defined in $[N_a - a] \times [N_{v_1}]$ for each n . The neighborhoods N_a, N_{v_1} are such that $\overline{N_a \times N_{v_1}} \subset U$, the open set in X_1 described in Lemma 3.1.

The functions $G_n(a, v_1)$ are not defined at $a = 0$. However, since, for each n and v_1 , there exists $\lim_{a \to 0} G_n(a, v_1) = -E_1' \, RT' \, [0, v]$, we can extend the definition of $G_n(a, v_1)$ to the points where $z = 0$ by defining:

$$G_n(0, v_1) = \lim_{a \to 0} G_n(a, v_1) = -E_1' \, RT' \, [0, v_1]$$

The extended function $G_n(a, v_1)$ is clearly continuous in the product space $N_a \times N_{v_1}$.

Now we show that in a certain neighborhood of $a = 0$ and a certain neighborhood of $v_1 = \theta_1$, the extended functions $G_n(a, v_1)$ converge uniformly. We prove first that there exists $M_0 > 0$ and neighborhoods N_a' , N_{v_1}' of $a = 0$ and $v_1 = \theta_1$, respectively, such that for all n ,

$$\| G_n(a, v_1) \| < M_0$$

for a and v_1 in N_a' , N_{v_1}' respectively. This statement is proved by induction. First

$$G_2(a, v_1) = -E_1' \, RT' \, [a, v_1]$$

Hence

$$\| G_1(a, v_1) \| = \| -E_1' RT' [a, v_1] \| \leq \| E_1' \| \| R \| \| T' [a, v_1] \|$$

By c) of Assumption 3.3, T' is a continuous mapping of a subset of $Z \times X$ into X. If B_o is a positive number satisfying the inequality $\| T' (\theta, \theta) \| \leq \frac{B_o}{2}$, then there exists spherical neighborhoods W_a, W_x of $a = 0$ and $x = \theta$ in Z and X respectively so that if a, x are in W_a, W_x, respectively, then

$$\| T' (a, x) \| < B_o$$

Hence

$$\| G_1(a, v_1) \| < \| E_1' \| \| R \| B_o$$

We define $M_o = \| E_1' \| \| R \| B_o$. We will show that M_o is the required bound. We define N_a', N_v' as follows:

1) $N'_{v_1} \subset N_{v_1} \cap W_x$; $N_a' \subset N_a \cap W_a$

2) N'_{v_1} and N_a' have the property that if $a \in N_a'$, $v_1 \in N'_{v_1}$, then

$$\| v_1 \| + |a|^{k-1} M_o < r$$

where r is the radius of W_x, defined above.

Now we assume that $\| G_n(a, v_1) \| < M_o$ for a, v_1 in N_a', N'_{v_1} respectively and prove that $\| G_{n+1}(a, v_1) \| < M_o$ for $a \in N_a'$, $v \in N'_{v_1}$.

$$\| G_{n+1}(a, v_1) \| = \| -E_1' RT' [a, v_1 + a^{k-1} G_n(a, v_1)] \|$$

Since $a \in N_a'$, $a \in W_a$. Also

$$\| v_1 + a^{k-1} G_n(a, v_1) \| \leq \| v_1 \| + |a|^{k-1} \| G_n(a, v_1) \| < r$$

and hence: $v_1 + a^{k-1} G_n(a, v_1) \in W_x$. Thus:

$$\| G_{n+1}(a, v_1) \| = \| -E_1' RT' [a, v_1 + a^{k-1} G_n(a, v_1)] \| \leq \| E_1' \| \| R \| \; B_o = M_o$$

for all $a \in N_a'$, $v \in N_{v_1}'$.

Now we show that the function $G_n(a, v_1)$ converges uniformly if a , v_1 are in sufficiently restricted neighborhoods N_a'' , N_{v_1}'' of 0 and θ_1 respectively. To show this, we add the following restrictions to a :

(2) $|a|$ is made so small that

$$|a| \left\{ \| E_1' \| \| R \| \; M[2 \| v_1 \| + |a|^{k-1} M_o] \right\} < \tfrac{1}{2}$$

This restriction together with the previous restrictions on a defines the neighborhood N_a'' . N_{v_1}'' is N_{v_1}' .

Now first

$$(3) \quad \| G_1(a, v_1) - G_2(a, v_1) \| \leq \| G_1(a, v_1) \| + \| G_2(a, v_1) \| \leq 2 M_o$$

We complete the demonstration by showing that:

$$\| G_{n+1}(a, v_1) - G_n(a, v_1) \| \leq \tfrac{1}{2} \| G_n(a, v_1) - G_{n-1}(a, v_1) \|$$

This is demonstrated as follows:

$$\| a^k G_{n+1}(a, v_1) - a^k G_n(a, v_1) \| = |a|^k \| G_{n+1}(a, v_1) - G_n(a, v_1) \|$$

$$= \| -E_1' RT[av_1 + a^k G_n(a, v_1)]$$

$$+ E_1' RT[av_1 + a^k G_{n-1}(a, v_1)] \|$$

$$< \| E_1' \| \| R \| \; M[2 |a| \| v_1 \| + |a|^k \| G_n(a, v_1) \| + |a|^k \| G_{n-1}(a, v_1) \|]$$

$$+ [|a|^k] [\| G_n(a, v_1) - G_{n-1}(a, v_1) \|]$$

by Assumption 3.3 b).

Dividing through by $|a|^k$, we obtain:

$$\| G_{n+1}(a, v_1) - G_n(a, v_1) \| < \|E_1'\| \, \|R\| \, M[2|a| \, \|v_1\| + 2 \, |a|^k M_0]$$

$$\| G_n(a, v_1) - G_{n-1}(a, v_1) \|$$

by restriction (1)

Inequality (1) holds for all $a \neq 0$ and $a \in N_a''$. Hence by the continuity of $G_n(a, v_1)$ for all n, we obtain the desired inequality:

$$(4) \qquad \| G_{n+1}(a, v_1) - G_n(a, v_1) \| \leqslant \tfrac{1}{2} \| G_n(a, v_1) - G_{n-1}(a, v_1) \|$$

Hence there exists $\lim_{n \to \infty} G_n(a, v_1) = G(a, v_1)$, a continuous function defined on $N_a'' \times N_{v_1}''$. Since

$$x_1'^{(n)} = a^k G_n(a, v_1)$$

then

$$x_1'^{(n)} = a^k G(a, v_1)$$

qed.

In the discussion which follows, we shall consider not V, the open set defined in the proof of Lemma 3.1 but a subset $M_a \times M_{v_1} = V_1 \subset V$ which has the property that

$$\overline{V}_1 \subset N_a'' \times N_{v_1}''$$

where N_a'', N_{v_1}'' are the neighborhoods defined in the proof of Lemma 3.3.

Now we study two cases:

<u>Case I</u>. The dimension of X_1, the null space of $(I - C)$, is 1.

For this case, we make the

Assumption 3.4. The $\lim_{a \to 0} \frac{E_1 RT(av_1, \theta)}{a^k}$, which is equal to $E_1 RT'[0, v_1]$ by c) of Assumption 3.3, is not equal to θ_1, if v_1 is a basis for X_1 with the property that v_1 is in V_1, the neighborhood defined above.

Lemma 3.4. The topological degree of the mapping $a^k E_1 RT'[0, v_1]$, (where a is a complex variable) relative to an arbitrary sphere with center 0_1 is k at the point 0_1.

Proof: This is a known fact.

Theorem 3.1. There exists a number $d_o > 0$ and a number $r_d > 0$ which depends on $d = d_o$ such that if S is a sphere in X_1 of radius $d \leq d_o$ and center θ_1 and y is such that $\|y\| < r_d$, then the topological degree at 0_1 relative to S of the mapping

$$E_1 RSy + E_1 RT[x_1 + F(y, x_1), y]$$

which takes X_1 into X_1, is k where k is the integer described in c) of Assumption 3.3.

Proof: We prove first that at the point θ_1 and relative to a sufficiently small sphere in X_1 with center θ_1, the mapping

$$E_1 RT[x_1 + F(\theta, x_1), \theta] \tag{1}$$

has topological degree k. We write the element x in the form av_1 where v_1 is the basis for X_1 mentioned in Assumption 3.4 and a is a scalar. Then by Lemma 3.3, the mapping (1) takes the form

$$E_1 RT[av_1 + a^k G(a, v_1), \theta]$$

For convenience, we write the mapping in the form

$$E_1 RT[av_1 + a^k G(a, v_1)]$$

Now by Assumption 3.3 c), we have:

$$E_1 RT[av_1 + a^k G(a, v_1)] = a^k E_1 RT'[a, v_1 + a^{k-1} G(a, v_1)]$$

Also by Assumption 3.3 c), T' is a continuous function. Hence if $\varepsilon > 0$, then there exists δ_ε such that if

$$|a| < \delta_\varepsilon$$

$$\| x - v_1 \| < \delta_\varepsilon$$

then

$$\| E_1 RT'[a, x] - E_1 RT'[0, v_1] \| < \varepsilon$$

We take $\varepsilon = \| E_1 RT'[0, v_1] \| \neq 0$ be Assumption 3.4. $\overline{V}$ is a closed, bounded subset of the space $Z \quad X_1$, hence $\overline{V}$ is compact. Since $G(a, v_1)$ is a continuous function defined on $\overline{V}$, then $G(a, v_1)$ has a maximum $c \geq 0$ on $\overline{V}$. Now we require that a shall contained in a spherical neighborhood M_a' of $a = 0$ with the following properties:

1) The radius of M_a' is $< \delta_\varepsilon$, where $\varepsilon = \| E_1 RT[0, v_1] \|$

2) $M_a' \subset M_a$.

3) If $a \in M_a'$, then $|a|^{k-1} c < \delta_\varepsilon$ where $\varepsilon = \| E_1 RT[0, v_1] \|$. Then

(2) $$\| a^k E_1 RT'[a, v_1 + a^{k-1} G(a, v_1)] - a^k E_1 RT'[0, v_1] \|$$

$$< |a|^k \varepsilon = |a|^k \| E_1 RT[0, v_1] \| = \| a^k E_1 RT'[0, v_1$$

for all $a \in M_a'$ except $a = 0$. In particular, inequality (2) holds for v_1 fixed and any sphere $S_1 = [a / |a| = d_1]$ that is contained in M_a'. Hence by Rouché's Theorem, the topological orders of

$$a^k E_1 RT' [0, v_1]$$

and

$$E_1 RT[x_1 + F(\theta, x_1), \theta]$$

are the same relative to θ_1 and a sphere $S_2 = [av_1 / a \in S_1]$. The number d_0 in the statement of the theorem is $\| av_1 \|$ where $av_1 \in S_2$.

Now by applying Rouché's Theorem again, we show that at θ_1 and relative to a sufficiently small sphere in X_1 with center θ_1, and for sufficiently small fixed y, there exists a topological degree for the mapping:

$$E_1 RSy + E_1 RT[x_1 + F(y, x_1), y]$$

acting on X_1, and that this topological degree is the same as the topological degree of the mapping:

$$E_1 RT[x_1 + F(\theta, x_1), \theta]$$

This demonstration, together with Lemma 3.4, completes the proof of our theorem.

As before, we write $x = av_1$ and keep v_1 fixed to the value given in Assumption 3.4. Now since $E_1 RT[x_1 + F(\theta, x_1), \theta]$ has a degree at θ_1 and relative to any sufficiently small sphere in X_1 with center θ_1, then for all x_1 on a given such fixed sphere $S_1 \subset M_a' \cdot M_{v_1}$:

$$\| E_1 RT[x_1 + F(\theta, x_1), \theta] \| > B_1 > 0$$

where B_1 is a constant. Also we note that the function

$$E_1 RSy + E_1 RT[x_1 + F(y, x_1), y]$$

is uniformly continuous, simultaneously in the two variables x and y. This is because, first, E_1, R, S are linear continuous transformations. Secondly, the function $F(y, x_1)$ is uniformly continuous simultaneously in the two variables x_1 and y because $F(y, x_1)$ is obtained by applying Theorem 2.11 and hence by conclusion C_3) of Theorem 2.11, $F(y, x_1)$ is uniformly continuous, simultaneously in both its variables. Finally, the transformation T has the proper continuity conditions in the bounded region $\bar{U}_1$ which we consider, by Assumption 3.3 b). Hence if $\epsilon > 0$, then there exists δ_ϵ such that if

$$\| y \| < \delta_\epsilon$$

then

$$\| E_1 RSy + E_1 RT[x_1 + F(y, x_1), y] - E_1 RT[x_1 + F(\theta, x_1), \theta] \| < \epsilon$$

for all $x_1 \in M_{a'} \cdot M_{v_1}$. Now take $\epsilon = B_1$. Then for $av_1 \in S_1$ and y such that $\| y \| < \delta_{B_1}$,

$$\| E_1 RSy + E_1 RT[x_1 + F(y, x_1), y] - E_1 RT[x_1 + F(\theta, x_1), \theta] \|$$
$$< \epsilon = B_1 < \| E_1 RT[x_1 + F(\theta, x_1), \theta] \|$$

Applying Rouche's Theorem, we obtain the desired result where $r_d = \delta_{B_1}$.

qed.

Case II. The dimension of X_1, the null space of $(I - C)$, is > 1. For this case, we make the following additional assumptions concerning the transformation T :

Assumption 3.5. The transformation T is split into a term of order k ($k \geq 2$) and a term of higher order, i.e.,

$$T(x, y) = T_k(x, y) + T_{k+1}(x, y)$$

and

a) T_k is homogeneous of degree k, i.e., if m is an arbitrary integer

$$T_k\left(\sum_{i=1}^{m} a_i \omega_i, \theta\right) = \sum_{i,j=1}^{m} \sum_{\substack{p,q=1 \\ p+q=k}}^{k} a_i^p a_j^q T_{pq}[\omega_i, \omega_j]$$

where the transformation T_{pq} are continuous mappings from $X \times X$ into X. In this work, we make the additional hypothesis that if $i = j$

$$T_{pq}(\omega_i, \omega_j) = T_k(\omega_i, \theta)$$

for all p and q.

b) There exists $\lim_{\omega \to \theta} \frac{\|T_{k+1}(\omega, \theta)\|}{\|\omega\|^k} = 0$.

Assumption 3.6. The element θ_1 is an isolated θ_1 - point of the transformation $E_1 - RT_k[x, \theta]$ acting on $x \in X_1$. We may express this assumption also in the following way: the only θ_1 - point of X_1 under the mapping $E_1 - RT_k[x, \theta]$ is θ_1. This seemingly stronger condition is actually equivalent to the first mentioned condition because of condition a) of Assumption 3.5, i.e.,

$$T_k[ax, \theta] = a^k T_k[x, \theta]$$

Lemma 3.5. The mapping $E_1 - RT_k[x, \theta]$ acting on X_1 has a topological degree defined at the point θ_1 and relative to an arbitrary sphere in X_1 having θ_1 as its center.

Proof: The proof follows immediately from the definition of topological degree and Assumption 3.5.

Theorem 3.2. There exists a number $d_0 > 0$ and a number $\delta_1 > 0$ which depends

on $d_2 \leq d_0$, such that if S is a sphere in X_1 of radius $d_2 \leq d_0$ and center θ_1 and y is such that $\|y\| < \rho d_2$, then the topological degree at θ_1 relative to S of the mapping

$$E_1 RSy + E_1 RT[x_1 + F(y, x_1), y]$$

which takes X_1 into X_1 is equal to the topological degree at θ_1 of the mapping

$$E_1 RT_k [x, \theta]$$

acting on X_1, relative to the same sphere. (The mapping $E_1 RT_k [x, 0]$ is equal to $E_1 RT_k \left[\sum_{i=1}^{n} a_i x_i, \theta\right]$ if $x_1, \dots, x_n$ is a basis for X_1. From Assumption 3.5, it follows that the mapping is of the form

$$\sum_{i=1}^{n} a_i^k E_1 R T_k [x_i, \theta] + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \sum_{\substack{p,q=1 \\ p+q=k}}^{k} a_i^p a_j^q T_{p,q} [x_i, x_j]$$

where the a_i's are the variables.)

Proof: We prove first that for a sufficiently small sphere, the mapping

(1) $$E_1 RT[x_1 + F(\theta, x_1), \theta]$$

has the same topological degree as

$$E_1 RT_k [x, \theta]$$

For conveneince, we write the mapping (1) in the form

$$E_1 RT[x_1 + F(\theta, x_1)]$$

Also we write x_1 in the form av_1 where a, v_1 are in M_a, M_{v_1}, the neighborhoods defined immediately after the proof of Lemma 3.3. By Lemma 3.3, if we write $T_k [av_1, \theta]$ as $T_k [av_1]$.

$$\| E_1 RT_k[av_1] - E_1 RT[av_1 + F(0, av_1)] \| =$$

$$\| E_1 RT_k[av_1] - E_1 RT[av_1 + a^k G(a, v_1)] \|$$

But

$$\| E_1 RT_k[av_1] - E_1 RT[av_1 + a^k G(a, v_1)] \| =$$

$$|a|^k \| E_1 RT_k[v_1] - E_1 RT_k[v_1 + a^{k-1} G(a, v_1)]$$

$$- \frac{1}{a^k} E_1 RT_{k+1}[av_1 + a^k G(a, v_1)] \|$$

$$= |a|^k \| E_1 RT_k[v_1] - E_1 RT_k[v_1] - \sum_{\substack{p,q=1 \\ p+q=k}}^{k-1} (a^{k-1})^q E_1 RT_{p,q}[v_1, G(a, v_1)]$$

$$- (a^{k-1})^k E_1 RT_k[g(a, v_1) - \frac{1}{a^k} E_1 RT_{k+1}[av_1 + a^k G(a, v_1)] \|$$

by Assumption 3.5.

Therefore

$$(2) \quad \| E_1 RT_k[av_1] - E_1 RT[av_1 + a^k G(a, v_1)] \|$$

$$\leq |a|^k \left\{ \sum_{p,q} |a^{(k-1)(q)}| \| E_1 RT_{p,q}[v_1, G(a, v_1)] \| + |a^{k-1}|^k \| E_1 RT_k[G(a, v_1)] \| \right.$$

$$\left. + \frac{1}{|a|^k} \| E_1 RT_{k+1}[av_1 + a^k G(a, v_1)] \| \right\}$$

Since $\| E_1 RT_k[av_1] - E_1 RT[av_1 + a^k G(a, v_1)] \| =$

$$|a|^k \| E_1 RT_k[v_1] - E_1 RT'[a, v_1 + a^{k-1} G(a, v_1)] \|$$

inequality (2) becomes, when divided by $|a|^k$:

$$(3) \quad \| E_1 RT_k[v_1] - E_1 RT'[a, v_1 + a^{k-1} G(a, v_1)] \|$$

$$\leq \sum_{p,q} |a^{k-1}|^q \| E_1 RT_{pq}[v_1, G(a, v_1)] \| + |a^{k-1}|^k \| E_1 RT_k[G(a, v_1)]$$

$$+ \frac{1}{|a|^k} \| E_1 RT_{k+1}[av_1 + a^k G(a, v_1)] \|$$

$G(a, v_1)$ is by Lemma 3.3, a continuous function of a and v_1, and $E_1 RT_k$ is a continuous transformation. Hence if

$$\| E_1 RT_k [G(a,\theta)] \| \leq M_1 > 0$$

then there exist spherical neighborhoods $M_a^{(2)} \subset M_a$ of $a = 0$ and $M_{v_1}^{(2)} \subset M_{v_1}$ of $v_1 = \theta_1$ such that if $a \in M_a^{(2)}$, $v_1 \in M_{v_1}^{(2)}$, then

$$\| E_1 RT_k [G(a, v_1)] \| < 2M_1$$

Also $$\frac{1}{|a|^k} \| E_1 RT_{k+1} [av_1 + a^k G_1(a,v_1)] \|$$

$$= \| v_1 + a^{k-1} G_1(a,v_1) \|^k \left\{ \frac{\| E_1 RT_{k+1} [av_1 + a^k G_1(a,v_1)] \|}{|a|^k \| v_1 + a^{k-1} G_1(a,v_1) \|^k} \right\}$$

(4) $$= \| v_1 + a^{k-1} G_1(a,v_1) \|^k \left\{ \frac{\| E_1 RT_{k+1} [av_1 + a^k G_1(a,v_1)] \|}{\| av_1 + a^k G_1(a,v_1) \|^k} \right\}$$

Since $G(a, v_1)$ is a continuous function of a and v_1, then there exists spherical neighborhoods $M_a^{(3)} \subset M_a^{(2)}$ and $M_{v_1}^{(3)} \subset M_{v_1}^{(2)}$ of $a = 0$ and $v_1 = \theta_1$ respectively, such that if $v_1 \in M_{v_1}^{(3)}$ and $a \in M_a^{(3)}$, then

$$\| v_1 + a^{k-1} G(a, v_1) \| < 2r$$

where r is the radius of $M_{v_1}^{(3)}$.

Finally, we deal with the term

$$\sum_{p,q} |a^{k-1}|^q \| E_1 RT_{p,q} [v_1, G(a,v_1)] \|$$

By Assumption 3.5 b), each function $T_{p,q}$ is continuous in v_1 and $G(a, v_1)$. Hence if

$$\| E_1 RT_{p,q} [\theta_1, G(0, \theta_1)] \| \leq M_{p,q} > 0$$

then there exist spherical neighborhoods $M_a^{(4)} \subset M_a^{(3)}$, $M_{v_1}^{(4)} \subset M_{v_1}^{(3)}$ of $a = 0$ and $v_1 = \theta_1$ respectively such that if $a \in M_a^{(4)}$, $v_1 \in M_{v_1}^{(4)}$, then

$$\| E_1 RT_{p,q}[v_1, G(a, v_1)] \| < 2M_{p,q}$$

for each $T_{p,q}$.

Summarizing these results, we have: there exist spherical neighborhoods P_a , P_{v_1} of $a = 0$, $v_1 = \theta_1$ respectively such that if $a \in P_a$, $v_1 \in P_{v_1}$, then

$$\| E_1 RT_k[G(a, v_1)] \| < 2M_1 \tag{5}$$

$$\| v_1 + a^{k-1} G(a, v_1)] \| < 2_n \tag{6}$$

$$\| E_1 RT_{p,q}[v_1, G(a, v_1)] \| < 2M_{p,q} \tag{7}$$

where M_1 , r, $M_{p,q}$ are positive constants. Hence by inequality (3) and equation (4) ,

$$\| E_1 RT_k[v_1] - E_1 RT'[a, v_1 + a^{k-1} G(a, v_1)] \|$$

$$\leq \sum_{p,q} |a^{k-1}|^q M_{p,q} + |a^{k-1}|^k 2M_1 + (2_n)^k \left\{ \frac{\| E_1 RT_{k+1}[av_1 + a^k G(a,v_1)] \|}{\| av_1 + a^k G(a,v_1) \|^k} \right\}$$

Now consider the sphere S in X_1 :

$$\| v_1 \| = d$$

where $d < r$, the radius of P_{v_1} .

Since $S = [v_1 / \| v_1 \| = d]$ is compact, then

$$\| E_1 RT_k[v_1] \|$$

has a minimum on this set. But this minimum m is > 0 by Assumption 3.6.

Now there exists a spherical neighborhood $P_a^{(1)} \subset P_a$ such that if $a \in P_a^{(1)}$, then

$$\sum_{p,q} |a^{k-1}|^q M_{p,q} + |a^{k-1}|^k 2M_1 < \frac{m}{2}$$

Also, by Assumption 3.5 b),

$$\lim_{\omega \to \theta} \frac{\|T_{k+1}[\omega, \theta]\|}{\|\omega\|^k} = 0$$

Hence there exists $\delta > 0$ such that if

$$\| av_1 + a^k G(a, v_1) \| < \delta$$

then

$$\frac{\|E_1 R T_{k+1}[av_1 + a^k G_1(a, v_1)]\|}{\|av_1 + a^k G_1(a, v_1)\|^k} < \frac{m}{2(2n)^k}$$

There exists a spherical neighborhood $P_a^{(2)} \subset P_a^{(1)}$ of $a = 0$ such that if $a \in P_a^{(2)}$, then

$$|a|(2r) < \delta$$

and hence by inequality (6):

$$\| av_1 + a^k G(a, v_1) \| < \delta$$

for $a \in P_a^{(2)}$ and $v_1 \in P_{v_1}$.

Hence if $a \in P_a^{(2)}$ and $\|v_1\| = d$, then

(8)
$$\begin{aligned} &\| E_1 RT[v_1] - E_1 RT'[a, v_1 + a^{k-1} G(a, v_1)] \| \\ &\leq \sum_{p,q} |a^{k-1}|^q M_{p,q} + |a^{k-1}|^k 2M_1 + (2n)^k \left\{ \frac{\|E_1 R T_{k+1}[av_1 + a^k G_1(a, v_1)]\|}{\|av_1 + a^k G_1(a, v_1)\|^k} \right\} \\ &< \frac{m}{2} + (2n)^k \left[\frac{m}{2(2n)^k} \right] = m \leq \|E_1 R T_k[v_1]\| \end{aligned}$$

Multiplying inequality (8) by $|a^k|$, we obtain:

(9) $\| E_1 RT_k[av_1] - E_1 RT[av_1 + a^k G(a, v_1)] \| < \| E_1 RT_k[av_1] \|$

for $a \in P_a^{(2)}$ and $\| v_1 \| = d$.

In particular, inequality (9) holds for $\| v_1 \| = d$ and a such that $|a| = d_1$ where $d_1 < r$, the radius of $P_a^{(2)}$. Hence by Lemma 3.5, the topological degree relative to the sphere

$$S_1 = [av_1 / \| v_1 \| = d \ , \ |a| = d_1]$$

at the point θ_1 of the mapping $E_1 RT_k[av_1]$ is the same as the topological degree of

$$E_1 RT[av_1 + a^k G(a, v_1)]$$

The number d_o in the statement of the theorem is $(d)(d_\theta)$. The topological degree of $E_1 RT[av_1 + a^k G(a, v_1)]$ is the same as the topological degree of

$$E_1 RSy + E_1 RT[av_1 + F(y, av_1), y]$$

by the same proof as is used for the proof of the analogous statement in Theorem 3.1.

qed.

It will be noted that this discussion up to this point is quite general, i.e., the statements proved so far hold regardless of the dimension of X_1 . Hence if Assumption 3.6 holds, Theorem 3.2 would apply if the dimension of X_1 were 1 . If the dimension of X_1 were 1 , Assumption 3.6 could be replaced by the simpler assumption

$$E_1 RT_k[v_1, \theta] \neq \theta$$

where v_1 is any basis for x_1 .

Now we have the general problem of determining the topological degree at θ_1 of the mapping

$$E_1 \, RT_k[x, \theta]$$

of X_1 into X_1. This is a mapping from complex Euclidean n-space into complex Euclidean n-space. Let $x_1, x_2, \cdots x_n$ be a basis for X_1 such that

$$\| x_i \| = d_2 \qquad i = 1, 2, \cdots, n$$

where d_2 is the number defined in the proof of Theorem 3.2. Then each element x in X_1 is of the form

$$x = \sum_{i=1}^{n} a_i x_i$$

where a_i is a complex number $(i = 1, 2, \cdots, n)$.

Then by Assumption 3.5, we may write

$$E_1 RT_k[x, \theta] = E_1 RT_k\Big[\sum_{i=1}^{n} a_i x_i, \theta\Big]$$

$$= \sum_{i=1}^{n} a_i^k \, E_1 RT_k[x_i, \theta] + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \sum_{\substack{p,q=1 \\ p+q=k}}^{k} a_i^p a_j^q \, E_1 RT_{p,q}[x_i, x_j]$$

The terms $E_1 RT_k[x_i, \theta]$ and $T_{pq}[x_i, x_j]$ are fixed elements of X_1. The a_i's are complex variables. Now if P_λ is the projection of X_1 onto the subspace with basis x_λ $(\lambda = 1, 2, \cdots, n)$ then the mapping may be written in the form

$$(1-\lambda) \sum_{i=1}^{n} a_i^k \, P_\lambda E_1 RT_k[x_i, \theta] + \sum_{\substack{i,j=1 \\ i \neq j}}^{n} \sum_{\substack{p,q=1 \\ p+q=k}}^{k} a_i^p a_j^q \, P_\lambda E_1 RT_{pq}[x_i, x_j] = b_\lambda x_\lambda$$

where $\lambda = 1, 2, \cdots, n$. That is, the point $(a_1, a_2, \cdots, a_n)$ in the complex Eudlidean n-space is mapped into the point $(b_1, b_2, \cdots, b_n)$ in the same n-space. Thus the general problem is to determine the topological degree at $(0, 0, \cdots, 0)$ of the mapping given by the equations $(1 - \lambda)$, i.e., to

determine the topological degree of a mapping which is given by n homogeneous forms in n variables.

In this thesis, we shall not deal with the general problem. We shall consider only the special case when $k = 2$ and $n = 2$. In this case, the equations $(1 - \lambda)$ become

$$a_1^2 P_1 E_1 R T_2 [x_1, \theta] + a_1 a_2 P_1 E_1 R T_{1,1} [x_1, x_2] + a_2^2 P_1 E_1 R T_2 [x_2, \theta] = b_1 x_1$$

$$a_1^2 P_2 E_1 R T_2 [x_1, \theta] + a_1 a_2 P_2 E_1 R T_{1,1} [x_1, x_2] + a_2^2 P_2 E_1 R T_2 [x_2, \theta] = b_2 x_2$$

For convenience, we write this mapping in the form

$$a_1 x^2 + \mathcal{B}_1 xy + C_1 y^2 = x' \tag{1}$$

$$a_2 x^2 + \mathcal{B}_2 xy + C_2 y^2 = y' \tag{2}$$

where a_i, $\mathcal{B}_i$, C_i are complex constants for $i = 1, 2$ and x, y are complex variables. By applying the spectral theorem for finite-dimensional transformations (See P. Halmos, [4], p. 127) we may write the transformation given by equations (1) and (2) in the form

$$A_1 x^2 + B_1 y^2 = x' \tag{3}$$

$$A_2 x^2 + B_2 xy + C_2 y^2 = y' \tag{4}$$

where A_1, A_2, B_1, B_2, C_1, C_2 are complex constants, and x, y represent complex variables.

Since the topological degree of the mapping given by equations (1) and (2) is not changed by applying the spectral theorem in this manner[5], we investigate the topological degree of the mapping given by equations (3) and (4). We make from

[5] This fact follows from the Theorem: Any real quadratic form can be brought to the diagonal form by using a unitary matrix with positive determinant.

now on the further

Assumption 3.7. If $A_1 \neq 0$ and $C_1 \neq 0$, then the following condition does not hold: the expression $\dfrac{A_1 C_2 - A_2 C_1}{B_2 \sqrt{-C_1 A_1}}$ is real and

$$(5) \qquad -1 \leq \frac{A_1 C_2 - A_2 C_1}{B_2 \sqrt{-C_1 A_1}} \leq 1$$

We prove now

Theorem 3.3. If the null space of $(I - C)$ has dimension 2, if $k = 2$, and if Assumptions 3.1, 3.2, 3.3, 3.5, 3.6, 3.7 are satisfied, then the topological degree at θ_1 relative to the sphere S, described in the statement of Theorem 3.2, of the mapping

$$E_1 RSy + E_1 RT[x_1 + F(y, x_1), y]$$

is four if y is restricted in the manner described in Theorem 3.2.

Proof: We shall prove that the topological degree of the mapping given by equations (3) and (4) is four. The proof of the theorem then follows from Theorem 3.2 and the derivation of equations (3) and (4). We note first that by Lemma 3.5, the mapping given by equations (3) and (4) has a topological degree at θ_1. We determine this topological degree by showing that the mapping given by equations (3) and (4) is homotopic in $X_1 - \theta_1$ to a mapping whose topological degree at θ_1 is known. It follows from the Deformationssatz, p. 424, A-H, that since the mappings are homotopic, their topological degrees are the same.

We assume first that $A_1 \neq 0$ and $C_1 \neq 0$. We will show that the mapping given by equations (3) and (4) is homotopic in $X_1 - \theta_1$ to the mapping

(6) $$A_1x^2 + C_1y^2 = x'$$

(7) $$A_2x^2 + C_2y^2 = y'$$

This is done by showing that the mapping

(8) $$A_1x^2 + C_1y^2 = x'$$

(9) $$A_2x^2 + (1 - t)\, B_2xy + C_2y^2 = y' \qquad 0 \leq t \leq 1$$

is a homotopy in $X_1 - O_1$. To prove this fact, we need only show that the equations

(10) $$A_1x^2 + C_1y^2 = 0$$

(11) $$A_2x^2 + (1 - t)\, B_2xy + C_2y^2 = 0$$

have only the solution $x = y = 0$.

Since $A_1 \neq 0$,

(12) $$x = \pm y \sqrt{-\frac{C_1}{A_1}}$$

Substituting from (12) into (11), we obtain:

$$A_2 y^2 \left[-\frac{C_1}{A_1}\right] + (1-t)\, B_2 y^2 \left[\pm\sqrt{-\frac{C_1}{A_1}}\right] + C_2 y^2 = 0$$

or

$$y^2 \left\{ -\frac{A_2 C_1}{A_1} + (1-t)\, B_2 \left(\pm\sqrt{-\frac{C_1}{A_1}}\right) + C_2 \right\} = 0$$

This equation has a non-zero solution in y only if

$$-\frac{A_2 C_1}{A_1} + (1-t)\, B_2 \left(\pm\sqrt{-\frac{C_1}{A_1}}\right) + C_2 = 0$$

or

$$-A_2C_1 \pm (1-t)B_2\sqrt{-C_1A_1} + A_1C_2 = 0$$

or

$$t = \frac{-A_2C_1 \pm B_2\sqrt{-C_1A_1} + A_1C_2}{\pm B_2\sqrt{-C_1A_1}} = 1 + \frac{A_1C_2 - A_2C_1}{\pm B_2\sqrt{-C_1A_1}}$$

Since $0 \leq t \leq 1$, then the mapping given by equations (8) and (9) can take a point different from (0, 0) into (0, 0) only if

$$0 \leq 1 + \frac{A_1 C_2 - A_2 C_1}{\pm B_2 \sqrt{-C_1 A_1}} \leq 1$$

or

$$-1 \leq \frac{A_1 C_2 - A_2 C_1}{\pm B_2 \sqrt{-C_1 A_1}} \leq 0$$

or

$$-1 \leq \frac{A_1 C_2 - A_2 C_1}{B_2 \sqrt{-C_1 A_1}} \leq 1$$

But this is the condition excluded by Assumption 3.7. Hence equations (8) and (9) describe a mapping which is a homotopy in $X_1 - 0_1$.

8 Thus we have dealt with the case $A_1 \neq 0$, $C_1 \neq 0$. Now suppose $C_1 = 0$. Then if the mapping given by (8) and (9) takes a point (x,y) different from (0,0) into (0,0), it is clear that from equation (8) that $x = 0$. Substituting the point (0, y) into equation (9), and setting $y' = 0$, we obtain

$$C_2 y^2 = 0$$

Since $y \neq 0$, $C_2 = 0$. But an examination of the equations (3) and (4) shows that if $C_1 = C_2 = 0$, then the point (0, y), where y is arbitrary is taken into (0, 0). Then no topological degree is defined for the mapping given by equations (3) and (4). This is a contradiction to Lemma 3.5. Hence if $C_1 = 0$, the mapping given by equations (8) and (9) maps no point different from (0, 0) into (0, 0).

If $A_1 = 0$, we obtain by similar reasoning the fact that the mapping given by equations (8) and (9) maps no point different from (0, 0) into (0, 0).

Thus our problem is reduced to that of the determination of the topological degree of the mapping given by equations (6) and (7). We show now

that the mapping given by equations (6) and (7) has the same topological degree as the mapping

(13) $A_1x^2 = x'$

(14) $A_2x^2 + C_2y^2 = y'$

This will be done by showing that the mapping

(15) $A_1x^2 + (1 - t)C_1y^2 = x'$ $\qquad (0 \leq t \leq 1)$

(16) $A_2x^2 + C_2y^2 = y'$

is a homotopy in the space with (0, 0) deleted except for a special case which will be dealt with in another manner. We may assume that A_1, C_1, A_2, C_2 are non-zero. Otherwise the mapping given by equations (6) and (7) already has the form of the mapping given equations (13) and (14). We solve the equations

(17) $A_1x^2 + (1 - t)C_1y^2 = 0$

(18) $A_2x^2 + C_2y^2 = 0$

Before doing so, we multiply equation (17) then by -1 if necessary, obtaining

(17') $A_1'x^2 + (1 - t)C_1'y^2 = 0$

so that the product $A_2C_1' > 0$. Now we investigate the solutions of the pair of equations

(19) $A_1'x^2 + (1 - t)C_1'y^2 = 0$

(20) $A_2x^2 + C_2y^2 = 0$

Now since $A_1' \neq 0$, then solving equations (19), we obtain

$$x = \pm y \sqrt{\frac{(t-1)C_1'}{A_1'}}$$

Then sutstituting in equation (20), we obtain:

$$\left[\frac{A_2(t-1)C_1'}{A_1'} + C_2 \right] y^2 = 0$$

This equation can have non-zero solutions in y only if

$$t \frac{A_2C_1'}{A_1'} - \frac{A_2C_1'}{A_1'} + C_2 = 0$$

or $$t(A_2C_1') - A_2C_1' + A_1'C_2 = 0$$

$$t = \frac{A_2C_1' - A_1'C_2}{A_2C_1'} = 1 - \frac{A_1'C_2}{A_2C_1'}$$

Since $0 \leq t \leq 1$, this means that the inequality

$$0 \leq \frac{A_1'C_2}{A_2C_1'} \leq 1$$

must hold. Or, since $A_2C_1' > 0$,

(21) $$0 \leq A_1'C_2 \leq A_2C_1'$$

Since A_1, A_2, C_1, C_2 are non-zero, inequality (21) must be of the form

$$A_2C_1' \geqslant A_1'C_2 > 0$$

Suppose

(22) $A_2C_1' > A_1'C_2$. We show that then the mapping

(23) $$A_2x^2 + (1 - t)C_2y^2 = y' \qquad 0 \leq t \leq 1$$

is a homotopy in the space with $(0, 0)$ deleted. Actually we show that if condition (22) holds, then the equations

(25) $$A_1x^2 + C_1y^2 = 0$$

(26) $A_2x^2 + (1 - t)\, C_2y^2 = 0 \qquad 0 \leq t \leq 1$

have only the solution $(0, 0)$. We actually deal with the equation

(27) $A_1' x^2 + C_1' y^2 = 0$

(28) $A_2x^2 + (1 - t)C_2y^2 = 0$

where, as before, equation (27) is obtained from equation (25) by multiplying equation (25) through by -1 if necessary so that the product $A_2C_1' > 0$.

Now solving equation (28), we obtain:

$$x = \pm y \sqrt{\frac{(t-1)C_2}{A_2}}$$

Substituting in equation (27), we obtain

$$\left(t\,\frac{A_1' C_2}{A_2} - \frac{A_1' C_2}{A_2} + C_1'\right) y^2 = 0$$

This equation can have a non-zero solution in y only if

$$t\,\frac{A_1' C_2}{A_2} - \frac{A_1' C_2}{A_2} + C_1' = 0$$

or

$$t = 1 - \frac{A_2 C_1'}{A_1 C_2}$$

Since $0 \leq t \leq 1$, this means that the equations

$$A_1' x^2 + C_1' y^2 = 0$$

$$A_2x^2 + (1 - t)\, C_2y^2 = 0$$

can have a solution different from $(0, 0)$ only if

$$0 \leq \frac{A_2 C_1'}{A_1' C_2} \leq 1$$

Since $A_2C_1' > 0$, this means that $A_1' C_2$ must be positive if the condition is to be satisfied. Hence the condition may be written

$$0 \leq A_2 C_1' \leq A_1' C_2$$

But by (22), $A_2 C_1' > A_1' C_2$. Hence equations (25) and (26) represent a homotopy in $X_1 - (0, 0)$.

Now suppose

$$A_2 C_1' = A_1' C_2$$

or

(29) $\frac{A_2}{A_1'} = \frac{C_2}{C_1'} = k$, some constant which is non-zero.

We show that in this case, the mapping represented by equations (6) and (7) does not have a topological degree at $(0, 0)$, relative to an arbitrary sphere with center $(0, 0)$. We write the equations

(30) $$A_1' x^2 + C_1' y^2 = 0$$

(31) $$A_2 x^2 + C_2 y^2 = 0$$

Since by equation (29), $A_2 = kA_1'$, $C_2 = k C_1'$, then equations (30) and (31) are the same equation since $k \neq 0$. Solving equation (30), we obtain

$$x = \pm y \sqrt{-\frac{C_1'}{A_1'}}$$

If r is an arbitrary positive number, then it is clear that there exists $y \neq 0$ such that

$$\sqrt{\left| \pm y \sqrt{-\frac{C_1'}{A_1'}} \right|^2 + |y|^2} = r$$

Since the mapping given by equations (30) and (31) takes the point $\left(y \sqrt{-\frac{C_1'}{A_1'}}, y \right)$ into $(0, 0)$, then it is clear that this mapping does not have a topological degree. But we have shown previously that the mapping given by equations (6) and (7) does have a topological degree. Hence equation (29) leads to a contradiction.

Thus we have shown that the mapping given by equations (3) and (4) has

the same topological degree as a mapping of the form

(32) $A_1x^2 = x'$

(33) $A_2x^2 + c_2y^2 = y'$

or

(34) $A_1x^2 + c_1y^2 = x'$

(35) $A_2x^2 = y'$

Finally, we show that the mapping given by the equations (32) and (33) has the same topological degree as the mapping

(36) $A_1x^2 = x'$

(37) $c_2y^2 = y'$

and that the mapping given by equations (34) and (35) has the same topological degree as the mapping

(38) $c_1y^2 = x'$

(39) $A_2x^2 = y'$

We demonstrate the first of these statements by showing that the mapping

$$A_1x^2 = x'$$

$$A_2(1 - t)x^2 + c_2y^2 = y'$$

is a homotopy in $E^2 - (0, 0)$. We set up the equations

(40) $A_1x^2 = 0$

(41) $$A_2(1 - t)x^2 + c_2 y^2 = 0$$

Since $A_1 \neq 0$, then equation (40) has only the solution $x = 0$. Substituting $x = 0$ into equation (41), we obtain:

$$c_2 y^2 = 0$$

Since $C_2 \neq 0$, this equation has only the solution $y = 0$. Hence equations (40) and (41) have only the solution $x = 0$, $y = 0$.

A similar argument shows that the mapping

$$A_1(1 - t)x^2 + c_1 y^2 = x'$$

$$A_2 x^2 = y'$$

is also a homotopy in $E^2 - (0, 0)$.

The mapping given by equations (35) and (37) is the product of the two mappings:

$$A_1 x^2 = x' \qquad x' = x''$$

$$y = y' \qquad c_2 y'^2 = y''$$

Each of these two mappings has topological degree 2. Since the topological degree of the product of two mappings is equal to the product of the topological degrees of the mappings (See J. Schander, [16], p. 670, also the proof of Theorem A1 in the Appendix and Lemma 5.2 of Chapter 5 in this thesis), the topological degree of the mapping given by equations (26) and (37) is four.

The mapping given by equations (38) and (39) is the product of the mappings

$$x = y' \qquad x' = x'' \qquad c_1 x''^2 = x'''$$

$$y = x' \qquad A_2 y'^2 = y'' \qquad y'' = y'''$$

The first of the mappings has topological degree one. The other two have topological degree two. Hence the product mapping has topological degree four.

We have shown that, in general, the topological degree of the considered mapping is equal to that of the mapping given by equations (6) and (7). Numerical values for this topological degree have been obtained by assuming that A_1, A_2, c_1, c_2 were all non-zero. It has already been proved that if $A_1 = 0$, then $A_2 \neq 0$, and if $c_1 = 0$, then $c_2 \neq 0$. Since either A_1 or C_1 must be non-zero and since either A_2 or c_2 must be non-zero (otherwise a topological degree would not be defined for the mapping given by equations (6) and (7).), it follows by the same arguments that we have already used that the topological degree of the mapping given by equations (6) and (7) is always equal to the topological degree of either the mapping given by equations (36) and (37) or the mapping given by equations (38) and (39). Hence the topological degree is always four. This completes the proof of Theorem 3.3.

Now we indicate a little the relationship between the topological index and the number of solutions of the corresponding equation by proving

Theorem 3.4. If an e-point p has index $n > 1$ under mapping f, then in an arbitrary neighborhood N_1 of the point o, there exists a point q such that there are m distinct q-points, where $m \geq \frac{n}{2}$, in a neignborhood N_2 of p, which depends upon N_1.

Proof: We prove first that there exists a point q for which there are at

least two q-points. Let S be a sphere with p as its center such that the index of p can be computed by using S. Let V be the solid sphere bounded by S. By the known property of topological index, there is a spherical neighborhood of o, say W, such that if point r is contained in W, then the topological degree of f relative to the sphere V at the point r is n. Suppose there is a solid sphere V_1 with o as its center such that V_1 is contained in $W \cap N_1$ and such that each point of V_1 is an image under f of a single point of V of index n. Since f is continuous, $f^{-1}[\text{int}(V_1)]$ is open; clearly p is an element of $f^{-1}[\text{int}(V_1)]$. Let σ be an n-simplex containing p and such that σ is contained in $V \cap f^{-1}[\text{int}(V_1)]$.

Now we define the index of p by using the boundary of σ. Distinct points of σ go, under f, into distinct points of V_1. The simplex is a closed, compact set. Hence, since f is continuous, $f(\sigma)$ is a closed, compact set. Hence the mapping

$$f : \sigma \longrightarrow f(\sigma)$$

is a homeomorphism (A-H, p. 95, Satz III). Hence the index of o-point p is ± 1. (A-H, p. 474, Satz VI). This is a contradiction to the hypothesis. Hence in each neighborhood of o, there is at least one point q_1 which is not the image of just one point in V of index n. The point q_1 must be the image of two or more points $t_1', t_1^2, \ldots, t_1^\ell$ where $2 \leq \ell \leq n$.

Applying the same argument as before to a neighborhood of q_1 which is contained in N_1, and using disjoint neighborhoods $U_1, U_2, \cdots, U_\ell$ of $t_1', t_1^2, \ldots t_1^\ell$ respectively, we obtain a point q_2 which is the image of at least $(\ell+1)$ points in $\bigcup_{i=1}^{\ell} U_i$. We cannot obtain more than $(\ell+1)$ points because first the index of some of the points $t_1', t_1^2, \ldots, t_1^\ell$ may be one and, secondly, because our method does not guarantee that there

is a point q_2 which is at once the image of two distinct points in more than one of the neighborhoods U_i of the points in these neighborhoods have index greater than one. Repetition of this argument gives the result stated in the theorem.

All of the discussion as far in this section has been for the purpose of dealing with the equation

$$(I - C)x + Sy + T(x, y) = \theta \tag{3.1}$$

in the neighborhoods of $x = \theta$ and $y = \theta$. That is, we have considered equation (3.1) in the neighborhood of the initial solution $x = \theta$, $y = \theta$.

We suppose now that an initial solution $x = x_0$, $y = y_0$ is given for equation (3.1), i.e., it is assumed that

$$(I - C)x_0 + Sy_0 + T(x_0, y_0) = \theta \tag{1}$$

where neither x_0 nor y_0 is necessarily equal to θ. Subtracting equation (1) from equation (3.1), we obtain:

$$(I - C)(x - x_0) + (Sy - Sy_0) + [T(x, y) - T(x_0, y_0)] = \theta \tag{2}$$

or

$$(I - C)(x - x_0) + S(y - y_0 + y_0) - Sy_0 + [T(x - x_0 + x_0, y - y_0 + y_0) - T(x_0, y_0)] = \theta \tag{3}$$

If $x - x_0$, $y - y_0$ are denoted by ξ, η respectively, equation (3) becomes

$$(I - C)(\xi) + S(\eta + y_0) - Sy_0 + [T(\xi + x_0, \eta + y_0) - T(x_0, y_0)] = \theta \tag{4}$$

Now the expression

$$T(\xi + x_0, \eta + y_0) - T(x_0, y_0)$$

is a mapping of the point (ξ, η) into X . We denote this transformation by $\mathcal{T}(\xi, \eta)$. From this definition of $\mathcal{T}$ it is clear that

a) $\mathcal{T}(\theta, \theta) = \theta$

Now we make

Assumption 3.8. The transformation $\mathcal{T}$ has the properties

b) There exists a maximal integer $k \geq 2$ such that if a is a scalar, then

$$\mathcal{T}(a\xi, \theta) = a^k \mathcal{T}'(a, \xi)$$

where $\mathcal{T}'$ is a continuous mapping of a subspace of $Z \times X$ into X .

In terms of the original transformation T , this means that

$$T(a\xi + x_0, y_0) - T(x_0, y_0) = a^k \mathcal{T}'(a, \xi)$$

c) There exists a positive number M and neighborhoods N_2 of $\xi = \theta$ and N_3 of $\eta = \theta$ such that if ξ, ξ_1 are elements of N_2 and η, η_1 are elements of N_3 , then

$$\|\mathcal{T}(\xi, \eta) - \mathcal{T}(\xi_1, \eta_1)\| < M\{\|\xi\| + \|\xi_1\| + \|\eta\| + \|\eta_1\|\}\{\|\xi - \xi_1\| + \|\eta - \eta_1\|\}$$

In terms of the original transformation T , this means that

$$\begin{aligned} &\|T(\xi + x_0, \eta + y_0) - T(x_0, y_0) - T(\xi_1 + x_0, \eta_1 + y_0) + T(x_0, y_0)\| \\ &= \|T(\xi + x_0, \eta + y_0) - T(\xi_1 + x_0, \eta_1 + y_0)\| < M\{\|\xi\| + \|\xi_1\| + \|\eta\| + \|\eta_1\|\}\{\|\xi - \xi_1\| + \|\eta - \eta_1\|\} \\ &= M\{\|x - x_0\| + \|x_1 - x_0\| + \|y - y_0\| + \|y_1 - y_0\|\}\{\|x - x_1\| + \|y - y_1\|\} \end{aligned}$$

where x , x_1, y , y_1 are equal to $x_0 + \xi$, $x_0 + \xi_1$, $y_0 + \eta$, $y_0 + \eta_1$, respectively.

Equation (4) now takes the form

(5) $$(1 - c)\xi + S(\eta + y_0) - Sy_0 + \mathcal{T}(\xi, \eta) = \theta$$

Because of properties a), b), and c) above, equation (5) is the same as equation (3.1) except that the term

$$S(\eta + y_0) - Sy_0$$

appears instead of the term Sy . This difference is not significant in the work, however. Hence the previous work applies to equation (5).

Chapter 4: An Application of the Preceding Results

We show now that the work of E. Schmidt [17] is a special case of the discussion in Chapter 3. Also, by applying the results of Chapter 3, we obtain an extension of a special case of Schmidt's work.

In Schmidt's work, the following equation is considered:

$$(4.1) \qquad x(s) - \int_a^b C(s,t)\,x(t)\,dt + W_{01}\left(x^s y\right) + \sum_{\substack{m,n \\ m+n\geq 2}} W_{mn}\left(x^s y\right) = 0$$

in which

$$(4.1a) \qquad W_{mn}\left(x^s y\right) = \sum_{i=1}^{\lambda} P^{(i)}_{\substack{\alpha_0 \alpha_1 \cdots \alpha_{p(i)} \\ \beta_0 \beta_1 \cdots \beta_{p(i)}}}\left(x^s y\right)$$

where λ is an arbitrary finite integer and

$$P^{(i)}_{\substack{\alpha_0 \alpha_1 \cdots \alpha_{p(i)} \\ \beta_0 \beta_1 \cdots \beta_{p(i)}}}\left(x^s y\right) = [x(s)]^{\alpha_0}[y(s)]^{\beta_0} \int_a^b \cdots \int_a^b f_{m;n_i}(s,t_1,\ldots,t_{p(i)})\, [x(t_1)]^{\alpha_1}[y(t_1)]^{\beta_1} \cdots [x(t_{p(i)})]^{\alpha_{p(i)}}[y(t_{p(i)})]^{\beta_{p(i)}}\, dt_1 \cdots dt_{p(i)}$$

where $\sum_{j=0}^{p(i)} \alpha_j = m$, $\sum_{j=0}^{p(i)} \beta_j = n$, and $f_{m;n_i}(s,t_1,\ldots,t_{p(i)})$ is continuous in the variables $s, t_1, \cdots, t_{p(i)}$.

(4.1 b) $C(s,t)$ is a continuous function in the variables s and t.

(4.1 c) The hypothesis of regular convergence holds on the series $\sum_{m+n\geq 2} W_{mn}\left(x^s y\right)$ i.e., there exist positive numbers h, k such that if $\|x\| < h$, $\|y\| < k$, then

$$\sum_{m,n} |\tilde{W}|_{mn} \|x\|^m \|y\|^n < \infty$$

where $\|x\|$, $\|y\|$ are the norms of x, y, respectively, as elements of the Banach space of continuous functions on the closed interval (a, b), and $|\tilde{W}|_{mn}$ is the norm of the continuous function $|W_{mn}|\binom{s}{t}$, which is obtained by

setting x and y equal to the function which is identically one on the closed interval (a, b) and taking the absolute values of the coefficient functions $f_{m;n;}$ in W_{mn} .

We show that equation (4.1) is a special case of equation (3.1) if X is the Banach space of complex-valued continuous functions on the closed interval (a, b) and the functions $x(s)$, $y(s)$ correspond to x, y, respectively of equation (3.1).

First, it is known that the mapping from $x(t)$ to $\int_a^b C(s,t)\,x(t)\,dt$ is a linear, completely continuous mapping of the Banach space of continuous functions on the interval (a, b) into itself. Hence the expression

$$x(s) - \int_a^b C(s,t)\,x(t)\,dt$$

corresponds exactly to the expression

$$(I - C)x$$

in equation (3.1).

Secondly, $W_{01}\binom{s}{x\,y}$ is of the form

$$A(s)\,y(s) + \int_a^b G_1(s,t)\,y(t)\,dt \tag{B}$$

where $A(s)$ is a continuous function of s . Hence the mapping from $y(t)$ into $W_{01}\binom{s}{x\,y}$ is a linear, continuous mapping of X into itself. Thus the expression (B) corresponds to Sy in equation (3.1).

Finally we show that

$$\sum_{m+n\geq 2} W_{mn}\binom{s}{x\,y}$$

has the properties of the mapping T of equation (3.1). We show that Assumptions 3.3 and 3.5 are satisfied.

From the definition of $W_{mn}\binom{s}{x\,y}$, it is clear that if we set $x(t) \equiv 0$, $y(t) \equiv 0$, then $W_{mn}\binom{s}{x\,y} \equiv 0$ for all m, n . Hence

$$\sum_{m+n\geq 2} W_{mn}\binom{s}{0\,0} = 0$$

Thus condition a) of Assumption 3.3 is satisfied.

If a is any complex number, then if $m + n \geq 2$ if $n \neq 0$

$$W_{mn}\left(\overset{S}{ax}\ 0\right) = 0$$

If $n = 0$, then $m \geq 2$ and

$$W_{mn}\left(\overset{S}{ax}\ 0\right) = a^m W_{mn}\left(\overset{S}{x}\ 0\right)$$

Hence if $\|ax\| < h$, then

$$\sum_{m+n\geq 2} W_{mn}\left(\overset{S}{ax}\ 0\right) = \sum_{\substack{m\geq 0\\ n=0}} a^m W_{mn}\left(\overset{S}{x}\ 0\right) = a^2 \left\{\sum_{\substack{m\geq 2\\ n=0}} a^{m-2} W_{mn}\left(\overset{S}{x}\ 0\right)\right\}$$

It is clear that, if we deal with scalars a such that $|a| < 1$,

$$\sum_{\substack{m\geq 2\\ n=0}} a^{m-2} W_{mn}\left(\overset{S}{x}\ 0\right)$$

is a continuous mapping from a subspace of $Z \times X$ into X since each $W_{mn}\left(\overset{S}{x}\ 0\right)$ is a continuous mapping of X into itself and by the regular convergence, Thus condition c) of Assumption 3.3 is satisfied.

Now we show that condition b) of Assumption 3.3 is satisfied. In this case

$$\|T(x_1,y_1) - T(x,y)\| = \left\|\sum_{m+n\geq 2} W_{mn}\left(\overset{S}{x_1}\ y_1\right) - \sum_{m+n\geq 2} W_{mn}\left(\overset{S}{x}\ y\right)\right\|$$

$$= \left\|\sum_{m,n} \int F_{mn}[x_1]^m[y_1]^n dt - \sum_{m,n} \int F_{mn}[x]^m[y]^n dt\right\|$$

where we denote $W_{mn}\left(\overset{S}{x}\ y\right)$ symbolically by

$$\int F_{mn}[x]^m[y]^n dt$$

instead of writing the expression given in (4.1 a).

$$\|T(x_1,y_1) - T(x,y)\| = \left\|\sum_{m,n} \int F_{mn}[x+(x_1-x)]^m[y+(y_1-y)]^n dt - \sum_{m,n} \int F_{mn}[x]^m[y]^n dt\right\|$$

Denoting $(x_1 - x)$ by Δx, $(y_1 - y)$ by Δy, we obtain:

$$\|T(x_1,y_1)-T(x,y)\| = \|\sum_{m,n}\int F_{mn}[x+\Delta x]^m[y+\Delta y]^n dt - \sum_{m,n}\int F_{mn}[x]^m[y]^n dt\|$$

$$\leq \|\sum_{m,n}\int F_{mn}[x+\Delta x]^m[y+\Delta y]^n dt - \sum_{m,n}\int F_{mn}[x]^m[y+\Delta y]^n dt\|$$

$$+\|\sum_{m,n}\int F_{mn}[x]^m[y+\Delta y]^n dt - \sum_{m,n}\int F_{mn}[x]^m[y]^n dt\|$$

$$(1) \quad = \|\sum_{m,n}\int F_{mn}\left\{[x+\Delta x]^m-[x]^m\right\}[y+\Delta y]^n dt\|$$

$$+\|\sum_{m,n}\int F_{mn}[x]^m\left\{[y+\Delta y]^n-[y]^n\right\} dt\|$$

(The symbolism we use permits us to write the expression this way). The terms in the series can be rearranged in this manner because of the hypothesis of regular convergence.

By the hypothesis of regular convergence, there exist positive numbers h and k such that if

$$\|x+\Delta x\| < h \qquad \|x\| < h \qquad \text{and} \qquad \|y+\Delta y\| < k$$

then

$$\sum_{\substack{m,n\\ m\geq 2}}\left\{\max_s \int_a^b |F_{mn}(s,t)|\,dt\right\}\|x+\Delta x\|^m\,\|y+\Delta y\|^n < \infty$$

(where, for convenience, we let the symbol t denote the variables $t_1, t_2, \cdots, t_p$) and

$$\sum_{\substack{m,n\\ m\geq 2}}\left\{\max_s \int_a^b |F_{mn}(s,t)|\,dt\right\}\|x\|^m\,\|y+\Delta y\|^n < \infty$$

Hence by the properties of absolute convergence

$$(2) \quad \sum_{\substack{m,n\\ m\geq 2}}\left\{\max_s \int_a^b |F_{mn}(s,t)|\,dt\right\}\left\{\|x+\Delta x\|^m-\|x\|^m\right\}\left\{\|y+\Delta y\|^n\right\} < \infty$$

Inequality (2) holds if $\|y+\Delta y\| < k$ and if $\|\Delta x\| < h$, $\|x+\Delta x\| < h$.

Now we consider a pair of constant functions $x(s)$ and $x_1(s)$ with the following properties:

a) $x(s)$ and $x_1(s)$ are positive constantx c_1 and c_2 which are less than h and less than k.

b) The number $c_1 - c = \Delta c$ is a positive number.

Then inequality (2) becomes:

$$(3)\quad \sum_{\substack{m,n\\ m\geq 2}} \left\{ \max_s \int_a^b |F_{mn}(s,t)|\,dt \right\} \left\{ (c_1+\Delta c)^m - (c)^m \right\} \left\{ \|y+\Delta y\|^n \right\} < \infty$$

or

$$(4)\quad \sum_{\substack{m,n\\ m\geq 2}} \left\{ \max_s \int_a^b |F_{mn}(s,t)|\,dt \right\} \left\{ \Delta c\,[B_m(c,\Delta c)] \right\} \left\{ \|y+\Delta y\|^n \right\} < \infty$$

where

$$B_m(c,\Delta c) = \frac{(c+\Delta c)^m - c^m}{\Delta c}$$

$$= m\,c^{m-1} + \frac{m(m-1)}{2!}\,c^{m-2}(\Delta c) + \frac{m(m-1)(m-2)}{3!}\,c^{m-3}(\Delta c)^2 + \cdots + (\Delta c)^{m-1}$$

Suppose that $\Delta c = r_1 > 0$

and $c = r_2 > 0$

Let r = minimum (r_1, r_2). Now consider arbitrary continuous functions $x_1(s)$, $x(s)$ such that $\|x_1\| < \frac{1}{3}\eta$, $\|x\| < \frac{1}{3}\eta$. Then $\|x_1 - x\| < \frac{1}{3}\eta$. Hence

$$(5)\quad \left\| \sum_{\substack{m,n\\ m\geq 2}} \int F_{mn} \left\{ [x+\Delta x]^m - [x]^m \right\} [y+\Delta y]^n\,dt \right\|$$

$$\leq \sum_{\substack{m,n\\ m\geq 2}} \left\{ \max_s \int_a^b |F_{mn}(s,t)|\,dt \right\} \left\{ [\|x\| + \|\Delta x\|]^m - [\|x\|]^m \right\} \left\{ \|y+\Delta y\|^n \right\}$$

This follows from the fact that a typical term in the expression

$$\sum_{\substack{m,n\\ m\geq 2}} \int F_{mn} \left\{ [x+\Delta x]^m - [x]^m \right\} \left\{ y+\Delta y \right\}^n dt$$

is the term

$$P_{\substack{d_0 d_1 \cdots d_p \\ \beta_0 \beta_1 \cdots \beta_p}} \left(x+\Delta x \overset{s}{,} y+\Delta y \right) - P_{\substack{d_0 d_1 \cdots d_p \\ \beta_0 \beta_1 \cdots \beta_p}} \left(x \overset{s}{,} y+\Delta y \right)$$

where $\sum_{i=0}^{p} d_i = m$ and $\sum_{j=0}^{p} \beta_j = n$. By (4.1 a) this expression

is equal to:

$$[(x+\Delta x)(s)]^{\alpha_0}[(y+\Delta y)(s)]^{\beta_0}\int_a^b\cdots\int_a^b f_{mn}(s,t_1,\ldots,t_p)$$
$$[(x+\Delta x)(t_1)]^{\alpha_1}[(y+\Delta y)(t_1)]^{\beta_1}\cdots[(x+\Delta x)(t_p)]^{\alpha_p}[(y+\Delta y)(t_p)]^{\beta_p}$$
$$dt_1\cdots dt_p$$
$$-[x(s)]^{\alpha_0}[(y+\Delta y)(s)]^{\beta_0}\int_a^b\cdots\int_a^b f_{mn}(s,t_1,\ldots,t_p)[x(t_1)]^{\alpha_1}[(y+\Delta y)(t_1)]^{\beta_1}$$
$$\cdots[x(t_p)]^{\alpha_p}[(y+\Delta y)(t_p)]^{\beta_p}dt_1\cdots dt_p$$
$$=\int_a^b\cdots\int_a^b f_{mn}(s,t_1,\ldots,t_p)\Big\{[(x+\Delta x)(s)]^{\alpha_0}[(x+\Delta x)(t_1)]^{\alpha_1}\cdots[(x+\Delta x)(t_p)]^{\alpha_p}$$
$$-[x(s)]^{\alpha_0}[x(t_1)]^{\alpha_1}\cdots[x(t_p)]^{\alpha_p}\Big\}\Big\{[(y+\Delta y)(s)]^{\beta_0}[(y+\Delta y)(t_1)]^{\beta_1}$$
$$\cdots[(y+\Delta y)(t_p)]^{\beta_p}\Big\}dt_1\cdots dt_p$$
$$=\int_a^b\cdots\int_a^b f_{mn}(s,t_1,\ldots,t_p)\Big\{[x(s)+\Delta x(s)]^{\alpha_0}[x(t_1)+\Delta x(t_1)]^{\alpha_1}\cdots[x(t_p)+\Delta x(t_p)]^{\alpha_p}$$
$$-[x(s)]^{\alpha_0}[x(t_1)]^{\alpha_1}\cdots[x(t_p)]^{\alpha_p}\Big\}\Big\{[(y+\Delta y)(s)]^{\beta_0}\cdots[(y+\Delta y)(t_p)]^{\beta_p}\Big\}dt_1\cdots dt_p$$
$$=\int_a^b\cdots\int_a^b f_{mn}(s,t_1,\ldots,t_p)\Big\{[x(s)]^{\alpha_0}\cdots[x(t_p)]^{\alpha_p}+\sum_{n,\ell}A_{n,\ell}(x)^n(\Delta x)^\ell$$
$$-[x(s)]^{\alpha_0}[x(t_1)]^{\alpha_1}\cdots[x(t_p)]^{\alpha_p}\Big\}\Big\{[(y+\Delta y)(s)]^{\beta_0}\cdots[(y+\Delta y)(t_p)]^{\beta_p}\Big\}dt_1\cdots dt_p$$

where each term in $\sum_{n,\ell}A_{n\ell}(x)^n(\Delta x)^\ell$ has a factor $[\Delta x(t)]^q$ where t represents some one of the variables $s, t_1, t_2, \cdots t_p$ and q is an integer ≥ 1. Also

$$\Big|\sum_{n,\ell}A_{n\ell}(x)^n(\Delta x)^\ell\Big| \leq m\|x\|^{m-1}\|\Delta x\|+\frac{m(m-1)}{2}\|x\|^{m-2}\|\Delta x\|^2$$
$$+\cdots+\frac{m(m-1)\cdots 2}{(m-1)!}\|x\|\,\|\Delta x\|^{m-1}+\|\Delta x\|^m$$
$$=\Big\{\|x\|+\|\Delta x\|\Big\}^m-\|x\|^m$$

Hence

$$\int_a^b \cdots \int_a^b f_{mn}(s,t_1,\ldots,t_p)\left\{[[(x+\Delta x)](s)]^{\alpha_0}[(x+\Delta x)(t_1)]^{\alpha_1}\cdots[(x+\Delta x)(t_p)]^{\alpha_p}\right.$$
$$\left. - [x(s)]^{\alpha_0}[x(t_1)]^{\alpha_1}\cdots[x(t_p)]^{\alpha_p}\right\}\left\{[(y+\Delta y)(s)]^{\beta_0}\cdots[(y+\Delta y)(t_p)]^{\beta_p}\right\}dt_1\cdots dt_p$$
$$\leq \left\{\max_s \int_a^b \cdots \int_a^b |f_{mn}(s,t_1,\ldots,t_p)|\,dt_1\cdots dt_p\right\}\left\{\|x+\Delta x\|^m - \|x\|^m\right\}\left\{\|y+\Delta y\|^n\right\}$$

Summing over m and n we obtain the inequality (5) .

Now

$$\sum_{\substack{m,n\\ m\geq 2}} \left\{\max_s \int_a^b |F_{mn}(s,t)|\,dt\right\}\left\{[\|x\|+\|\Delta x\|]^m - [\|x\|]^m\right\}\left\{\|y+\Delta y\|^n\right\}$$
$$= \sum_{\substack{m,n\\ m\geq 2}} \left\{\max_s \int_a^b |F_{mn}(s,t)|\,dt\right\}\left\{(\|\Delta x\|\, B_m(\|x\|,\|\Delta x\|)\right\}\left\{\|y+\Delta y\|^n\right\}$$
$$= \|x\|\,\|\Delta x\| \sum_{\substack{m,n\\ m\geq 2}} \left\{\max_s \int_a^b |F_{mn}(s,t)|\,dt\right\}\left\{m\|x\|^{m-2}\right\}\left\{\|y+\Delta y\|^n\right\}$$
$$+ \|\Delta x\|^2 \sum_{\substack{m,n\\ m\geq 2}} \left\{\max_s \int_a^b |F_{mn}(s,t)|\,dt\right\}\left\{\frac{m(m-1)}{2!}\|x\|^{m-2}\right.$$
$$\left. + \frac{m(m-1)(m-2)}{3!}\|x\|^{m-3}\|\Delta x\| + \cdots + \|\Delta x\|^{m-3}\right\}\left\{\|y+\Delta y\|^n\right\}$$
$$< \|\Delta x\|\,\|x\| \sum_{\substack{m,n\\ m\geq 2}} \left\{\max_s \int_a^b |F_{mn}(s,t)|\,dt\right\}[m c^{m-2}]\left\{\|y+\Delta y\|^n\right\}$$
$$+ \|\Delta x\|^2 \sum_{\substack{m,n\\ m\geq 2}} \left\{\max_s \int_a^b |F_{mn}(s,t)|\,dt\right\}\left\{\frac{m(m-1)}{2!}c^{m-2} + \right.$$
$$\left. \frac{m(m-1)(m-2)}{3!}c^{m-3}\Delta c + \cdots + (\Delta c)^{m-2}\right\}\left[\|y+\Delta y\|^n\right]$$

The series

$$(6)\qquad \sum_{\substack{m,n\\ m\geq 2}} \left\{\max_s \int_a^b |F_{mn}(s,t)|\,dt\right\}\left\{m c^{m-2}\right\}\left\{\|y+\Delta y\|^n\right\} = M_1$$

and the series

$$(7)\qquad \sum_{\substack{m,n\\ m\geq 2}} \left\{\max_s \int_a^b |F_{mn}(s,t)|\,dt\right\}\left\{\frac{m(m-1)}{2!}c^{m-2} + \frac{m(m-1)(m-2)}{3!}c^{m-3}\Delta c + \right.$$
$$\left. \cdots + (\Delta c)^{m-2}\right\}\left\{\|y+\Delta y\|^n\right\} = M_2$$

converge because the original series

$$(8)\quad \sum_{\substack{m,n\\ m\geq 2}} \left\{ \max_s \int_a^b |F_{mn}(s,t)|\,dt \right\} \left\{ [c+\Delta c]^m - c^m \right\} \left\{ \|y+\Delta y\|^n \right\}$$

converges absolutely. Hence the subseries (5) and (6) also converge. Thus we conclude: if $x(s)$, $x_1(s)$ are such that $\|x_1\| < \frac{1}{3}\eta$, $\|x\| < \frac{1}{3}\eta$ $\|y\| < k$, $\|y_1\| < k$ then

$$(9)\quad \left\| \sum_{\substack{m,n\\ m\geq 2}} \int_a^b F_{mn} \left\{ [x+\Delta x]^m - [x]^m \right\} \left\{ y+\Delta y \right\}^n dt \right\| \leq \|\Delta x\| \|x\| M_1 + M_2 \|\Delta x\|^2$$

where M_1, M_2 are positive constants. Similarly, there exists η' such that if $\|y\| < \frac{1}{3}\eta'$, $\|y_1\| < \frac{1}{3}\eta'$ and $\|x\| < \frac{1}{3}\eta$, $\|x_1\| < \frac{1}{3}\eta$ then

$$\left\| \sum_{\substack{m,n\\ m\geq 2}} \int_a^b F_{mn} [x]^m \left\{ [y+\Delta y]^n - [y]^n \right\} dt \right\| < M_3 \|\Delta y\| \|y\| + M_4 \|\Delta y\|^2$$

where M_3, M_4 are positive constants.

Now also

$$\left\| \sum_{\substack{m,n\\ m=1\\ n\geq 2}} \int_a^b F_{mn}(s,t) \left\{ [x+\Delta x]^m - [x]^m \right\} \left\{ \|y+\Delta y\|^n \right\} dt \right\|$$

$$= \left\| \sum_{n\geq 2} \int_a^b F_{mn}(s,t) [\Delta x] [y+\Delta y]^n dt \right\|$$

$$\leq M_5 \|\Delta x\| \|y+\Delta y\| = M_5 \|\Delta x\| \|y_1\|$$

where M_5 is a positive constant, by the same type of argument as before.

Similarly,

$$\left\| \sum_{\substack{m,n\\ n=1\\ m\geq 2}} \int_a^b F_{mn}(s,t) \left\{ [x+\Delta x]^m \right\} \left\{ [y+\Delta y]^n - [y]^n \right\} dt \right\|$$

$$\leq M_6 \|\Delta y\| \|x+\Delta x\| = M_6 \|\Delta y\| \|x_1\|$$

where M_6 is a positive constant.

Finally,

$$\left\| \int_a^b F_{11}(s,t) [\Delta x] [y+\Delta y]\,dt \right\| + \left\| \int_a^b F_{11} [x] [\Delta y]\,dt \right\|$$

$$\leq M_7 \|\Delta x\| \|y_1\| + M_8 \|x\| \|\Delta y\|$$

where M_7, M_8 are positive constants.

Hence

$$\left\| \sum_{m,n} \int_a^b F_{mn} \left\{ [x+\Delta x]^m - [x]^m \right\} \left\{ [y+\Delta y]^n \right\} dt \right\| + \left\| \sum_{m,n} \int_a^b F_{mn} \left\{ [x]^m \right\} \left\{ [y+\Delta y]^n - [y]^n \right\} dt \right\|$$

$$\leq M_1 \|x\| \|\Delta x\| + M_2 \|\Delta x\|^2 + M_3 \|\Delta y\| \|y\| + M_4 \|\Delta y\|^2 + M_5 \|\Delta x\| \|y\| + M_6 \|\Delta y\| \|x_1\| + M_7 \|\Delta x\| \|y_1\| + M_8 \|x\| \|\Delta y\|$$

$$\leq \sum_{i=1}^{8} M_i \left[\|x_1\| + \|x\| + \|y_1\| + \|y\| \right] \left[\|x_1 - x\| + \|y_1 - y\| \right]$$

Combining this result with inequality (1) of this discussion, we have:

$$\| T(x_1, y_1) - T(x, y) \| \leq \sum_{i=1}^{8} M_i \left[\|x_1\| + \|x\| + \|y_1\| + \|y\| \right] \left[\|x_1 - x\| + \|y_1 - y\| \right]$$

which is the desired result.

Now we show that Assumption 3.5 is satisfied. We show that if the transformation $W_{20}(x^s y)$ is a non-zero transformation, then $k = 2$. In this case, $W_{20}(x^s y) + W_{11}(x^s y) + W_{02}(x^s y)$ corresponds to $T_2(x, y)$ and $\sum_{m+n \geq 3} W_{mn}(x^s y)$ corresponds to $T_3(x, y)$. Hence it is clear that parts a) and b) of Assumption 3.5 are satisfied.

Similarly, if $m = \ell$ is the minimal integer for which the transformation is non-zero, then $k = \ell$, and

$$T_k(x, y) = \sum_{m+n=\ell} W_{mn}(x^s y)$$

and

$$T_{k+1}(x, y) = T - T_k = \sum_{m+n \geq \ell+1} W_{mn}(x^s y) + \sum_{m+n < \ell} W_{mn}(x^s y)$$

Again, it is clear that Assumption 3.5 is satisfied.

It can be proved that Assumption 3.6 of Chapter 3 becomes, for the problem considered by Schmidt, exactly Schmidt's assumption: "$L_k \neq 0$". (See

E. Schmidt [17], § 9).

We obtain now an extension of a special case of Schmidt's work by considering the following functional equation:

$$x(s) + \int_a^b G(s,t)\left\{ F[x(t), y(t)]\right\} dt = 0 \tag{4.2}$$

If $G(s, t)$, $F(x, y)$ are given continuous functions of their arguments, we wish to solve equation (4.2) in the sense of determining the functions $x(s)$ which satisfy equation (4.2) if the function $y(t)$ is given.

We assume that equation (4.2) has an initial solution $x_0(t)$, $y_0(t)$, i.e., there exist functions $x_0(t)$, $y_0(t)$ continuous in the closed interval (a, b) such that

$$x_0(s) + \int_a^b G(s,t)\left\{ F[x_0(t), y_0(t)]\right\} dt = 0 \tag{10}$$

Subtracting equation (1) from equation (4.2), we obtain:

$$x(s) - x_0(s) + \int_a^b G(s,t)\left\{ F[x(t), y(t)] - F[x_0(t), y_0(t)]\right\} dt = 0 \tag{4.3}$$

(It should be noted that the derivation of equation (4.3) is a special case of the discussion of solutions "away from zero" at the end of Chapter 3.)

We obtain a result which is an extension of Schmidt's work in the following sense: Schmidt deals only with infinite series. Hence in order to apply Schmidt's work to the problem of solving equation (4.3), it is necessary to assume that $F(x, y)$ is an analytic function of the two variables x, y . We shall obtain a result which holds if the function $F(x, y)$ has just third or fourth derivatives. We introduce first the following constants. Let ε be a positive number and let A , B , C , D be the following numbers:

$$A = \min_{a \le t \le b} x_0(t) - \varepsilon \qquad B = \max_{a \le t \le b} x_0(t) + \varepsilon$$

$$C = \min_{a \le t \le b} y_0(t) - \varepsilon \qquad D = \max_{a \le t \le b} y_0(t) + \varepsilon$$

Theorem 4.1. If the third derivatives of the function $F(x, y)$ exist and have Lipschitz conditions on them for

$$A \le x \le B$$

$$C \le y \le D$$

and if the number $F_{xx}(x_0, y_0)$ is non-zero, then equation (4.3) satisfies Assumptions 3.1, 3.2, 3.3 and 3.5 of Chapter 3. Hence the discussion of Case II in Chapter 3 applies to equation (4.3) and consequently to equation (4.2).

Proof: By hypotheses,

$$F(x,y) - F(x_0,y_0) = \Delta x\, F_x(x_0,y_0) + \Delta y\, F_y(x_0,y_0) + \frac{(\Delta x)^2}{2!} F_{xx}(x_0,y_0) + (\Delta x)(\Delta y) F_{xy}(x_0,y_0) + \frac{(\Delta y)^2}{2!} F_{yy}(x_0,y_0) + R_3$$

where R_3 is the remainder in the Taylor's expansion and $\Delta x = x - x_0$, $\Delta y = y - y_0$. R_3 may be written in the form

$$R_3 = \frac{1}{2!} \int_0^1 [(\Delta x)^3 F_{xxx}(x_0+\tau\Delta x, y_0+\tau\Delta y) + 3(\Delta x)^2 F_{xxy}(x_0+\tau\Delta x, y_0+\tau\Delta y) + 3(\Delta x)(\Delta y)^2 F_{xyy}(x_0+\tau\Delta x, y_0+\tau\Delta y) + (\Delta y)^3 F_{yyy}(x_0+\tau\Delta x, y_0+\tau\Delta y)](1-\tau)\,d\tau$$

Hence, equation (4.3) takes the form

$$\Delta x + \int_a^b G(s,t) \left\{ \Delta x\, F_x(x_0,y_0) + \Delta y F_y(x_0,y_0) + \frac{(\Delta x)^2}{2!} F_{xx}(x_0,y_0) + (\Delta x)(\Delta y) F_{xy}(x_0,y_0) + \frac{(\Delta y)^2}{2!} F_{yy}(x_0,y_0) + R_3 \right\} dt = 0$$

By hypothesis, the number $F_{xx}(x_0, y_0)$ is non-zero. Hence it is clear that Assumptions 3.1, 3.2, 3.5 and parts a) and c) of Assumption 3.3 are satisfied. We must prove that part b) of Assumption 3.3 is satisfied.

In this case, Δx and Δy are the elements x, y described in Chapter 3, and

$$T(\Delta x, \Delta y) = \int_a^b G(s,t) \left\{ \frac{(\Delta x)^2}{2!} F_{xx}(x_0,y_0) + (\Delta x)(\Delta y) F_{xy}(x_0,y_0) + \frac{(\Delta y)^2}{2!} F_{yy}(x_0,y_0) + R_3 \right\} dt$$

$$T_2(\Delta x, \Delta y) = \int_a^b G(s,t) \left\{ \frac{(\Delta x)^2}{2!} F_{xx}(x_0,y_0) + (\Delta x)(\Delta y) F_{xy}(x_0,y_0) + \frac{(\Delta y)^2}{2!} F_{yy}(x_0,y_0) \right\} dt$$

$$T_3(\Delta x, \Delta y) = \int_a^b G(s,t) R_3 \, dt$$

We show first that $T_3(\Delta x, \Delta y)$ satisfies part b) of Assumption 3.3. For convenience, we introduce the notation $\xi = \Delta x$, $\eta = \Delta y$. Then

$$T_3(\xi,\eta) = \int_a^b \frac{G(s,t)}{2!} \left\{ \int_0^1 \xi^3 F_{xxx}(x_0+\tau\xi, y_0+\tau\eta) + 3\xi^2\eta F_{xxy}(x_0+\tau\xi, y_0+\tau\eta) + 3\xi\eta^2 F_{xyy}(x_0+\tau\xi, y_0+\tau\eta) + \eta^3 F_{yyy}(x_0+\tau\xi, y_0+\tau\eta) \right] (1-\tau) d\tau \Big\} dt$$

$$T_3(\xi_1,\eta_1) = \int_a^b \frac{G(s,t)}{2!} \left\{ \int_0^1 [\xi_1^3 F_{xxx}(x_0+\tau\xi_1, y_0+\tau\eta_1) + 3\xi_1^2\eta_1 F_{xxy}(x_0+\tau\xi_1, y_0+\tau\eta_1) + 3\xi_1\eta_1^2 F_{xyy}(x_0+\tau\xi_1, y_0+\tau\eta_1) + \eta_1^3 F_{yyy}(x_0+\tau\xi_1, y_0+\tau\eta_1)] (1-\tau) d\tau \right\} dt$$

We show that the differences of the corresponding terms in $T_3(\xi,\eta)$ and $T_3(\xi_1,\eta_1)$ all satisfy part b) of Assumption 3.3.

$$\int_a^b \frac{G(s,t)}{2!} \left\{ \int_0^1 [\xi^3 F_{xxx}(x_0+\tau\xi, y_0+\tau\eta)][1-\tau] d\tau \right\} dt - \int_a^b \frac{G(s,t)}{2!} \left\{ \int_0^1 [\xi_1^3 F_{xxx}(x_0+\tau\xi_1, y_0+\tau\eta_1)][1-\tau] d\tau \right\} dt$$

$$= \int_a^b \frac{G(s,t)}{2!} \left\{ \int_0^1 [\xi^3 - \xi_1^3][F_{xxx}(x_0+\tau\xi, y+\tau\eta)][1-\tau] d\tau \right\} dt + \int_a^b \frac{G(s,t)}{2!} \left\{ \int_0^1 \xi_1^3 [F_{xxx}(x_0+\tau\xi, y_0+\tau\eta) - F_{xxx}(x_0+\tau\xi_1, y_0+\tau\eta_1)] [1-\tau] d\tau \right\} dt$$

But $\int_a^b \frac{G(s,t)}{2!} \left\{ \int_0^1 \xi_1^3 \left[F_{xxx}(x_0+\tau\xi, y_0+\tau\eta) - F_{xxx}(x_0+\tau\xi_1, y_0+\tau\eta_1) \right][1-\tau]\,d\tau \right\} dt$

$\leq \int_a^b \frac{G(s,t)}{2!} \|\xi_1\|^3 \left\{ \int_0^1 \tau K \left[\|\xi-\xi_1\| + \|\eta-\eta_1\| \right][1-\tau]\,d\tau \right\} dt$

(where K is the Lipschitz constant for F_{xxx}, given by hypotheses).

$\leq \int_a^b \frac{G(s,t)}{2!} \|\xi_1\|^3 K \left[\|\xi-\xi_1\| + \|\eta-\eta_1\| \right] dt$

$\leq \left\{ \max_{a\leq s\leq b} \left[\int_a^b \frac{G(s,t)}{2!}\,dt \right] \right\} \|\xi_1\|^3 K \left[\|\xi-\xi_1\| + \|\eta-\eta_1\| \right]$

Let $M_1 = \left[\max_{a\leq s\leq b} \int_a^b \frac{G(s,t)}{2!}\,dt \right]$. Then we obtain:

$\leq M_1 h^2 K \left[\|\xi-\xi_1\| + \|\eta-\eta_1\| \right] \|\xi_1\|$

$\leq M_1 h^2 K \left\{ \|\xi_1\| + \|\xi\| \right\} \left\{ \|\xi-\xi_1\| + \|\eta-\eta_1\| \right\}$

$< M_1 h^2 K \left\{ \|\xi_1\| + \|\xi\| + \|\eta_1\| + \|\eta\| \right\} \left\{ \|\xi-\xi_1\| + \|\eta-\eta_1\| \right\}$

Also

$\int_a^b \frac{G(s,t)}{2!} \left\{ \int_0^1 \left[\xi^3 - \xi_1^3 \right] \left[F_{xxx}(x_0+\tau\xi, y_0+\tau\eta) \right][1-\tau]\,d\tau \right\} dt$

$\leq M_1 \left\{ \max_{a\leq t\leq b} \left| \int_0^1 \left[F_{xxx}(x_0+\tau\xi, y_0+\tau\eta) \right][1-\tau]\,d\tau \right| \right\} \|\xi^3 - \xi_1^3\|$

Let $M_2 = \max_{a\leq t\leq b} \left| \int_0^1 \left[F_{xxx}(x_0+\tau\xi, y_0+\tau\eta) \right][1-\tau]\,d\tau \right|$. Then the inequality becomes:

$\leq M_1 M_2 \|\xi^3 - \xi_1^3\|$

$\leq M_1 M_2 \|\xi-\xi_1\| \|\xi^2 + \xi_1\xi + \xi_1^2\|$

$\leq M_1 M_2 \|\xi-\xi_1\| \left\{ \|\xi_1+\xi\| \|\xi_1\| + \|\xi^2\| \right\}$

$\leq M_1 M_2 \|\xi-\xi_1\| \left\{ h \|\xi_1+\xi\| + h \|\xi\| \right\}$

$\leq 2 M_1 M_2 h \|\xi-\xi_1\| \left\{ \|\xi_1\| + \|\xi\| \right\}$

Now $$\int_a^b \frac{G(s,t)}{2!}\left\{\int_0^1 3\left[\xi^2\eta\, F_{xxy}(x_0+\tau\xi, y_0+\tau\eta) - \xi_1^2\eta_1 F_{xxy}(x_0+\tau\xi_1, y_0+\tau\eta_1)\right][1-\tau]\,d\tau\right\}dt$$

$$= \int_a^b \frac{3}{2!} G(s,t)\left\{\int_0^1 (\xi^2\eta - \xi_1^2\eta_1)[F_{xxy}(x_0+\tau\xi, y_0+\tau\eta)][1-\tau]\,d\tau\right\}dt$$

$$+ \int_a^b \frac{3}{2!} G(s,t)\left\{\int_0^1 \xi_1^2\eta_1[F_{xxy}(x_0+\tau\xi, y_0+\tau\eta) - F_{xxy}(x_0+\tau\xi_1, y_0+\tau\eta_1)]\right.$$
$$\left.[1-\tau]\,d\tau\right\}dt$$

The second term is dealt with in the same manner as in the preceding proof, i.e., the Lipschitz condition on F_{xxy} is utilized. Hence we consider the term:

$$\int_a^b \frac{3}{2} G(s,t)\left\{\int_0^1 (\xi^2\eta - \xi_1^2\eta_1)[F_{xxy}(x_0+\tau\xi, y_0+\tau\eta)][1-\tau]\,d\tau\right\}dt$$

$$\leq M_1\left\{\max_{a\leq t\leq b}\left|\int_0^1 [F_{xxy}(x_0+\tau\xi, y_0+\tau\eta)][1-\tau]\right|\right\}\|\xi^2\eta - \xi_1^2\eta_1\|$$

Let $M_3 = \max_{a\leq t\leq b}\left|\int_0^1 [F_{xxy}(x_0+\tau\xi, y_0+\tau\eta)][1-\tau]\,d\tau\right|$. Then the inequality becomes:

$$\leq \frac{3}{2} M_1 M_3 \|\xi^2\eta - \xi_1^2\eta_1\| \leq \frac{3}{2} M_1 M_3\left[\|\xi^2\eta - \xi_1^2\eta\| + \|\xi_1^2\eta - \xi_1^2\eta_1\|\right]$$

$$\leq \frac{3}{2} M_1 M_3\left[\|\eta\|\,\|\xi^2 - \xi_1^2\| + \|\xi_1^2\|\,\|\eta - \eta_1\|\right]$$

$$\leq \frac{3}{2} M_1 M_3 D\left[\|\xi - \xi_1\|\,\|\xi + \xi_1\| + \|\xi_1\|\,\|\eta - \eta_1\|\right]$$

where $D = \max(h, k)$

$$\leq \frac{3}{2} M_1 M_3 D\left\{\|\xi\| + \|\xi_1\|\right\}\left\{\|\xi - \xi_1\| + \|\eta - \eta_1\|\right\}$$

The analogous demonstration for the remaining two terms in

$$T_3(\xi,\eta) - T_3(\xi_1,\eta_1)$$

are carried out in the same manner as for the first two terms except that the roles of ξ and η are interchanged in the proofs.

The proof that T_2 satisfies part b) of Assumption 3.3 if of the same

type except that it is shorter.

This completes the proof of Theorem 4.1.

Theorem 4.2. If the second derivatives of the function $F(x,y)$ exist and have Lipschitz conditions on them, and if the number $F_x(x_0, y_0)$ is not equal to zero, then equation (4.3) satisfies Assumptions 3.1, 3.2, 3.3. If, in addition, the null space of the transformation

$$\Delta x + \int_a^b G(s,t) F_x(x_0,y_0) \Delta x \, dt$$

acting on Δx, has dimension one, then the discussion of Case 1 in Chapter 3 applies to equation (4.3) and consequently to equation (4.2).

Proof: It is proved that Assumptions 3.1, 3.2, and 3.3 are satisfied by using arguments similar to those used in the proof of Theorem 4.1.

It may be remarked here that equation (4.3) is a generalization of the integral equation treated by R. Iglisch [7] (See § 3, p. 107 of [7], equation (31)).

Chapter 5: An Extension of the Leray-Schander Definition of Degree of a Mapping in Banach Space

In this section, we shall define a multiplicity for certain mappings in Banach space of the form

$$I - C + T$$

where I is the identity transformation, C is linear, completely continuous, and T is a kth order transformation $(k \geq 2)$. It will be shown that if the Leray-Schander topological degree (See Leray and Schander [11]) is defined for transformation $I - C + T$, then the Leray-Schander topological degree and the multiplicity given by our definition are equal or differ by a factor of (-1). Thus we obtain an extension of the Leray-Schander definition.

We consider the equation in Banach space X

$$(I - C)x + T(x) = y \tag{5.1}$$

This is a special case of equation (3.1), i.e., I is the identity transformation, C is linear, completely continuous, the transformation T satisfies Assumption 3.3 and is, in this case, independent of y, and

$$Sy = -y$$

Now the theory of Chapter 3 may be applied to equation (5.1). In particular we recall that if y_0 is an element of a sufficiently small neighborhood of $y = \theta$, then the problem of determining the elements x which are in a sufficiently small neighborhood of $x = \theta$ and which satisfy equation (5.1) when

$y = y_0$ is reduced to the problem of solving an equation in Euclidean n-space which, in this case, has the form:

$$E_1 RT[x_1 + F(x_1, y_0)] - E Ry_0 = \theta \tag{5.2}$$

(Equation (5.2) is written in the notation of Chapter 3. Cf. equation (3.7) in Chapter 3.)

We approach thus problem by studying the topological degree of the mapping given by the left side of equation (5.2) when y_0 is fixed.

More generally, if (x_2, y_2) is an initial solution of equation (5.1), and if T satisfies Assumption 3.8, the problem of obtaining solutions x for equation (5.1) when y is in a sufficiently small neighborhood of y_2 is reduced to a study of the topological degree at θ_1 of the mapping in X_1

$$E_1 R\mathcal{T}[\xi_1 + F(\eta, \xi)] - E_1 R\eta \tag{5.3}$$

where $\xi = x - x_2$ and $\eta = y - y_2$ and $\mathcal{T}(\xi) = T(x_2 + \xi) - T(x_2)$.

This suggests

Definition 5.1. The multiplicity of the mapping $I - C + T$ relative to y sufficiently close to y_2 and relative to a sufficiently small sphere S with center x_2 is the topological degree of the mapping (5.3) at θ_1 and relative to a sphere in X_1 with center θ_1 and radius equal to the radius of S.

We remark here that the multiplicity (Vielfachheit) considered by R. Iglisch [8] is a special case of the multiplicity we consider. Iglisch's definition is based on a consideration of an integral equation of the type that Schmidt deals with. Our definition is based on a consideration of the generalized equation in Banach space.

The multiplicity given by this definition has some of the properties of a topological degree. First, if the topological degree at y of a mapping H, relative to a sphere S, is different from zero, then there exists an element x of S such that

$$H(x) = y$$

From the corresponding statement for the topological degree of a mapping from Euclidean n-space into Euclidean n-space, and the definition of the mapping (5.3), it follows that this condition holds for the multiplicity we have defined.

Secondly, the topological degree is the same relative to all y in a sufficiently small neighborhood of y_0. If the transformation T satisfies the conditions of Assumption 3.4 or of Assumptions 3.5 and 3.6, then this property holds. (See Theorems 3.1 and 3.2 of Chapter 3).

We wish to investigate the relationship between the multiplicity defined here and the topological degree of a mapping in Banach space as defined by Leray and Schander (See Leray and Schander [11]). First, we justify Definition 5.1 by exhibiting a transformation to which our definition of multiplicity applies and for which the work of Leray and Schander does not hold. The Leray-Schander topological degree is defined for mappings of the form $I + D$ where I is the identity transformation and D is completely continuous but not necessarily linear. Hence, we must exhibit a transformation $I - C + T$ such that T is not completely continuous but for which our definition of multiplicity holds.

We consider the Banach space Y of complex-valued continuous functions defined on the interval $(0, 1)$ where, if $f(t) \in Y$, then

$$\|f(t)\| = \max_{0 \le t \le 1} |f(t)|$$

The transformation C is the finite dimensional (hence completely continuous) mapping which takes $f(t)$ into the function

$$g(t) = [f(1)]t$$

C is clearly linear and continuous. Also C maps Y into a one-dimensional linear subspace L of Y. From the definition of C, it follows that the null space X_1 of $(I - C)$ is L, hence is one-dimensional. The transformation T is defined as:

$$T : f(t) \longrightarrow [f(t)]^2$$

T is continuous. $T(\theta) = \theta$. Also the transformation T is not completely continuous. Since

$$T[a\, f(t)] = a^2 [f(t)]^2 = a^2\, T[f(t)] ,$$

$T'[a, f(t)]$ is simply $T[f(t)]$. Finally:

$$\| T(f_1) - T(f_2) \| = \| f_1^2 - f_2^2 \| = \max_{0 \le t \le 1} | [f_1(t)]^2 - [f_2(t)]^2 |$$

$$\le \left\{ \max_{0 \le t \le 1} | f_1(t) - f_2(t) | \right\} \left\{ \max_{0 \le t \le 1} | f_1(t) + f_2(t) | \right\}$$

$$= \| f_1 - f_2 \| \, \| f_1 + f_2 \| \le \left\{ \| f_1 - f_2 \| \right\} \left\{ \| f_1 \| + \| f_2 \| \right\}$$

Thus the transformation $I - C + T$ satisfies the conditions of Assumptions 3.1, 3.2, and 3.3. Moreover, in this case, Assumption 3.4 is satisfied because

$$E_1' R T' [0, v_1] = E_1' R[q]$$

where g is the function $g(t) = t^2$. This is from the definition of T'

in this case and because $f_0(t) = t$ is a basis for the subspace L. We show that $E_1' R[q(t)] \neq \theta$. This proves that Assumption 3.4 is satisfied. We show first that the index of $I - C$ is 1. This is proved by showing that $(I - C)^n = (I - C)$ when $n > 1$.

$$(I-C)^2 [f(t)] = (I - 2C + C^2)[f(t)] = f(t) - 2f(1)t + C[f(1)t]$$
$$= f(t) - 2f(1)t + f(1)t$$
$$= f(t) - f(1)t$$
$$= (I-C)[f(t)]$$

The proof is completed by induction.

Thus, in this case, $X_m = L$. Now $g(t) = t^2$ is not an element of $L = X_m$. Hence $E_1' g(t) \neq \theta$. Hence, since by Theorem 2.9, X_m, X^m are invariant subspaces of R, and R is one-to-one,

$$R E_1' g(t) \neq \theta$$

and $R E_1' g(t)$ is an element of X^m. Hence

$$E_1' R g(t) \neq \theta$$

We prove now that if the Leray-Schander topological degree is defined for a transformation $(I - C + T)$, then the Leray-Schander degree and the multiplicity of $(I - C + T)$ as given by Definition 5.1 are equal or differ by a factor of (-1). For our proof, we require two lemmas concerning the Leray-Schander topological degree.

Lemma 5.1. Let $\Phi = I + F$ and $\Psi = I + G$ be two mappings of subsets of Banach space X into X such that

a) I is the identity mapping on X; F, G are completely continuous mappings of subsets of X into X.

b) ω is a bounded open set with boundary ω' such that F is defined on

$\bar{\omega} = \omega \cup \omega'$ and G is defined on the closure S of a bounded open set containing $(I - F)(\bar{\omega})$.

We denote by d a bounded component of $X - \Phi(\omega')$. If the point p is not in $\Psi\Phi(\omega')$ and is not in $\Psi[\text{Boundary } \Phi(\omega)]$ and if $[\Psi\Phi, \omega, p]$ is defined, then only a finite number of the terms $[\Phi, \omega, d] \cdot [\Psi, d, p]$ are non-zero and:

$$[\Psi\Phi, \omega, p] = \sum_{(d)} [\Phi, \omega, d][\Psi, d, p]$$

where $\sum_{(d)}$ means the summation over all the bounded components d.

$[\Psi\Phi, \omega, p]$ is the Leray-Schander topological degree at the point p of the mapping $\Psi\Phi$, relative to ω and $[\Phi, \omega, d], [\Psi, d, p]$ have similar meanings.

Proof: This lemma is known. See Leray [10]. (For a detailed proof of the lemma, see Theorem A1 of the appendix to this thesis.)

In order to apply Lemma 5.1, it is necessary to consider a domain such that there is just one non-zero term in

$$\sum_{(d)} [\Phi, \omega, d][\Psi, d, c]$$

Lemma 5.2. If $\Phi(\theta) = \theta$ and θ is an isolated θ-point of Ψ, i.e., there exists a neighborhood $U(\theta)$ of θ such that if x is an element of U and $\Psi(x) = \theta$, then $x = \theta$; then there exists a bounded connected open set ω containing θ and such that there is at most one non-zero term in

$$\sum_{(d)} [\Phi, \omega, d][\Psi, d, \theta]$$

Proof: Let ω be a spherical neighborhood of θ such that ω is contained in $\Phi^{-1}(U)$. Such a ω exists because U is open, Φ is continuous,

$\bar{\omega} = \omega \vee \omega'$ and G is defined on the closure S of a bounded open set containing $(I - F)(\bar{\omega})$.

We denote by d a bounded component of $X - \Phi(\omega')$. If the point p is not in $\Psi\Phi(\omega')$ and is not in Ψ [Boundary $\Phi(\omega)$] and if $[\Psi\Phi, \omega, p]$ is defined, then only a finite number of the terms $[\Phi, \omega, d] \cdot [\Psi, d, p]$ are non-zero and:

$$[\Psi\Phi, \omega, p] = \sum_{(d)} [\Phi, \omega, d][\Psi, d, p]$$

where $\sum_{(d)}$ means the summation over all the bounded components d.

$[\Psi\Phi, \omega, p]$ is the Leray-Schander topological degree at the point p of the mapping $\Psi\Phi$, relative to ω and $[\Phi, \omega, d]$, $[\Psi, d, p]$ have similar meanings.

Proof: This lemma is known. See Leray [10]. (For a detailed proof of the lemma, see Theorem A1 of the appendix to this thesis.)

In order to apply Lemma 5.1, it is necessary to consider a domain such that there is just one non-zero term in

$$\sum_{(d)} [\Phi, \omega, d][\Psi, d, c]$$

Lemma 5.2. If $\Phi(\theta) = \theta$ and θ is an isolated θ-point of Ψ, i.e., there exists a neighborhood $U(\theta)$ of θ such that if x is an element of U and $\Psi(x) = \theta$, then $x = \theta$; then there exists a bounded connected open set ω containing θ and such that there is at most one non-zero term in

$$\sum_{(d)} [\Phi, \omega, d][\Psi, d, \theta]$$

Proof: Let ω be a spherical neighborhood of θ such that ω is contained in $\Phi^{-1}(U)$. Such a ω exists because U is open, Φ is continuous,

and $\Phi(\theta) = \theta$. $\Phi(\omega)$ is obviously contained in U. Now consider $X - \Phi(\omega')$. Since $\Phi(\omega')$ is closed, $X - \Phi(\omega')$ is an open set. This set has disjoint components $D_1, D_2, D_3, \cdots, D_n, \cdots$ which are also open since Banach space X is locally connected. We consider the set of these D_i such that $[\Phi, \omega, D_i] \neq 0$ and call this set $\{D_i\}$. Each such D_i is contained in $\Phi(\omega)$ from the properties of topological degree. Since

$$\bigcup_i D_i \subset \Phi(\omega) \subset U(\theta)$$

it follows from the definition of $U(\theta)$ that only D_i such that θ is an element of D_i can be mapped so that $\Psi(D_i) \supset \theta$. Hence, at most one of the terms in

$$\sum_{(d)} [\Phi, \omega, d][\Psi, d, \theta]$$

is non-zero because at most one of the domains D_i contains θ which fact implies that at most one of the terms $[\Psi, d, \theta]$ is non-zero.

qed.

Theorem 5.1. Given a mapping $I - C + T$ of Banach space X into itself such that the Leray-Schander topological degree is defined for $I - C + T$ (i.e., we assume that T is completely continuous) and such that the multiplicity of Definition 5.1 is defined for $I - C + T$. If transformation T satisfies Assumption 3.4 or Assumptions 3.5 and 3.6 of Chapter 3, then

$$T_L(I - C + T) = T_D(I - C + T)$$

or

$$T_L(I - C + T) = -T_D(I - C + T)$$

where T_L is the Leray-Schander topological degree at θ, relative to a sufficiently small sphere S with center θ, and T_D is the multiplicity of Definition 5.1 relative to the same point and sphere.

Proof: By Theorem 2.9, $R = I + S$ where I is the identity transformation and S is linear, completely continuous. Moreover, R is one-to-one. Hence R has Leray-Schander topological degree $+1$ or -1 relative to θ and S. (See pp. 56-59, Leray and Schander [11]). Hence by Lemma 5.1 and 5.2, if S is made sufficiently small:

$$T_L[R(I-C+T)] = (T_L[R])(T_L[I-C+T]) = (\pm 1)\, T_L[I-C+T]$$

We will show that

$$T_L[R(I-C+T)] = T_D[I-C+T]$$

This will complete the proof of the theorem.

We consider the equation:

$$R(I - C + T)x = Ry$$

which is equivalent in the sense of Definition 3.1 to the equation

$$(I - C + T)x = y$$

because there exists R^{-1} (See Theorem 2.9, Chapter 2.) We carry out essentially the same procedure as used in Chapter 3. Denoting Ry by Z and applying Theorem 2.10, we obtain

(1) $$x - E_1 x + RT_x = Z$$

Applying transformations E_1, E_1' of Theorem 2.7 to equation (1), we obtain:

(2) $$E_1 RTx = E_1 Z \qquad \text{or} \qquad E_1 RT[x_1 + x_1'] = E_1 Z = Z_1$$

and

(3) $$x_1' + E_1' RT[x_1 + x_1'] = E_1' Z = Z_1'$$

By Lemma 3.1, Theorem 2.11 can be applied to (3) and we obtain

(4) $$x_1' = F(x_1, Z_1')$$

Substituting from (4) into (2), we obtain:

$$E_1 RT[x_1 + r(x_1, Z_1')] = E_1 Z = Z_1 \tag{5}$$

Now we define two mappings Φ and Υ whose product is $R(I - C + T)$.

We define the mapping Φ as:

$$\Phi(x) = \Phi(x_1 + x_1') = x_1 + x_1' + E_1' RT[x_1 + x_1'] = x_1 + Z_1'$$

The mapping Υ is defined as:

$$\begin{aligned}\Upsilon(x_1 + z_1') &= (I - E_1)(x_1 + z_1') + E_1 RT\left\{E_1(x_1 + z_1') + F[E_1(x_1 + z_1'), E_1'(x_1 + z_1')]\right\} \\ &= z_1' + E_1 RT[x_1 + F(x_1, z_1')] \\ &= z_1' + z_1 = z\end{aligned}$$

It is clear from the definitions of Υ and Φ that

$$\Upsilon \Phi = R(I - C + T)$$

<u>Lemma 1</u>. There exists a topological index [in the sense of Leray and Schander] of Φ relative to θ and this topological index is $+1$.

Proof: We show first that the identity mapping I is the differential at at θ of the mapping Φ. Using the definition of differential given by Hildebrandt and Graves (See p. 136, Hildebrandt and Graves [6]), we show that:

$$\lim_{\alpha \to 0} \left[\frac{\Phi(\theta + \alpha dx) - \Phi(\theta)}{\alpha}\right] = I(dx) = dx$$

This is shown as follows:

$$\begin{aligned}\left\| \frac{\Phi(\theta + \alpha dx) - \Phi(\theta)}{\alpha} - dx \right\| &= \left\| \frac{\alpha dx + E_1' RT[\alpha dx]}{\alpha} - dx \right\| \\ &= \left\| \frac{E_1' RT[\alpha dx]}{\alpha} \right\| \\ &\leq \frac{1}{|\alpha|} \|E_1'\| \|R\| M \|\alpha dx\|^2 \\ &= \|E_1'\| \|R\| M |\alpha| \|dx\|^2\end{aligned}$$

But $\lim_{\alpha \to 0} \|E_1'\| \|R\| M |\alpha| \|dx\|^2 = 0$

Hence I is the differential at θ of the mapping Φ. The identity has topological index +1. (See Leray and Schander, p. 55 [11]). Now we apply Lemma 3, § 1 of Rothe [14]. By this lemma, the mapping Φ has topological index +1.

qed.

Lemma 2. The Leray-Schander topological degree at θ of the mapping Υ relative to a sufficiently small sphere S with center θ is equal to the topological degree at θ of the mapping

$$E_1 RT[x_1 + F(x_1, Z_1')] ,$$

relative to the intersection of S with X.

Proof: In the mapping Υ, the mapping $E_1 RT[x_1 + F(x_1, Z_1')]$ is a finite dimensional mapping into X_1. This mapping is continuous; also it is bounded because $F(x_1, Z_1')$ is bounded and the mapping $E_1 RT$ is bounded in the neighborhood of θ which is considered. Thus Υ is a layer mapping according to the definition of Rothe, p. 374 [15]. Hence by the definition of Leray-Schander topological degree, the Leray-Schander topological degree at θ of Υ, relative to a sphere with center θ is the topological degree of Υ regarded as a mapping of X_1 into X_1. If we denote this mapping by Υ/X_1 it is clear from the definition of Υ that

$$\Upsilon/X_1 = E_1 R T [x_1 + F(x_1, z_1')]$$

Now the topological degree of $\Upsilon/X_1 = E_1 RT[x_1 + F(x_1, Z_1')]$ is, by Definition 5.1, the topological degree T_D of $I - C + T$.

It is clear that Lemma 5.1 may be applied to these mappings Φ and Ψ. Moreover, since $\Phi(\theta) = \theta$ from the definition of Φ and since θ is an isolated θ-point of Ψ if Assumption 3.4 holds or if Assumptions 3.5 and 3.6 hold, the hypotheses of Lemma 5.2 are satisfied. Hence, if we consider a sufficiently small sphere ω with θ as its center, then for topological degrees T_L, T_D relative to ω and 0:

$$T_L[R(I-C+T)] = T_L[\Psi\Phi] = \{T_L[\Psi]\}\{T_L[\Phi]\}$$
$$= \{T_D(I-C+T)\}\{+1\}$$
$$= T_D(I-C+T)$$

But $T_L[R(I-C+T)] = \pm[T_L(I-C+T)]$

Hence

$$T_D(I-C+T) = T_L(I-C+T)$$

or

$$T_D(I-C+T) = (-1)[T_L(I-C+T)]$$

qed.

Definition 5.1 gives a multiplicity for $I - C + T$ relative to the point y_2 and a sufficiently small sphere with center x_2 if (x_2, y_2) is a solution of the equation:

$$(I - C + T)x = y$$

The elements x_2, y_2 are not necessarily equal to θ. A direct consequence of Theorem 5.1 is the fact that this multiplicity is equal, except for a factor of (± 1), to the Leray-Schander topological degree at θ, relative to a sufficiently small sphere with center θ, of the mapping

$$I - C + T,$$

where the transformation $\mathcal{T}$ is the transformation defined in Chapter 3, provided that the Leray-Schander topological degree described exists.

Bibliography

1. Alexandroff and Hopf, Topologie, Erster Band, Juliano Springer, Berlin, 1935. (Lithoprint, J. W. Edwards, Ann Arbor, 1945).

2. S. Banach, Theorie des Operations Lineaires, 1932.

3. R. Caccioppoli, Sulle corrispondenze funzionali inverse diramate: teoria generale e applicazioni ad alcune equazioni funzionali non lineari e al problema di Plateau, Rendiconti Della Accademia Nazionale dei Lincei, vol. 24 (1936), pp. 258-263.

4. P. Halmos, Finite Dimensional Vector Spaces, Princeton University Press, Princeton, 1942.

5. T. H. Hildebrandt, Uber vollstetige lineare Transformationen, Acta Mathematica, vol. 51 (1928), pp. 311-318.

6. T. H. Hildebrandt and L. M. Graves, Implicit Functions and their Differentials in General Analysis, Am. Math. Soc. Trans. Vol. XXIX , (1927) pp. 127-153.

7. R. Iglisch, Reele Losungsfelder der elliptischen Differentialgleichung $\Delta u = F(u)$ und nichtlinearer Integralgleichungen, Mathematische Annalen, vol. 101 (1929) pp. 98-119.

8. R. Iglisch, Uber die Velfachheit einer Losung in der Theorie der nichtlinearen Integralgleichungen von E. Schmidt, Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 39, (1930 pp. 65-72.

9. K. Knopp, Theory of Functions, Part II, 4th German Edition, Translated by F. Bagemihl, Dover Publications, New York, 1947.

10. J. Leray, Note, Topologie des espaces abstrait de M. Banach, Comptes Rendus de l'Academie des Sciences, t. 200, 25 Mars 1935, pp. 1082-1084.

11. J. Leray and J. Schander, Topologie et equations fonctionnelles, Ann. Ecole Norm. vol. 51 (1934), pp. 45-78.

12. A. Leapounoff, Sur les figures d'equilibre peu differentes des ellipsoides d'une masse liquide homogene douee d'un nouvement de rotation. Premiere Partie. Etude generale du probleme. Memoire presente a l'Academie Imperiale des Sciences le 21 mars 1906, St. Petersbourg 1906, pp. 1-225.

13. F. Riesz, Uber lineare Funktionalgleichungen, Acta Mathematica, vol. 41 (1918), pp. 71-98.

14. E. H. Rothe, Topological proofs of uniqueness theorems in the theory of differential and integral equations, Bulletin, Am. Math. Soc., vol. 45 (1939) pp. 606-613.

15. E. H. Rothe, The Theory of topological order in some linear topological spaces, Iowa State College Journal of Science, Vol. XIII, No. 4, July, 1939, pp. 373-390.

16. J. Schander, Uber den Zusammenhang zwischen der Eindentigkeit und Losbarkeit partieller Differentialgleichungen zwerter Ordmung vom elliptischen Typus.

17. E. Schmidt, Zur Theorie des linearen und nichtlinearen Integralgleichungen, Mathematische Annalen, vol. 65, (1907-8), pp. 370-399.

Appendix

We prove here two facts which are known but for which there are, to the writer's knowledge, no complete proofs in the literature. The first of these is the theorem stated as Lemma 5.1 in Chapter 5 of this thesis. It is as follows:

Theorem A1. Let $\Phi = I + F$ and $\Psi = I + G$ be two mappings of subsets of Banach space X into X such that;

a) I is the identity mapping on X ; F, G are completely continuous mappings of subsets of X into X .

b) ω is a bounded open set with boundary ω' such that F is defined on $\overline{\omega} = \omega \cup \omega'$ and G is defined on a point set containing the closure S of a bounded open set which contains $(I + F)(\overline{\omega})$. We denote by d a bounded component of $X - \Phi(\omega')$. If the point p is not in $\Psi(\omega')$ and is not in Ψ [Boundary of $\Phi(\omega)$] and if $[\Psi\Phi, \omega, p]$ is defined, then only a finite number of the terms $[\Phi, \omega, d][\Psi, d, p]$ are non-zero and:

$$(1) \qquad [\Psi\Phi, \omega, p] = \sum_{(d)} [\Phi, \omega, d][\Psi, d, p]$$

where $\sum_{(d)}$ means the summation over all the bounded components d . $[\Psi\Phi, \omega, p]$ is the Leray Schander topological degree at p of the mapping $\Psi\Phi$ relative to ω , and $[\Phi, \omega, d]$, $[\Psi, d, p]$ have similar meanings.

Proof: An outline of the proof of this theorem may be found in Leray [10]. The detailed proof is as follows. We prove the theorem first for the finite dimensional case, i.e., we prove that if Φ, Ψ are continuous mappings

from R_1^n to R_2^n and from R_2^n to R_3^n respectively where R_1^n, R_2^n, R_3^n are n-dimensional real Euclidean spaces; if ω_n is a bounded open set in R_1^n, p is a point which is not in $\psi\phi(\omega_n)$ and is not in $\psi[B d_y \phi(\omega_n)]$ and if $[\psi\phi, \omega_n, p]$ is defined, then only a finite number of the terms $[\phi, \omega_n, d]\cdot[\psi, d, p]$ are non-zero and

$$[\psi\phi, \omega_n, p] = \sum_{(d)} [\phi, \omega_n, d][\psi, d, p] \tag{2}$$

where d represents a bounded component of $R_2^n - \phi(\omega_n')$, $[\psi\phi, \omega_n, d]$ is the topological degree at p of $\psi\phi$ relative to ω_n and $[\phi, \omega_n, d]$, $[\psi, d, p]$ have similar meanings.

First since ω_n' is a bounded closed set in R_1^n it is compact. Hence $\phi(\omega_n')$ is closed. Since R_2^n is locally connected, it follows then that the components of $R_2^n - \phi(\omega_n')$ are open sets. Only the bounded components d occur in the sum

$$\sum_{(d)} [\phi, \omega, d][\psi, d, p] \tag{3}$$

because of the following

Lemma A1. If $[\phi, \omega_n, d] \neq 0$, then d is bounded.

Proof: The argument above shows that $\phi(\omega_n)$ is bounded. Let $M > 0$ be a bound for $\phi(\omega_n)$. If x is an element of R_2^n such that $\|x\| > M$, then

$$[\phi, \omega_n, x] = 0$$

For if $[\phi, \omega_n, x] \neq 0$, then there exists y, an element of ω_n such that $\phi(y) = x$. But this is impossible because

$$\|\phi(y)\| \leq M$$

Thus if d is unbounded, hence has in it elements x such that $\|x\| > M$,

we must have

$$[\phi, \omega_n, d] = 0$$

qed.

Now suppose ϕ, γ are simplicial mappings relative to simplicial subdivisions of R_1^n, R_2^n, R_3^n. We suppose that the point p is contained in the interior of one of the simplices of the subdivision of R_3^n so that a topological degree at p of γ and $\gamma\phi$ is defined. If the simplicial subdivision of R_2^n used in the definition of ϕ and the simplicial subdivision of R_2^n used in the definition of γ are the same, then equation (2) follows from the definition of topological degree for simplicial mappings. Since the topological degree of $\gamma\phi$ is defined, it is clear that the sum on the right in equation (2) has only a finite number of non-zero terms.

Since equation (2) holds for simplicial mappings, it holds also for continuous mappings. In order to prove this statement, we remark first that the topological degree of a continuous mapping is, by definition, the topological degree of a simplicial mapping which is a sufficiently good approximation of the continuous mapping. Let ϕ, γ be continuous mappings. Then by the Approximationssatz (A-H, p. 319) there exist simplicial mappings ϕ_m, γ_m such that

$$\| \phi(x) - \phi_m(x) \| < \frac{1}{m} \qquad \text{for } x \in \overline{\omega_n}$$

and $$\| \gamma(x) - \gamma_m(x) \| < \frac{1}{m} \qquad \text{for } x \in S \text{, a sphere containing } \phi(\omega_n)$$

Then if $r = \left(\rho[p, \gamma\phi(\omega_n)]\right)$, there exists an integer n such that if $n > n_0$

$$\| \gamma\phi(x) - \gamma_m\phi_m(x) \| < \frac{n}{4} \qquad \text{for } x \in \overline{\omega_n}$$

Hence by Rouche's Theorem (Theorem 2.12 in Chapter 2)

(d) $$[\gamma\phi, \omega_n, p] = [\gamma_m \phi_m, \omega_n, p]$$

Now if $[b]$ denotes the set of points b such that

$$\gamma(b) = p$$

then since $[b]$ is closed, the set $B = [b] \cap S$ is closed and compact. Let 2δ be the distance between $\phi(\omega_n')$ and B. Then the set of points whose distance from B is $< \delta$ is the union of a finite number of connected open sets V_ν. (A generalized version of this statement is proved in Assertion 4 in the proof of Lemma A2; so we defer the proof until then.)

Finally for γ_m close enough to γ, the sets V_ν contain all the points $b^* \in S$ such that

$$\gamma_m(b^*) = p$$

(This statement is proved in more general form in Assertion 5 in the proof of Lemma A2. Consequently we defer the proof until then.)

Hence from the definition of V_ν, it follows that if m is sufficiently large

$$\sum_{(d^*)} [\phi_m, \omega_n, d^*][\gamma_m, d^*, p] = \sum_\nu [\phi_m, \omega_n, V_\nu][\gamma_m, V_\nu, p]$$

where we denote by d^* a bounded component of $R_2^n - \phi_n(\omega_n')$

Also

$$\sum_{(d)} [\phi, \omega_n, d][\gamma, d, p] = \sum_\nu [\phi, \omega_n, V_\nu][\gamma, V_\nu, p]$$

by the definition of V_ν. Finally, by Rouche's Theorem

$$\sum_\nu [\phi_m, \omega_n, V_\nu][\gamma_m, V_\nu, p] = \sum_\nu [\phi, \omega_n, V_\nu][\gamma, V_\nu, p]$$

Hence

(4) $$\sum_{(d^*)} [\phi_m, \omega_n, d^*][\gamma_m, d^*, p] = \sum_{(d)} [\phi, \omega_n, d][\gamma, d, p]$$

From (α) and (4) and the statement for simplicial mappings, it follows that

$$(5) \qquad [\Psi\Phi, \omega_n, p] = \sum_{(d)} [\Phi, \omega_n, d][\Psi, d, p]$$

Now from the definition of topological degree for layer mappings (See Rothe [14]) it follows that equation (1) holds if Φ and Ψ are layer mappings because equation (1) holds in the finite case. The proof for arbitrary Φ and Ψ of the form $I + F$ and $I + G$, respectively, where F, G are completely continuous, follows from the definition of topological degree for Φ and Ψ and from the following lemma:

Lemma A2. If $\{\Phi_n\}, \{\Psi_n\}$ are sequences of transformations which converge uniformly to and and if the following equation holds for each n :

$$(1^*) \qquad [\Psi_n \Phi_n, \omega, p] = \sum_{(d^*)} [\Phi_n, \omega, d^*][\Psi_n, d^*, p]$$

then equation (1) holds for Φ and Ψ.

The proof of Theorem A1 is a consequence of the following application of Lemma A2: we let $\{\Phi_n\}$, $\{\Psi_n\}$ be sequences of layer mappings which converge uniformly to Φ and Ψ respectively. As remarked above, equation (1) holds for layer mappings, i.e., equation (1^*) holds if Φ_n, Ψ_n are layer mappings. Hence the proof of Theorem A1 is complete if we prove Lemma A2.

Proof of Lemma A2. We prove first that if n is sufficiently large, then

$$(\alpha) \qquad [\Psi\Phi, \omega, p] = [\Psi_n, \Phi_n, \omega, p]$$

In proving this statement, we prove first two assertions

Assertion 1. If $\Phi_n = I + F_n$ where F_n is completely continuous, then M, the closure of the set

$$[\bigcup_n F_n(\bar{\omega})] \cup [F(\bar{\omega})]$$

is sequentially compact in itself.

Proof. Since each F_m is a completely continuous transformation, each F_m is sequentially compact because $\bar{\omega}$ is, by hypothesis, bounded. Now we prove that $\bigcup_m F_m(\bar{\omega})$ is a sequentially compact set. Consider an infinite set $\{x_\nu\} \subset \bigcup_m F_m(\bar{\omega})$. We must show that this set has a limit point. If an infinite subset, say $\{x_{\nu_j}\}$ of $\{x_\nu\}$ is contained in some definite $F_{m_0}(\bar{\omega})$ then since each $F_m(\bar{\omega})$ is compact, the set $\{x_{\nu_j}\}$ has a limit point and the proof is complete. If no $F_m(\bar{\omega})$ contains an infinite subset of $\{x_\nu\}$ then there exists a subsequence $\{x_{\nu_\kappa}\}$ such that

$$x_{\nu_\kappa} \in F_{m_\kappa}(\bar{\omega})$$

Since $x_{\nu_\kappa} \in F_{m_\kappa}(\bar{\omega})$, then there exists $x_{m_\kappa} \in F(\bar{\omega})$ such that

$$\|x_{m_\kappa} - x_{\nu_\kappa}\| < \varepsilon_{m_\kappa} \tag{1}$$

where $\{\varepsilon_{m_\kappa}\}$ is a sequence of real numbers such that $\lim_{\kappa \to \infty} \varepsilon_{m_\kappa} = 0$ This follows from the definition of F_m. If $x_{\nu_\kappa} = F_{m_\kappa}(y_{m_\kappa})$ where $y_{m_\kappa} \in \bar{\omega}$ then $x_{m_\kappa} = F(y_{m_\kappa})$

Now consider the set $\{x_{m_\kappa}\} \subset F(\bar{\omega})$. Since $F(\bar{\omega})$ is compact, $\{x_{m_\kappa}\}$ has a limit point x_0 if the set $\{x_{m_\kappa}\}$ has an infinite set of distinct elements. If x_0 is a limit point of $\{x_{m_\kappa}\}$, then it is a limit point of $\{x_{\nu_\kappa}\}$ by inequality (1) above. If $\{x_{m_\kappa}\}$ consists of only a finite set of distinct elements, then at least one of these elements is a limit point of $\{x_{m_\kappa}\}$ by (1) above. Thus $\bigcup_m F_m(\bar{\omega})$ is sequentially compact. Since F is completely continuous, $F(\bar{\omega})$ is sequentially compact. Hence M, the closure of

$$[\bigcup_m F_m(\bar{\omega})] \cup [F(\bar{\omega})]$$

is sequentially compact in itself.

qed.

Now by hypothesis, the point p is not an element of $\Psi\Phi(\bar{\omega})$.

Since ω' is closed and since $\Psi\Phi$ is of the form:

$$\Psi\Phi = I + H$$

where I is the identity and H is a completely continuous mapping, the set $\Psi\Phi(\omega')$ is closed and hence there exists between p and $\Psi\Phi(\omega')$ a distance

$$\rho[p, \Psi\Phi(\omega')] = \eta > 0$$

Now instead of dealing with the sequences

$$\{\Phi_m\} = \{I + F_m\}$$

and

$$\{\Psi_m\} = \{I + G_m\}$$

we deal with subsequences which will be denoted by $\{\Phi_{m_i}\}$ and $\{\Psi_{m_i}\}$ and which have the property that

$$\|F(x) - F_{m_i}(x)\| < \frac{1}{i}$$

for x , any element of $\bar{\omega}$. Also

$$\|G(x) - G_{m_i}(x)\| < \frac{1}{i}$$

for y , any element of S where S is the set defined in condition b) of the statement of Theorem A1.

It will be noted that since S is the closure of an open set containing $F(\bar{\omega})$, S contains all the sets $F_m(\bar{\omega})$ for n sufficiently large. Moreover, since $M \cdot S$ is a closed subset of M , then from Assertion 1 , it follows that $M \cdot S$ is sequentially compact in itself.

Assertion 2. If m_1 is sufficiently large, then

$$\|\Psi\Phi(x) - \Psi_{m_i}\Phi_{m_i}(x)\| < \frac{\eta}{4}$$

where x is any element of $\bar{\omega}$ and $\eta = \rho[p, \Psi\Phi(\omega')]$

Proof:

$$(1)\quad \|\Psi\Phi(x) - \Psi_{m_i}\Phi_{m_i}(x)\| = \|(I+G)(I+F)(x) - (I+G_{m_i})(I+F_{m_i})(x)\|$$
$$\leq \|I(x) + G(x) + F(x) + GF(x) - I(x) - G_{m_i}(x) - F_{m_i}(x) - G_{m_i}F_{m_i}(x)\|$$
$$\leq \|G(x) - G_{m_i}(x)\| + \|F(x) - F_{m_i}(x)\| + \|GF(x) - G_{m_i}F_{m_i}(x)\|$$

Now $\| F(x) - F_{m_i}(x) \| < \frac{1}{i}$ for x, any element of $\bar{\omega}$ and $\| G(x) - G_{m_i}(x) \| < \frac{1}{i}$ for $x \in M \cap S$. So we consider

$$\| GF(x) - G_{m_i} F_{m_i}(x) \|$$

$$(2) \quad \| GF(x) - G_{m_i} F_{m_i}(x) \| \leq \| GF(x) - GF_{m_i}(x) \| + \| GF_{m_i}(x) - G_{m_i} F_{m_i}(x) \|$$

Now $\| GF_{m_i}(x) - G_{m_i} F_{m_i}(x) \| < \frac{1}{i}$ for x any element of $\bar{\omega}$ and i so large that $F_{m_i}(x)$ is contained in $M \cap S$. So we must deal with the term

$$\| GF(x) - GF_{m_i}(x) \|$$

Since $M \cap S$ is sequentially compact in itself and G is continuous, then G is uniformly continuous on the set $M \quad S$. This implies that there exists $\delta_i > 0$ such that if x_1, x_2 are elements of $M \cap S$ and if $\| x_1 - x_2 \| < \delta_i$, then

$$\| G(x_1) - G(x_2) \| < \frac{1}{i}$$

Let m_{j_0} be an integer such that $\frac{1}{j_0} < \delta_i$. Then

$$\| F(x) - F_{m_{j_0}}(x) \| < \frac{1}{j_0} < \delta_i$$

for $x \in \bar{\omega}$.

Hence

$$\| GF(x) - GF_{m_{j_0}}(x) \| < \frac{1}{i}$$

for $x \in \bar{\omega}$.

Returning to inequality (2), we see that for $m > \begin{cases} m_i \\ m_{j_0} \end{cases}$

$$\| GF(x) - G_m F_m(x) \| < \frac{2}{i} \qquad \text{for } x \in \bar{\omega}.$$

Hence inequality (1) becomes for $m > m_i$, and $m > m_{j_0}$

$$\| \Psi\Phi(x) - \Psi_m \Phi_m(x) \| < \frac{4}{i}$$

for $x \in \bar{\omega}$. Thus in order to prove the lemma, we need only take i large enough so that

$$\frac{4}{i} < \frac{\eta}{4}$$

or

$$\frac{1}{i} < \frac{\eta}{16}$$

qed.

The inequality of Assertion (2) holds in particular for all x in ω'. Hence by Rouche's Théorem for the Leray Schander topological degree (See Rothe [14]. Theorem 4, p. 380) the topological degree at p of $\Psi\Phi$, relative to $\overline{\omega}$, is the same as the topological degree at p of $\Psi_m\Phi_m$ relative to $\overline{\omega}$, i.e.,

$$[\Psi\Phi, \omega, p] = [\Psi_m\Phi_m, \omega, p]$$

if m is sufficiently large.

This completes the proof of statement (α) on page 5.

Now we prove that if m is sufficiently large, then

$$(\beta) \qquad \sum_{(\delta^*)} [\Phi_m, \omega, \delta^*][\Psi_m, \delta^*, p] = \sum_{(\delta)} [\Phi, \omega, \delta][\Psi, \delta, p]$$

where δ^* denotes a bounded component of $\chi - \Phi_m(\omega')$. The proof of Lemma A2 follows from (α) and (β).

Assertion 3. If p is a fixed point, then the set [b] of point b such that $\Psi(b) = p$ is such that [b] $\cap$ S is a closed compact set.

Proof: The set [b] is closed because Ψ is continuous, p is a closed set, and the inverse image of a closed set under a continuous mapping is closed. Since S is closed, then [b] $\cap$ S is closed.

To prove that [b] $\cap$ S is compact, we must prove: if $\{b_\nu\}$ is an arbitrary infinite set in [b] $\cap$ S, then $\{b_\nu\}$ has a limit point b_0.

$$\Psi(b_\nu) = I(b_\nu) + G_1(b_\nu) = p$$

$$(1) \qquad \therefore \; b_\nu = p - G_1(b_\nu)$$

Since $\{b_\nu\} \subset S$ which is a bounded set, then $\{G_1(b_\nu)\}$ is sequentially compact. Hence there exists subset $\{b_{\nu_n}\} \subset \{b_\nu\}$ such that $\lim_n G_1(b_{\nu_n}) = b'$ Hence by equation (1), $\lim_n b_{\nu_n} = p - b'$. Thus $\{b_\nu\}$ has limit point $p - b'$.

qed.

Now let B designate the set $[b] \cap S$. By the hypothesis of Theorem A1, p is not in $\bigcup \Phi(\omega')$. Hence

$$B \cap \Phi(\omega') = \Lambda$$

Since B is closed and compact and $\Phi(\omega')$ is closed, there exists a minimal distance between B and $\Phi(\omega')$ which distance we designate by $2l$

Assertion 4. The set of points whose distance from B is $< l$ is the union of a finite number of connected open sets.

Proof: About each point of B, take an open sphere of radius l. The union U of these spheres is the set of points whose distance from B is $< l$. It is clear that U is an open set. Consider the components V_ν of U. Each V_ν is open because the Banach space X is locally connected. Moreover, each V_ν contains at least one point of B because each component V_ν contains at least one of the spheres. Now the number of components V_ν is finite. This is proved as follows: Suppose the set $[V_\nu]$ is infinite. For each V_ν, let p_ν be a point in $V_\nu \cap B$. The set $\{p_\nu\}$ is infinite because there is an infinite set of components which are disjoint by definition. Also the set $\{p_\nu\}$ has no limit point because such a limit point would be in B since B is closed. Hence the limit point would be in some V_ν. This is impossible from the definition of $\{p_\nu\}$. Hence the infinite set $\{p_\nu\}$ contained in B which is compact, has no limit point. Contradiction. Hence the required set is the set

qed.

Remark. In defining the set $\{p_\nu\}$, it is necessary to use the Axiom of Choice.

Now if Φ_m is such that

$$\| \Phi_m(x) - \Phi(x) \| < l$$

for all $x \in \overline{\omega}$, then the connected open sets $V_1, V_2, \cdots, V_k$ are disjoint from $\Phi_m(\omega')$, i.e.,

$$V_\nu \cap \Phi_m(\omega') = \phi \qquad \text{for } \nu = 1, 2, \cdots, k .$$

This follows from the definition of V_ν and the fact that the distance from B to $\Phi(\omega')$ is 2ℓ.

Assertion 5. For Ψ_{m_i} close enough to Ψ, the sets V_ν contain all the points $b^* \in S$ such that $\Psi_m(b^*) = p$.

Proof: Suppose that for infinitely many $\Psi_{m_{i_j}}$, there exists $b^*_{m_{i_j}} \in S$ such that $b^*_{m_{i_j}} \in V_\nu$ for $\nu = 1, 2, \cdots, k$ and $\Psi_{m_{i_j}}(b^*_{m_{i_j}}) = p$
Since $\| \Psi_{m_{i_j}}(x) - \Psi(x) \| < \frac{1}{i}$ for $x \in S$, then

$$\| p - \Psi(b^*_{m_{i_j}}) \| < \frac{1}{i} \tag{1}$$

Now we consider two cases: if the set of points $\{b^*_{m_{i_j}}\}$ is finite, then for $m_{i_j} > m_1$ where m_1 is some integer, $\Psi(b^*_{m_{i_j}}) = p$ by inequality (1) above. Then $b^*_{m_{i_j}}$ for $m_{i_j} > m_1$ is either in some V_ν or $b^*_{m_{i_j}}$ is not in S. Both of these conditions are impossible because of the assumption. Hence the set $\{b^*_{m_{i_j}}\}$ cannot be finite.

If the set of points $\{b^*_{m_{i_j}}\}$ is infinite, then

$$\Psi(b^*_{m_{i_j}}) = (I + G)(b^*_{m_{i_j}}) = b^*_{m_{i_j}} + G(b^*_{m_{i_j}}) \tag{2}$$

Since $\{b^*_{m_{i_j}}\}$ is contained in S and hence is a bounded set, then $G(b^*_{m_{i_j}})$ is compact. Hence there exists $\{b^*_{m_{i_{j'}}}\} \subset \{b^*_{m_{i_j}}\}$ and $b_0 \in S$ such that

$$\lim_{j' \to \infty} G(b^*_{m_{i_{j'}}}) = b$$

Now $\{\Psi(b^*_{m_{i_{j'}}})\}$ has p as a limit by inequality (1) above. Hence, since by equation (2)

$$\Psi(b^*_{m_{i_{j'}}}) = b^*_{m_{i_{j'}}} + G(b^*_{m_{i_{j'}}})$$

it follows that $\lim_{j' \to \infty} b^*_{m_i j'} = p - b_0$ Now since for all j', $b^*_{m_i j'} \notin \bigcup_{\nu=1}^{k} V_\nu$, then $p - b_0 \notin \bigcup_{\nu=1}^{k} V_\nu$. But $p - b_0 \in S$ because S is closed and $b^*_{m_i j'} \in S$ for all j'. By inequality (1) and the continuity of Ψ

$$\Psi(p - b_0) = p$$

Thus $p - b_0 \in S$, $p - b_0 \notin \bigcup_{\nu=1}^{k} V_\nu$ and $\Psi(p - b_0) = p$

This is a contradiction to the definition of the set B.

qed.

Hence for m_i sufficiently large,

$$(3) \quad \sum_{(d^*)} [\Phi_{m_i}, \omega, d^*][\Psi_{m_i}, d^*, p] = \sum_{\nu} [\Phi_{m_i}, \omega, V_\nu][\Psi_{m_i}, V_\nu, p]$$

by the definition of V_ν and Assertion 5. Also

$$(4) \quad \sum_{(d)} [\Phi, \omega, d][\Psi, d, p] = \sum_{\nu} [\Phi, \omega, V_\nu][\Psi, V_\nu, p]$$

by the definition of V. Finally, by Rouche's Theorem

$$(5) \quad \sum_{\nu} [\Phi_{m_i}, \omega, V_\nu][\Psi_{m_i}, V_\nu, p] = \sum_{\nu} [\Phi_{m_i}, \omega, V_\nu][\Psi, V_\nu, p]$$

Statement (β) on page 9 follows from (3), (4), and (5). Thus the proof of Lemma A2 and hence the proof of Theorem A1 is complete.

The second theorem which we prove is a lemma which is stated in Leray and Schander [11]. Since this lemma is necessary for the definition of topological degree of a mappint in Banach space given by Leray and Schander and since no complete proof of the lemma is given in [11], we include a proof here although the lemma is not used directly in this thesis[6].

[6] The proof here was given to the writer in outline form by Professor E.H. Rothe

We consider ω_{n+p} a bounded open set in (n+p)-dimensional Euclidean space E^{n+p}. Let E^n be an n-dimensional linear manifold of E^{n+p} which is such that

$$E^n \cap \omega_{n+p} = \omega_n \neq \emptyset$$

and let b be a point in E^n. Let ω'_{n+p} be the point set boundary of ω_{n+p} and let ω'_n be the boundary of $\overline{\omega}_n$ in E^n. Then $\omega'_n \subset \omega'_{n+p}$

Now we consider a continuous mapping Φ_{n+p} taking $\overline{\omega}_{n+p}$ into E^{n+p}. We make the following assumptions concerning Φ_{n+p}:

A) The point b is not an element of $\Phi_{n+p}(\omega'_{n+p})$ and, a fortiori, is not an element of $\Phi_{n+p}(\omega'_n)$.

B) Every point of $\overline{\omega}_{n+p}$ is moved by Φ_{n+p} parallel to the linear manifold E_1^n i.e., Φ_{n+p} is a layer mapping with respect to E^n.

If we write $\Phi_n = \Phi_{n+p} / E^n$ then by assumption B), the mapping Φ_n takes $\overline{\omega}_n$ into E^n. Also by assumption B), the topological degrees $[\Phi_{n+p}, \omega_{n+p}, b]$ and $[\Phi, \omega_n, b]$ are defined. The theorem which we prove is:

Theorem A2. $[\Phi_{n+p}, \omega_{n+p}, b] = [\Phi_n, \omega_n, b]$.

Proof: By definition, the topological degree of a mapping is the topological degree of a certain simplicial mapping which is a sufficiently good approximation of the original mapping (See A-H, pp. 473-474). If ε_1 is an arbitrary positive number, then by Satz II and Satz III, p. 319, A-H, there exists a simplicial mapping of the required type Φ'_{n+p} which is a layer mapping with respect to E^n, which is within ε of Φ_{n+p} and which is, a fortiori, such that $\Phi'_n = \Phi'_{n+p} / E^n$ is a simplicial mapping which is within ε of Φ_n.

We choose Φ'_{n+p} and Φ'_n so that their topological degrees are equal to those of Φ_{n+p} and Φ_n respectively. Throughout the following discussion, we shall deal with Φ'_{n+p} and Φ'_n. For convenience, we drop

the prime notation and call these simplicial mappings Φ_{n+p} and Φ_n.

It is obvicously sufficient to prove the theorem for the case: $p = 1$. Now suppose $\epsilon > 0$ is a positive number such that

(1) $$\epsilon < \rho\left[\Phi_{n+1}(\omega'_{n+1}), b\right]$$

where $\rho\left[\Phi_{n+1}(\omega'_{n+1}), b\right]$ is the distance between the set $\Phi_{n+1}(\omega'_{n+1})$ and the point b. Let S be an $(n+1)$-sphere of radius r containing $\bar{\omega}_{n+1}$ and $\Phi_{n+1}(\bar{\omega}_{n+1})$ such that the distance in E^{n+1} between $\bar{\omega}_{n+1} \cup [\Phi_{n+1}(\bar{\omega}_{n+1})]$ and the boundary of S is $> \epsilon$. Let k_n be the simplicial complex consisting of an n-dimensional Euclidean simplex σ^n in E^n which contains S.

We show now that there exists a simplicial subdivision Z^n of K_n with the following properties:

α) The diameters of all the simplices are less than $\epsilon/4$ where ϵ is the number described in inequlaity (1) above.

β) If x is an element of the set $[\Phi^{-1}_{n+1}(b)] \cup [b]$, then x is in the interior of one of the n-simplices σ_i^n of Z^n and in each n-simplex, there exists at most one point x of the set $[\Phi^{-1}_{n+1}(b)] \cup [b]$. (The set $[\Phi^{-1}_{n+1}(b)]$ is a finite set. This is one of the conditions on the simplicial mapping Φ_{n+1} [See A-H, pp. 473-474.]

To show that this simplicial subdivision exists, we prove:

Lemma A3. Given a finite simplicial complex K_n of homogeneous dimension n (in the sense of A-H, pp. 126-127) such that all the simplices of K_n are Euclidean simplices and given a finite set of points T such that if x is an element of T, then x is contained in the interior of an n-dimensional simplex of K_n; then there exists a simplicial subdivision Z^n of K_n such that the diameters of the simplices of Z^n are all $\epsilon/4$ where ϵ is an

arbitrary positive number and such that if x is an element of T, then x is in the interior of one of the n-simplices of Z^n, and there is at most one point $x \in T$ in each n-simplex of Z^n.

Proof: We prove that if σ^n is a Euclidean n-simplex of k_n, then there exists a simplicial subdivision of σ^n such that if $x \in T$ is in the interior of σ^n, then it is in the interior of one of the n-simplices of the simplicial subdivision of σ^n and such that the diameter of each of the simplices of the subdivision is $\leq \left[\frac{n}{n+1} + \frac{1}{2(n+1)}\right] d$ where d is the diameter of σ^n. The proof of the lemma follows from this statement because if the statement is applied to each n-simplex of K_n and then to each n-simplex of the resulting subdivision and this process is repeated sufficiently often, then a simplicial subdivision of K_n can be obtained which is so fine that in each n-simplex of the subdivision, there is at most one point $x \in T$. This follows from the hypothesis that T is a finite set.

We deal first with the case in which the simplex σ^n contains just one point q_1 of the set T. Consider the "eigentliche" barycentric subdivision (See A-H, pp. 133-136) of the simplex σ^n. If the point q_1 is in the interior of an n-simplex of this simplicial subdivision of σ^n then the desired result is obtained. The simplices of the subdivision have diameter $\leq \left(\frac{n}{n+1}\right) d$ by A-H, Satz IIa, p. 136.

Now suppose the point q_1 with coordinates $(x_{11}, x_{12}, \cdots, x_{1n})$ is contained in the boundary of one of the n-simplices of the subdivision. Let the point O with coordinates $(x_{o1}, x_{o2}, \ldots\ldots, x_{on})$ be the central point of the subdivision. If q_1 is contained in the boundary of one of the n-simplices, this means, from the definition of barycentric subdivision, that q_1 is contained in at most $n-1$ $(n-1)$-dimensional hyperplanes which can be described by the equations:

$$(1\text{-}i) \qquad \begin{vmatrix} x_1 & x_2 & \cdots & x_n & 1 \\ x_{01} & x_{02} & \cdots & x_{0n} & 1 \\ x_{11}^{i} & x_{12}^{i} & \cdots & x_{1n}^{i} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n-11}^{i} & x_{n-12}^{i} & \cdots & x_{n-1n}^{i} & 1 \end{vmatrix} = 0$$

where $i = 1, 2, 3, \cdots, r$ and $1 \leq n \leq n-1$. The $x_1, x_2, \cdots x_n$ are the coordinates of a variable point, and $(x_{m1}^{i}, x_{m2}^{i}, \ldots, x_{mn}^{i})$ are the coordinates of a fixed point which is in the hyperplane and which is contained in the boundary of the simplex σ^n $(m = 1, 2, \ldots, n-1)$.

All of these hyperplanes contain the point $q_1 = (x_{11}, x_{12}, \ldots, x_{1n})$. This fact is expressed by the equations:

$$(2\text{-}i) \qquad \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} & 1 \\ x_{01} & x_{02} & \cdots & x_{0n} & 1 \\ x_{11}^{i} & x_{12}^{i} & \cdots & x_{1n}^{i} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n-11}^{i} & x_{n-12}^{i} & \cdots & x_{n-1n}^{i} & 1 \end{vmatrix} = 0$$

for $i = 1, 2, \cdots, n$.

The point q_1 is not contained in the other k $(n-1)$-dimensional hyperplane of the barycentric subdivision. Let these hyperplanes be described by the equations:

$$(3\text{-}j) \qquad \begin{vmatrix} x_1 & x_2 & \cdots & x_n & 1 \\ x_{01} & x_{02} & \cdots & x_{0n} & 1 \\ x_{11}^{j} & x_{12}^{j} & \cdots & x_{1n}^{j} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n-1,1}^{j} & x_{n-1,2}^{j} & \cdots & x_{n-1,n}^{j} & 1 \end{vmatrix} = 0$$

where $j = 1, 2, \cdots, k$ and $(x_{m1}^{j}, x_{m2}^{j}, \ldots, x_{mn}^{j})$ are the coordinates of a fixed point which is in the hyperplane and which is contained in the boundary of the simplex σ^n $(m = 1, 2, \cdots n-1)$. Then the fact that q_1 is not in these hyperplanes is described by the inequalities:

$$(4\text{-}j) \qquad \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} & 1 \\ x_{01} & x_{02} & \cdots & x_{0n} & 1 \\ x_{11}^{j} & x_{12}^{j} & \cdots & x_{1n}^{j} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n-1,1}^{j} & x_{n-1,2}^{j} & \cdots & x_{n-1,n}^{j} & 1 \end{vmatrix} \neq 0$$

where $j = 1, 2, \cdots, k$.

Now since the determinants in equations (4-j) are continuous functions of $x_{01}, x_{02}, \ldots, x_{0n}$ then there exists a neighborhood N_0 of the point 0 of radius $< \left[\frac{1}{2(n+1)}\right]\delta$ such that if the point $(y_1, y_2, \cdots, y_n) \in N_0$, then:

$$(5\text{-}j) \qquad \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} & 1 \\ y_1 & y_2 & \cdots & y_n & 1 \\ x_{11}^{j} & x_{12}^{j} & \cdots & x_{1n}^{j} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n-1\,1}^{j} & x_{n-1\,2}^{j} & \cdots & x_{n-1\,n}^{j} & 1 \end{vmatrix} \neq 0$$

where $j = 1, 2, 3, \cdots k$.

On the other hand, since each of the determinants in equations (2-i) is a linear, continuous function of $x_{01}, x_{02}, \cdots, x_{0n}$ which is not identically zero, then in each neighborhood N of $(x_{01}, x_{02}, \cdots, x_{0n})$ there exist infinitely many points $(y_1, y_2, \cdots, y_n)$ such that

$$(6\text{-}i) \qquad \begin{vmatrix} x_{11} & x_{12} & \cdots & x_{1n} & 1 \\ y_1 & y_2 & \cdots & y_n & 1 \\ x_{11}^{i} & x_{12}^{i} & \cdots & x_{1n}^{i} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n-1\,1}^{i} & x_{n-1\,2}^{i} & \cdots & x_{n-1\,n}^{i} & 1 \end{vmatrix} \neq 0$$

for $i = 1, 2, \cdots, n$. That is, the point $(x_{11}, x_{12}, \cdots, x_{1n})$ is not in the planes

$$(7\text{-}i) \qquad \begin{vmatrix} x_1 & x_2 & \cdots & x_n & 1 \\ y_1 & y_2 & \cdots & y_n & 1 \\ x_{11}^{i} & x_{12}^{i} & \cdots & x_{1n}^{i} & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_{n-1\,1}^{i} & x_{n-1\,2}^{i} & \cdots & x_{n-1\,n}^{i} & 1 \end{vmatrix} = 0$$

Now we consider a point $(y_1, y_2, \cdots, y_n)$ in the neighborhood N_0 which satisfies the inequalities (S-j) and (t-i). Then the point $(x_{01}, x_{02}, \cdots, x_{0n})$ is not in any of the hyperplanes of the simplicial subdivision which has $(y_1, y_2, \cdots, y_n)$ as its center. On the other hand, since the point $(y_1, y_2, \cdots, y_n)$ has distance less than $\left[\frac{1}{2(n+1)}\right] d$ from the point $(x_{01}, x_{02}, \cdots, x_{0n})$, then the diameters of the simplices of this subdivision must be less than

$$\left[\frac{n}{n+1} + \frac{1}{2(n+1)}\right] d$$

This completes the proof for the case in which the simplex σ^n contains just one point $x \in T$ in its interior.

Now suppose that the simplex σ^n contains in its interior the points $q_1, q_2, \cdots, q_k$ $(k > 1)$ of the set T. In the same way as before, we show that there exist neighborhoods N_i $(i = 1, 2, \cdots, k)$ of radius $< \frac{d}{2(n+1)}$ of the point 0 such that if 0_i is any point in the neighborhood N_i and if the point q_i is not in any hyperplane of the subdivision defined relative to the point 0, then the point q_i is not in any hyperplane of the subdivision defined relative to the original point 0. Also, in the open set

$$\bigcap_{i=1}^{k} N_i$$

a point $(y_1, y_2, \cdots, y_n)$ may be chosen such that the points $q_1, \cdots, q_k$ are not in the hyperplanes of the subdivision defined relative to $(y_1, y_2, \cdots y_n)$ if $q_1, q_2, \cdots q_k$ are in the corresponding hyperplanes of the subdivision defined relative to the point 0. This result is obtained by using the same type of argument as in the preceding case.

Thus there exists a point $(y_1, y_2, \cdots, y_n)$ which satisfies the required conditions. If this point is used as the center of the subdivision, the desired result is obtained.

qed.

Now we define an $(n+1)$ dimensional complex K_{n+1}, which contains K_n, in the following way. Let E_a^n and E_b^n be the two n-dimensional hyperplanes which are parallel to E^n and at distance m from E^n where m is a positive integer such that

$$m \epsilon > 2n$$

where r is the radius of the sphere S.

Suppose coordinates $x_1, x_2, \cdots, x_n, x_{n+1}$ in E^{n+1} are so chosen that the equation of E^n is $x_{n+1} = 0$. Let the equations of E_a^n and E_b^n be $x_{n+1} = a$ and $x_{n+1} = b$ respectively. Now if the vertices of the simplex $\sigma^n = \overline{u_0 u_1 \cdots u_n}$ are $(x_{01}, x_{02}, \cdots, x_{0n-1}, 0)$ $(x_{11}, x_{12}, \cdots, x_{1n-1}, 0), \cdots, (x_{n1}, x_{n2}, \cdots, x_{nn-1}, 0)$ then the sets of points

$$(x_{01}, x_{02}, \cdots, x_{0n-1}, a), \quad (x_{11}, x_{12}, \cdots, x_{1n-1}, a), \cdots, (x_{n1}, x_{n2}, \cdots x_{nn-1}, a)$$

and

$$(x_{01}, x_{02}, \cdots, x_{0n-1}, b), \quad (x_{11}, x_{12}, \cdots, x_{1n-1}, b), \cdots, (x_{n0}, x_{n2}, \cdots, x_{nn-1}, b)$$

are the vertices of Euclidean n-simplices $\overline{v_0 v_1 \cdots v_n}$ and $\overline{w_0 w_1 \cdots w_n}$ in E_a^n and E_b^n respectively.

Now by considering the complexes

$$\sum_{j=0}^{n} \overline{u_0 u_1 \cdots u_j v_j \cdots v_n} \quad \text{and} \quad \sum_{j=0}^{n} \overline{w_0 w_1 \cdots w_j u_j \cdots u_n}$$

(these complexes are called prisms) associated with the simplex $\overline{u_0 u_1 \cdots u_n}$ and taking the union of all $(n+1)$-simplices in these prisms, we obtain an $(n+1)$-dimensional cell complex K_{n+1} which contains S and a fortiori contains $\overline{\omega_{n+1}} \cup \left[\Phi_{n+1}(\overline{\omega}_{n+1}) \right]$

Now we obtain a simplicial subdivision of K_{n+1} and define a simplicial mapping f_{n+1} relative to this simplicial subdivision which is a sufficiently good approximation of Φ_{n+1} so that its topological degree at b is the same as the topological degree of Φ_{n+1} at b. The mapping f_{n+1} is a

layer mapping relative to E^n and the mapping $f_{n+1} / E^n = f_n$ has the same topological degree at b as Φ_n has at b. We then show that the topological degrees of f_{n+1} and f_n are the same.

In order to obtain the simplicial subdivision of K_{n+1}, we consider the set $\{E^n_\mu\}$ of n-dimensional hyperplanes in E^{n+1} which we parallel to E^n and which are at distance

$$\mu\,\varepsilon/4$$

for E^n where μ is an integer such that $\mu = 0, 1, 2, \cdots 4m$. Let E_1^n and $E_1^{n'}$ be the two planes which are at distance $\varepsilon/4$ from E^n. In the same way as for the planes E^n, E_a^n, and E_b^n, we define the prisms

$$\sum_{j=0}^{n} \overline{u_0 u_1 \cdots u_j v_j \cdots v_n}$$

and

$$\sum_{j=0}^{n} \overline{w_0 w_1 \cdots w_j^* u_j u_{j+1} \cdots u_n}$$

for each simplex $\overline{u_0 u_1 \cdots u_n}$ in Z^n where $\overline{v_0 v_1 \cdots v_n}$ and $\overline{w_0 w_1 \cdots w_n}$ are the corresponding cimplices in E_1^n and $E_1^{n'}$ respectively.

If E_p^n is one of the hyperplanes in the set $\{E^n_\mu\}$, then in the same way, we construct the prisms which have n-dimensional faces in E_p^n and in $E_{p_2}^n$, the two hyperplanes in the set $\{E^n_\mu\}$ which are at distance $\varepsilon/4$ from E_p^n if these prisms have not already been constructed. Carrying out this construction for each E_p^n, we obtain a simplicial subdivision Z^{n+1} of K_{n+1}. Thus we obtain subdivisions of subsets of E^n and E^{n+1}, called Z^n and Z^{n+1} respectively such that both simplicial subdivisions consist entirely of simplices whose diameters are less than $\varepsilon/2$.

Now by assumption B), we deal with a continuous mapping Φ_{n+1} which is a "layer mapping with respect to E^n," i.e.,

$$\Phi_{n+1} = i + s$$

where i is the identity mapping and if $x \in E^{n+1}$, then

$$\rho(x) \in E^n$$

Thus all points of E^{n+1} are moved parallel to E^n by the mapping

We prove the theorem for the case in which ω_{n+1} is a polyhedron which is the union of a finite numbe of the simplices of the subdivision Z^{n+1} of a subset of E^{n+1} described above. The proof in the general case is then a direct consequence of Definition 2.1 in Chapter 2.

In order to determine $[\Phi_{n+1}, \omega_{n+1}, b]$, we deal with a simplicial approximation f_{n+1}, relative to Z^{n+1}, of Φ_{n+1}. The existance of a simplicial approxiamtion f_{n+1} relative to Z^{n+1} such that

$$(2) \qquad \max_{x \in \overline{\omega}_{n+1}} \| \Phi_{n+1}(x) - f_{n+1}(x) \| < \varepsilon/2$$

and the fact that f_{n+1} is a layer mapping with respect to E^n because Φ_{n+1} is a layer mapping with respect to E^n follows from A-H, p. 319, Satz II and Satz III.

Since f_{n+1} is a layer mapping with respect to E^n, then

$$f_{n+1} / E_n = f_n$$

is a sinplicial mapping relative to Z^n. Moreover, it follows from (2) that:

$$(3) \qquad \max_{x \in \overline{\omega_n}} \| \Phi_n(x) - f_n(x) \| < \varepsilon/2$$

Hence by inequality (1):

$$\max_{x \in \overline{\omega_n}} \| \Phi_n(x) - f_n(x) \| < \frac{\rho[\Phi_n(\omega_n'), b]}{2}$$

Hence by Rouché's Theorem (Theorem 2.12, Chapter 2)

$$(4) \qquad [f_n, \omega_n, b] = [\Phi_n, \omega_n, b]$$

Now we compute $[f_n, \omega_n, b]$. If σ_i^n is an n-simplex of E^n containing a point x_i such that $f_n(x_i) = b_1$ then

$$(5) \qquad [f_n, \omega_n, b] = \sum_{i=1}^{n} (\operatorname{Sign} \sigma_i^n)(1)$$

where $(\text{sign } \sigma_i^n)$ is the sign of the determinant of the linear mapping f_n / σ_i^n of σ_i^n onto the simplex containing b. Since σ_i^n is a face of the (n+1)-simplex σ_i^{n+1} (The definition of Z^n and Z^{n+1} is such that every σ_i^n in Z^n is the face of some (n+1)-simplex in Z^{n+1}.), then by definition $f_n / \sigma_i^n = f_{n+1} / \sigma_i^{n+1}$. Since f_{n+1} is a layer mapping with respect to E^n, and if $(x_1, x_2, \cdots, x_n)$, $(x_1, x_2, \cdots, x_n, x_{n+1})$ are coordinates for the vector space associated with E^n and E^{n+1} respectively, then the linear mapping f_{n+1} / σ_i^{n+1} can be described as follows:

$$x_1 = x_1 + \sum_{j=1}^{n} a_{1j} x_j$$

$$x_2 = x_2 + \sum_{j=1}^{n} a_{2j} x_j$$

$$\cdots \quad \cdots$$

$$x_n = x_n + \sum_{j=1}^{n} a_{nj} x_j$$

$$x_{n+1} = x_{n+1}$$

The determinant of this mapping is

$$D_{n+1}^{i} = \begin{vmatrix} a_{11}+1 & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22}+1 & \cdots & a_{2n} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn+1} & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{vmatrix}$$

Now the mapping $f_{n+1} / \sigma_i^n = f_n / \sigma_i^n$ can be described as:

$$x_1 = x_1 + \sum_{j=1}^{n} a_{1j} x_j$$

$$x_2 = x_2 + \sum_{j=1}^{n} a_{2j} x_j$$

$$x_n = x_n + \sum_{j=1}^{n} a_{nj} x_j$$

The determinant of this mapping is:

$$D_n^i = \begin{vmatrix} a_{11}+1 & a_{12} & a_{13} & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22}+1 & a_{23} & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdot & \cdot & a_{nn+1} \end{vmatrix}$$

Hence

(6) $$(\operatorname{sign} \sigma_i^n) = (\operatorname{sign} D_n^i)$$

It follows from (4), (5), and (6) that

(7) $$[\Phi_n, \omega_n, b] = \sum_{i=1}^{n} (\operatorname{sign} D_n^i)(1)$$

Now we compute $[\Phi_{n+1}, \omega_{n+1}, b]$. From the theory of topological degree, it is known that there exists a spherical neighborhood N of b of radius Such that if $c \in N$, then

(8) $$[\Phi_{n+1}, \omega_{n+1}, b] = [\Phi_{n+1}, \omega_{n+1}, c]$$

In the work which follows, we choose ε_1 so that

$$\varepsilon_1 < \varepsilon/4$$

We consider a point c which is in N and which is contained in the interior of some simplex of the subdivision Z^{n+1} of a subset of E^{n+1} which has the n-simplex containing b as a face. The point c is defined as follows: suppose the point b is contained in the n-simplex

$$\overline{u_0 u_1 \cdots u_n}$$

Then there are two (n+1)-simplices

$$\overline{u_0 u_1 \cdots u_n v_n} \qquad \overline{w_0 u_0 u_1 \cdots u_n}$$

such that the simplex $\overline{u_0 u_1 \cdots u_n}$ is a face of each of these simplices. This follows from the definition of the simplicial subdivision Z^{n+1}.

Suppose the point b in the interior of $\overline{u_0 u_1 \cdots u_n}$ can be written:

$$b = \sum_{i=0}^{n} \eta_i u_i$$

Since b is in the interior of $\overline{u_0 u_1 \cdots u_n}$, $\eta_i > 0$ for $i = 0, 1, 2, \cdots, n$ and $\sum_{i=0}^{n} = 1$. Then the point c is defined as:

$$c = \sum_{i=0}^{n} \frac{\eta_i u_i}{\sum_{i=0}^{n} \eta_i + \eta_n'} + \frac{\eta_n'}{\sum_{i=0}^{n} \eta_i + \eta_n'} v_n$$

where $\eta_n' > 0$ and η_n' is sufficiently small so that

$$\rho(b, c) < \varepsilon.$$

Since $\eta_n' > 0$ and $\eta_i > 0$ for $i = 0, 1, 2, \cdots, n$, the point c is in the interior of the simplex

$$\overline{u_0 u_1 u_2 \cdots u_n v_n}$$

Lemma A4. If the n-simplex $\sigma^n = \overline{u_0 u_1 \cdots u_n}$ is such that the point b is in the interior of $f_n(\sigma^n)$, then the point c is in the interior of where $\sigma^{n+1} = \overline{u_0 u_1 \cdots u_n v_n}$ and conversely.

Proof: If b is contained in the interior of $f_n(\sigma^n)$, then

$$b = \sum_{i=0}^{n} \eta_i f_n(u_i)$$

where $\eta_i > 0$ for $i = 0, 1, 2, \cdots, n$. Using the definition given above, we wish to locate the point c. From the definition, the point c is in the (n+1)-simplex having the n-simplex

$$\overline{f_n(u_0) f_n(u_1) \cdots f_n(u_n)}$$

as a face. Since f_{n+1} is a layer mapping, it follows that this (n+1)-simplex must be

$$\overline{f_n(u_0) f_n(u_1) \cdots f_n(u_n) f_{n+1}(v_n)}$$

Hence

$$c = \sum_{i=0}^{n} \frac{\eta_i f_n(u_i)}{\sum_{i=0}^{n} \eta_i + \eta_n'} + \frac{\eta_n'}{\sum_{i=0}^{n} \eta_i + \eta_n'} f_{n+1}(v_n)$$

where $\eta_n' > 0$ and η_n' is chosen so that

$$\rho(b,c) < \varepsilon_1$$

Hence c is in the interior of $f_{n+1}(\sigma^{n+1})$

Conversely, if c is in the interior of $f_{n+1}(\sigma^{n+1})$, then

$$c = \sum_{i=0}^{n} \eta_i f(u_i) + \eta_n' f(v_n)$$

where $\eta_i > 0$ for $i = 0, 1, 2, \cdots, n$ and $\eta_n' > 0$. Also $\sum_{i=0}^{n} \eta_i + \eta_n' = 1$

Then by the definition of c and b,

$$b = \sum_{i=0}^{n} \eta_i'' f(u_i)$$

(9)

where η_i'' for $i = 0, 1, 2, \cdots n$, $\sum_{i=0}^{n} \eta_i'' = 1$ and

$$\frac{\eta_i''}{\sum_{\lambda=0}^{n} \eta_\lambda'' + \eta_n'} = \eta_i \qquad \text{for } i = 0, 1, 2, \cdots, n$$

From equation (9), it is clear then that b is in the interior of

qed.

Now we compute $[\Phi_{n+1}, \omega_{n+1}, c]$. First

$$[\Phi_{n+1}, \omega_{n+1}, c] = [f_{n+1}, \omega_{n+1}, c] \tag{10}$$

This follows from inequality (2), the fact that

$$\| \Phi_{n+1}(x) - c \| > \tfrac{3}{4} \varepsilon$$

for $x \in \omega_{n+1}'$ and by application of Rouché's Theorem. If c is in the interior of $f_{n+1}(\sigma_i^{n+1})$, then the n-face σ_i^n of σ_i^{n+1} which is in E^n (Such an n-face exists by definition of c and because f_{n+1} is a layer mapping). is such that b is in the interior of $f_n(\sigma_i^n)$. This follows from Lemma A4. Thus

$$[\Phi_{n+1}, \omega_{n+1}, c] = [f_{n+1}, \omega_{n+1}, c] = \sum_{i=1}^{n} (\text{sign } \sigma_i^{n+1})(1) \tag{11}$$

But inspection of the determinants D_n^i and D_{n+1}^i shows that

$$\text{sign}\ (\sigma_i^{n+1}) = \text{sign}\ (D_{n+1}^i) = (-1)^{2n}(\text{sign}\ D_n^i) = (\text{sign}\ D_n^i)$$

Hence

$$[f_{n+1}, \omega_{n+1}, c] = \sum_{i=1}^{n} (\text{sign}\ D_n^i)(1) \tag{12}$$

Hence from (7), (8), (11), and (12), it follows that:

$$[\Phi_{n+1}, \omega_{n+1}, b] = [\Phi_n, \omega_n, b] \tag{13}$$

Thus the proof of the theorem is complete.

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